(a) Prove that $\sin^4\theta + \cos^4\theta = 1 - \dfrac{1}{2}\sin^2 2\theta$.

Solution

$$\begin{aligned}
\sin^4\theta + \cos^4\theta &= \sin^4\theta + \cos^4\theta + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta \\
&= (\sin^2\theta + \cos^2\theta) - \frac{1}{2}(2\sin\theta\cos\theta)^2 \\
&= 1 - \frac{1}{2}\sin^2 2\theta
\end{aligned}$$

(b) Hence show that $\sin^4\dfrac{\pi}{16} + \sin^4\dfrac{3\pi}{16} + \sin^4\dfrac{5\pi}{16} + \sin^4\dfrac{7\pi}{16} = \dfrac{3}{2}$.

Solution

$$\begin{aligned}
\sin^4\frac{\pi}{16} + \sin^4\frac{3\pi}{16} + \sin^4\frac{5\pi}{16} + \sin^4\frac{7\pi}{16} &= \sin^4\frac{\pi}{16} + \sin^4\frac{3\pi}{16} + \cos^4\frac{3\pi}{16} + \cos^4\frac{\pi}{16} \\
&= 1 - \frac{1}{2}\sin^2\frac{\pi}{8} + 1 - \frac{1}{2}\sin^2\frac{3\pi}{8} \\
&= 2 - \frac{1}{2}\left(\sin^2\frac{\pi}{8} + \cos^2\frac{\pi}{8}\right) \\
&= \frac{3}{2}
\end{aligned}$$