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Tight wavelet frames on local fields

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Summary: In this paper, some algorithms for constructing tight wavelet frames on local fields using the unitary extension principles are suggested. We present a sufficient condition for finite number of functions to form a tight wavelet frame and establish general principles for constructing tight wavelet frames on local fields.

1 Introduction

Tight wavelet frames are different from the orthonormal wavelets because of redundancy. By sacrificing orthonormality and allowing redundancy, the tight wavelet frames become much easier to construct than the orthonormal wavelets. Tight wavelet frames provide representations of signals and images in applications, where redundancy of the representation is preferred and the perfect reconstruction property of the associated filter bank algorithm, as in the case of orthonormal wavelets, is kept (see [10]). The main tools for construction and characterization of wavelet frames are the several extension principles, the unitary extension principle (UEP) and oblique extension principle (OEP) as well as their generalized versions, the mixed unitary extension principle (MUEP) and the mixed oblique extension principle (MOEP). They give sufficient conditions for constructing tight and dual wavelet frames for any given refinable function which generates a multiresolution analysis (MRA). These essential methods were firstly introduced by Ron and Shen in [19, 20] and in the fundamental work of Daubechies et al. [8] for scalar refinable functions $f \in L^2(\mathbb{R}^d)$. The resulting tight wavelet frames are based on a multiresolution analysis, and the generators are often called *mother framelets*. The theory of tight wavelet frames has been extensively studied and well developed over the recent years. To mention only a few references on tight wavelet frames, the reader is referred to [4, 5, 15, 17] and many references therein.

In recent years there has been a considerable interest in the problem of constructing wavelet bases on locally compact Abelian groups, for example, Dahlke introduced multiresolution analysis and wavelets on locally compact Abelian groups [6], Lang [14] constructed compactly supported orthogonal wavelets on the locally compact Cantor

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dyadic group $\mathcal{C}$ by following the procedure of Daubechies [7] (or see also [9]) via scaling filters and these wavelets turn out to be certain lacunary Walsh series on the real line. Later on, Farkov [11] extended the results of Lang [14] on the wavelet analysis on the Cantor dyadic group $\mathcal{C}$ to the locally compact Abelian group G_p which is defined for an integer $p \geq 2$ and coincides with $\mathcal{C}$ when $p = 2$. Concerning the construction of wavelets on half-line $\mathbb{R}^+$, Farkov [12] has given the general construction of all compactly supported orthogonal p-wavelets in $L^2(\mathbb{R}^+)$ and proved necessary and sufficient conditions for scaling filters with p^n many terms ($p, n \geq 2$) to generate a p-MRA analysis in $L^2(\mathbb{R}^+)$. These studies were continued by Shah and Debnath in [21, 22] where they have given some new algorithms for constructing the corresponding wavelet frames and wavelet packets on the positive half-line $\mathbb{R}^+$. On the other hand, R.L. Benedetto and J.J. Benedetto [3] developed a wavelet theory for local fields and related groups. Jiang et al. [13] pointed out a method for constructing orthogonal wavelets on local field $\mathbb{K}$ with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(\mathbb{K})$. Subsequently, the tight wavelet frames on local fields were constructed by Li and Jiang in [16]. They have established necessary condition and sufficient conditions for tight wavelet frame on local fields in frequency domain. Recently, Behera and Jahan [1] have constructed wavelet packets and wavelet frame packets on local field $\mathbb{K}$ of positive characteristic and show how to construct an orthonormal basis from a Riesz basis. Further, they have given the characterization of scaling functions associated with given multiresolution analysis of positive characteristic on local field $\mathbb{K}$ in [2].

Although there are many results for tight wavelet frames on $\mathbb{R}^d$ using extension principles, the counterparts on local field are not yet reported. So this paper is concerned with tight wavelet frames on local field $\mathbb{K}$ using unitary extension principle. Therefore, the main aim of this work is to develop a constructive procedure for constructing tight wavelet frames on local field $\mathbb{K}$ of positive characteristic by following the procedure of Daubechies et al. [8] via extension principles.

This paper is structured as follows. In Section 2, we discuss some preliminary facts about local fields and introduce the concept of MRA based wavelet frames on local field $\mathbb{K}$ of positive characteristic. In Section 3, we present a general construction scheme for tight wavelet frames in $L^2(\mathbb{K})$ in terms of extension principles.

2 Preliminaries on local fields

Let $\mathbb{K}$ be a field and a topological space. Then $\mathbb{K}$ is called a *local field* if both $\mathbb{K}^+$ and $\mathbb{K}^*$ are locally compact Abelian groups, where $\mathbb{K}^+$ and $\mathbb{K}^*$ denote the additive and multiplicative groups of $\mathbb{K}$, respectively. If $\mathbb{K}$ is any field and is endowed with the discrete topology, then $\mathbb{K}$ is a local field. Further, if $\mathbb{K}$ is connected, then $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. If $\mathbb{K}$ is not connected, then it is totally disconnected. Hence by a local field, we mean a field $\mathbb{K}$ which is locally compact, non-discrete and totally disconnected. The p-adic fields are examples of local fields. More details are referred to [18] and [23]. In the rest of this paper, we use $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let $\mathbb{K}$ be a fixed local field. Then there is an integer $q = \mathrm{p}^r$, where p is a fixed prime element of $\mathbb{K}$ and r is a positive integer, and a norm $|.|$ on $\mathbb{K}$ such that for all $x \in \mathbb{K}$ we have $|x| \geq 0$ and for each $x \in \mathbb{K} \setminus \{0\}$ we get $|x| = q^k$ for some integer k. This norm is non-Archimedean, that is $|x+y| \leq \max\{|x|,|y|\}$ for all $x, y \in \mathbb{K}$ and $|x+y| = \max\{|x|,|y|\}$ whenever $|x| \neq |y|$. Let dx be the Haar measure on the locally compact, topological group $(\mathbb{K},+)$. This measure is normalized so that $\int_{\mathcal{D}} dx = 1$, where $\mathcal{D} = \{x \in \mathbb{K} : |x| \leq 1\}$ is the *ring of integers* in $\mathbb{K}$. Define $\mathcal{B} = \{x \in \mathbb{K} : |x| < 1\}$. The set $\mathcal{B}$ is called the *prime ideal* in $\mathbb{K}$. The prime ideal in $\mathbb{K}$ is the unique maximal ideal in $\mathcal{D}$ and hence as result $\mathcal{B}$ is both principal and prime. Therefore, for such an ideal $\mathcal{B}$ in $\mathcal{D}$, we have $\mathcal{B} = \langle \mathrm{p} \rangle = \mathrm{p}\mathcal{D}$.

Let $\mathcal{D}^* = \mathcal{D} \setminus \mathcal{B} = \{x \in \mathbb{K} : |x| = 1\}$. Then, it is easy to verify that $\mathcal{D}^*$ is a group of units in $\mathbb{K}^*$ and if $x \neq 0$, then we may write $x = \mathrm{p}^k x', x' \in \mathcal{D}^*$. Moreover, each $\mathcal{B}^k = \mathrm{p}^k\mathcal{D} = \left\{x \in \mathbb{K} : |x| < q^{-k}\right\}$ is a compact subgroup of $\mathbb{K}^+$ and usually known as the *fractional ideals* of $\mathbb{K}^+$ (see [18]). Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathcal{B}$ in $\mathcal{D}$, then every element $x \in \mathbb{K}$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathrm{p}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on $\mathbb{K}^+$ that is trivial on $\mathcal{D}$ but is non-trivial on $\mathcal{B}^{-1}$. Therefore, χ is constant on cosets of $\mathcal{D}$ so if $y \in \mathcal{B}^k$, then $\chi_y(x) = \chi(yx), x \in \mathbb{K}$. Suppose that χ_u is any character on $\mathbb{K}^+$, then clearly the restriction $\chi_u|\mathcal{D}$ is also a character on $\mathcal{D}$. Therefore, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathcal{D}$ in $\mathbb{K}^+$, then it is proved in [23] that the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathcal{D}$ is a complete orthonormal system on $\mathcal{D}$.

We now impose a natural order on the sequence $\{u(n)\}_{n\in\mathbb{N}_0}$. Since $\mathcal{D}/\mathcal{B} \cong GF(q) = \Gamma$, where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$ (see [23]). We choose a set $\{1 = \epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{c-1}\} \subset \mathcal{D}^*$ such that $\operatorname{span}\{1 = \epsilon_0, \epsilon_1, \epsilon_2, \ldots, \epsilon_{c-1}\} \cong GF(q)$. For $n \in \mathbb{N}_0$ such that

$$0 \leq n < q, \;\; n = a_0 + a_1 p + \cdots + a_{c-1} p^{c-1}, \; 0 \leq a_k < p \;\text{ and }\; k = 0, 1, \ldots, c-1,$$

we define

$$u(n) = (a_0 + a_1\epsilon_1 + \cdots + a_{c-1}\epsilon_{c-1})\mathrm{p}^{-1}.$$

Also, for $n = b_0 + b_1 q + \cdots + b_s q^s, n \geq 0, 0 \leq b_k < q$, we set

$$u(n) = u(b_0) + \mathrm{p}^{-1}u(b_1) + \cdots + \mathrm{p}^{-s}u(b_s).$$

Then, it is easy to verify that (see [23])

$$\begin{aligned} \{u(k) : k \in \mathbb{N}_0\} &= \{-u(k) : k \in \mathbb{N}_0\}, \\ \{u(k) + u(\ell) : k \in \mathbb{N}_0\} &= \{u(k) : k \in \mathbb{N}_0\}, \text{ for } \ell \in \mathbb{N}_0 \end{aligned}$$

and $u(n) = 0 \Leftrightarrow n = 0$. Further, hereafter we will denote $\chi_{u(n)}$ by $\chi_n, n \geq 0$. We also denote the test function space on $\mathbb{K}$ by Ω, i.e., each function f in Ω is a finite linear combination of functions of the form $\mathbf{1}_k(x-h), h \in \mathbb{K}, k \in \mathbb{Z}$, where $\mathbf{1}_k$ is the characteristic function of $\mathcal{B}^k$. Then, it is clear that Ω is dense in $L^p(\mathbb{K}), 1 \leq p < \infty$, and each function in Ω is of compact support and so is its Fourier transform.

The Fourier transform of a function $f \in L^1(\mathbb{K})$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\, \overline{\chi_\xi(x)} dx.$$

Note that

$$\hat{f}(\xi) = \int_{\mathbb{K}} f(x)\, \overline{\chi_\xi(x)} dx = \int_{\mathbb{K}} f(x)\, \chi(-\xi x) dx.$$

The properties of the Fourier transform on the local field $\mathbb{K}$ are quite similar to those of the Fourier analysis on the real line (see [18,23]). In particular, if $f \in L^1(\mathbb{K}) \cap L^2(\mathbb{K})$, then $\hat{f} \in L^2(\mathbb{K})$ and

$$\|\hat{f}\|_2 = \|f\|_2.$$

For any function $\varphi \in L^2(\mathbb{K})$, let V_0 be the closed shift invariant space generated by $\{\varphi(x - u(k)) : k \in \mathbb{Z}^+\}$ and $V_j = \{\varphi(\mathfrak{p}^{-j}.) : \varphi \in V_0\}$, $j \in \mathbb{Z}$. Then, it is claimed in [13] that the closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ forms an MRA in $L^2(\mathbb{K})$. The generator φ of the MRA is known as a *scaling function* or a *refinable function*. Recall that an MRA is a family of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{K})$ that satisfies: (i) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$, (ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{K})$ and (iii) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$ (see [3]).

For given $\Psi := \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{K})$, define the wavelet system

$$X(\Psi) := \left\{\psi_{\ell,j,k} : 1 \le \ell \le L; j \in \mathbb{Z}, k \in \mathbb{N}_0\right\} \tag{2.1}$$

where $\psi^\ell_{j,k} = q^{j/2}\psi^\ell(\mathfrak{p}^j. - u(k))$. The wavelet system $X(\Psi)$ is called a *wavelet frame*, if there exist positive numbers $0 < A \le B < \infty$ such that for all $f \in L^2(\mathbb{K})$

$$A\|f\|^2 \le \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle f, \psi_{\ell,j,k}\rangle|^2 \le B\|f\|^2. \tag{2.2}$$

The largest constant A and the smallest constant B satisfying (2.2) are called the *lower and upper wavelet frame bound*, respectively. A wavelet frame is a *tight wavelet frame* if A and B are chosen so that $A = B$ and then the set $\Psi := \{\psi_1, \ldots, \psi_L\}$ is called a set of *generators* for the corresponding tight wavelet frame. Furthermore, the wavelet frame is called a *Parseval wavelet frame* if $A = B = 1$, i.e.,

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} |\langle f, \psi_{\ell,j,k}\rangle|^2 = \|f\|^2, \ \ \forall\, f \in L^2(\mathbb{K}) \tag{2.3}$$

and in this case, every function $f \in L^2(\mathbb{K})$ can be written as

$$f(x) = \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{N}_0} \langle f, \psi_{\ell,j,k}\rangle \psi_{\ell,j,k}(x).$$

A finite family $\Psi := \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{K})$ that satisfies (2.3) is called an *MRA tight wavelet frame*, with frame bound equal to 1, associated with refinable function φ that generates the subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{K})$ in the sense of increasing property (i) of the MRA, if $\Psi \subset V_1$.

In order to obtain a fast wavelet frame transform, tight wavelet frames are generally derived from refinable functions via a multiresolution analysis. We say that $\varphi \in L^2(\mathbb{K})$ is a *refinable function*, if it satisfies an equation of the type

$$\varphi(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} h_k \, \varphi\big(\mathfrak{p}^{-1}x - u(k)\big) \tag{2.4}$$

for some $\{h_k : k \in \mathbb{N}_0\} \in l^2(\mathbb{N}_0)$. The functional equation (2.4) is known as the *refinement equation*. The Fourier transform of (2.4) yields

$$\hat{\varphi}(\xi) = m_0(\mathfrak{p}\xi)\, \hat{\varphi}(\mathfrak{p}\xi), \tag{2.5}$$

where

$$m_0(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} h_k \, \overline{\chi_k(\xi)}, \tag{2.6}$$

is an integral-periodic function in $L^2(\mathcal{D})$ and is often called the *refinement symbol* of φ. Observe that $\chi_k(0) = \hat{\varphi}(0) = 1$. Therefore by letting $\xi = 0$ in (2.5) and (2.6), we obtain $\sum_{k \in \mathbb{N}_0} h_k = 1$. Further, it is proved in [1, Theorem 5.1] that a function $\varphi \in L^2(\mathbb{K})$ generates an MRA in $L^2(\mathbb{K})$ if and only if

$$\sum_{k \in \mathbb{N}_0} \big|\hat{\varphi}\big(\xi + u(k)\big)\big|^2 = 1, \quad \text{for } a.e.\ \xi \in \mathcal{D}, \tag{2.7}$$

and

$$\lim_{j \to \infty} \big|\hat{\varphi}(\mathfrak{p}^j \xi)\big| = 1, \ \text{ for } a.e.\ \xi \in \mathbb{K}. \tag{2.8}$$

Let the refinable function $\varphi \in L^2(\mathbb{K})$ generates an MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{K})$ and $\Psi := \{\psi_1, \ldots, \psi_L\} \subset V_1$, then

$$\psi_\ell(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} h_k^\ell \, \varphi\big(\mathfrak{p}^{-1}x - u(k)\big), \ \ \ell = 1, \ldots, L. \tag{2.9}$$

The Fourier transform of (2.9) gives

$$\hat{\psi}_\ell(\xi) = m_\ell(\mathfrak{p}\xi)\, \hat{\varphi}(\mathfrak{p}\xi), \tag{2.10}$$

where

$$m_\ell(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} h_k^\ell \, \overline{\chi_k(\xi)}, \ \ \ell = 1, \ldots, L \tag{2.11}$$

are the integral-periodic function in $L^2(\mathcal{D})$ and are called the *framelet symbols* or *wavelet masks*. For notational convenience, refinement mask together with wavelet masks $\{m_0, m_1, \ldots, m_L\}$ is also called *combined MRA masks* (see [8]).

With $m_\ell(\xi), \ell = 0, 1, \ldots, L$ as the wavelet masks, we formulate the matrix

$$\begin{aligned}&\mathcal{M}(\xi)\\ &= \begin{pmatrix} m_0\big(\mathfrak{p}\xi + \mathfrak{p}u(0)\big) & m_1\big(\xi + \mathfrak{p}u(0)\big) & \cdots & m_L\big(\mathfrak{p}\xi + \mathfrak{p}u(0)\big) \\ m_0\big(\mathfrak{p}\xi + \mathfrak{p}u(1)\big) & m_1\big(\mathfrak{p}\xi + \mathfrak{p}u(1)\big) & \cdots & m_L\big(\mathfrak{p}\xi + \mathfrak{p}u(1)\big) \\ \vdots & \vdots & \ddots & \vdots \\ m_0\big(\mathfrak{p}\xi + \mathfrak{p}u(q-1)\big) & m_1\big(\mathfrak{p}\xi + \mathfrak{p}u(q-1)\big) & \cdots & m_L\big(\mathfrak{p}\xi + \mathfrak{p}u(q-1)\big) \end{pmatrix},\end{aligned} \tag{2.12}$$

and it is well-known (see [8]) that for constructing tight frames via multiresolution analysis the matrix $\mathcal{M}(\xi)$ plays an important role. Infact, the equality in

$$\mathcal{M}(\xi)\mathcal{M}^*(\xi) = I_q, \ \text{ for } a.e.\, \xi \in \mathcal{D} \tag{2.13}$$

is equivalent to that for any function $f \in L^2(\mathbb{K})$, there exists exact formulae of decomposition and reconstruction.

3 Tight wavelet frames on local field $\mathbb{K}$

The following theorem, the main result of our paper, shows that a unitary matrix leads to a tight wavelet frame on local field $\mathbb{K}$.

Theorem 3.1 *Suppose that the refinable function φ and the framelet symbols $m_0, m_1, \dots, m_L$ satisfy* (2.5)–(2.10)*. Furthermore, if the matrix $\mathcal{M}(\xi)$ satisfy* (2.13)*, then the wavelet system $X(\Psi)$ given by* (2.1) *is a tight wavelet frame for $L^2(\mathbb{K})$ with frame bound 1.*

We split the proof of Theorem 3.1 into several lemmas.

Lemma 3.2 *If the framelet symbols $m_\ell, \ell = 0, 1, \dots, L$ satisfy the condition* (2.13)*. Then for any $\xi \in \mathbb{K}$, we have*

$$\sum_{k=0}^{q-1} |m_\ell(\mathfrak{p}\xi + \mathfrak{p}u(k))|^2 \le 1, \ \text{for } \ell = 0, 1, \dots, L. \tag{3.1}$$

Proof: Without the lose of generality, it suffices to prove inequality (3.1) only for $\ell = 0$. Let

$$\mathcal{M}_0(\xi) = \begin{pmatrix} m_1\big(\mathfrak{p}\xi + \mathfrak{p}u(0)\big) & m_2\big(\mathfrak{p}\xi + \mathfrak{p}u(0)\big) & \dots & m_L\big(\mathfrak{p}\xi + \mathfrak{p}u(0)\big) \\ m_1\big(\mathfrak{p}\xi + \mathfrak{p}u(1)\big) & m_2\big(\mathfrak{p}\xi + \mathfrak{p}u(1)\big) & \dots & m_L\big(\mathfrak{p}\xi + \mathfrak{p}u(1)\big) \\ \vdots & \vdots & \ddots & \vdots \\ m_1\big(\mathfrak{p}\xi + \mathfrak{p}u(q-1)\big) & m_2\big(\mathfrak{p}\xi + \mathfrak{p}u(q-1)\big) & \dots & m_L\big(\mathfrak{p}\xi + \mathfrak{p}u(q-1)\big) \end{pmatrix}. \tag{3.2}$$

Taking $\alpha = \Big(m_0\big(\mathfrak{p}\xi + \mathfrak{p}u(0)\big), m_0\big(\mathfrak{p}\xi + \mathfrak{p}u(1)\big), \dots, m_0\big(\mathfrak{p}\xi + \mathfrak{p}u(q-1)\big)\Big)^T$, we can rewrite (2.13) as

$$\mathbb{M}(\xi) = \mathcal{M}_0(\xi)\mathcal{M}^*{}_0(\xi) = I_q - \alpha\alpha^T = (\beta_1, \beta_2, \dots, \beta_q), \tag{3.3}$$

where

$$
\begin{aligned}
\beta_1 &= \Big(1-\big|m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(0)\big)\big|^2,\ -m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(1)\big)\overline{m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(0)\big)},\\
&\qquad \ldots,-m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(q-1)\big)\overline{m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(0)\big)}\Big)^T\\
\beta_2 &= \Big(-m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(0)\big)\overline{m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(1)\big)},\ 1-\big|m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(1)\big)\big|^2,\\
&\qquad \ldots,-m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(q-1)\big)\overline{m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(1)\big)}\Big)^T\\
&\ \vdots\\
\beta_q &= \Big(-m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(0)\big)\overline{m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(q-1)\big)},\\
&\qquad -m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(1)\big)\overline{m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(q-1)\big)},\\
&\qquad \ldots,1-\big|m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(q-1)\big)\big|^2\Big)^T.
\end{aligned}
$$

The q-eigenvalues of the Hermitian matrix $\mathbb{M}(\xi)$ are given by

$$
\gamma_1(\xi)=\gamma_2(\xi)=\cdots=\gamma_{q-1}(\xi)=1,\ \gamma_q(\xi)=1-\sum_{k=0}^{q-1}\big|m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(k)\big)\big|^2. \tag{3.4}
$$

Since $\mathbb{M}(\xi)$ is a positive definite matrix, hence $\gamma_q(\xi)\geq 0$, which is (3.1) for $\ell=0$. This completes the proof. □

We further assume that:

$$
\sum_{k=0}^{q-1}\big|m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(k)\big)\big|^2\leq 1,\ \ a.e. \tag{3.5}
$$

and $\mathbb{M}(\xi)$ is as in (3.3) with q-eigenvalues given by (3.4), then the unit eigen-vectors of the matrix $\mathbb{M}(\xi)$ can be represented by

$$
\begin{aligned}
\delta_1 &= \frac{1}{\Omega_1}\Big(-\overline{m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(1)\big)},-\overline{m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(0)\big)},\ 0,\ldots,0\Big)^T\\
\delta_k &= \frac{1}{\Omega_k}\Big(-m_0(\mathfrak{p}\xi+\mathfrak{p}u(0))\overline{m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(k)\big)},\ldots,\\
&\qquad -m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(k-1)\big)\overline{m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(k)\big)},\\
&\qquad \sum_{t=0}^{k-1}|m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(t)\big)|^2,\ 0,\ldots,0\Big)^T,\quad k=2,3,\ldots,q-1,\\
&\ \vdots\\
\delta_q &= \frac{1}{\Omega_q}\Big(m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(0)\big),m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(1)\big),\ldots,m_0\big(\mathfrak{p}\xi+\mathfrak{p}u(q-1)\big)\Big)^T
\end{aligned}
$$

where

$$\Omega_1^2 = \left|m_0\big(\mathrm{p}\xi+\mathrm{p}u(0)\big)\right|^2+\left|m_0\big(\mathrm{p}\xi+\mathrm{p}u(1)\big)\right|^2,$$

$$\Omega_k^2 = \left|m_0\big(\mathrm{p}\xi+\mathrm{p}u(k)\big)\right|^2\sum_{t=0}^{k-1}\left|m_0\big(\mathrm{p}\xi+\mathrm{p}u(t)\big)\right|^2+\left(\sum_{t=0}^{k-1}\left|m_0\big(\mathrm{p}\xi+\mathrm{p}u(t)\big)\right|^2\right)^2, \quad k=2,\ldots,q-1,$$

$$\vdots$$

$$\Omega_q^2 = \sum_{t=0}^{q-1}\left|m_0\big(\mathrm{p}\xi+\mathrm{p}u(t)\big)\right|^2.$$

Thus, we have

$$\mathbb{M}(\xi)=\mathcal{P}(\xi)\Lambda(\xi)\mathcal{P}^*(\xi)$$

where

$$\mathcal{P}(\xi)=\big(\delta_1,\delta_2,\ldots,\delta_q\big), \quad \Lambda(\xi)=\mathrm{diag}\big(\gamma_1,\gamma_2,\ldots,\gamma_q\big). \tag{3.6}$$

Using (2.8), (3.5) and the fact that $\hat{\varphi}(\xi)$ is continuous at $\xi=0$, it is easy to verify that

$$\sum_{k\in\mathbb{N}_0}\left|\hat{\varphi}\big(\xi+u(k)\big)\right|^2\le 1, \ \ a.e.\ \xi\in\mathcal{D}. \tag{3.7}$$

Lemma 3.3 *Let $\varphi\in L^2(\mathbb{K})$ be a refinable function with refinement mask $m_0(\xi)$ such that condition* (3.5) *is satisfied. Then, $P_j=\sum_{k\in\mathbb{N}_0}\left|\langle f,\varphi_{j,k}\rangle\right|^2<\infty$, for any function $f\in L^2(\mathbb{K})$ and*

$$(i) \quad \lim_{j\to\infty}P_j=\|f\|^2; \qquad (ii) \quad \lim_{j\to-\infty}P_j=0$$

where $\varphi_{j,k}(x)=q^{j/2}\varphi\big(\mathrm{p}^{-j}x-u(k)\big), j\in\mathbb{Z}, k\in\mathbb{N}_0$.

Proof: Implementation of Plancherel and Parseval formulae yields

$$\begin{aligned}
&\sum_{k\in\mathbb{N}_0}\left|\langle f,\varphi_{j,k}\rangle\right|^2\\
&\quad = q^j\sum_{k\in\mathbb{N}_0}\left|\int_{\mathbb{K}}\hat{f}(\xi)\overline{\hat{\varphi}(\mathrm{p}^j\xi)}\chi_k(\mathrm{p}^j\xi)d\xi\right|^2\\
&\quad = q^j\sum_{k\in\mathbb{N}_0}\left|\int_{\mathcal{B}^{-j}}\left[\sum_{n\in\mathbb{N}_0}\hat{f}\big(\xi+\mathrm{p}^{-j}u(n)\big)\overline{\hat{\varphi}\big(\mathrm{p}^j(\xi+\mathrm{p}^{-j}u(n))\big)}\right]\chi_k(\mathrm{p}^j\xi)d\xi\right|^2\\
&\quad = \int_{\mathcal{B}^{-j}}\left|\sum_{n\in\mathbb{N}_0}\hat{f}\big(\xi+\mathrm{p}^{-j}u(n)\big)\overline{\hat{\varphi}\big(\mathrm{p}^j(\xi+\mathrm{p}^{-j}u(n))\big)}\right|^2d\xi\\
&\quad = \|F_j\|^2,
\end{aligned} \tag{3.8}$$

where

$$F_j(\xi) = \sum_{n\in\mathbb{N}_0} \hat{f}\big(\xi + \mathfrak{p}^{-j}u(n)\big)\overline{\hat{\varphi}\big(\mathfrak{p}^j(\xi + \mathfrak{p}^{-j}u(n))\big)}.$$

Now let us introduce the following sequences of functions

$$\hat{g}_j(\xi) = \begin{cases} \hat{f}(\xi), & \text{if } \xi \in \mathcal{B}^{-j}, \\ 0, & \text{if } \xi \notin \mathcal{B}^{-j}, \end{cases} \qquad h_j = f - g_j, \quad j \in \mathbb{N}_0$$

$$G_j(\xi) = \sum_{n\in\mathbb{N}_0} \hat{g}_j\big(\xi + \mathfrak{p}^{-j}u(n)\big)\overline{\hat{\varphi}\big(\mathfrak{p}^j(\xi + \mathfrak{p}^{-j}u(n))\big)},$$

$$H_j(\xi) = \sum_{n\in\mathbb{N}_0} \hat{h}_j\big(\xi + \mathfrak{p}^{-j}u(n)\big)\overline{\hat{\varphi}\big(\mathfrak{p}^j(\xi + \mathfrak{p}^{-j}u(n))\big)}.$$

It is obvious that as $j \to \infty$, $\|G_j\|_2 = \|\hat{f}\|_2$. Further, by (3.7), we have

$$\begin{aligned} \|H_j\|^2 &= \int_{\mathcal{B}^{-j}} \left| \sum_{n\in\mathbb{N}_0} \hat{h}_j\big(\xi + \mathfrak{p}^{-j}u(n)\big)\overline{\hat{\varphi}\big(\mathfrak{p}^j(\xi + \mathfrak{p}^{-j}u(n))\big)} \right|^2 d\xi \\ &\le \int_{\mathcal{B}^{-j}} \Big| \sum_{n\in\mathbb{N}_0} \hat{h}_j\big(\xi + \mathfrak{p}^{-j}u(n)\big)\Big|^2 \sum_{n\in\mathbb{N}_0} \big|\hat{\varphi}\big(\mathfrak{p}^j\xi + u(n)\big)\big|^2 d\xi \\ &\le \int_{\mathcal{B}^{-j}} \Big| \sum_{n\in\mathbb{N}_0} \hat{h}_j\big(\xi + \mathfrak{p}^{-j}u(n)\big)\Big|^2 d\xi \\ &\le \|\hat{h}_j\|^2 \to 0, \text{ as } j \to \infty, \end{aligned} \tag{3.9}$$

since

$$\|G_j\|_2 - \|H_j\|_2 \le \|F_j\|_2 = \|G_j + H_j\|_2 \le \|G_j\|_2 + \|H_j\|_2,$$

It follows from (3.8) and (3.9) that

$$\sum_{k\in\mathbb{N}_0} \big|\langle f, \varphi_{j,k}\rangle\big|^2 = \|F_j\|_2^2 \to \|\hat{f}\|_2^2 = \|f\|_2^2, \text{ as } j \to \infty.$$

Thus, relation (i) is proved.

In order to prove (ii), we consider

$$f_1(x) = \begin{cases} f(x), & \text{if } x \in \mathcal{B}^{-1}, \\ 0, & \text{if } x \notin \mathcal{B}^{-1}. \end{cases}$$

Clearly, $f_1 \in \Omega$ and recall that the space Ω of all finite linear combinations of functions of the form $\mathbf{1}_k(.-h)$ is dense in $L^2(\mathbb{K})$. Therefore, for an arbitrary $\varepsilon > 0$ $(\varepsilon < \frac{1}{q})$, we have $\|f - f_1\|_2 < \varepsilon$. Since

$$\begin{aligned}
\sum_{k\in\mathbb{N}_0} \left|\langle f, \varphi_{j,k}\rangle\right|^2 &= \sum_{k\in\mathbb{N}_0} \left|\langle f - f_1, \varphi_{j,k}\rangle + \langle f_1, \varphi_{j,k}\rangle\right|^2 \\
&\le q \sum_{k\in\mathbb{N}_0} \left|\langle f_1, \varphi_{j,k}\rangle\right|^2 + q \sum_{k\in\mathbb{N}_0} \left|\langle f - f_1, \varphi_{j,k}\rangle\right|^2 \\
&\le q \sum_{k\in\mathbb{N}_0} \left|\langle f_1, \varphi_{j,k}\rangle\right|^2 + \|f - f_1\|_2 \\
&\le q \sum_{k\in\mathbb{N}_0} \left|\langle f_1, \varphi_{j,k}\rangle\right|^2 + \varepsilon.
\end{aligned}$$

Therefore, we need only to prove that

$$\lim_{j\to-\infty} \sum_{k\in\mathbb{N}_0} \left|\langle f_1, \varphi_{j,k}\rangle\right|^2 = 0.$$

We have

$$\begin{aligned}
\sum_{k\in\mathbb{N}_0} \left|\langle f_1, \varphi_{j,k}\rangle\right|^2 &= \sum_{k\in\mathbb{N}_0} \left|\int_{|x|\le q} f(x)\varphi_{j,k}(x)dx\right|^2 \\
&= \sum_{k\in\mathbb{N}_0} \left|\int_{|x|\le q} f(x) q^{j/2}\overline{\varphi\big(\mathrm{p}^{-j}x - u(k)\big)}dx\right|^2 \\
&\le q^j \sum_{k\in\mathbb{N}_0} \left(\int_{|x|\le q} |f(x)|\left|\varphi\big(\mathrm{p}^{-j}x - u(k)\big)\right| dx\right)^2 \\
&\le q^j \|f\|^2 \sum_{k\in\mathbb{N}_0} \left(\int_{|x|\le q} \left|\varphi\big(\mathrm{p}^{-j}x - u(k)\big)\right| dx\right)^2 \\
&\le q^j \|f\|^2 \sum_{k\in\mathbb{N}_0} q \int_{|x|\le q} \left|\varphi\big(\mathrm{p}^{-j}x - u(k)\big)\right|^2 dx \\
&= \|f\|^2 q \sum_{k\in\mathbb{N}_0} \int_{|y+u(k)|\le q^{j+1}} |\varphi(y)|^2 dy \\
&= \|f\|^2 q \sum_{k\in\mathbb{N}_0} \int_{\Gamma_{j,k}} |\varphi(y)|^2 dy
\end{aligned}$$

where $\Gamma_{j,k} = \{y : |y + u(k)| \le q^{j+1}\}$. Let $\Gamma_j = \bigcup_{k\in\mathbb{N}_0} \Gamma_{j,k}$. Therefore, it is easy to verify that for sufficiently small j, the collection $\{\Gamma_{j,k} : k \in \mathbb{N}_0\}$ is a disjoint collection, since $\{u(k) : k \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathcal{D}$ in $\mathbb{K}$ and $\{u(k) : k \in \mathbb{N}_0\} = \{-u(k) : k \in \mathbb{N}_0\}$. Therefore, for j small enough, we have

$$\begin{aligned}\sum_{k\in\mathbb{N}_0} \left|\langle f_1, \varphi_{j,k}\rangle\right|^2 &\le \|f\|^2 q \sum_{k\in\mathbb{N}_0} \int_{\Gamma_j} |\varphi(y)|^2 dy \\ &= \|f\|^2 q \int_{\mathbb{K}} \mathbf{1}_{\Gamma_j}(y)|\varphi(y)|^2 dy. \end{aligned} \tag{3.10}$$

Note that, if $y \neq u(k), k \in \mathbb{N}_0$, then $\mathbf{1}_{\Gamma_j}(y) \to 0$ as $j \to -\infty$. In fact, there exists $J \in \mathbb{Z}$ such that $\mathbf{1}_{\Gamma_j}(y) = 0$ if $j < J$. Then, by the application of Dominated Convergence Theorem, the right hand side of (3.10) tends to 0 as $j \to -\infty$ and hence as a result, we get the desired result, i.e., $\lim_{j\to-\infty} P_j = 0$. □

Lemma 3.4 *If* (2.13) *holds, then for any* $f \in L^2(\mathbb{K})$ *and* $J \in \mathbb{Z}$, *we have*

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0} \left|\langle f, \psi_{\ell,j,k}\rangle\right|^2 = \sum_{k\in\mathbb{N}_0} \left|\langle f, \varphi_{J,k}\rangle\right|^2 + \sum_{\ell=1}^{L}\sum_{j\ge J}\sum_{k\in\mathbb{N}_0} \left|\langle f, \psi_{\ell,j,k}\rangle\right|^2 < \infty. \tag{3.11}$$

Proof: It follows from (2.13) that

$$\sum_{\ell=0}^{L} \left|m_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)\right|^2 = 1, \;\; k = 0, 1, 2, \ldots, q-1$$

and for $k_1 \neq k_2,\ 0 \le k_1, k_2 \le q-1, k_1, k_2 \in \mathbb{N}_0$,

$$\sum_{\ell=0}^{L} m_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(k_1)\big)\overline{m_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(k_2)\big)} = 0. \tag{3.12}$$

Let

$$\Delta_t = \sum_{k\in\mathbb{N}_0} \hat{f}\big(\xi + \mathfrak{p}^{-r-1}u(k) + (t-1)\mathfrak{p}^{-r}\big)\overline{\hat{\varphi}\big(\mathfrak{p}^{r+1}\xi + u(k) + (t-1)\mathfrak{p}\big)}, \;\; 1 \le t \le q.$$

By analogy with (3.8), for any $r \in \mathbb{Z}$, we obtain

$$\begin{aligned}&\sum_{k\in\mathbb{N}_0} \left|\langle f, \varphi_{r,k}\rangle\right|^2 + \sum_{\ell=1}^{L}\sum_{k\in\mathbb{N}_0} \left|\langle f, \psi_{\ell,r,k}\rangle\right|^2 \\ &= \int_{\mathcal{B}^{-r}} \Big|\sum_{k\in\mathbb{N}_0} \hat{f}\big(\xi + \mathfrak{p}^{-r}u(k)\big)\overline{\hat{\varphi}\big(\mathfrak{p}^r(\xi + \mathfrak{p}^{-r}u(k))\big)}\Big|^2 d\xi \\ &\quad + \sum_{\ell=1}^{L}\int_{\mathcal{B}^{-r}} \Big|\sum_{k\in\mathbb{N}_0} \hat{f}\big(\xi + \mathfrak{p}^{-r}u(k)\big)\overline{\hat{\psi}_\ell\big(\mathfrak{p}^r(\xi + \mathfrak{p}^{-r}u(k))\big)}\Big|^2 d\xi\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell=0}^{L} \int_{\mathcal{B}^{-r}} \Big| \sum_{k\in\mathbb{N}_0} \hat{f}\big(\xi + \mathrm{p}^{-r}u(k)\big) \overline{m_\ell\big(\mathrm{p}^{r+1}(\xi + \mathrm{p}^{-r}u(k))\big)} \\
&\qquad\qquad \cdot \overline{\hat{\varphi}_\ell\big(\mathrm{p}^{r+1}(\xi + \mathrm{p}^{-r}u(k))\big)} \Big|^2 d\xi \\
&= \sum_{t=1}^{q} \left[\sum_{\ell=0}^{L} \int_{\mathcal{B}^{-r}} \left| \Delta_t \overline{m_\ell\big(\mathrm{p}^{r+1}\xi + \mathrm{p}(t-1)u(k)\big)} \right|^2 d\xi \right] \\
&\quad + \sum_{t=1}^{q} \sum_{s=1, t\neq s}^{q} \left[\sum_{\ell=0}^{L} \int_{\mathcal{B}^{-r}} \Delta_t \overline{m_\ell\big(\mathrm{p}^{r+1}\xi + \mathrm{p}(t-1)u(k)\big)}\, \overline{\Delta_s} \right. \\
&\qquad\qquad \left. \cdot\, m_\ell\big(\mathrm{p}^{r+1}\xi + \mathrm{p}(s-1)u(k)\big) d\xi \right] \\
&= \sum_{t=1}^{q} \left[\int_{\mathcal{B}^{-r}} |\Delta_t|^2 d\xi \right] \\
&= \sum_{k\in\mathbb{N}_0} \left| \langle f, \varphi_{r+1,k} \rangle \right|^2 < \infty.
\end{aligned}
$$

Using Lemma 3.3, we obtain Lemma 3.4. □

Theorem 3.1 is an easy consequence of Lemmas 3.2–3.4.

It is immediate from Theorem 3.1 that the construction of wavelet frames from a given refinable function on local field $\mathbb{K}$ is to seek solutions $m_\ell(\xi)$, $\ell = 1, 2, \ldots, L$, satisfying equation (2.13). In the following, we will give the solution structure.

Theorem 3.5 *Let $m_0(\xi)$ be the refinement mask of a refinable function φ of an MRA and satisfies inequality* (3.5)*, then there exist integral-periodic functions $m_1, m_2, \ldots, m_L$ satisfying* (2.13) *for $L = q$. Moreover, any solution of* (2.13) *can be represented in the form of the first row of the matrix*

$$\mathcal{M}_0(\xi) = \mathcal{P}(\xi)\sqrt{\Lambda(\xi)}\,\mathcal{Q}(\xi)$$

where $\mathcal{Q}(\xi)$ is an arbitrary unitary matrix of order $q \times q$ whose elements are integral-periodic measurable components.

Proof: The conclusion of this result is clear. In fact, $\mathbb{M}(\xi) = \mathcal{M}_0(\xi)\mathcal{M}_0^*(\xi)$ is a Hermitian matrix. Hence

$$\mathbb{M}(\xi) = \mathcal{M}_0(\xi)\mathcal{M}_0^*(\xi) = \mathcal{P}(\xi)\sqrt{\Lambda(\xi)}\,\mathcal{Q}(\xi)\big(\mathcal{P}(\xi)\sqrt{\Lambda(\xi)}\,\mathcal{Q}(\xi)\big)^*$$

where $\mathcal{Q}(\xi)$ is an arbitrary unitary matrix of order $q \times q$. Thus, in view of (3.3) and (3.6), we obtain

$$\mathcal{M}_0(\xi) = \mathcal{P}(\xi)\sqrt{\Lambda(\xi)}\,\mathcal{Q}(\xi).$$

This completes the proof. □

Remark 3.6 When $L > q$ to describe all possible solution to (2.13), we have to take an arbitrary unitary matrix $\mathcal{Q}(\xi)$ with integral-periodic elements and $\Lambda'(\xi)$ can be extension of $\sqrt{\Lambda(\xi)}$ by means of filling all new column with zero.

Example 3.7 Let

$$m_0(\xi) = \frac{1}{q}\sum_{k=0}^{q-1}\overline{\chi_k(\xi)} = \begin{cases} 1, & |\xi| \le q^{-1}, \\ 0, & q^{-1} < |\xi| \le 1, \end{cases}$$

be the *refinement mask* of the characteristic function φ of $\mathcal{D}$ whose refinement equation is given by

$$\varphi(x) = \sum_{k=0}^{q-1}\varphi\big(\mathfrak{p}^{-1}x - u(k)\big).$$

Define the integral-periodic functions $m_\ell(\xi), \ell = 1,2,\dots,L$ as follows:

$$m_\ell(\xi) = \frac{1}{\sqrt{2q}}\Big[\overline{\chi_\ell(\xi)} - \overline{\chi_{\ell-1}(\xi)}\Big], \;\; \ell = 1,2,\dots,L.$$

Then, clearly the matrix $\mathcal{M}(\xi) = \big[m_\ell\big(\mathfrak{p}\xi + \mathfrak{p}u(k)\big)\big]$ formed by $m_\ell(\xi)$, $1 \le \ell \le L$ satisfy the UEP condition (2.13) and hence by Theorem 3.1, the system

$$X(\Psi) = \Big\{\psi_{\ell,j,k} : 1 \le \ell \le L; j \in \mathbb{Z}, k \in \mathbb{N}_0\Big\}$$

generated by the functions

$$\psi_\ell(x) = \sqrt{\frac{q}{2}}\big[\varphi\big(\mathfrak{p}^{-1}x - u(\ell)\big) - \varphi\big(\mathfrak{p}^{-1}x - u(\ell-1)\big)\big], \;\; \ell = 1,2,\dots,L$$

forms a tight wavelet frames for $L^2(\mathbb{K})$.

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