Discrete potential theory

based on the lectures by Nicola Arcozzi

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Contents

Chapter 0

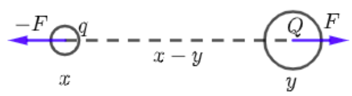
Introduction to potential theory

0.1 Basic concepts

Let q and Q be two point charges with coordinates x and y respectively and suppose $x \neq y$, so that their euclidean distance, given by $|x-y|$, is positive. By Coulomb's law, the electrostatic force between q and Q is given by

$$F = kqQ\frac{x-y}{|x-y|^3},$$

(see figure 1) where $k = \frac{1}{4\pi\varepsilon_0}$ is Coulomb's constant.



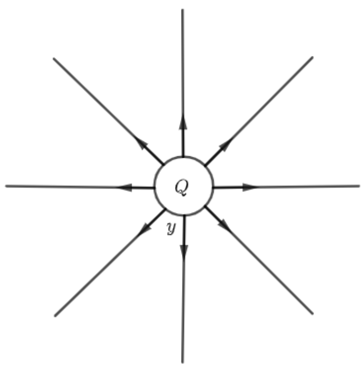


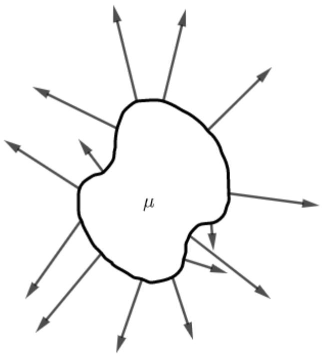
Figure 1: The electrostatic force between q and Q (on the left).

Figure 2: The electrostatic field generated by $Q > 0$ (on the right).

Then, the electrostatic field generated by the point charge Q is defined as the function

$$E(x) = kQ\frac{x-y}{|x-y|^3}$$

(see figure 2).



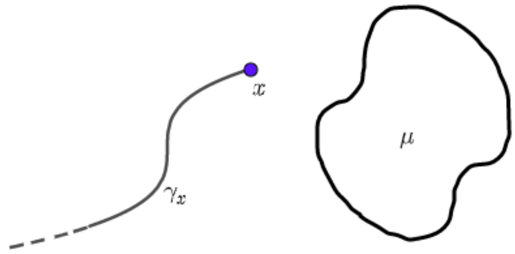


Figure 3: The electrostatic field generated by a distribution of charges $\mu > 0$ (on the left).

Figure 4: Physical interpretation of the potential V^μ (on the right).

The field generated by a finite number of charges, say $Q_1, \ldots, Q_N$ ($N > 1$) at positions $y_1, \ldots, y_N$ respectively, is the sum of the electrostatic fields generated by each single charge:

$$E(x) = k \sum_{j=1}^{N} Q_j \frac{x - y_j}{|x - y_j|^3}.$$

More generally, if μ is a distribution of charges (a compactly supported signed measure on $\mathbb{R}^3$), then the electrostatic field generated by μ is given by

$$E^\mu(x) = k \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} d\mu(y).$$

(see figure 3).

Since $E(x)$ is radial, it is conservative, with $-\nabla \left(\frac{1}{|x|}\right) = \frac{x}{|x|^3}$. Hence, the electrostatic field generated by the distribution of charges μ is the gradient of the **potential**: $E^\mu = -\nabla V^\mu$, with

$$V^\mu(x) = k \int_{\mathbb{R}^3} \frac{d\mu(y)}{|x - y|}.$$

The definition of potential has a physical interpretation: for $x \in \mathbb{R}^3$ fixed, consider a smooth curve γ_x joining ∞ to x. Then, $V^\mu(x)$ is the work required to move a charge $q = +1$ from ∞ to x against the electrostatic field generated by μ under the assumption $V^\mu(\infty) := \lim_{|x| \to +\infty} V^\mu(x) = 0$ (see figure 4):

$$V^\mu(x) = -\int_{\gamma_x} E^\mu \cdot dy.$$

Finally, we focus on the energy $\mathcal{E}$ stored in a distribution of charges. Consider N charges $Q_1, \ldots, Q_N$ at positions $x_1, \ldots, x_N$ respectively. The energy of that distribution is that needed to arrange the configuration moving the charges one by one from ∞ to their position. So,

$$\frac{1}{2}\mathcal{E} = -Q_2 \int_{\gamma_{x_2}} E^{Q_1}(y)\cdot dy - Q_3 \int_{\gamma_{x_3}} E^{Q_1Q_2}(y)\cdot dy - \ldots - Q_N \int_{\gamma_{x_N}} E^{Q_1\ldots Q_{N-1}}(y)\cdot dy =$$
$$= V^{Q_1}(x_2) + \ldots + V^{Q_1\ldots Q_{N-1}}(x_N) = k \sum_{1\le j<l\le N} \frac{Q_jQ_l}{|x_j - x_l|} = \frac{1}{2}k\sum_{j\neq l}\frac{Q_jQ_l}{|x_j - x_l|}.$$

In the case of a continuous distribution of charges μ,

$$\mathcal{E}(\mu) := k\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{|x-y|}d\mu(x)d\mu(y).$$

0.1.1 Some applications to calculus

In this section, we provide an evidence of Poisson's equations. Let V^μ be the potential of the electrostatic field generated by the distribution of charges μ supported on a compact set $A \subset \mathbb{R}^3$ having smooth boundary [1], then V^μ is characterized by

$$\begin{cases} \Delta V^\mu = -\frac{\mu}{\varepsilon_0}, \\ V^\mu(\infty) = 0, \end{cases} \tag{P}$$

where the equation $\Delta V^\mu = -\frac{\mu}{\varepsilon_0}$ must be intended in some distributional sense.

Recall that, by divergence theorem, if $U : \bar{A} \to \mathbb{R}^3$ is a field, then

$$\int_{\partial A} U\cdot \hat{n} d\sigma = \int_A div(U)dA, \tag{1}$$

where $\hat{n}$ is the outward pointing unit normal vector on ∂A, $d\sigma$ is the surface measure on ∂A, $div(U) = \nabla\cdot U = \frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2} + \frac{\partial U}{\partial x_2}$ and dA is the volume measure on A. If $0 \notin \bar{A}$, then applying (1) to $U(x) = \nabla\left(\frac{1}{|x|}\right) = -\frac{x}{|x|^3}$ and using the fact that $div(\nabla U) = \Delta U$, we have

$$\int_{\partial A}\nabla\left(\frac{1}{|x|}\right)\cdot\hat{n}d\sigma = \int_A \Delta\left(\frac{1}{|x|}\right)dx_1dx_2dx_3 = 0 \tag{2}$$

since $\Delta\left(\frac{1}{|x|}\right) = 0$ on $\mathbb{R}^2\setminus\{0\}$. If $0 \in A$, then we consider a ball $B_\varepsilon = B(0,\varepsilon)$ and observe that the same argument above leads to

$$0 = \int_{A\setminus B_\varepsilon}\Delta\left(\frac{1}{|x|}\right)dx_1dx_2dx_3 \underset{(1)}{=} \int_{\partial A}\nabla\left(\frac{1}{|x|}\right)\cdot\hat{n}d\sigma + \int_{\partial B_\varepsilon}\nabla\left(\frac{1}{|x|}\right)\cdot\hat{n}d\sigma =$$
$$\underset{\substack{\text{polar}\\\text{coord.}}}{=} \int_{\partial A}\nabla\left(\frac{1}{|x|}\right)\cdot\hat{n}d\sigma + \frac{1}{\varepsilon^2}4\pi\varepsilon^2 = 4\pi + \int_{\partial A}\nabla\left(\frac{1}{|x|}\right)\cdot\hat{n}d\sigma,$$

[1] Even if this theorem can be further generalized to situations in which the boundary is not necessarily smooth.

so that

$$\int_{\partial A} \nabla \left(\frac{1}{|x|}\right) \cdot \hat{n} d\sigma = -4\pi. \tag{3}$$

(2) and (2.2) together give

$$\int_{\partial A} \nabla \left(\frac{1}{|x|}\right) \cdot \hat{n} d\sigma = \begin{cases} 0 & \text{if } 0 \notin \bar{A}, \\ -4\pi & \text{if } 0 \in A \end{cases} =: -4\pi\delta_0(A),$$

in some distributional sense (so that we don't care of what happens if $0 \in \partial A$). Using Gauss theorem again:

$$\int_A \Delta \left(\frac{1}{|x|}\right) dA = \int_{\partial A} \nabla \left(\frac{1}{|x|}\right) \cdot \hat{n} d\sigma = -4\pi\delta_0(A),$$

so that (using the identity $-4\pi\delta_0(A) = -4\pi \int_A \delta_0 dA$ that must be interpreted in a distributional sense), one has

$$\Delta \left(\frac{1}{|x|}\right) = -4\pi\delta_0(x) \qquad \Longrightarrow \qquad \Delta \left(\frac{k}{|x|}\right) = -\frac{\delta_0}{\varepsilon_0}.$$

Writing $\mu = \mu * \delta_0 = \delta_0 * \mu$ (in a distributional sense),

$$V^\mu(x) = k \int_{\mathbb{R}^3} \frac{1}{|x-y|} d\mu(x) = k \left(\mu * \frac{1}{|\cdot|}\right)(x),$$

so that

$$\Delta(V^\mu) = \mu * \left(\Delta \left(\frac{1}{|\cdot|}\right)\right) = \mu * (-4\pi k \delta_0) = -4\pi k \mu * \delta_0 = -4\pi k \mu = -\frac{\mu}{\varepsilon_0}.$$

0.1.2 Getting rid of vectors and derivatives

We want to dispense with vectors and derivatives. First, observe that the energy integral $\mathcal{E}(\mu) = k \iint \frac{1}{|x-y|} d\mu(x) d\mu(y)$ can be written in several different ways:

$$\begin{aligned}
\mathcal{E}(\mu) = k \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{d\mu(x) d\mu(y)}{|x-y|} &= \int_{\mathbb{R}^3} \underbrace{k \int_{\mathbb{R}^3} \frac{d\mu(y)}{|x-y|}}_{= V^\mu(x)} d\mu(x) = \int_{\mathbb{R}^3} V^\mu(x) d\mu(x) = \\
&= -\varepsilon_0 \int_{\mathbb{R}^3} V^\mu(x) \Delta V^\mu(x) dx = \varepsilon_0 \int_{\mathbb{R}^3} |\nabla V^\mu(x)|^2 dx = \varepsilon_0 \int_{\mathbb{R}^3} |E^\mu(x)|^2 dx = \\
&= \varepsilon_0 \int_{\mathbb{R}^3} |(-\Delta)^{1/2} V^\mu(x)|^2 dx,
\end{aligned}$$

where the fractional powers of Δ are defined via the Fourier transform as follows: recall that the Fourier transform is defined for a function $\varphi \in \mathcal{S}(\mathbb{R}^3) := \Big\{ g \in \mathcal{C}^\infty(\mathbb{R}^3) \; : \; \sup_{x \in \mathbb{R}^3} |x^\alpha D^\beta g(x)| < \infty \;\; \forall \alpha, \beta \in \mathbb{N}^n \Big\}$ as

$$\hat{\varphi}(\xi) := \int_{\mathbb{R}^3} \varphi(x) e^{-2\pi i \xi \cdot x} dx,$$

so that for any *tempered distribution* $u \in \mathcal{S}'$ it is well defined another tempered distribution $\hat{u} = \mathcal{F}(u)$ characterized by $\hat{u}(\varphi) = u(\hat{\varphi})$ $(\varphi \in \mathcal{S})$. The operator $\mathcal{F} : u \in \mathcal{S}' \to \hat{u} \in \mathcal{S}'$ is a surjective isomorphism and it is still called the **Fourier transform**. Now, for $s \in \mathbb{R}$, define

$$(-\Delta)^s u := \mathcal{F}^{-1}((4\pi^2|\cdot|^2)^s \hat{u}),$$

so we get the definition of $(-\Delta)^{1/2}$ by choosing $s = 1/2$.

If we define, for $f : \mathbb{R}^3 \to \mathbb{C}$, the **Riesz potential** I_1 of f as

$$I_1 f(x) := \frac{1}{2\pi^2} \int_{\mathbb{R}^3} f(y) \frac{d\mu(y)}{|x-y|^2}, \tag{4}$$

then it can be proved that $(-\Delta)^{1/2} I_1 f = f$, so that

$$I_1 f = (-\Delta)^{-1/2} f,$$

where $(-\Delta)^{-1/2}$ is also the inverse of the operator $(-\Delta)^{1/2}$. For a measure μ, (4) becomes

$$I_1 \mu := \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \frac{d\mu(y)}{|x-y|^2}.$$

Observe that most of what we have defined so far can be written in terms of μ and I_1 alone: for instance,

$$V^\mu(x) = \frac{1}{\varepsilon_0}(-\Delta)^{-1}\mu(x) = \frac{1}{\varepsilon_0} I_1 I_1 \mu(x)$$

and

$$\mathcal{E}(\mu) = \varepsilon_0 \int_{\mathbb{R}^3} |(-\Delta)^{1/2} V^\mu(x)|^2 dx = \frac{1}{\varepsilon_0} \int_{\mathbb{R}^3} |I_1 \mu(x)|^2 dx.$$

0.1.3 Capacity of conductors

A conductor is a set where charges are free to move. In most cases, it is a compact set, not necessarily connected, although we fictionally assume that charges are free to pass from one connected component to the others.

Let $M > 0$ be the total amount of charge on a conductor C, set $\|\mu\| := \mu(C) = M$, where, for simplicity, μ is assumed to be a positive measure on $\mathbb{R}^3$. The charges in C will start moving in C under Coulomb's force, until they reach an equilibrium configuration μ^c, which is proven to exist and to be unique (see figure 5).

For such a distribution, one has

$$E^{\mu^c} = 0 \quad on \quad supp(\mu^c),$$

otherwise the charges would still move under the action of the electrostatic force. It is easy to see that, even if C is not connected, V^{μ^c} must be constant on $supp(\mu^c)$. Actually, V^{μ^c} must be constant on the whole of C.

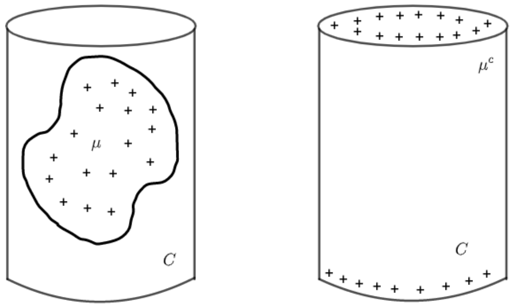


Figure 5: The equilibrium configuration.

The **capacity** of C is defined as

$$Cap(C) := \max\Big\{\ \|\mu\|\ :\ V^{\mu^c} = 1\ on\ C\Big\},$$

the maximum amount of charge M for which $V^{\mu^c} = 1$ on $supp(\mu^c)$.

Gauss proved that the equilibrium distribution possesses simultaneously a number of properties, some of them of extremal importance:

(1) $V^{\mu^c} \leq 1$ on $\mathbb{R}^3$;

(2) $\mathcal{E}(\mu^c) \leq \mathcal{E}(\mu)$.

Chapter 1

Axiomatic non-linear potential theory

1.1 Kernels, potentials and capacity

Let $p \in (1, +\infty)$, X be a locally compact metric space and (M, m) be a measure space.

Definition 1.1.1 (Lower semicontinuous function)**.** Recall that a function $f : X \to [0, +\infty]$ is **lower semicontinuous** (LSC for short) if

$$\liminf_{x \to x_0} f(x) \geq f(x_0) \qquad \forall x_0 \in X.$$

Example 1.1.2. Any characteristic function $f(x) = \chi_{(a,b)}(x)$ is LSC. See figure 1.1.

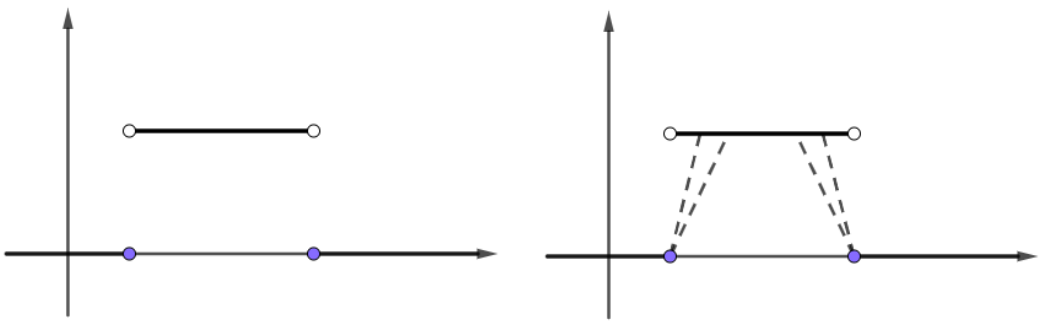


Figure 1.1: An example of LSC function (on the left).

Figure 1.2: LSC functions can be approximated pointwise from below by compactly supported continuous functions (on the right).

We will use the following characterizations of LSC functions:

Proposition 1.1.3. *(i) A function $h \geq 0$ is LSC on X if and only if there exists a sequence $\{h_j\}_j \subset \mathcal{C}_c^0(X) = \{f : X \to \mathbb{R} \ : \ f$ is continuous and $supp(f)$ is compact$\}$ such that $h_j \nearrow h$ as $j \to +\infty$ pointwise.*

(ii) h is LSC on X if and only if $\{x \in X \ : \ h(x) > \alpha\}$ is open for all $\alpha \in \mathbb{R}$.

(iii) If h is LSC on a compact set K, then h attains its minimum on K.

Definition 1.1.4 (Kernel function)**.** A **kernel** on $X \times M$ is a function $K : X \times M \to [0, +\infty]$ such that

(a) for all $x \in X$, the function $K(x, \cdot) : M \to [0, +\infty]$ is measurable;

(b) for all $\alpha \in M$, the function $K(\cdot, \alpha) : X \to [0, +\infty]$ is LSC on X.

Definition 1.1.5 (K and $\check{K}$)**.** Let $\mu \geq 0$ be a Borel measure on X and $f \geq 0$ be an $m-$measurable function on M. We define

$$Kf(x) := \int_M f(\alpha)K(x, \alpha)dm(\alpha) \quad \text{and} \quad \check{K}\mu(\alpha) := \int_X K(x, \alpha)d\mu(x).$$

Remark 1.1.6. 1. Kf and $\check{K}\mu$ are everywhere well defined up to allow them to take $+\infty$ as a value.

2. Both K and $\check{K}$ are generalizations of the Riesz potentials. In fact,

$$I_1 f(x) = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} f(y) \frac{1}{|x-y|^2} d\mu(y)$$

and

$$I_1 \mu(x) = \frac{1}{2\pi^2} \int_{\mathbb{R}^3} \frac{1}{|x-y|^2} d\mu(y),$$

that is, $X = M = \mathbb{R}^3$ and $K(x, y) = \frac{1}{|x-y|^2}$.

We also define the **energy** as

$$\mathcal{E}(\mu) := \int_M [\check{K}\mu(\alpha)]^{p'} dm(\alpha),$$

where p' is the conjugate exponent of p (that is, the positive number characterized by $\frac{1}{p} + \frac{1}{p'} = 1$), and the **potential** as

$$V^\mu(x) = (K(\check{K}\mu))^{p-1}(x).$$

Remark 1.1.7. The potential V^μ is non-linear.

Definition 1.1.8 (Capacity)**.** Let $E \subseteq X$. The **capacity** of E is defined as

$$Cap(E) := \inf\Big\{ \|f\|^p_{L^p(dm)} \ : \ f \geq 0, \ Kf \geq 1 \ on \ E \Big\}.$$

Proposition 1.1.9. *(i) $Cap(\varnothing) = 0$;*

(ii) $A \subseteq B \Longrightarrow Cap(A) \leq Cap(B)$;

(iii) $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{P}(X) \Longrightarrow Cap\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} Cap(E_k)$.

Proof. Item (iii) is the only non-trivial point of the assertion. Let $\varepsilon > 0$. For all $k = 1, 2, \ldots$, let f_k be a function such that $f_k \geq 0$ and $K f_k \geq 1$ on E_k, with

$$\|f_k\|_{L^p(dm)}^p \leq Cap(E_k) + 2^{-k}\varepsilon.$$

The function $f := \max\{f_k\}_{k=1}^{\infty}$ is clearly m-measurable and non-negative. Also, for all k, $Kf \geq Kf_k \geq 1$ on E_k, so that $Kf \geq 1$ on $\bigcup_k E_k$. Finally, using the fact that $\max\{a_k^p\}_k = (\max\{a_k\}_k)^p \leq (\sum_k a_k)^p$ $(p > 1)$ and the monotone convergence theorem, we get:

$$Cap\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \int_M f^p dm \leq \int_M \sum_{k=1}^{\infty} f_k^p dm = \sum_{k=1}^{\infty} \int_M f_k^p dm \leq \sum_{K=1}^{\infty} Cap(E_k) + \varepsilon.$$

□

1.2 Tilings of rectangles and finite rooted subdyadic trees

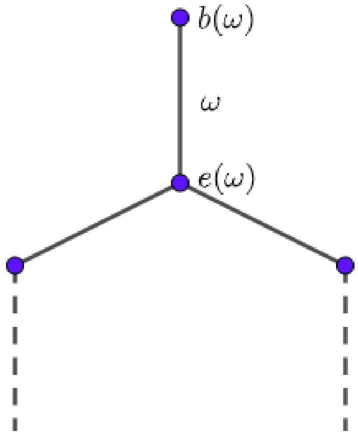
In this section, we deal with the model of finite trees. We consider a rooted finite subdyadic tree $T = (V(T), E(T))$. This means that the set $E(T)$ of its edges has a distinguished **root-edge** ω, one of which endpoints, the pre-root vertex $b(\omega)$ is endpoint ω alone (see figure 1.3).

Each vertex $x \in V(T)$ is endpoint of no more than three edges and it is a **leaf** if $x \neq b(\alpha)$ for all $\alpha \in E(T)$ (see figure 1.4). The set of the leaves of a finite tree T is called the **boundary of** T and it is denoted with ∂T.

The fact that T has a root edge ω and a pre-root vertex $b(\omega)$ induces a partial order on vertices and edges. If $\alpha \in E(T)$, we denote with $b(\alpha)$ the upper vertex (which is the one closest to $b(\omega)$) and with $e(\alpha)$ the lower vertex. With this notation, for all α, one has $e(\alpha) \leq \alpha \leq b(\alpha)$ (see figure 1.5).

Remark 1.2.1. An alternative point of view that allows one to define rooted finite subdyadic trees is that of considering them as subtrees of a rooted dyadic tree $T_2 = (V(T_2), E(T_2))$, which is a rooted tree such that each vertex, except $b(\omega)$, is the endpoint of exactly 3 edges (see figure 1.6 for an example). Alternatively, any finite subdyadic tree can be seen as the subtree $T = T(F)$ generated by any finite subset $F \subset V(T_2)$. Of course, F can always be assumed to be such that $F = \partial(T(F))$.

Rooted finite subdyadic trees are naturally related to finite square tilings of rectangles, which provide another representation (or model) of them. How do we associate the square tiling of a rectangle to a finite subdyadic rooted tree and vice versa? We see this though an example: consider the finite rooted subdyadic tree of figure 1.7 and



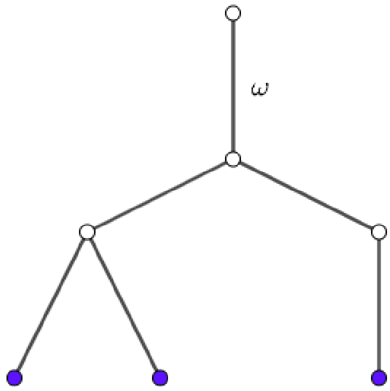


Figure 1.3: The root-edge ω (on the left).

Figure 1.4: The leaves are the colored vertices. Observe that each vertex is endpoint of 1, 2 or 3 edges (on the right).

interpret each of its vertices, except $b(\omega)$, as the center of one of the squares that define the tiling (see figure 1.7). Analogously, any finite square tiling of a rectangle defines naturally a rooted finite subdyadic tree.

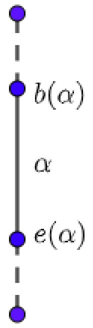
Exercise 1.2.2. *Consider the tiling of figure 1.7 and suppose that the height of the rectangle is* 1. *Calculate the length of the basis and the lengths of the sides of the three lower squares.*

Call a, b and c the lengths of the three lower squares' sides, as in figure 1.8. Then, we have to calculate a, b, c and $a+b+c$. By comparing the lengths of the squares and using the fact that the height of the rectangle is 1, it is clear that

$$\begin{cases} (a+b+c)+a=1, \\ b=c, \\ b+(b+c)=a. \end{cases}$$

This system admits a unique solution: $a=\frac{3}{8}$, $b=c=\frac{1}{8}$, which gives $a+b+c=\frac{5}{8}$.

So, we saw that to any rooted finite subdyadic tree it is possible to associate the square-tiling of a rectangle. If we require these rectangles to have height 1, then to each rooted finite subdyadic tree it is possible to associate a number as well, that is the length of the basis of the rectangle whose square-tiling represents the tree. From now on, when representing a rooted finite subdyadic tree with the square-tiling of a



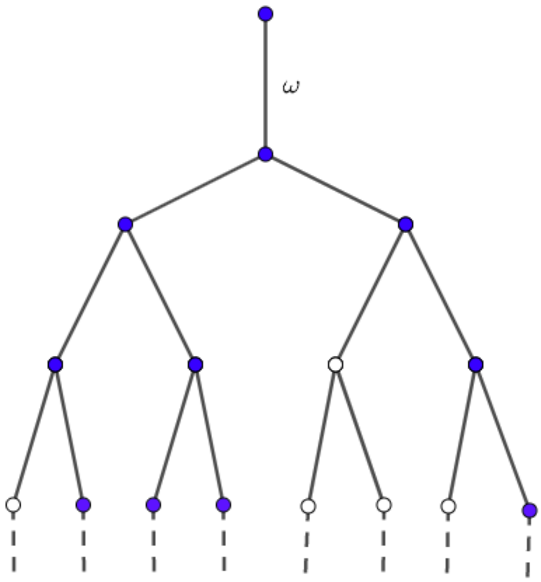


Figure 1.5: $e(\alpha) \leq \alpha \leq b(\alpha)$ (on the left).

Figure 1.6: A rooted finite subdyadic tree as a subtree T of a rooted dyadic tree T_2 (on the right).

rectangle we will always assume that the length of their heights is 1.

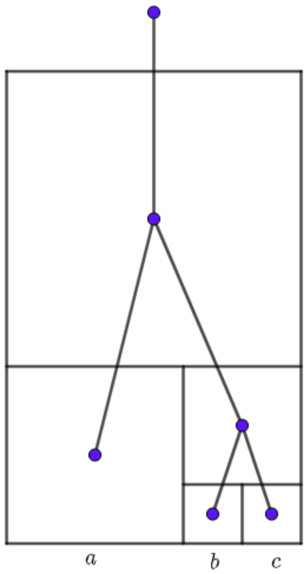
Let T be a finite rooted subdyadic tree with root-edge ω, then

$$Qap_\omega(\partial T)$$

denotes the length of the basis of the rectangle associated to T. Also, when ω is obvious or irrelevant to the context, we write $Qap(\partial T)$ instead of $Qap_\omega(\partial T)$.

Now, we describe a procedure to "unite" two rooted finite subdyadic trees into one: consider two rooted finite subdyadic trees T_+ and T_- and let ω_+ and ω_- be their root-edges respectively, then consider T as the rooted finite subdyadic tree obtained by juxtaposing $b(\omega_+)$ and $b(\omega_-)$ and adding a new root-edge ω, so that $e(\omega) = b(\omega_+) = b(\omega_-)$ (see figure 1.9).

Clearly, $\partial T = \partial T_+ \cup \partial T_-$. We want to find a relation between $Qap_\omega(\partial T)$ and $Qap_{\omega_\pm}(\partial T_\pm)$. For, consider the tilings related to T_+ and T_- respectively. To get the tiling of T, juxtapose the two tilings of T_+ and T_- along their right and left sides respectively, then juxtapose a square on the top (see figure 1.10).
This juxtaposition alone is not enough to get the tiling of the union of the two trees. In fact, a rescaling is needed to turn the height of the rectangle in figure 1.10 equal to 1. Before the rescaling, this rectangle has basis $Qap_{\omega_+}(\partial T_+) + Qap_{\omega_-}(\partial T_-)$ and height



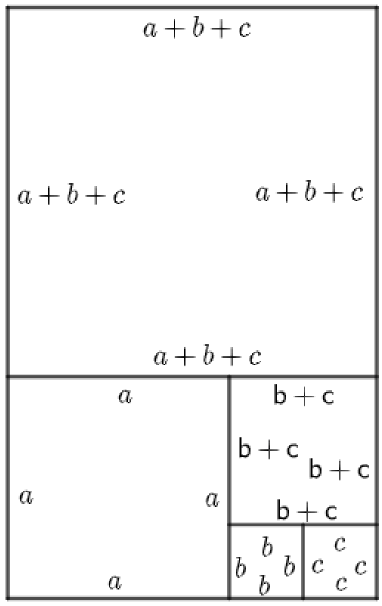


Figure 1.7: From trees to tilings and vice-versa (on the left).

Figure 1.8: The tree of Example 1.2.2 (on the right).

$1+Qap_{\omega_+}(\partial T_+)+Qap_{\omega_-}(\partial T_-)$. After the rescaling the height becomes 1, and the basis $Qap_\omega(\partial T)$ is given by the proportion

$$\frac{Qap_{\omega_+}(\partial T_+)+Qap_{\omega_-}(\partial T_-)}{1+Qap_{\omega_+}(\partial T_+)+Qap_{\omega_-}(\partial T_-)}=\frac{Qap_\omega(\partial T)}{1},$$

that is:

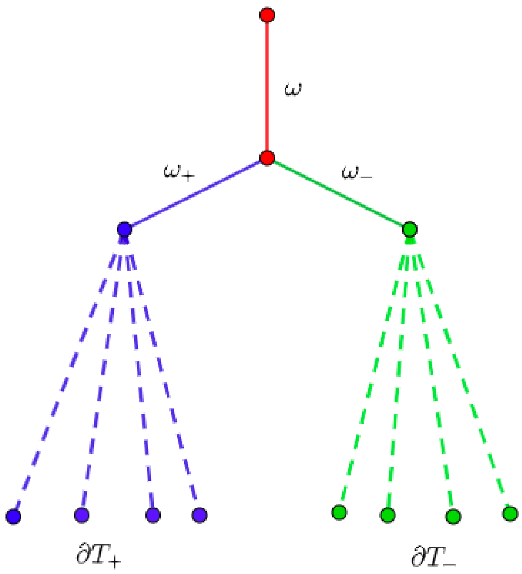
$$Qap_\omega(\partial T)=\frac{Qap_{\omega_+}(\partial T_+)+Qap_{\omega_-}(\partial T_-)}{1+Qap_{\omega_+}(\partial T_+)+Qap_{\omega_-}(\partial T_-)}.$$

Rewriting this expression:

Proposition 1.2.3. *Let T_+ and T_- be two rooted finite subdyadic trees with ω_+ and ω_- as root-edges respectively. Let T be the tree obtained with the procedure described above and let ω be its root-edge. Then,*

$$Qap_\omega(\partial T)=\frac{1}{1+\frac{1}{Qap_{\omega_+}(\partial T_+)+Qap_{\omega_-}(\partial T_-)}}.$$

Finally, we prove that $Qap_\omega(\partial T)=Cap(\partial T)$ up to choosing the metric space X, the measure space M and $p\in(1,+\infty)$ correctly. Of course, $X=\partial T$ which is trivially a metric space. Then, we take $M=E(T)$ with the counting measure. For all $x\in\partial T$



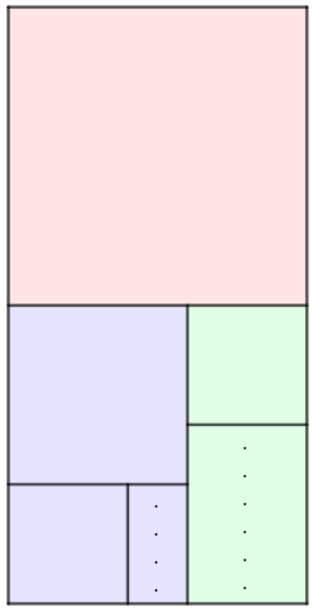


Figure 1.9: How to get a finite rooted subdyadic tree starting by two different ones (on the right).

Figure 1.10: The tiling of the "union" of two finite rooted subdyadic trees in figure 1.9 (on the left).

and all $\alpha \in E(T)$, define

$$K(x,\alpha) := \chi_{[b(\omega),x]}(\alpha) = \begin{cases} 1 & \alpha \in [b(\omega), x], \\ 0 & \text{otherwise}, \end{cases}$$

where $[b(\omega), x]$ is the set of the edges connecting $b(\omega)$ to the leaf x. From now on, we use the notation $o := b(\omega)$.

Take $f : E(T) \to [0, +\infty]$, then

$$Kf(x) = \int_{E(T)} f(\alpha) K(x,\alpha) dm = \sum_{\alpha \in E(T)} f(\alpha)\chi_{[o,x]}(\alpha) = \sum_{\alpha \in [o,x]} f(\alpha),$$

as dm denotes the counting measure on $E(T)$. We use $p = 2$, so that

$$\begin{aligned} Cap(\partial T) &= \inf \Big\{ \|f\|^2_{\ell^2} \ : \ f \geq 0,\ Kf(x) \geq 1 \ for \ x \in \partial T \Big\} = \\ &= \inf \Big\{ \sum_{\alpha \in E(T)} f(\alpha)^2 \ : \ f \geq 0,\ Kf(x) \geq 1 \ for \ x \in \partial T \Big\}. \end{aligned} \tag{1.1}$$

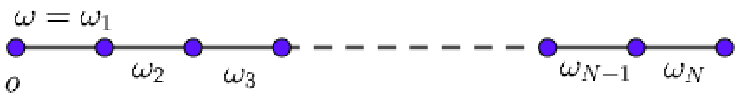


Figure 1.11: The trivial tree.

Example 1.2.4. Consider the trivial tree in figure 1.11, we want to calculate $Cap(\partial T)$ and prove that, in this case, it is equal to $Qap_\omega(\partial T)$. By (1.1), it is clear that

$$Cap(\partial T) = \inf\Big\{ \sum_{\alpha\in E(T)} f(\alpha)^2 \ :\ f \geq 0 \ and \ Kf(x) \geq 1 \ if \ x \in \partial T \Big\} =$$
$$= \inf\Big\{ \sum_{j=1}^{N} f(\omega_j)^2 \ :\ f \geq 0 \ and \ \sum_{j=1}^{N} f(\omega_j) \geq 1 \Big\}.$$

Let $x_j := f(\omega_j)$, we want to find x_j ($j = 1, \dots, N$) such that $F(x_1, \dots, x_N) = \sum_{j=1}^{\infty} x_j^2$ is minimal on the constraint $g(x_1, \dots, x_N) = \sum_{j=1}^{N} x_j \geq 1$ (the case $N = 2$ is represented in figure 1.12). We start observing that the infimum defining $Cap(\partial T)$ is actually a

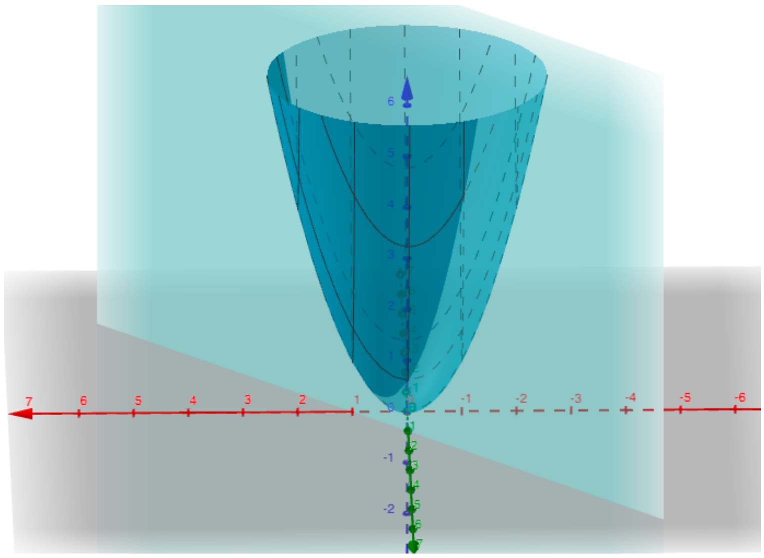


Figure 1.12: The constrained optimization problem of Example 1.2.4 for $N = 2$.

minimum and the minimum is attained on the constraint $g(x_1, \dots, x_N) = 1$. So, we approach the problem using Lagrange multipliers, that is we have to solve

$$\begin{cases} \nabla F(x_1, \dots, x_N) = \lambda \nabla g(x_1, \dots, x_N), \\ g(x_1, \dots, x_N) = 1 \end{cases} \implies \begin{cases} 2x_j = \lambda & \text{for } j = 1, \dots, N, \\ x_1 + \ldots + x_N = 1, \end{cases}$$

which admits the solution: $x_j = f(\omega_j) = \frac{1}{n}$ for all $j = 1, \dots, N$. In particular,

$$Cap(\partial T) = \sum_{j=1}^{N} \frac{1}{N^2} = \frac{1}{N}.$$

Finally, observe that the tiling that corresponds to the trivial tree of figure 1.12 is a rectangle decomposed into N equal squares. Since the height of this rectangle has to be 1, it is obvious that

$$Qap_\omega(\partial T) = \frac{1}{N} = Cap(\partial T).$$

Remark 1.2.5. We conclude this section with several remarks concerning (1.1).

1. The condition $f \geq 0$ can be dropped. In fact, if f does not satisfy it, then replacing f with $\max\{f, 0\}$, which is still a function in ℓ^2, one gets $\sum_{\alpha \in E(T)} f(\alpha)^2$ smaller, while enlarging the value of $\sum_{\alpha \in [o,x]} f(\alpha)$.

2. The infimum is actually a minimum, by Weierstrass theorem.

3. The condition $\sum_{\alpha \in [o,x]} f(\alpha) \geq 1$ can be replaced with the stronger $\sum_{\alpha \in [o,x]} f(\alpha) = 1$. In fact, if $f(\alpha_0) > 0$ for $\alpha_0 \in [o, x]$, then replacing $f(\alpha_0)$ with a smaller value, the sum $\sum_{\alpha \in [o,x]} f(\alpha)^2$ gets smaller. So, if $\sum_{\alpha \in [o,x]} f(\alpha) > 1$, by replacing some of the $f(\alpha)$ with smaller values one gets $\sum_{\alpha \in [o,x]} f(\alpha) = 1$ with smaller $\sum_{\alpha \in [o,x]} f(\alpha)^2$.

4. Clearly, each function f that contributes to $Cap(\partial T)$ is supported on the tree generated by ∂T.

5. Consider an edge α such that $e(\alpha)$ is endpoint of exactly three edges: α, α_+ and α_- as in figure 1.13. If f is a (the, as we shall see) argument of the minimum

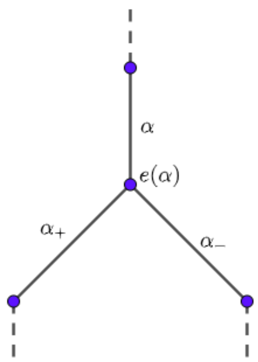


Figure 1.13: Remark 1.2.5 5.

of (1.1), then $f(\alpha) = f(\alpha_+) + f(\alpha_-)$. To see why, suppose to change the values of f only on α, α_+ and α_- leaving $f(\alpha) + f(\alpha_\pm)$ unchanged. Let g be the new function obtained, i.e. for some $t \in \mathbb{R}$,

$$\begin{cases} g(\alpha) = f(\alpha) + t, \\ g(\alpha_\pm) = f(\alpha_\pm) - t, \\ g(\beta) = f(\beta) & \text{for all } \beta \neq \alpha, \alpha_\pm. \end{cases} \tag{1.2}$$

Then,

$$\sum_{\beta \in E(T)} g(\beta)^2 = \sum_{\substack{\beta \in E(T) \\ \beta \neq \alpha, \alpha_+, \alpha_-}} f(\beta)^2 + \underbrace{(f(\alpha) + t)^2 + (f(\alpha_+) - t)^2 + (f(\alpha_-) - t)^2}_{=: \varphi(t)}.$$

Clearly, $\varphi(t)$ has a minimum in $t = \frac{f(\alpha_+) + f(\alpha_-) - f(\alpha)}{3}$. Plugging t into (1.2), we get

$$\begin{cases} g(\alpha) = \frac{2f(\alpha) + f(\alpha_+) + f(\alpha_-)}{3}, \\ g(\alpha_\pm) = \frac{f(\alpha) + 2f(\alpha_\pm) - f(\alpha_\mp)}{3}, \end{cases}$$

which satisfies $g(\alpha_+) + g(\alpha_-) = g(\alpha)$.

6. The extremal f is unique. In fact, if f_1 and f_2 are both extremals for (1.1), then $\frac{f_1 + f_2}{2} \in \ell^2$ and

$$\begin{cases} \sum_{\alpha \in E(T)} f_1(\alpha)^2 = m = \sum_{\alpha \in E(T)} f_2(\alpha)^2, \\ \sum_{\alpha \in [o,x]} f_1(\alpha) = \sum_{\alpha \in [o,x]} f_2(\alpha) = 1 & \text{for all } x \in \partial T. \end{cases}$$

Hence,

$$K\left(\frac{f_1 + f_2}{2}\right)(x) = \sum_{\alpha \in [o,x]} \frac{f_1(\alpha) + f_2(\alpha)}{2} = 1$$

and, by Jensen's inequality

$$\sum_{\alpha \in E(T)} \left(\frac{f_1(\alpha) + f_2(\alpha)}{2}\right)^2 \leq \frac{1}{2}\left(\sum_{\alpha \in E(T)} f_1(\alpha)^2 + \sum_{\alpha \in E(T)} f_2(\alpha)^2\right) = m,$$

which contradicts the minimality of both f_1 and f_2, as Jensen's inequality can be strict by choosing f_1 and f_2 properly.

7. Let f be the argument of the minimum in (1.1). We can think of the minimizer f as a measure on ∂T: if we set

$$\partial S(\alpha) := \left\{ x \in \partial T \ : \ \alpha \in [o, x] \right\},$$

then

$$\mu(\partial S(\alpha)) := f(\alpha) \tag{1.3}$$

defines a measure on ∂T, which is called the **equilibrium measure**.

Theorem 1.2.6. *Let T be a finite rooted subdyadic tree with root-edge ω. Then,*

$$Cap(\partial T) = Qap_\omega(\partial T).$$

Proof. Let f be the extremal in (1.1). By Remark 1.2.5 5., if $x, y \in \partial T$ are two leaves which are endpoints of two edges α_x and α_y such that $b(\alpha_x) = b(\alpha_y) = e(\alpha)$ for some $\alpha \in [o, x] \cap [o, y]$ (see figure 1.14), then, by (1.3) and Remark (1.2.5) item 5.,

$$\mu(\partial S(\alpha)) = \mu(\partial S(\alpha_x)) + \mu(\partial S(\alpha_y)).$$

It is clear, that the basis of the rectangle R whose square-tiling is related to T is given

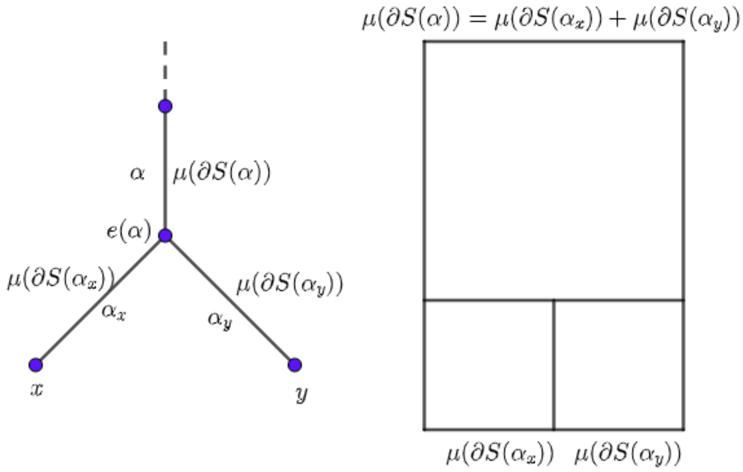


Figure 1.14: The situation in the Proof of Theorem 1.2.6 and the tiling related to that branch of the tree.

by

$$Qap_\omega(\partial T) = \sum_{x \in \partial T} \mu(\partial S(\alpha_x)),$$

where $\alpha_x \in E(T)$ is the unique edge such that $e(\alpha_x) = x$. Also, since the height of the same rectangle is 1, it is clear that the area of R is equal to

$$A(R) = 1 \cdot \sum_{x \in \partial T} \mu(\partial S(\alpha_x)) = \sum_{x \in \partial T} \mu(\partial S(\alpha_x)) = Qap_\omega(\partial T).$$

However, the same area can be calculated by summing together the areas of the squares which give the tiling of R. These areas are exactly $\mu(\partial S(\alpha))^2 = f(\alpha)^2$, which summed together give $Cap(\partial T)$, since f is the minimizer. $\square$

1.3 Topics of functional analysis

1.3.1 The weak-$*$ topology on $\mathcal{M}_+(X)$

Define

$$\mathcal{M}(X) := \Big\{\mu \text{ } \textit{Borel signed measures s.t.} \text{ } \|\mu\| := |\mu|(X) < \infty\Big\},$$

where $|\mu|(X)$ is the **total variation** of μ, that is

$$|\mu|(X) = \sup\Big\{\sum_{j=1}^{\infty}|\mu(E_j)| \; : \; E_j \textit{ is Borel} - \textit{measurable and } X = \bigsqcup_{j=1}^{\infty} E_j\Big\}.$$

It can be proved that $\mu \in \mathcal{M}(X) \to \|\mu\| \in [0, +\infty)$ defines a Banach norm on $\mathcal{M}(X)$.

Set

$$\mathcal{M}_+(X) := \Big\{\mu \text{ } \textit{Borel positive measures s.t.} \text{ } \|\mu\| := \mu(X) < \infty\Big\}$$

and

$$\mathcal{C}_0(X) := \Big\{f : X \to \mathbb{R} \; : \; f \in \mathcal{C}^0(X) \textit{ and } \lim_{x\to+\infty} f(x) = 0\Big\},$$

where $\mathcal{C}^0(X)$ is the space of the continuous functions on X taking values in $\mathbb{R}$ and $\lim_{x\to+\infty} f(x) = 0$ means that for all $\varepsilon > 0$ there exists $K_\varepsilon \subseteq X$ compact such that $|f(x)| < \varepsilon$ if $x \in X \setminus K_\varepsilon$.

A well known result of functional analysis relates $\mathcal{C}_0$ to the topological dual of $\mathcal{M}$.

Theorem 1.3.1 (Riesz representation). *$\mathcal{C}_0(X)^* = \mathcal{M}(X)$ under the duality pairing*

$$\mu(\varphi) := \langle\varphi, \mu\rangle = \int_X \varphi d\mu$$

for all $\varphi \in \mathcal{C}_0(X)$ and for all $\mu \in \mathcal{M}(X)$.

Given a sequence $\{\mu_j\}_j \subset \mathcal{M}(X)$ (resp. $\subset \mathcal{M}_+(X)$), we say that $\{\mu_j\}_j$ converges weakly-$*$ to $\mu \in \mathcal{M}(X)$ (resp. $\in \mathcal{M}_+(X)$) and write $\mu_j \xrightarrow[j\to+\infty]{*} \mu$ if for all $\varphi \in \mathcal{C}_0(X)$, $\lim_{j\to+\infty} \mu_j(\varphi) = \mu(\varphi)$.

We conclude this section with a compactness result involving the topology induced by the weak-$*$ convergence, which happens to coincide with the smaller topology with respect to which all the evaluation linear functionals $\Lambda_\varphi : \mu \in \mathcal{M}(X) \to \mu(\varphi) \in \mathbb{R}$ are continuous [1].

Theorem 1.3.2 (Banach-Alaoglu). *Let V be a Banach space, then the unit ball B_{V^*} of V^* is weak-$*$ compact. This is equivalent to claiming that*

$$\forall\{f_j\}_j \subset B_{V^*}, \quad \exists\{f_{j_k}\}_k \subseteq \{f_j\}_j \textit{ and } \exists f \in V^* \quad \textit{s.t.} \quad f_{j_k} \xrightarrow[k\to+\infty]{*} f.$$

[1] The euclidean topology is the one considered in this work, when no other topology is specified, on $\mathbb{R}^n$ or $\mathbb{C}^n$

1.3.2 Uniform convexity

Recall that a subset C of a vector space X is called **convex** if for all $t \in [0,1]$ and all $x, y \in C$, $tx + (1-t)y \in C$.

Definition 1.3.3 (Uniform convexity)**.** A Banach space $(X, \|\cdot\|)$ is called **uniformly convex** if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ satisfying $\|x\|, \|y\| \geq 1+\delta$ and $\left\|\frac{x+y}{2}\right\| < 1$ one has $\|x-y\| \leq \varepsilon$.

Example 1.3.4. For all $p \in (1, +\infty)$, $L^p(dm)$ is uniformly convex.

Theorem 1.3.5. *Let X be a uniformly convex Banach space. Let $C \subseteq X$ be closed and convex. Then, there exists a unique $c \in C$ such that $\|c\| = \min\{\|x\| \ : \ x \in C\}$.*

Lemma 1.3.6. *Let $(V, \|\cdot\|)$ be a uniformly convex Banach space and $\{x_j\}_j \subset V$ be a sequence such that $\lim_{j\to+\infty} \|x_j\| = 1$ and $\liminf_{j,k\to+\infty} \left\|\frac{x_j+x_k}{2}\right\| \geq 1$. Then, there exists $x_0 \in V$ such that $x_j \xrightarrow[j\to+\infty]{V} x_0$.*

Proof. Fix $\varepsilon > 0$. For $\delta > 0$ to be chosen, there exists $j_0 = j_0(\delta) > 0$ such that for all $j, k \geq j_0$, $\|x_j\| < 1+\delta$ and $\left\|\frac{x_j+x_k}{2}\right\| \geq 1-\delta$. In particular, the sequence $\left\{\frac{x_j}{1-\delta}\right\}_j$ satisfies the axioms of Definition 1.3.3, so that up to choose δ small enough, one has $\|x_j - x_k\| \leq \varepsilon$, that is $\{x_j\}_j$ is a Cauchy sequence of V, which is Banach. This concludes the proof. $\square$

1.4 Back to potential theory

Proposition 1.4.1. *Let $p \in (1, +\infty)$.*

(i) The mapping $x \in X \mapsto Kf(x)$ is LSC on X for all $f \in L^p(dm)$;

(ii) the mapping $\mu \in \mathcal{M}_+(X) \mapsto \check{K}\mu(\alpha)$ is LSC on $\mathcal{M}_+(X)$ for all $\alpha \in M$;

(iii) for all $f \in L^p(dm)$, the mapping

$$\mu \in \mathcal{M}_+(X) \mapsto \mathcal{E}(\mu; f) := \int_M \check{K}\mu(\alpha) f(\alpha) dm(\alpha) = \int_X Kf(x) d\mu(x)$$

is LSC on $\mathcal{M}_+(X)$ with respect to the weak-$*$ topology.

Proof. (i) Let $x_0 \in X$ and $\{x_k\}_k \subset X$ be such that $\lim_{k\to+\infty} x_k = x$ in X and $\lim_{k\to+\infty} Kf(x_k) = \liminf_{x\to x_0} Kf(x)$. Then,

$$Kf(x_0) = \int_M K(x_0,\alpha) f(\alpha) dm(\alpha) \underset{\substack{K(\cdot,\alpha) \\ \text{is LSC}}}{\leq} \int_M \liminf_{k\to+\infty} K(x_k,\alpha) f(\alpha) dm(\alpha) \leq$$

$$\underset{\text{(Fatou)}}{\leq} \liminf_{k\to+\infty} \int_M K(x_k,\alpha) f(\alpha) dm(\alpha) = \liminf_{k\to+\infty} Kf(x_k) = \lim_{k\to+\infty} Kf(x_k) =$$

$$= \liminf_{x\to x_0} Kf(x).$$

(ii) We use Proposition 1.1.3 (i). Let $\{h_j^{(\alpha)}\}_j \subset \mathcal{C}_c^0(X)$ be such that $h_j^{(\alpha)} \nearrow K(\cdot, \alpha)$ pointwise. Let $\{\mu_l\}_l \subset \mathcal{M}_+(X)$ be such that $\mu_l \xrightarrow[l\to+\infty]{*} \mu \in \mathcal{M}_+(X)$. Then, by the monotone convergence theorem, for all l

$$\check{K}\mu_l(\alpha) = \int_X K(x,\alpha)d\mu_l(x) = \int_X \lim_{j\to+\infty} h_j^{(\alpha)}(x)d\mu_l(x) = \lim_{j\to+\infty}\int_X h_j^{(\alpha)}(x)d\mu_l(x) \geq$$
$$\geq \int_X h_j^{(\alpha)}(x)d\mu_l(x)$$

for all j. Since μ_l converges to μ in the weak-$*$ topology, taking the $\liminf_{l\to\infty}$ at both sides of the inequality above,

$$\liminf_{l\to+\infty} \check{K}\mu_l(\alpha) \geq \liminf_{l\to+\infty}\int_X h_j^{(\alpha)}(x)d\mu_l(x) = \int_X h_j^{(\alpha)}(x)d\mu(x)$$

for all j. Then, by the monotone convergence theorem,

$$\liminf_{l\to+\infty} \check{K}\mu_l(\alpha) \geq \lim_{k\to+\infty}\int_X h_j^{(\alpha)}(x)d\mu(x) = \int_X K(x,\alpha)d\mu(x) = \check{K}\mu(\alpha).$$

(iii) Let $\mu_l \xrightarrow[l\to+\infty]{*} \mu$ in $\mathcal{M}_+(X)$. Then, by (ii)

$$\liminf_{l\to+\infty} \mathcal{E}(\mu_l; f) = \liminf_{l\to+\infty}\int_M f(\alpha)\check{K}\mu_l(\alpha)dm(\alpha) \underset{\text{(Fatou)}}{\geq}$$
$$\geq \int_M f(\alpha)\liminf_{l\to+\infty}\check{K}\mu_l(\alpha)dm(\alpha) \underset{(ii)}{\geq} \int_M f(\alpha)\check{K}\mu(\alpha)dm(\alpha) = \mathcal{E}(\mu; f).$$

□

We prove that the capacity is outer regular, that is: for all $E \subseteq X$

$$Cap(E) = \inf\Big\{Cap(U) \ : \ U \supseteq E, \ U \ open\Big\}. \tag{1.4}$$

Proposition 1.4.2. *Cap is outer regular.*

Proof. Let $E \subseteq X$. We have to prove (1.4). Observe that the assertion is obvious if $Cap(E) = +\infty$. Hence, we suppose $Cap(E) < +\infty$.

($\leq$) follows directly from the subadditivity of Cap.

($\geq$) Fix $\varepsilon > 0$ and let $f \in L_+^p(dm)$ be such that $Kf \geq 1$ on E and $\|f\|_{L^p(dm)}^p \leq Cap(E) + \varepsilon$. By Proposition 1.1.3 (ii) and Proposition 1.4.1 (i), the set

$$U = \{x \in X \ : \ Kf(x) > 1 - \varepsilon\}$$

is open. Then, since $K\left(\frac{f}{1-\varepsilon}\right) = \frac{Kf}{1-\varepsilon} > 1$ on U, we have

$$Cap(U) \leq \left\|\frac{f}{1-\varepsilon}\right\|_{L^p(dm)}^p = \frac{1}{(1-\varepsilon)^p}\|f\|_{L^p(dm)}^p \leq \frac{Cap(E)+\varepsilon}{(1-\varepsilon)^p}$$

for all $\varepsilon > 0$. The assertion follows taking $\varepsilon \to 0^+$.

$\square$

Remark 1.4.3. The following inequality, which is called the **weak capacity inequality** is tautological: for all $\lambda > 0$ and all $f \in L^p_+(dm)$,

$$\lambda^p Cap\left(\left\{Kf \geq \lambda\right\}\right) \leq \int_M f^p dm.$$

The holy Graal of this theory is a strong version of this inequality, which seems to be holding only in specific circumstances: for all $f \in L^p_+(dm)$,

$$\int_0^{+\infty} Cap\left(\left\{Kf \geq \lambda\right\}\right) d(\lambda^p) \lesssim \int_M f^p dm,$$

where $\lesssim$ stands for "$\leq$ up to some constant independent on f".

Now, we give a characterization of sets of capacity 0 in terms of the existence of functions L^p_+ which are infinite on these sets.

Proposition 1.4.4. *Let $E \subseteq X$. $Cap(E) = 0$ if and only if there exists $f \in L^p_+(dm)$ such that $Kf(x) = +\infty$ for all $x \in E$.*

Proof. ($\Leftarrow$) If $f \in L^p_+(dm)$ and satisfies $Kf(x) = +\infty$ for all $x \in E$, then for all $N > 0$ one has $K(f/N)(x) \geq 1$ for all $x \in E$. Hence, $Cap(E) \leq \left\|\frac{f}{N}\right\|^p_{L^p(dm)} \xrightarrow[N\to 0]{} 0$.

($\Rightarrow$) If $Cap(E) = 0$, then for all $N > 0$ there exists $f_N \in L^p_+(dm)$ such that $Kf_N \geq 1$ on E and $\|f_N\|_{L^p(dm)} \leq 2^{-N}$. The function $f := \sum_{N=1}^{+\infty} f_N$ satisfies $\|f\|^p_{L^p(dm)} \leq \sum_N \|f_N\|_{L^p(dm)} = 1$ and clearly $Kf = +\infty$ on E.

$\square$

The following Egorov-type theorem holds:

Theorem 1.4.5 (Egorov)**.** *Let $\{f_j\}_j \subset L^p(dm)$ be a Cauchy sequence. Suppose that $f_j \xrightarrow[j\to+\infty]{L^p(dm)} f \in L^p(dm)$. Then, for all $\varepsilon > 0$ there exists U_ε open such that $Cap(U_\varepsilon) < \varepsilon$ and a subsequence $\{f_{j_k}\}_k \subseteq \{f_j\}_j$ such that $Kf_{j_k} \xrightarrow[k\to+\infty]{} Kf$ uniformly on $X \setminus U_\varepsilon$.*

Next, we focus on the notion of non-negligibility in potential theory in terms of capacity. We say that a property $P(x)$ holds **quasi-everywhere** on X if $Cap(\{x \in X \ : \ P(x) \ fails\}) = 0$. Define

$$\Omega_E := \left\{f \in L^p_+(dm) \ : \ Kf \geq 1 \ on \ E\right\}$$

and

$$\widetilde{\Omega_E} := \left\{f \in L^p_+(dm) \ : \ Kf \geq 1 \ q.e. \ on \ E\right\}.$$

Proposition 1.4.6. $\widetilde{\Omega_E} = \overline{\Omega_E}^{L^p(dm)}$ *for all $p \in (1, +\infty)$.*

Proof. ($\supseteq$) It is enough to prove that $\widetilde{\Omega_E}$ is closed in Ω_E. For, let $\{f_j\}_j \subseteq L^p_+(dm)$ be such that $Kf_j \geq 1$ on $X \setminus E_j$ with $Cap(E_j) = 0$ for all j and $f_j \xrightarrow[j\to+\infty]{L^p(dm)} f \in L^p_+(dm)$. We must show that $Kf \geq 1$ on $X \setminus E_0$ with $Cap(E_0) = 0$. By Theorem 1.4.5, for all $N > 0$ there exists $\{f^{(N)}_{j_k}\}_k \subset \{f_j\}_j$ such that $Kf^{(N)}_{j_k} \xrightarrow[k\to+\infty]{} Kf$ uniformly on $X \setminus F_N$ with $Cap(F_N) < 2^{-N}$. In particular, $Kf \geq 1$ on $X \setminus E_0 \cup \bigcup_N F_N$ and $Cap(E_0) \leq \sum_N Cap(E_N) = 0$ gives that $Cap(E_0) = 0$.

($\subseteq$) If $f \in L^p_+(dm)$ and $Kf \geq 1$ q.e. on E, then $Kf \geq 1$ on $E \setminus E_0$ for some $E_0 \subseteq X$ with $Cap(E_0) = 0$. By Proposition 1.4.4, for all $N > 0$ there exists $f_N \in L^p_+(dm)$ such that $Kf_N = +\infty$ on E_0 and $\|f_N\|_{L^p(dm)} \leq \frac{1}{N}$. So,

$$\|f_N + f - f\|_{L^p(dm)} = \|f_N\|_{L^p(dm)} \leq \frac{1}{N} \xrightarrow[N\to+\infty]{} 0,$$

so that $f_N + f \xrightarrow[N\to+\infty]{L^p(dm)} f$ and clearly $f + f_N \in L^p_+(dm)$. Moreover,

$$K(f + f_N)(x) = Kf_N(x) + Kf(x) = +\infty$$

on E_0 and

$$K(f + f_N)(x) = Kf(x) + Kf_N(x) \geq 1 + \underbrace{Kf_N(x)}_{\geq 0} \geq 1$$

on $E \setminus E_0$. Hence, $K(f_N + f)(x) \geq 1$ on E. In particular, $\{f + f_N\}_N \subset \Omega_E$ and converges to f in $L^p(dm)$.

□

Theorem 1.4.7. *Suppose that $Cap(E) < \infty$ and $p \in (1, +\infty)$. Then, there exists a unique $f^E \in L^p_+(dm)$ such that $Kf^E \geq 1$ q.e. on E and $Cap(E) = \|f^E\|^p_{L^p(dm)}$.*

Such f^E is called the **equilibrium function**.

Proof. $\overline{\Omega_E}^{L^p(dm)}$ is closed (obviously) and convex in $L^p_+(dm)$, since if $t \in [0,1]$ and $f_1, f_2 \in \overline{\Omega_E}^{L^p(dm)}$, then for all $x \in E$

$$K(tf_1 + (1-t)f_2)(x) = t\underbrace{Kf_1(x)}_{\geq 1} + (1-t)\underbrace{Kf_2(x)}_{\geq 1} \geq 1.$$

Since $L^p(dm)$ is uniformly convex for $p \in (1, +\infty)$, the assertion follows directly by Theorem 1.3.5. □

1.5 Another digression on trees

1.5.1 The metric on a tree

Consider a general rooted tree $T = (V(T), E(T))$ with root-edge ω such that for all $x \in V(T)$

$$card\Big(\Big\{\alpha \in E(T)\ :\ x \text{ } is \text{ } endpoint \text{ } of \text{ } \alpha\Big\}\Big) < \infty.$$

We write $T = (T, \omega)$. General trees like these may not be finite, in the which case the definition of ∂T as the set of the leaves of T is not satisfactory and must be modified.

First, we define a sort of "euclidean" metric on $V(T)$ as follows: to each $\alpha \in E(T)$ we associate the number $2^{-d(e(\alpha),b(\omega))} =: 2^{-|\alpha|-1}$, where

$$d(x, y) = card(\{\alpha \in [x, y]\}), \tag{1.5}$$

$[x, y]$ denoting a path connecting x to y which contains the less number of edges as possible.

Definition 1.5.1 (Distance on $V(T)$)**.** For all $x, y \in V(T)$ set $\rho(x, y) := \sum_{\alpha\in[x,y]} 2^{-|\alpha|-1}$.

Remark 1.5.2. ρ is a metric on $V(T)$ and $\rho(x, y) < 1$ for all $x, y \in V(T)$.

Once a metric is defined, one can consider the completion $\bar{T}$ of $V(T)$ with respect to ρ. Clearly, we can write $\bar{T} = V(T) \cup [\bar{T} \setminus V(T)]$. This latest set contains the points of $\bar{T}$ which are interpreted as "leaves at infinity", in the sense that they are represented by paths starting from $b(\omega)$ and containing infinite edges.

Then,

$$\partial T = \partial_\omega T := \{x \in V(T)\ :\ x \text{ } is \text{ } a \text{ } leaf \text{ } and \text{ } x \neq b(\omega)\} \cup (\bar{T} \setminus V(T)).$$

In most applications either $\bar{T}\setminus V(T) = \varnothing$ (that is, T is a finite tree) or $\partial T = \bar{T}\setminus V(T)$ (T is a tree with no leaves other than $b(\omega)$).

If $\zeta \in \bar{T} \setminus V(T)$, we set $[\zeta, b(\omega)] = \{edges \text{ } in \text{ } the \text{ } line \text{ } \zeta\}$.

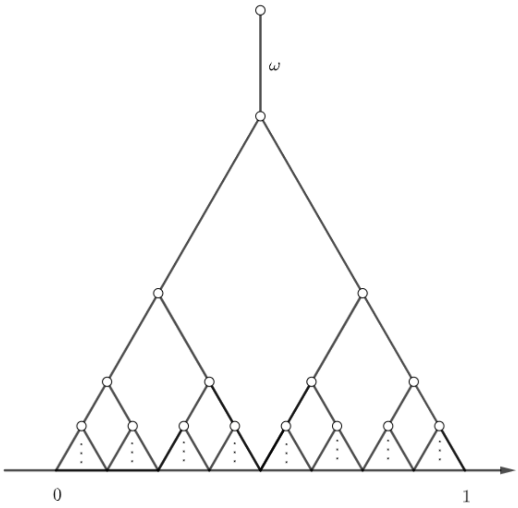
For $x, y \in V(T)$, their **confluent** $x \wedge y \in V(T)$ is the vertex defined by

$$[b(\omega), x] \cap [b(\omega), y] = [b(\omega), x \wedge y].$$

in the case in which $x = \zeta, y = \zeta' \in \bar{T} \setminus V(T)$, then the confluent $\zeta \wedge \zeta'$ of ζ and ζ' is defined by the relation

$$[b(\omega), \zeta] \cap [b(\omega), \zeta'] = [b(\omega), \zeta \wedge \zeta'].$$

Remark 1.5.3. We provide two models that represents a rooted dyadic tree T and its boundary ∂T.



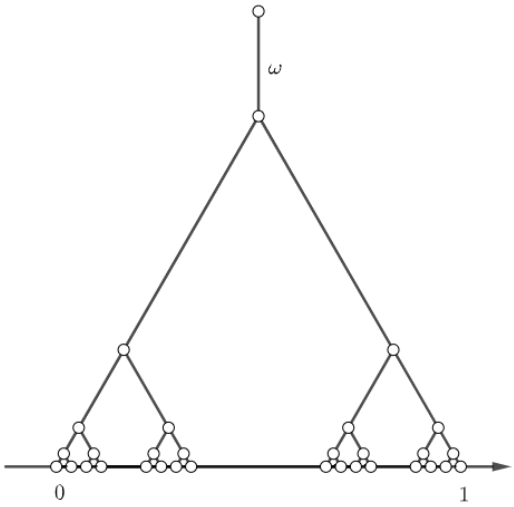


Figure 1.15: The first model of rooted dyadic tree (on the left).

Figure 1.16: The second model of rooted dyadic tree (on the right).

1. The first model is that represented in figure 1.15. In this model, each $\zeta \in \bar{T} \setminus V(T)$ is interpreted as a path joining $b(\omega)$ to a point $x(\zeta) \in [0,1]$ and the lengths of all the edges are given by 2^{-N} for some $N > 0$.

   We see that this model is not topologically faithful. In fact, even if every such element ζ is uniquely associated to a sequence of $\{\omega\} \times \{0,1\}^{\mathbb{N}}$, there are points of $[0,1]$ that "are the endpoints" of some ζ_1 and ζ_2 for $\zeta_1 \neq \zeta_2$. For instance, 1/2 "is the endpoint" of

   $$\zeta_1 = (\omega, 1, 0, 0, \dots) \quad \text{and} \quad \zeta_2 = (\omega, 0, 1, 1, 1, \dots),$$

   where the sequences are defined as follows: starting from the vertex ω, we assign to the sequence the values 0 and 1 depending on whether we move to the left or to the right. For instance, the first sequence means: *at* $b(\omega)$ *choose the right edge, at the end of it choose the left edge, and so on.*

   Clearly, following the two paths, the "reached point" will be 1/2. However, the two boundary points do not have distance 0.

2. The second model of the rooted dyadic tree T is obtained via the Cantor set C (see figure 1.16). With this representation, it is possible to associate to each $\zeta \in \partial T$ a point $c(\zeta) \in C$ via a homeomorphism.

1.5.2 The capacity of a rooted tree

Consider (T, ω) any rooted tree. We defined a metric space associated to $\bar{T}$, that is $X = (\bar{T}, \rho)$. Take $M = (E(T), \sigma)$ as a measure space, where $\sigma : \alpha \in E(T) \mapsto \sigma(\alpha) \geq 0$ is the measure on $E(T)$. For instance, in Remark 1.5.3 1., we put $\sigma(\alpha) = 2^{-N}$ if α was an edge of the N-th generation. Put

$$K(x, \alpha) := \chi_{[b(\omega),x]}(\alpha) = \begin{cases} 1 & \text{if } \alpha \in [b(\omega), x], \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all $f : E(T) \to [0, +\infty)$,

$$Kf(x) = \sum_{\alpha \in [b(\omega),x]} f(\alpha)\sigma(\alpha).$$

If $\mu \geq 0$ is a Borel measure on $(\bar{T}, \rho)$,

$$\check{K}\mu(\alpha) := \mu(S(\alpha)),$$

where $S(\alpha)$ is the set of all the vertices and the boundary points of T that are $\leq \alpha$ (roughly speaking, the $x \in \partial T$ that are below α with $\alpha \in [b(\omega), x]$).

Let $E \subseteq \bar{T}$. In this framework, for some $p \in (1, +\infty)$,

$$Cap(E) = \inf\Big\{ \sum_{\alpha \in E(T)} f(\alpha)^p \sigma(\alpha) \ : \ f \geq 0, \sum_{\alpha \in [b(\omega),x]} f(\alpha)\sigma(\alpha) \geq 1 \ \ \forall x \in E \Big\}.$$

1.5.3 A digression on hyperbolic geometry

On the complex unit disk $\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| < 1\}$ we can define two metrics: the euclidean metric, having $ds^2 = |dz|^2$, and the hyperbolic metric, characterized by $d_h s^2 = 4\frac{|dz|^2}{(1-|z|^2)^2}$. In this section, we give an idea of why this two metrics are somehow related to measures on rooted dyadic trees.

The euclidean length of a smooth curve $\gamma : [0, 1] \to \mathbb{D}$ is defined by:

$$\ell(\gamma) = \int_\gamma |dz| = \int_0^1 |\dot{\gamma}(t)| dt,$$

while its hyperbolic length is given by

$$\ell_h(\gamma) = \int_\gamma \frac{2|dz|}{1 - |z|^2} = \int_0^1 \frac{2|\dot{\gamma}(t)|}{1 - |\gamma(t)|^2} dt.$$

The geodesics on $\mathbb{D}$ with respect to the euclidean metric are segments joining two of its points, while the hyperbolic geodesics are represented in figure 1.17.

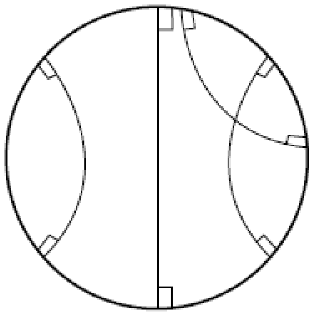


Figure 1.17: The geodesics of the unit disk with respect to the hyperbolic metric are arcs of euclidean circles orthogonal to the boundary of $\mathbb{D}$ or its diameters.

Remark 1.5.4. Observe that euclidean geodesics of $\mathbb{D}$ have always finite length, while the same is not true in the hyperbolic setting. To see this, consider the segment $\gamma(t) = t$ for $t \in [0,1]$, which corresponds to the radius of $\mathcal{D}$ that lies on the positive real axis. Clearly, $\ell(\gamma) = 1$, while $\ell_h(\gamma) = +\infty$. This is due to the singularity that $d_h s^2$ has in correspondence of $\partial\mathbb{D}$.

Consider a rooted dyadic tree. Metaphorically, the measure $\sigma(\alpha) = 1$ for all $\alpha \in E(T)$ corresponds to the hyperbolic metric on $\mathbb{D}$, while the measure $\sigma(\alpha) = 2^{-|\alpha|-1}$ defined in the previous paragraphs, corresponds to the euclidean measure on the disc. Figure 1.18 shows how $\mathbb{D}$ is a model for the rooted dyadic tree in this context.

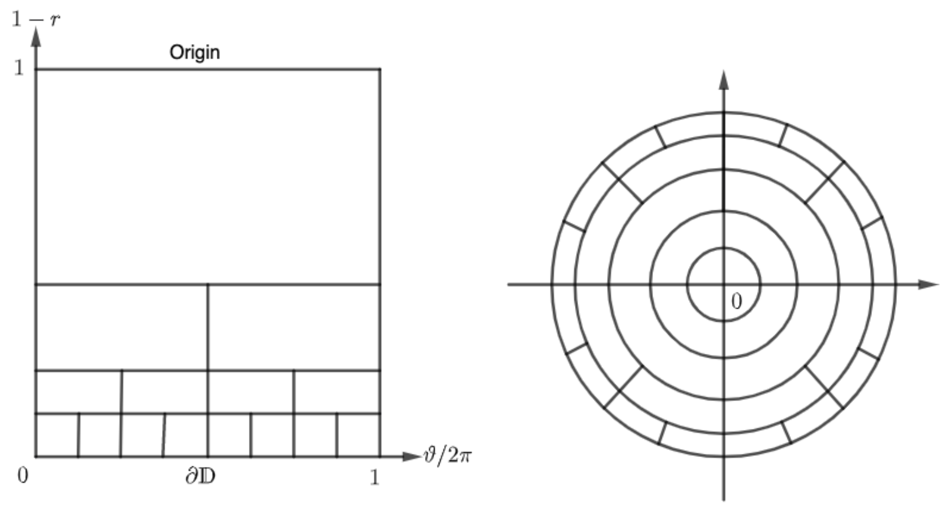


Figure 1.18: The tiling of the rectangle on the left is represented on the unit disk. On the other hand, the unit disk on the right is represented as the rectangle on the left. Observe that euclidean balls on the right become segments on the left.

1.5.4 An example of minimizing function that does not belong to Ω_E

It may happen that the minimizing function f belongs to $\overline{\Omega_E}^{L^p(dm)} \setminus \Omega_E$. For instance, take the infinite rooted trivial tree of figure 1.19, whose boundary is given by one line, say $\partial T = \{[b(\omega), x]\}$. Choose $p = 2$ and $\sigma(\alpha) = 1$ for all $\alpha \in E(T)$.

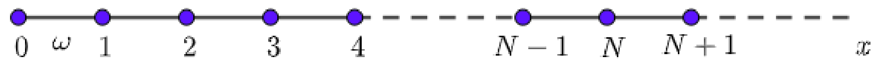


Figure 1.19: The infinite rooted trivial tree.

Clearly, for all $N > 0$ the functions

$$f_N([j, j+1]) = \begin{cases} \frac{1}{N} & \text{if } 0 \leq j < N, \\ 0 & \text{if } j \geq N \end{cases}$$

define non-negative sequences of $\ell^2(\mathbb{N})$ and $\sum_{\alpha \in [b(\omega),x]} f_N(\alpha) = 1$ for all $N > 0$. Therefore, f contributes to the calculation of $Cap(\partial T)$. In particular, for all $N > 0$,

$$\sum_{\alpha \in [b(\omega),x]} f_N(\alpha)^2 = \sum_{j=0}^{N-1} \frac{1}{N^2} = \frac{1}{N} \xrightarrow[N \to +\infty]{} 0,$$

which means that $Cap(\partial T) = 0$. However, the only non-negative sequence $g(\alpha)$ such that $\sum_{\alpha \in [b(\omega),x]} g(\alpha)^2 = 0$ is the zero sequence, that cannot be admissible for the calculation of $Cap(\partial T)$, since it does not satisfy $\sum_{\alpha \in E(T)} g(\alpha) \geq 1$. Observe that $f_N \xrightarrow[N \to +\infty]{} 0$. In particular, the minimizer 0 belongs to $\overline{\Omega_{\partial T}}^{L^p(dm)} \setminus \Omega_{\partial T}$

1.6 The problem of capacitability

In the previous sections, we proved the outer regularity of Cap. As far as the inner regularity is concerned, however, it happens that its validity is far more subtle and it is not always holding.

Definition 1.6.1 (Capacitable sets)**.** A subset $E \subseteq X$ is called **capacitable** if $Cap(E)$ is inner regular, that is if

$$Cap(E) = \sup \Big\{ Cap(K) \ : \ K \subseteq E, \ K \ compact \Big\}.$$

Theorem 1.6.2 (Choquet)**.** *Suppose that (X, ρ) is a locally compact, separable, complete metric space. Let $C : \mathcal{P}(X) \to [0, +\infty]$ be such that* *(a)* $C(\varnothing) = 0$;

(b) $E \subseteq F \Longrightarrow C(E) \leq C(F)$;

(c) if $\{K_j\}_j$ *are compact subsets such that* $K_1 \supseteq K_2 \supseteq K_3 \supseteq \dots$ *and* $K := \bigcap_j K_j$ *then* $C(K_j) \searrow C(K)$ *as* $j \to +\infty$;

(d) $E_1 \subseteq E_2 \subseteq E_3 \subseteq \dots$, $E = \bigcup_j E_j \Longrightarrow C(E_j) \nearrow C(E)$ *as* $j \to +\infty$.

Then, for all the Borel subsets $E \subseteq X$,

$$C(E) = \sup \Big\{ C(K) \; : \; K \subseteq E, \; K \textit{ compact} \Big\}.$$

We check that Cap satisfies all the assumptions of Theorem 1.6.2: (a) and (b) have been already proved in the previous sections. To prove (c), let K_j and K be as in Theorem 1.6.2 (c). Let U be open and such that $K \subseteq U$, then

$$K_j \subseteq U \quad \forall j \geq j_0$$

for some j_0. In fact, $U \cup \bigcup_{j=1}^{\infty} (X \setminus K_j)$ is an open cover of K_1, so that there exists $N > 0$ such that $K_j \subseteq K_1 \subseteq U \cup (X \setminus K_1) \cup \ldots \cup (X \setminus K_N)$ (for all j). In particular, for all $j > j_0 = N$ it must be $K_j \subset U$.

Thus, for all $j \geq j_0$

$$Cap(K) \leq Cap(K_j) \leq Cap(U)$$

and (c) follows taking the infimum on the open sets containing K (and, thus K_j for all $j \geq j_0$ for some $j_0 = j_0(U)$).

To prove (d) we use Lemma 1.3.6.

Proposition 1.6.3. *Cap satisfies Theorem 1.6.2 (d). Moreover, if* $Cap(E) < \infty$, *then* $f^{E_j} \xrightarrow[j \to +\infty]{L^p(dm)} f$.

Proof. If $\sup_j Cap(E_j) = +\infty$, then

$$\lim_{j \to +\infty} Cap(E_j) = \sup_j Cap(E_j) = +\infty,$$

since $\{Cap(E_j)\}_j$ is an unbounded non-decreasing sequence of real numbers. On the other hand, by subadditivity $Cap(E) \geq Cap(E_j)$ for all j, so that $Cap(E) = +\infty$ and the assertion follows.

Also, if $\sup_j Cap(E_j) = 0$, then for all j there exists $f_j \in L^p(dm)$ such that $\|f_j\|_{L^p(dm)} \leq 2^{-j}$ with $f(x) = +\infty$ for all $x \in E_j$. Then, the function $f = \sum_j f_j$ satisfies $\|f\|_{L^p(dm)} \leq \sum_j \|f_j\|_{L^p(dm)} = 1$ and $f(x) = +\infty$ on $E = \bigcup_j E_j$, which means that $Cap(E) = 0$.

Finally, suppose that $0 < \sup_j Cap(E_j) < \infty$. For all $j > k$, since $E_k \subseteq E_j$, $Kf^{E_j} \geq 1$ q.e. on E_k. By the convexity of $\overline{\Omega_{E_k}}^{L^p(dm)}$, $\frac{f^{E_j}+f^{E_k}}{2} \in \overline{\Omega_{E_k}}^{L^p(dm)}$, so that:

$$Cap(E_k) \leq \int_M \left(\frac{f^{E_j} + f^{E_k}}{2}\right)^p dm. \tag{1.6}$$

$Cap(E_k) \xrightarrow[k \to +\infty]{} A := \sup\{Cap(E_l) \; : \; l \geq 1\} < \infty$ and the convergence is granted as we are taking the limit of a non-decreasing sequence of real numbers. On the other hand, $\int_M (f^{E_j})^p dm, \int_M (f^{E_k})^p dm \leq A$ since the two integrals equal $Cap(E_j)$ and $Cap(E_k)$ respectively (which are $\leq A$ by subadditivity). By Jensen's inequality applied to (1.6)

$$\lim_{j,k \to +\infty} \left\| \frac{f^{E_j} + f^{E_k}}{2} \right\|^p_{L^p(dm)} = A.$$

Also, by the definitions of the functions f^{E_j}, $\lim_{j\to+\infty} \left\| f^{E_j} \right\|^p_{L^p(dm)} = \lim_{j\to+\infty} Cap(E_j) = A$. Therefore, the assumptions of Lemma 1.3.6 are verified and we conclude that there exists $g \in L^p(dm)$ such that $\lim_{j\to+\infty} f^{E_j} = g$ in $L^p(dm)$. Since each f^{E_j} belongs to $\overline{\Omega_E}^{L^p(dm)}$, their limit is itself an element of $\overline{\Omega_E}^{L^p(dm)}$, hence $g \in \overline{\Omega_E}^{L^p(dm)}$. This means that $Kg \geq 1$ q.e. on E, so that

$$Cap(E) \leq \|g\|^p_{L^p(dm)} = \lim_{j\to+\infty} \left\| f^{E_j} \right\|^p_{L^p(dm)} = \lim_{j\to+\infty} Cap(E_j). \tag{1.7}$$

On the other hand, by subadditivity $Cap(E) \geq Cap(E_j)$ for all j. Hence,

$$\lim_{j\to+\infty} Cap(E_j) = Cap(E).$$

Plugging this into (1.7), we find that $\|g\|^p_{L^p(dm)} = Cap(E)$, but the minimizer is unique, so that $g = f^E$ in $L^p(dm)$. □

Hence, under the hypothesis on X of Theorem 1.6.2, Cap is inner regular.

1.7 Capacity and potentials via measures

Theorem 1.7.1 (Min/max)**.** *Let U be a topological vector space and V be a vector space. Let $\mathcal{X} \subseteq U$ be convex and compact and $\mathcal{Y} \subseteq V$ be convex. Let $f : \mathcal{X} \times \mathcal{Y} \to (-\infty, +\infty]$ be such that $f(\cdot, y)$ is convex and LSC for all $y \in \mathcal{Y}$ and $f(x, \cdot)$ is concave. Then,*

$$\min_{x\in\mathcal{X}} \sup_{y\in\mathcal{Y}} f(x,y) = \sup_{y\in\mathcal{Y}} \min_{x\in\mathcal{X}} f(x,y).$$

Remark 1.7.2. The concavity assumption on f is fundamental for this theorem to hold. In fact, the function $f(x, y) = (x-y)^2$ defined on $[0, 1]\times[0, 1]$ satisfies all the hypothesis of Theorem 1.7.1, but the concavity of $f(x, \cdot)$ for all $x \in [0, 1]$. For this function, one has

$$\min_{x\in[0,1]} \sup_{y\in[0,1]} (x-y)^2 = \min_{x\in[0,1]} \max\{x^2, (1-x)^2\} = \frac{1}{2},$$

while

$$\sup_{y\in[0,1]} \min_{x\in[0,1]} (x-y)^2 = \sup_{y\in[0,1]} 0 = 0.$$

We apply the min/max theorem to the following situation: consider $1 < p < +\infty$ and $K \subseteq X$ a compact. Let

$$\mathcal{X} := \Big\{ \mu \in \mathcal{M}_+(K) \ : \ \mu(K) = 1 \Big\}$$

equipped with the weak-$*$ topology. Since convex combinations of measures of $\mathcal{X}$ are still measures of $\mathcal{X}$, it follows easily that $\mathcal{X}$ is convex. It is also compact, in fact if $\{\mu_j\}_j \subseteq \mathcal{X}$ is bounded then, by Banach-Alaoglu theorem, there exists a subsequence $\{\mu_{j_k}\}_k$ such that $\mu_{j_k} \xrightarrow[k\to\infty]{*} \mu$ for some measure $\mu \in \mathcal{M}_+(X)$. Since, by weak-$*$ convergence,

$$\mu(K) = \int_K d\mu = \lim_{k\to+\infty} \int_K d\mu_{j_k} = \lim_{k\to+\infty} \underbrace{\mu_{j_k}(K)}_{\equiv 1} = 1,$$

$\mu \in \mathcal{X}$. This proves the compactness of $\mathcal{X}$.

Consider the convex set [2]

$$\mathcal{Y} := \Big\{ f \in L^p_+(dm) \ : \ \|f\|_{L^p(dm)} \leq 1 \Big\}$$

and take $f(\mu, f) = \mathcal{E}(\mu; f)$ defined as in Proposition 1.4.1 (iii). Recall that this function is LSC with respect to its μ variable and it is linear with respect to both its variables, that implies its convexity and concavity.

Then, by Theorem 1.7.1, we have

$$\min_{\mu\in\mathcal{X}} \sup_{f\in\mathcal{Y}} \mathcal{E}(\mu; f) = \sup_{f\in\mathcal{Y}} \min_{\mu\in\mathcal{X}} \mathcal{E}(\mu; f).$$

We use this fact to prove the following result.

Theorem 1.7.3. *If $K \subseteq X$ is compact, then*

$$Cap(K) = \sup\Big\{ \frac{\mu(K)^p}{\mathcal{E}(\mu)^{p-1}} \ : \ \mu \geq 0,\ supp(\mu) \subseteq K \Big\}. \tag{1.8}$$

(1.8) is called the **dual definition of capacity**.

Before proving this result, we observe that for all $f \in \Omega_K$ (recall that this means that $f \in L^p_+(dm)$ with $Kf(x) \geq 1$ on K), since $Kf \geq 0$ on X,

$$\begin{aligned} \mu(K) &\leq \int_K Kf(x)d\mu(x) \leq \int_X Kf(x)d\mu(x) = \int_M f(\alpha)\check{K}\mu(\alpha)dm(\alpha) \leq \\ &\leq \|f\|_{L^p(dm)} \left(\int_M (\check{K}\mu(\alpha))^{p'} dm(\alpha) \right)^{1/p'} = \|f\|_{L^p(dm)}\, \mathcal{E}(\mu)^{1/p'}, \end{aligned}$$

by Hölder's inequality. Taking the infimum over such functions f, we get

$$\mu(K) \leq Cap(K)^{1/p} \mathcal{E}(\mu)^{1/p'}. \tag{1.9}$$

[2] Recall that the closed unit ball of $L^p(dm)$ is convex, so that also its intersection with $\{f \geq 0\}$ is convex.

Proof of Theorem 1.7.3. With the notation above, by duality

$$\sup_{f\in\mathcal{Y}}\mathcal{E}(\mu;f)=\sup_{f\in\mathcal{Y}}\int_M f(\alpha)\check{K}\mu(\alpha)dm(\alpha)=\left\|\check{K}\mu\right\|_{L^{p'}(dm)},$$

so that, since $\check{K}\mu$ is homogeneous of degree 1 [3],

$$\min_{\mu\in\mathcal{X}}\sup_{f\in\mathcal{Y}}\mathcal{E}(\mu;f)=\min_{\mu\in\mathcal{X}}\left\|\check{K}\mu\right\|_{L^{p'}(dm)}\underset{\substack{(\mathcal{E}(\mu)=\\ =\|\check{K}\mu\|^{p'}_{L^{p'}(dm)})}}{=}\min_{\mu\in\mathcal{X}}\mathcal{E}(\mu)^{1/p'}=\min_{\substack{\mu\in\mathcal{M}_+(K)\\ \mu(K)=1}}\mathcal{E}(\mu)^{1/p'}=$$
$$=\min_{\mu\in\mathcal{M}_+(K)}\frac{\mathcal{E}(\mu)^{1/p'}}{\mu(K)}=\left(\max_{\mu\in\mathcal{M}_+(K)}\frac{\mu(K)^p}{\mathcal{E}(\mu)^{p-1}}\right)^{-1/p}, \tag{1.10}$$

where we used the defining relation between p' and p.

On the other hand, consider

$$\min_{\mu\in\mathcal{X}}\mathcal{E}(\mu;f)=\min_{\mu\in\mathcal{X}}\int_X Kf(x)d\mu(x). \tag{1.11}$$

Observe that for all $\mu\in\mathcal{X}$, we have

$$\int_X Kf(x)d\mu(x)\geq\int_K Kf(x)d\mu(x)\geq\min_{x\in K}Kf(x)\cdot\mu(K)=\min_{x\in K}Kf(x),$$

where we used the non-negativity of $Kf(x)$ and μ, together with the fact that Kf has a minimum on the compact K by Proposition 1.1.3 (iii). Moreover, if x_0 is such a minimum, then $\delta_{x_0}\in\mathcal{X}$ and

$$\int_X Kf(x)d\delta_{x_0}(x)=Kf(x_0)=\min_{x\in K}Kf(x),$$

so that

$$\min_{\mu\in\mathcal{X}}\int_X Kf(x)d\mu(x)=\min_{x\in K}Kf(x).$$

For this reason, we have

$$\sup_{f\in\mathcal{Y}}\min_{\mu\in\mathcal{X}}\mathcal{E}(\mu;f)=\sup_{f\in\mathcal{Y}}\min_{x\in K}Kf(x)=\sup_{f\in L^p_+(dm)}\frac{\min_{x\in K}Kf(x)}{\|f\|_{L^p(dm)}}=\frac{1}{\inf_{f\in L^p_+(dm)}\frac{\|f\|_{L^p(dm)}}{\min_{x\in K}Kf(x)}}=$$
$$=\frac{1}{\inf\left\{\|f\|_{L^p(dm)}\ :\ f\geq 0,\ Kf(x)\geq 1\ \forall x\in K\right\}}=\frac{1}{Cap(K)^{1/p}} \tag{1.12}$$

and the assertion follows using Theorem (1.7.1) on (1.10) and comparing it with (1.12). □

[3] This implies that $\mathcal{E}(\mu)$ is homogeneous of degree p' by the very definition of $\mathcal{E}(\mu)$.

Corollary 1.7.4. *If $E \subseteq X$ is capacitable, then*

$$Cap(E) = \sup_{\substack{\mu \geq 0 \\ supp(\mu) \subseteq E}} \frac{\mu(E)^p}{\mathcal{E}(\mu)^{p-1}}.$$

Proof. Just use the previous theorem and the inner regularity of $Cap(E)$:

$$\sup_{\substack{\mu \geq 0 \\ \mathrm{supp}(\mu) \subseteq E}} \frac{\mu(E)^p}{\mathcal{E}(\mu)^{p-1}} = \sup_{\substack{K \subseteq E \\ K \text{ compact}}} \frac{\mu(K)^p}{\mathcal{E}(\mu)^{p-1}} = \sup_{\substack{K \subseteq E \\ K \text{ compact}}} Cap(K) = Cap(E).$$

$\square$

Theorem 1.7.5. *Let $K \subseteq X$ be compact and $1 < p < \infty$. Then, there exists a unique $\mu^K \in \mathcal{M}_+(K)$ such that*

(i) $f^K = (\check{K}\mu^K)^{p'-1}$*;*

(ii) the following identities hold:

$$Cap(K) = \mu^K(K) = \int_M (f^K)^p dm = \int_M (\check{K}\mu^K)^{p'} dm = \mathcal{E}(\mu^K) = \int_X V^{\mu^K} d\mu^K$$

where we recall that $V^{\mu^K} = K(\check{K}\mu^K)^{p'-1}$.

Proof. By Theorem 1.7.3, there exists $\{\mu_j\}_j \subseteq \mathcal{M}_+(K)$ such that $\left\|\check{K}\mu_j\right\|_{L^{p'}(dm)} = 1 \; \forall j$ and $\mu_j(K) \xrightarrow[j \to +\infty]{} Cap(K)^{1/p}$. This last convergence gives the boundedness of $\{\mu_j\}_j$, so that $\mu_j \xrightarrow[j \to +\infty]{*} \mu \in \mathcal{M}_+(K)$ up to subsequences, by Banach-Alaoglu theorem.

By Theorem 1.4.1 (ii) and by the definition of $\mathcal{E}(\mu)$, we have

$$1 = \liminf_{j \to +\infty} \left\|\check{K}\mu_j\right\|_{L^{p'}(dm)} \geq \left\|\check{K}\mu\right\|_{L^{p'}(dm)} = \mathcal{E}(\mu)^{1/p'} \tag{1.13}$$

and, by weak convergence,

$$\mu(K) = \int_K d\mu = \lim_{j \to +\infty} \int_K d\mu_j = \lim_{j \to +\infty} \mu_j(K) = Cap(K)^{1/p}. \tag{1.14}$$

Putting things together,

$$Cap(K)^{1/p} \underset{(1.14)}{=} \mu(K) \underset{(1.13)}{\leq} \frac{\mu(K)}{\mathcal{E}(\mu)^{1/p'}}.$$

Since, the other inequality follows obviously by the dual definition of capacity, we have

$$Cap(K) = \frac{\mu(K)^p}{\mathcal{E}(\mu)^{p/p'}} = \frac{\mu(K)^p}{\mathcal{E}(\mu)^{p-1}}.$$

Observe that for all $\lambda > 0$, $(\lambda\mu)(K) = \lambda \cdot \mu(K)$ obviously, while $\check{K}\mu$ is homogeneous of degree $\lambda^{p'}$, so that $\forall \lambda > 0$

$$Cap(K) = \frac{\mu(K)^p}{\mathcal{E}(\mu)^{p-1}} = \frac{\lambda^p \mu(K)^p}{\lambda^p \mathcal{E}(\mu)^{p-1}} = \frac{(\lambda\mu)(K)^p}{\mathcal{E}(\lambda\mu)^{p-1}}.$$

We claim that, up to choose $\lambda > 0$ properly, we have $Cap(K) = (\lambda\mu)(K)$. This is true, because by (1.14), for $\lambda = Cap(K)^{1/p'}$, we have

$$Cap(K) = Cap(K)^{1/p'+1/p} = Cap(K)^{1/p'} Cap(K)^{1/p} = Cap(K)^{1/p'} \mu(K)$$

and, for this value of λ, we also have $\mathcal{E}(\lambda\mu) = (\lambda\mu)(K)$.

Let $\mu^K := \lambda\mu$ for such a $\lambda > 0$ and suppose to know that $Kf^K \geq 1$ μ^K-a.e. on K [4]. Then, by Hölder's inequality,

$$\begin{aligned} Cap(K) = \mu^K(K) &= \int_K d\mu^K(x) \leq \int_X d\mu^K(x) \leq \int_X (Kf^K)(x) d\mu^K(x) = \\ &= \int_M f^K(\alpha) \check{K}(\mu^K)(\alpha) dm(\alpha) \leq \underbrace{\|f^K\|_{L^p(dm)}}_{= Cap(K)^{1/p}} \underbrace{\|\check{K}\mu^K\|_{L^{p'}(dm)}}_{= Cap(K)^{1/p'}} = \\ &= Cap(K)^{1/p} Cap(K)^{1/p'} = Cap(K). \end{aligned}$$

Hence, Hölder's inequality is actually an equality, but this can be possible if and only if $(f^K)^p = (x \cdot \check{K}(\mu^K))^{p'}$ m-a.e. on M for some constant c. Since the integrals of these functions are both equal to $Cap(K)$, it must be $c = 1$ and both (i) and (ii) follow.

It remains to prove that $Kf^K \geq 1$ μ^K-a.e. on K. For, consider $S = \{x \,:\, Kf^K(x) < 1\}$, which is a Borel set, and take $F \subseteq S$ compact, then

$$\mu^K(F) \underset{(1.9)}{\leq} Cap(F)^{1/p} \mathcal{E}(\mu^K|_F)^{1/p'} = 0 \tag{1.15}$$

since $F \subseteq S$. But S is a Borel set, because of the lower semicontinuity of Kf, and μ^K is a Borel measure (hence, inner regular), so that $\mu^K(S) = \sup\{\mu^K(F) \,:\, F \subseteq S,\ F\ compact\} = 0$ and the assertion follows. $\square$

This theorem extends to any $E \subseteq X$ under further assumptions on M, X and K:

Theorem 1.7.6. *Suppose that*

(a) M is locally compact and m is a Borel measure on M;

(b) for all $f \in \mathcal{C}^0_C(M)$, $Kf \in \mathcal{C}_0(X)$.

Let $E \subseteq X$ such that $Cap(E) < \infty$. Then, there exists a unique measure μ^E with $supp(\mu^E) \subseteq \bar{E}$ such that $f^E = (\check{K}\mu^E)^{p'-1}$. Moreover, $Kf^E \geq 1$ q.e. on E and $Kf^E(x) \leq 1$ for all $x \in supp(\mu^E)$.

[4] We only know that $Kf^K \geq 1$ q.e. on K, which is a weaker condition.

1.8 Dyadic Riesz-Bessel potentials

In this section, we compare the definitions of capacity of compacts subsets of the boundary of a dyadic tree provided by different choices of kernels.

Recalling the definition of $d(x, y)$ as given in (1.5), consider the measure m on ∂T defined by

$$m(\partial S(\alpha)) := e^{-d(b(\alpha),o)}.$$

With the notation we introduced previously, let $X = M = \partial T$, with metric on X given by $\rho(x, y) = 2^{-d(x \wedge y, o)}$ and measure on M given by m. Fixed a parameter $s \in (0, 1)$, define the kernel $K_s : \partial T \times \partial T \to [0, +\infty]$ as

$$K_s(x, y) := \frac{1}{\rho(x, y)^s}.$$

Denote with Qap_s the capacity of a compact subset $K \subseteq \partial T$ with these choices of X, M and K_s, explicitly:

$$Qap_s(K) = \sup \left\{ \frac{\mu(K)^2}{\widetilde{\mathcal{E}}_s(\mu)} \ : \ \mathrm{supp}(\mu) \subseteq K \right\},$$

where

$$\widetilde{\mathcal{E}}_s(\mu) = \int_{\partial T} |\check{K}_s \mu(x)|^2 dm(x) = \int_{\partial T} \left(\int_{\partial T} \frac{d\mu(y)}{\rho(x, y)^s} \right)^2 dm(x). \tag{1.16}$$

The exponent $p = 2$ is chosen in order to calculate the approximate behavior of $\widetilde{\mathcal{E}}_s(\mu)$ easier: proceeding from (1.16):

$$\begin{aligned} \widetilde{\mathcal{E}}_s(\mu) &= \int_{\partial T} \int_{\partial T} \frac{d\mu(y)}{\rho(x, y)^s} \int_{\partial T} \frac{d\mu(z)}{\rho(x, z)^s} dm(x) = \\ &= \int_{\partial T} \int_{\partial T} d\mu(y) d\mu(z) \underbrace{\int_{\partial T} \frac{dm(x)}{\rho(x, z)^s \rho(x, y)^s}}_{\text{Does not depend on } \mu.} . \end{aligned} \tag{1.17}$$

We evaluate explicitly the last integral, which only depends on the quantities defining the geometry of ∂T. We divide the calculation into three cases.

1. To each vertex t such that $e(\omega) \leq t \leq z \wedge y$ corresponds those elements of ∂T that "can be reached" by a line through t. Let $E_{1,t}$ denote the set of these boundary points and let $x_1 = x_1(t) \in E_{1,t}$ be one of these paths. It is clear that in this situation $t = y \wedge x_1 = z \wedge x_1$ and for some $C > 0$,

$$m(E_{1,t}) = C \cdot 2^{-d(t,o)} \approx 2^{-|t|},$$

where, in fact, $\approx$ means that the equality holds up to some multiplying constant. Since

$$\rho(x_1, y)^s = \rho(x_1, z)^s = 2^{-s|t|},$$

the contribution of $E_{1,t}$ to the integral $\int_{\partial T} \frac{dm(x)}{\rho(x,z)^s \rho(x,y)^s}$ is approximately given by

$$\frac{m(E_{1,t})}{\rho(x_1,z)^s \rho(x_1,y)^s} \approx \frac{2^{-|t|}}{2^{-2s|t|}} = 2^{(2s-1)|t|}.$$

Therefore, the total contribution of $\bigcup_{e(\omega)\leq t\leq z\wedge y} E_{1,t}$ to the integral above is approximately

$$\sum_{e(\omega)\leq t\leq z\wedge y} 2^{(2s-1)|t|} \approx \begin{cases} 1 & \text{if } 0<s<1/2, \\ |z\wedge y| & \text{if } s=1/2, \\ 2^{(2s-1)|z\wedge y|} & \text{if } 1/2<s<1 \end{cases} =: H_s(z,y). \tag{1.18}$$

Observe that $H_s = H_s(z\wedge y)$.

2. Next, we consider all the points of ∂T that corresponds to lines containing any vertex t such that $z\wedge y \leq t \leq z$. Let $E_{2,t}$ denote the set of these boundary points and $x_2 \in E_{2,t}$. As before,
$$m(E_{2,t}) \approx 2^{-|t|},$$
while this time $x_2\wedge z = t$ and $x_2 \wedge y = z\wedge y$, so that the contribution that $E_{2,t}$ gives to integral is approximately $\frac{2^{-|t|}}{2^{-s|t|}\cdot 2^{-s|z\wedge y|}}$ and the total contribution that $\bigcup_{z\wedge y\leq t\leq z} E_{2,t}$ gives to the integral is approxmiately
$$\begin{aligned}\sum_{z\wedge y\leq t\leq z} \frac{2^{-|t|}}{2^{-s|t|}\cdot 2^{-s|z\wedge y|}} &= 2^{s|z\wedge y|}\sum_{z\wedge y\leq t\leq z} 2^{-(1-s)|t|} \approx 2^{-(1-s)|z\wedge y|}2^{s|z\wedge y|} = \\ &= 2^{(2s-1)|z\wedge y|},\end{aligned} \tag{1.19}$$
which can be approximated further as in (1.18).

3. The same argument as above proves that the same estimates holds for the boundary points in $E_{3,t}$, the subset of ∂T related to the vertices t satisfying $z\wedge y\leq t\leq y$. Therefore, the total contribution of these boundary points to the integral is approximately given by (1.19).

To proceed the calculation from (1.17), we divide the three cases:

- if $0<s<1/2$, then
$$\widetilde{\mathcal{E}}_s(\mu) \approx \int_{\partial T}\int_{\partial T} 1\cdot d\mu(y)d\mu(z) = \mu(\partial T)^2.$$
This case is not very interesting, because if $K\subseteq \partial T$ is any compact subset of ∂T, then, using the calculation above for measures μ supported in K, one gets
$$Cap(K) = \sup\left\{\frac{\mu(K)^2}{\widetilde{\mathcal{E}}_s(\mu)} \ :\ \text{supp}(\mu)\subseteq K\right\} \approx \sup\left\{\frac{\mu(K)^2}{\mu(K)^2} \ :\ \text{supp}(\mu)\subseteq K\right\} = 1,$$
i.e. all the non-empty compact subsets K of ∂T have finite non-zero capacity.

- If $s = 1/2$, then

$$\widetilde{\mathcal{E}}_{1/2}(\mu) \approx \int_{\partial T}\int_{\partial T} |z \wedge y| d\mu(y) d\mu(z) = \int_{\partial T}\int_{\partial T} \sum_{\alpha \in [o, z\wedge y]} 1 d\mu(y) d\mu(z) =$$
$$= \sum_{\alpha \in [o, z\wedge y]} \int\int_{\{(z,y)\ :\ z\in \partial S(\alpha),\ y \in \partial S(\alpha)\}} d\mu(y) d\mu(z) = \sum_{\alpha \in [o, z\wedge y]} \mu(\partial S(\alpha))^2.$$

- If $1/2 < s < 1$, then

$$\widetilde{\mathcal{E}}_s(\mu) \approx \sum_{\alpha \in [o, z\wedge y]} 2^{(2s-1)|\alpha|} \mu(\partial S(\alpha))^2.$$

The calculation above refers to the case in which $X = M = \partial T$ with metric ρ and measure m defined as at the beginning of this section and potential given by $K_s(x, y) = \rho(x, y)^{-s}$ ($s \in (0, 1)$). But we introduced another potential theory on ∂T for $p = 2$, that is the one given by $X = \partial T$ as before, $M = E(T)$ with measure $\sigma = \sigma(\alpha)$ and potential $H : (x, \alpha) \in X \times M \mapsto H(x, \alpha) \in \{0, 1\}$ given by

$$H(x, \alpha) = \chi_{[o,x]}(\alpha) = \begin{cases} 1 & \text{if } \alpha \in [o, x], \\ 0 & \text{if } \alpha \notin [o, x] \end{cases} = \begin{cases} 1 & \text{if } x \in \partial(S(\alpha)), \\ 0 & \text{otherwise} \end{cases}.$$

Recall that in this case, for each function $f : M \to [0, +\infty]$ and measure μ on X

$$Hf(x) = \sum_{\alpha \in [o,x]} f(\alpha)\sigma(\alpha),$$

$$\check{H}\mu(\alpha) = \mu(\partial S(\alpha))$$

and

$$\mathcal{E}(\mu) = \sum_{\alpha \in E(T)} \check{H}(\mu)(\alpha)^2 \sigma(\alpha) = \sum_{\alpha \in E(T)} \mu(\partial S(\alpha))^2 \sigma(\alpha).$$

Clearly, $\mathcal{E}(\mu) \approx \widetilde{\mathcal{E}}_s(\mu)$ for $\sigma(\alpha) = 2^{(2s-1)|\alpha|}$, so that for all $s \in (0, 1)$

$$Cap(K) \approx Qap_s(K)$$

for all $K \subseteq \partial T$ compact. In conclusion, the definitions of capacity provided by the two different kernels, have the same behavior on compact subsets.

Chapter 2

Potential theory on trees

In the previous chapters we used dyadic trees as examples for discrete potential theory. In this chapter, we go further into potential theory of rooted trees, with the notation previously introduced.

2.1 Recap

In this section, we recall potential theory for rooted dyadic trees, recalling the main facts we saw for general rooted trees. Let $1 < p < \infty$ and T be any rooted tree.

Let $X = (\partial T, \rho)$ be the metric space ∂T with distance ρ and $M = (E(T), \sigma)$ be the measure space of the edges of T with measure σ. Here, ∂T is defined exactly as in the dyadic case, in the obvious way.

We can define a potential on $X \times M$ as always: let ω be the root of T and $o = b(\omega)$, we define for all $x \in \partial T$ and $\alpha \in E(T)$,

$$K(x, \alpha) := \chi_{[o,x]}(\alpha) = \chi_{\partial S(\alpha)}(x) = \begin{cases} 1 & \text{if } \alpha \in [o, x], \\ 0 & \text{otherwise.} \end{cases}$$

Under this definition,

$$Kf(x) = \sum_{\alpha \in [o,x]} f(\alpha)\sigma(\alpha) \quad \textit{and} \quad \check{K}\mu(\alpha) = \mu(\partial S(\alpha))$$

for all $f \in \ell^p_+(\sigma)$ and for all $\mu \in \mathcal{M}_+(\partial T)$. We also defined

$$\mathcal{E}(\mu) := \int_M (\check{K}\mu)^{p'} d\sigma = \sum_{\alpha \in E(T)} \mu(\partial S(\alpha))^{p'} \sigma(\alpha)$$

the energy and

$$V^\mu(x) = K(\check{K}\mu)^{p'-1}(x) = \sum_{\alpha \in [o,x]} (\check{K}\mu)^{p'-1}(\alpha)\sigma(\alpha) = \sum_{\alpha \in [o,x]} \mu(\partial S(\alpha))^{p'-1}\sigma(\alpha)$$

the potential. Then, the capacity of a compact subset $K \subseteq \partial T$ is defined as

$$
\begin{aligned}
Cap(K) &= \inf \Big\{ \|f\|_{\ell^p_+(\sigma)} \ : \ Kf(x) \geq 1 \ for \ all \ x \in K \Big\} = \\
&= \max \Big\{ \frac{\mu(K)^p}{\mathcal{E}(\mu)^{p-1}} \ : \ \mu \in \mathcal{M}_+(K) \Big\}.
\end{aligned}
$$

We also know that we can associate to any subset $E \subseteq \partial T$ a measure $\mu^E \in \mathcal{M}_+(E)$, called the equilibrium measure, with the property that the function

$$f^E(\alpha) := (\check{K}\mu^E)^{p'-1}(\alpha) = \big[\mu^E(\partial S(\alpha))\big]^{p'-1}$$

satisfies

$$\big\|f^E\big\|_{\ell^p_+(\sigma)} = \inf \Big\{ \|f\|_{\ell^p_+(\sigma)} \ : \ Kf(x) \geq 1 \ for \ quasi \ all \ x \in K \Big\}.$$

Moreover,

$$Cap(E) = \mu^E(\bar{E}) = \mathcal{E}(\mu^E) = \int V^{\mu^E}(x) d\mu^E(x)$$

and $V^{\mu^E}(x) \leq 1$ for all $x \in \text{supp}(\mu^E)$, while $V^{\mu^E}(x) \geq 1$ quasi-everywhere on E.

We conclude this section with a couple of remarks in the form of two propositions:

Proposition 2.1.1. $V^{\mu^E} \leq 1$ *on* $\bar{T} = \partial T \cup V(T)$.

Proposition 2.1.2. $V^{\mu^E} < 1$ *on* $\bar{T} \setminus \bar{E}$.

Proof. Seeking a contradiction, let $x \in \text{supp}(\mu^E)$ be such that $\mu^E(x) = 1$. If we set $y = x \wedge \text{supp}(\mu^E)$ to be the vertex in $[o, x]$ closest to x such that $\mu^E(S(\alpha_x)) > 0$, where α_x is the only edge such that $y = e(\alpha_x)$, then $V^{\mu^E}(y) = V^{\mu^E}(x) = 1$. Hence, for all $w \leq y$, $V^{\mu^E}(w) > 1$, which contradicts Proposition 2.1.1. □

2.2 p-harmonicity for potentials

The potential V^μ associated to a positive measure μ is defined by

$$V^\mu(x) = \sum_{\alpha \in [o,x]} \mu(\partial S(\alpha))^{p'-1} \sigma(\alpha).$$

If μ is a bounded Borel signed measure on ∂T, on the other hand, we set

$$V^\mu(x) = \sum_{\alpha \in [o,x]} \sigma(\alpha) \mu(\partial S(\alpha)) \cdot |\mu(\partial S(\alpha))|^{p'-2}.$$

In any case, we set $f^\mu(\alpha) := \sigma(\alpha)\mu(\partial S(\alpha)) \cdot |\mu(\partial S(\alpha))|^{p'-2}$ and $t^{p-1} := t \cdot |t|^{p-2}$. It is easy to see that f^μ is equivalently defined by $\mu(\partial S(\alpha)) = f^\mu(\alpha) \cdot |f^\mu(\alpha)|^{p-2}$.

Let β_j ($j = 1, \ldots, N$) be the edges having $b(\beta_j) = e(\alpha)$. Clearly,

$$\mu(\partial S(\alpha)) = \sum_{j=1}^{N} \mu(\partial S(\beta_j)) \tag{2.1}$$

and (2.1) implies

$$f^\mu(\alpha)|f^\mu(\alpha)|^{p-2} = \sum_{j=1}^{N} f^\mu(\beta_j)|f^\mu(\beta_j)|^{p-2}. \tag{2.2}$$

Also, it is clear by the definition of V^μ that

$$f^\mu(\alpha)\sigma(\alpha) = V^\mu(e(\alpha)) - V^\mu(b(\alpha)). \tag{2.3}$$

Plugging (2.3) into (2.2), and using the definition of t^{p-1} given above, we get the p-harmonic equation with weight σ on T for the potential V^μ:

$$\left(\frac{V^\mu(e(\alpha)) - V^\mu(b(\alpha))}{\sigma(\alpha)}\right)^{p-1} = \sum_{j=1}^{N} \left(\frac{V^\mu(e(\beta_j)) - V^\mu(b(\beta_j))}{\sigma(\beta_j)}\right)^{p-1}.$$

We focus on the case $p = 2$ and proceed with the calculation:

$$\begin{aligned} \frac{V^\mu(e(\alpha)) - V^\mu(b(\alpha))}{\sigma(\alpha)} &= \sum_{j=1}^{N} \frac{V^\mu(e(\beta_j)) - V^\mu(b(\beta_j))}{\sigma(\beta_j)} \underset{(b(\beta_j)=e(\alpha))}{=} \\ &= \sum_{j=1}^{N} \frac{V^\mu(e(\beta_j)) - V^\mu(e(\alpha))}{\sigma(\beta_j)}. \end{aligned}$$

Rearranging:

$$V^\mu(e(\alpha)) \left(\frac{1}{\sigma(\alpha)} + \sum_{j=1}^{N} \frac{1}{\sigma(\beta_j)} \right) = \frac{V^\mu(b(\alpha))}{\sigma(\alpha)} + \sum_{j=1}^{N} \frac{1}{\sigma(\beta_j)} V^\mu(b(\beta_j)). \tag{2.4}$$

Set $x := e(\alpha)$, write $y \sim x$ if $[x, y] \in E(T)$ and define

$$c(x, y) := \frac{1 \big/ \sigma([x, y])}{\sum_{z \sim x} 1 \big/ \sigma([x, z])}.$$

Then, $\sum_{y \sim x} c(x, y) = 1$ and equation (2.4) reads as a mean-value property:

$$V^\mu(x) = \sum_{y \sim x} c(x, y) V^\mu(y). \tag{2.5}$$

Equation (2.5) can be also written as follows:

$$\Delta_{T,c} V^\mu(x) := \sum_{y \sim x} c(x, y)[V^\mu(y) - V^\mu(x)] = 0.$$

Since the operator $\Delta_{T,c}$ is a discrete **Laplace operator**, this latest relation tells that V^μ is harmonic with respect to $\Delta_{T,c}$.

In the following tabular, we compare the whole theory on T_N with the variational calculus on appropriate subsets of $\mathbb{R}^n$ (for instance, balls) stressing the functionals involved.

	$\mathbb{R}^n$	T_N
$p=2$	$\Delta u = 0$, that is the Euler-Lagrange equation associated to the functional $\int \lvert\nabla u\rvert^2$	$\Delta_{T,c} u = 0$, that is the Euler-Lagrange equation associated to the functional $\sum_{\alpha \in E(T_N)} [u(e(\alpha)) - u(b(\alpha))]^2 \sigma(\alpha)$
$p \neq 2$	The function that minimizes the energy functional $\int \lvert\nabla u\rvert^p$ satisfies $div[\nabla u \lvert\nabla u\rvert^{p-1}] = 0$	In this case, the functional is $\sum_{\alpha \in E(T_N)} [u(e(\alpha)) - u(b(\alpha))]^p \sigma(\alpha)$

2.3 A recursive relation

In this section we prove the following recursive relation for the capacity of a given rooted tree (T, ω). Referring to figure 2.1, if $\alpha_1, \dots, \alpha_N$ are the edges such that $b(\alpha_j) = e(\omega)$ $(j = 1, \dots, N)$, $E = \partial T$ and $E_j = \partial S(\alpha_j)$, then

$$Cap_\omega(E) = \frac{\sum_{j=1}^N Cap_{\alpha_j}(E_j)}{\left[1 + \sigma(\omega)\left(\sum_{j=1}^N Cap_{\alpha_j}(E_j)\right)^{p'-1}\right]^{p-1}}. \qquad (2.6)$$

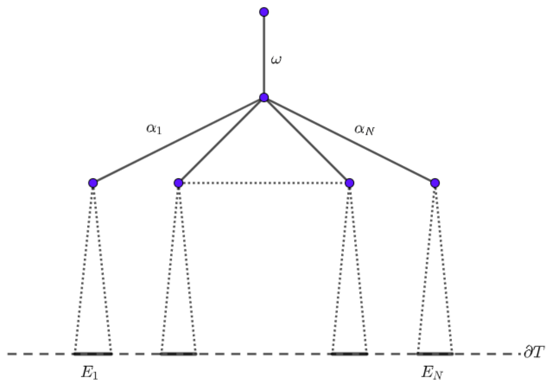


Figure 2.1

Observe that we already proved (2.6) for $p = 2$, $\sigma \equiv 1$ and $T = T_2$ the rooted dyadic tree.

To prove (2.6), let $\mu = \mu^E$. Then, for quasi-every $x \in E$,

$$1 = V^\mu(x) = \sum_{\alpha \in [o,x]} \sigma(\alpha)\mu(\partial S(\alpha))^{p'-1} = \sigma(\omega) \underbrace{[\mu(\partial T)]^{p'-1}}_{= Cap_\omega(E)^{p'-1}} + \sum_{\alpha \in [e(\omega),x]} \sigma(\alpha)\mu(\partial S(\alpha))^{p'-1},$$

that gives

$$1 - \sigma(\omega) Cap_\omega(E)^{p'-1} = \sum_{\alpha \in [e(\omega),x]} \sigma(\alpha)\mu(\partial S(\alpha))^{p'-1}.$$

Therefore,

$$1 = \sum_{\alpha \in [e(\omega),x]} \sigma(\alpha) \frac{\mu(\partial S(\alpha))}{[1 - \sigma(\omega) Cap_\omega(E)^{p'-1}]^{p-1}}.$$

It is easy to see that for all $j = 1, \ldots, N$, the measure $\mu|_{\partial S(\alpha_j)} \cdot \frac{1}{[1-\sigma(\omega)Cap_\omega(E)^{p'-1}]^{p-1}}$ is the equilibrium measure for $Cap_{\alpha_j}(E_j)$, so that

$$Cap_{\alpha_j}(E_j) = \mu^{E_j}(E_j) = \frac{\mu(E_j)}{[1 - \sigma(\omega) Cap_\omega(E)^{p'-1}]^{p-1}}$$

and, summing on j,

$$\sum_{j=1}^{N} Cap_{\alpha_j}(E_j) = \frac{\mu(E)}{[1 - \sigma(\omega) Cap_\omega(E)^{p'-1}]^{p'-1}}. \tag{2.7}$$

The assertion follows inverting (2.7).

We apply the recursive formula in two situations:

Example 2.3.1. (1) Let $\sigma \equiv 1$ and T_N be the infinite N-adic tree with $N \geq 2$, that is the infinite tree such that every vertex x is endpoint of exactly $N+1$ edges. Let $\alpha_1, \ldots, \alpha_N$ be the edges such that $b(\alpha_j) = e(\omega)$. It is clear that $Cap_\omega(\partial T_N) = Cap_{\alpha_j}(\partial S(\alpha_j))$ for all $j = 1, \ldots, N$, so that (2.6) becomes

$$Cap_\omega(\partial T_N) = \frac{N \cdot Cap_\omega(\partial T_N)}{\left[1 + \sigma(\omega)\left(N \cdot Cap_\omega(\partial T_N)\right)^{p'-1}\right]^{p-1}}$$

Solving the equation for $Cap_\omega(\partial T_N)$, we get

$$Cap_\omega(\partial T_N) = \frac{(N^{p'-1} - 1)^{p-1}}{N}.$$

Example 2.3.2. Take the dyadic case, $N = 2$, with $p = 2$. Then, $Cap_\omega(\partial T_2) = \frac{1}{2}$.

(2) Next, consider a finite N-adic tree of depth M, say $T_{N,M}$. Let $C(k) := Cap_\alpha(\partial S(\alpha))$, where α is an edge at level k. Then,

$$C(k) = \begin{cases} 1 & \text{if } k = 0, \\ \frac{N \cdot C(k-1)}{\left[1 + (N \cdot C(k-1))^{p'-1}\right]^{p-1}} & \text{otherwise.} \end{cases}$$

2.4 Desymmetrization

Consider the infinite rooted dyadic tree $T = T_2$ and think of ∂T as $\partial\mathbb{D} = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. In this framework, for a given $E \subseteq \partial T$, the expression $\sqrt{E}$ makes sense and define a *rescaled version of* E. Therefore, let $E \subseteq \partial T$ and consider its two rescaled copies $E_\pm := \pm\sqrt{E} \subseteq \partial S(\alpha_\pm)$, where $\alpha_\pm$ are the edges such that $b(\alpha_\pm) = e(\omega)$ (see figure 2.2). Clearly,

$$|E| = |E_+| + |E_-|.$$

However, by the recursive formula and using the fact that $Cap_\omega(E) = Cap_{\alpha_\pm}(E_\pm)$, we have:

$$Cap_\omega(E_+ \cup E_-) = \frac{Cap_{\omega_+}(E_+) + Cap_{\omega_-}(E_-)}{1 + Cap_{\omega_+}(E_+) + Cap_{\omega_-}(E_-)} = \frac{2Cap_\omega(E)}{1 + 2Cap_\omega(E)} \geq Cap_\omega(E),$$

where we used the fact that $Cap_\omega(E) \leq Cap_\omega(\partial T) = 1/2$, and $\frac{2t}{1+2t} \geq t$ if $0 \leq t \leq 1/2$, which is the case.

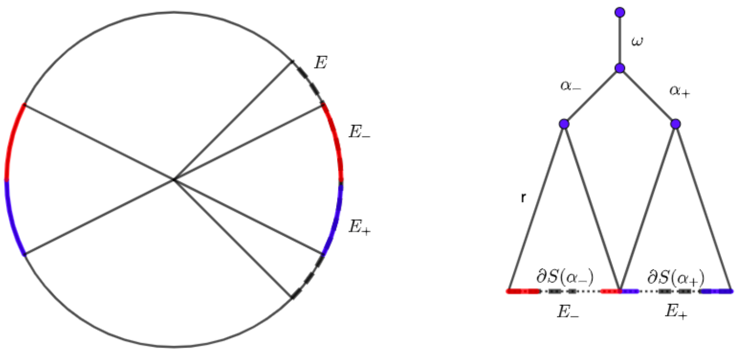


Figure 2.2: The definitions of E and $E_\pm$ as subsets of ∂D and of T_2 respectively.

Chapter 3

Trace inequalities

In the classical Sobolev spaces theory, fixed $1 < p < \infty$, one defines

$$W^{1,p}(\mathbb{R}^n) := \left\{ u \in L^1_{loc}(\mathbb{R}^n) \ : \ \|u\|^p_{W^{1,p}} = \|u\|^p_p + \|\nabla u\|^p_p < \infty \right\},$$

where ∇u is the weak gradient of u. Given a submanifold L of dimension $m < n$, the trace of $u \in W^{1,p}$ is the function $Tr(u)$ such that

$$\int_L |Tr(u)|^p d\mathcal{H}^m \lesssim \|u\|^p_p ,$$

where $d\mathcal{H}^m$ denotes the Hausdorff measure on L, and the question is characterizing all the measures $\mu \geq 0$ such that

$$\int_L |u|^p \lesssim \|u\|^p_p .$$

Now, in the framework of rooted trees, the equivalent problem goes as follows: one fixes $1 < p < \infty$ and takes as usual $X = \bar{T}$ and $M = (E(T), \sigma)$, where (T, ω) is a given rooted tree. The question becomes characterizing all measures $\mu \geq 0$ such that $\forall \varphi : E(T) \to [0, +\infty)$,

$$\int_{\bar{T}} (I(\varphi\sigma))^p d\mu \lesssim \sum_{\alpha \in E(T)} \varphi^p(\alpha)\sigma(\alpha), \tag{3.1}$$

where $I(\varphi\sigma)(x) = \sum_{\alpha \in [o,x]} \varphi(\alpha)\sigma(\alpha)$. The minimal constant under which (3.1) holds is denoted with $[\mu]$ and inequality (3.1) is called the **weighted Hardy inequality**.

3.1 Geometric characterization of $[\mu]$

We give a geometric characterization of $[\mu]$ that involves both capacity and μ. Set $[\mu]_c$ as the smallest positive constant such that

$$\mu\left(\bigsqcup_{j=1}^{N} S(\alpha_j)\right) \lesssim Cap\left(\bigsqcup_{j=1}^{N} S(\alpha_j)\right) \tag{3.2}$$

holds for any choice of $\alpha_1, \ldots, \alpha_N \in E(T)$ $(N \in \mathbb{N}_{\neq 0})$.

To prove the main result of this section, we need a strong capacity inequality.

Proposition 3.1.1 (Strong capacity inequality)**.** *For all $f \geq 0$ on $E(T)$,*

$$\sum_{k=-\infty}^{+\infty} 2^{kp} Cap\left(\left\{x \ : \ If(x) \geq 2^k\right\}\right) \leq C(p) \sum_{\alpha \in E(T)} f^p(\alpha)\pi(\alpha) \tag{3.3}$$

for some $C(p) \geq 1$.

Proof. For all $k \in \mathbb{Z}$, set

$$\Omega_k := \left\{x \ : \ If(x) = \sum_{\alpha \in [o,x]} f(\alpha) > 2^k\right\}.$$

Clearly, since If is non-decreasing, $\Omega_{k+1} \subseteq \Omega_k$. We only prove the assertion in the case in which the inclusions are all strict. For, define $f_k := f \cdot \chi_{\Omega_k \setminus \Omega_{k+1}}$. Let $x \in \Omega_k$ and $y \in \Omega_{k+1}$ and consider the geodesic from y to $o = b(\omega)$ through $\alpha_x = [P(x), x]$, being $P(x)$ the only vertex such that $\alpha_x \in E(T)$. Then,

$$If_k(y) = \sum_{\alpha \in [P(x),y]} f(\alpha) = If(y) - If(P(x)) \geq 2^{k+1} - 2^k = 2^k.$$

Hence, $2^{-k} f_k$ is admissible for the calculation of $Cap(\Omega_{k+1})$, in particular:

$$Cap(\Omega_{k+1}) \leq \sum_{\alpha \in \Omega_k \setminus \Omega_{k+1}} (2^{-k} f_k)^p(\alpha)\pi(\alpha) = 2^p \sum_{\alpha \in \Omega_k \setminus \Omega_{k+1}} (2^{-(k+1)} f_k)^p(\alpha)\pi(\alpha),$$

which gives $Cap(\omega_k) \leq 2^p \sum_{\alpha \in \Omega_k \setminus \Omega_{k-1}} (2^{-k} f_k)^p(\alpha)\pi(\alpha)$. Hence,

$$\sum_{k=-\infty}^{+\infty} 2^{kp} \cdot Cap(\Omega_k) \leq 2^p \sum_{k=-\infty}^{+\infty} \sum_{\alpha \in \Omega_{k-1} \setminus \Omega_k} f_k(\alpha)^p \pi(\alpha) = 2^p \sum_{\alpha \in E(T)} f(\alpha)^p \pi(\alpha).$$

□

Theorem 3.1.2. *Suppose that μ satisfies (3.1). Then, $[\mu] \approx [\mu]_c$.*

Proof. We first prove that $[\mu]_c \leq [\mu]$, for we check that the inequality (3.2) holds with $[\mu]$ as a constant. Let $E \subseteq \partial T$. Recall that, in this setting,

$$Cap(E) = \inf\left\{\sum_\alpha \varphi^p \sigma \ : \ I(\varphi\sigma) \geq 1 \ on \ E\right\}$$

and change the notation as follows: set $g := \varphi\sigma$ and $\pi := \sigma^{1-p}$, so that $\varphi^p \sigma = (g\sigma^{-1})^p \sigma = g^p \sigma^{1-p} = g^p \pi$ and

$$Cap(E) = \inf\left\{\sum_\alpha g^p \pi \ : \ I(g) \geq 1 \ on \ E\right\}.$$

Next, we consider $f := f^{\bigsqcup_{j=1}^N S(\alpha_j)}$ the q.e. minimizer of $Cap(E)$. Then, $(If)|_{\bigsqcup_{j=1}^N S(\alpha_j)} = 1$ and, therefore,

$$\mu\left(\bigsqcup_{j=1}^N S(\alpha_j)\right) \leq \int_{\bar{T}} (If)^p d\mu \underset{(3.1)}{\leq} [\mu] \sum_\alpha f^p \pi = [\mu] Cap\left(\bigsqcup_{j=1}^N S(\alpha_j)\right).$$

This proves that $[\mu]_c \leq [\mu]$. We need to check the other inequality up-to-some-constant. For, we use (3.3):

$$\int_{\bar{T}} (If)^p d\mu \leq \sum_k (2^{k+1})^p \mu\left(\left\{x \ : \ 2^k \leq If(x) < 2^{k+1}\right\}\right) \leq 2^p \sum_k 2^{kp} \mu\left(\left\{x \ : \ If(x) \geq 2^k\right\}\right) \leq$$
$$\leq 2^p [\mu]_c \sum_k Cap\left(\left\{x \ : \ If(x) \geq 2^k\right\}\right) \underset{(3.3)}{\leq} [\mu]_c 2^p C(p) \sum_\alpha f^p(\alpha)\pi(\alpha),$$

where we used the fact that the sets $\{x \ : \ If(x) \geq 2^k\}$ have form $\bigsqcup_{j=1}^N S(\alpha_j)$, since If is non-decreasing. The assertion follows by the minimality of $[\mu]$. □

Remark 3.1.3. Inequality (3.2) bears advantages and disadvantages. In fact, while the measure μ appears on the left hand-side of (3.2) only, the sets $\bigsqcup_j S(\alpha_j)$ appearing on the right hand-side are usually too complicated for an explicit calculation of capacity.

3.2 Mass-energy characterization

In this section, we characterize $[\mu]$ in terms of the minimal constant with respect to which the inequality

$$\sum_{\beta \leq \alpha} \mu(S(\beta))^{p'} \pi(\beta)^{1-p'} \lesssim \mu(S(\alpha)) \tag{3.4}$$

holds for all $\alpha \in E(T)$. Let this constant be $[\mu]_{m/e}^{p'-1}$. The left hand-side of (3.4) represents a form of **local energy** for $S(\alpha)$, while the right hand-side represents its **local mass**.

Remark 3.2.1. Inequality (3.4) bears advantages and disadvantages as well as (3.2). In fact, $S(\alpha)$ appears in (3.4) for one α only, but μ appears at both of its sides.

Theorem 3.2.2. $[\mu] \approx [\mu]_{m/e}$.

To prove this theorem, we need some property of the maximal function, defined for all $g : E(T) \to \mathbb{R}$ by

$$\mathcal{M}_\mu g(\alpha) := \max_{\beta \geq \alpha} \frac{1}{\mu(S(\beta))} \int_{S(\beta)} |g| d\mu.$$

Proposition 3.2.3. *For all $g, h : E(T) \to \mathbb{R}$ measurable,*

(i) $\mathcal{M}_\mu(cg) = c\mathcal{M}_\mu(g)$ for all $c > 0$;

(ii) $\mathcal{M}_\mu(g+h) \leq \mathcal{M}_\mu(g) + \mathcal{M}_\mu(h)$.

The following version of Marcinkiewicz interpolation theorem is used to study the boundedness properties of $\mathcal{M}_\mu$ in this framework:

Theorem 3.2.4. *Let $T : L^1(X) + L^\infty(X) \to L^1(Y) + L^\infty(Y)$ be a 1-homogeneous and subadditive operator such that, for some $C_1, C_\infty > 0$,*

(a) $\|Tg\|_{L^\infty(Y)} \leq C_\infty \|g\|_{L^\infty(X)}$ *for all $g \in L^\infty(X)$ (strong-type (∞,∞) operator);*

(b) $t\mu(\{y \ : \ Tg(y) > t\}) \leq C_1 \|g\|_{L^1(X)}$ *for all $t > 0$ and all $g \in L^1(X)$ (weak-type $(1,1)$ operator).*

Then, for all $1 < p < \infty$, $T : L^p(X) \to L^p(Y)$ is bounded with $\|T\|_{L^p(X)\to L^p(Y)} \leq \left(\frac{2^{p-1}p}{p-1}\right)^{1/p} C_1^{1/p} C_\infty^{1/p'}$.

Proof. Observe that, as a consequence of Fubini's theorem, for all $p \in (1, +\infty)$ and all $f \in L^p(\mathbb{R}^n)$, $\|f\|_p^p = p\int_0^{+\infty} t^{p-1}\mu(\{x \ : \ |f(x)| > t\})dt$. In fact,

$$p\int_0^{+\infty} t^{p-1}\mu(\{x \ : \ |f(x)| > t\})dt = p\int_0^{+\infty}\int_{\{x \ : \ |f(x)|>t\}} d\mu dt = \int_{\mathbb{R}^n}\int_0^{|f(x)|} pt^{p-1} dt d\mu =$$
$$= \int_{\mathbb{R}^n} |f(x)|^p d\mu = \|f\|_p^p.$$

Each function $h \in L^p(X)$ can be decomposed into a sum $h = h_0 + h_1$, where $h_0 \in L^p(X)$ and $h_1 \in L^\infty(X)$ as follows: for fixed $c, t > 0$ to be chosen, define

$$A_0 := \{x \in \mathbb{R}^n \ : \ |h(x)| > ct\} \quad \textit{and} \quad A_1 = \{x \in \mathbb{R}^n \ : \ |h(x)| \leq ct\},$$

then set $h_0 := h \cdot \chi_{A_0}$ and $h_1 := h \cdot \chi_{A_1}$. In our case, T is a strong-type (∞,∞) operator, so that

$$\|Th_1\|_{L^\infty(Y)} \leq C_\infty \|h\|_{L^\infty(X)} \leq C_\infty ct = \frac{t}{2}$$

if we choose $c = \frac{1}{2C_\infty}$. Hence, in this case,

$$\mu\left(\left\{x \ : \ |Th_1(x)| > \frac{t}{2}\right\}\right) = 0.$$

So, choosing $c = \frac{1}{2C_\infty}$, using the fact that $\{x : |Th| > t\} \subseteq \{x : |Th_0| > t/2\} \cup \{x :$

$|Th_1| > t/2\}$ and Tonelli's theorem,

$$
\begin{aligned}
\|Th\|_p^p &= p\int_0^{+\infty} t^{p-1}\mu(\{x\ :\ |Th(x)|>t\})dt \leq \\
&\leq p\int_0^{+\infty} t^{p-1}\left(\mu\left(\left\{x\ :\ |Th_0(x)|>\frac{t}{2}\right\}\right)+\mu\left(\left\{x\ :\ |Th_1(x)|>\frac{t}{2}\right\}\right)\right)dt = \\
&= p\int_0^{+\infty} t^{p-1}\mu\left(\left\{x\ :\ |Th_0(x)|>\frac{t}{2}\right\}\right)dt \underset{weak-(1,1)}{\leq} p\int_0^{+\infty} t^{p-1}\frac{C_0}{t}\|h_0\|_{L^1(X)}dt = \\
&= pC_0\int_0^{\infty} t^{p-2}\int_{\mathbb{R}^n}|h_0(x)|d\mu(x)dt = pC_0\int_0^{\infty} t^{p-2}\int_{A_0}|h(x)|d\mu(x)dt = \\
&= pC_0\int_{\mathbb{R}^n}|h(x)|\int_0^{2C_\infty|h(x)|} t^{p-2}dtd\mu(x) = \frac{pC_0(2C_\infty)^{p-1}}{(p-1)}\int_{\mathbb{R}^n}|h(x)|^{1+p-1}d\mu(x).
\end{aligned}
$$

Therefore,

$$\|Th\|_p^p \leq \frac{2^{p-1}p}{p-1}C_1C_\infty^{1-p}\|h\|_{L^p(X)}^p,$$

which is the assertion. □

We hit the proof of Theorem 3.2.2:

Proof of Theorem 3.2.2. We use a duality argument. The condition $\int_{\bar{T}}(If)^p d\mu \leq [\mu]\sum_\alpha f(\alpha)^p\pi(\alpha)$ is equivalent to the boundedness of $I : \ell^p(\pi) \to L^p(\mu)$ with $\|I\|_{\ell^p(\pi)\to L^p(\mu)} = [\mu]^{1/p}$. Consider the adjoint operator $I^* : L^{p'}(\mu) \to \ell^{p'}(\pi^{1-p'})$ with respect to the $L^2(\mu)$ duality pairing between $L^p(\mu)$ and $L^{p'}(\mu)$ and to the ℓ^2 duality pairing between ℓ^p and $\ell^{p'}$. Then, for $g : \bar{T} \to \mathbb{R}$, we claim that

$$I^*g(\alpha) = \int_{S(\alpha)} g(x)d\mu(x) \tag{3.5}$$

holds. For, for all $f \in \ell^p(\pi)$,

$$
\begin{aligned}
\langle I^*g, f\rangle_{\ell^2} = \langle g, If\rangle_{L^2(\mu)} &= \int_{\bar{T}} g(x)\sum_{\alpha\in[o,x]} f(\alpha)d\mu(x) = \sum_{\alpha\in[o,x]} f(\alpha)\int_{\{x\ :\ \alpha\in[o,x]\}} g(x)d\mu(x) = \\
&= \sum_{\alpha\in[o,x]} f(\alpha)\int_{S(\alpha)} g(x)d\mu(x),
\end{aligned}
$$

which is (3.5). By the theory of adjoint operators, we know that

$$[\mu]^{1/p} = \|I_\mu\|_{\ell^p(\pi)\to L^p(\mu)} = \left\|I_\mu^*\right\|_{L^{p'}(\mu)\to\ell^{p'}(\pi^{1-p'})},$$

so that inequality (3.1) with constant $[\mu]$ is equivalent to

$$\sum_{\alpha\in E(T)} (I^*g(\alpha))^{p'}\pi(\alpha)^{1-p'} \leq [\mu]^{p'-1}\int_{\bar{T}} g(x)^{p'}d\mu(x) \tag{3.6}$$

for all $g \geq 0$. We write the left hand-side of (3.6) as an average:

$$\sum_{\alpha}(I^*g(\alpha))^{p'}\pi(\alpha)^{1-p'} = \sum_{\alpha}\left(\frac{I^*g(\alpha)}{\mu(S(\alpha))}\right)^{p'}\mu(S(\alpha))^{p'}\pi(\alpha)^{1-p'},$$

where, using (3.5), we recognize the maximal operator $\mathcal{M}_\mu g$, and prove the following stronger inequality holds:

$$\sum_{\alpha}\mathcal{M}_\mu g(\alpha)^{p'}\mu(S(\alpha))^{p'}\pi(\alpha)^{1-p'} \leq [\mu]^{p'-1}\int_{\bar{T}} g(x)^{p'}d\mu(x). \tag{3.7}$$

Since (3.7) is equivalent to the boundedness of $\mathcal{M}_\mu : L^{p'}(\mu) \to L^{p'}(E(T),\eta)$, where $\eta(\alpha) := \mu(S(\alpha))^{p'}\pi(\alpha)^{1-p'}$, we want to use Marcinkiewicz interpolation theorem on $\mathcal{M}_\mu$, with $X = (\bar{T},\mu)$ and $Y = (E(T),\eta)$. We check that $\mathcal{M}_\mu$ satisfies all its assumptions. For all $h \in L^\infty(\mu)$ and for all $\alpha \in E(T)$,

$$\mathcal{M}_\mu h(\alpha) = \max_{\beta\geq\alpha}\frac{1}{\mu(S(\beta))}\int_{S(\beta)} hd\mu \leq \max_{\beta\geq\alpha}\frac{1}{\mu(S(\beta))}\int_{S(\beta)} d\mu\, \|h\|_{L^\infty(X)} = \|h\|_{L^\infty(X)},$$

so that $\mathcal{M}_\mu$ is bounded as an operator from $L^\infty(\mu)$ to $\ell^\infty(\eta)$. It remains to check that $\mathcal{M}_\mu$ is a weak-type $(1,1)$ operator.

Recall that, by the assumptions,

$$\sum_{\beta\leq\alpha}\mu(S(\beta))^{p'}\pi(\beta)^{1-p'} \leq [\mu]_{m/e}^{p'-1}\mu(S(\alpha)),$$

take $g \in L^1(\mu)$, $g \geq 0$ and $t > 0$. For $\alpha_1, \alpha_2, \ldots$ properly chosen with the property $\frac{1}{\mu(S(\alpha_j))}\int_{S(\alpha_j)} gd\mu > t$, we can write

$$\Big\{\alpha \,:\, \mathcal{M}_\mu g(\alpha) > t\Big\} = \bigsqcup_{j=1}^{\infty}\underbrace{\Big\{\alpha \,:\, \alpha \leq \alpha_j\Big\}}_{=:\, E_{\alpha_j}},$$

so that

$$\begin{aligned} t\eta\left(\{\alpha \,:\, \mathcal{M}_\mu g(\alpha) > t\}\right) &= t\sum_j \eta(E_{\alpha_j}) = t\sum_j\sum_{\alpha\leq\alpha_j}(\mu(S(\alpha_j)))^{p'-1}\pi^{1-p'}(\alpha) \leq \\ &\leq [\mu]_{m/e}^{p'-1}\sum_j t(\mu(S(\alpha_j))) \leq [\mu]_{m/e}^{p'-1}\sum_j\int_{S(\alpha_j)} gd\mu \leq [\mu]_{m/e}^{p'-1}\int_{\bar{T}} gd\mu. \end{aligned}$$

Hence, $\mathcal{M}_\mu$ is of weak-type (1,1) and, by Marchinkiewicz interpolation theorem, it is also a bounded operator from $L^{p'}(\mu)$ to $L^{p'}(E(T),\eta)$. In particular, (3.7) holds with bounding constant $C' \lesssim [\mu]_{m/e}^{p'-1}$. This gives the assertion. □

3.3 Applications to the Dirichlet space

Consider the following seminorm on $Hol(\mathbb{D};\mathbb{C})$, where $\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| < 1\}$:

$$\{f\}^2 := \frac{1}{\pi}\int_{\mathbb{D}} |f'(z)|^2 dm(z),$$

being dm the Lebesgue measure on $\mathbb{D}$.

Definition 3.3.1. The Dirichlet space on $\mathbb{D}$ is the space of holomorphic functions on $\mathbb{D}$ having finite their $\{\cdot\}$ seminorm:

$$\mathcal{D} := \Big\{ f \in Hol(\mathbb{D};\mathbb{C}) \ : \ \{f\} < \infty \Big\}.$$

Remark 3.3.2. The Dirichlet space is a normed space under the norm

$$\|f\|_{\mathcal{D}}^2 := \{f\}^2 + |f(0)|^2.$$

Remark 3.3.3. Let $f \in \mathcal{D}$ with $f = u + iv$, where $u = \Re(f)$ and $v = \Im(f)$. Since the Jacobian of f is

$$Jf = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix},$$

has determinant given by

$$\det(Jf) = u_x^2 + u_y^2 = |\partial_x f|^2 = |f'|^2,$$

$\{f\}^2$ can be interpreted in terms of the area of $f(\mathbb{D})$: $\{f\}^2 = \frac{1}{\pi}|f(\mathbb{D})|$ counting multiplicities [1].

Remark 3.3.4. We denote with $Aut(\mathbb{D})$ the space of the automorphisms of $\mathbb{D}$ to itself, which is completely characterized: if $\varphi : \mathbb{D} \to \mathbb{D}$ is such an automorphism, then there exist $a \in \mathbb{D}$ and $t \in \mathbb{R}$ such that

$$\varphi(z) = e^{it}\frac{a-z}{1-\bar{a}z}. \tag{3.8}$$

These mappings are called **Möbius maps** and are all conformal mappings. We point out some geometric properties of $\mathcal{D}$ regarding these maps.

(1) For all $f \in \mathcal{D}$ and all $\varphi \in Aut(\mathbb{D})$, $\{f \circ \varphi\}^2 = \{f\}^2$.

(2) The hyperbolic metric $ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$ on $\mathbb{D}$ is φ-invariant for all $\varphi \in Aut(\mathbb{D})$:

$$\frac{2d\varphi(z)}{1-|\varphi(z)|^2} = \frac{2|dz|^2}{1-|z|^2}.$$

[1] If f is not injective, $f(\mathbb{D})$ may present overlapping having non-zero area.

We want to characterize the measures $\mu \geq 0$ on $\mathbb{D}$ such that the following weighted Poincaré-type inequality holds for all $f \in \mathcal{D}$:

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) \lesssim \left(\frac{1}{\pi} \int_{\mathbb{D}} |f'(z)|^2 dm(z) + |f(0)|^2 \right). \tag{3.9}$$

Denote with $C(\mu)$ the smallest positive constant such that (3.9) holds.

Consider the identification of the infinite rooted dyadic tree T_2 with $\partial\mathbb{D}$, choose $p = 2$ and consider the kernel $K(s,t) = \frac{1}{|s-t|^{1/2}}$, where $|s-t|$ is the Euclidean distance on ∂D [2].

Consider the inequality:

$$\mu\left(\bigcup_{j=1}^{N} S(I_j)\right) \lesssim Cap\left(\bigcup_{j=1}^{N} I_j\right), \tag{3.10}$$

where $I_1, \ldots, I_N$ are any disjoint closed arcs of $\partial\mathbb{D}$ and each $S(I_j)$ is the part of $\mathbb{D}$ surrounded by I_j and the hyberbolic geodesic connecting its endpoints [3], and let $cap(\mu)$ be the best constant with respect to which (3.10) holds. Then, we have the following characterization of $C(\mu)$:

Theorem 3.3.5 ([2])**.** *$C(\mu) \approx cap(\mu)$.*

Consider the double representation of T_2 as dyadic tilings of the unit square and of $\mathbb{D}$ given in figure 1.18. If $f \in \mathcal{D}$ is such that $f(0) = 0$, then

$$f(z) = \int_{\Gamma_z} f'(w)dw,$$

where $\Gamma_z : [0,1] \to \mathbb{D}$ is any smooth path connecting z to 0. $\Gamma_z([0,1])$ is a closed path in $\mathbb{D}$ which intersects a certain number of boxes of the dyadic tiling of $\mathbb{D}$. Let $Q(0)$ be the box containing 0 and $Q(z)$ be the box of $\mathbb{D}$ containing z. Then,

$$f(z) = \int_{\Gamma_z} f'(w)dw = \sum_{\alpha} \int_{\Gamma_z \cap Q_\alpha} f'(w)dw \approx \sum_{\alpha} f'(c(\alpha))(1 - |c(\alpha)|),$$

where the sums are taken over the α such that Q_α is a box which intersects $\Gamma_z([0,1])$ and $c(\alpha)$ is the point of Q_α corresponding to the center of the tile in $[0,1]^2$ which corresponds to Q_α. Proceeding with the calculation:

$$f(z) \approx \sum_{\alpha} f'(c(\alpha))(1 - |c(\alpha)|) = I\varphi(Q(z)),$$

[2]That is the length of the shortest arc connecting $s, t \in \partial\mathbb{D}$.

[3]Recall that hyperbolic geodesics are either arcs of diameters or arcs of circumferences that are orthogonal to ∂D.

where

$$\varphi(\alpha) := |f'(c(\alpha))|(1-|c(\alpha)|).$$

Heuristically, starting from the left hand-side of (3.9), we get

$$\int_{\mathbb{D}} |f(z)|^2 d\mu(z) = \sum_{\beta} \int_{Q_\beta} |f(z)|^2 d\mu(z) \approx \sum_{\beta} (I\varphi(Q_\beta))^2 \mu(Q_\beta),$$

while the right hand-side gives:

$$\frac{1}{\pi}\int_{\mathbb{D}} |f'|^2 dm = \sum_{\beta} \int_{Q_\beta} |f'|^2 dm \approx \sum_{\beta} |f'(c(\beta))|^2 \underbrace{|Q_\beta|}_{\approx (1-|Q_\beta|)^2},$$

so that we find that (3.9) is also equivalent to the dyadic Hardy inequality:

$$\sum_{\beta} (I\varphi(Q_\beta))^2 \mu(Q_\beta) \lesssim \sum_{\beta} \varphi(Q_\beta)^2. \tag{3.11}$$

Let $\langle\mu\rangle$ be the smallest constant with respect to which (3.11) holds.

Theorem 3.3.6. *μ is a Carleson-Stegenga measure if and only if the map $\beta \mapsto \mu(Q_\beta)$ satisfies (3.11) with $\langle\mu\rangle \approx C(\mu)$.*

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