VCE 2017 Mathematical Methods Exam 1 Solutions

Question 1

a. $f:(-2,\infty)\to\mathbb{R}$, $f(x)=\dfrac{x}{x+2}$

Using the quotient rule:

$$f'(x)=\frac{(1)(x+2)-(1)(x)}{(x+2)^2}$$
$$=\frac{x+2-x}{(x+2)^2}$$
$$=\frac{2}{(x+2)^2}$$

b. $g(x)=\left(2-x^3\right)^3$

Using the chain rule:

$$g'(x)=3\left(2-x^3\right)^2\left(-3x^2\right)$$
$$=-9x^2\left(2-x^3\right)^2$$

Therefore at $x=1$:

$$g'(1)=-9(1)^2\left(2-(1)^3\right)^2$$
$$=-9(1)(1)$$
$$=-9$$

Question 2

$y=x\log_e(3x)$

a. Using the product rule:

$$\frac{dy}{dx}=(1)\left(\log_e(3x)\right)+(x)\left(\frac{3}{3x}\right)$$
$$=\log_e(3x)+1$$

b. $\log_e(3x)+1=\dfrac{d}{dx}\left(x\log_e(3x)\right)$

$$\int_1^2\left(\log_e(3x)+1\right)dx=\left[x\log_e(3x)\right]_1^2$$
$$=2\log_e(6)-1\log_e(3)$$
$$=2\log_e(6)-\log_e(3)$$
$$=\log_e(36)-\log_e(3)$$
$$=\log_e(12)$$

Question 3

$f:[-3,0]\to\mathbb{R},\ f(x)=(x+2)^2(x-1)$

a. Expanding $f(x)$:

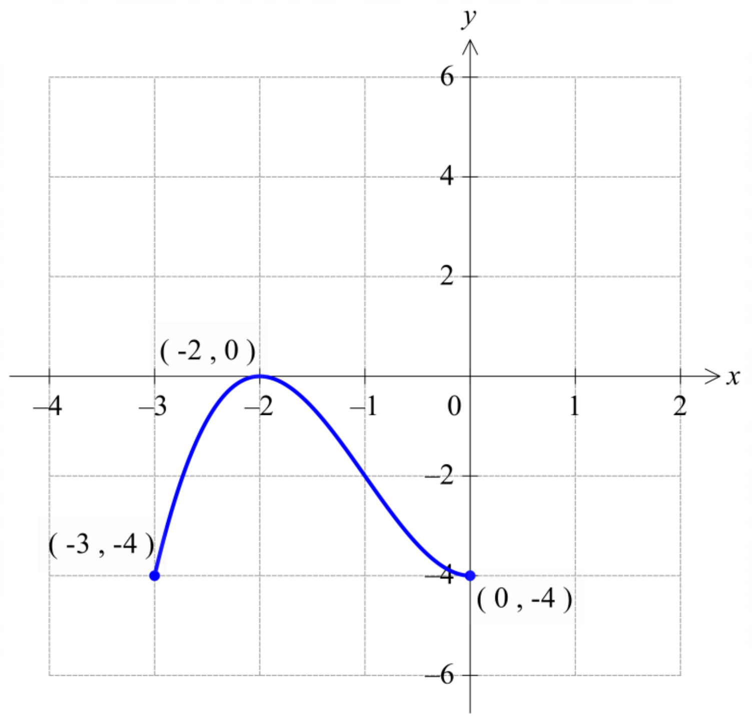
$$(x+2)^2(x-1)=\left(x^2+4x+4\right)(x-1)$$
$$=x^3-x^2+4x^2-4x+4x-4$$
$$=x^3+3x^2-4\text{, as required}$$

b. Endpoints:

$$f(-3)=(-3+2)^2(-3-1)$$
$$=(-1)^2(-4)$$
$$=-4 \text{ so endpoint at } (-3,-4)$$

$$f(0)=(0+2)^2(0-1)$$
$$=(2)^2(-1)$$
$$=-4 \text{ so endpoint at } (0,-4)$$

Also, $f(x)$ is a cubic with a repeated root at $x=-2$, so there is a maximum turning point at $(-2,0)$.



Question 4

$p=\dfrac{1}{4}$

$$\text{sd}\left(\hat{P}\right)\le\frac{1}{100}$$
$$\sqrt{\frac{p(1-p)}{n}}\le\frac{1}{100}$$
$$\sqrt{\frac{\frac{1}{4}\times\frac{3}{4}}{n}}\le\frac{1}{100}$$
$$\frac{3}{16n}\le\frac{1}{10000}$$

$n\ge1875$ so the smallest integer value of n is 1875.

Question 5

a. Let C be the event that Jac types in the correct password. Then $\Pr(C)=\frac{2}{5}$.

$$\begin{aligned}\Pr(\text{does not log on succcessfully})&=\left(\Pr(C')\right)^3\\&=\left(1-\Pr(C)\right)^3\\&=\left(1-\frac{2}{5}\right)^3\\&=\left(\frac{3}{5}\right)^3\\&=\frac{27}{125}\end{aligned}$$

b. Easy way:

$$\begin{aligned}\Pr(\text{log on succcessfully})&=1-\Pr(\text{does not log on succcessfully})\\&=1-\frac{27}{125}\\&=\frac{98}{125}\end{aligned}$$

Harder way:
Jac logs on successfully if he succeeds on the first attempt, or fails on first then succeeds, or fails twice then succeeds.

$$\begin{aligned}\Pr(\text{logs on successfully})&=\Pr(C)+\Pr(C')\Pr(C)+\left(\Pr(C')\right)^2\Pr(C)\\&=\frac{2}{5}+\frac{3}{5}\times\frac{2}{5}+\left(\frac{3}{5}\right)^2\times\frac{2}{5}\\&=\frac{2}{5}+\frac{6}{25}+\frac{9}{25}\times\frac{2}{5}\\&=\frac{2}{5}+\frac{6}{25}+\frac{18}{125}\\&=\frac{98}{125}\end{aligned}$$

c. Jac logs on successfully if he succeeds on the second or third attempt if he fails on first then succeeds, or fails twice then succeeds.

$$\begin{aligned}\Pr(\text{logs on successfully on second or third attempt})&=\Pr(C')\Pr(C)+\left(\Pr(C')\right)\Pr(C)\\&=\frac{3}{5}\times\frac{2}{5}+\left(\frac{3}{5}\right)^2\times\frac{2}{5}\\&=\frac{6}{25}+\frac{9}{25}\times\frac{2}{5}\\&=\frac{6}{25}+\frac{18}{125}\\&=\frac{48}{125}\end{aligned}$$

Question 6

$\left(\tan(\theta)-1\right)\left(\sin(\theta)-\sqrt{3}\cos(\theta)\right)\left(\sin(\theta)+\sqrt{3}\cos(\theta)\right)=0$

a. The null factor law states either of the three brackets must equal zero in order for the entire expression to equal zero:

$\tan(\theta)=1$ or $\sin(\theta)-\sqrt{3}\cos(\theta)=0$ or $\sin(\theta)+\sqrt{3}\cos(\theta)=0$

$$\sin(\theta)=\sqrt{3}\cos(\theta) \qquad \sin(\theta)=-\sqrt{3}\cos(\theta)$$

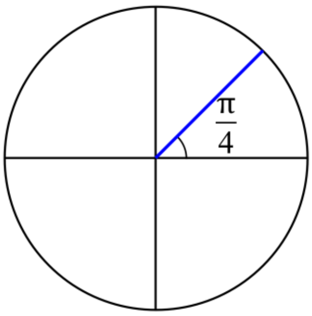
$$\frac{\sin(\theta)}{\cos(\theta)}=\sqrt{3} \qquad \frac{\sin(\theta)}{\cos(\theta)}=-\sqrt{3}$$

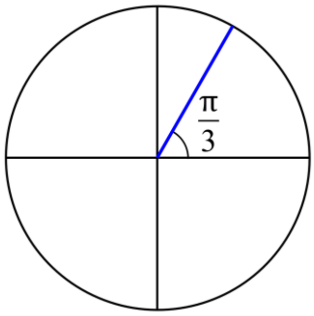
$$\tan(\theta)=\sqrt{3} \qquad \tan(\theta)=-\sqrt{3}$$

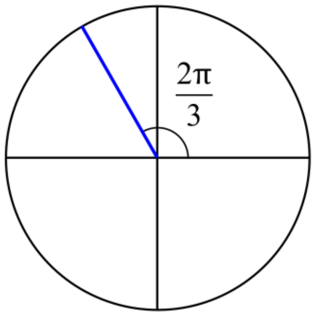
The possible values of $\tan(\theta)$ are 1, $\sqrt{3}$ and $-\sqrt{3}$.

b. Solving over $0\le\theta\le\pi$:

$\tan(\theta)=1$ or $\tan(\theta)=\sqrt{3}$ or $\tan(\theta)=-\sqrt{3}$



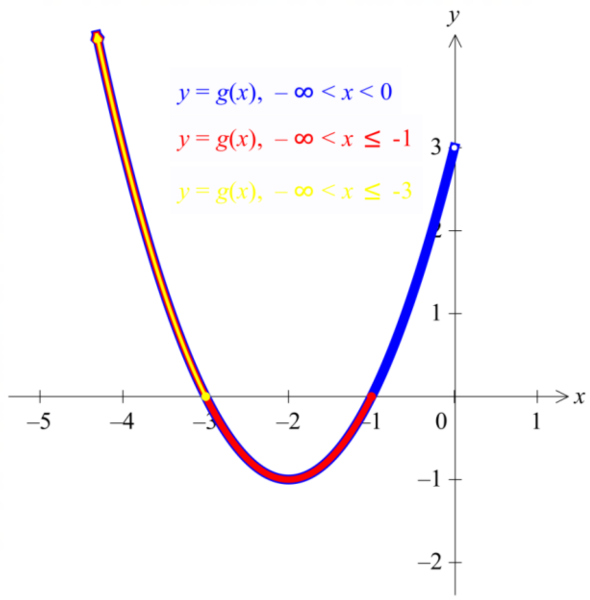




So all possible solutions for $\left(\tan(\theta)-1\right)\left(\sin(\theta)-\sqrt{3}\cos(\theta)\right)\left(\sin(\theta)+\sqrt{3}\cos(\theta)\right)=0$ over $0\le\theta\le\pi$ are $\theta=\frac{\pi}{4}$, $\theta=\frac{\pi}{3}$ and $\theta=\frac{2\pi}{3}$.

Question 7

$f:[0,\infty)\to\mathbb{R},\ f(x)=\sqrt{x+1}$



a. $f(0)=\sqrt{1}=1$ and f is an increasing function over $[0,\infty)$ so ran $f=[1,\infty)$.

b. $g:(-\infty,c]\to\mathbb{R},\ g(x)=x^2+4x+3$

i. Solve $g(x)=0$:

$$x^2+4x+3=0$$
$$(x+3)(x+1)=0$$
$$x=-3 \text{ or } x=-1$$

But if $c=-1$, $\text{ran}\,g=[-1,\infty)$. So $c=-3$ is the largest possible value of c for $\text{ran}\,g\subseteq\text{dom}\,f$.

ii. For $c=-3$, $\text{ran}\,g=[0,\infty)$. So $\text{dom}\,f=\text{ran}\,g=[0,\infty)$, therefore $\text{ran}\,f(g(x))=[1,\infty)$.

c. $h:\mathbb{R}\to\mathbb{R},\ h(x)=x^2+3$

$\text{ran}\,h=[3,\infty)$.

$$f(3)=\sqrt{3+1}$$
$$=2$$

Therefore $\text{ran}\,f(h(x))=[2,\infty)$.

Question 8

$\Pr(A|B)=\frac{1}{5}$, $\Pr(B|A)=\frac{1}{4}$, $\Pr(A\cap B)=p$

a. $\Pr(B|A)=\frac{\Pr(B\cap A)}{\Pr(A)}=\frac{\Pr(A\cap B)}{\Pr(A)}$

$$\frac{1}{4}=\frac{p}{\Pr(A)}$$

$\Pr(A)=4p$

b. $\Pr(A|B)=\frac{\Pr(A\cap B)}{\Pr(B)}$

$$\frac{1}{5}=\frac{p}{\Pr(B)}$$

$\Pr(B)=5p$

$$\begin{aligned}\Pr(A'\cap B') &= 1-\Pr(A\cup B)\\ &= 1-\left(\Pr(A)+\Pr(B)-\Pr(A\cap B)\right)\\ &= 1-(4p+5p-p)\\ &= 1-8p\end{aligned}$$

c. $\Pr(A\cup B)\le\frac{1}{5}$

$8p\le\frac{1}{5}$

$p\le\frac{1}{40}$

But $p\in(0,1]$ so $p\in\left(0,\frac{1}{40}\right]$.

$p\neq 0$ because $\Pr(A)\neq 0$, $\Pr(B)\neq 0$, $\Pr(A|B)\neq 0$ and $\Pr(B|A)\neq 0$.

Question 9

$f:[0,1]\to\mathbb{R},\ f(x)=\sqrt{x}(1-x)$

a. Area is given by the definite integral

$$A=\int_0^1\sqrt{x}(1-x)\,dx$$
$$=\int_0^1 x^{\frac{1}{2}}(1-x)\,dx$$
$$=\int_0^1\left(x^{\frac{1}{2}}-x^{\frac{3}{2}}\right)dx$$
$$=\left[\frac{x^{\frac{3}{2}}}{3/2}-\frac{x^{\frac{5}{2}}}{5/2}\right]_0^1$$
$$=\left[\frac{2x^{\frac{3}{2}}}{3}-\frac{2x^{\frac{5}{2}}}{5}\right]_0^1$$
$$=\frac{2}{3}-\frac{2}{5}$$
$$=\frac{10-6}{15}$$
$$=\frac{4}{15}$$

b. Finding the derivative of $f(x)$ using the product rule:

$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}(1-x)+(-1)\sqrt{x}$$
$$=\frac{1}{2\sqrt{x}}(1-x)-\sqrt{x}$$
$$=\frac{1-x}{2\sqrt{x}}-\sqrt{x}$$
$$=\frac{1-x-2\sqrt{x}\sqrt{x}}{2\sqrt{x}}$$
$$=\frac{1-x-2x}{2\sqrt{x}}$$
$$=\frac{1-3x}{2\sqrt{x}}\text{, as required}$$

c. If $\theta=45^\circ$ then the gradient is $\tan(45^\circ)=1$.

When AC has gradient 1, BC has gradient -1 (as AC and CB are perpendicular).

$$f'(x)=1$$
$$\frac{1-3x}{2\sqrt{x}}=-1$$
$$1-3x=-2\sqrt{x}$$
$$(1-3x)^2=4x$$
$$1-6x+9x^2=4x$$
$$9x^2-10x+1=0$$
$$(9x-1)(x-1)=0$$

$x=1$ as $x=\frac{1}{9}$ is for the other tangent

$$f(1)=\sqrt{1}(1-1)$$
$$=1\times 0$$
$$=0$$

So the line BC has gradient $m=-1$ and passes through the point $(1,0)$.

Therefore the equation of the line BC is

$$y-y_1=m(x-x_1)$$
$$y-0=-1(x-1)$$
$$y=-x+1$$
$$y=1-x$$

d. AC has gradient 1.

$$f'(x) = 1$$
$$\frac{1-3x}{2\sqrt{x}} = 1$$
$$1-3x = 2\sqrt{x}$$
$$(1-3x)^2 = 4x$$
$$1-6x+9x^2 = 4x$$
$$9x^2 - 10x + 1 = 0$$
$$(9x-1)(x-1) = 0$$
$$x = \frac{1}{9} \text{ as } x \in (0,1)$$
$$f\left(\frac{1}{9}\right) = \sqrt{\frac{1}{9}}\left(1-\frac{1}{9}\right)$$
$$= \frac{1}{3} \times \frac{8}{9}$$
$$= \frac{8}{27}$$

So the line AC has gradient $m = 1$ and passes through the point $\left(\frac{1}{9}, \frac{8}{27}\right)$.

Therefore the equation of the line AC is

$$y - y_1 = m(x - x_1)$$
$$y - \frac{8}{27} = 1\left(x - \frac{1}{9}\right)$$
$$y - \frac{8}{27} = x - \frac{1}{9}$$
$$y = x - \frac{1}{9} + \frac{8}{27}$$
$$y = x + \frac{5}{27}$$

C is the intersection of AC and CB:

$$y = x + \frac{5}{27} \cdots \boxed{1}$$
$$y = 1 - x \cdots \boxed{2}$$

$\boxed{1} + \boxed{2}$:

$$2y = \frac{32}{27}$$
$$y = \frac{16}{27}$$

Sub $y = \frac{16}{27} \rightarrow \boxed{2}$:

$$\frac{16}{27} = 1 - x$$
$$x = 1 - \frac{16}{27}$$
$$= \frac{11}{27}$$

Therefore $C = \left(\frac{11}{27}, \frac{16}{27}\right)$.