COMPLEX RELATIONAL CONVERSIONS.

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Let Rel here denote the category of binary relations. We will denote relation composition by the symbol $\centerdot$ and use the symbol $\rightleftharpoons$ to denote arrows in Rel, in the sense that the notation $\rho\colon X \rightleftharpoons Z$ in Rel denotes an binary relation ρ between X and Z, i.e. from the object X of Rel to the object Z of Rel.

Departing from usual set theoretic notations, we will write $x\colon X$ to denote the claim that the item x belongs to the collection X and will write $xz\colon \rho$ to denote the claim that a binary relation $\rho\colon X \rightleftharpoons Z$ links items $x\colon X$ and $z\colon Z$. And we will note $\mathcal{P}W$ the collection of sub-collections of W.

We here wish to classify the involutive and contravariant endofunctors of Rel fixing objects. The first obvious observation is that the conversion of binary relations is such an involutive and contravariant endofunctor of Rel fixing objects. For reasons that will make more sense at the end of this document, we we will say that the conversion of binary relations is the real conversion and we will call complex conversions the involutive and contravariant endofunctors of Rel fixing objects.

Let therefore $\diamond\colon \mathrm{Rel} \to \mathrm{Rel}$ be such a complex conversion. As the notation may be somewhat confusing we here observe that, as a covariant functor, the complex conversion $\diamond$ should be noted $\diamond\colon \mathrm{Rel}^\circ \to \mathrm{Rel}$ (or $\diamond\colon \mathrm{Rel} \to \mathrm{Rel}^\circ$), but, as a contravariant functor, we indeed note it $\diamond\colon \mathrm{Rel} \to \mathrm{Rel}$ as an arrow in the category whose objects are categories and whose arrows are covariant or contravariant functors.

For the sake of straightforwardness of argumentation, we will not abide by a strict orthodoxy of universality in our argumentation, and we will therefore select and distinguish an object $\star$ in Rel that happens to concretely represent a collection that happens to be a singleton. We will homonymously name $\star$ the unique item of the singleton $\star$.

The monoid $\mathrm{Rel}\,\star$ of endomorphisms of $\star$ contains only two distinct items, namely $\mathbf{1}_\star$ and another item we will name $\mathbf{0}_\star$. We know, by involutivity of complex conversions, that $\mathbf{0}_\star^{\diamond\diamond} = \mathbf{0}_\star$. As the complex conversion $\diamond$ fixes objects, it fixes the categorical unit $\mathbf{1}_\star$ of the object $\star$. This implies that $\mathbf{1}_\star^\diamond = \mathbf{1}_\star \neq \mathbf{0}_\star = \mathbf{0}_\star^{\diamond\diamond}$ and therefore implies that $\mathbf{1}_\star \neq \mathbf{0}_\star^\diamond$. This leaves only one option: $\mathbf{0}_\star^\diamond = \mathbf{0}_\star$. This proves that the complex conversion $\diamond$ fixes the whole of the monoid $\mathrm{Rel}\,\star$.

Date: 16th February 2022.

We now consider an object W of Rel, which represents a collection. For any sub-collection $A\colon \mathcal{P}W$ of W, we may define two binary relations $\overrightarrow{A}: \star \rightleftharpoons W$ and $\overleftarrow{A}: W \rightleftharpoons \star$ characterised as follows:

$$\star\alpha\colon \overrightarrow{A} \iff \alpha\colon A \qquad \alpha\star\colon \overleftarrow{A} \iff \alpha\colon A$$

Let $\rho\colon X \rightleftharpoons Z$ be an arrow of Rel. We define a relation $\rho^{\ddagger}: \mathcal{P}Z \rightleftharpoons \mathcal{P}X$ as follows:

$$V\,U\colon \rho^{\ddagger} \iff \mathbf{0}_{\star} = \overrightarrow{U} \cdot \rho \cdot \overleftarrow{V}$$

For a sub-collection $U\colon \mathcal{P}W$ of W, the relation $\left(\overrightarrow{U}\right)^{\diamond}: W \rightleftharpoons \star$ induces and thus defines a sub-collection $U^{\overleftarrow{\diamond}}: \mathcal{P}W$ as follows:

$$\omega\colon U^{\overleftarrow{\diamond}} \iff \omega\star\colon \left(\overrightarrow{U}\right)^{\diamond}$$

We analogously define $V^{\overrightarrow{\diamond}}: \mathcal{P}W$ for a sub-collection $V\colon \mathcal{P}W$ of W as follows:

$$\omega\colon V^{\overrightarrow{\diamond}} \iff \star\omega\colon \left(\overleftarrow{V}\right)^{\diamond}$$

We therefore have:

$$\omega\star\colon \overleftarrow{U^{\overleftarrow{\diamond}}} \iff \omega\star\colon \left(\overrightarrow{U}\right)^{\diamond} \qquad \star\,\omega\colon \overrightarrow{V^{\overrightarrow{\diamond}}} \iff \star\omega\colon \left(\overleftarrow{V}\right)^{\diamond}$$

Which proves that:

$$\overleftarrow{U^{\overleftarrow{\diamond}}} = \left(\overrightarrow{U}\right)^{\diamond} \qquad \overrightarrow{V^{\overrightarrow{\diamond}}} = \left(\overleftarrow{V}\right)^{\diamond}$$

By contravariance, and given that the complex conversion $\diamond$ fixes $\mathbf{0}_{\star}$, we have:

$$\mathbf{0}_{\star} = \overrightarrow{U} \cdot \rho \cdot \overleftarrow{V} \iff \mathbf{0}_{\star} = \left(\overleftarrow{V}\right)^{\diamond} \cdot \rho^{\diamond} \cdot \left(\overrightarrow{U}\right)^{\diamond} \iff \mathbf{0}_{\star} = \overrightarrow{V^{\overrightarrow{\diamond}}} \cdot \rho \cdot \overleftarrow{U^{\overleftarrow{\diamond}}}$$

$$V\,U\colon \rho^{\ddagger} \iff U^{\overleftarrow{\diamond}}\,V^{\overrightarrow{\diamond}}: \rho^{\diamond\ddagger}$$

As we do know that the complex conversion $\diamond$ fixes any categorical unit $\mathbf{1}_W$, we have:

$$V\,U\colon \mathbf{1}_W^{\ddagger} \iff U^{\overleftarrow{\diamond}}\,V^{\overrightarrow{\diamond}}: \mathbf{1}_W^{\ddagger}$$

We may now synthetise notations further by defining the binary relations $\overleftarrow{\between}$ and $\overrightarrow{\between}$ via the following characterisations:

$$U' \mathrel{\overleftarrow{\between}} U \iff U'^{\overleftarrow{\diamond}}\,U : \mathbf{1}_W^{\ddagger} \qquad V \mathrel{\overrightarrow{\between}} V' \iff V\,V'^{\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger}$$

We now define two model theoretic structures $\overleftarrow{\mathcal{P}}W$ and $\overrightarrow{\mathcal{P}}W$ on the domain $\mathcal{P}W$ endowed with the structural symbolic relation $\between$ as follows:

$$\overleftarrow{\mathcal{P}}W \models U' \between U \iff U' \mathrel{\overleftarrow{\between}} U \qquad \overrightarrow{\mathcal{P}}W \models V \between V' \iff V \mathrel{\overrightarrow{\between}} V'$$

We know that $\diamond$ is involutive. Therefore, for any $U\colon \mathcal{P}W$ and any $V\colon \mathcal{P}W$, we have:

$$\overrightarrow{U} = \left(\overrightarrow{U}\right)^{\diamond\diamond} = \left(\overleftarrow{U^{\overleftarrow{\diamond}}}\right)^{\diamond} = \overrightarrow{U^{\overleftarrow{\diamond}\overrightarrow{\diamond}}} \qquad \overleftarrow{V} = \left(\overleftarrow{V}\right)^{\diamond\diamond} = \left(\overrightarrow{V^{\overrightarrow{\diamond}}}\right)^{\diamond} = \overleftarrow{V^{\overrightarrow{\diamond}\overleftarrow{\diamond}}}$$

This proves that:

$$U = U^{\overleftarrow{\diamond}\overrightarrow{\diamond}} \qquad V = V^{\overrightarrow{\diamond}\overleftarrow{\diamond}}$$

And we may now observe that $\overleftarrow{\diamond}: \mathcal{P}W \to \mathcal{P}W$ induces an embedding of model theoretic structures $\overleftarrow{\mathcal{P}}W \to \overrightarrow{\mathcal{P}}W$. We may first unpack our definitions:

$$\overleftarrow{\mathcal{P}}W \models U' \mathrel{)\!(} U \iff U' \overleftarrow{\mathrel{)\!(}}\, U \iff U'^{\overleftarrow{\diamond}}\, U : \mathbf{1}_W^{\ddagger}$$
$$\overrightarrow{\mathcal{P}}W \models U'^{\overleftarrow{\diamond}} \mathrel{)\!(} U^{\overleftarrow{\diamond}} \iff U'^{\overleftarrow{\diamond}} \overrightarrow{\mathrel{)\!(}}\, U^{\overleftarrow{\diamond}} \iff U'^{\overleftarrow{\diamond}}\, U^{\overleftarrow{\diamond}\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger}$$

And then observe, given that $U = U^{\overleftarrow{\diamond}\overrightarrow{\diamond}}$ holds, that the the embedding property holds too:

$$\overleftarrow{\mathcal{P}}W \models U' \mathrel{)\!(} U \iff U'^{\overleftarrow{\diamond}}\, U : \mathbf{1}_W^{\ddagger} \iff U'^{\overleftarrow{\diamond}}\, U^{\overleftarrow{\diamond}\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger} \iff \overrightarrow{\mathcal{P}}W \models U'^{\overleftarrow{\diamond}} \mathrel{)\!(} U^{\overleftarrow{\diamond}}$$

We may anologously prove that $\overrightarrow{\diamond} \colon \mathcal{P}W \to \mathcal{P}W$ induces an embedding of model theoretic structures $\overrightarrow{\mathcal{P}}W \to \overleftarrow{\mathcal{P}}W$. We analogously unpack our definitions:

$$\overrightarrow{\mathcal{P}}W \models V \mathrel{)\!(} V' \iff V \overrightarrow{\mathrel{)\!(}}\, V' \iff V\, V'^{\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger}$$
$$\overleftarrow{\mathcal{P}}W \models V^{\overrightarrow{\diamond}} \mathrel{)\!(} V'^{\overrightarrow{\diamond}} \iff V^{\overrightarrow{\diamond}} \overleftarrow{\mathrel{)\!(}}\, V'^{\overrightarrow{\diamond}} \iff V^{\overrightarrow{\diamond}\overleftarrow{\diamond}}\, V'^{\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger}$$

And then analogously observe, given that $V = V^{\overrightarrow{\diamond}\overleftarrow{\diamond}}$ holds, that the the embedding property holds too:

$$\overleftarrow{\mathcal{P}}W \models U' \mathrel{)\!(} U \iff U'^{\overleftarrow{\diamond}}\, U : \mathbf{1}_W^{\ddagger} \iff U'^{\overleftarrow{\diamond}}\, U^{\overleftarrow{\diamond}\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger} \iff \overrightarrow{\mathcal{P}}W \models U'^{\overleftarrow{\diamond}} \mathrel{)\!(} U^{\overleftarrow{\diamond}}$$

As we do know that the identities $U = U^{\overleftarrow{\diamond}\overrightarrow{\diamond}}$ and $V = V^{\overrightarrow{\diamond}\overleftarrow{\diamond}}$ universally hold, we then do know that the underlying mappings of the embeddings $\overrightarrow{\diamond} \colon \overrightarrow{\mathcal{P}}W \to \overleftarrow{\mathcal{P}}W$ and $\overleftarrow{\diamond} \colon \overleftarrow{\mathcal{P}}W \to \overrightarrow{\mathcal{P}}W$ are mappings that are inverse to one another. Which proves that these embeddings are isomorphisms of model theoretic structures for the structural symbolic relation $\mathrel{)\!(}$.

Now, we also know that the common domain $\mathcal{P}W$ of both the structures $\overrightarrow{\mathcal{P}}W$ and $\overleftarrow{\mathcal{P}}W$ is canonically endowed with the ordering of inclusion $\subseteq$ on $\mathcal{P}W$. And we may now enrich the model theoretic structures $\overrightarrow{\mathcal{P}}W$ and $\overleftarrow{\mathcal{P}}W$ with an additional structural relation $\leqslant$ which is interpreted as follows:

$$\overrightarrow{\mathcal{P}}W \models U \leqslant V \iff U \subseteq V \iff \overleftarrow{\mathcal{P}}W \models U \leqslant V$$

We will now show that the inverse mappings $\overrightarrow{\diamond} \colon \mathcal{P}W \to \mathcal{P}W$ and $\overleftarrow{\diamond} \colon \mathcal{P}W \to \mathcal{P}W$ again induce isomorphisms of ordered structures $\overrightarrow{\diamond} \colon \overrightarrow{\mathcal{P}}W \to \overleftarrow{\mathcal{P}}W$ and $\overleftarrow{\diamond} \colon \overleftarrow{\mathcal{P}}W \to \overrightarrow{\mathcal{P}}W$.

For two sub-collections A and B of W, the statement $\mathbf{0}_\star = \overrightarrow{B} \centerdot \mathbf{1}_W \centerdot \overleftarrow{A}$ is equivalent to the claim that $A$and B are disjoint sub-collections of W. Indeed, as Rel$\,\star$ contains only two items, namely $\mathbf{0}_\star$ and $\mathbf{1}_\star$, we know that the following equivalences hold:

$$\begin{aligned}
\mathbf{0}_\star \neq \overrightarrow{B} \centerdot \mathbf{1}_W \centerdot \overleftarrow{A} &\iff \mathbf{1}_\star = \overrightarrow{B} \centerdot \mathbf{1}_W \centerdot \overleftarrow{A} \\
&\iff \exists a, b \colon W, \begin{cases} \star\, b \colon \overrightarrow{B} \\ b\, a \colon \mathbf{1}_W \\ a\, \star \colon \overleftarrow{A} \end{cases} \\
&\iff \exists w \colon W,\ w \colon B \wedge w \colon A
\end{aligned}$$

This indeed proves that the statement $\mathbf{0}_\star = \overrightarrow{B} \centerdot \mathbf{1}_W \centerdot \overleftarrow{A}$ boils down to the claim that there is no item $w \colon W$ common to both B and A. In other words, B and A are disjoint sub-collections of W.

By definition of $\mathbf{1}_W^{\ddagger}$, the statement $A\,B\colon \mathbf{1}_W^{\ddagger}$ then is equivalent to the claim that A and B are disjoint. An observation which allows us to rewrite $\subseteq$ as follows in terms of $\mathbf{1}_W^{\ddagger}$:

$$A \subseteq B \iff \forall R : \mathcal{P}W,\ R\,B : \mathbf{1}_W^{\ddagger} \to R\,A : \mathbf{1}_W^{\ddagger}$$

Indeed: If $A \subseteq B$, it is a trivial observation that when R is not disjoint from A, there is a $w : W$ belonging to both R and A, and, as A is included in B, we then know that w is in B too, and hence that R may not be disjoint from A; and one get the proof in the direction $\implies$ by contraposing that conclusion. In the direction $\impliedby$, one may select R to be the complement of B and we then know that that complement is disjoint from A, which implies that A is included in B.

From this rewriting of $\subseteq$ in terms of $\mathbf{1}_W^{\ddagger}$, we may derive the following:

$$\begin{aligned} A^{\overrightarrow{\diamond}} \subseteq B^{\overrightarrow{\diamond}} &\iff \forall R : \mathcal{P}W,\ R\,B^{\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger} \to R\,A^{\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger} \\ &\iff \forall R : \mathcal{P}W,\ R \overrightarrow{\between} B \to R \overrightarrow{\between} A \end{aligned}$$

But it just so happens that $\overrightarrow{\between}$ is symmetric. Recall that we have $V\,U\colon \mathbf{1}_W^{\ddagger} \iff U^{\overleftarrow{\diamond}}\,V^{\overrightarrow{\diamond}}\colon \mathbf{1}_W^{\ddagger}$. From which we may derive, with $U \coloneqq V^{\overrightarrow{\diamond}}$ and hence $V = V^{\overrightarrow{\diamond}\overleftarrow{\diamond}} = U^{\overleftarrow{\diamond}}$, the following equivalence:

$$V \overrightarrow{\between} V' \iff V\,V'^{\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger} \iff U^{\overleftarrow{\diamond}}\,V'^{\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger} \iff V'\,U : \mathbf{1}_W^{\ddagger} \iff V'\,V^{\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger} \iff V' \overrightarrow{\between} V$$

This symmetry allows us to rewrite $A^{\overrightarrow{\diamond}} \subseteq B^{\overrightarrow{\diamond}}$ as follows:

$$\begin{aligned} A^{\overrightarrow{\diamond}} \subseteq B^{\overrightarrow{\diamond}} &\iff \forall R : \mathcal{P}W,\ B \overrightarrow{\between} R \to A \overrightarrow{\between} R \\ &\iff \forall R : \mathcal{P}W,\ B\,R^{\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger} \to A\,R^{\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger} \end{aligned}$$

However, we do know that $\overrightarrow{\diamond} : \mathcal{P}W \to \mathcal{P}W$ is an invertible mapping and hence surjective. As $R^{\overrightarrow{\diamond}}$ reaches all the possible $S : \mathcal{P}W$, we may then rewrite the above as follows:

$$\begin{aligned} A^{\overrightarrow{\diamond}} \subseteq B^{\overrightarrow{\diamond}} &\iff \forall S : \mathcal{P}W,\ B\,S\colon \mathbf{1}_W^{\ddagger} \to A\,S\colon \mathbf{1}_W^{\ddagger} \\ &\iff \forall S : \mathcal{P}W,\ S\,B\colon \mathbf{1}_W^{\ddagger} \to S\,A\colon \mathbf{1}_W^{\ddagger} \\ &\iff A \subseteq B \end{aligned}$$

Model theoretically, we may rephrase the above as follows:

$$\overrightarrow{\mathcal{P}}W \models A \leqslant B \iff A \subseteq B \iff A^{\overrightarrow{\diamond}} \subseteq B^{\overrightarrow{\diamond}} \iff \overleftarrow{\mathcal{P}}W \models A^{\overrightarrow{\diamond}} \leqslant B^{\overrightarrow{\diamond}}$$

The invertible mapping $\overrightarrow{\diamond} : \mathcal{P}W \to \mathcal{P}W$ therefore induces an embedding $\overrightarrow{\mathcal{P}}W \to \overleftarrow{\mathcal{P}}W$ of model theoretic structures for the structural symbolic relations $\between$ and $\leqslant$. And, as the underlying mapping of this embedding is invertible, it is an isomorphism.

Moreover, the ordered structures on $\overrightarrow{\mathcal{P}}W$ and $\overleftarrow{\mathcal{P}}W$ are those of boolean algebras since $(\mathcal{P}W, \subseteq)$ is the boolean algebra structure they share. The mapping $\overrightarrow{\diamond} : \mathcal{P}W \to \mathcal{P}W$ therefore induces an isomorphism $\overrightarrow{\diamond} : \overrightarrow{\mathcal{P}}W \to \overleftarrow{\mathcal{P}}W$ of boolean algebras. By Stone's theorem, this isomorphism of boolean algebras is induced by an invertible mapping on their atoms. Invertible mapping which we will homonymously noted $\overrightarrow{\diamond} : W \to W$.

We analogously obtain the mapping $\overleftarrow{\diamond} : W \to W$ inducing $\overleftarrow{\diamond} : \mathcal{P}W \to \mathcal{P}W$. And, as $\overrightarrow{\diamond} : \mathcal{P}W \to \mathcal{P}W$ and $\overleftarrow{\diamond} : \mathcal{P}W \to \mathcal{P}W$ are inverse isomorphisms of boolean algebras, the mappings $\overrightarrow{\diamond} : W \to W$ and $\overleftarrow{\diamond} : W \to W$ are also inverse to one another by Stone's theorem.

Let now v be an item of W, $u\colon W$ its image by $\overrightarrow{\diamond}\colon W \to W$, and $\overline{u}\colon \mathcal{P}W$ the complement of u in W. We here identify atoms with the singleton they induce. We have $\overline{u}\,u\colon \mathbf{1}_W^{\ddagger}$ and hence $\overline{u}\,v^{\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger}$. Hence $\overline{u} \overrightarrow{)\!(} v$ which we may model theoretically rewrite as $\overrightarrow{\mathcal{P}}W \models \overline{u})\!(v$. But we have seen that the structural relation $)\!($ of $\overrightarrow{\mathcal{P}}W$ is symmetric, and we therefore know that $\overrightarrow{\mathcal{P}}W \models v)\!(\overline{u}$ and hence that $v\,\overline{u}^{\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger}$. This proves that $\overline{u}^{\overrightarrow{\diamond}}$ is included in the complement $\overline{v}$ of v. As we have seen that $\overleftarrow{\diamond}\colon \mathcal{P}W \to \mathcal{P}W$ is iso-tone for the ordering $\subseteq$ of inclusion on $\mathcal{P}W$, we then know that $\overline{u} = \overline{u}^{\overrightarrow{\diamond}\overleftarrow{\diamond}} \subseteq \overline{v}^{\overleftarrow{\diamond}}$. And, consequently, we know that the complement $\overline{\overline{v}^{\overleftarrow{\diamond}}}$ of $\overline{v}^{\overleftarrow{\diamond}}$ is included in u. But we also do know that $v\,\overline{v}\colon \mathbf{1}_W^{\ddagger}$ and, as $v = v^{\overrightarrow{\diamond}\overleftarrow{\diamond}} = u^{\overleftarrow{\diamond}}$, that $\overleftarrow{\mathcal{P}}W \models u)\!(\overline{v}$. Which implies, as $\overleftarrow{\diamond}\colon \overleftarrow{\mathcal{P}}W \to \overrightarrow{\mathcal{P}}W$ is an isomorphism for the structural relation $)\!($, that $\overrightarrow{\mathcal{P}}W \models u^{\overleftarrow{\diamond}})\!(\overline{v}^{\overleftarrow{\diamond}}$. As the structural relation $)\!($ of $\overrightarrow{\mathcal{P}}W$ is known to be symmetric, we have $\overrightarrow{\mathcal{P}}W \models \overline{v}^{\overleftarrow{\diamond}})\!(u^{\overleftarrow{\diamond}}$ and thus $\overline{v}^{\overleftarrow{\diamond}}\,u^{\overleftarrow{\diamond}\overrightarrow{\diamond}} : \mathbf{1}_W^{\ddagger}$. Which implies that $u = u^{\overleftarrow{\diamond}\overrightarrow{\diamond}}$ is included in the complement $\overline{\overline{v}^{\overleftarrow{\diamond}}}$ of $\overline{v}^{\overleftarrow{\diamond}}$. We have thus proven that both $u \subseteq \overline{\overline{v}^{\overleftarrow{\diamond}}}$ and that $\overline{\overline{v}^{\overleftarrow{\diamond}}} \subseteq u$. We therefore know that $\overline{\overline{v}^{\overleftarrow{\diamond}}} = u$ and hence that $\overline{u} = \overline{v}^{\overleftarrow{\diamond}}$

We now wish to prove that $\overleftarrow{\mathcal{P}}W \models v)\!(u$ cannot hold. If it doesn't hold, the statement $v^{\overleftarrow{\diamond}}\,u\colon \mathbf{1}_W^{\ddagger}$ would then also not hold, and, $v^{\overleftarrow{\diamond}}$ and u being both atoms of the boolean algebra $\mathcal{P}W$, we would have $v^{\overleftarrow{\diamond}} = u$. And as we also have $u = v^{\overrightarrow{\diamond}}$, we would know that $v^{\overleftarrow{\diamond}}$ and $v^{\overrightarrow{\diamond}}$ coincide. The mappings $\overleftarrow{\diamond} : W \to W$ and $\overrightarrow{\diamond} : W \to W$ would then not only be inverses to one another but the same mapping $\diamond : W \to W$ admitting itself as its inverse. The mapping would be involutive in the sense that we would have $\mathbf{1}_W = \diamond \centerdot \diamond$. Which is what we want to prove.

Let us therefore suppose, for the purpose of ad absurdo reasoning, that $\overleftarrow{\mathcal{P}}W \models v)\!(u$ does hold. We would then know that $v^{\overleftarrow{\diamond}}\,u\colon \mathbf{1}_W^{\ddagger}$. Which would imply that $v^{\overleftarrow{\diamond}} \subseteq \overline{u} = \overline{v}^{\overleftarrow{\diamond}}$. As we have seen that $\overrightarrow{\diamond}\colon \mathcal{P}W \to \mathcal{P}W$ is iso-tone for the ordering $\subseteq$ of inclusion on $\mathcal{P}W$, we would then know that $v = v^{\overleftarrow{\diamond}\overrightarrow{\diamond}} \subseteq \overline{v}^{\overleftarrow{\diamond}\overrightarrow{\diamond}} = \overline{v}$. But $v \subseteq \overline{v}$ is an impossibility. This proves ad abusrdo that $\overleftarrow{\mathcal{P}}W \models v)\!(u$ may not hold and that the mappings $\overleftarrow{\diamond} : W \to W$ and $\overrightarrow{\diamond} : W \to W$ then coincide into an involutive mapping $\diamond : W \to W$.

We now sum up.

Any complex conversion $\diamond : \mathrm{Rel} \to \mathrm{Rel}$ induces on any object W of the category Rel an involutive mapping $\diamond_W : W \to W$ as above.

It will be shown in future versions of this document that:

(1) The real conversion of binary relations (i.e. the contravariant endofunctor $\circ : \mathrm{Rel} \to \mathrm{Rel}$ that maps any binary relation ρ to the binary relation ρ° characterised by $xz\colon \rho^\circ \iff zx : \rho$) is characterised by the fact that all the involutive mappings $\diamond_W : W \to W$ are the identities $\mathbf{1}_W : W \to W$ of W.
(2) The data of such involutive mappings for all objects of Rel uniquely characterises a complex conversion.
(3) One may embed the category Rel into a category CRel of so-called complex relations endowed with a canonical involutive and contravariant endofunctor called the conversion such that its trace on Rel is the real conversion

and such that any complex conversion on Rel may be induced as the trace of the conversion of CRel of a fully faithful embedding of Rel $\to$ CRel.

Further developments would include a characterisation of the real conversion in terms of pseudo-inverse à la Moore-Penrose and à la Wagner-Preston. This will yield a first order axiomatisation of the conversion of binary relations in the language of categories.

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