Preface

We are thankful to Almighty ALLAH Who gave us an opportunity to write the book , named 'Applied Mathematics-I ' as Textbook of Diploma of Associate Engineer (DAE), intending to cover the new syllabus for the first year students

Throughout the book, emphasis is on correct methods of computation, transposition of formulae, logical layout of solutions, neatness and clarity of arrangement of material , systematic use of all the normal mathematical and other tables.

This book covers topics including algebra, trigonometry, vectors & scalars, matrices & determinants and mensuration.

Normally students face difficulty in solving complicated problems because they do not make a systematic attempt. We have attempted to help the students to overcome the difficulty by providing detailed instructions for an orderly approach. Difficult procedures and types of problems appearing in the exercise are illustrated by carefully explained examples. In the presentation of these illustrated examples, we have avoided unnecessary explanations. It is hoped that this book will help to give students a good foundation in old and new techniques.

Students are reminded that in order to acquire a proper understanding of the subject and its application, it is necessary to learn a number of sound basic rules and methods.No scientific or engineering subject can be fully comprehended and satisfactorily studied without a sound mathematical background.

We would like to express , sincere and thanks to Mr.Jawad Ahmed Qureshi Chief Operating Officer TEVTA,Engr. Mr.Azhar Iqbal Shad G.M Academic,Engr.Mazhar Abbas Naqvi Manager (Curriculum) and Engr. Syed Muhammad Waqar ud- Din Deputy Director(technical) Curriculum Section Acade mics Wing, who took keen interest and inspired us for the completion of this task.

We made every effort to make the book valuable both for students and teachers , however we shall gratefully welcome to receive any suggestion for the further improvement of the book.

Authors.

A Textbook of

APPLIED
MATHEMATICS-I

for **Math–113**

First year
Diploma of Associate Engineer
(DAE)

Including:
Objective Type
and
Short Questions

Authors/MDS

Sanaullah Khan
Associate Professor
Govt. College of Tech.
RailwayRoad Lahore

Tahir Hameed
Assistant Professor
Govt. College of Tech.
Raiwind Road, Lahore

Nasir-ud-Din Mahmood
Assistant Professor
Govt. College of Tech.
for Printing&Graphic Arts

Reviewed by(MEC)

Muhammad Javaid Iqbal
Associate Professor
Govt. College of Tech.
RailwayRoad Lahore

Syed Tanvir Haider
Assistant Professor
Govt. College of Tech.
Raiwind Road Lahore

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PAPER A From Chapter 1 to 7

Chapter 1
Quadratic Equations

1.1 Equation:

An equation is a statement of equality '=' between two expression for particular values of the variable. For example

$5x + 6 = 2$, x is the variable (unknown)

The equations can be divided into the following two kinds:

Conditional Equation:

It is an equation in which two algebraic expressions are equal for particular value/s of the variable e.g.,

a) $2x = 3$ is true only for $x = 3/2$

b) $x^2 + x - 6 = 0$ is true only for $x = 2, -3$

Note: for simplicity a conditional equation is called an equation.

Identity:

It is an equation which holds good for all value of the variable e.g;

a) $(a + b)\, x \equiv ax + bx$ is an identity and its two sides are equal for all values of x.

b) $(x + 3)(x + 4) \equiv x^2 + 7x + 12$ is also an identity which is true for all values of x.

For convenience, the symbol '=' shall be used both for equation and identity.

1.2 Degree of an Equation:

The degree of an equation is the highest sum of powers of the variables in one of the term of the equation. For example

$2x + 5 = 0$	1st degree equation in single variable
$3x + 7y = 8$	1st degree equation in two variables
$2x^2 - 7x + 8 = 0$	2nd degree equation in single variable
$2xy - 7x + 3y = 2$	2nd degree equation in two variables
$x^3 - 2x^2 + 7x + 4 = 0$	3rd degree equation in single variable
$x^2y + xy + x = 2$	3rd degree equation in two variables

1.3 Polynomial Equation of Degree n:

An equation of the form

$$a_nx^n + a_{n-1}x^{n-1} + \text{----------------} + a_3x^3 + a_2x^2 + a_1x + a_0 = 0\text{--------------}(1)$$

Where n is a non-negative integer and a_n, a_{n-1}, -------------, a_3, a_2, a_1, a_0 are real constants, is called polynomial equation of degree n. Note that the degree of the equation in the single variable is the highest power of x which appear in the equation. Thus

$3x^4 + 2x^3 + 7 = 0$

$x^4 + x^3 + x^2 + x + 1 = 0$, $x^4 = 0$

are all fourth-degree polynomial equations.

By the techniques of higher mathematics, it may be shown that nth degree equation of the form (1) has exactly n solutions (roots). These roots may be real, complex or a mixture of both. Further it may be shown that if such an equation has complex roots, they occur in pairs of conjugates complex numbers. In other words it cannot have an odd number of complex roots.

A number of the roots may be equal. Thus all four roots of $x^4 = 0$ are equal which are zero, and the four roots of $x^4 - 2x^2 + 1 = 0$

Comprise two pairs of equal roots (1, 1, -1, -1).

1.4 Linear and Cubic Equation:

The equation of first degree is **called linear equation.**

For example,

i) $x + 5 = 1$ (in single variable)

ii) $x + y = 4$ (in two variables)

The equation of third degree is **called cubic equation.**

For example,

i) $a_3x^3 + a_2x^2 + a_1x + a0 = 0$ (in single variable)

ii) $9x^3 + 5x^2 + 3x = 0$ (in single variable)

iii) $x^2y + xy + y = 8$ (in two variables)

1.5 Quadratic Equation:

The equation of second degree is called quadratic equation. The word quadratic comes from the Latin for "square", since the highest power of the unknown that appears in the equation is square. For example

$2x^2 - 3x + 7 = 0$ (in single variable)

$xy - 2x + y = 9$ (in two variable)

Standard form of quadratic equation

The standard form of the quadratic equation is $ax^2 + bx + c = 0$, where a, b and c are constants with $a \neq 0$.

If $b \neq 0$ then this equation is called **complete quadratic equation** in x.

If $b = 0$ then it is called a **pure or incomplete quadratic equation** in x.

For example, $5x^2 + 6x + 2 = 0$ is a complete quadratic equation is x.

and $3x^2 - 4 = 0$ is a pure or incomplete quadratic equation.

1.6 Roots of the Equation:

The value of the variable which satisfies the equation is called the root of the equation. A quadratic equation has two roots and hence there will be two values of the variable which satisfy the quadratic equation. For example the roots of $x^2 + x - 6 = 0$ are 2 and -3.

1.7 Methods of Solving Quadratic Equation:

There are three methods for solving a quadratic equation:

i) By factorization

ii) By completing the square

iii) By using quadratic formula

i) Solution by Factorization:

Method:

Step I: Write the equation in standard form.

Step II: Factorize the quadratic equation on the left hand side if possible.

Step III: The left hand side will be the product of two linear factors. Then equate each of the linear factor to zero and solve for values of x. These values of x give the solution of the equation.

Example 1:

Solve the equation $3x^2 + 5x = 2$

Solution:

$$3x^2 + 5x = 2$$

Write in standard form $3x^2 + 5x - 2 = 0$

Factorize the left hand side $3x^2 + 6x - x - 2 = 0$

$3x(x + 2) - 1(x + 2) = 0$

$(3x - 1)(x + 2) = 0$

Equate each of the linear factor to zero.

$3x - 1 = 0$ or $x + 2 = 0$

$3x = 1$ or $x = -2$

$x = \frac{1}{3}$

$x = \frac{1}{3}$, - 2 are the roots of the Equation.

Solution Set = $\{\frac{1}{3}, -2\}$

Example 2:

Solve the equation $6x^2 - 5x = 4$

Solution:

$6x^2 - 5x = 4$

$6x^2 - 5x - 4 = 0$

$6x^2 - 8x + 3x - 4 = 0$

$2x(3x - 4) + 1(3x - 4) = 0$

$(2x + 1)(3x - 4) = 0$

$\therefore$ Either $2x + 1 = 0$ or $3x - 4 = 0$

Which gives $2x = -1$ which gives $3x = 4$

$\Rightarrow x = -\frac{1}{2}$ $\Rightarrow x = \frac{4}{3}$

$\therefore$ Required Solution Set = $\left\{-\frac{1}{2}, \frac{4}{3}\right\}$

ii) Solution of quadratic equation by Completing the Square

Method:

Step I: Write the quadratic equation is standard form.

Step II: Divide both sides of the equation by the co-efficient of x^2 if it is not already 1.

Step III: Shift the constant term to the R.H.S.

Step IV: Add the square of one-half of the co-efficient of x to both side.

Step V: Write the L.H.S as complete square and simplify the R.H.S.

Step VI: Take the square root on both sides and solve for x.

Example 3:

Solve the equation $3x^2 = 15 - 4x$ by completing the square.

Solution: $3x^2 = 15 - 4x$

Step I Write in standard form: $3x^2 + 4x - 15 = 0$

Step II Dividing by 3 to both sides: $x^2 + \frac{4}{3}x - 5 = 0$

Step III Shift constant term to R.H.S: $x^2 + \frac{4}{3}x = 5$

Step IV Adding the square of one half of the co-efficient of x. i.e., $\left(\frac{4}{6}\right)^2$ on both sides:

$$x^2 + \frac{4}{3}x + \left(\frac{4}{6}\right)^2 = 5 + \left(\frac{4}{6}\right)^2$$

Step V: Write the L.H.S. as complete square and simplify the R.H.S. :

$$\left(x + \frac{4}{6}\right)^2 = 5 + \frac{16}{36}$$

$$= \frac{180 + 16}{36}$$

$$\left(x + \frac{4}{6}\right)^2 = \frac{196}{36}$$

Step VI: Taking square root of both sides and Solve for x

$$\sqrt{\left(x + \frac{4}{6}\right)^2} = \sqrt{\frac{196}{36}}$$

$$x + \frac{4}{6} = \pm\frac{14}{6}$$

$$x + \frac{4}{6} = \pm\frac{7}{3}$$

$$x + \frac{4}{6} = \frac{7}{3}, \qquad x + \frac{4}{6} = -\frac{7}{3}$$

$$\Rightarrow \quad x = \frac{7}{3} - \frac{4}{6} \qquad \Rightarrow \quad x = -\frac{7}{3} - \frac{4}{6}$$

$$x = \frac{10}{6}, \qquad x = \frac{-18}{6}$$

$$x = \frac{5}{3}, \qquad x = -3$$

Hence, the solution set = $\left\{-3, \frac{5}{3}\right\}$

Example 4:

Solve the equation $a^2 x^2 = ab\,x + 2b^2$ by completing the square.

Solution:

$a^2 x^2 = ab\,x + 2b^2$

$a^2 x^2 - ab\,x - 2b^2 = 0$

Dividing both sides by a^2, we have

$$x^2 - \frac{bx}{a} - \frac{2b^2}{a^2} = 0$$

$$x^2 - \frac{bx}{a} = \frac{2b^2}{a^2}$$

Adding the square of one half of the co-efficient of x i.e., $\left(-\frac{b}{2a}\right)^2$ on both sides.

$$x^2 - \frac{bx}{a} + \left(-\frac{b}{2a}\right)^2 = \frac{2b^2}{a^2} + \left(-\frac{b}{2a}\right)^2$$

$$\left(x - \frac{b}{2a}\right)^2 = \frac{2b^2}{a^2} + \frac{b^2}{4a^2}$$

$$\left(x - \frac{b}{2a}\right)^2 = \frac{8b^2 + b^2}{4a^2}$$

$$\left(x - \frac{b}{2a}\right)^2 = \frac{9b^2}{4a^2}$$

Taking square root on both sides

$$x - \frac{b}{2a} = \pm\frac{3b}{2a}$$

$$x - \frac{b}{2a} = \frac{3b}{2a} \qquad\qquad x - \frac{b}{2a} = \frac{3b}{2a}$$

$$\Rightarrow \quad x = \frac{b}{2a} + \frac{3b}{2a} \qquad\qquad \Rightarrow \quad x = \frac{b}{2a} - \frac{3b}{2a}$$

$$\Rightarrow \quad x = \frac{b + 3b}{2a} \qquad\qquad \Rightarrow \quad x = \frac{b - 3b}{2a}$$

$$\Rightarrow \quad x = \frac{4b}{2a} \qquad\qquad \Rightarrow \quad x = -\frac{2b}{2a}$$

$$\Rightarrow \quad x = \frac{2b}{a} \qquad\qquad \Rightarrow \quad x = -\frac{b}{a}$$

Solution Set = $\left\{\frac{2b}{a}, -\frac{b}{a}\right\}$

iii) Derivation of Quadratic formula

Consider the standard form of quadratic equation $ax^2 + bx + c = 0$.
Solve this equation by completing the square.

$$ax^2 + bx + c = 0$$

Dividing both sides by a

$$x^2 + \frac{b}{a}x + \frac{c}{a} = o$$

Take the constant term to the R.H.S

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

To complete the square on L.H.S. add $\left(\frac{b}{2a}\right)^2$ to both sides.

$$X^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Taking square root of both sides

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

which is called the **Quadratic formula.**

Where, a = co-efficient of x^2 , b = coefficient of x , c = constant term

Actually, the Quadratic formula is the general solution of the quadratic equation $ax^2 + bx + c = 0$

Note: $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$ are also called roots of the quadratic equation

Method:

To solve the quadratic equation by Using Quadratic formula:

Step I: Write the Quadratic Equation in Standard form.

Step II: By comparing this equation with standard form $ax^2 + bx + c = 0$ to identify the values of a , b , c.

Step III: Putting these values of a, b, c in Quadratic formula

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and solve for x.

Example 5:

Solve the equation $3x^2 + 5x = 2$

Solution:

$$3x^2 + 5x = 2$$

$$3x^2 + 5x - 2 = 0$$

Composing with the standard form $ax^2 + bx + c = 0$, we have a = 3 , b = 5, c = -2.

Putting these values in Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{(5)^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{-5 \pm \sqrt{25 + 24}}{6}$$

$$x = \frac{-5 \pm 7}{6}$$

$$x = \frac{-5 + 7}{6} \qquad\qquad x = \frac{-5 - 7}{6}$$

$$x = \frac{2}{6} \qquad\qquad x = \frac{-12}{6}$$

$$x = \frac{1}{3} \qquad\qquad x = -2$$

$$\text{Sol. Set} = \left\{\frac{1}{3},\ 2\right\}$$

Example 6:

Solve the equation $15x^2 - 2ax - a^2 = 0$ by using Quadratic formula:

Solution:

$$15x^2 - 2ax - a^2 = 0$$

Comparing this equation with General Quadratic Equation

Here, $a = 15$, $b = -2a$, $c = -a^2$

Putting these values in Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2a) \pm \sqrt{(-2a)^2 - 4(15)(-a^2)}}{2(15)}$$

$$= \frac{-(-2a) \pm \sqrt{4a^2 + 60a^2}}{30}$$

$$= \frac{2a \pm 8a}{30}$$

$$x = \frac{2a + 8a}{30} \qquad\qquad x = \frac{2a - 8a}{30}$$

$$x = \frac{10a}{30} \qquad\qquad x = \frac{-6a}{30}$$

$$x = \frac{a}{3} \qquad\qquad x = -\frac{a}{5}$$

$$\text{Sol. Set} = \left\{\frac{a}{3},\ -\frac{a}{5}\right\}$$

Example 7:

Solve the equation $\frac{1}{2x - 5} + \frac{5}{2x - 1} = 2$ by using Quadratic formula.

Solution:

$$\frac{1}{2x - 5} + \frac{5}{2x - 1} = 2$$

Multiplying throughout by (2x – 5)(2x – 1), we get

$$(2x - 1) + 5(2x - 5) = 2\,(2x - 5)\,(2x - 1)$$
$$2x - 1 + 10x - 25 = 8x^2 - 24x + 10$$
$$8x^2 - 36x + 36 = 0$$
$$2x^2 - 9x + 9 = 0$$

Comparing this equation with General Quadratic Equation

Here, a = 2 , b = -9 , c = 9

Putting these values in the Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{(-9) \pm \sqrt{(-9)^2 - 4(2)(9)}}{2(2)}$$
$$= \frac{9 \pm \sqrt{81 - 72}}{4}$$
$$= \frac{9 \pm 3}{4}$$

$$x = \frac{9+3}{4} \qquad x = \frac{9-3}{4}$$
$$x = \frac{12}{4} \qquad x = \frac{6}{4}$$
$$x = 3 \qquad x = -\frac{3}{2}$$

$$\text{Sol. Set } \left\{3, -\frac{3}{2}\right\}$$

Exercise 1.1

Q.1. Solve the following equations by factorization.

(i). $x^2 + 7x = 8$ (ii). $3x^2 + 7x + 4 = 0$

(iii). $x^2 - 3x = 2x - 6$ (iv). $3x^2 - 1 = \frac{1}{5}(1 - x)$

(v). $(2x + 3)(x + 1) = 1$ (vi). $\frac{1}{2x - 5} + \frac{5}{2x - 1} = 2$

(vii). $\dfrac{4}{x-1} - \dfrac{5}{x+2} = \dfrac{3}{x}$ (viii). $\dfrac{1}{a+b+x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{x}$

(ix). $abx^2 + (b^2 - ac)x - bc = 0$ (x). $(a+b)x^2 + (a + 2b + c)x + (b + c) = 0$

(xi). $\dfrac{a}{ax-1} + \dfrac{b}{bx-1} = a + b$ (xii). $\dfrac{x+2}{x-1} + 2\dfrac{2}{3} = \dfrac{x+3}{x-2}$

Q.2. Solve the following equations by the method of completing the square.

(i). $x^2 - 6x + 8 = 0$ (ii). $32 - 3x^2 = 10x$

(iii). $(x - 2)(x + 3) = 2(x + 11)$ (iv). $x^2 + (a + b)x + ab = 0$

(v). $x + \dfrac{1}{x} = \dfrac{10}{3}$ (vi). $\dfrac{10}{x-5} + \dfrac{10}{x+5} = \dfrac{5}{6}$

(vii). $2x^2 - 5bx = 3b^2$ (viii). $x^2 - 2ax + a^2 - b^2 = 0$

Q.3 Solve the following equations by using quadratic formula.

(i). $2x^2 + 3x - 9 = 0$ (ii). $(x + 1)^2 = 3x + 14$

(iii). $\dfrac{1}{x+1} + \dfrac{1}{x+2} + \dfrac{1}{x+3} = \dfrac{3}{x}$ (iv). $x^2 - 3\left(x + \frac{25}{4}\right) = 9x - \frac{25}{2}$

(v). $x^2 + (m-n)x - 2(m-n)^2 = 0$ (vi) $mx^2 + (1 + m)x + 1 = 0$

(vii) $abx^2 + (2b - 3a)x - 6 = 0$ (viii). $x^2 + (b - a)x - ab = 0$

(ix) $\dfrac{x}{x+1} + \dfrac{x+1}{x+2} + \dfrac{x+2}{x+3} = 3$

Q.4 The sum of a number and its square is 56 . Find the number.

Q.5 A projectile is fired vertically into the air. The distance (in meter) above the ground as a function of time (in seconds) is given by $s = 300 - 100t - 16t^2$. When will the projectile hit the ground?

Q.6 The hypotenuse of a right triangle is 18 meters. If one side is 4 meters longer than the other side, what is the length of the shorter side ?

Answers 1.1

Q.1. (i). $\{1, -8\}$ (ii). $\left\{-1, \dfrac{-4}{3}\right\}$ (iii). $\{2, 3\}$

(iv). $\{\frac{3}{5}, -\frac{2}{3}\}$ (v). $\left\{-2, -\dfrac{1}{2}\right\}$ (vi). $\left\{-\frac{3}{2}, 3\right\}$

(vii). $\left\{-\frac{1}{2}, 3\right\}$ (viii). {-a, -b} (ix). $\{-\frac{b}{a}, \frac{c}{b}\}$

(x). $\{-1, -\frac{b+c}{a+b}\}$ (xi). $\{\frac{a+b}{ab}, \frac{2}{a+b}\}$ (xii). $\{\frac{13}{4}, \frac{1}{2}\}$

Q.2. (i). {2, 4} (ii). $\{2, -\frac{16}{3}\}$ (iii). $\{\frac{1 \pm \sqrt{113}}{2}\}$

(iv). {– a, – b} (v). $\{3, \frac{1}{3}\}$ (vi). {-1, 25}

(vii). $\{3b, -\frac{b}{2}\}$ (viii) { (a + b), (a – b)}

Q.3. (i). $\{\frac{3}{2}, -3\}$ (ii). $\{\frac{1 \pm \sqrt{53}}{2}\}$ (iii). $\{\frac{-11 \pm \sqrt{13}}{6}\}$

(iv). $\{\frac{25}{2}, -\frac{1}{2}\}$ (v). $\{m-n, -2(m-n)\}$ (vi) $\left\{-1, \frac{-1}{m}\right\}$

(vii). $\{-\frac{2}{a}, \frac{3}{b}\}$ (viii). $\{-b, a\}$ (ix) $\left\{\frac{-6+\sqrt{3}}{3}, \frac{-6-\sqrt{3}}{3}\right\}$

Q.4. 7 , - 8 Q.5. 8.465 seconds Q.6. 10.6 m

1.8 Classification of Numbers

1. The Set N of Natural Numbers:

Whose elements are the counting, or natural numbers:

N = {1, 2, 3, - - - - - - - - }

2. The Set Z of Integers:

Whose elements are the positive and negative whole numbers and zero:

Z = {- - - - - - , -2, -1, 0, 1, 2, - - - - - - }

3. The Set Q of Rational Numbers: Whose elements are all those numbers that can be represented as the quotient of two integers $\frac{a}{b}$, where $b \neq 0$. Among the elements of Q are such numbers as $-\frac{3}{4}, \frac{18}{27}, \frac{5}{1}, -\frac{9}{1}$. In symbol

$$Q = \left\{\frac{a}{b} \mid a, b \in Z, b \neq 0\right\}$$

Equivalently, rational numbers are numbers with terminating or repeating decimal representation, such as

1.125, 1.52222, 1.56666, 0.3333

4. The Set Q′ of Irrational Numbers:

Whose elements are the numbers with decimal representations that are non-terminating and non-repeating. Among the elements of this set are such numbers as $\sqrt{2},\ -\sqrt{7},\ \pi$.

An irrational number cannot be represented in the form $\frac{a}{b}$, where $a, b \in Z$. In symbols,

$Q' = \{\text{irrational numbers}\}$

5. The Set R of Real Numbers:

Which is the set of all rational and irrational numbers:

$$R = \{x \mid x \in Q \cup Q'\}$$

6. The set I of Imaginary Numbers:

Whose numbers can be represented in the form x + yi, where x and y are real numbers, $y \neq 0$ and $i = \sqrt{-1}$

$$I = \{x + yi \mid x, y \in R, y \neq 0, i = \sqrt{-1}\}$$

If x = 0, then the imaginary number is called a pure imaginary number.

An imaginary number is defined as, a number whose square is a negative i.e, $\sqrt{-1},\ \sqrt{-3},\ \sqrt{-5}$

7. The set C of Complex Numbers:

Whose members can be represented in the form x + y i, where x and y real numbers and $i = \sqrt{-1}$:

$$C = \{x + yi \mid x, y \in R, i = \sqrt{-1}\}$$

With this familiar identification, the foregoing sets of numbers are related as indicated in Fig. 1.

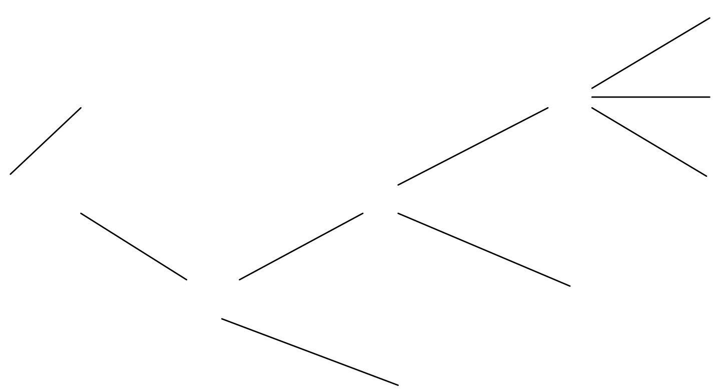


Fig. 1

Hence, it is clear that $N \subseteq Z \subseteq Q \subseteq R \subseteq C$

1.9 Nature of the roots of the Equation $ax^2 + bx + c = 0$

The two roots of the Quadratic equation $ax^2 + bx + c = 0$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression $b^2 - 4ac$ which appear under radical sign is called the Discriminant (Disc.) of the quadratic equation. i.e., Disc = $b^2 - 4ac$

The expression $b^2 - 4ac$ discriminates the nature of the roots, whether they are real, rational, irrational or imaginary. There are three possibilities.

(i) $b^2 - 4ac < 0$ (ii) $b^2 - 4ac = 0$ (iii) $b^2 - 4ac > 0$

(i) If $b^2 - 4ac < 0$, then roots will be imaginary and unequal.
(ii) If $b^2 - 4ac = 0$, then roots will be real, equal and rational.
(This means the left hand side of the equation is a perfect square).
(iii) If $b^2 - 4ac > 0$, then two cases arises:
(a) $b^2 - 4ac$ is a perfect square, the roots are real, rational and unequal.
(This mean the equation can be solved by the factorization).
(b) $b^2 - 4ac$ is not a perfect square, then roots are real, irrational and unequal.

Example 1:

Find the nature of the roots of the given equation

$$9x^2 + 6x + 1 = 0$$

Solution:

$$9x^2 + 6x + 1 = 0$$

Here $a = 9$, $b = 6$, $c = 1$

Therefore , Discriminant $= b^2 - 4ac$
$= (6)^2 - 4(9)(1)$
$= 36 - 36$
$= 0$

Because $b^2 - 4ac = 0$

$\therefore$ roots are equal, real and rational.

Example 2:

Find the nature of the roots of the Equation

$$3x^2 - 13x + 9 = 0$$

Solution:

$$3x^2 - 13x + 9 = 0$$

Here $a = 3$, $b = -13$, $c = 9$

Discriminant $= b^2 - 4ac$
$= (-13)^2 - 4(3)(9)$
$= 169 - 108 = 61$

Disc $= b^2 - 4ac = 61$ which is positive

Hence the roots are real, unequal and irrational.

Example 3:

For what value of "K" the roots of $Kx^2 + 4x + (K - 3) = 0$ are equal.

Solution:

$$Kx^2 + 4x + (K - 3) = 0$$

Here $a = K$, $b = 4$, $c = K - 3$

Disc $= b^2 - 4ac$
$= (4)^2 - 4(K)(K - 3)$
$= 16 - 4K^2 + 12K$

The roots are equal if $b^2 - 4ac = 0$

$$\text{i.e.}\quad 16 - 4K^2 + 12K = 0$$
$$4K^2 - 3K - 4 = 0$$
$$K^2 - 4K + K - 4 = 0$$
$$K(K-4) + 1(K-4) = 0$$

Or $K = 4, -1$

Hence roots will be equal if $K = 4, -1$

Example 4:

Show that the roots of the equation

$$2(a+b)x^2 - 2(a+b+c)x + c = 0 \text{ are real}$$

Solution: $2(a+b)x^2 - 2(a+b+c)x + c = 0$

Here, $a = 2(a+b)$, $b = -2(a+b+c)$, $c = c$

Discriminant $= b^2 - 4ac$

$$= [-2(a+b+c)]^2 - 4[2(a+b)c]$$
$$= 4(a^2 + b^2 + c^2 + 2ab + 2bc + 2ac) - 8(ac + bc)$$
$$= 4(a^2 + b^2 + c^2 + 2ab + \cancel{2bc} + \cancel{2ac} - \cancel{2ac} - \cancel{2bc})$$
$$= 4(a^2 + b^2 + c^2 + 2ab)$$
$$= 4[(a^2 + b^2 + 2ab) + c^2]$$
$$= 4[(a+b)^2 + c^2]$$

Since each term is positive, hence

Disc > 0 Hence , the roots are real.

Example 5:

For what value of K the roots of equation $2x^2 + 5x + k = 0$ will be rational.

Solution:

$2x^2 + 5x + k = 0$

Here, $a = 2$, $b = 5$, $c = k$

The roots of the equation are rational if

Disc $= b^2 - 4ac = 0$

So, $5^2 - 4(2)k = 0$

$25 - 8k = 0$

$k = \dfrac{25}{8}$ Ans

Exercise 1.2

Q1. Find the nature of the roots of the following equations

(i) $2x^2 + 3x + 1 = 0$ (ii) $6x^2 = 7x + 5$

(iii) $3x^2 + 7x - 2 = 0$ (iv) $\sqrt{2}\,x^2 + 3x - \sqrt{8} = 0$

Q2. For what value of K the roots of the given equations are equal.

(i) $x^2 + 3(K+1)x + 4K + 5 = 0$ (ii) $x^2 + 2(K-2)x - 8k = 0$

(iii) $(3K+6)x^2 + 6x + K = 0$ (iv) $(K+2)x^2 - 2Kx + K - 1 = 0$

Q3. Show that the roots of the equations

(i) $a^2(mx+c)^2 + b^2x^2 = a^2b^2$ will be equal if $c^2 = b^2 + a^2m^2$

(ii) $(mx + c)^2 = 4ax$ will be equal if $c = \frac{a}{m}$

(iii) $x^2 + (mx + c)^2 = a^2$ has equal roots if $c^2 = a^2 (1 + m^2)$.

Q4. If the roots of $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal then prove that $a^3 + b^3 + c^3 = 3abc$

Q5. Show that the roots of the following equations are real

(i) $x^2 - 2\left(m+\frac{1}{m}\right)x + 3 = 0$

(ii) $x^2 - 2ax + a^2 = b^2 + c^2$

(iii) $(b^2 - 4ac)x^2 + 4(a + c)x - 4 = 0$

Q6. Show that the roots of the following equations are rational

(i) $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$

(ii) $(a + 2b)x^2 + 2(a + b + c)x + (a + 2c) = 0$

(iii) $(a + b)x^2 - ax - b) = 0$

(iv) $p x^2 - (p - q) x - q = 0$

Q7. For what value of 'K' the equation $(4-k) x^2 + 2(k+2) x + 8k + 1 = 0$ will be a perfect square.
(Hint : The equation will be perfect square if Disc. $b^2 - 4ac = 0$)

Answers 1.2

Q1. (i) Real, rational, unequal (ii) unequal, real and rational
(iii) ir-rational, unequal, real (iv) Real, unequal, ir-rational

Q2. (i) $1, \frac{-11}{9}$ (ii) -2 (iii) $1, -3$ (iv) 2

Q7. 0, 3

1.10 Sum and Product of the Roots

(Relation between the roots and Co-efficient of $ax^2 + bx + c = 0$)
The roots of the equation $ax^2 + bx + c = 0$ are

$$\propto = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Sum of roots

Add the two roots

$$\propto + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b\ -b}{2a}$$

$$= \frac{-2b}{2a} \qquad = -\frac{b}{a}$$

Hence, sum of roots = $\propto + \beta = \dfrac{-\text{Co-efficient of x}}{\text{Co-efficient of } x^2}$

Product of roots

$$\propto \beta = \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right) x \left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right) \quad =$$

$$= \frac{(-b)^2 - (\sqrt{b^2 - 4ac}\ \ )^2}{2\ a^2}$$

$$= \frac{b^2-b^2+4ac}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$a\beta = \frac{c}{a}$$

i.e. product of roots = $\propto \beta = \dfrac{-\text{Constant term}}{\text{Co-efficient of } x^2}$

Example 1:

Find the sum and the Product of the roots in the Equation $2x^2 + 4 = 7x$

Solution:

$2x^2 + 4 = 7x$

$2x^2 - 7x + 4 = 0$

Here a = 2, b = -7, c = 4

Sum of the roots = $-\dfrac{b}{a} = -\left(-\dfrac{7}{2}\right) = \dfrac{7}{2}$

Product of roots = $\dfrac{c}{a} = \dfrac{4}{2} = 2$

Example 2:

Find the value of "K" if sum of roots of

$(2k - 1)x^2 + (4K - 1)x + (K + 3) = 0$ is $\dfrac{5}{2}$

Solution:

$(2k - 1)x^2 + (4K - 1)x + (K + 3) = 0$

Here a = (2k – 1), b = 4K – 1, c = K + 3

Sum of roots $= -\frac{b}{a}$

$\frac{5}{2} = -\frac{(4K - 1)}{(2K - 1)}$ $\quad \because$ Sum of roots $= \frac{5}{2}$

$5\,(2K - 1) = -2\,(4K - 1)$

$10K - 5 = -8K + 2$

$10K + 8K = 5 + 5$

$18K = 7$

$K = \frac{7}{18}$

Example 3:

If one root of $4x^2 - 3x + K = 0$ is 3 times the other, find the value of "K".

Solution:

Given Equation is $4x^2 - 3x + K = 0$

Let one root be α, then other will be 3α.

Sum of roots $= -\frac{a}{b}$

$\alpha + 3\alpha = -\frac{(-3)}{4}$

$4\alpha = \frac{3}{4}$

$\alpha = \frac{3}{16}$

Product of roots $= \frac{c}{a}$

$\alpha(3\alpha) = \frac{K}{4}$

$3\alpha^2 = \frac{K}{4}$

$K = 12\alpha^2$

Putting the value of $\alpha = \frac{3}{16}$ we have

$K = 12\left(\frac{3}{16}\right)^2$

$= \frac{12x9}{256} = \frac{27}{64}$

Exercise 1.3

Q1. Without solving, find the sum and the product of the roots of the following equations.

(i) $x^2 - x + 1 = 0$ (ii) $2y^2 + 5y - 1 = 0$

(iii) $x^2 - 9 = 0$ (iv) $2x^2 + 4 = 7x$

(v) $5x^2 + x - 7 = 0$

Q2. Find the value of k, given that

(i) The product of the roots of the equation $(k + 1)x^2 + (4k + 3)x + (k - 1) = 0$ is $\frac{7}{2}$

(ii) The sum of the roots of the equation $3x^2 + k\,x + 5 = 0$ will be equal to the product of its roots.

(iii) The sum of the roots of the equation $4x^2 + kx - 7 = 0$ is 3.

Q3. (i) If the difference of the roots of $x^2 - 7x + k - 4 = 0$ is 5, find the value of k and the roots.

(ii) If the difference of the roots of $6x^2 - 23x + c = 0$ is $\frac{5}{6}$, find the value of k and the roots.

Q4. If α, β are the roots of $ax^2 + bx + c = 0$ find the value of

(i) $\alpha^3 + \beta^3$ (ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ (iii) $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} = 0$

(iv) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ (v) $\frac{\alpha}{\beta} - \frac{\beta}{\alpha}$

(v)

Q5. If p, q are the roots of $2x^2 - 6x + 3 = 0$ find the value of $(p^3 + q^3) - 3pq\,(p^2 + q^2) - 3pq\,(p + q)$

Q6. The roots of the equation $px^2 + qx + q = 0$ are α and β,

Prove that $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}} = 0$

Q7. Find the condition that one root of the equation $px^2 + qx + r = 0$ is square of the other.

Q8. Find the value of k given that if one root of $9x^2 - 15x + k = 0$ exceeds the other by 3. Also find the roots.

Q9. If α, β are the roots of the equation $px^2 + qx + r = 0$ then find the values of

(i) $\alpha^2 + \beta^2$ (ii) $(\alpha - \beta)^2$ (iii) $\alpha^3\beta + \alpha\beta^3$

Answers 1.3

Q1. (i) 1, 1 (ii) $-\frac{5}{2}, -\frac{1}{2}$ (iii) 0, - 9 (iv) $\frac{7}{2}, 2$ (v) $-\frac{1}{5}, -\frac{7}{5}$

Q2.(i) $\frac{7}{18}$ (ii) $-\frac{9}{5}$ (iii) - 12

Q3.(i) K =10, roots = 6, 1 (ii) $\propto = \frac{7}{3},\ \beta = \frac{3}{2};$ c = 21

Q4. (i) $\frac{-b^3+3abc}{a^3}$ (ii) $\frac{b^2-2ac}{c^2}$ (iii) $-\frac{b}{\sqrt{ac}}$ (iv) $\frac{3abc-b^3}{a^2c}$ (v) $\frac{-b\sqrt{b^2-4ac}}{ac}$

Q5. - 27 **Q7.** Pr (p + r)+q^3 = 3pqr **Q8.** K = - 14, roots are $-\frac{2}{3},\ \frac{7}{3}$

Q9. (i) $\frac{q^2-2pr}{p^2}$ (ii) $\frac{q^2-4pr}{p^2}$ (iii) $\frac{r(q^2-2pr)}{p^3}$

1.11 Formation of Quadratic Equation from the given roots :

Let α,β be the roots of the Equation $ax^2 + bx + c = 0$

The sum of roots $= \alpha + \beta = -\frac{b}{a}$ ……… (I)

Product of roots $= \propto . \beta = \frac{c}{a}$ ……… (II)

The equation is $ax^2 + bx + c = 0$

Divide this equation by a $\Rightarrow$ $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Or $x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0$

From I and II this equation becomes

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Or x^2 – (Sum of roots) x + Product of roots = 0

Or $x^2 - (S)x + (P) = 0$

is the required equation, where $S = \alpha + \beta$ and $P = \alpha\,\beta$

Alternate method:-

Let α,β be the roots of the equation $a x^2 + b x + c = 0$

i.e., $x = \alpha$ and $x = \beta$

$\Rightarrow$ $x - \alpha = 0$ and $x - \beta = 0$

$\Rightarrow$ $(x - \alpha)(x - \beta) = 0$

$$x^2 - \alpha x - \beta x + \alpha\beta = 0$$

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Or $\quad x^2 -$ (Sum of roots) x + Product of roots = 0

Or $\quad x^2 - Sx + P = 0$

is the required equation, where $S = \alpha + \beta$ and $P = \alpha\beta$

Example 4:

Form a quadratic Equation whose roots are $3\sqrt{5}, -3\sqrt{5}$

Solution:

Roots of the required Equation are $3\sqrt{5}$ and $-3\sqrt{5}$

Therefore S = Sum of roots = $3\sqrt{5} - 3\sqrt{5}$

$S = 0$

P = Product of roots = $(3\sqrt{5})(-3\sqrt{5}) = -9\,(5)$

$P = -45$

Required equation is

$x^2 -$ (Sum of roots) x + (Product of roots) = 0

Or $\quad x^2 - Sx + P = 0$

$x^2 - 0(x) + (-45) = 0$

$x^2 - 0 - 45 = 0$

$x^2 - 45 = 0$

Example 5:

If α, β are the roots of the equation $ax^2 + bx + c = 0$, find the equation whose **roots are $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$.**

Solution:

Because α, β are the roots of the Equation $ax^2 + bx + c = 0$

The sum of roots = $\alpha + \beta = -\dfrac{b}{a}$

Product of roots = $\alpha\ \beta = \dfrac{b}{a}$

Roots of the required equation are $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$

Therefore ,

S = sum of roots of required equation = $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

$$= \frac{\alpha^2 + \beta^2}{\alpha\beta} \because (\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

$$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{\left(-\frac{b}{a}\right)^2 - 2\frac{c}{a}}{\alpha\beta}$$

$$= \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c}{a}} \quad = \frac{b^2 - 2ac}{a^2} \text{ x } \frac{a}{c}$$

$$S = \frac{b^2 - 2ac}{ac}$$

P = Product of roots of required equation $= \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} \quad = \frac{\alpha\beta}{\beta\alpha}$

P $= 1$

Required equation is: $x^2 - Sx + P = 0$

$$x^2 + \left(\frac{b^2 - 2ac}{ac}\right) x + 1 = 0$$

$$acx^2 - (b^2 - 2ac)x + ac = 0$$

Exercise 1.4

Q1. Form quadratic equations with the following given numbers as its roots.

(i) $2, -3$ (ii) $3 + i$, $3 - i$ (iii) $2+\sqrt{3},\ 2-\sqrt{3}$

(iv) $-3+\sqrt{5},\ -3-\sqrt{5}$ (v) $4 + 5i$, $4 - 5i$

Q2. Find the quadratic equation with roots

(i)Equal numerically but opposite in sign to those of the roots of the equation $3x^2 + 5x - 7 = 0$

(ii)Twice the roots of the equation $5x^2 + 3x + 2 = 0$

(iii)Exceeding by '2' than those of the roots of $4x^2 + 5x + 6 = 0$

Q3. Form the quadratic equation whose roots are less by '1' than those of $3x^2 - 4x - 1 = 0$

Q4. Form the quadratic equation whose roots are the square of the roots of the equation $2x^2 - 3x - 5 = 0$

Q5. Find the equation whose roots are reciprocal of the roots of the equation $px^2 - qx + r = 0$

Q6. If $\propto, \beta$ are the roots of the equation $x^2 - 4x + 2 = 0$ find the equation whose roots are

(i) α^2 , β^2 (ii) α^3 , β^3 (iii) $\propto + \frac{1}{\propto}, \beta + \frac{1}{\beta}$

(iv) $\alpha + 2$, $\beta + 2$

Q7. If α , β are the roots of $ax^2 + bx + c = 0$ form an equation whose roots are

(i) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (ii) $\frac{\alpha^2}{\beta}, \frac{\beta^2}{\alpha}$ (iii) $\frac{\alpha+1}{\alpha}, \frac{\beta+1}{\beta}$

Answers 1.4

Q1. (i) $x^2 + x - 6 = 0$ (ii) $x^2 - 6x + 10 = 0$
(iii) $x^2 - 4x + 1 = 0$ (iv) $x^2 + 6x + 4 = 0$ (v) $x^2 - 5x + 41 = 0$

Q2. (i) $3x^2 - 5x - 7 = 0$ (ii) $5x^2 - 6x + 8 = 0$
(iii) $4x^2 - 11x + 12 = 0$

Q3. $3x^2 + 2x - 2 = 0$ **Q4 .** $4x^2 - 29x + 25 = 0$

Q5. $rx^2 - qx + p = 0$ **Q6.** (i) $x^2 - 12x + 4 = 0$ (ii) $x^2 - 40x + 8 = 0$
(iii) $2x^2 - 12x + 17 = 0$ (iv) $x^2 - 8x + 14 = 0$

Q7. (i) $acx^2 - (b^2 - 2ac)x + ac = 0$ (ii) $a^2cx^2 + (b^3 - 3abc)x + ac^2 = 0$
(iii) $cx^2 - (2c - b)x + (a - b + c) = 0$

Summary

Quadratic Equation:

An equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, where $a, b, c \in R$ and x is a variable, is called a quadratic equation.

If α, β are its roots then

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}, \ \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Nature of Roots:

(i) If $b^2 - 4ac > 0$ the roots are real and distinct.
(ii) If $b^2 - 4ac = 0$ the roots are real and equal.
(iii) If $b^2 - 4ac < 0$ the roots are imaginary.
(iv) If $b^2 - 4ac$ is a perfect square, roots will be rational, otherwise irrational.

Relation between Roots and Co-efficients

If α and β be the roots of the equation $ax^2 + bx + c = 0$

Then sum of roots = $\alpha + \beta = \frac{-b}{a}$

Product of roots $= \alpha\beta = \frac{c}{a}$

Formation of Equation

If α and β be the roots of the equation $ax^2 + bx + c = 0$ then we have

x^2 – (sum of roots)x + (product of roots) = 0

Short Questions

Write Short answers of the following questions:

Solve the following quadratic equations by factorization

Q.1 $x^2 + 7x + 12 = 0$

Q2. $x^2 - x = 2$

Q3. $x(x + 7) = (2x - 1)(x + 4)$

Q4. $6x^2 - 5x = 4$

Q5. $3x^2 + 5x = 2$

Q6. $2x^2 + x = 1$

Q7. $m x^2 + (1 + m) x + 1 = 0$

Solve the following equations by completing the square:

Q8. $x^2 - 2x - 899 = 0$

Q9. $2x^2 + 12x - 110 = 0$

Q10. $x^2 + 5x - 6 = 0$

Q11. $x^2 - 6x + 8 = 0$

Solve the following equations by quadratic formula :

Q12. $4x^2 + 7x - 1 = 0$

Q13. $9x^2 - x - 8 = 0$

Q14. $X^2 - 3x - 18 = 0$

Q15. $X^2 - 3x = 2x - 6$

Q16. $3x^2 - 5x - 2 = 0$

Q17. $16x^2 + 8x + 1 = 0$

Q18 Define discriminant

Discuss the nature of the roots of the equation:

Q19 $2x^2 - 7x + 3 = 0$

Q20. $x^2 - 5x - 2 = 0$

Q21. $x^2 + x + 1 = 0$

Q22. $x^2 - 2\sqrt{2}\,x + 2 = 0$

Q23. $9x^2 + 6x + 1 = 0$

Q24. $3x^2 - 13x + 9 = 0$

For what value of K the roots of the following equations are equal:

Q25 $Kx^2 + 4x + 3 = 0$

Q26. $2x^2 + 5x + K = 0$

Q27 Prove that the roots of the equation

$(a + b)x^2 - ax - b = 0$ are rational

Q28 Write relation between the roots and the coefficients of the quadratic equation

$ax^2 + bx + c = 0$

Q.29 If the sum of the roots of $4x^2 + kx - 7 = 0$ is 3, Find the value of k.

Q.30 Find the value of K if the sum of the roots of equation

$(2k - 1)x^2 + (4k - 1)x + (K + 3) = 0$ is 5/2

Find the sum and product of the roots of following equations:

Q31 $7x^2 - 5x + 4 = 0$

Q32. $x^2 - 9 = 0$

Q33. $9x^2 + 6x + 1 = 0$

Q34. For what value of k the sum of roots of equation $3x^2 + kx + 5 = 0$

may be equal to the product of roots?

Q35. If α, β are the roots of $x^2 - px - p - c = 0$ then prove that $(1+\alpha)(1+\beta) = 1 - c$

Write the quadratic equation for the following equations whose roots are :

Q.36 -2, -3

Q37. $i\sqrt{3}$, $-i\sqrt{3}$

Q38. $-2+\sqrt{3}$, $-2-\sqrt{3}$

Q.39 Form the quadratic equation whose roots are equal numerically but opposite in sign to those of $3x^2-7x-6=0$

If α, β are the roots of the equation $x^2 - 4x + 2 = 0$ find equation whose roots are:

Q40. $\frac{1}{\alpha}, \frac{1}{\beta}$

Q41 $-\alpha$, $-\beta$

Answers

Q1. {- 3 , - 4} Q2 {- 1 , 2} Q3 { 2 , -2} Q4 {4/3 , - ½} Q5 {1 , -6}

Q6 {-1 , ½} Q7 { -1 , - 1/m} Q8 {-29 , 31} Q9 {-11 , 5} Q10 {1 , – 6}

Q11 {2 , 4} Q12. {1 , -6} Q13 $\{\frac{-7-\sqrt{65}}{8}$, $\frac{-7+\sqrt{65}}{8}\}$ Q14 {-8/9 , 1}

Q15 {6 , -3} Q16 {2 , 3} Q17 {2 , -1/3} Q18 {-1/4 }

Q19. Roots are rational , real and unequal Q20 Roots are irrational , real and unequal

Q21 Roots are imaginary Q22 Roots are equal and real

Q23 Roots are equal and real Q24 Roots are unequal , real and irrational

Q25. K = 4/3 Q26. K = 5 Q29 K = -12 Q30. K = 7/18

Q31 S = 5/7 , P = 4/7 Q32 S = 0 , P = - 9 Q33 S = -2/3 , 1/9

Q34 K = - 5 Q36 $x^2 + 5x + 6 = 0$ Q37 $x^2 + 3 = 0$ Q38 $x^2 + 4x + 1 = 0$

Q39 $3x^2 + 7x - 2 = 0$ Q40 $2x^2 - 4x + 1 = 0$ Q41 $x^2 + 4x + 2 = 0$

Objective Type Questions

Q1. Each question has four possible answers .Choose the correct answer and encircle it .

__1. The standard form of a quadratic equation is:
(a) $ax^2 + bx = 0$ (b) $ax^2 = 0$
(c) $ax^2 + bx + c = 0$ (d) $ax^2 + c = 0$

__2. The roots of the equation $x^2 + 4x - 21 = 0$ are:
(a) (7, 3) (b) (–7, 3)
(c) (–7, –3) (d) (7, –3)

__3. To make $x^2 - 5x$ a complete square we should add:
(a) 25 (b) $\frac{25}{4}$ (c) $\frac{25}{9}$ (d) $\frac{25}{16}$

__4. The factors of $x^2 - 7x + 12 = 0$ are:
(a) $(x - 4)(x + 3)$ (b) $(x - 4)(x - 3)$
(c) $(x + 4)(x + 3)$ (d) $(x + 4)(x - 3)$

__5. The quadratic formula is:
(a) $\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$ (b) $\frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$
(c) $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (d) $\frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$

__6. A second degree equation is known as:
(a) Linear (b) Quadratic
(c) Cubic (e) None of these

__7. Factors of $x^3 - 1$ are:
(a) $(x - 1)(x^2 - x - 1)$ (b) $(x - 1)(x^2 + x + 1)$
(c) $(x - 1)(x^2 + x - 1)$ (d) $(x - 1)(x^2 - x + 1)$

__8. To make $49x^2 + 5x$ a complete square we must add:
(a) $\left(\frac{5}{14}\right)^2$ (b) $\left(\frac{14}{5}\right)^2$
(c) $\left(\frac{5}{7}\right)^2$ (d) $\left(\frac{7}{5}\right)^2$

__9. $lx^2 + mx + n = 0$ will be a pure quadratic equation if:
(a) $l = 0$ (b) $m = 0$
(c) $n = 0$ (d) Both l, $m = 0$

__10. If the discrimnant $b^2 - 4ac$ is negative, the roots are:
(a) Real (b) Rational
(c) Irrational (d) Imaginary

__11. If the discriminant $b^2 - 4ac$ is a perfect square, its roots will be:
(a) Imaginary (b) Rational
(c) Equal (d) Irrational

__12. The product of roots of $2x^2 - 3x - 5 = 0$ is:

(a) $-\frac{5}{2}$ (b) $\frac{5}{2}$

(c) $\frac{2}{5}$ (d) $-\frac{2}{5}$

__13. The sum of roots of $2x^2 - 3x - 5 = 0$ is:

(a) $-\frac{3}{2}$ (b) $\frac{3}{2}$

(c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

__14. If 2 and – 5 are the roots of the equation, then the equations is:

(a) $x^2 + 3x + 10 = 0$ (b) $x^2 - 3x - 10 = 0$

(c) $x^2 + 3x - 10 = 0$ (d) $2x^2 - 5x + 1 = 0$

__15. If ± 3 are the roots of the equation, then the equation is:

(a) $x^2 - 3 = 0$ (b) $x^2 - 9 = 0$

(c) $x^2 + 3 = 0$ (d) $x^2 + 9 = 0$

__16. If 'S' is the sum and 'P' is the product of roots, then equation is:

(a) $x^2 + Sx + P = 0$ (b) $x^2 + Sx - P = 0$

(c) $x^2 - Sx + P = 0$ (d) $x^2 - Sx - P = 0$

__17. Roots of the equation $x^2 + x - 1 = 0$ are:

(a) Equal (b) Irrational

(c) Imaginary (d) Rational

__18. If the discriminant of an equation is zero, then the roots will be:

(a) Imaginary (b) Real

(c) Equal (d) Irrational

__19. Sum of the roots of $ax^2 - bx + c = 0$ is:

(a) $-\frac{c}{a}$ (b) $\frac{c}{a}$

(c) $-\frac{b}{a}$ (d) $\frac{b}{a}$

__20. Product of roots of ax2 + bx – c = 0 is:

(a) $\frac{c}{a}$ (b) $-\frac{c}{a}$ (c) $\frac{a}{b}$ (d) $-\frac{a}{b}$

Answers

1.	c	2.	b	3.	b	4.	b	5.	c
6.	b	7.	b	8.	a	9.	b	10.	d
11.	b	12.	a	13.	b	14.	c	15.	b
16.	c	17.	b	18.	c	19.	d	20.	b

Chapter 2
Sequences and Series

2.1 Introduction:

The INVENTOR of chess asked the King of the Kingdom that he may be rewarded in lieu of his INVENTION with one grain of wheat for the first square of the board, two grains for the second, four grains for the third, eight grains for the fourth, and so on for the sixty four squares. Fortunately, this apparently modest request was examined before it was granted. By the twentieth square, the reward would have amounted to more than a million grains of wheat; by the sixty-fourth square the number called for would have been astronomical and the bulk would have for exceeded all the grains in the kingdom.

The basis of this story a sequence of numbers that have a mathematical relationship --- has a great many important applications. Many of them are beyond the scope of this book, but we shall explore the means of dealing with a number of practical, and often entertaining, problems of this type.

2.2 Sequences:

A set of numbers arranged in order by some fixed rule is called as sequences.

For example

(i) 2, 4, 6, 8, 10, 12, 14 - - - - - - - - - - - -

(ii) 1, 3, 5, 7, 9, - - - - - - - - - - - -

(iii) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16},$ - - - - - - - - - - - -

In sequence $a_1, a_2, a_3, \ldots\ldots a_n, \ldots\ldots$ a_1 is the first term, a_2 is the second term, a_3 is the third and so on.

A sequence is called **finite sequence** if it has finite terms e.g., 2, 4, 6, 8, 10, 12, 14, 16.

A sequence is called **infinite sequence** if it has infinite terms, e.g., , 4, 6, 8, 10, 12, 14, - - - - - - - -

2.3 Progression:

If a sequence of number is such that each term can be obtained from the preceding one by the operation of some law, the sequence is called a progression.

Note:- Each progression is a sequence but each sequence may or may not be a progression

2.4 Arithmetic Sequence:

A sequence in which each term after the first term is obtained from the preceding term by adding a fixed number, is called as an arithmetic sequence or Arithmetic Progression, it is denoted by A.P.

e.g., (i) 2, 4, 6, 8, 10, 12, - - - - - - - - -

(ii) 1, 3, 5, 7, 9, 11, - - - - - - - - - -

Common Difference:

The fixed number in above definition is called as common difference. It is denoted by d. it is obtained by subtracting the preceding terms from the next term i.e; $a_n - a_{n-1}$; $n>1$.

For example

2, 4, 6, 8, 4, - - - - - - - -

d = Common difference = $a_2 - a_1 = 4 - 2 = 2$

Or d = Common difference = $a_3 - a_2 = 6 - 4 = 2$

The General Form of an Arithmetic Progression:

Let "a" be the first term and "d" be the common difference, then General form of an arithmetic progression is

$a, a + d, a + 2d,$ - - - - - - - - - - $a + (n - 1)d$

2.5 nth term or General term(or, last term)of an Arithmetic Progression:

If "a" be the first term and "d" be the common difference then

a_1 = first term = $a = a + (1 - 1)d$

a_2 = 2nd term = $a + d = a + (2 - 1)d$

a_3 = 3rd term = $a + 2d = a + (3 - 1)d$

a_4 = 4th term = $a + 3d = a + (4 - 1)d$

- -

- -

- -

a_n = nth term = $a + (n - 1)d$

$\boxed{a_n = \text{nth term} = a + (n - 1)d}$

in which $a = 1^{st}$ term

d = common difference

n = number of terms

Example 1:

Find the 7th term of A.P. in which the first term is 7 and the common difference is – 3.

Solution:

a_7 = 7th term = ?

$a_1 = 7$

$d = 3$

Putting these values in $a_n = a + (n - 1)d$

$a_7 = 7 + (7 - 1)(-3)$

$$= 7 + 6(-3)$$
$$= 7 - 18$$
$$\boxed{a_7 = -11}$$

Example 2:

Find the 9th term of the A.P. $-\frac{5}{4}, -\frac{1}{4}, \frac{3}{4}, \ldots\ldots$

Solution:

$$a_1 = -\frac{5}{4}$$

$$d = -\frac{1}{4} - \left(-\frac{5}{4}\right) = \frac{-1}{4} + \frac{5}{4}$$

$$= \frac{-1+5}{4} = \frac{4}{4} = 1$$

9th term $=$ $a_9 = ?$

$$a_n = a + (n-1)d$$

$$a_9 = -\frac{5}{4} + (9-1).1$$

$$= \frac{-5}{4} + 8$$

$$a_9 = \frac{-5+32}{4} = \frac{27}{4}$$

Example 3:

Find the sequence whose general term is $\frac{n(n-1)}{2}$

Solution:

Here $a_n = \frac{n(n-1)}{2}$

Put n = 1, $a_1 = \frac{1(1-1)}{2} = \frac{0}{2} = 0$

Put n = 2, $a_2 = \frac{2(2-1)}{2} = \frac{2(1)}{2} = 1$

Put n = 3, $a_3 = \frac{3(3-1)}{2} = \frac{3(2)}{2} = 3$

Put n = 4, $a_4 = \frac{4(4-1)}{2} = \frac{4(3)}{2} = 6$

Put n = 5, $a_5 = \frac{5(5-1)}{2} = \frac{5(4)}{2} = 10$

Therefore the required sequence is 0, 1, 3, 6, 10, - - - - - - - -

Exercise 2.1

Q.1 Find the terms indicated in each of the following A.P.

(i) 1, 4, 7, - - - - - - - - 7th term

(ii) 7, 17, 27, - - - - - - - - 13th term

(iii) $\frac{1}{6}, \frac{1}{4}, \frac{1}{3}$, - - - - - - - - 20th term

Q.2 Find the first four terms of A.P. in which first term is 7 and common difference is – 4.

Q.3 Find the number of terms in an A.P. in which $a_1 = 5$, d = 25 and $a_n = 130$.

Q.4 (i) Which term in the arithmetic progression 4, 1, -2 …. is – 77?

(ii) Which term in the arithmetic progression 17, 13, 9 …. is – 19?

Q.5 Find the 7th term of an A.P. whose 4th term is 5 and the common difference is –2.

Q.6 What is the first term of the eight term A.P. in which the common difference is 6 and the 8th term is 17.

Q.7 Find the 20th term of the A.P. whose 3rd term is 7 and 8th term is 17.

Q.8 If the 12th term of an A.P. is 19 and 17th term in 29, Find the first term and the common difference.

Q.9 The 9th term of an A.P. is 30 and the 17th term is 50. Find the first three terms.

Q.10 Find the sequence whose nth term is 4n + 5. Also prove that the sequence is in A.P.

Q.11 If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P, show that $b = \frac{2ac}{a+c}$

Q.12 If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P, show that the common difference is $\frac{a-c}{2ac}$

Answers 2.1

Q.1 (i) $a_7 = 19$ (ii) $a_{13} = 127$ (iii) $a_{20} = \frac{7}{4}$

Q.2 7, 3, –1, –5 Q.3 n = 6

Q.4 (i) n = 28 (ii) n = 10 Q.5 $a_{17} = -21$

Q.6 $a = -25$ Q.7 $a_{20} = 41$

Q.8 $a = -3, d = 2$ Q.9 $10, \frac{25}{2}, 15$

Q.10 9, 13, 17, - - - - - - and the difference between consecutive terms is equal. So the sequence is an A.P.

2.6 Arithmetic Means (A.Ms):

If a, A, b are three consecutive terms in an Arithmetic Progression, Then A is called the Arithmetic Mean (A.M) of a and b.

i.e. if a, A, b are in A.P. then

$$A - a = b - A$$
$$A + A = a + b$$
$$2A = a + b$$
$$A = \frac{a+b}{2}$$

The arithmetic mean of two numbers is equal to one half the sum of the two numbers.

Example 1:

Find the A.M. between $\sqrt{5} - 4$ and $\sqrt{5} + 4$

Solution:

Here $a = \sqrt{5} - 4$, $b = \sqrt{5} + 4$

$$\text{A.M.} = A = \frac{a+b}{2}$$
$$A = \frac{\sqrt{5} - 4 + \sqrt{5} + 4}{2}$$
$$= \frac{2\sqrt{5}}{2} = \sqrt{5}$$

2.7 n Arithmetic Means between a and b:

The number A_1, A_2, A_3 - - - - - An are said to be n arithmetic means between a and b if a, A, A_2, A_3, - - - - - - An, b are in A.P. We may obtain the arithmetic means between two numbers by using $a_n = a + (n-1)d$ to find d, and the means can then be computed.

Example 2:

Insert three A.M's between – 18 and 4.

Solution:

Let A_1, A_2, A_3 be the required A.M's between –18 and 4, then

$-18, A_1, A_2, A_3, 4$ are in A.P.

Here $a = -18$, $n = 5$, $a_5 = 4$, $d = ?$

Using $a_n = a + (n-1)d$

$a_5 = -18 + (5 - 1)d$
$4 = -18 + 4d$
$4d = 4 + 18$
$4d = 22$
$d = \dfrac{11}{2}$

Therefore A_1 = 2nd term $= a + d$

$= -18 + \dfrac{11}{2} = \dfrac{-25}{2}$

A_2 = 3rd term $= A_1 + d$ $= \dfrac{-25}{2} + \dfrac{11}{2}$

$= \dfrac{-25+11}{2}$

$A_2 = \dfrac{-14}{2} = -7$

Thus the required A.M's are $\dfrac{-25}{2}, -7, \dfrac{-3}{2}$

Example 3:

Insert n A.M's between a and b.

Solution:

Let, $A_1, A_2, A_3, \ldots\ldots A_n$ be the n. A.M's between a and b.
Then a, $A_1, A_2, A_3, \ldots\ldots$ b, are in A.P.
Let, d be the common difference
So, $a = a$, $n = n + 2$, $d = ?$ $a_n = b$

$a_n = a + (n - 1)d$
$b = a + (n + 2 - 1)d$
$b = a + (n + 1)d$
$b - a = (n + 1)d$
$d = \dfrac{b-a}{n+1}$

$A_1 = a + d = a + \dfrac{b-a}{n+1} = \dfrac{a(n+1)+b-a}{n+1} = \dfrac{an+a+b-a}{n+1}$

$A_1 = \dfrac{an+b}{n+1}$

$A_2 = A_1 + d = \dfrac{an+b}{n+1} + \dfrac{b-a}{n+1} = \dfrac{an+a+b-a}{n+1}$

$$A_2 = \frac{(n-1)a + 2b}{n+1}$$

$$A_n = \frac{[n-(n-1)]a + nb}{n+1} = \frac{(n-n+1)a + nb}{n+1} \;\ldots\; \frac{a+nb}{n+1}$$

Thus n A.M's between a and b are:

$$\frac{an+b}{n+1}, \frac{(n-1)a+2b}{n+1}, \frac{(n-2)a+3b}{n+1} \;\ldots\; \frac{a+nb}{n+1}$$

Exercise 2.2

Q.1 Find the A.M. between
(i) 17 and –3 (ii) – 5 and 40
(iii) $2+\sqrt{3}$ and $2-\sqrt{3}$ (iv) x + b and x – b

Q.2 Insert two A.M's between – 5 and 40.

Q.3 Insert four A.M's between $\frac{\sqrt{2}}{2}$ and $\frac{3\sqrt{2}}{2}$

Q.4 Insert five A.M's between 10 and 25.

Q.5 Insert six A.M's between 12 and –9.

Q.6 If 5 , 8 are two A.M's between a and b , find a and b

Q.7 Find the value of n so that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the A.M's between a and b.

Q.8 Find the value of x if x + 1, 4x + 1 and 8x – 1 are the consecutive terms of an arithmetic progression.

Q.9 Show that the sum of n A.M's between a and b is equal to n times their single A.M.

Answer 2.2

Q.1 (i) 7 (ii) $\frac{35}{2}$ (iii) 2 (iv) x

Q.2 10, 25 Q.3 $\frac{7\sqrt{2}}{10}, \frac{9\sqrt{2}}{10}, \frac{11\sqrt{2}}{10}, \frac{13\sqrt{2}}{10}$

Q.4 $\frac{25}{2}, 15, \frac{35}{2}, 20, \frac{45}{2}$ Q.5 9, 6, 3, 0 –3, –6

Q.6 a= 2 , b = 11 Q.7 n = 0 Q.8 x = 2

Series:

The sum of the terms of a sequence is called as "series". For example: 1, 4, 9, 16, - - - - - - is a sequence.

Sum of the terms of sequence i.e., 1 + 4 + 9 + 16 - - - - - represent a series.

2.8 Arithmetic Series:

The sum of the terms of an Arithmetic sequence is called as Arithmetic series. For example:

7, 17, 27, 37, 47, - - - - - - - - - - is an A.P.

7 + 17 + 27 + 37 + 47 + - - - - - - - - is Arithmetic series.

The sum of n terms of an Arithmetic Sequence:

The general form of an arithmetic sequence is a, a + d, a + 2d, - - - - - - - - a + (n – 1)d.

Let Sn denoted the sum of n terms of an Arithmetic sequence.

Then $\quad Sn = a + (a + d)+(a + 2d) + \text{- - - - -} + [a +(n - 1)d]$

Let nth term $\quad [a + (n - 1)d] = \ell$

The above series can be written as

$Sn = a+(a + d) + (a + 2d) + \text{- - - - - -} + \ell$

Or, $\quad Sn = a + (a + d) + (a + 2d) + \text{- - - - - -} + (\ell - 2d) + (\ell - d) + \ell$ -----(I)

Writing 1 in reverse order, we have

$Sn = \ell + (\ell - d) + (\ell - 2d) + \text{- - -}(a + 2d) + (a + d) + a$…… (II)

Adding I and II

$$2Sn = (a + \ell) + (a + \ell) + (a + \ell) + \text{- - - -} + (a + \ell)$$

$$2Sn = n\,(a + \ell)$$

$$Sn = \frac{n}{2}(a + \ell) \qquad \text{But } \ell = a + (n - 1)d$$

$$Sn = \frac{n}{2}[a + (a + (n - 1)d)]$$

$$= \frac{n}{2}[a + a + (n - 1)d]$$

$$\boxed{Sn = \frac{n}{2}[2a + (n - 1)d]}$$

is the formula for the sum of n terms of an arithmetic sequence.

Example 1:

Find the sum of the series 3 + 11 + 19 + - - - - to 16 terms.

Solution:

Here $\quad a = 3,\ d = 11 - 3 = 8,\ \ n = 16$

Using formula $S_n = \frac{n}{2}[2a + (n-1)d]$

$$S_{16} = \frac{16}{2}[2(3) + (16-1)8]$$
$$= 8[6 + 15(8)]$$
$$= 8[6 + 120]$$
$$S_{16} = 8 \times 126 = 1008$$

Example 2:

Find the sum of all natural numbers from 1 to 500 which are divisible by 3.

Solution:

The sequence of numbers divisible by 3 is
3, 6, 9, 12, - - - - - 498 (which is in A.P.)
Here $a = 3$, $d = 6 - 3 = 3$, $n = ?$ $a_n = 498$
First we find n
For this using $a_n = a + (n-1)d$

$$498 = 3 + (n-1)(3)$$
$$498 = 3 + 3n - 3$$
$$3n = 498$$
$$n = 166$$

Now $a = 3$, $d = 3$, $n = 166$, $Sn = ?$

$$Sn = \frac{n}{2}[2a + (n-1)d]$$
$$S_{166} = \frac{166}{2}[2(3) + (166-1)3]$$
$$= 83[6 + 165(3)] = 83(6 + 495)$$
$$= 83 \times 501$$
$$S_{166} = 41583$$

Example 3:

` If the sum of n terms of an A.P. is $2n + 3n^2$. Find the nth term.

Solution:

We have $S_n = 2n + 3n^2$

$$S_{n-1} = 2(n-1) + 3(n-1)^2$$
$$= 2(n-1) + 3(n^2 - 2n + 1)$$
$$= 2n - 2 + 3n^2 - 6n + 3$$
$$S_{n-1} = 3n^2 - 4n + 1$$

nth term $= a_n = S_n - S_{n-1}$

$$= 2n + 3n^2 - (3n^2 - 4n + 1)$$
$$= 2n + 3n^2 - 3n^2 + 4n - 1$$
$$a_n = 6n - 1$$

Example 4:

The sum of three numbers in an A.P. is 12 and the sum of their cubes is 408. Find them.

Solution:

Let the required numbers be

$a - d, a, a + d$

According to 1st condition:

$$(a - d) + a + (a + d) = 12$$
$$a - d + a + a + d = 12$$
$$3a = 12$$
$$\boxed{a = 4}$$

According to 2nd given condition:

$$(a - d)^3 + a^3 + (a + d)^3 = 408$$
$$a^3 - d^3 - 3a^2d + 3ad^2 + a^3 + a^3 + d^3 + d^3 + 3a^2d + 3ad^2 = 408$$
$$3a^3 + 6ad^2 = 408$$
$$3(4)^3 + 6(4)\,d^2 = 408$$
$$24d^2 = 408 - 192$$
$$d^2 = 9$$
$$\Rightarrow \qquad d = \pm 3$$

When $a = 4$ $d = 3$ then number are $a - d, a, a + d$

i.e. $4 - 3, 4, 4 + 3$ i.e. 1, 4 , 7

when $a = 4$, $d = -3$ then numbers are $a - d, a, a + d$

$4 - (-3), 4, 4 + (-3)$

$4 + 3, 4, 4 - 3$ i.e., 7, 4 , 1

Hence the required numbers are 1, 4, 7 or 7, 4, 1

Note:

The problem containing three or more numbers in A.P. whose sum in given it is often to assume the number as follows.

If the required numbers in A.P are odd i.e. 3, 5, 7 etc. Then take 'a'(first term) as the middle number and d as the common difference.

Thus three numbers are $a - d, a, a + d$. If the required numbers in A.P are even i.e. 2, 4, 6, etc. then take $a - d, a + d$ as the middle numbers and 2d as the common difference.

Thus four numbers are $a - 3d, a - d, a + d, a + 3d$ and six numbers are:

$a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$ etc.

Example 5:

A man buys a used car for \$600 and agrees to pay \$100 down and \$100 per month plus interest at 6 percent on the outstanding indebtedness until the car paid for. How much will the car cost him?

Solution:

The rate of 6 percent per year is 0.5 percent per month.

Hence, when the purchaser makes his first payment, he will owe 1 month's interest.

The interest on \$500 = (500)(0.005) = \$2.50

The purchaser will pay in the second month = \$102.50

Since the purchaser pays \$100 on the principal, his interest from month to month is reduced by 0.5 percent of \$100, which is \$0.50 per month.

The final payment will be \$100 plus interest on 100 for 1 month, which is = \$100.50

Hence his payments on \$500 constitute an arithmetic progression

$$102.50 + - - - - - - + 100.50$$

Here $a = 102.40$, $\ell = 100.50$ and $n = 5$

Therefore by the formula

$$S = \frac{n}{2}(a + \ell)$$
$$= \frac{5}{2}(102.50 + 100.50)$$
$$= \frac{5}{2}(203) = \$507.50$$

Thus, the total cost of the car will be \$ 607.50

Exercise 2.3

Q.1 Sum the series:

(i) $5 + 8 + 11 + 14 + \ldots\ldots$ to n terms.

(ii) $51 + 50 + 49 + \ldots\ldots + 21$.

(iii) $5 + 3 + 1 - 1 - \ldots\ldots$ to 10 terms.

(iv) $\frac{1}{1-\sqrt{x}} + \frac{1}{1-x} + \frac{1}{1+\sqrt{x}} + \text{--------}$ to n terms

Q.2 The nth term of a series in $4n + 1$. Find the sum of its 1st n terms and also the sum of its first hundred terms.

Q.3 Find the sum of the first 200 odd positive integers.

Q.4 Find the sum of all the integral multiples of 3 between 4 and 97

Q.5 How many terms of the series:

(i) $-9 - 6 - 3 \ldots\ldots\ldots$ amount to 66?

(ii) $5 + 7 + 9 + \ldots\ldots$ amount to 192?

Q.6 Obtain the sum of all the integers in the first 1000 positive integers which are neither divisible by 5 nor 2.

Q.7 The sum of n terms of a series is $7n^2 + 8n$. Show that it is an A.P and find its common difference.

Q.8 Sum the series
$1 + 3 - 5 + 7 + 9 - 11 + 13 + 15 - 17$ …… to 3n terms.

Q.9 If S_1, S_2, S_3 be sums to n, 2n, 3n terms of an arithmetic progression, Show that $S_3 = 3(S_2 - S_1)$.

Q.10 The sum of three numbers in A.P. is 24, and their product is 440. Find the numbers.

Q.11 Find four numbers in A.P. whose sum is 24 and the sum of whose square is 164.

Q.12 Find the five number in A.P. whose sum is 30 and the sum of whose square is 190.0

Q.13 How many bricks will be there in a pile if there are 27 bricks in the bottom row, 25 in the second row, etc., and one in the top row?

Q.14 A machine costs Rs. 3200, depreciates 25 percent the first year, 21 percent of the original value the second year, 17 percent of the original value of the third year, and so on for 6 years. What is its value at the end of 6 years.

Answers 2.3

Q.1 (i) $S_n = \frac{n}{2}[3n + 7]$ (ii) $n = 31$, $S_n = 1116$

(iii) $S_{10} = 40$ (iv) $\frac{n}{2}\left[\frac{2+(3-n)\sqrt{x}}{1-x}\right]$

Q.2 $S_n = n(2n + 3)$, $S_{100} = 20300$ Q.3 $S_{200} = 40{,}000$
Q.4 1581 Q.5 (i) $n = 11$ (ii) $n = 12$
Q.6 2000,000 Q.7 $d = 14$ Q.8 $S_{3n} = n(3n - 4)$
Q.10 5, 8, 11 or 11, 8, 5 Q.11 3, 5, 7, 9 or 9, 7, 5, 3
Q.12 4, 5, 6, 7, 8 or 8, 7, 6, 5, 4 Q.13 196 Q.14 Rs. 320.00

2.9 Geometric Sequence or Progression (G.P):

A geometric progression is a sequence of numbers each term of which after the first is obtained by multiplying the preceding term by a constant number called the common ratio. Common ratio is denoted by ‘r’.

Example:

(i) 2, 4, 8, 16, 32, …… is G.P
because each number is obtained by multiplying the preceding number by 2.

(ii) 2, 4, 8, ……

(iii) 4, 12, 36, ……

Note:- In geometric progression, the ratio between any two consecutive terms remains constant and is obtained by dividing the next term with the preceeding term, i.e., $r = \dfrac{a_n}{a_{n-1}}$, $n > 1$

2.10 nth term or General term(or, last term) of a Geometric Progression (G.P):

If a is the first term and r is the common ratio then the general form of G.P is a, ar, ar^2, ar^3, ……

If $a_1 = 1^{st}$ term $= a$
$a_2 = 2^{nd}$ term $= ar$
$a_3 = 3^{rd}$ term $= ar^2$
- - - - - - - - - - - - - -
- - - - - - - - - - - - - -
a_n = nth term $= ar^{n-1}$

Which is the nth term of G.P in which:
a = 1st term
r = common ratio
n = number of terms
a_n = nth term = last term

Example 2:

Write the sequence in which

$a = 5, \ r = 3, \ n = 5$
$a_1 = a = 5$
$a_2 = ar = 5(3) = 15$
$a_3 = a_2r = 15(3) = 45$
$a_4 = a_3r = 45(3) = 135$
$a_5 = a_4r = 135(3) = 405$

Therefore, the required sequence is: 5, 15, 45, 135, 405

Example 3:

Find 4th term in the G.P. 5, 10, 20, ……

Solution:

$a = 5, \ r = \dfrac{10}{5} = 2, \ a_n = ?$

$a_n = ar^{n-1}$

$a_4 = a_4 = 5(2)^{4-1} = 5 \times 8 = 40$

Example 4:

Find n in the G.P. 4, –2, 1, …… if $a_n = \dfrac{1}{16}$

Solution: **Since** 4, –2, 1, ……

Here, $a = 4, r = -2, 4 = -\frac{1}{2}, a_n = \frac{1}{16}$

$a_n = ar^{n-1}$

$\frac{1}{16} = 4(-\frac{1}{2})^{n-1}$

let $\frac{1}{16} \times 4 = (-\frac{1}{2})^{n-1} = \frac{1}{64} = (-\frac{1}{2})^{n-1}$

$(-\frac{1}{2})^6 = (-\frac{1}{2})^{n-1}$

$\Rightarrow \quad n - 1 = 6 \quad$ or $\quad n = 6 + 1 = 7$

Example 5:

Find the G.P. of which the third term is 4 and 6th is –32.

Solution:

Here $a_3 = 4, \ a_6 = -32$

$a_n = ar^{n-1}$

$a_3 = ar^{3-1}, a_6 = ar^{6-1}$

$4 = ar^2$ …… (i) $\qquad -32 = ar^5$ …… (ii)

Dividing (i) by (ii)

$\frac{ar^2}{ar^5} = \frac{4}{-32}$ or $\frac{1}{r^3} = \frac{-1}{8}$

$r^3 = -8 = (-2)^3$

$\Rightarrow r = -2$

Example 6:

The population of a town increases at the rate of 10% annually. Its present population is 2,00,000 what will be its population at the end of 5 years?

Solution:

Let, present population = a = 2,00,000 (given)

The increase of population at the end of 1st year

$= a(10/100) = a(0.1)$

Total population at the end of 1st year $= a + a(0.1) = a(1.1)$

Total population at the end of 2nd year $= a(1.1)(1.1) = a(1.1)^2$

The population at the end of 5years is the 6^{th} terms of G.P

$a, a(1.1), a(1.1)^2 \ldots\ldots$

Here $a = 2{,}00{,}000$, $r = 1.1$, $n = 6$, $a_6 = ?$

Since, $a_n = ar^{n-1}$

$a_6 = 2{,}00{,}000\ (1.1)^5 = 2{,}00{,}000\ (1.61051) = 322102$

Example 7:

The value of an auto mobile depreciate at the rate of 15% per year. What will be the value of an automobile 3 years hence which is now purchased for Rs. 45,000?

Solution:

a = 45,000 = Purchased value of automobile

The amount depreciate at the end of 1st year = a (15/100) = 0.15a

The value of automobile at the end of 1st year = a – 0.15a

= a(1 – 0.15) = a(0.85)

The value of automobile at the end of 2nd year = a(0.85)(1 – 0.15)

= a(0.85)(0.85)

= $a(0.85)^2$

The value of automobile at the end of 3rd year = $a(0.85)^3$

= $45{,}000(0.85)^3$

= 45,000 (0.614125)

= 27635.63 rupees

Exercise 2.4

Q.1 Write the next five terms of the following G.Ps.

(i) 2, 10, ………. (ii) 27, 9, 3, ……….

(iii) $1, \frac{1}{2}, \frac{1}{4}, \ldots\ldots\ldots$

Q.2 Find the term indicated in each of the following G.Ps.

(i) $1, 3^3, 3^6$, ………., 6th term

(ii) i, –1, –i, 1, ………., 13th term

(iii) $\sqrt{2}, \sqrt{6}, 3\sqrt{2}, \ldots\ldots\ldots$, 15th term

(iv) $\frac{1}{3}, -\frac{1}{9}, \frac{1}{27}$ ………., 6th term

Q.3 Find the nth term of the G.P.

(i) a = 8, $r = \frac{3}{2}$, n = 5

(ii) a = –1, r = –4, n = 6

(iii) a = 3, r = –2, n = 10

Q.4 Write down the finite geometric sequence which satisfies the given condition.

(i) a = 3, r = 5, n = 6

(ii) First term = 2, second term = – 6, n = 5

(iii) Third term = 9, sixth term = $\frac{1}{3}$, n = 8

(iv) Fifth term = 9, eight term = 243, n = 8

Q.5 If $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are G.P ,show that the common ratio is $\pm\sqrt{\frac{a}{b}}$

Q.6 If the second term of a G.P is 2 and the 11^{th} term is $\frac{1}{256}$, what is the first term? What is the nth term.

Q.7 Find the 10th term of a G.P if 2nd term 43 and 4th term is 9.

Q.8 What is the first term of a six term geometric progression in which the ratio is $\sqrt{3}$ and the sixth term is 27?

Q.9 A business concern pays profit at the rate of 15% compounded annually. If an amount of Rs. 2,00,000 is invested with the concern now, what total amount will become payable at the end of 5 years?

Q.10 A rubber bell is dropped from a height of 16dm, it continuously rebounds to $\frac{3}{4}$ of the distance of its previous fall. How high does it rebound its fourth time?

Q.11 Find three consecutive numbers in G.P whose sum is 26 and their product is 216.

Q.12 If the sum of the four numbers consecutive numbers of a G.P is 80 and A.M between second and fourth of them is 30.Find the terms.

Answers 2.4

1. (i) 50, 250, 1250, 6250, 31250 (ii) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$

   (iii) $\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$

2. (i) $(27)^5$ (ii) i (iii) $\sqrt{2}(3)^7$ (iv) $-\frac{1}{729}$

3. (i) $\frac{81}{2}$ (ii) 1024 (iii) -1536

4. (i) 3, 15, 75, 375, 1875, 9375

   (ii) 2, –6, 18, –54, 162

   (iii) $81, 27, 9, 3, 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}$

   (iv) $\frac{1}{9}, \frac{1}{3}, 1, 3, 9, 27, 81, 243$

6. $a = 4, a_n = 4\left(\frac{1}{2}\right)^{n-1}$ 7. 243

8. $a = \sqrt{3}$ 9. 402271.44 10. 5.06 dm

11. 2 , 6 , 18 or 18 , 6 , 2 12. 2 , 6 , 18 , 54

2.11 Geometric Mean:

When three quantities are in G.P., the middle one is called the Geometric Mean (G.M.) between the other two. Thus G will be the G.M. between a and b if a, G, b are in G.P.

To Find G.M between a and b:

Let, G be the G.M. between a and b

Then a. G. b are in G.P

$\therefore \quad \frac{G}{a} = \frac{b}{G} \Rightarrow G^2 = ab$

$G = \pm\sqrt{ab}$

Hence the G.M. between two quantities is equal to the square root of their product.

Example 1:

Find the G.M. between 8 and 72.

Solution:

$G = \pm\sqrt{ab}$

$G = \pm\sqrt{8 \text{ x } 72} = \pm\sqrt{8 \text{ x } 8 \text{ x } 9} = \pm 8 \text{x } 3$

$G = \pm 24$

2.12 n G.Ms Between a and b:

The numbers $G_1, G_2, G_3 \ldots\ldots G_n$ are said to be n G.Ms between a and b if $a, G_1, G_2, G_3 \ldots\ldots G_n, b$ are in G.P.

In order to obtain the G.M's between a and b, we use the formula $a_n = ar^{n-1}$ to find the value of r and then the G.M's can be computed.

To Insert n G.M's Between Two Numbers a and b

Let, $G_1, G_2, G_3 \ldots\ldots G_n$ be n G.Ms between a and b

Here $a = a$, $a_n = b$, $n = n + 2$, $r = ?$

$$\begin{aligned}
a_n &= ar^{n-1} \\
b &= ar^{n-2-1} = ar^{n-1} \\
b/a &= r^{n-1} \\
\Rightarrow r &= (b/a)^{1/(n-1)}
\end{aligned}$$

So,
$$\begin{aligned}
G_1 &= ar = a(b/a)^{1/(n+1)} \\
G_2 &= G_1 r = a(b/a)^{1/(n+1)}(b/a)^{1/(n+1)} = a(b/a)^{2/(n+1)} \\
G_3 &= G_2 r = a(b/a)^{2/(n+1)}(b/a)^{1/(n+1)} = a(b/a)^{3/(n+1)} \\
G_n &= a(b/a)^{n/(n+1)}
\end{aligned}$$

Example 2:

Find three G.M's between 2 and 32.

Solution:

Let, $G_1, G_2, G_3 \ldots\ldots G_n$ be n G.Ms between 2 and 32
Then 2, G_1, G_2, G_3 32 are in G.P.
Here a = 2, $a_n = 32$, r = ? n = 5

$$a_n = ar^{n-1}$$
$$32 = 2(r)^{5-1} = 2r^4$$
$$16 = r^4$$
$$2^4 = r^4$$
$$\Rightarrow r = 2$$

So, $G_1 = ar = 2(2) = 4$
$G_2 = G_1 r = 4(2) = 8$
$G_3 = G_2 r = 8(2) = 16$

Thus three G.M's between 2 and 32 are 4, 8, 16.

Example 3:

Insert 6 G.M's between 2 and 256.

Solution:

Let, $G_1, G_2, G_3, G_4, G_5, G_6$ be six G.M's between 2 and 256.
Then 2, $G_1, G_2, G_3, G_4, G_5, G_6$ 256 are in G.P.
Here a = 2, $a_n = 32$, r = ? n = 5

$$a_n = ar^{n-1}$$
$$256 = 2(r)^{8-1} = 2r^7$$
$$128 = r^7$$
$$(2)^7 = r^2$$
$$\Rightarrow r = 2$$

So, $G_1 = ar = 2(2) = 4$
$G_2 = G_1 r = 4(2) = 8$
$G_3 = G_2 r = 8(2) = 16$
$G_4 = G_3 r = 16(2) = 32$
$G_5 = G_4 r = 32(2) = 64$
$G_6 = G_5 r = 64(2) = 128$

Hence, required G.M's are 4, 8, 16, 32, 64, 128.

Example 4:

The A.M between two numbers is 10 and their G.M is 8.
Determine the numbers.

Solution: A.M $= \dfrac{a+b}{2} = 10$

$a + b = 20 \ldots\ldots\ldots$ (1)

G.M. $= \sqrt{ab} = 8$

$\therefore$ $ab = 64 \ldots\ldots\ldots$ (2)

from (2) $b = \dfrac{64}{a}$, Put in (1)

$a + \frac{64}{2} = 20$

$a^2 + 64 = 20a$

$(a - 16)(a - 4) = 0$

$\Rightarrow a = 16$ or $a = 4$

When, $a = 16$, $b = \frac{64}{16} = 4$

When, $a = 4$, $b = \frac{64}{16} = 16$

Hence the numbers are 4 and 16.

Exercise 2.5

Q1. Find G.M between

(i) 4, 64 (ii) $\frac{1}{3}$, 243 (iii) $\frac{8}{9}$, $\frac{8}{9}$

Q2. Insert two G.M's between $\sqrt{2}$ and 2.

Q3. Insert three G.M's between 256 and 1.

Q4. Insert four G.M's between 9 and $\frac{1}{27}$.

Q5. Show that A.M of two unequal positive quantities is greater than this G.M.

Q6. For what value of n is $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ the G.M between a and b, where a and b are not zero simultaneously.

Q7. Prove that the product of n G.M's between a and b is equal to the n power of the single G.M between them.

Q8. The A.M of two positive integral numbers exceeds their (positive)G.M by 2 and their sum is 20.Find the numbers.

Answers 2.5

Q1. (i) ± 16 (ii) ± 9 (iii) $\pm \frac{8}{9}$

Q2. $2^{2/3}, 2^{5/6}$ Q3. 64, 16, 4 Q4. 3, 1, $\frac{1}{3}, \frac{1}{9}$

Q6. $n = -\frac{1}{2}$ Q8. 16 , 4 or 4 , 16

2.13 Geometric Series

A geometric series is the sum of the terms of a geometric sequence.

If a, ar, ar^2, …… + ar^{n-1} is a geometric sequence.

Then $a + ar + ar^2 + \ldots\ldots + ar^{n-1}$ is a geometric series.

Sum of n Terms of a Geometric Series

Let, S_n be the sum of geometric series

i.e. $S_n = a + ar + ar^2 + \ldots\ldots + ar^{n-1}$ ……………….. (1)

Multiplying by r on both sides

$rS_n = ar + ar^2 + ar^3 + \ldots\ldots + ar^{n-1} + ar^n$……………….. (2)

Subtracting (2) from (1), we get

$S_n - r\,S_n = a - a\,r^n$

$(1 - r)S_n = a(1 - r^n)$

$$S_n = \frac{a(1-r^n)}{1-r} \;;\; r \neq 1$$

For convenience, we use :

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{if } |r| < 1$$

and $$S_n = \frac{a(r^n-1)}{r-1} \quad \text{if } |r| > 1$$

Example 1:

Sum the series $\frac{2}{3}, -1, \frac{3}{2}, \ldots\ldots\ldots$ to 7 terms

Solution

Here $a = \frac{2}{3}$, $r = \frac{-1}{\frac{2}{3}} = \frac{-3}{2}$

$$S_n = \frac{a(1-r^n)}{1-r} \text{ (because } r < 1)$$

$$S_7 = \frac{\frac{2}{3}\left[1-\left(-\frac{3}{2}\right)^7\right]}{1-\left(-\frac{3}{2}\right)} = \frac{\frac{2}{3}\left[1+\frac{2187}{128}\right]}{\frac{5}{2}}$$

$$S_7 = \frac{2}{3}\left(\frac{2315}{128}\right)\frac{2}{5} = \frac{463}{96}$$

Example 2:

Sum to 5 terms the series 1 + 3 + 9 + ……..

Solution:

The given series is a G.P.

in which a = 1, r = $\frac{a_2}{a_1} = \frac{3}{1} = 3,\ n = 5$

$\therefore \quad S_n = \frac{a(r^4 - 1)}{r - 1}$ (because r > 1)

$$S_n = \frac{1[(3)^5 - 1]}{3 - 1} = \frac{243 - 1}{2} = \frac{242}{2} = 121$$

Example 3:

Find S_n for the series 2 + 4 + 8 + ……….. + 2^n.

$\therefore$ Since r = r > 1

$$\therefore \quad S_n = \frac{a(r^n - 1)}{r - 1} = \frac{2(2^n - 1)}{2 - 1} = 2^{n+1} - 2$$

Example 4:

How many terms of the series

$$\frac{2}{3} - \frac{1}{3} + \frac{1}{2} + \ \ldots\ldots\ldots \text{ amount to } \frac{55}{72}$$

Solution:

$$S_n = \frac{55}{72},\ n = ?\ \ a = \frac{2}{9},\ \ r = \frac{-\frac{1}{3}}{\frac{2}{9}} = \frac{-3}{2}$$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$\frac{55}{72} = \frac{\frac{2}{9}\left[1 - \left(-\frac{3}{2}\right)^n\right]}{1 - \left(-\frac{3}{2}\right)} = \frac{\frac{2}{9}\left[1 - \left(-\frac{3}{2}\right)^n\right]}{\frac{3}{2}}$$

$$\frac{55}{72} = \frac{4}{45}\left[1 - \left(-\frac{3}{2}\right)^n\right]$$

$$\frac{45 \text{x} 55}{72 \text{x} 4} = 1 - \left(-\frac{3}{2}\right)^n \qquad \Rightarrow \frac{275}{32} = 1 - \left(-\frac{3}{2}\right)^n$$

$$\left(-\frac{3}{2}\right)^n = 1 - \frac{275}{32} = \frac{243}{32} = \left(-\frac{3}{2}\right)^5$$

$\Rightarrow n = 5$

Example 5:

Sum the series:

(i) 0.2 + .22 + .222 + …… to n terms

(ii) $(x + y)(x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + ……$ to n terms.

Solutions:

(i) 0.2 + .22 + .222 + …… to n terms

Let, S_n = .2 + .22 + .222 + …… to n terms

= 2[.1 + .11 + .111 + …… to n terms]

Multiplying and dividing by 9

$$S_n = \frac{2}{9}[.9 + .99 + .999 + …… \text{ to n terms}]$$

$$= \frac{2}{9}[(1 - 1) + (1 - .01) + (0.1 - .001) + …… \text{ to n terms}]$$

$$= \frac{2}{9}[(1+1+1+……\text{n terms})–(0.1+.01+.001+…..\text{to n terms})]$$

$a = .1 \quad r = \frac{.01}{.1} = 0.1 = \frac{1}{10}$

$a = \frac{1}{10}$

We use $S_n = \frac{a(1-r^n)}{1-r}$

$$S_n = \frac{2}{9}\left[n - \frac{\frac{1}{10}\left\{1-\left(\frac{1}{10}\right)^n\right\}}{1-\frac{1}{10}}\right] = \frac{2}{9}\left[n - \frac{1}{9}\left\{1-\frac{1}{10^n}\right\}\right]$$

Solution (ii)

Let, $S_n = (x + y) + (x^2 + xy + y^2) + (x^3+x^2y+xy^2+y^3) + ….$ to n term.

Multiplying and dividing by (x – y)

$$S_n = \frac{1}{(x-y)}[(x+y)(x-y)+(x-y)(x^2+xy+y^2)+(x-y)(x^3+x^2y+xy^2+y^3)+ …$$

$$S_n = \frac{1}{(x-y)}[(x^2 - y^2)+ (x^3 - y^3)+ (x^4 - y^4)+ …. \text{ to n term}]$$

$$S_n = \frac{1}{(x-y)}\left[(x^2 + x^3 + x^4 + \ldots \text{ to n term}) - (y^2 + x^3 + y^4 + \ldots \text{ to n term})\right]$$

We use $S_n = \dfrac{a(r^n - 1)}{r-1}$

$$S_n = \frac{1}{(x-y)}\left[\frac{x^2(x^n-1)}{x-1} - \frac{y^2(y^n-1)}{y-1}\right]$$

Example 6:

The sum of the first 10 terms of a G.P. is equal to 244 times the sum of first 5 terms. Find common ratio.

Solution:

Here, n = 10, n = 5, r = ?

So, $S_n = \dfrac{a(1-r^n)}{1-r}$

$$S_{10} = \frac{a(1-r^{10})}{1-r}, \quad S_5 = \frac{a(1-r^5)}{1-r}$$

By the Given condition:

$S_{10} = 244S_5$

$$\frac{a(1-r^{10})}{1-r} = 244\left[\frac{a(1-r^5)}{1-r}\right]$$

$\Rightarrow 1 - r^{10} = 244(1 - r^5)$

$(1)^2 - (r^2)^5 = 244(1 - r^5)$

$(1 - r^5)(1 + r^5) = 244(1 - r^5) \quad \Rightarrow (1 - r^5)[1 + r^5 - 244] = 0$

$\Rightarrow 1 + r^5 - 244 = 0$ or $1 - r^5 = 0$

$1 + r^5 = 244 = 0$ $\quad r^5 = 1$

$r^5 = 243 \quad \Rightarrow \boxed{r = 3}$ $\quad r = 1$ which not possible

Example 7:

Given n = 6, $r = \dfrac{2}{3}$, $S_n = \dfrac{665}{144}$ find a.

Solution:

Formula $S_n = \dfrac{a(r^n - 1)}{r-1}$ $\quad \because |r| > 1$

$$\frac{665}{144} = \frac{a\left[1-\left(\frac{2}{3}\right)^6\right]}{1-\frac{2}{3}}$$

$$= \frac{a\left[1-\frac{64}{729}\right]}{\frac{1}{3}}$$

$$\frac{665}{144} = a\left[\frac{665}{243}\right]$$

$$a = \frac{665}{144} \text{ x } \frac{243}{665}$$

$$\boxed{a = \frac{27}{16}}$$

Example 8:

If a man deposits \$ 200 at the beginning of each year in a bank that pays 4 percent compounded annually, how much will be to his credit at the end of 6 years?

Solution:

The man deposits \$ 200 at the beginning of each year.
The bank pays 4% compounded interest annually
At the end of first year the principle amount or credit becomes
= 200(1.04)
At the beginning of second year the principle amount or credit is
= 200 + 200 (1.04)
At the end of second year the principle amount or credit becomes

$$= 200(1.04) + 200(1.04)^2$$
$$= 200(1.04 + 1.04^2)$$

So at the end of 6 years the principle amount or credit becomes
$= 200\ (1.04 + 1.04^2 + \ldots$ sum upto 6 times)
Consider, $1.04 + 1.04^2 + - - - - - - - -$ 6 terms.

$a = 1.04,\ r = 1.04$, and $n = 6$

By the formula

$$S_n = \frac{a(r^n - 1)}{r - 1} \qquad \because |r| > 1$$

$$S_6 = \frac{1.04(1.04^6 - 1)}{1.04 - 1}$$

$$= \frac{1.04(1.2653 - 1)}{0.04}$$

$$= \frac{1.04 \text{ x } 0.2653}{0.04}$$

$$= 6.8983$$

Hence at the end of 6 years the credit is $= 200(6.8983)$
$= \$1379.66$

Exercise 2.6

Q1. Find the sum of each of the following series:

(i) $1+\frac{1}{3}+\frac{1}{9}+$ - - - - - - - to 6 terms

(ii) $x + x^2 + x^3$ - - - - - - - to 20 terms.

(iii) $\frac{1}{8}+\frac{1}{4}+\frac{1}{2}+ - - - - - - - - + 64$

(iv) $3 + 3^2 + 3^3 + - - - - - - - - - - + 3^n$

Q2. How many terms of the series?

$\frac{2}{3}-1+\frac{3}{2}-\frac{9}{4}+$ - - - - - - - amount to $-\frac{133}{48}$

Q3. Sum the series.

(i) $.3 + .33 + .333 +$ - - - - - - - to n terms.

(ii) $3 + 33 + 333 +$ - - - - - - - to n terms.

(iii) $1 + (1 + x)r + (1+ x+x^2)r^2+(1+x+x^2+x^3)r^3 + - - -$ to n terms.

Q4. What is the sum of the geometric series for which $a = 2$, $n = 5$, $l = a_n = 32$?

Q5. A rubber ball is dropped from a height of 4.8 dm. It continuously rebounds, each time rebounding $\frac{3}{4}$ of the distance of the preceding fall. How much distance has it traveled when it strikes the ground for the sixth time?

Q6. The first term of geometric progression is $\frac{1}{2}$ and the 10th term is 256, using formula find sum of its 12 terms.

Q7. What is first term of a six term G.P. in which the common ratio is $\sqrt{3}$ and the sixth term is 27 find also the sum of the first three terms.

Answers 2.6

1. (i) $\frac{364}{243}$ (ii) $\frac{x(1-x^{20})}{1-x}$ (iii) 1023/8

(iv) $\dfrac{3(3^n - 1)}{2}$

2. n = 6

3. (i) $\dfrac{1}{3}\left[n - \dfrac{1}{9}\left(1 - \dfrac{1}{10^n}\right)\right]$ (ii) $\dfrac{1}{3}\left[\dfrac{10(10^n - 1)}{9} - n\right]$

(iii) $\dfrac{1}{(1-x)}\left[\dfrac{1-r^n}{1-r} - \dfrac{x(1-r^n x^n}{1-rx}\right]$

4. 62 5. 26.76 6. $\dfrac{4085}{2}$ 7. $\sqrt{3};\dfrac{\sqrt{3}(3\sqrt{3}-1)}{\sqrt{3}-1}$

2.14 Infinite Geometric Sequence:

A geometric sequence in which the number of terms are infinite is called as infinite geometric sequence.

For example:

(i) $2, \dfrac{4}{3}, \dfrac{8}{9}, \dfrac{16}{27}$, - - - - - -

(ii) 2, 4, 8, 16, 32, - - - - - - -

Infinite Series:

Consider a geometric sequence a, ar, ar^2, - - - - - - - to n terms.

Let S_n denote the sum of n terms then $S_n = a + ar + ar^2 +$ - - - - - - to n terms.

Formula $S_n = \dfrac{a(1-r^n)}{1-r}$ $|r| < 1$

Taking limit as n → ∞ on both sides

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\frac{(1-r^n)}{1-r}$$

$$= \lim_{n\to\infty} a\left[\frac{1}{1-r} - \frac{r^n}{1-r}\right]$$

$$= \lim_{n\to\infty}\left(\frac{a}{1-r}\right) - \lim_{n\to\infty}\frac{ar^n}{1-r}$$

as n → ∞ , r^n → 0

Therefore $S\infty = \dfrac{a}{1-r} - 0$

$$\boxed{S\infty = \frac{a}{1-r}}$$

$\therefore$ the formula for the sum of infinite terms of G.P.

Convergent Series:

An infinite series is said to be the convergent series when its sum tends to a finite and definite limit.

For example:

$\frac{2}{3} + \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \text{-------}$ is a series

Here $a = \frac{2}{3}$, $r = \frac{1}{3} + \frac{2}{3} = \frac{1}{2} < 1$

$$S\infty = \frac{a}{1-r}$$

$$= \frac{\frac{2}{3}}{1-\frac{1}{3}} = \frac{\frac{2}{3}}{\frac{1}{2}}$$

$$= \frac{2}{3} \text{ x } \frac{1}{2} = \frac{4}{3}$$

Hence the series is convergent.

Divergent Series:

When the sum of an infinite series is infinite, it is said to be the Divergent series.

For example:

$2 + 4 + 8 + 16 + 32 + \text{--------}$

Here $a = 2$, $r =$, $2 > 1$

Therefore we use formula

$$S_n = \frac{a(r^n - 1)}{r-1} = \frac{2(2^n - 1)}{2-1}$$

$$S_n = 2^{n+1} - 2$$

$$\underset{n\to\infty}{\text{limit}}\, S_n = \underset{n\to\infty}{\text{limit}}\ (2^{n+1} - 2)$$

$$S\infty = 2^{\infty+1} - 2$$

$$= \infty \text{ as } n \to \infty,\ 2^{n+1} \to \infty$$

Hence the series is a divergent series.

2.14 Recurring Decimals:

When we attempt to express a common fraction such as $\frac{3}{8}$ or as $\frac{4}{11}$ as a decimal fraction, the decimal always either terminates or ultimately repeats.

Thus $\frac{3}{8} = 0.375$ (Decimal terminate)

$\frac{4}{11} = 0.363636$ (Decimal repeats)

We can express the recurring decimal fraction $0.\overline{36}$ (or $0.\dot{3}\dot{6}$) as a common fraction.

The bar ($0.\overline{36}$) means that the numbers appearing under it are repeated endlessly. i.e. $0.\overline{36}$ means 0.363636 - - - - - - -

Thus a non-terminating decimal fraction in which some digits are repeated again and again in the same order in its decimal parts is called a recurring decimal fraction.

Example 1:

Find the fraction equivalent to the recurring decimals $0.\overline{123}$.

Solution:

Let $S = 0.\overline{123}$

$= 0.123\ 123\ 123 - - - - - - - - - - - \infty$

$= 0.123 + 0.000123 + 0.000000123 - - - - - - \infty$

$$= \frac{123}{1000} \frac{123}{1000\ 000} \frac{123}{1000\ 000\ 000} + - - - - - - \infty$$

$$= \frac{123}{10^3} + \frac{123}{10^6} + \frac{123}{10^9} + - - - - - - \infty$$

Here $a = \frac{123}{10^3}$, $r = \frac{1}{10^3}$

$$S = \frac{a}{1-r}$$

$$= \frac{\frac{123}{10^3}}{1-\frac{1}{10^3}} = \frac{\frac{123}{1000}}{1-\frac{1}{1000}} = \frac{\frac{123}{1000}}{\frac{1000-1}{1000}}$$

$=$

$$= \frac{123}{1000} \text{x} \frac{1000}{999} = \frac{123}{999}$$
$$= \frac{41}{333}$$

Example 2:

Find the sum of infinite geometric series in which a = 128,

$$r = -\frac{1}{2}.$$

Solution:

$$a = 128, r = -\frac{1}{2}$$

Using $S\infty = \frac{a}{1-r}$

$$S\infty = \frac{128}{1-\left(\frac{1}{2}\right)} = \frac{128}{1+\frac{1}{2}}$$

$$= \frac{128}{\frac{3}{2}} = 128\text{x}\frac{2}{3}$$

$$S\infty = \frac{256}{3}$$

Exercise 2.7

Q.1 Find the sum of the following infinite geometric series

(i) $\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \text{- - - - - - -}$

(ii) $2 + \sqrt{2} + 1 + \text{- - - - - - -}$

Q.2 Find the sum of the following infinite geometric series

(i) $a = 3, \ r = \frac{2}{3}$ (ii) $a = 3, \ r = \frac{3}{4}$

Q.3 Which of the following series are (i) divergent (ii) convergent

(i) 1 + 4 + 16 + 64 + - - - - - - - - -

(ii) $6 + 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \text{- - - - - - -}$

(iii) 6 + 12 + 24 + 48 + - - - - - - - - - -

Q.4 Find the fractions equivalent to the recurring decimals.

(i) $0.\overline{36}$ (ii) $2.\overline{43}$ (iii) $0.\overline{836}$

Q.5 Find the sum to infinity of the series $1 + (1 + k)r + 1 + k + k^2)r^2 + (1 + k + k^2 + k^3)r^3 +$ - - - - - - - - r and k being proper fraction.

Q.6 If $y = x + x^2 + x^3 + \text{- - - - - - - -} \infty$ and if x is positive and less than unity show that $x = \frac{y}{1+y}$

Q.7 What distance a ball travel before coming to rest if it is dropped from a height of 6 dm and after each fall it rebounds $\frac{2}{3}$ of the distance it fell.

Q.8 The sum of an infinite geometric series in 15 and the sum of the squares of its terms is 45. Find the series.

Answers 2.7

Q.1 (i) $S\infty = \frac{1}{4}$ (ii) $S\infty = \frac{2\sqrt{2}}{\sqrt{2}-1}$

Q.2 (i) 9 (ii) 12

Q.3 (i) Divergent (ii) Convergent (iii) Divergent

Q.4 (i) $\frac{4}{11}$ (ii) $\frac{241}{99}$ (iii) $\frac{5}{6}$

Q.5 $\frac{1}{(1-r)(1-Kr)}$

Q.7 30 dm.

Q.8 $5 + \frac{10}{5} + \frac{20}{9} + \text{- - - - - - -}$

Summary

1. nth term of General Term of an Arithmetic progression.
   $\boxed{a_n = a + (n-1)d}$
2. Arithmetic means between a and b
   $\boxed{A = \frac{a+b}{2}}$
3. Sum of the First n terms of an arithmetic series.
   (i) $S_n = \frac{n}{2}[2a + (n-1)d]$
   (ii) $S_n = \frac{n}{2}(a + l)$ when last term is given.
4. General or nth term of a G.P
   $a_n = ar^{n-1}$
5. Geometric means between a and b
   $G = \pm\sqrt{ab}$
6. Sum of n terms of a Geometric Series
   $S_n = \frac{a(1-r^n)}{1-r}$ if $r < 1$
   $S_n = \frac{a(r^n-1)}{r-1}$ if $r > 1$
7. Sum of an infinite Geometric Series
   $\boxed{S\infty = \frac{a}{1-r}}$

Short questions

Write the short answers of the following

Q.1: Define a sequence.

Q.2: Define finite sequence.

Q.3: Define infinite sequence.

Q.4: Define common difference.

Q.5: Write the nth term of arithmetic progression.

Q.6: Find the 7^{th} term of A.P. in which the first term is 7 and the common difference is –3.

Q.7: Find the 7^{th} term of an AP 1, 4, 7, ……………..

Q.8: Find the sequence whose general term in 4n +1.

Q.9: Define a series.

Q.10: Write the formula to find the sum of n term of an arithmetic sequence.

Q.11: Find the sum of the series 3 + 11 + 19 + ……to 16 terms.

Q.12: Find the sum of the series 5 + 8 + 11 + 14 + …….to n terms.

Q.13: Define arithmetic means (AMs).

Q.14: Find the A.M. between $\sqrt{5} - 4$ and $\sqrt{5} + 4$.

Q.15: Define a common ratio.

Q.16: Write the nth term of a geometric progressions .

Q.17: Find the term indicated in the following G.P. 1, 3^3, 3^6, ………… 6^{th} terms.

Q.18: write down the geometric sequence in which first term is 2 and the second term is – 6 and n = 5.

Q.19: Write the formula of sum of the first n terms of an geometric sequence for $|r| < 1$ and for $|r| > 1$

Q.20: Define geometric means.

Q.21: Find the G.M. between (i) 8 and 72 (ii) $\frac{4}{3}$, 243.

Q.22: Sum to 5 term the series 1 + 3 +9 + ……………

Q.23: Find the sum of the following series: $1 + \frac{1}{3} + \frac{1}{9} + \ldots\ldots$ to 6 terms.

Q.24: Find the sum of infinite geometric series in which a=128, $r = -\frac{1}{2}$

Q.25: Find the sum of following infinite geometric series $2+\sqrt{2}+1+\ldots\ldots\ldots\ldots$

Answers

Q6 $a_7 = -11$ Q7 $a_7 = 19$ Q8 5 , 9 , 13,………

Q11 1008 Q12 $\frac{n}{2}[7+3n]$ Q14 $\sqrt{5}$

Q17 $(27)^5$ Q18 2 , -6 , 18 , -54 , 162 Q21 (i) ± 24 (ii) ± 9

Q22 121 Q23 $\frac{364}{243}$ Q24 $\frac{256}{3}$ Q25 $\frac{4}{2-\sqrt{2}}$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. The nth term of an A.P. whose 1st term is ‘a’ and common difference is ‘d’ is:
(a) $2a + (n + 1)d$ (b) $a + (n + 1)d$
(c) $a + (n - 1)d$ (d) $a + (d - 1)n$

__2. The nth term of an A.P. 1, 4, 7, …… is:
(a) 17 (b) 19 (c) 21 (d) 23

__3. If a, b, c are in A.P. then:
(a) $b - a = c - b$ (b) $\frac{b}{a} = \frac{c}{b}$
(c) $a + b = b + c$ (d) $\frac{a}{c} = \frac{b}{a}$

__4. The 10th term is 7, 17, 27, …… is:
(a) 97 (b) 98 (c) 99 (d) 100

__5. The sum of n terms of an A.P. with ‘a’ as 1st term and ‘d’ as common difference is:
(a) $\frac{n}{2}[a + (n - 1)d]$ (b) $\frac{n}{2}[2a + (n - 1)d]$
(c) $\frac{n}{2}[a + (n + 1)d]$ (d) $\frac{n}{2}[2a - (n - 1)d]$

__6. Arithmetic mean between $x - \sqrt{3}$ and $x + \sqrt{3}$ is:
(a) x (b) 2x (c) 3 (d) –3

__7. If $S_n = (n^2 + n + 1)$ then its 4th term will be:
(a) 21 (b) 40 (c) 41 (d) 101

__8. Arithmetic mean between – 7 and 7 is:
(a) $\frac{7}{2}$ (b) $-\frac{7}{2}$ (c) 0 (d) 14

__9. The sum of the series 1 + 2 + 3 + …… + 100 is:
(a) 100 (b) 5000 (c) 5050 (d) 500

__10. The nth term of a G.P a, ar, ar^2, …… is:
(a) ar^2 (b) ar^{n+1} (c) $\frac{1}{a} r^{n-1}$ (d) ar^{n-1}

__11. The 5th term of a G.P $1, \frac{1}{2}, \frac{1}{4}, \ldots\ldots$ is:
(a) $\frac{1}{8}$ (b) $-\frac{1}{8}$ (c) $\frac{1}{16}$ (d) $\frac{1}{32}$

__12. The 6th term of G.P $1, \sqrt{2}, \sqrt{4}, \ldots\ldots$ is:

(a) $4\sqrt{2}$ (b) 4 (c) $\sqrt{2}$ (d)

__13. The G.M. between a and b is:

(a) $\pm ab$ (b) ab (c) $\pm\sqrt{ab}$ (d) $\sqrt{ab}$

__14. If x, y, z are in G.P. then:

(a) $2y = x + z$ (b) $2y = xz$ (c) $y^2 = xz$ (d) $z^2 = xy$

__15. Geometric mean between 3 and 27 is:

(a) -9 (b) 12 (c) 15 (d) ± 9

__16. The sum of n terms of a geometric series:

$a + ar + ar^2 + \ldots\ldots; |r| < 1$

(a) $\dfrac{ar^{n-1}}{r-1}$ (b) $\dfrac{a(1-r^n)}{1-r}$

(c) $\dfrac{ar^{n+1}}{1-r}$ (d) $\dfrac{a(r^n-1)}{1-r}$

__17. The sum of 6 terms of the series 1 + 2 + 4 + …… is:

(a) 63 (b) 64 (c) 65 (d) 66

__18. The sum of 5 terms of the series 1 – 2 + 4 ……:

(a) 16 (b) 11 (c) –11 (d) –16

__19. The sum of infinite terms of a G.P. a, ar_1, ar_1^2, …… if |r| < 1 is:

(a) $\dfrac{a}{1-r}$ (b) $\dfrac{a(1-r^n)}{1-r}$

(c) ar^{n-1} (d) None of these

__20. The sum of infinite geometric series $1 + \frac{1}{3} + \frac{1}{9} + \ldots\ldots$ is:

(a) $\dfrac{2}{3}$ (b) $-\dfrac{2}{3}$ (c) $\dfrac{3}{2}$ (d) $-\dfrac{3}{2}$

Answers

1.	c	2.	b	3.	a	4.	a	5.	b
6.	a	7.	a	8.	c	9.	c	10.	d
11.	c	12.	a	13.	c	14.	c	15.	d
16.	b	17.	a	18.	b	19.	a	20.	c

Chapter 3
Binomial Theorem

3.1 Introduction:

An algebraic expression containing two terms is called a binomial expression, Bi means two and nom means term. Thus the general type of a binomial is $a + b$, $x - 2$, $3x + 4$ etc. The expression of a binomial raised to a small positive power can be solved by ordinary multiplication , but for large power the actual multiplication is laborious and for fractional power actual multiplication is not possible. By means of binomial theorem, this work reduced to a shorter form. This theorem was first established by Sir Isaac Newton.

3.2 Factorial of a Positive Integer:

If n is a positive integer, then the factorial of '*n*' denoted by $n!$ or $\underline{|n}$ and is defined as the product of n +ve integers from n to 1 (or 1 to n)

i.e., $n! = n(n-1)(n-2) \ldots\ldots$ 3.2.1

For example,

$$4! = 4.3.2.1 = 24$$

and $$6! = 6.5.4.3.2.1 = 720$$

one important relationship concerning factorials is that

$$(n+1)! = (n+1)\, n! \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_} \quad (1)$$

for instance,

$$5! = 5.4.3.2.1$$
$$= 5(4.3.2.1)$$
$$5! = 5.4!$$

Obviously, $1! = 1$ and this permits to define from equation (1)

$$n! = \frac{(n+1)!}{n+1}$$

Substitute 0 for n, we obtain

$$0! = \frac{(0+1)!}{0+1} = \frac{1!}{1} = \frac{1}{1}$$

$$0! = 1$$

3.3 Combination:

Each of the groups or selections which can be made out of a given number of things by taking some or all of them at a time is called combination.

In combination the order in which things occur is not considered e.g.; combination of a, b, c taken two at a time are ab, bc, ca.

The numbers $\binom{n}{r}$ **or** ${}^{n}c_{r}$

The numbers of the combination of n different objects taken 'r' at a time is denoted by $\binom{n}{r}$ **or** ${}^{n}c_{r}$ and is defined as,

$$\binom{n}{r} = \frac{n!}{r!\ (n-r)!}$$

e.g,
$$\binom{6}{4} = \frac{6!}{4!(6-4)!}$$
$$= \frac{6 \times 5 \times 4!}{4! \times 2!} = \frac{6 \times 5}{2 \times 1} = 15$$

Example 1: Expand $\binom{7}{3}$

Solution. $\binom{7}{3} = \frac{7!}{3!(7-3)!}$

$$= \frac{7.6.5.4!}{3.2.1.4!}$$
$$= 35$$

This can also be expand as

$$\binom{7}{3} = \frac{7.6.5}{3.2.1} = 35$$

If we want to expand $\binom{7}{5}$, then

$$\binom{7}{5} = \frac{7.6.5.4.3}{5.4.3.2.1} = 21$$

Procedure: Expand the above number as the lower number and the lower number expand till 1.

Method 2

For expansion of $\binom{n}{r}$ we can apply the method:

a. If r is less than (n – r) then take r factors in the numerator from n to downward and r factors in the denominator ending to 1.

b. If n – r is less than r, then take (n – r) factors in the numerator from n to downward and take (n – r) factors in the denominator ending to 1. For example, to expand $\binom{7}{5}$ again, here 7 – 5 = 2 is less than 5, so take two factors in numerator and two in the denominator as,

$$\binom{7}{5} = \frac{7.6}{2.1} = 21$$

Some Important Results

(i). $\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \times n!} = 1$

(ii) $\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \times 0!} = \frac{n!}{n! \times 1} = 1$

(iii) $\binom{n}{r} = \binom{n}{n-r}$

For example

$$\binom{4}{0} = \binom{4}{4} = 1 \text{ as } \quad \frac{4!}{0!(4-0)!} = \frac{4!}{4!\,0!}$$

$$1 = 1$$

$$\binom{4}{3} = \binom{4}{1} = 4 \quad \text{as} \quad \frac{4!}{3!.1!} = \frac{4!}{1!.\ 3!}$$

$$\frac{4.\ 3!}{3!.1!} = \frac{4.\ 3!}{1!.\ 3!}$$

$$4 = 4$$

Note: The numbers $\binom{n}{r}$ **or** ${}^{n}c_{r}$ are also called binomial co-efficients

3.4 The Binomial Theorem:

The rule or formula for expansion of $(a + b)^n$, where n is any positive integral power , is called binomial theorem .

For any positive integral n

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2} + \binom{n}{3}a^{n-3}b^3 \dots\dots$$

$$+\binom{n}{r}a^{n-r}b^r \dots\dots\dots.. + \binom{n}{n}b^n \text{---------------------}(1)$$

or briefly, $$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

Remarks:- The coefficients of the successive terms are

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, ----, \binom{n}{r} ----, \binom{n}{n}$$

and are called **Binomial coefficients.**

Note : Sum of binomial coefficients is 2^n

Another form of the Binomial theorem:

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots\ldots +$$

$$\frac{n(n-1)(n-2)------(n-r+1)}{r!}a^{n-r}b^r + \ldots\ldots\ldots + b^n \text{------------------ (2)}$$

Note: Since,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

So, $$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \times n!} = 1$$

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n(n-1)!}{1!(n-1)!} = \frac{n}{1!}$$

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-2)!}{2!(n-2)!} = \frac{n(n-1)}{2!}$$

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)(n-3)!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{3!}$$

$$\binom{n}{r} = \frac{n(n-1)(n-2)!\ldots\ldots(n-r+1)(n-r)!}{r!(n-r)!}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r!}$$

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \times 0!} = \frac{n!}{n! \times 1} = 1$$

The following points can be observed in the expansion of $(a + b)^n$

1. There are $(n + 1)$ terms in the expansion.
2. The 1^{st} term is a^n and $(n + 1)$th term or the last term is b^n
3. The exponent of 'a' decreases from n to zero.
4. The exponent of 'b' increases from zero to n.
5. The sum of the exponents of a and b in any term is equal to index n.
6. The co-efficients of the term equidistant from the beginning and end of the expansion are equal as $\binom{n}{r} = \binom{n}{n-r}$

3.5 General Term:

The term $\binom{n}{r} a^{n-r} b^r$ in the expansion of binomial theorem is called the General term or $(r + 1)$th term. It is denoted by T_{r+1}. Hence

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

Note: The General term is used to find out the specified term or the required co-efficient of the term in the binomial expansion

Example 2: Expand $(x + y)^4$ by binomial theorem:

Solution:

$$(x + y)^4 = x^4 + \binom{4}{1} x^{4-1} y + \binom{4}{2} x^{4-2} y^2 + \binom{4}{3} x^{4-3} y^3 + y^4$$

$$= x^4 + 4x^3y + \frac{4 \times 3}{2 \times 1} x^2y^2 + \frac{4 \times 3 \times 2}{3 \times 2 \times 1} xy^3 + y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Example 3: Expand by binomial theorem $\left(a - \frac{1}{a}\right)^6$

Solution:

$$\left(a - \frac{1}{a}\right)^6 = a^6 + \binom{6}{1} a^{6-1}\left(-\frac{1}{a}\right)^1 + \binom{6}{2} a^{6-2}\left(-\frac{1}{a}\right)^2 + \binom{6}{3} a^{6-3}\left(-\frac{1}{a}\right)^3 +$$

$$\binom{6}{4} a^{6-4}\left(-\frac{1}{a}\right)^4 + \binom{6}{5} a^{6-5}\left(-\frac{1}{a}\right)^5 + \binom{6}{6} a^{6-6}\left(-\frac{1}{a}\right)^6$$

$$= a^6 + 6a^5\left(-\frac{1}{a}\right) + \frac{6 \times 5}{2 \times 1} a^4\left(-\frac{1}{a^2}\right) + \frac{6 \times 5 \times 4}{3 \times 2 \times 1} a^3\left(-\frac{1}{a^3}\right) +$$

$$\frac{6 \times 5 \times 4 \times 3}{4 \times 3 \times 2 \times 1}\, a^2\left(-\frac{1}{a^4}\right)+\frac{6 \times 5 \times 4 \times 3 \times 2}{5 \times 4 \times 3 \times 2 \times 1}\, a\left(-\frac{1}{a^5}\right)^5+\left(-\frac{1}{a^6}\right)$$

$$= a^6 - 6a^4 + 15a^2 - 20 + \frac{15}{a^2} - \frac{6}{a^5} + \frac{1}{a^6}$$

Example 4: Expand $\left(\frac{x^2}{2} - \frac{2}{x}\right)^4$

Solution:

$$\left(\frac{x^2}{2} - \frac{2}{x}\right)^4 = \left(\frac{x^2}{2}\right)^4 + \binom{4}{1}\left(\frac{x^2}{2}\right)^{4-1}\left(\frac{-2}{x}\right)^1 + \binom{4}{2}\left(\frac{x^2}{2}\right)^{4-2}\left(\frac{-2}{x}\right)^2$$

$$+ \binom{4}{3}\left(\frac{x^2}{2}\right)^{4-3}\left(\frac{-2}{x}\right)^3 + \binom{4}{4}\left(\frac{x^2}{2}\right)^{4-4}\left(\frac{-2}{x}\right)^4$$

$$= \frac{x^4}{16} + 4\left(\frac{x^2}{2}\right)^3\left(-\frac{2}{x}\right) + \frac{4\,.\,3}{2\,.\,1}\left(\frac{x^2}{2}\right)^2\left(\frac{4}{x^2}\right) +$$

$$\frac{4\,.\,3\,.\,2}{3\,.\,2\,.\,1}\left(\frac{x^2}{2}\right)\left(-\frac{8}{x^3}\right) + \frac{16}{x^4}$$

$$= \frac{x^8}{16} - 4.\frac{x^8}{8}.\frac{2}{x} + 6.\frac{x^4}{4}.\frac{4}{x^2} - 4\frac{x^2}{2}.\frac{8}{x^3} + \frac{16}{x^4}$$

$$= \frac{x^8}{16} - x^5 + 6x^2 - \frac{16}{x} + \frac{16}{x^4}$$

Example 5: Expand $(1.04)^5$ by the binomial formula and find its value to two decimal places.

Solution:

$$(1.04)^5 = (1 + 0.04)^5$$

$$(1 + 0.04)^5 = (1)^5 + \binom{5}{1}(1)^{5-1}(0.04) + \binom{5}{2}(1)^{5-2}(0.04)^2 + \binom{5}{3}$$

$$(1)^{5-3}(0.04)^3 + \binom{5}{4}(1)^{5-4}(0.04)^4 + (0.04)^5$$

$$= 1. + 0.2 + 0.016 + 0.00064 + 0.000128$$
$$+ 0.000\,000\,1024$$
$$= 1.22$$

Example 6: Find the eighth term in the expansion of $\left(2x^2 - \frac{1}{x^2}\right)^{12}$

Solution: $\left(2x^2 - \frac{1}{x^2}\right)^{12}$

The General term is, $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

Here $T_8 = ?$ $a = 2x^2$, $b = -\frac{1}{x^2}$, $n = 12$, $r = 7$,

Therefore, $T_{7+1} = \binom{12}{7}\left(2x^2\right)^{12-7}\left(-\frac{1}{x^2}\right)^7$

$$T_8 = \frac{12\,.\,11\,.\,10\,.\,9\,.\,8\,.\,7\,.\,6}{7\,.\,6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1}\left(2x^2\right)^5 \frac{(-1)^7}{x^{14}}$$

$$T_8 = 793 \text{ x } 32x^{10} \frac{(-1)}{x^{14}}$$

$$T_8 = -\frac{25\,344}{x^4}$$

Eighth term $= T_8 = -\dfrac{25\,344}{x^4}$

.

3.6 Middle Term in the Expansion $(a + b)^n$

In the expansion of $(a + b)^n$, there are $(n + 1)$ terms.

Case I :

If n is even then $(n + 1)$ will be odd, so $\left(\frac{n}{2} + 1\right)$th term will be the only one middle term in the expension .

For example, if $n = 8$ (even), number of terms will be 9 (odd), therefore, $\left(\frac{8}{2} + 1\right) = 5^{th}$ will be middle term.

Case II:

If n is odd then $(n + 1)$ will be even, in this case there will not be a single middle term, but $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+1}{2} + 1\right)$th term will be the two middle terms in the expension.

For example, for $n = 9$ (odd), number of terms is 10 i.e. $\left(\frac{9+1}{2}\right)$th and $\left(\frac{9+1}{2} + 1\right)$th i.e. 5^{th} and 6^{th} terms are taken as middle terms and these middle terms are found by using the formula for the general term.

Example 7: Find the middle term of $\left(1-\frac{x^2}{2}\right)^{14}$.

Solution:

We have n = 14, then number of terms is 15.

$\therefore \left(\frac{14}{2}+1\right)$ i.e. 8^{th} will be middle term.

$a = 1$, $b = -\frac{x^2}{2}$, $n = 14$, $r = 7$, $T_8 = ?$

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$T_{7+1} = \binom{14}{7}(1)^{14-7}\left(-\frac{x^2}{2}\right)^7 = \frac{14!}{7!7!}(-1)^7 \frac{x^{14}}{2^7}$$

$$T_8 = \frac{14.13.12.11.10.9.8.7!}{7.6.5.4.3.2.1\ 7!} \frac{(-1)}{128}.x^{14}$$

$$T_8 = -(2)(13)(11)(2)(3)\frac{1}{128}.\ x^{14}$$

$$T_8 = -\frac{429}{16}x^{14}$$

Example 8 : Find the coefficient of x^{19} in $(2x^3 - 3x)^9$.

Solution:

Here, $a = 2x^3$, $b = -3x$, $n = 9$

First we find r.

Since $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

$$= \binom{9}{r}\left(2x^3\right)^{9-r}(-3x)^r$$

$$= \binom{9}{r} 2^{9-r}(-3)^r x^{27-3r}.x^r$$

$$= \binom{9}{r} 2^{9-r}(-3)^r .x^{27-2r} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

But we require x^{19}, so put

$19 = 27 - 2r$

$2r = 8$

$r = 4$

Putting the value of r in equation (1)

$$T_{4+1} = \binom{9}{r} 2^{9-4} (-3)^4 x^{19}$$

$$= \frac{9.8.7.6.5}{4.3.2.1} . 2^5 . 3^4 x^{19}$$

$$= 630 \text{ x } 32 \text{ x } 81 x^{19}$$

$$T_5 = 1632960 x^{19}$$

Hence the coefficient of x^{19} is 1632960

Example 9: Find the term independent of x in the expansion of $\left(2x^2 + \frac{1}{x}\right)^9$.

Solution:

Let T_{r+1} be the term independent of x.

We have $a = 2x^2$, $b = \frac{1}{x}$, $n = 9$

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r = \binom{9}{r} (2x^2)^{9-r} \left(\frac{1}{x}\right)^r$$

$$T_{r+1} = \binom{9}{r} 2^{9-r} . x^{18-2r} . x^r$$

$$T_{r+1} = \binom{9}{r} 2^{9-r} . x^{18-3r} \text{ .... (1)}$$

Since T_{r+1} is the term independent of x i.e. x^0.

$\therefore$ power of x must be zero.

i.e. $18 - 3r = 0 \Rightarrow r = 6$

put in (1)

$$T_{r+1} = \binom{9}{6} 2^{9-6} . x^0 = \frac{!9}{!6!3^{2^3}} .1$$

$$= \frac{{}^3\cancel{9} . \cancel{8}^4 . 7.6!}{6! . 3 . \cancel{2} . 1} . 8 . 1 = 672$$

Exercise 3.1

1. Expand the following by the binomial formula.

(i) $\left(x + \frac{1}{x}\right)^4$ (ii) $\left(\frac{2x}{3} - \frac{3}{2x}\right)^5$ (iii) $\left(\frac{x}{2} - \frac{2}{y}\right)^4$

(iv) $(2x - y)^5$ (v) $\left(2a - \frac{x}{a}\right)^7$ (vi) $\left(\frac{x}{y} - \frac{y}{x}\right)^4$

(vii) $(-x + y^{-1})^4$

2. Compute to two decimal places of decimal by use of binomial formula.

(i) $(1.02)^4$ (ii) $(0.98)^6$ (iii) $(2.03)^5$

3. Find the value of

(i) $(x + y)^5 + (x - y)^5$ (ii) $(x + \sqrt{2})^4 + (x - \sqrt{2})^4$

4. Expanding the following in ascending powers of x

(i) $(1 - x + x^2)^4$ (ii) $(2 + x - x^2)^4$

5. Find

(i) the 5^{th} term in the expansion of $\left(2x^2 - \frac{3}{x}\right)^{10}$

(ii) the 6^{th} term in the expansion of $\left(x^2 + \frac{y}{2}\right)^{15}$

(iii) the 8^{th} term in the expansion of $\left(\sqrt{x} + \frac{2}{\sqrt{x}}\right)^{12}$

(iv) the 7^{th} term in the expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

6. Find the middle term of the following expansions

(i) $\left(3x^2 + \frac{1}{2x}\right)^{10}$ (ii) $\left(\frac{a}{2} - \frac{b}{3}\right)^{11}$ (iii) $\left(2x + \frac{1}{x}\right)^7$

7. Find the specified term in the expansion of

(i) $\left(2x^2 - \frac{3}{x}\right)^{10}$: term involving x^5

(ii) $\left(2x^2 - \dfrac{1}{2x}\right)^{10}$: term involving x^5

(iii) $\left(x^3 + \dfrac{1}{x}\right)^{7}$: term involving x^9

(iv) $\left(\dfrac{x}{2} - \dfrac{4}{x}\right)^{8}$: term involving x^2

(v) $\left(\dfrac{p^2}{2} + 6q^2\right)^{12}$: term involving q^8

8. Find the coefficient of

(i) x^5 in the expansion of $\left(2x^2 - \dfrac{3}{x}\right)^{10}$

(ii) x^{20} in the expansion of $\left(2x^2 + \dfrac{1}{2x}\right)^{16}$

(iii) x^5 in the expansion of $\left(2x^2 - \dfrac{1}{3x}\right)^{10}$

(iv) b^6 in the expansion of $\left(\dfrac{a^2}{2} + 2b^2\right)^{10}$

9. Find the constant term in the expansion of

(i) $\left(x^2 - \dfrac{1}{x}\right)^{9}$ (ii) $\left(\sqrt{x} + \dfrac{1}{3x^2}\right)^{10}$

10. Find the term independent of x in the expansion of the following

(i) $\left(2x^2 - \dfrac{1}{x}\right)^{12}$ (ii) $\left(2x^2 + \dfrac{1}{x}\right)^{9}$

Answers 3.1

1. (i) $x^4 + 4x^2 + 6 + \dfrac{4}{x^2} + \dfrac{1}{x^4}$

(ii) $\dfrac{32}{243}x^5 - \dfrac{40}{27}x^3 + \dfrac{20}{3}x - \dfrac{15}{x} + \dfrac{135}{8x^3} - \dfrac{243}{32x^5}$

(iii) $\dfrac{x^4}{16} - \dfrac{x^3}{y} + \dfrac{6x^2}{y^2} - \dfrac{6x}{y^3} + \dfrac{16}{y^4}$

(iv) $32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$

(v) $128a^7 - 448a^5x + 672a^3x^2 - 560ax^3 + 280\frac{x^4}{a} -$

$84\frac{x^5}{a^3} + 14\frac{x^6}{a^5} - \frac{x^7}{a^7}$

(vi) $\frac{x^8}{y^8} - 8\frac{x^6}{y^6} + 28\frac{x^2}{y^2} - 56\frac{x^2}{y^2} + 70 - 56\frac{y^2}{x^2} + 28\frac{y^4}{x^4} - 8\frac{y^6}{x^6} + \frac{y^8}{x^8}$

(vii) $x^4 - 4x^3y^{-1} + 6x^2y^{-2} - 4xy^{-3} + y^{-4}$

2. (i) 1.14 (ii) 0.88 (iii) 34.47
3. (i) $2x^5 + 20x^3y^2 + 10xy^4$ (ii) $2x^4 + 24x^2 + 8$
4. (i) $1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$
   (ii) $16 + 32x - 8x^2 - 40x^3 + x^4 + 20x^5 - 2x^6 - 4x^7 + x^8$
5. (i) $1088640x^8$ (ii) $\frac{3003}{32}x^{20}y^5$ (iii) $\frac{101376}{x}$ (iv) $\frac{10500}{x^3}$
6. (i) $1913.625x^5$ (ii) $-\frac{77a^6b^5}{2592} + \frac{77a^5b^6}{3888}$ (iii) $\frac{280}{x} + 560x$
7. (i) $-1959552x^5$ (ii) $-252x^5$ (iii) $35x^9$ (iv) $-112x^2$
   (v) $\frac{880}{9}p^{16}q^8$
8. (i) -1959552 (ii) 46590 (iii) 33.185 (iv) $\frac{15}{2}a^{14}$
9. (i) 84 (ii) 5
10. (i) 7920 (ii) 672

3.7 Binomial Series

Since by the Binomial formula for positive integer n, we have

$$(a + b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots\ldots\ldots\ldots + b^n \qquad \ldots\ldots\ldots (2)$$

put a = 1 and b = x, then the above form becomes:

$$(1 + x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \ldots\ldots + x^n$$

if n is –ve integer or a fractional number (-ve or +ve), then

$$(1+x)^n = 1 + \frac{n}{1!}x + \frac{n(n-1)}{2!}x^2 + \ldots\ldots\ldots\ldots\ldots\infty \qquad (3).$$

The series on the R.H.S of equation (3) is called binomial series.
This series is valid only when x is numerically less than unity i.e.,$|x| < 1$otherwise the expression will not be valid.

Note: The first term in the expression must be unity. For example, when n is not a positive integer (negative or fraction) to expand $(a + x)^n$, we shall have to write it as, $(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n$ and then apply the binomial series, where $\left|\frac{x}{a}\right|$ must be less than 1.

3.8 Application of the Binomial Series; Approximations:

The binomial series can be used to find expression approximately equal to the given expressions under given conditions.

Example 1: If x is very small, so that its square and higher powers can be neglected then prove that

$$\frac{1+x}{1-x} = 1 + 2x$$

Solution:

$\frac{1+x}{1-x}$ this can be written as $(1 + x)(1 - x)^{-1}$

$= (1 + x)(1 + x + x^2 + \ldots\ldots\ldots$ higher powers of x)

$= 1 + x + x +$ neglecting higher powers of x.

$= 1 + 2x$

Example 2: Find to four places of decimal the value of $(1.02)^8$

Solution:

$$(1.02)^8 = (1 + 0.02)^8$$
$$= (1 + 0.02)^8$$
$$= 1 + \frac{8}{1}(0.02) + \frac{8.7}{2.1}(0.02)^2 + \frac{8.7.6}{3.2.1}(0.02)^3 + \ldots$$
$$= 1 + 0.16 + 0.0112 + 0.000448 + \ldots$$
$$= 1.1716$$

Example 3: Write and simplify the first four terms in the expansion of $(1 - 2x)^{-1}$.

Solution:

$(1 - 2x)^{-1}$

$= [1 + (-2x)]^{-1}$

Using $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + - - - - -$

$$= 1 + (-1)(-2x) + \frac{(-1)(-1-1)}{2!}(-2x)^2 + -----$$

$$\frac{(-1)(-1-1)(-1-2)}{3!}(-2x)^3 + -----$$

$$= 1 + 2x + \frac{(-1)(-2)}{2.1}4x^2 + \frac{(-1)(-2)(-3)}{3.2.1}(-8x^3) + -----$$

$$= 1 + 2x + 4x^2 + 8x^3 + ----$$

Example 4: Write the first three terms in the expansion of $(2 + x)^{-3}$

Solution :

$$(2+x)^{-3} = (2)^{-3}\left(1 + \frac{x}{2}\right)^{-3}$$

$$= (2)^{-3}\left[1 + (-3)\left(\frac{x}{2}\right) + \frac{(-3)(-3-1)}{2!}\left(\frac{x}{2}\right)^2 + \cdots\right]$$

$$= \frac{1}{8}\left[1 - \frac{3}{2}x + 3x^2 + ---\right]$$

Root Extraction:

The second application of the binomial series is that of finding the root of any quantity.

Example 5: Find square root of 24 correct to 5 places of decimals.
Solution:

$$\sqrt{24} = (25-1)^{1/2}$$

$$= (25)^{1/2}\left(1-\frac{1}{25}\right)^{1/2}$$

$$= 5\left(1-\frac{1}{5^2}\right)^{1/2}$$

$$= 5\left[1+\frac{1}{2}\left(-\frac{1}{5^2}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{5^2}\right)^2 + --\right]$$

$$= 5\left[1-\frac{1}{2.5^2}-\frac{1}{2^3.5^4}-\frac{1}{2^4.5^6}- ----\right]$$

$$= 5\,[1 - (0.02 + 0.0002 + 0.000004 + ---)]$$

$= 4.89898$

Example 6: evaluate $\sqrt[3]{29}$ to the nearest hundredth.
Solution :

$$\sqrt[3]{29} = (27+2)^{1/3} = \left[27\left(1+\frac{2}{27}\right)\right]^{1/3} = 3\left[1+\frac{2}{27}\right]^{1/3} + \ldots\ldots$$

$$= 3\left[1+\frac{1}{3}\left(\frac{2}{27}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1.2}\left(\frac{2}{27}\right)^2+\ldots\ldots\ldots\right]$$

$$= 3\left[1+\frac{2}{81}+\frac{1}{2}\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(\frac{2}{27}\right)^2+\ldots\ldots\ldots\right]$$

$$= 3\,[1+0.0247-0.0006\ldots\ldots\ldots\ldots\ldots]$$

$$= 3\,[1.0212] = 3.07$$

Exercise 3.2

Q1: Expand upto four terms.

(i) $(1-3x)^{1/3}$ (ii) $(1-2x)^{-3/4}$ (iii) $(1+x)^{-3}$

(iv) $\dfrac{1}{\sqrt{1+x}}$ (v) $(4+x)^{1/2}$ (vi) $(2+x)^{-3}$

Q2: Using the binomial expansion, calculate to the nearest hundredth.

(i) $\sqrt[4]{65}$ (ii) $\sqrt{17}$ (iii) $(1.01)^{-7}$

(iv) $\sqrt{28}$ (v) $\sqrt{40}$ (vi) $\sqrt{80}$

Q3: Find the coefficient of x^5 in the expansion of

(i) $\dfrac{(1+x)^2}{(1-x)^2}$ (ii) $\dfrac{(1+x)^2}{(1-x)^3}$

Q4: If x is nearly equal to unity, prove that

$$\frac{mx^n - n\,x^m}{x^n - x^m} = \frac{1}{1-x}$$

Answers 3.2

Q1: (i) $1-x\ -x\ -\frac{5}{3}x^3+----$ (ii) $1+\frac{3}{2}x+\frac{21}{8}x^2+\frac{77}{16}x^3+----$

(iii) $1-3x+6x^2-10^3----$ (iv) $1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{5}{16}x^3+----$

(v) $2+\frac{x}{2}-\frac{x^2}{64}+\frac{x^3}{512}+----$ (vi) $\frac{1}{8}\left[1-\frac{3}{2}x+\frac{3}{2}x^2-\frac{5}{4}x^3\right]$

Q2: (i) 2.84 (ii) 4.12 (iii) 0.93
(iv) 5.29 (v) 6.32 (vi) 8.94

Q3: (i) 20 (ii) 61

Summary

Binomial Theorem

An expression consisting of two terms only is called a binomial expression. If n is a positive index, then

1. The general term in the binomial expansion is $T_{r-1} = {}^nC_r a^{n-r} b^r$
2. The number of terms in the expansion of $(a + b)^n$ is n + 1.
3. The sum of the binomial coefficients in the expansion of $(a + b)^n$ is 2^n. i.e. ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$
4. The sum of the even terms in the expansion of $(a + b)^n$ is equal to the sum of odd terms.
5. When n is even, then the only middle term is the $\left(\frac{n+2}{2}\right)$th term.
6. When n is odd, then there are two middle terms viz $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th terms.

Note: If n is not a positive index.

i.e. $(a + b)^n = a^n\left(1+\frac{n}{a}\right)^n$

$$= a^n\left[1+n\left(\frac{b}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+--------\right]$$

1. Here n is a negative or a fraction, the quantities ${}^nC_1$, ${}^nC_2$ ------- here no meaning at all. Hence co-efficients can not be represented as ${}^nC_1$, ${}^nC_2$ ---------.
2. The number of terms in the expansion is infinite as n is a negative or fraction.

Short Questions

Write the short answers of the following

Expand by Bi-nomial theorem Q.No. 1 to 4

Q.1 $(2x - 3y)^4$ Q.2 $\left(\frac{x}{y} + \frac{y}{x}\right)^4$

Q.3 $\left(\frac{x}{2} - \frac{2}{y}\right)^4$ **Q.4** $\left(x + \frac{1}{x}\right)^4$

Q5 State Bi-nomial Theorem for positive integer n

Q.6 State Bi-nomial Theorem for n negative and rational.

Calculate the following by Binomial Theorem up to two decimal places.

Q.7 $(1.02)^{10}$ **Q.8** $(1.04)^5$

Q.9 Find the 7^{th} term in the expansion of $\left(x - \frac{1}{x}\right)^9$

Q.10 Find the 6^{th} term in the expansion of $(x + 3y)^{10}$

Q.11 Find 5^{th} term in the expansion of $(2x - \frac{x^2}{4})^7$

Expand to three term

Q.12 $(1 + 2x)^{-2}$ **Q.13** $\frac{1}{(1 + x)^2}$

Q.14 $\frac{1}{\sqrt{1+x}}$ **Q.15** $(4 - 3x)^{1/2}$

Q.16 Using the Binomial series calculate $\sqrt[3]{65}$ to the nearest hundredth.

Which will be the middle term/terms in the expansion of

Q.17 $(2x+3)^{12}$ **Q. 18** $(x+\frac{3}{x})^{15}$?

Q19 Which term is the middle term or terms in the Binomial expansion of $(a+b)^n$

(i) When n is even (ii) When n is odd

Answers

Q1. $16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4$

Q.2 $\frac{x^4}{y^4} + 4\frac{x^2}{y^2} + 6 + 4\frac{y^2}{x^2} + 4\frac{y^4}{x^4}$ **Q.3** $\frac{x^4}{16} - \frac{x^3}{y} + \frac{6x^2}{y^2} - \frac{16x}{y^3} + \frac{16}{y^4}$

Q.4 $(x)^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}$ **Q.7** 1.22 **Q.** 8 1.22

Q.9 $\frac{84}{x^3}$ Q.10 $61236\,x^5y^5$ **Q.11** $\frac{35}{32}x^{11}$

Q.12 $1 - 4x + 12x^2 + \ldots\ldots\ldots\ldots$ **Q.13** $1 - 2x + 3x^2 + \ldots\ldots\ldots\ldots$

Q.14 $1 - \frac{x}{2} + \frac{3}{8}x^2 + \ldots\ldots\ldots\ldots$ **Q.15** $2 - \frac{3x}{4} - \frac{9x^2}{64} + \ldots\ldots\ldots\ldots$

Q.16 4.02 **Q.17** $T_7 = \binom{12}{6}(2x)^6(3)^6$

Q.18 $T_8 = \binom{15}{7}(3)^7x$ and $T_9 = \binom{15}{8}\frac{(3)^8}{x}$

Q.19 (i) $\left(\frac{n}{2}+1\right)$ (ii) $\left(\frac{n+1}{2}\right)$ and $\left(\frac{n+1}{2}+1\right)$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. Third term of $(x + y)^4$ is:
(a) $4x^2y^2$ (b) $4x^3y$ c) $6x^2y^2$ (d) $6x^3y$

__2. The number of terms in the expansion $(a + b)^{13}$ are:
(a) 12 (b) 13 (c) 14 (d) 15

__3. The value of $\binom{n}{r}$ is:

(a) $\dfrac{n!}{r!(n-r)!}$ (b) $\dfrac{n}{r(n-r)}$ (c) $\dfrac{n!}{r!(n-r)}$ (d) $\dfrac{n!}{(n-r)!}$

__4. The second last term in the expansion of $(a + b)^7$ is:
(a) $7a^6b$ (b) $7ab^6$
(c) $7b^7$ (d) 15

__5. $\binom{6}{4}$ will have the value:

(a) 10 (b) 15 (c) 20 (d) 25

__6. $\binom{3}{0}$ will have the value:

(a) 0 (b) 1 (c) 2 (d) 3

__7. In the expansion of $(a + b)^n$ the general term is:

(a) $\binom{n}{r}a^rb^r$ (b) $\binom{n}{r}a^{n-r}b^r$

(c) $\binom{n}{r-1}a^{n-r+1}b^{r-1}$ (d) $\binom{n}{r}a^{n-r-1}b^{r-1}$

__8. In the expansion of $(a + b)^n$ the term $\binom{n}{r}a^{n-r}b^r$ will be:

(a) nth term (b) rth term
(c) (r + 1)th term (d) None of these

__9. In the expansion of $(a + b)^n$ the rth term is:
(a) ${}^nC_ra^rb^r$ (b) ${}^nC_ra^{n-r}b^r$

(c) $^nC_ra^{n-r+1}b^{r-1}$ (d) $^nC_ra^{n-r-1}b^{r-1}$

__10. In the expansion of $(1 + x)^n$ the co-efficient of 3^{rd} term is:

(a) $\binom{n}{0}$ (b) $\binom{n}{1}$ (c) $\binom{n}{2}$ (d) $\binom{n}{3}$

__11. In the expansion of $(a + b)^n$ the sum of the exponents of a and b in any term is:

(a) n (b) n – 1 (c) n + 1 (d) None of these

__12. The middle term in the expansion of $(a + b)^6$ is:

(a) $15a^4b^2$ (b) $20a^3b^3$ (c) $15a^2b^4$ (d) $6ab^5$

__13. The value of $\binom{n}{n}$ is equal to:

(a) Zero (b) 1 (c) n (d) -n

__14. The expansion of $(1 + x)^{-1}$ is:

(a) $1 - x - x^2 - x^3 + \ldots$

(b) $1 - x + x^2 - x^3 + \ldots$

(c) $1 - \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$

(d) $1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots$

__15. The expansion of $(1 - x)^{-1}$ is:

(a) $1 + x + x^2 + x^3 + \ldots$

(b) $1 - x + x^2 - x^3 + \ldots$

(c) $1 + \frac{1}{1!}x - \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$

(d) $1 - \frac{1}{1!}x + \frac{1}{2!}x^3 - \frac{1}{3!}x^3 + \ldots$

__16. Binomial series for $(1 + x)^n$ is valid only when:

(a) $x < 1$ (b) $x < -1$

(c) $|x| < 1$ (d) None of these

__17. The value of $\binom{2n}{n}$ is:

(a) $\frac{2n}{n!\,n!}$ (b) $\frac{(2n)!}{n!\,n!}$

(c) $\frac{(2n)!}{n!}$ (d) $\frac{(2n)!}{n(n-1!)}$

__18. The middle term of $\left(\frac{x}{y}-\frac{y}{x}\right)^4$ is:

(a) $\frac{4x^2}{y^2}$ (b) 6 (c) 8 (d) $\frac{4x}{y}$

Answers

1.	c	2.	c	3.	a	4.	b	5.	b
6.	b	7.	b	8.	c	9.	c	10.	c
11.	a	12.	b	13.	b	14.	b	15.	a
16.	b	17.	d	18.	c	19.	b	20.	b

Chapter 4
Partial Fractions

4.1 Introduction: A fraction is a symbol indicating the division of integers. For example, $\frac{13}{9}$, $\frac{2}{3}$ are fractions and are called Common Fraction. The dividend (upper number) is called the numerator N(x) and the divisor (lower number) is called the denominator, D(x).

From the previous study of elementary algebra we have learnt how the sum of different fractions can be found by taking L.C.M. and then add all the fractions. For example

i) $\frac{1}{x-1} + \frac{2}{x+2} = \frac{3x}{(x-1)(x+2)}$

ii) $\frac{2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x-2} = \frac{9x^2+5x-3}{(x+1)^2(x-2)}$

Here we study the reverse process, i.e., we split up a single fraction into a number of fractions whose denominators are the factors of denominator of that fraction. These fractions are called **Partial fractions.**

4.2 Partial fractions :

To express a single rational fraction into the sum of two or more single rational fractions is called **Partial fraction resolution**.

For example,

$$\frac{2x + x^2 - 1}{x(x^2 - 1)} = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}$$

$\frac{2x + x^2 - 1}{x(x^2 - 1)}$ is the resultant fraction and $\frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}$ are its partial fractions.

4.3 Polynomial:

Any expression of the form $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ where $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are real constants, if $a_n \neq 0$ then P(x) is called polynomial of degree n.

4.4 Rational fraction:

We know that $\frac{p}{q}$, $q \neq 0$ is called a rational number. Similarly the quotient of two polynomials $\frac{N(x)}{D(x)}$ where $D(x) \neq 0$, with no common factors, is called a rational fraction. A rational fraction is of two types:

4.5 Proper Fraction:

A rational fraction $\dfrac{N(x)}{D(x)}$ is called a proper fraction if the degree of numerator N(x) is less than the degree of Denominator D(x).

For example

(i) $\dfrac{9x^2 - 9x + 6}{(x-1)(2x-1)(x+2)}$

(ii) $\dfrac{6x + 27}{3x^3 - 9x}$

4.6 Improper Fraction:

A rational fraction $\dfrac{N(x)}{D(x)}$ is called an improper fraction if the degree of the Numerator N(x) is greater than or equal to the degree of the Denominator D(x)

For example

(i) $\dfrac{2x^3 - 5x^2 - 3x - 10}{x^2 - 1}$

(ii) $\dfrac{6x^3 - 5x^2 - 7}{3x^2 - 2x - 1}$

<u>Note:</u> An improper fraction can be expressed, by division, as the sum of a polynomial and a proper fraction.

For example:

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x + 3) + \frac{8x - 4}{x^2 - 2x - 1}$$

Which is obtained as, divide $6x^2 + 5x^2 - 7$ by $3x^2 - 2x - 1$ then we get a polynomial (2x+3) and a proper fraction $\dfrac{8x - 4}{x^2 - 2x - 1}$

4.7 Process of Finding Partial Fraction:

A proper fraction $\dfrac{N(x)}{D(x)}$ can be resolved into partial fractions as:

(I) If in the denominator D(x) a linear factor (ax + b) occurs and is non-repeating, its partial fraction will be of the form $\dfrac{A}{ax + b}$,where A is a constant whose value is to be determined.

(II) If in the denominator D(x) a linear factor (ax + b) occurs n times, i.e., $(ax + b)^n$, then there will be n partial fractions of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\frac{A_3}{(ax+b)^3}+.....+\frac{A_n}{(ax+b)^n}$$

,where A_1, A_2, A_3 - - - - - - - - - - A_n are constants whose values are to be determined

(III) If in the denominator D(x) a quadratic factor $ax^2 + bx + c$ occurs and is non-repeating, its partial fraction will be of the form $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants whose values are to be determined.

(IV) If in the denominator a quadratic factor $ax^2 + bx + c$ occurs n times, i.e., $(ax^2 + bx + c)^n$,then there will be n partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+$$

$$----------+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Where A_1, A_2, A_3 - - - - - - - - A_n and B_1, B_2, B_3 - - - - - - - B_n are constants whose values are to be determined.

Note: The evaluation of the coefficients of the partial fractions is based on the following theorem:

If two polynomials are equal for all values of the variables, then the coefficients having same degree on both sides are equal, for example , if

$$px^2 + qx + a = 2x^2 - 3x + 5 \quad \forall\ x \text{ , then}$$

$$p = 2, \quad q = -3 \quad \text{and} \quad a = 5.$$

4.8 Type I

When the factors of the denominator are all linear and distinct i.e., non repeating.

Example 1:

Resolve $\frac{7x-25}{(x-3)(x-4)}$ into partial fractions.

Solution:

$$\frac{7x-25}{(x-3)(x-4)}=\frac{A}{x-3}+\frac{B}{x-4} \text{ ------------------(1)}$$

Multiplying both sides by L.C.M. i.e., (x – 3)(x – 4), we get

$$7x - 25 = A(x - 4) + B(x - 3) \text{ -------------- (2)}$$

$$7x - 25 = Ax - 4A + Bx - 3B$$

$$7x - 25 = Ax + Bx - 4A - 3B$$
$$7x - 25 = (A + B)x - 4A - 3B$$

Comparing the co-efficients of like powers of x on both sides, we have

$$7 = A + B \text{ and}$$
$$-25 = -4A - 3B$$

Solving these equation we get

$A = 4$ and $B = 3$

Hence the required partial fractions are:

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

Alternative Method:

Since $7x - 25 = A(x - 4) + B(x - 3)$

Put $x - 4 = 0, \Rightarrow x = 4$ in equation (2)

$$7(4) - 25 = A(4 - 4) + B(4 - 3)$$
$$28 - 25 = 0 + B(1)$$
$$B = 3$$

Put $x - 3 = 0 \Rightarrow x = 3$ in equation (2)

$$7(3) - 25 = A(3 - 4) + B(3 - 3)$$
$$21 - 25 = A(-1) + 0$$
$$-4 = -A$$
$$A = 4$$

Hence the required partial fractions are

$$\frac{7x - 25}{(x - 3)(x - 4)} = \frac{4}{x-3} + \frac{3}{x-4}$$

Note : The R.H.S of equation (1) is the identity equation of L.H.S

Example 2:

write the identity equation of $\dfrac{7x-25}{(x-3)(x-4)}$

Solution : The identity equation of $\dfrac{7x-25}{(x-3)(x-4)}$ is

$$\frac{7x-25}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

Example 3:

Resolve into partial fraction: $\dfrac{1}{x^2 - 1}$

Solutios: $\dfrac{1}{x^2 - 1} = \dfrac{A}{x - 1} + \dfrac{B}{x + 1}$

$1 = A(x + 1) + B(x - 1) \qquad (1)$

Put $x - 1 = 0, \Rightarrow x = 1$ in equation (1)

$1 = A(1 + 1) + B(1 - 1) \Rightarrow A = \dfrac{1}{2}$

Put $x + 1 = 0, \Rightarrow x = -1$ in equation (1)

$1 = A(-1+1) + B(-1-1)$

$1 = -2B, \Rightarrow B = \dfrac{1}{2}$

$$\frac{1}{x^2 - 1} = \frac{1}{2(x - 1)} - \frac{1}{2(x + 1)}$$

Example 4:

Resolve into partial fractions $\dfrac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1}$

Solution:

This is an improper fraction first we convert it into a polynomial and a proper fraction by division.

$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = (2x + 3) + \frac{8x - 4}{x^2 - 2x - 1}$$

Let $\dfrac{8x - 4}{x^2 - 2x - 1} = \dfrac{8x - 4}{(3x + 1)} = \dfrac{A}{x - 1} + \dfrac{B}{3x + 1}$

Multiplying both sides by $(x - 1)(3x + 1)$ we get

$$8x - 4 = A(3x + 1) + B(x - 1) \qquad (I)$$

Put $x - 1 = 0, \Rightarrow x = 1$ in (I), we get

The value of A

$$8(1) - 4 = A(3(1) + 1) + B(1 - 1)$$

$$8 - 4 = A(3 + 1) + 0$$

$$4 = 4A$$

$$\Rightarrow \quad A = 1$$

Put $3x + 1 = 0 \Rightarrow x = -\dfrac{1}{3}$ in (I)

$$8\left(-\frac{1}{3}\right)-4 = B\left(-\frac{1}{3}-1\right)$$

$$-\frac{8}{3}-4 = \left(-\frac{4}{3}\right)$$

$$-\frac{20}{3} = -\frac{4}{3}\ B$$

$$\Rightarrow \quad B = \frac{20}{3} \text{ x } \frac{3}{4} = 5$$

Hence the required partial fractions are

$$\frac{6x^3+5x^2-7}{3x^2-2x-1} = (2x+3)+\frac{1}{x-1}+\frac{5}{3x+1}$$

Example 5:

Resolve into partial fraction $\dfrac{8x-8}{x^3-2x^2-8x}$

Solution: $\dfrac{8x-8}{x^3-2x^2-8x} = \dfrac{8x-8}{x(x^2-2x-8)} = \dfrac{8x-8}{x(x-4)(x+2)}$

Let $\dfrac{8x-8}{x^3-2x^2-8x} = \dfrac{A}{x}+\dfrac{B}{x-4}+\dfrac{C}{x+2}$

Multiplying both sides by L.C.M. i.e., $x(x-4)(x+2)$

$$8x-8 = A(x-4)(x+2)+Bx(x+2)+Cx(x-4)$$

(I)

Put $x = 0$ in equation (I), we have

$$8(0)-8 = A(0-4)(0+2)+B(0)(0+2)+C(0)(0-4)$$

$$-8 = -8A+0+0$$

$$\Rightarrow \quad A = 1$$

Put $x-4 = 0 \quad \Rightarrow \quad x = 4$ in Equation (I), we have

$$8(4)-8 = B(4)(4+2)$$

$$32-8 = 24B$$

$$24 = 24B$$

$$\Rightarrow \quad B = 1$$

Put $x+2 = 0 \Rightarrow x = -2$ in Eq. (I), we have

$$8(-2)-8 = C(-2)(-2-4)$$

$$-16-8 = C(-2)(-6)$$

$$-24 = 12C$$

$$\Rightarrow \quad C = -2$$

Hence the required partial fractions

$$\frac{8x-8}{x^3-2x^2-8x}=\frac{1}{x}-\frac{1}{x-4}-\frac{2}{x+2}$$

Exercise 4.1

Resolve into partial fraction:

Q.1 $\dfrac{2x+3}{(x-2)(x+5)}$ Q.2 $\dfrac{2x+5}{x^2+5x+6}$

Q.3 $\dfrac{3x^2-2x-5}{(x-2)(x+2)(x+3)}$ Q.4 $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$

Q.5 $\dfrac{x}{(x-a)(x-b)(x-c)}$ Q.6 $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$

Q.7 $\dfrac{2x^3-x^2+1}{(x+3)(x-1)(x+5)}$ Q.8 $\dfrac{1}{(1-x)(1-2x)(1-3x)}$

Q.9 $\dfrac{6x+27}{4x^3-9x}$ Q.10 $\dfrac{9x^2-9x+6}{(x-1)(2x-1)(x+2)}$

Q.11 $\dfrac{x^4}{(x-1)(x-2)(x-3)}$ Q.12 $\dfrac{2x^3+x^2-x-3}{x(x-1)(2x+3)}$

Answers 4.1

Q.1 $\dfrac{1}{x-2}+\dfrac{1}{x+5}$ Q.2 $\dfrac{1}{x+2}+\dfrac{1}{x+3}$

Q.3 $\dfrac{3}{20(x-2)}-\dfrac{11}{4(x-2)}+\dfrac{28}{5(x+3)}$

Q.4 $1+\dfrac{3}{x-4}-\dfrac{24}{x-5}+\dfrac{30}{x-6}$

Q.5 $\dfrac{a}{(a-b)(a-c)(x-a)}+\dfrac{b}{(b-a)(b-c)(x-b)}+\dfrac{c}{(c-b)(c-a)(x-c)}$

Q.6 $\dfrac{a^2}{(a-b)(a-c)(1-ax)}+\dfrac{b^2}{(b-a)(b-c)(1-bx)}+\dfrac{c^2}{(c-b)(c-a)(1-cx)}$

Q.7 $2+\dfrac{31}{4(x+3)}+\dfrac{1}{12(x-1)}-\dfrac{137}{6(x+5)}$

Q.8 $\dfrac{1}{2(1-x)}-\dfrac{4}{(1-2x)}+\dfrac{9}{2(1-3x)}$

Q.9 $\dfrac{3}{x}+\dfrac{4}{2x-3}+\dfrac{2}{2x+3}$

Q.10 $\dfrac{2}{x-1}-\dfrac{3}{2x-1}+\dfrac{4}{x+12}$

Q.11 $x+6+\dfrac{1}{2(x-1)}-\dfrac{16}{x-2}+\dfrac{81}{2(x-3)}$

Q.12 $1+\dfrac{1}{x}-\dfrac{1}{5(x-1)}-\dfrac{8}{5(2x+3)}$

4.9 Type II:

When the factors of the denominator are all linear but some are repeated.

Example 1:

Resolve into partial fractions: $\dfrac{x^2-3x+1}{(x-1)^2(x-2)}$

Solution:

$$\frac{x^2-3x+1}{(x-1)^2(x-2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}$$

Multiplying both sides by L.C.M. i.e., $(x-1)^2(x-2)$, we get

$x^2-3x+1=A(x-1)(x-2)+B(x-2)+C(x-1)^2$ (I)

Putting $x-1=0 \Rightarrow x=1$ in (I), then

$(1)^2-3(1)+1=B(1-2)$

$1-3+1=-B$

$-1=-B$

$\Rightarrow B=1$

Putting $x-2=0 \Rightarrow x=2$ in (I), then

$(2)^2-3(2)+1=C(2-1)^2$

$4-6+1=C(1)^2$

$\Rightarrow -1=C$

Now $x^2-3x+1=A(x^2-3x+2)+B(x-2)+C(x^2-2x+1)$

Comparing the co-efficient of like powers of x on both sides, we get

$A+C=1$

$A=1-C$

$= 1 - (-1)$
$= 1 + 1 = 2$
$\Rightarrow \quad A = 2$

Hence the required partial fractions are

$$\frac{x^2 - 3x + 1}{(x-1)^2(x-2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{1}{x-2}$$

Example 2:

Resolve into partial fraction $\dfrac{1}{x^4(x+1)}$

Solution

$$\frac{1}{x^4(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+1}$$

Where A, B, C, D and E are constants. To find these constants multiplying both sides by L.C.M. i.e., $x^4(x + 1)$, we get

$1 = A(x^3)(x + 1) + Bx^2(x + 1) + Cx(x + 1) + D(x + 1) + Ex^4$ (I)

Putting $x = -1$ in Eq. (I)

$1 = E(-1)^4$
$\Rightarrow \quad E = 1$

Putting x = 0 in Eq. (I), we have

$1 = D(0 + 1)$
$1 = D$
$\Rightarrow \quad D = 1$

$1 = A(x^4 + x^3)+B(x^3 + x^2)+C(x^2 + x)+D(x + 1) + Ex$

Comparing the co-efficient of like powers of x on both sides.

Co-efficient of x^3 : $A + B = 0$ ……… (i)

Co-efficient of x^2 : $B + C = 0$ ……… (ii)

Co-efficient of x : $C + D = 0$ ……… (iii)

Putting the value of D = 1 in (iii)

$C + 1 = 0$
$\Rightarrow \quad C = -1$

Putting this value in (ii), we get

$B - 1 = 0$
$\Rightarrow \quad B = 1$

Putting B = 1 in (i), we have

$A + 1 = 0$
$\Rightarrow \quad A = -1$

Hence the required partial fraction are

$$\frac{1}{x^4(x+1)} = \frac{-1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x+1}$$

Example 3:

Resolve into partial fractions $\dfrac{4+7x}{(2+3x)(1+x)^2}$

Solution:

$$\frac{4+7x}{(2+3x)(1+x)^2} = \frac{A}{2+3x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$$

Multiplying both sides by L.C.M. i.e., $(2 + 3x)(1 + x)^2$

We get $4 + 7x = A(1 + x)^2 + B(2 + 3x)(1 + x) + C(2 + 3x)$ …. (I)

Put $2 + 3x = 0 \Rightarrow x = -\frac{2}{3}$ in (I)

Then $4+7\left(-\frac{2}{3}\right) = A\left(1-\frac{2}{3}\right)^2$

$$4-\frac{14}{3} = A\left(-\frac{1}{3}\right)^2$$

$$-\frac{2}{3} = \frac{1}{9}A$$

$\Rightarrow$ $A = \frac{-2}{3} \times \frac{9}{1} = -6$

$A = -6$

Put $1 + x = 0 \Rightarrow x = -1$ in eq. (I), we get

$4 + 7(-1) = C(2 - 3)$

$4 - 7 = C(-1)$

$-3 = -C$

$\Rightarrow$ $C = 3$

$4 + 7x = A(x^2 + 2x + 1) + B(2 + 5x + 3x^2) + C(2 + 3x)$

Comparing the co-efficient of x^2 on both sides

$A + 3B = 0$

$-6 + 3B = 0$

$3B = 6$

$\Rightarrow$ $B = 2$

Hence the required partial fraction will be

$$\frac{-6}{2+3x} + \frac{2}{1+x} + \frac{3}{(1+x)^2}$$

Exercise 4.2

Resolve into partial fraction:

Q.1 $\dfrac{x+4}{(x-2)^2(x+1)}$ Q2. $\dfrac{1}{(x+1)(x^2-1)}$

Q.3 $\dfrac{4x^3}{(x+1)^2(x^2-1)}$ Q.4 $\dfrac{2x+1}{(x+3)(x-1)(x+2)^2}$

Q.5 $\dfrac{6x^2-11x-32}{(x+6)(x+1)^2}$ Q.6 $\dfrac{x^2-x-3}{(x-1)^3}$

Q.7 $\dfrac{5x^2+36x-27}{x^4-6x^3+9x^2}$ Q.8 $\dfrac{4x^2-13x}{(x+3)(x-2)^2}$

Q.9 $\dfrac{x^4+1}{x^2(x-1)}$ Q.10 $\dfrac{x^3-8x^2+17x+1}{(x-3)^3}$

Q.11 $\dfrac{x^2}{(x-1)^3(x+2)}$ Q.12 $\dfrac{2x+1}{(x+2)(x-3)^2}$

Answers4.2

Q.1 $-\dfrac{1}{3(x-2)}+\dfrac{2}{(x-2)^2}+\dfrac{1}{3(x+1)}$

Q.2 $\dfrac{1}{4(x-1)}-\dfrac{1}{4(x+1)}-\dfrac{1}{2(x+1)^2}$

Q.3 $\dfrac{1}{2(x-1)}+\dfrac{7}{2(x+1)}-\dfrac{5}{(x+1)^2}+\dfrac{2}{(x+1)^3}$

Q.4 $\dfrac{5}{4(x+3)}+\dfrac{1}{12(x-1)}-\dfrac{4}{3(x+2)}+\dfrac{1}{(x+2)^2}$

Q.5 $\dfrac{10}{x+6}-\dfrac{4}{x+1}-\dfrac{3}{(x-1)^2}$

Q.6 $\dfrac{1}{x-1}+\dfrac{1}{(x-1)^2}-\dfrac{3}{(x-1)^3}$

Q.7 $\dfrac{2}{x}-\dfrac{3}{x^2}-\dfrac{2}{(x-3)}+\dfrac{14}{(x-3)^2}$

Q.8 $\dfrac{3}{x+3}+\dfrac{1}{x-2}-\dfrac{2}{(x-2)^2}$

Q.9 $x + 1 - \dfrac{1}{x} - \dfrac{1}{x^2} + \dfrac{2}{x-1}$

Q.10 $1 + \dfrac{1}{x-3} - \dfrac{4}{(x-3)^2} + \dfrac{7}{(x-3)^3}$

Q.11 $\dfrac{4}{27(x-1)} + \dfrac{5}{9(x-1)^2} + \dfrac{1}{3(x-1)^3} - \dfrac{4}{27(x+2)}$

Q.12 $-\dfrac{3}{25(x+2)} + \dfrac{3}{25(x-3)} + \dfrac{7}{5(x-3)^2}$

4.10 Type III:

When the denominator contains ir-reducible quadratic factors which are non-repeated.

Example 1:

Resolve into partial fractions $\dfrac{9x-7}{(x+3)(x^2+1)}$

Solution:

$$\frac{9x-7}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$

Multiplying both sides by L.C.M. i.e., $(x + 3)(x^2 + 1)$, we get

$9x - 7 = A(x^2 + 1) + (Bx + C)(x + 3)$ (I)

Put $x + 3 = 0 \Rightarrow x = -3$ in Eq. (I), we have

$9(-3) - 7 = A((-3)^2 + 1) + (B(-3) + C)(-3 + 3)$

$-27 - 7 = 10A + 0$

$A = -\dfrac{34}{10} \Rightarrow \boxed{A = -\dfrac{17}{5}}$

$9x - 7 = A(x^2 + 1) + B(x^2 + 3x) + C(x + 3)$

Comparing the co-efficient of like powers of x on both sides

$A + B = 0$

$3B + C = 9$

Putting value of A in Eq. (i)

$-\dfrac{17}{5} + B = 0 \Rightarrow \boxed{B = \dfrac{17}{5}}$

From Eq. (iii)

$C = 9 - 3B = 9 - 3\left(\dfrac{17}{4}\right)$

$= 9 - \dfrac{51}{5} \Rightarrow \boxed{C = -\dfrac{6}{5}}$

Hence the required partial fraction are

$$\frac{-17}{5(x+3)}+\frac{17x-6}{5(x^2+1)}$$

Example 2:

Resolve into partial fraction $\dfrac{x^2+1}{x^4+x^2+1}$

Solution:

Let $\dfrac{x^2+1}{x^4+x^2+1}=\dfrac{x^2+1}{(x^2-x+1)(x^2+x+1)}$

$$\frac{x^2+1}{(x^2-x+1)(x^2+x+1)}=\frac{Ax+B}{(x^2-x+1)}+\frac{Cx+D}{(x^2+x+1)}$$

Multiplying both sides by L.C.M. i.e., $(x^2-x+1)(x^2+x+1)$

$x^2+1=(Ax+B)(x^2+x+1)+(Cx+D)(x^2-x+1)$

Comparing the co-efficient of like powers of x, we have

Co-efficient of x^3 : $A+C=0$ ………… (i)

Co-efficient of x^2 : $A+B-C+D=1$ ……….. (ii)

Co-efficient of x : $A+B+C-D=0$ ……….. (iii)

Constant $B+D=1$ ……….. (iv)

Subtract (iv) from (ii) we have

$A-C=0$ …………… (v)

$A=C$ …………… (vi)

Adding (i) and (v), we have

$A=0$

Putting $A=0$ in (vi), we have

$C=0$

Putting the value of A and C in (iii), we have

$B-D=0$ …………… (vii)

Adding (iv) and (vii)

$$2B=1 \quad\Rightarrow\quad B=\frac{1}{2}$$

from (vii) $B=D$, therefore

$$D=\frac{1}{2}$$

Hence the required partial fraction are

$$\frac{0x+\frac{1}{2}}{(x^2-x+1)}+\frac{0x+\frac{1}{2}}{(x^2+x+1)}$$

i.e., $$\frac{1}{2(x^2-x+1)}+\frac{1}{2(x^2+x+1)}$$

Exercise 4.3

Resolve into partial fraction:

Q.1 $\dfrac{x^2+3x-1}{(x-2)(x^2+5)}$ Q.2 $\dfrac{x^2-x+2}{(x+1)(x^2+3)}$

Q.3 $\dfrac{3x+7}{(x+3)(x^2+1)}$ Q.4 $\dfrac{1}{(x^3+1)}$

Q.5 $\dfrac{1}{(x+1)(x^2+1)}$ Q.6 $\dfrac{3x+7}{(x^2+x+1)(x^2-4)}$

Q.7 $\dfrac{3x^2-x+1}{(x+1)(x^2-x+3)}$ Q.8 $\dfrac{x+a}{x^2(x-a)(x^2+a^2)}$

Q.9 $\dfrac{x^5}{x^4-1}$ Q.10 $\dfrac{x^2+x+1}{(x^2-x-2)(x^2-2)}$

Q.11 $\dfrac{1}{x^3-1}$ Q.12 $\dfrac{x^2+3x+3}{(x^2-1)(x^2+4)}$

Answers 4.3

Q.1 $\dfrac{1}{x-2}+\dfrac{3}{x^2+5}$ Q.2 $\dfrac{1}{x+1}-\dfrac{1}{x^2+3}$

Q.3 $-\dfrac{1}{5(x+3)}+\dfrac{x+12}{5(x^2+1)}$ Q.4 $\dfrac{1}{3(x+1)}-\dfrac{(x-2)}{3(x^2-x+1)}$

Q.5 $\dfrac{1}{2(x+1)}-\dfrac{x-1}{2(x^2+1)}$

Q.6 $\dfrac{13}{28(X-2)}-\dfrac{1}{12(X+2)}-\dfrac{8X+31}{21(X^2+X+1)}$

Q.7 $\dfrac{1}{x+1}+\dfrac{2x-2}{x^2-x+3}$

Q.8 $\dfrac{1}{a^3}\left[\dfrac{1}{X-a}+\dfrac{x}{X^2+a^2}-\dfrac{2}{X}-\dfrac{a}{X^2}\right]$

Q.9 $x+\dfrac{1}{4(x-1)}+\dfrac{1}{4(x+1)}-\dfrac{x}{2(x^2+1)}$

Q.10 $\dfrac{1}{3(x+1)}+\dfrac{7}{6(x-2)}-\dfrac{3x+2}{2(x^2-2)}$

Q.11 $\dfrac{1}{3(x-1)}-\dfrac{x+2}{3(x^2+x+1)}$

Q.12 $\dfrac{7}{10(x-1)}-\dfrac{1}{10(x+1)}-\dfrac{3x-1}{5(x^2+4)}$

4.11 Type IV: Quadratic repeated factors

When the denominator has repeated Quadratic factors.

Example 1:

Resolve into partial fraction $\dfrac{x^2}{(1-x)(1+x^2)^2}$

Solution:

$$\frac{x^2}{(1-x)(1+x^2)^2}=\frac{A}{1-x}+\frac{Bx+C}{(1+x^2)}+\frac{Dx+E}{(1+x^2)^2}$$

Multiplying both sides by L.C.M. i.e., $(1-x)(1+x^2)^2$ on both sides, we have

$x^2 = A(1+x^2)^2+(Bx+C)(1-x)(1+x^2)+(Dx+E)(1-x)$ ……(i)

$x^2 = A(1+2x^2+x^4)+(Bx+C)(1-x+x^2-x^3)+(Dx+E)(1-x)$

Put $1-x=0 \Rightarrow x=1$ in eq. (i), we have

$(1)^2 = A(1+(1)^2)^2$

$1 = 4A \Rightarrow \boxed{A=\dfrac{1}{4}}$

$x^2 = A(1+2x^2+x^4)+B(x-x^2+x^3-x^4)+C(1-x+x^2-x^3)$
$+D(x-x^2)+E(1-x)$ …………………… (ii)

Comparing the co-efficients of like powers of x on both sides in Equation (II), we have

Co-efficient of x^4	:	$A-B=0$ …………….	(i)
Co-efficient of x^3	:	$B-C=0$ …………….	(ii)
Co-efficient of x^2	:	$2A-B+C-D=1$ …….	(iii)
Co-efficient of x	:	$B-C+D-E=0$ …….	(iv)
Co-efficient term	:	$A+C+E=0$ …………….	(v)
from (i),		$B=A$	

$$\Rightarrow \quad B=\frac{1}{4} \quad \because \quad A=\frac{1}{4}$$

from (i) $B = C$

$$\Rightarrow \quad C = \frac{1}{4} \qquad \because \quad C = \frac{1}{4}$$

from (iii) $D = 2A - B + C - 1$

$$= 2\left(\frac{1}{4}\right) - \frac{1}{4} + \frac{1}{4} - 1$$

$$\Rightarrow \quad \boxed{D = -\frac{1}{2}}$$

from (v) $E = -A - C$

$$E = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

Hence the required partial fractions are by putting the values of A, B, C, D, E,

$$\frac{\frac{1}{4}}{1-x} + \frac{\frac{1}{4}x + \frac{1}{4}}{1+x^2} + \frac{-\frac{1}{2}x - \frac{1}{2}}{(1+x^2)^2}$$

$$\frac{1}{4(1-x)} + \frac{(x+1)}{4(1+x^2)} - \frac{x+1}{2(1+x^2)^2}$$

Example 2:

Resolve into partial fractions $\dfrac{x^2 + x + 2}{x^2(x^2+3)^2}$

Solution:

Let $$\frac{x^2 + x + 2}{x^2(x^2+3)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 3} + \frac{Ex + F}{(x^2+3)^2}$$

Multiplying both sides by L.C.M. i.e., $x^2(x^2+3)^2$, we have

$$x^2 + x + 2 = Ax(x^2+3)^2 + B(x^2+3)^2$$
$$+(cx + D)x^2(x^2+3) + (Ex + F)(x^2)$$

Putting x = 0 on both sides, we have

$$2 = B\,(0+3)^2$$
$$2 = 9B \quad \Rightarrow \quad \boxed{B = \frac{2}{9}}$$

Now $$x^2 + x + 2 = Ax(x^4 + 6x^2 + 9) + B(x^4 + 6x^2 + 9)$$
$$+C(x^5 + 3x^2) + D(x^4 + 3x^2) + E(x^3) + Fx^2$$

$$x^2 + x + 2 = (A + C)x^5 + (B + D)x^4 + (6A + 3C + E)x^3$$

$$+(6B+3D+F)x^2+(x+9B)$$

Comparing the co-efficient of like powers of x on both sides of Eq. (I), we have

Co-efficient of x^5 : $A + C = 0$ ……… (i)

Co-efficient of x^4 : $B - D = 0$ ……… (ii)

Co-efficient of x^3 : $6A + 3C + E = 0$ ……… (iii)

Co-efficient of x^2 : $6B + 3D + F = 1$ ……… (iv)

Co-efficient of x : $9A = 1$ ……… (v)

Co-efficient term : $9B = 1$ ……… (vi)

from (v) $9A = 1$

$\Rightarrow$ $\boxed{A = \frac{1}{9}}$

from (i) $A + C = 0$

$C = -A$

$\Rightarrow$ $\boxed{C = -\frac{1}{9}}$

from (i) $B + D = 0$

$D = -B$

$\Rightarrow$ $\boxed{D = -\frac{2}{9}}$

from (iii) $6A + 3C + E =$

$$6\left(\frac{1}{9}\right)+3\left(-\frac{1}{9}\right)+E=0$$

$$E=\frac{3}{9}-\frac{6}{9}$$

$\Rightarrow$ $\boxed{E = -\frac{1}{3}}$

from (iv) $6B + 3D + F = 1$

$$F = 1 - 6B - 3D$$

$$=1-6\left(\frac{2}{9}\right)-3\left(\frac{2}{9}\right)$$

$$=1-\frac{12}{9}+\frac{6}{9}$$

$$\Rightarrow \quad \boxed{F=\frac{1}{3}}$$

Hence the required partial fractions are

$$\frac{\frac{1}{9}}{x}+\frac{\frac{2}{9}}{x^2}+\frac{-\frac{1}{9}x-\frac{2}{9}}{x^2+3}+\frac{-\frac{1}{3}x+\frac{1}{3}}{(x^2+3)^2}$$

$$=\frac{1}{9x}+\frac{2}{9x^2}-\frac{x+2}{9(x^2+3)}-\frac{x-1}{3(x^2+3)^2}$$

Exercise 4.4

Resolve into Partial Fraction:

Q.1 $\dfrac{7}{(x+1)(x^2+2)^2}$

Q.2 $\dfrac{x^2}{(1+x)(1+x^2)^2}$

Q.3 $\dfrac{5x^2+3x+9}{x(x^2+3)^2}$

Q.4 $\dfrac{4x^4+3x^3+6x^2+5x}{(x-1)(x^2+x+1)^2}$

Q.5 $\dfrac{2x^4-3x^2-4x}{(x+1)(x^2+2)^2}$

Q.6 $\dfrac{x^3-15x^2-8x-7}{(2x-5)(1+x^2)^2}$

Q.7 $\dfrac{49}{(x-2)(x^2+3)^2}$

Q.8 $\dfrac{8x^2}{(1-x^2)(1+x^2)^2}$

Q.9 $\dfrac{x^4+x^3+2x^2-7}{(x+2)(x^2+x+1)^2}$

Q.10 $\dfrac{x^2+2}{(x^2+1)(x^2+4)^2}$

Q.11 $\dfrac{1}{x^4+x^2+1}$

Answers 4.4

Q.1 $\dfrac{7}{9(X+1)}-\dfrac{7X-7}{9(X^2+2)}-\dfrac{7X-7}{3(X^2+2)^2}$

Q.2 $\dfrac{1}{4(1+x)}-\dfrac{x-1}{4(1+x^2)}+\dfrac{x-1}{2(1+x^2)^2}$

Q.3 $\dfrac{1}{x}-\dfrac{x}{x^2+3}+\dfrac{2x+3}{(x^2+3)^2}$

Q.4 $\dfrac{2}{X-1}+\dfrac{2X-1}{X^2+X+1}+\dfrac{3}{(X^2+X+1)^2}$

Q.5 $\dfrac{1}{3(x+1)}+\dfrac{5(x-1)}{3(x^2+2)}-\dfrac{2(3x-1)}{(x^2+1)^2}$

Q.6 $-\dfrac{2}{2x-5}+\dfrac{x+3}{1+x^2}+\dfrac{x-2}{(1+x^2)^2}$

Q.7 $\dfrac{1}{x-2}-\dfrac{x+2}{x^2+3}-\dfrac{7x+14}{(x^2+3)^2}$

Q.8 $\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{2}{1+x^2}-\dfrac{4}{(1+x^2)^2}$

Q.9 $\dfrac{1}{x+2}+\dfrac{2x-3}{(x^2+x+1)^2}-\dfrac{1}{x^2+x+1}$

Q.10 $\dfrac{1}{9(x^2+1)}-\dfrac{1}{9(x^2+4)}+\dfrac{2}{3(x^2+4)^2}$

Q.11 $-\dfrac{(x-1)}{2(x^2-x+1)}+\dfrac{(x+1)}{2(x^2+x+1)}$

Summary

Let N(x) ≠ and D(x) ≠ 0 be two polynomials. The $\dfrac{N(x)}{D(x)}$ is called a proper fraction if the degree of N(x) is smaller than the degree of D(x).

For example: $\dfrac{x-1}{x^2+5x+6}$ is a proper fraction.

Also $\dfrac{N(x^1)}{D(x)}$ is called an improper fraction of the degree of N(x) is greater than or equal to the degree of D(x).

For example: $\dfrac{x^5}{x^4-1}$ is an improper fraction.

In such problems we divide N(x) by D(x) obtaining a quotient Q(x) and a remainder R(x) whose degree is smaller than that of D(x).

Thus $\dfrac{N(x)'}{D(x)}=Q(x)+\dfrac{R(x)}{D(x)}$ where $\dfrac{R(x)'}{D(x)}$ is proper fraction.

Types of proper fraction into partial fractions.

Type 1:

Linear and distinct factors in the D(x)

$$\frac{x-a}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

Type 2:

Linear repeated factors in D(x)

$$\frac{x-a}{(x+a)(x^2+b^2)} = \frac{A}{x+a} + \frac{Bx+C}{x^2+b^2}$$

Type 3:

Quadratic Factors in the D(x)

$$\frac{x-a}{(x+a)(x^2+b)^2} = \frac{A}{x+a} + \frac{Bx+C}{x^2+b^2}$$

Type 4:

Quadratic repeated factors in D(x):

$$\frac{x-a}{(x^2+a^2)(x^2+b^2)} = \frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{x^2+b^2} + \frac{Ex+F}{(x^2+b^2)^2}$$

Short Questions:

Write the short answers of the following:

Q.1: What is partial fractions?

Q.2: Define proper fraction and give example.

Q.3: Define improper fraction and given one example:

Q.4: Resolve into partial fractions $\frac{2x}{(x - 2)(x + 5)}$

Q.5: Resolve into partial fractions: $\frac{1}{x^2 - x}$

Q.6: Resolve $\frac{7x + 25}{(x + 3)(x + 4)}$ into partial fraction.

Q.7: Resolve $\frac{1}{x^2 - 1}$ into partial fraction:

Q.8: Resolve $\frac{x^2 + 1}{(x + 1)(x - 1)}$ into partial fractions.

Q.9: Write an identity equation of $\frac{8x^2}{(1 - x^2)(1 + x^2)^2}$

Q.10: Write an identity equation of $\frac{2x + 5}{x^2 + 5x + 6}$

Q.11: Write identity equation of $\frac{x - 5}{(x + 1)(x^2 + 3)}$

Q.12: Write an identity equation of $\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1}$

Q.13: Write an identity equation of $\frac{(x - 1)(x - 2)(x - 3)}{(x - 4)(x - 5)(x - 6)}$

Q.14: Write an identity equation of $\frac{x^5}{x^4 - 1}$

Q.15: Write an identity equation of $\frac{2x^4 - 3x^2 - 4x}{(x + 1)(x^2 + 2)^2}$

Q16. Form of partial fraction of $\dfrac{1}{(x+1)(x-2)}$ is __________.

Q.17. Form of partial fraction of $\dfrac{1}{(x+1)^2(x-2)}$ is __________.

Q.18. Form of partial fraction of $\dfrac{1}{(x^2+1)(x-2)}$ is __________.

Q.19. Form of partial fraction of $\dfrac{1}{(x^2+1)(x-4)^2}$ is __________.

Q.20. Form of partial fraction of $\dfrac{1}{(x^3-1)(x^2+1)}$ is __________.

Answers

Q4. $\dfrac{4}{7(x-2)} - \dfrac{10}{7(x+5)}$

Q5. $\dfrac{-1}{x} + \dfrac{1}{x-1}$

Q6. $\dfrac{4}{x+3} + \dfrac{3}{x+4}$

Q7. $\dfrac{1}{x^2-1} = \dfrac{1}{2(x-1)} - \dfrac{1}{2(x+1)}$

Q8. $1 + \dfrac{1}{x+1} + \dfrac{1}{x-1}$

Q9. $\dfrac{A}{1-x} + \dfrac{B}{1+x} + \dfrac{Cx+D}{1+x^2} + \dfrac{Ex+F}{(1+x^2)^2}$

Q10. $\dfrac{A}{x+2} + \dfrac{B}{x+3}$

Q11. $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+3}$

Q12. $(2x+3) + \dfrac{A}{x-1} + \dfrac{B}{3x+1}$

Q13. $1 + \dfrac{A}{4-4} + \dfrac{B}{x-5} + \dfrac{C}{x-6}$

Q14. $x + \dfrac{A}{x-1} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1}$

Q15. $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+2} + \dfrac{Dx+E}{(x^2+2)^2}$

Q16. $\dfrac{A}{x+1} + \dfrac{B}{x-2}$

Q17. $\dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-2}$

Q18. $\dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-2}$

Q19. $\dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2}$

Q20. $\dfrac{A}{(x-1)} + \dfrac{Bx+C}{(x^2+x+1)} + \dfrac{Dx+E}{x^2+1}$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. If the degree of numerator N(x) is equal or greater than the degree of denominator D(x), then the fraction is:
(a) proper (b) improper
(c) Neither proper non-improper (d) Both proper and improper

__2. If the degree of numerator is less than the degree of denominator, then the fraction is:
(a) Proper (b) Improper
(c) Neither proper non-improper (d) Both proper and improper

__3. The fraction $\dfrac{2x+5}{x^2+5x+6}$ is known as:
(a) Proper (b) Improper
(c) Both proper and improper (d) None of these

__4. The number of partial fractions of $\dfrac{6x+27}{4x^3-9x}$ are:
(a) 2 (b) 3
(c) 4 (d) None of these

__5. The number of partial fractions of $\dfrac{x^3-3x^2+1}{(x-1)(x+1)(x^2-1)}$ are:
(a) 2 (b) 3
(c) 4 (d) 5

__6. The equivalent partial fraction of $\dfrac{x+11}{(x+1)(x-3)^2}$ is:
(a) $\dfrac{A}{x+1}+\dfrac{B}{(x-3)^2}$ (b) $\dfrac{A}{x+1}+\dfrac{B}{x-3}$
(c) $\dfrac{A}{x+1}+\dfrac{B}{x-3}+\dfrac{C}{(x-3)^2}$ (d) $\dfrac{A}{x+1}+\dfrac{Bx+C}{(x-3)^2}$

__7. The equivalent partial fraction of $\dfrac{x^4}{(x^2+1)(x^2+3)}$ is:
(a) $\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+3}$ (b) $\dfrac{Ax+B}{x^2+1}+\dfrac{Cx}{x^2+3}$
(c) $1+\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+3}$ (d) $\dfrac{Ax}{x^2+1}+\dfrac{Bx}{x^2+3}$

__8. Partial fraction of $\frac{2}{x(x+1)}$ is:

(a) $\frac{2}{x} - \frac{1}{x+1}$ (b) $\frac{1}{x} - \frac{2}{x+1}$

(c) $\frac{2}{x} - \frac{2}{x+1}$ (d) $\frac{2}{x} + \frac{2}{x+1}$

__9. Partial fraction of $\frac{2x+3}{(x-2)(x+5)}$ is called:

(a) $\frac{2}{x-2} + \frac{1}{x+5}$ (b) $\frac{3}{x-2} + \frac{1}{x+5}$

(c) $\frac{2}{x-2} + \frac{3}{x+5}$ (d) $\frac{1}{x-2} + \frac{1}{x+5}$

__10. The fraction $\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$ is called:

(a) Proper (ii) Improper

(c) Both proper and Improper (iv) None of these

Answers:

1.	b	2.	a	3.	a	4.	b	5.	c
6.	c	7.	c	8.	c	9.	d	10.	B

Chapter 5
Fundamentals of Trigonometry

5.1 Introduction:

The word “trigonometry” is a Greek word. Its mean “measurement of a triangle”. Therefore trigonometry is that branch of mathematics concerned with the measurement of sides and angle of a plane triangle and the investigations of the various relations which exist among them. Today the subject of trigonometry also includes another distinct branch which concerns itself with properties relations between and behavior of trigonometric functions.

The importance of trigonometry will be immediately realized when its applications in solving problem of mensuration, mechanics physics, surveying and astronomy are encountered.

5.2 Types of Trigonometry:

There are two types of trigonometry

(1) Plane Trigonometry (2) Spherical Trigonometry

1. Plane Trigonometry

Plane trigonometry is concerned with angles, triangles and other figures which lie in a plane.

2. Spherical Trigonometry

Spherical Trigonometry is concerned with the spherical triangles, that is, triangles lies on a sphere and sides of which are circular arcs.

5.3 Angle:

An angle is defined as the union of two non-collinear rays which have a common end-points.

An angle is also defined as it measures the rotation of a line from one position to another about a fixed point on it.

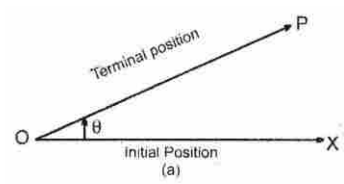


Fig. 4.1

In figure 5.1(a)the first position OX is called initial line (position)and second position OP is called terminal line or generating line(position) of $\angle XOP$.

If the terminal side resolves in anticlockwise direction the angle described is positive as shown in figure (i)

If terminal side resolves in clockwise direction, the angle described is negative as shown in figure (ii)

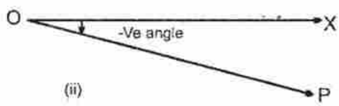


Fig. 4.3

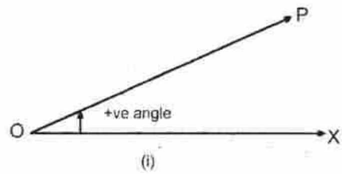


Fig. 4.2

5.4 Quadrants:

Two mutually perpendiculars straight lines xox□ and yoy□ divide the plane into four equal parts, each part is called quadrant.
Thus XOY, X′OY, X′OY′ and XOY′ are called the Ist, IInd, IIIrd and IVth quadrants respectively.

In first quadrant the angle vary from 0^o to 90^o in anti-clockwise direction and from -270^o to -360^o in clockwise direction.

In second quadrant the angle vary from 90^o to 180^o in anti-clockwise direction and -180^o to -270^o in clockwise direction.

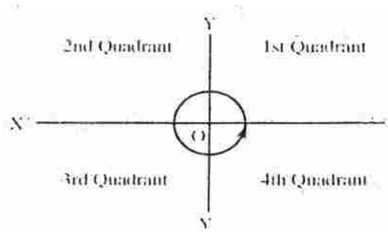


Fig. 4.4

In third quadrant the angle vary from 180^o to 270^o in anti-clockwise direction and from -90^o to -180^o in clockwise direction.

In fourth quadrant the angle vary from 270^o to 360^o in anti-clockwise direction and from -0^o to -90^o in clockwise direction.

5.5 Measurement of Angles:

The size of any angle is determined by the amount of rotations. In trigonometry two systems of measuring angles are used.

(i) Sexagesimal or English system (Degree)
(ii) Circular measure system (Radian)

(i) Sexagesimal or English System (Degree)

The sexagesimal system is older and is more commonly used. The name derive from the Latin for “sixty”. The fundamental unit of angle measure in the sexagesimal system is the degree of arc. By definition, when a circle is divided into 360 equal parts, then

One degree = $\frac{1}{360}$th part of a circle.

Therefore, one full circle = 360 degrees.

The symbol of degrees is denoted by $(\ )^0$.

Thus an angle of 20 degrees may be written as 20^o.
Since there are four right angles in a complete circle .

One right angle = $\frac{1}{4}$ circle = $\frac{1}{4}(360^o) = 90^o$

The degree is further subdivided in two ways, depending upon whether we work in the common sexagesimal system or the decimal sexagesimal system. In the common sexagesimal system, the degree is subdivided into 60 equal parts, called minutes, denoted by the symbol$(\ )'$, and the minute is further subdivided into 60 equal parts, called second, indicated by the symbol $(\ )''$. Therefore

1 minute = 60 seconds
1 degree = 60 minutes = 3600 seconds
1 circle = 360 degrees = 21600 minutes = 12,96,000 sec.

In the decimal sexagesimal system, angles smaller than 1^o are expressed as decimal fractions of a degree. Thus one-tenth $\left(\frac{1}{10}\right)$ of a degree is expressed as 0.1^o in the decimal sexagesimal system and as 6′ in the common sexagesimal system; one-hundredth $\left(\frac{1}{100}\right)$ of a degree is 0.01^o in the decimal system and 36" in the common system; and $47\frac{1}{9}$ degrees comes out $(47.111\ \ldots)^o$ in the decimal system and $47^o6'40''$ in the common system.

(ii) Circular measure system (Radian)

This system is comparatively recent. The unit used in this system is called a Radian. The Radian is define "The angle subtended at the centre of a circle by an arc equal in length to the radius of the circle." As shown in fig., Arc AB is equal in length to the radius $\overline{OB}$ of the circle. The subtended, $\angle AOB$ is then one radian.

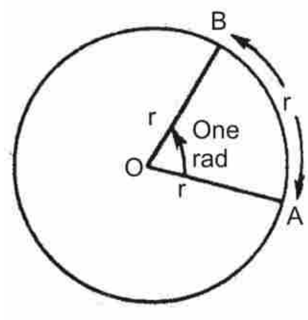


Fig. 4.5

i.e. m $\angle AOB$ = 1 radian.

5.6 Relation between Degree and Radian Measure:

Consider a circle of radius r, then the circumference of the circle is $2\pi r$. By definition of radian,

An arc of length ′r′ subtends an angle = 1 radian

$\therefore$ An arc of length $2\pi r$ subtends an angle = 2π radian

Also an arc of length $2\pi r$ subtends an angle = 360^o

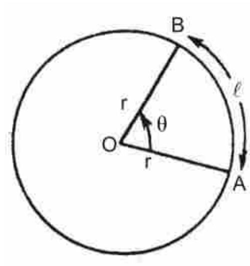


Fig. 4.6

Then 2π radians = 360°

Or π radians = 180°

$$1 \text{ radians} = \frac{180^o}{\pi}$$

$$1 \text{ radians} = \frac{180}{3.1416}$$

Or 1 radians = 57.3°

Therefore to convert radians into degree,

we multiply the number of radians by $\frac{180^o}{\pi}$ or 57.3.

Now Again

$$360^o = 2\pi \text{ radians}$$

$$1^o = \frac{2\pi r}{360^o} \text{ radians}$$

$$1^o = \frac{\pi}{180^o}$$

Or $$1^o = \frac{3.1416}{180^o}$$

$$1^o = 0.01745 \text{ radians}$$

Therefore, to convert degree into radians, we multiply the number of degrees by $\frac{\pi}{180}$ or 0.0175.

Note: One complete revolution = 360° = 2π radius.

5.7 Relation between Length of a Circular Arc and the Radian Measure of its Central Angle:

Let "*l*" be the length of a circular arc $\overline{AB}$ of a circle of radius r, and θ be its central angle measure in radians. Then the ratio of *l* to the circumference $2\pi r$ of the circle is the same as the ratio of θ to 2π.

Therefore

$$l : 2\pi r = \theta : 2\pi$$

Or $$\frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

$$\frac{l}{r} = \theta$$

$$l = \theta r$$, where θ is in radian

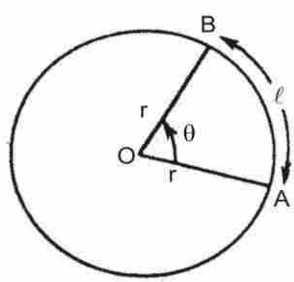


Fig. 4.7

Fig. 5.7

Note:

If the angle will be given in degree measure we have to convert it into Radian measure before applying the formula.

Example 1:

Convert 120^o into Radian Measure.

Solution:

120^o

$$120^o = 120 \text{ x } \frac{\pi}{180}$$

$$= \frac{2\pi}{3} = \frac{2(3.1416)}{3} = 2.09 \text{ rad.}$$

Example 2:

Convert 37° 25´ 38" into Radian measure.

Solution:

$$37^o\ 25'\ 28'' = 37^o + \frac{25}{60} + \frac{28}{3600}$$

$$= 37^o + \frac{5^o}{12} + \frac{7^o}{900}$$

$$= 37^o + \frac{382^o}{900}$$

$$= 37 + \frac{181^o}{450} = \frac{16831^o}{450} = \frac{16831}{450} \text{ x } \frac{\pi}{180}$$

$$= \frac{16831(3.14160)}{81000} = \frac{52876.26}{81000}$$

$$37^o\ 25'\ 28'' = 0.65 \text{ radians}$$

Example 3:

Express in Degrees:

(i) $\frac{5\pi}{3}$ rad (ii) 2.5793 rad (iii) $\frac{\pi}{6}$ rad (iv) $\frac{\pi}{3}$ rad

Solution:

(i) Since 1 rad $= \frac{180}{\pi}$ deg

$$\therefore \frac{5\pi}{3} \text{ rad} = \frac{5\pi}{3} \text{ x } \frac{180}{\pi} \text{ deg} = 5(60) \text{ deg} = 300^o$$

(ii) 2.5793 rad $= 2.5793\ (57^o .29578)$ $\therefore$ 1 rad $= 57^o .295778$

$= 147^o .78301 = 147.78$ (two decimal places)

(iii) $\frac{\pi}{6}$ rad $= \frac{\pi}{6} \text{ x } \frac{12}{\pi}$ deg $= 30^o$

(iv) $\frac{\pi}{3}$ rad $= \frac{\pi}{3} \text{ x } \frac{12}{\pi}$ deg $= 60^o$

Example 4:

What is the length of an arc of a circle of radius 5 cm. whose central angle is of 140^o?

Solution: l = length of an arc = ?

r = radius = 5 cm

$\theta = 140^o$

Since 1 deg = 0.01745 rad

$\therefore\ \theta = 140 \text{ x } 0.01745$ rad $= 2.443$ rad

$\therefore\ l = r\theta$

$l = (5)(2.443) = 12.215$ cm

Example 5:

A curve on a highway is laid out an arc of a circle of radius 620m. How long is the arc that subtends a central angle of 32^o?

Solution: r = 620m l = ? $\theta = 32^o = 32 \text{ x } \frac{\pi}{180}$ rad

$l = 620 \text{ x } 32 \text{ x } \frac{\pi}{180} = 346.41$ m

Example 6:

A railway Train is traveling on a curve of half a kilometer radius at the rate of 20 km per hour through what angle had it turned in 10 seconds?

Solution:

Radius = r = $\frac{1}{2}$ km, θ = ?

We know s = vt

v = velocity of Train = 20km/hour = $\frac{20}{3600}$ km/sec.

$v = \frac{1}{180}$ km/sec

l = Distance traveled by train in 10 seconds = $\frac{1}{180}$ x10 km/sec

$l = \frac{1}{18}$

Since $l = r\theta$

$\Rightarrow \quad \frac{1}{18} = \frac{1}{2}\theta$

$\theta = \frac{2}{18} = \frac{1}{9}$ rad

Example7

The moon subtends an angle of 0.5^o as observed from the Earth. Its distance from the earth is 384400 km. Find the length of the diameter of the Moon.

Solution:

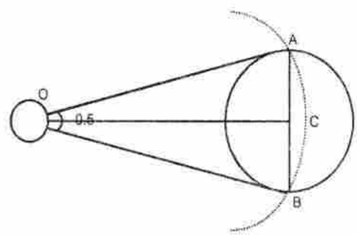


Fig. 4.8

l = AB = diameter of the Moon = ?

as angle 0.5 is very small.

i.e. AB (arc length) consider as a straight line AB

θ = $0.5^o = 0.5 \times 0.01745$ rad = 0.008725 rad

r = OC = d = distance between the earth and the moon

r = OC = 3844000 km

Since $l = r\theta$

l= 384400 x 0.008725 = 3353.89 km

Exercise 5.1

1. Convert the following to Radian measure

   (i) 210^o (ii) 540^o (iii) $42^o\ 36'\ 12''$ (iv) $24^o32'30''$

2. Convert the following to degree measure:

   (i) $\frac{5\pi}{4}$ rad (ii) $\frac{2\pi}{3}$ rad (iii) 5.52 rad (iv) 1.30 rad

3. Find the missing element l, r, θ when:

   (i) $l = 8.4$ cm, $\theta = 2.8$ rad

   (ii) $l = 12.2$ cm, r = 5cm

   (iii) r = 620m, $\theta = 32^o$

4. How far a part are two cities on the equator whose longitudes are 10^oE and 50^oW? (Radius of the Earth is 6400km)

5. A space man land on the moon and observes that the Earth´s diameter subtends an angle of $1^o\ 54'$ at his place of landing. If the Earth´s radius is 6400km, find the distance between the Earth and the Moon.
6. The sun is about 1.496×10^8 km away from the Earth. If the angle subtended by the sun on the surface of the earth is 9.3×10^{-3} radians approximately. What is the diameter of the sun?
7. A horse moves in a circle, at one end of a rope 27cm long, the other end being fixed. How far does the horse move when the rope traces an angle of 70^o at the centre.
8. Lahore is 68km from Gujranwala. Find the angle subtended at the centre of the earth by the road. Joining these two cities, earth being regarded as a sphere of 6400km radius.
9. A circular wire of radius 6 cm is cut straightened and then bend so as to lie along the circumference of a hoop of radius 24 cm. find measure of the angle which it subtend at the centre of the hoop
10. A pendulum 5 meters long swings through an angle of 4.5^0. through what distance does the bob moves ?
11. A flywheel rotates at 300 rev/min. If the radius is 6 cm. through what total distance does a point on the rim travel in 30 seconds ?

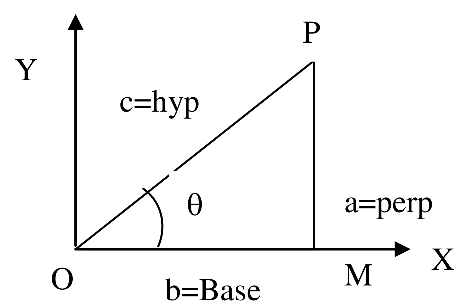
Answers 5.1

1	(i) 3.66 rad	(ii) 3π	(iii) 0.74 rad	(iv) 0.42 rad
2.	(i) 225^o	(ii) 120^o	(iii) $316^o\ 16'\ 19''$	(iv) $74^o\ 29'\ 4''$
3.	(i) r = 3cm	(ii) $\theta = 2.443$ rad	(iii) $l = 346.4$ meters	
4.	6704.76 km	5. 386240 km		
6.	1.39×10^6 km	7. 33 m	8. $36'\ 43''$	9. $\pi/2$ or 90^0
10.	0.39 m	11. 5657 cm		

5.8 Trigonometric Function and Ratios:

Let the initial line OX revolves and trace out an angle θ. Take a point P on the final line. Draw perpendicular PM from P on OX:
$\angle XOP = \theta$, where θ may be in degree or radians.
Now OMP is a right angled triangle,
We can form the six ratios as follows:

$$\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{b}{a}, \frac{c}{b}, \frac{c}{a}$$



In fact these ratios depend only on the size of the angle and not on the triangle formed. Therefore these ratios called **Trigonometic ratios or**

trigonometric functions of angle θ Fig. 5.10

and defined as below: θ

$$\textbf{Sin } \theta = \frac{a}{c} = \frac{MP}{OP} = \frac{\text{Perpendicular}}{\text{Hypotenuse}}$$

$$\text{Cos } \theta = \frac{b}{c} = \frac{OM}{OP} = \frac{\text{Base}}{\text{Hypotenuse}}$$

$$\tan \theta = \frac{a}{b} = \frac{MP}{OM} = \frac{\text{Perpendicular}}{\text{Base}}$$

$$\text{Cot } \theta = \frac{b}{a} = \frac{OM}{MP} = \frac{\text{Base}}{\text{Perpendicular}}$$

$$\text{Sec } \theta = \frac{c}{b} = \frac{OP}{OM} = \frac{\text{Hypotenuse}}{\text{Perpendicular}}$$

$$\text{Cosec } \theta = \frac{c}{a} = \frac{OP}{PM} = \frac{\text{Hyponenuse}}{\text{Perpendicular}}$$

5.9 Reciprocal Functions:

From the above definition of trigonometric functions, we observe that

(i) $\text{Sin } \theta = \dfrac{1}{\text{Cosec } \theta}$ or, $\text{Cosec } \theta = \dfrac{1}{\text{Sin } \theta}$ i.e. Sin θ and Cosec θ are reciprocal of each other.

(ii) $\text{Cos } \theta = \dfrac{1}{\text{Sec } \theta}$ or, $\text{Sec } \theta = \dfrac{1}{\text{Cos } \theta}$ i.e. Cos θ and Sec θ are reciprocals of each other.

(iii) $\tan \theta = \dfrac{1}{\text{Cot } \theta}$ or, $\text{Cot } \theta = \dfrac{1}{\tan \theta}$ i.e. tan θ and Cot θ are reciprocals of each other.

We can also see that;

$$\tan \theta = \frac{\text{Sin } \theta}{\text{Cos } \theta} \quad \text{and} \quad \text{Cot } \theta = \frac{\text{Cos } \theta}{\text{Sin } \theta}$$

5.10 Rectangular Co-ordinates and Sign Convention:

In plane geometry the position of a point can be fixed by measuring its perpendicular distance from each of two perpendicular called co-ordinate axes. The horizontal line (x-axis) is also called abscissa and the vertical line(y-axis) is called as ordinate.

Distance measured from the point O in the direction OX and OY are regarded as positive, while in the direction of OX′ and OY′ are considered negative.

Thus in the given figure OM_1, OM_4, MP_1 and M_2P_2 are positive, while OM_2, OM_3, M_3P_3 and M_4P_4 are negative.

The terminal line i.e., OP_1, OP_2, OP_3, and OP_4 are positive in all the quadrants.

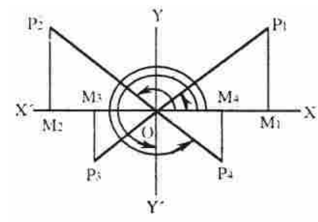
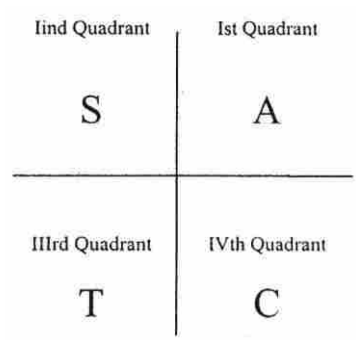


Fig. 4.11

5.11 Signs of Trigonometric Functions:

The trigonometric ratios discussed above have different signs in different quadrants. Also from the above discussion we see that OM and MP changes their sign in different quadrants. We can remember the sign of trigonometric function by "ACTS" Rule or CAST rule. In "CAST" C stands for cosine A stands for All and S stands for Sine and T stands for Tangent.



First Quadrant:

In first quadrant sign of all the trigonometric functions are positive i.e., sin, cos, tan, Cot, Sec, Cosec all are positive.

Second Quadrant:

In second quadrant sine and its inverse cosec are positive. The remaining four trigonometric function i.e., cos, tan, cot, sec are negative.

Third Quadrant:

In third quadrant tan and its reciprocal cot are positive the remaining four function i.e., Sin, cos, sec and cosec are negative.

Fourth Quadrant:

In fourth quadrant cos and its reciprocal sec are positive, the remaining four functions i.e., sin, tan, cot and cosec are negative.

5.12 Trigonometric Ratios of Particular Angles:

1. Trigonometric Ratios of 30^o or $\frac{\pi}{6}$:

Let the initial line OX revolve and trace out an angle of 30^o. Take a point P on the final line. Draw PQ perpendicular from P on OX. In 30^o right angled triangle, the side opposite to the 30^o angle is one-half the length of the hypotenuse, i.e., if PQ = 1 unit then OP will be 2 units.

From fig. OPQ is a right angled triangle

$\therefore$ By Pythagorean theorem, we have

$(OP)^2 = (OQ)^2 + (PQ)^2$

$(2)^2 = (OQ)^2 + (1)^2$

$4 = (OQ)^2 + 1$

$(OQ)^2 = 3$

$(OQ) = \sqrt{3}$

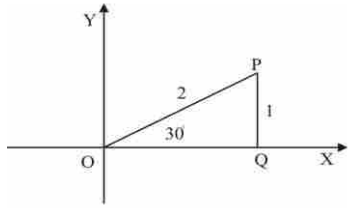


Fig. 4.12

Therefore $\quad \text{Sin } 30^o = \dfrac{\text{Prep.}}{\text{Hyp}} = \dfrac{PQ}{OP} = \dfrac{1}{2}$

$$\text{Cos } 30^o = \frac{\text{Base}}{\text{Hyp}} = \frac{OQ}{OP} = \frac{\sqrt{3}}{2}$$

$$\tan 30^o = \frac{\text{Prep.}}{\text{Base}} = \frac{PQ}{OQ} = \frac{1}{\sqrt{3}}$$

$$\text{Cot } 30^o = \frac{\text{Base}}{\text{Prep.}} = \frac{OQ}{PQ} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\text{Sec } 30^o = \frac{\text{Hyp.}}{\text{Base}} = \frac{OP}{OQ} = \frac{2}{\sqrt{3}}$$

$$\text{Cosec } 30^o = \frac{\text{Hyp.}}{\text{Prep.}} = \frac{OP}{PQ} = \frac{2}{1} = 2$$

2. Trigonometric ratios of 45^o Or $\dfrac{\pi}{4}$

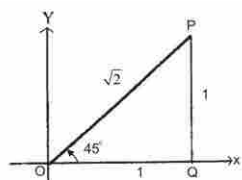


Fig. 4.13

Let the initial line OX revolve and trace out an angle of 45^o. Take a point P on the final line. Draw PQ perpendicular from P on OX . In 45^o right angled triangle the length of the perpendicular is equal to the length of the base

i.e., if PQ = 1 unit. then OQ = 1 unit

From figure by Pythagorean theorem.

$(OP)^2 = (OQ)^2 + (PQ)^2$

$(OP)^2 = (1)^2 + (1)^2 = 1 + 1 = 2$

$OP = \sqrt{2}$

Therefore $\quad \text{Sin } 45^o = \dfrac{\text{Prep.}}{\text{Hyp}} = \dfrac{PQ}{OP} = \dfrac{1}{\sqrt{2}}$

$$\text{Cos } 45^o = \frac{\text{Base}}{\text{Hyp}} = \frac{OQ}{OP} = \frac{1}{\sqrt{2}}$$

$$\tan 45^o = \frac{\text{Prep.}}{\text{Base}} = \frac{PQ}{OQ} = \frac{1}{1} = 1$$

$$\text{Cot } 45^o = \frac{\text{Base}}{\text{Prep.}} = \frac{OQ}{PQ} = \frac{1}{1} = 1$$

$$\text{Sec } 45^o = \frac{\text{Hyp.}}{\text{Base}} = \frac{OP}{OQ} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\text{Cosec } 45^o = \frac{\text{Hyp.}}{\text{Prep.}} = \frac{OP}{PQ} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

3. Trigonometric Ratios of 60^o or $\frac{\pi}{3}$

Let the initial line OX revolve and trace out an angle of 60^o. Take a point P on the final line. Draw PQ perpendicular from P on OX. In 60^o right angle triangle the length of the base is one-half of the Hypotenuse.

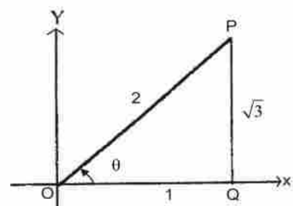


Fig. 4.14

i.e., OQ = Base = 1 unit

then, OP= Hyp = 2 units

from figure by Pythagorean Theorem:

$$(OP)^2 = (OQ)^2 + (PQ)^2$$
$$(2)^2 = (1)^2 + (PQ)^2$$
$$4 = 1\ (PQ)^2$$
$$(PQ)^2 = 3$$
$$PQ = \sqrt{3}$$

Therefore $$\text{Sin } 60^o = \frac{\text{Prep.}}{\text{Hyp}} = \frac{PQ}{OP} = \frac{\sqrt{3}}{2}$$

$$\text{Cos } 60^o = \frac{\text{Base}}{\text{Hyp}} = \frac{OQ}{OP} = \frac{1}{2}$$

$$\tan 60^o = \frac{\text{Prep.}}{\text{Base}} = \frac{PQ}{OQ} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\text{Cot } 60^o = \frac{\text{Base}}{\text{Prep.}} = \frac{OQ}{PQ} = \frac{1}{\sqrt{3}}$$

$$\text{Sec } 60^o = \frac{\text{Hyp.}}{\text{Base}} = \frac{OP}{OQ} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\text{Cosec } 60^o = \frac{\text{Hyp.}}{\text{Prep.}} = \frac{OP}{PQ} = \frac{2}{\sqrt{3}} =$$

Trigonometric ratios of 0^o

Let the initial line revolve and trace out a small angle nearly equal to zero 0^o. Take a point P on the final line. Draw PM perpen-dicular on OX.

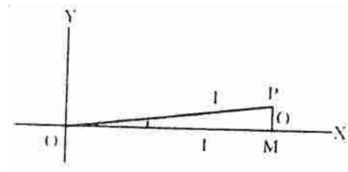


Fig. 4.15

PM = 0

and OP = 1, OM = 1

(Because they just coincide x-axis)

Therefore from figure.

$$\text{Sin } 0^o = \frac{\text{Prep.}}{\text{Hyp}} = \frac{PM}{OP} = \frac{0}{1} = 0$$

$$\text{Cos } 0^o = \frac{\text{Base}}{\text{Hyp}} = \frac{OM}{OP} = \frac{1}{1} = 1$$

$$\tan 0^o = \frac{\text{Prep.}}{\text{Base}} = \frac{PM}{OM} = \frac{0}{1} = 0$$

$$\text{Cot } 0^o = \frac{\text{Base}}{\text{Prep.}} = \frac{OM}{PM} = \frac{1}{0} = \infty$$

$$\text{Sec } 0^o = \frac{\text{Hyp.}}{\text{Base}} = \frac{OP}{OM} = \frac{1}{1} = 1$$

$$\text{Cosec } 0^o = \frac{\text{Hyp.}}{\text{Prep.}} = \frac{OP}{PM} = \frac{1}{0} = \infty$$

Trigonometric Ratio of 90^o

Let initial line revolve and trace out an angle nearly equal to 90^o. Take a point P on the final line. Draw PQ perpendicular from P on OX.

OQ = 0, OP = 1, PQ = 1 (Because they just coincide y-axis).

Therefore $\text{Sin } 90^o = \frac{\text{Prep.}}{\text{Hyp}} = \frac{PQ}{OP} = \frac{1}{1} = 1$

$$\text{Cos } 90^o = \frac{\text{Base}}{\text{Hyp}} = \frac{OQ}{OP} = \frac{0}{1} = 0$$

$$\tan 90^o = \frac{\text{Prep.}}{\text{Base}} = \frac{PQ}{OQ} = \frac{1}{0} = \infty$$

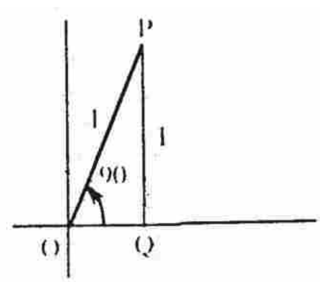


Fig. 4.16

$$\text{Cot } 90^o = \frac{\text{Base}}{\text{Prep.}} = \frac{OQ}{PQ} = \frac{0}{1} = 0$$

Fig. 5.16

$$\text{Sec } 90^o = \frac{\text{Hyp.}}{\text{Base}} = \frac{OP}{OQ} = \frac{1}{0} = \infty$$

$$\text{Cosec } 90^o = \frac{\text{Hyp.}}{\text{Prep.}} = \frac{OP}{PQ} = \frac{1}{1} = 1$$

Table for Trigonometrical Ratios of Special angle

$\frac{\text{Angles}}{\text{Ratios}}$	0^o	30^o Or $\frac{\pi}{6}$	45^o Or $\frac{\pi}{4}$	60^o Or $\frac{\pi}{3}$	90^o Or $\frac{\pi}{2}$
Sin θ	$\sqrt{\frac{0}{4}} = 0$	$\sqrt{\frac{1}{4}} = \frac{1}{2}$	$\sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$	$\sqrt{\frac{4}{4}} = 1$
Cos θ	$\sqrt{\frac{4}{4}} = 1$	$\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$	$\sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$	$\sqrt{\frac{1}{4}} = \frac{1}{2}$	$\sqrt{\frac{0}{4}} = 0$
Tan θ	$\sqrt{\frac{0}{4}} = 0$	$\sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$	$\sqrt{\frac{2}{2}} = 1$	$\sqrt{\frac{3}{1}} = \sqrt{3}$	$\sqrt{\frac{4}{0}} = \infty$

Example 1:

If $\cos\,\theta = \frac{5}{13}$ and the terminal side of the angle lies in the first quadrant find the values of the other five trigonometric ratio of θ.

Solution:

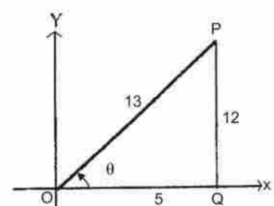


Fig. 4.17

In this cause $\cos\,\theta = \frac{5}{13}$

$$\cos\,\theta = \frac{\text{Base}}{\text{Hyp}} = \frac{5}{13}$$

From Fig. $(OP)^2 = (OQ)^2 + (PQ)^2$

$$(13)^2 = (5)^2 + (PQ)^2$$
$$169 = 25 + (PQ)^2$$
$$(PQ)^2 = 169 - 25$$
$$= 144$$
$$PQ = \pm 12$$

Because θ lies in the first quadrant

i.e., $\sin\,\theta = \frac{12}{13}$ $\because$ All the trigonometric ratios will be positive.

$$\text{Cos } \theta = \frac{5}{13}$$

$$\tan \theta = \frac{12}{5}, \qquad \sec \theta = \frac{13}{5}$$

$$\cot \theta = \frac{5}{12}, \qquad \text{cosec } \theta = \frac{13}{12}$$

Example 2:

Prove that $\cos 90^o - \cos 30^o = -2 \sin 60^o \sin 30^o$

Solution:

$$\text{L.H.S} = \cos 90^o - \cos 30^o$$

$$= 0 - \frac{\sqrt{3}}{2}$$

$$\text{L.H.S} = -\frac{\sqrt{3}}{2}$$

$$\text{R.H.S} = -2 \sin 60^o \sin 30^o$$

$$= -2.\frac{\sqrt{3}}{2} \text{ x } \frac{1}{2}$$

$$\text{R.H.S} = -\frac{\sqrt{3}}{2}$$

Hence L.H.S = R.H.S

Example 3:

Verify that $\sin^2 30^o + \sin^2 60^o + \tan^2 45^o = 2$

Solution:

$$\text{L.H.S} = \sin^2 30^o + \sin^2 60^o + \tan^2 45^o$$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + (1)^2$$

$$= \frac{1}{4} + \frac{3}{4} + 1$$

$$= \frac{1+3+4}{4}$$

$$= \frac{8}{4}$$

$$\text{L.H.S} = 2 = \text{R.H.S}$$

Exercise 5.2

Q.1 If $\sin\theta = \frac{2}{3}$, and the terminal side of the angle lies in the second quadrant, find the remaining trigonometric ratios of θ.

Q.2 If $\sin\theta = \frac{3}{8}$, and the terminal side of the angle lies in the second quadrant, find the remaining trigonometric ratios.

Q.3 If $\cos\theta = -\frac{\sqrt{3}}{2}$, and the terminal side of the angle lies in the third quadrant, find the remaining trigonometric ratios of θ.

Q.4 If $\tan\theta = \frac{3}{4}$, and the terminal side of the angle lies in the third quadrant, find the remaining trigonometric ratios of θ.

Q.5 If $\tan\theta = -\frac{1}{3}$, and the terminal side of the angle lies in the second quadrant, find the remaining trigonometric ratios of θ.

Q.6 If $\cot\theta = \frac{4}{3}$, and the terminal side of the angle is not in the first quadrant, find the trigonometric ratios of 0.

Q.7 If $\cot\theta = \frac{2}{3}$, and the terminal side of the angle does not lies in the first quadrant, find the trigonometric ratios of θ.

Q.8 If $\sin\theta = \frac{4}{5}$,and $\frac{\pi}{2} < \theta < \pi$ find the trigonometric ratios of θ

Q.9 If $\sin\theta = \frac{7}{25}$, find $\cos\theta$, if angle θ is an acute angle.

Q.10 If $\sin\theta = \frac{5}{6}$, find $\cos\theta$, if angle θ is an obtuse angle.

Q.11 Prove that:

(i) $\sin\frac{\pi}{3}\cos\frac{\pi}{6} + \cos\frac{\pi}{3}\sin\frac{\pi}{6} = \sin\frac{\pi}{2}$

(ii) $4\tan 60^\circ \tan 30^\circ \tan 45^\circ \sin 30^\circ \cos 60^\circ = 1$

(iii) $2\sin 45^\circ + \frac{1}{2}\operatorname{cosec} 45^\circ = \frac{3}{\sqrt{2}}$

(iv) $\cos 90^\circ - \cos 30^\circ = -2\sin 60^\circ \sin 30^\circ$

(v) $\sin^2\frac{\pi}{6}+\sin^2\frac{\pi}{3}+\tan^2\frac{\pi}{4}=2$

Q.12 $\sin^2\frac{\pi}{6}:\sin^2\frac{\pi}{4}:\sin^2\frac{\pi}{3}:\sin^2\frac{\pi}{2}=1:2:3:4$

Q.13 Evaluate

(i) $\cos 30^o \cos 60^o - \sin 30^o \sin 60^o$

(ii) $\dfrac{\tan 60^o - \tan 30^o}{1+\tan 60^o \tan 30^o}$

Answers 5.2

Q.1 Sin $\theta = \frac{2}{3}$ Cot $\theta = -\frac{\sqrt{5}}{2}$

$COS\theta = -\frac{\sqrt{5}}{3}$ sec θ = - $\frac{3}{\sqrt{5}}$

tan $\theta = -\frac{2}{\sqrt{5}}$ Cosec $\theta = \frac{3}{2}$

Q.2 Sin $\theta = \frac{3}{8}$ Cot $\theta = -\frac{\sqrt{55}}{3}$

Cos $\theta = -\frac{\sqrt{55}}{8}$ Sec $\theta = -\frac{8}{\sqrt{55}}$

tan $\theta = -\frac{3}{\sqrt{55}}$ Cosec $\theta = \frac{8}{3}$

Q.3 Sin $\theta = \frac{-1}{2}$ Cot $\theta = \sqrt{3}$

Cos $\theta = -\frac{\sqrt{3}}{2}$ Sec $\theta = -\frac{2}{\sqrt{3}}$

tan $\theta = \frac{1}{\sqrt{3}}$ Cosec $\theta = -2$

Q.4 Sin $\theta = -\frac{3}{5}$ Cot $\theta = \frac{4}{3}$

Cos $\theta = -\frac{4}{5}$ Sec $\theta = -\frac{5}{4}$

tan $\theta = \frac{3}{4}$ Cosec $\theta = -\frac{5}{3}$

Q.5 $\text{Sin } \theta = \frac{1}{\sqrt{10}}$ $\text{Cot } \theta = -3$

$\text{Cos } \theta = -\frac{3}{\sqrt{10}}$ $\text{Sec } \theta = -\frac{\sqrt{10}}{3}$

$\tan \theta = -\frac{1}{3}$ $\text{Cosec } \theta = \sqrt{10}$

Q.6 $\text{Sin } \theta = -\frac{3}{5}$ $\text{Cot } \theta = \frac{4}{3}$

$\text{Cos } \theta = -\frac{4}{5}$ $\text{Sec } \theta = -\frac{5}{4}$

$\tan \theta = \frac{3}{4}$ $\text{Cosec } \theta = -\frac{5}{3}$

Q.7 $\text{Sin } \theta = \frac{-3}{\sqrt{10}}$ $\text{Cot } \theta = \frac{2}{3}$

$\text{Cos } \theta = -\frac{2}{\sqrt{13}}$ $\text{Sec } \theta = -\frac{\sqrt{13}}{2}$

$\tan \theta = \frac{3}{2}$ $\text{Cosec } \theta = -\frac{\sqrt{13}}{3}$

Q.8 $\text{Sin } \theta = \frac{4}{5}$ $\text{Cot } \theta = -\frac{3}{4}$

$\text{Cos } \theta = -\frac{3}{4}$ $\text{Sec } \theta = -\frac{5}{3}$

$\tan \theta = -\frac{4}{3}$ $\text{Cosec } \theta = \frac{5}{4}$

Q.9 $\cos \theta = \frac{24}{25}$

Q.10 $\cos \theta - \frac{\sqrt{11}}{6}$

Q.13 (i) 0 (ii) $\frac{1}{\sqrt{3}}$

5.13 Fundamental Identities:

For any real number θ, we shall derive the following three fundamental identities

(i) $\text{Cos}^2\theta + \text{Sin}^2\theta = 1$
(ii) $\text{Sec}^2\theta = 1 + \tan^2\theta$
(iii) $\text{Cosec}^2\theta = 1 + \text{Cot}^2\theta$

Proof :

Consider an angle $\angle XOP = \theta$ in the standard position. Take a point P on the terminal line of the angle θ. Draw PQ perpendicular from P on OX.

From fig., ΔOPQ is a right angled triangle. By pythagoruse theorem

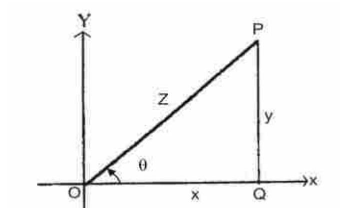


Fig. 4.18

$$(OP)^2 = (OQ)^2 + (PQ)^2$$

Or, $z^2 = x^2 + y^2$

(i) Dividing both sides by z^2

then $$\frac{z^2}{z^2} = \frac{x^2}{z^2} + \frac{y^2}{z^2}$$

$$1 = \left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2$$

$$1 = (\text{Cos}\,\theta)^2 + (\text{Sin}\,\theta)^2$$

$$1 = \text{Cos}^2\theta + \text{Sin}^2\theta$$

or, $$\text{Cos}^2\theta + \text{Sin}^2\theta = 1$$

(ii) Dividing both sides of Eq. (i) by x^2, we have

$$\frac{z^2}{x^2} = \frac{x^2}{x^2} + \frac{y^2}{x^2}$$

$$\left(\frac{z}{x}\right)^2 = 1 + \left(\frac{y}{x}\right)^2$$

$$(\text{Sec}\,\theta)^2 = 1 + (\tan\theta)^2$$

$$\text{Sec}^2\theta = 1 + \tan^2\theta$$

(iii) Again, dividing both sides of Eq (i) by y^2, we have

$$\frac{z^2}{y^2} = \frac{x^2}{y^2} + \frac{y^2}{y^2}$$

$$\left(\frac{z}{y}\right)^2 = \left(\frac{x}{y}\right)^2 + 1$$

$$(\text{Cosec } \theta)^2 = (\text{Cot } \theta)^2 + 1$$
$$\text{Cosec}^2 \theta = \text{Cot}^2 \theta + 1$$
$$\text{Cosec}^2 \theta = 1 + \text{Cot}^2 \theta$$

Example 1:

Prove that $\dfrac{\text{Sin x}}{\text{Cosec x}} + \dfrac{\text{Cos x}}{\text{sec x}} = 1$

Solution:

$$\text{L.S.H.} = \frac{\text{Sin x}}{\text{Cosec x}} + \frac{\text{Cos x}}{\text{sec x}}$$
$$= \text{Sin x}.\frac{1}{\text{Cosec x}} + \text{Cos x}.\frac{1}{\text{sec x}} \because \quad \frac{1}{\text{Cosec x}} = \text{Sin x}$$
$$= \text{Sin x. Sin x} + \text{Cos x. Cos x} \quad \because \quad \frac{1}{\text{Sec x}} = \text{Cos x}$$
$$= \text{Sin}^2 \text{ x} + \text{Cos}^2 \text{ x}$$
$$= 1$$
$$= \text{R.H.S}$$

Example 2:

Prove that $\dfrac{\text{Sec x} - \text{Cos x}}{1 + \text{Cos x}} = \text{Sec x} - 1$

Solution:

$$\text{L.H.S} = \frac{\text{Sec x} - \text{Cos x}}{1 + \text{Cos x}}$$
$$= \frac{\frac{1}{\text{Cos x}} - \text{Cos x}}{1 + \text{Cos x}}$$
$$= \frac{\frac{1 - \text{Cos}^2 \text{ x}}{\text{Cos x}}}{1 + \text{Cos x}} = \frac{1 - \text{Cos}^2 \text{ x}}{\text{Cos x}(1 + \text{Cos x})}$$
$$= \frac{(1 - \text{Cos x})(1 + \text{Cos x})}{\text{Cos x}(1 + \text{Cos x})}$$
$$= \frac{1 - \text{Cos x}}{\text{Cos x}} = \frac{1}{\text{Cos x}} - \frac{\text{Cos x}}{\text{Cos x}}$$
$$= \text{Sex x} - 1 = \text{R.H.S.}$$

Example 3:

prove that $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$

Solution: L.H.S. $= \sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}$

$$= \sqrt{\frac{(1-\sin\theta)(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}}$$

$$= \sqrt{\frac{(1-\sin\theta)^2}{1-\sin^2\theta}} \qquad = \sqrt{\frac{(1-\sin\theta)^2}{\cos^2\theta}}$$

$$= \frac{(1-\sin\theta)}{\cos\theta} \qquad = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta - \tan\theta \qquad = \text{R.H.S.}$$

Exercise 5.3

Prove the following Identities:

Q.1 $1 - 2\,\text{Sin}^2\,\theta = 2\,\text{Cos}^2\,\theta - 1$

Q.2 $\text{Cos}^4\,\theta - \text{Sin}^4\,\theta = 1 - 2\,\text{Sin}^2\theta$

Q.3 $\dfrac{1}{\text{Cosec}^2\,\theta} + \dfrac{1}{\text{Sec}^2\,\theta} = 1$

Q.4 $\dfrac{1}{\tan\theta + \text{Cot}\,\theta} = \text{Sin}\,\theta.\text{Cos}\,\theta$

Q.5 $(\text{Sec}\,\theta - \tan\theta)^2 = \dfrac{1-\text{Sin}\,\theta}{1+\text{Sin}\,\theta}$

Q.6 $(\text{Cosec}\,\theta - \text{Cot}\,\theta)^2 = \dfrac{1-\text{Cos}\,\theta}{1+\text{Cos}\,\theta}$

Q.7 $(1-\text{Sin}^2\theta)(1+\tan^2\theta)=1$

Q.8 $\dfrac{1}{1+\text{Sin}\,\theta} + \dfrac{1}{1-\text{Sin}\,\theta} = 2\text{Sec}^2\theta$

Q.9 $\sqrt{\dfrac{1-\text{Sin}\,\theta}{1+\text{Sin}\,\theta}} = \text{Sec}\theta - \tan\theta$

Q.10 $\sqrt{\dfrac{1+\text{Cos}\,\theta}{1-\text{Cos}\,\theta}} = \text{Cosec}\,\theta + \cot\theta$

Q.11 $\dfrac{1-\tan A}{1+\tan A}=\dfrac{\text{Cot } A-1}{\text{Cot } A+1}$

Q.12 $\dfrac{\text{Cot}^2\theta-1}{\text{Cot}^2\theta+1}=2\text{Cos}^2\theta-1$

Q.13 $\dfrac{\tan\theta}{1-\text{Cot }\theta}+\dfrac{\text{Cot }\theta}{1-\tan\theta}=\text{Sec }\theta\ \text{Cosec }\theta+1$

Q.14 $\dfrac{\text{Sec }\theta-\tan\theta}{\text{Sec }\theta+\tan\theta}=1-2\text{Sec }\theta\tan\theta+2\tan^2\theta$

Q.15 $\dfrac{1+\tan^2\theta}{1+\cot^2\theta}=\dfrac{(1-\tan\theta)^2}{(1-\cot\theta)^2}$

Q.16 $\text{Cosec } A+\text{Cot } A=\dfrac{1}{\text{Cosec } A-\text{Cot } A}$

Q.17 $\dfrac{1}{\text{Sec }\theta+\tan\theta}=\dfrac{1-\text{Sin }\theta}{\text{Cos }\theta}=\text{Sec } x-\tan x$

Q.18 $(1-\tan\theta)^2+(1-\text{Cot }\theta)^2=(\text{Sec }\theta-\text{Cosec }\theta)^2$

Q.19 $\dfrac{\text{Cos}^3 t-\text{Sin}^3 t}{\text{Cos } t-\text{Sin } t}=1+\text{Sin } t\ \text{Cos } t$

Q.20 $\text{Sec}^2 A+\tan^2 A=(1-\text{Sin}^4 A)\ \text{Sec}^4 A$

Q.21 $\dfrac{\text{Sec } x-\text{Cos } x}{1+\text{Cos } x}=\text{Sec } x-1$

Q.22 $\dfrac{1+\text{Sin }\theta+\text{Cos }\theta}{1+\text{Sin }\theta-\text{Cos }\theta}=\dfrac{\text{Sin }\theta}{1-\text{Cos }\theta}$

Q.23 $\dfrac{\text{Sin } x+\text{Cos } x}{\tan^2 x-1}=\dfrac{\text{Cos}^2 x}{\text{Sin } x-\text{Cos } x}$

Q.24 $(1+\text{Sin }\theta)(1-\text{Sin }\theta)=\dfrac{1}{\text{Sec}^2\theta}$

Q.25 $\dfrac{\tan\theta}{\text{Sec }\theta-1}+\dfrac{\tan\theta}{\text{Sec }\theta+1}=2\text{Cosec }\theta$

Q.26 $\dfrac{\cot\theta\cos\theta}{\cot\theta+\cos\theta}=\dfrac{\cot\theta-\cos\theta}{\cot\theta\cos\theta}$

Q.27 If $m=\tan\theta+\text{Sin }\theta$ and $n=\tan\theta-\text{Sin }\theta$ than prove that $m^2-n^2=4\sqrt{m\,n}$

5.14 Graph of Trigonometric Functions:

In order to graph a function = f(x), we give number of values of x and obtain the corresponding values of y. The several ordered pairs (x, y) are obtained we plotted these points by a curve we get the required graph.

5.14.1 Graph of Sine Let, y = Sinx where, $0^o \le x \le 360^o$

Or where, $0 \le x \le 2\pi$

1. Variations

Quadrants

	1st	2nd	3^{rd}	4^{th}
x Sinx	0 to 90^o + ve, Increase from 0 to 1	90^o to 180^o + ve, decrease from 1 to 0	180^o to 270^o $-$ve, decrease from 0 to -1	270^o to 360^o $-$ve, Increases from -1 to 0

2. Table:

x	0	30^o	60^o	90^o	120^o	150^o	180^o
Sinx	0	0.5	0.87	1	0.87	0.5	0

x	210^o	240^o	270^o	300^o	330^o	360^o
Sinx	-0.50	$-.87$	-1	$-.87$	$-.5$	0

3. Graph in Figure (5.19):

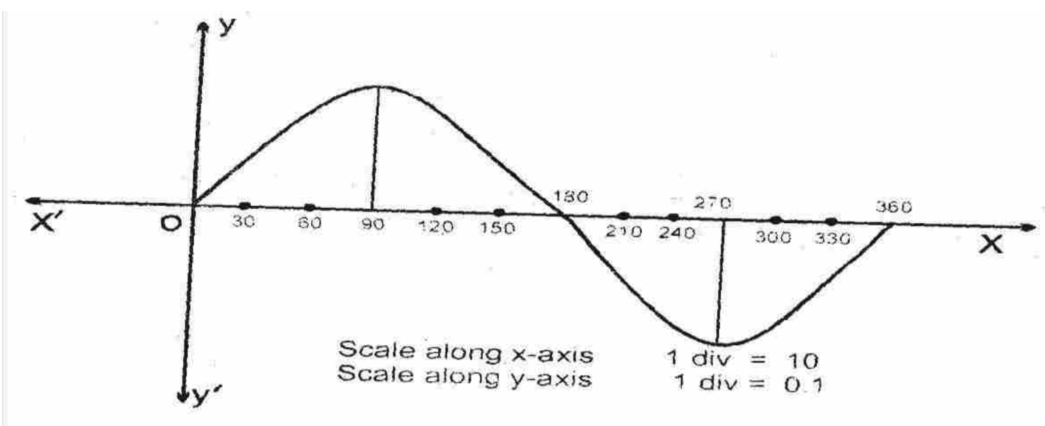


Fig. 4.19

5.14.2 Graph of Cosine

Let, y = Cosx where, $0^o \le x \le 360^o$

Or where, $0 \le x \le 2\pi$

1. Variations

Quadrants

	1st	2nd	3^{rd}	4^{th}
x y = Cosx	0 to 90^o + ve, decrease from 1 to 0	90^o to 180^o $-$ ve, decrease from 0 to -1	180^o to 270^o $-$ve, increase from -1 to 0	270^o to 360^o + ve, increases from 0 to 1

2. Table:

x	0	30^o	60^o	90^o	120^o	150^o	180^o
y = Cosx	1	0.87	0.5	0	-0.5	-0.87	-1

x	210^o	240^o	270^o	300^o	330^o	360^o
y = Cosx	-0.87	-0.5	0	0.5	0.87	1

3. Graph in Figure (5.20):

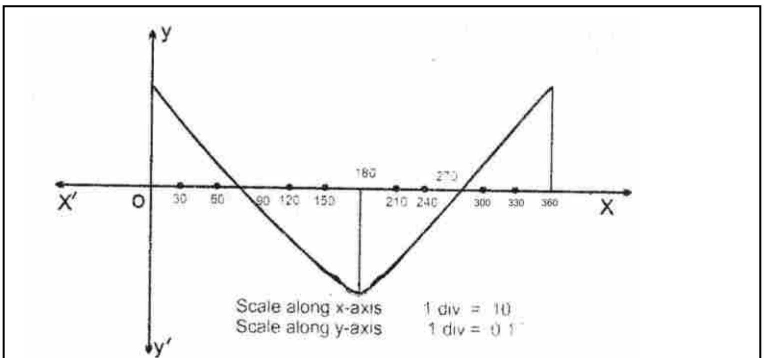


Fig. 4.20

Fig. 5.20

5.14.3 Graph of tanx

Let, y = tanx where, $0^o \le x \le 360^o$

Or where, $0 \le x \le 2\pi$

1. Variations

Quadrants

	1^{st}	2nd	3rd	4^{th}
x y = tanx	0 to 90^o + ve, Increase from 0 to $\propto$	90^o to 180^o $-$ve, increase from $-\propto$ to 0	180^o to 270^o + ve, increase from 0 to $\propto$	270^o to 360^o $-$ve, increases from $-\propto$ to 0

2. Table:

x	0	30^o	60^o	90^o	120^o	150^o	180^o
y = tanx	0	0.58	1.73	$\propto$	-1.73	-0.58	0

x	210^o	240^o	270^o	300^o	330^o	360^o
y = tanx	+ .58	1.73	$-\infty, +\infty$	-1.73	-2.58	0

3. Graph in Figure (5.21):

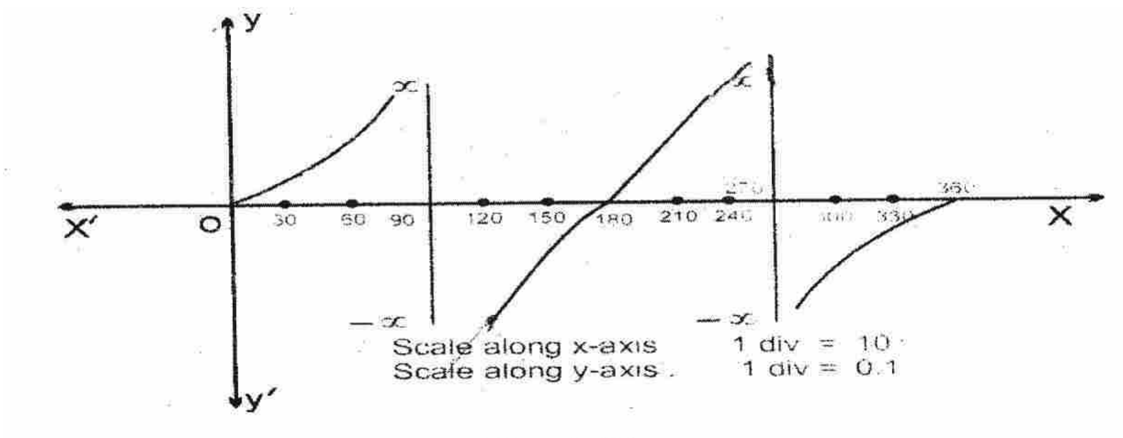


Fig. 5.21

5.14.4 Graph of Cotx:

Let, y = cotx where, $0^o \le x \le 360^o$

1. Variations

Quadrants

	1^{st}	2nd	3rd	4^{th}
x y = Cotx	0 to 90^o + ve, Increase from ∞ to 0	90^o to 180^o $-$ve, increase from 0 to $-\infty$	180^o to 270^o + ve, increase from ∞ to 0	270^o to 360^o $-$ve, increases from 0 to $-\infty$

2. Table:

x	0	30^o	60^o	90^o	120^o	150^o	180^o
y = Cotx	∞	1.73	0.58	0	-0.58	-1.73	$-\infty$

x	210^o	240^o	270^o	300^o	330^o	360^o
y = Cotx	1.73	0.58	0	-0.58	-1.73	∞

3. Graph in Figure (5.22):

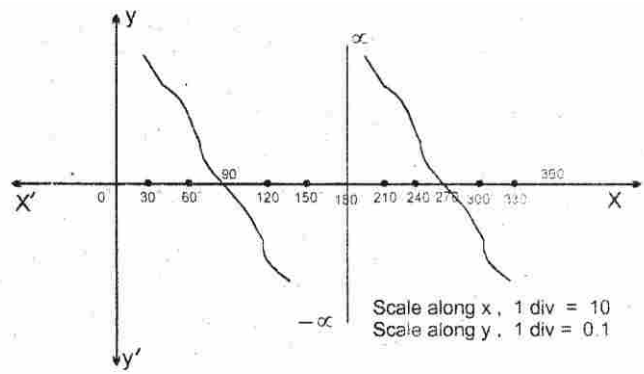


Fig. 4.22

5.14.5 Graph of Secx:

Let, y = secx where, $0^o \le x \le 360^o$

1. Variations

Quadrants

	1^{st}	2nd	3rd	4^{th}
x y = Secx	0 to 90^o + ve, Increase from 1 to $\propto$	90^o to 180^o $-$ve, increase from $-\propto$ to -1	180^o to 270^o $-$ve, increase from -1 to $-\propto$	270^o to 360^o + ve, increases from $-\propto$ to 1

2. Table:

x	0	30^o	60^o	90^o	120^o	150^o	180^o
y = Secx	1	1.15	2	$+\propto$	-2	1.15	1

x	210^o	240^o	270^o	300^o	330^o	360^o
y = Secx	-1.15	-2	$\propto$	2	1.15	1

3. Graph in Figure (5.23):

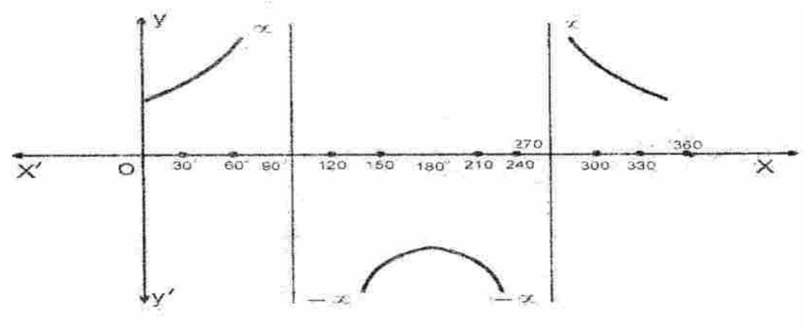


Fig. 4.23

5.14.6 Graph of Cosecx: Let, $y = \text{Cosec}x$

where, $0^o \le x \le 360^o$

1. Variations

Quadrants

	1^{st}	2nd	3rd	4^{th}
x y = Cosecx	0 to 90^o + ve, Increase from $\propto$ to 1	90^o to 180^o + ve, increase from 1 to $\propto$	180^o to 270^o $-$ve, increase from $-\propto$ to -1	270^o to 360^o $-$ ve, increases from -1 to $-\propto$

2. Table:

x	0	30^o	60^o	90^o	120^o	150^o	180^o
y = Cosecx	$\propto$	2	1.15	1	1.15	2	$\propto$

x	210^o	240^o	270^o	300^o	330^o	360^o
y = Cosecx	-2	-1.15	-1	-1.15	-2	$-\propto$

3. Graph in Figure (5.24):

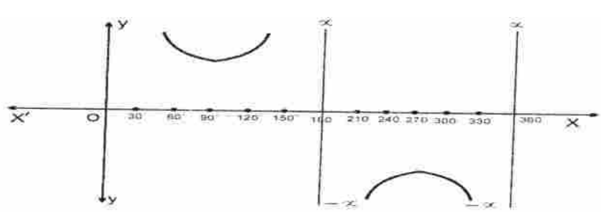


Fig. 4.24

Fig.5.24

Exercise 5.4

Q.1 Draw the graph of tan 2A as A varies from 0 to π .

Q.2 Plot the graph of 1 – Sin x as x varies from 0 to 2π .

Q.3 Draw the graphs for its complete period.

(i) $y = \frac{1}{2}\text{Sin } 2x$ (ii) $y = \text{Sin } 2x$

(iii) $y = \frac{1}{2}\text{Cos } 2x$

Summary

Trigonometry means measurement of triangles.

1. Radian is an angle subtended at the center of a circle by an arc of the circle equal in length to its radius.

   i.e. π Radian = 180 degree

   1 rad = $57^o\ 17'\ 45''$

   1 degree = 0.01745 radian

2. Length of arc of the circle, $l = s = r\theta$

3. Trigonometric functions are defined as:

$$\text{Sin}\theta = \frac{AP}{OP}, \text{Cosec}\theta = \frac{OP}{AP}$$

$$\text{Cos}\theta = \frac{OA}{OP}, \text{Sec}\theta = \frac{OP}{OA}$$

$$\tan\theta = \frac{AP}{OA}, \text{Cot}\theta = \frac{OA}{AP}$$

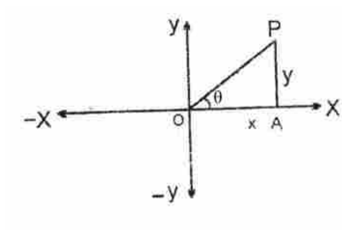


Fig. 4.25

4. Relation between trigonometric ratios:

(i) $\text{Sec}\theta = \frac{1}{\text{Cos}\theta}$ (ii) $\text{Cosec}\theta = \frac{1}{\text{Sin}\theta}$

(iii) $\text{Cot}\theta = \frac{1}{\tan\theta}$ (iv) $\text{Cos}\theta = \frac{1}{\text{Sec}\theta}$

(v) $\text{Sin}\theta = \frac{1}{\text{Cosec}\theta}$ (vi) $\tan\theta = \frac{1}{\text{Cot}\theta}$

(vii) $\text{Sin}^2\theta + \text{Cos}^2\theta = 1$ (viii) $\text{Sec}^2\theta = 1 + \tan^2\theta$

(ix) $\text{Cosec}^2\theta = 1 + \text{Cot}^2\theta$

5. Signs of the trigonometric functions in the Four Quadrants.

Quadrant	I	II	III	IV
Positive	All +ve	$\text{Sin}\,\theta$, $\text{Cosec}\,\theta$	$\tan\theta$, $\cot\theta$	$\text{Cos}\,\theta$, $\text{Sec}\,\theta$
Negative	Nil	$\text{Cos}\,\theta$ $\text{Sec}\,\theta$ $\tan\theta$ $\text{Cot}\,\theta$	$\text{Cos}\,\theta$ $\text{Sec}\,\theta$ $\text{Sin}\,\theta$ $\text{Cosec}\,\theta$	$\text{Sin}\,\theta$ $\text{Cosec}\,\theta$ $\tan\theta$ $\text{Cot}\,\theta$

Short Questions

Write the short answers of the following:

Q.1: Define degree and radians measure

Q.2: Convert into radius measure.

(a) 120^{o}, (b) $22\frac{1}{2}^{o}$, (c) $12^{o}\ 40'$, (d) $42^{o}\ 36'\ 12''$

Q.3: Convert into degree measure

(a) $\frac{\bar{\wedge}}{2}$ rad, (b) 0.726 rad. (c) $\frac{2\bar{\wedge}}{3}$ rad.

Q.4: Prove that $\ell = r\theta$

Q.5: What is the length of an arc of a circle of radius 5 cm whose central angle is 140^{o} ?

Q.6: Find the length of the equatorial arc subtending an angle 1^{o} at the centre of the earth taking the radius of earth as 6400 KM.

Q.7: Find the length of the arc cut off on a circle of radius 3 cm by a central angle of 2 radius.

Q.8: Find the radius of the circle when $\ell = 8.4$ cm, $\theta = 2.8$ rad

Q.9: If a minute hand of a clock is 10 cm long, how far does the tip of the hand move in 30 minutes?

Q.10 Find x, if $\tan^2 45^o - \cos^2 60^o = x \sin 45^o \cos 45^o . \tan 60^o$.

Q.11: Find r when $l = 33$ cm. $\theta = 6$ radian

Q12: Prove that $2\sin 45^o + \frac{1}{2}\ \text{cosec}\ 45^o = \frac{3}{\sqrt{2}}$

Q.13: Prove that $\tan^2 30^o + \tan^2 45^o + \tan^2 60^o = \frac{13}{3}$

Q.12: Prove that $\dfrac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}} = \sqrt{3}$

Q.13: prove that $\cos 30^o \, \text{Cos}\, 60^o - \sin 30^o \sin 60^o = 0$

Q.14: Prove that $\text{Cos}\, 90^o - \cos 30^o = -2 \sin 60^o \sin 30^o$

Q.15: Prove that $\text{Sin}^2\theta + \cos^2\theta = 1$

Q.16: Prove that: $1 + \tan^2\theta = \sec^2\theta$

Q.17: Prove that $1 + \cot^2\theta = \text{Cosec}^2\theta$

Q.18: Prove that: $(1 + \text{Sin}\,\theta)(1 - \text{Sin}\,\theta) = \dfrac{1}{\text{Sec}^2\theta}$

Q.19: Show that: $\text{Cot}^4\theta + \text{Cot}^2\theta = \text{Cosec}^4\theta - \text{cosec}^2\theta$

Q.20: Prove that: $\text{Cos}\theta + \tan\theta\, \text{Sin}\theta = \text{Sec}\,\theta$

Q.21: Prove that $1 - 2\,\text{Sin}^2\theta = 2\,\text{Cos}^2\theta - 1$

Q.22: $\text{Cos}^4\theta - \sin^4\theta = 1 - 2\sin^2\theta$

Q.23: $\dfrac{1}{1 + \text{Sin}\,\theta} + \dfrac{1}{1 - \text{SIn}\theta} = 2\,\text{Sec}2\theta$

Answers

2. (a) 2.09 rad (b) 0.39 rad (c) 0.22 rad (d) 0.74 radius

3. (a) 90^o (b) $41^0\, 35'\, 48''$ (c) 120 degree

5. 12.21 cm. 6. 111.7 Km 7. 6 cm

8. 3cm. 9. 31.4 cm 10. $\dfrac{\sqrt{3}}{2}$ 11. 5.5 cm.

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. One degree is equal to:

(a) π rad (b) $\frac{\pi}{180}$ rad

(c) $\frac{180}{\pi}$ rad (d) $\frac{\pi}{360}$ rad

__2. 15^o is equal to:

(a) $\frac{\pi}{6}$ rad (b) $\frac{\pi}{3}$ rad

(c) $\frac{\pi}{12}$ rad (d) $\frac{\pi}{15}$ rad

__3. 75^o is equal to

(a) $\frac{\pi}{12}$ rad (b) $\frac{2\pi}{3}$ rad

(c) $\frac{4\pi}{3}$ rad (d) $\frac{5\pi}{12}$ rad

__4. One radian is equal to:

(a) 90^o (b) $\left(\frac{90}{\pi}\right)^o$

(c) 180^o (d) $\left(\frac{180}{\pi}\right)^o$

__5. The degree measure of one radian is approximately equal to:

(a) 57.3 (b) 57.2

(c) 57.1 (d) 57.0

__6. $\frac{2\pi}{3}$ radians are equal to:

(a) 60^o (b) 90^o

(c) 120^o (d) 150^o

__7. The terminal side of θ lies in 4^{th} quadrant, sign of the $\sin\theta$ will be:

(a) Positive (b) Negative

(c) Both +ve and – ve (d) None of these

__8. The terminal side of θ lies in 4^{th} quadrant, both $\sin\theta$ and $\tan\theta$ are:

(a) $\sin\theta > 0, \tan\theta > 0$ (b) $\sin\theta > 0, \tan\theta < 0$

(c) $\sin\theta < 0, \tan\theta < 0$ (d) $\sin\theta < 0, \tan\theta > 0$

__9. A circle is equal to 2π rad and also to 360^o, then:

(a) $360^o = 2\pi$ rad (b) $360^o = \frac{3}{4}\pi$ rad

(c) $360^o = \frac{\pi}{6}$rad (d) None of a, b & c

__10. π rad is equal to:

(a) 360^o (b) 270^o

(c) 180^o (d) 90^o

__11. The relation between are l, radius r and central angle θ rad is:

(a) $l = \frac{\theta}{r}$ (b) $l = \frac{r}{\theta}$

(c) $l = r\,\theta$ (d) $l = r^2\,\theta$

__12. If l = 12cm and r = 3 cm, then θ is equal to:

(a) 36 rad (b) 4 rad

(c) $\frac{1}{4}$rad (d) 18 rad

__13. An angle subtended at the centre of a circle by an arc equal to the radius of the circle is called:

(a) Right angle (b) Degree

(c) Radian (d) Acute angle

__14. The radian measure of the angle described by a wheel in 5 revolution is:

(a) 5π (b) 10π

(c) 15π (d) 20π

__15. If an are of a circle has length l and subtends an angle θ, then radius ´r´ will be:

(a) $\frac{\theta}{l}$ (b) $\frac{l}{\theta}$

(c) $l\,\theta$ (d) $l + \theta$

__16. If sin x = $\frac{\sqrt{3}}{2}$ and the terminal ray of x lies in 1st quadrant, then cosx is equal to:

(a) $\frac{1}{\sqrt{2}}$ (b) $-\frac{1}{2}$

(c) $\frac{1}{2}$ (d) $-\frac{1}{\sqrt{2}}$

__17. If $\sin\theta = \frac{3}{5}$ and the terminal side of the angle lies in 2nd quadrant, then $\tan\theta$ is equal to:

(a) $\frac{4}{5}$ (b) $-\frac{4}{5}$ (c) $\frac{5}{4}$ (d) $-\frac{3}{4}$

__18. If $\sin\theta$ is +ve and $\cos\theta$ is –ve, then the terminal side of the angle lies in:

(a) 1st quad (b) 2nd quad (c) 3rd quad (d) 4th quad

__19. If $\sin\theta$ is +ve and $\tan\theta$ is-ve, then the terminal side of the angle lies in

(a) 1st quad (b) 2nd quad
(c) 3rd quad (4) 4th quad

__20. If $\sin\theta = \frac{2}{\sqrt{7}}$ and $\cos\theta = -\frac{1}{\sqrt{7}}$, then $\cot\theta$ is equal to:

(a) 2 (b) -1
(c) $-\frac{1}{2}$ (d) -2

21. $\sec^2\theta + \text{cosec}^2\theta$ is equal to:

(a) $\sec^2\theta\,\text{cosec}^2\theta$ (b) $\sin\theta\ \cos\theta$
(c) $2\sec^2\theta$ (d) $2\,\text{cosec}^2\theta$

Answers

1.	b	2.	c	3.	d	4.	d	5.	a
6.	c	7.	b	8.	c	9.	a	10.	c
11.	c	12.	b	13.	c	14.	b	15.	b
16.	c	17.	b	18.	b	19.	d	20.	c
21.	a								

Chapter 7
Solution of Triangles

7.1 Solution of Triangles:

A triangle has six parts in which three angles usually denoted by α, β, γ and the three sides opposite to α, β, γ denoted by a, b, c respectively. These are called the elements of Triangle. If any three out of six elements at least one side are given them the remaining three elements can be determined by the use of trigonometric functions and their tables.

This process of finding the elements of triangle is called the solution of the triangle.

First we discuss the solution of right angled triangles i.e. triangles which have one angle given equal to a right angle.

In solving right angled triangle γ denotes the right angle. We shall use the following cases

Case-I:

When the hypotenuse and one Side is given.

Let a & c be the given side and hypotenuse respectively. Then angle α can be found by the relation.

$$\text{Sin } \alpha = \frac{a}{c}$$

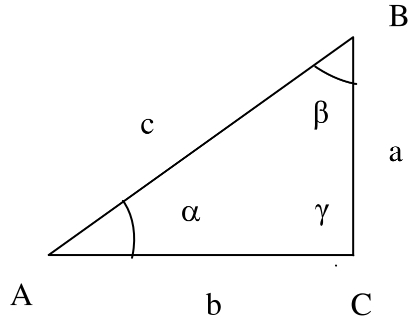


Fig.7.1

Also angle β and side "b" can be obtained by the relations

$\beta = 90^o - \alpha$ and $\text{Cos } \alpha = \frac{b}{c}$

Case-II:

When the two sides a and b are given. Here we use the following relations to find α , β & c.

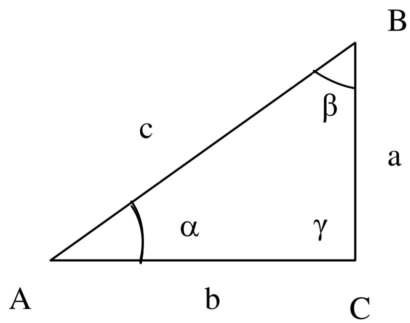


Fig.7.2

$\text{Tan } \alpha = \frac{a}{b}$, $\beta = 90^o - \alpha$,

$c = \sqrt{a^2 + b^2}$

Case-III:

When an angle α and one of the sides b , is given. The sides a , c and β are found from the following relations.

Tan $\alpha = \frac{a}{b}$ and Cos $\alpha = \frac{b}{c}$, $\beta = 90^o - \alpha$

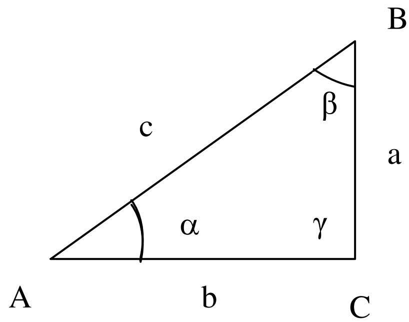


Fig.7.3

Case-IV:

When an angle α and the hypotenuse 'c' is given.The sides a , b and β can be found from the following relations.

Sin $\alpha = \frac{a}{c}$, Cos $\alpha = \frac{b}{c}$ and $\beta = 90^o - \alpha$

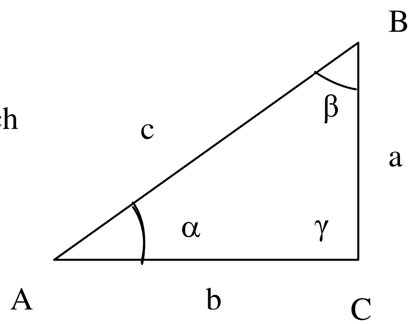


Fig.7.4

Example-1:

Solve the right triangle ABC in which $\alpha = 34^o\ 17'$, b=31.75, $\gamma = 90^o$

Solution:

Given that

$\alpha = 34^o\ 17'$, b = 31.75, $\gamma = 90^o$

We have to find

a = ? c = ? β = ?

$$\text{Tan } \alpha = \frac{a}{b}$$

$$\text{Tan } 34^o\ 17' = \frac{a}{31.75}$$

$\Rightarrow$ a = 31.75 tan $34^o\ 17'$

a = 31.75 (0.6817) = 21.64

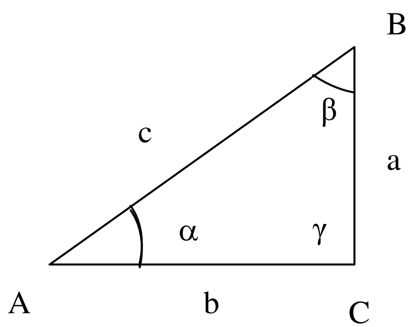


Fig.7.5

Also $\cos\alpha = \frac{b}{c}$

$$\Rightarrow \quad \text{Cos } 34^o\ 17' = \frac{31.75}{c}$$

$$C = \frac{31.75}{\text{Cos } 34^o\ 17'} \quad \Rightarrow \quad \beta = 90^o - 34^o\ 17' = 55^o\ 43'$$

Example 2:

Solve the right Δ ABC in which $\gamma = 90^o$, a = 450, b = 340

Solution:

$a = 450, \quad b = 340, \quad \gamma = 90^o,$

$c = ? = \alpha = ? \quad \beta = ?$

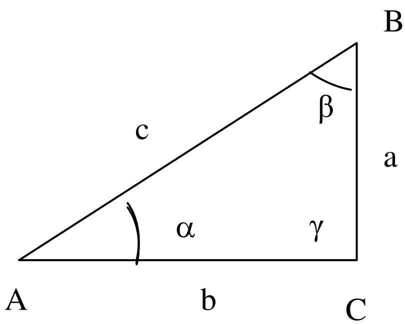


Fig. 7.6

$$\text{Tan}\,\alpha = \frac{a}{b}$$

$$\text{Tan}\,\alpha = \frac{450}{340} = 1.3231$$

$$\Rightarrow \quad \alpha = 52^o\ 56'$$

$$\beta = 90^o - \alpha = 90^o - 52^o\ 56' = 37^o\ 4'$$

By Pythagoras theorem:

$$C^2 = a^2 + b^2 = (450)^2 + (340)^2 = 318100$$

$$C = 564$$

Exercise 7.1

Solve the right triangle ABC in which $\gamma = 90^o$

(1)	$a = 250,$	$\alpha = 42^o\ 25'$	(2)	$a = 482,$	$\alpha = 35^o\ 36'$
(3)	$a = 5\ ,$	$c = 13$	(4)	$b = 312,$	$\alpha = 23^o\ 42'$
(5)	$a = 212,$	$\beta = 40^o\ 55'$	(6)	$c = 232,$	$\beta = 52^o\ 46'$
(7)	$c = 540,$	$a = 380$			

Answers 7.1

1.	$\beta = 47^o\ 35'$	$b = 273.63$	$c = 370.64$
2.	$\beta = 54^o\ 24'$	$b = 673.25$	$c = 828.01$
3.	$b = 12$	$\alpha = 22^o\ 37'$	$\beta = 67^o\ 23'$
4.	$a = 136.96$	$c = 340.72$	$\beta = 66^o\ 18'$
5.	$\alpha = 49^o\ 05'$	$b = 183.74$	$c = 280.5$
6.	$a = 184.72$	$b = 140.37$	$\alpha = 37^o\ 14'$
7.	$b = 383.61$	$\alpha = 44^o\ 44'$	$\beta = 45^o\ 16'$

7.2 Application of Right Angled Triangles

(Measurement of Heights and Distances)

Sometimes we deal with problems in which we have to find heights and distances of inaccessible objects.

The solution of these problems are generally the same as that of solving the right triangles.

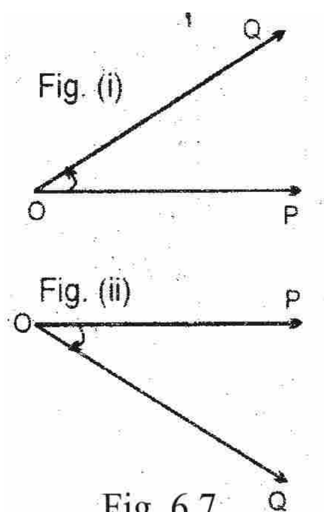


Fig. 6.7

7.3 Angle of Elevation and Depression:

If O be the eye of the observer, Q the position of the object and OP a horizontal line through O then:

i. If Q be above OP, then <POQ is called angle of elevation is shown in Figure (1)

ii. If Q be below OP, then <POQ is called angle of depression is shown in Figure (2)

Example 1:

Find the distance of man from the foot of tower 100m high if the angle of elevation of its top as observed by the man is $52^o\ 30'$.

Solution:

Let, A be the position of man and B be the foot of tower BC. Height of tower = BC = 100m in right Δ AB

$$\text{Tan } 52^o\ 32' = \frac{BC}{AB}$$

$$1.3032 = \frac{100}{AB} \Rightarrow AB = \frac{100}{1.3032} = 78.73m$$

AB = distance of man from the foot of tower = 76.73m

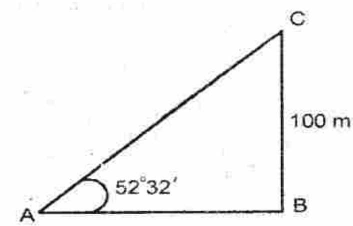


Fig. 6.8

Example 2:

From the two successive positions on the straight road 1000 meters apart man observes that the angle of elevation of the top a directly ahead of him are of $12^o\ 10'$ and $42^o\ 35'$. How high is the tower above the road.

Solution:

Let, A and D be the two successive positions of a man on the road.

AD = 1000m (Given)

Let BC = height of tower = h = ?

And DB = xm

In Δ ABC

$$\tan 12^o\ 10' = \frac{BC}{AB}$$

$$0.2156 = \frac{h}{(x + 1000)}$$

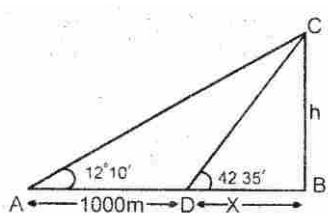


Fig. 6.9

Fig. 7.9

h = 0.2156 (x + 1000) ………….(1)

In Δ DBC

$$\tan 42^o\ 35' = \frac{BC}{DB} = \frac{h}{x}$$

$$0.9190 = \frac{h}{x}$$

$$x = \frac{h}{0.9190}$$

Put in (1)

$$h = 0.2156\left[\frac{h}{0.9190} + 100\right]$$

$$h = \frac{0.2156}{0.9190}h + \frac{(0.2156)(100)}{0.9190}$$

$$h = 0.2346h + 215.6$$

$$h = 0.2346h = 2156$$

$$0.7654h = 215.6$$

$$h = \frac{2156}{.7654} = 28.168$$

Example 3:

Measure of the angle of elevation of the top of a flag staff observed from a point 200 meters from its foot is

Solution:

Let height of flog staff = BC = h = ?

A = point of observation

In right ΔABC

$$\tan 30^o = \frac{h}{200} \Rightarrow \quad h = 200\ (0.577)$$

$$h = 115.4m$$

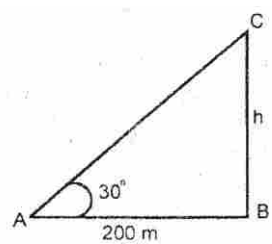


Fig. 6.10

Example 4:

Find the measures of the angle of elevation of the top of a tree 400 meters high, when observed from a point 250 meters away from the foot of the base.

Solution: Given that:

Height of tree = BC = 400m

AB = 250m

Let

$\angle$BAC= α = ?

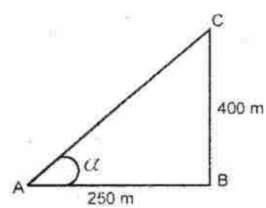


Fig. 6.11

$\angle$BAC= angle of elevation of top of the tree

$$\tan\alpha = \frac{BC}{AB} = \frac{400}{250} = 1.6$$

$$\alpha = \tan^{-1}(1.6) = 58^o$$

Example 5:

The measure of the angle of depression of an airport as observed by a pilot while flying at a height of 5000 meters is $40^{o}\ 32'$. How far is the pilot from a point directly over the airport?

Solution:

The pilot is at the height of C

BC = 5000m

From right $\angle ABC$

$$\tan 40^{o}\ 32' = \frac{5000}{x}$$

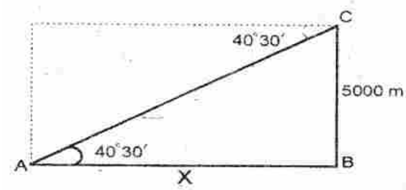


Fig. 6.12
Fig. 7.12

$$x = \frac{5000}{\tan 40^{o}32'} = \frac{5000}{0.8551} = 584736\text{m}$$

Example 6: From a point on the ground the measure of angle of elevation of the top of tower is 30^{o}. On walking 100 meters towards the tower the measure of the angle is found to be of 45^{o}. Find the height of the tower.

Solution:

Let BC = height of tower

= h = ?

And DB = x m

AD = 100 m

AB = 100 + x

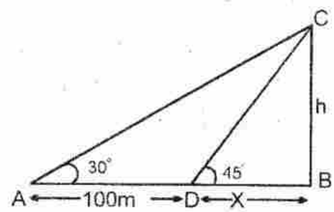


Fig. 6.13

In right ΔABC

$$\text{Tan}30^{o} = \frac{BC}{AB}$$

$$\frac{1}{\sqrt{3}} = \frac{h}{100+x}$$

$100 + x = \sqrt{3}\,h$ …………………. (1)

In right ΔBDC

$$\tan 45^{o} = \frac{h}{x}$$

$$1 = \frac{h}{x}$$

$x = h$ ……………….. (2)

Put $x = h$ in (1)

$100 + h = \sqrt{3}\,h$

$1.7321h - h = 100$

$$h = \frac{100}{0.7321} = 136.60\text{m}$$

Example 7:

A pole being broken by the wind, its top struck ground at an angle of 30^o and at a distance of 10m from the foot of the pole. Find the whole height of the pole.

Solution:

Let BC = h = height of pole = ?

AD = CD

In right ΔABD

$$\text{Tan}30^o = \frac{BD}{10}$$

BD = $10\tan30^o = 10(0.5774) = 5.77$m

Also

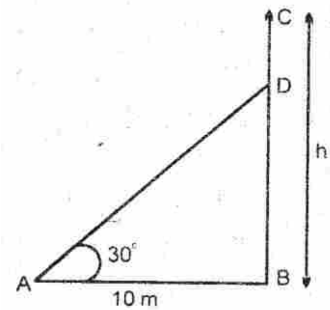


Fig. 6.14

$$\text{Cos}30^o = \frac{AB}{AD} \Rightarrow AD = \frac{10}{\text{Cos}30^o} = \frac{10}{0.8660} = 11.55\text{m}$$

Height of pole = h = BD + AD

$\therefore$ AD = CD

h = 11.55 + 5.77 = 17.32m

Exercise 7.2

Q1. How far is a man from the foot of tower 150 meters high, if the measure of the angle of elevation of its top as observed by him is $40^o\,30'$.

Q2. The shadow of a building is 220 meters when the measure of the angle of elevation of the sun is 35^o. Find the height of the building.

Q3. The measure of the angle of elevation of a kite is 35. The string of the kite is 340 meters long. If the sag in the string is 10 meters. Find the height of the kite.

Q4. A man 18dm. tall observes that the angle of elevation of the top of a tree at a distance of 12m from the man is 32^0.What is the height of the tree?

Q5. On walking 300 meters towards a tower in a horizontal line through its base, the measure of the angle of elevation of the top changes from 30^o to 60^o. Find the height of the tower.

Q6. The measure of the angle of elevation of the top of a cliff is 25. On walking 100 meters straight towards the cliff, the measure of the angle of elevation of the top is 48^o. Find the height of the cliff.

Q7. From two points A and B. 50 meters apart and in the line with a tree, the measures of the angles of elevation of the top of the tree are 30^o and 40^o respectively. Find the height of the tree.

Q8. Two men on the opposite sides of a tower observe that the measures of the angles of elevation of the tower as observed by them separately are 15^o and 25^o respectively. If the height of the tower is 150 meters. Find the distance between the observers.

Q9. From a light-house, angles of depression of two ships on opposite of the light-house are observed to be 30^o and 45^o. If the height of the light house be 300m. Find the distance between the ships of the line joining them passes through foot of light-hosue.

Q10. The measure of angle elevation of the top of a tower is 30^o from a point on the ground. On retreating 100 meters, the measure of the angle of elevation is found to be 15^o. Find the height of the tower.

Q11. From the top of a hill 200 meters high, the angles of depression of the top and bottom of a tower are observed to be 30^o and 60^o respectively. Find the height of the tower.

Q12. A television antenna is on the roof of a building. From a point on the ground 36m from the building, the angle of elevation of the top and the bottom of the antenna are 51^o and 42^o respectively. How tall is the antenna?

Q13. A ladder 20 meter long reaches the distance of 20 meters, from the top of a building. At the foot of the ladder the measure of the angle of elevation of the top of the building is 60^o. Find the height of the building.

Q14. A man standing on the bank of a canal observes that the measure of the angle of elevation of a tree is 60^0.On retreating 40m from the bank , he finds the measure the angle of elevation of the tree as 30^0.Find the height of the tree and the width of the canal.

Q15. Two buildings A and B are 100m apart. The angle of elevation from the top of the building A to the top of the building B is 20^0. The angle of elevation from the base of the building B to the top of the building A is 50^0. Find the height of the building B.

Answers 7.2

(1)	175.63m	(2)	154.05m	(3)	h = 189.29m
(4)	9.6m	(5)	h = 259.81m	(6)	h = 80.37m
(7)	h = 17.10m	(8)	881.58m	(9)	819.6m
(10)	49.98m	(11)	133.3m	(12)	12.1m
(13)	h = 30m	(14)	34.64m ; 20m	(15)	155.5 m

7.4 Law of Sines:

In any triangle, the length of the sides are proportional to the sines of measures of the angle opposite to those sides. It means

$$\frac{a}{\text{Sin}\alpha} = \frac{b}{\text{Sin}\beta} = \frac{c}{\text{Sin}\gamma}$$

Proof: Let one angle of the triangle say β be acute, then γ will be either acute, obtuse or right as in figure 1, 2, 3.

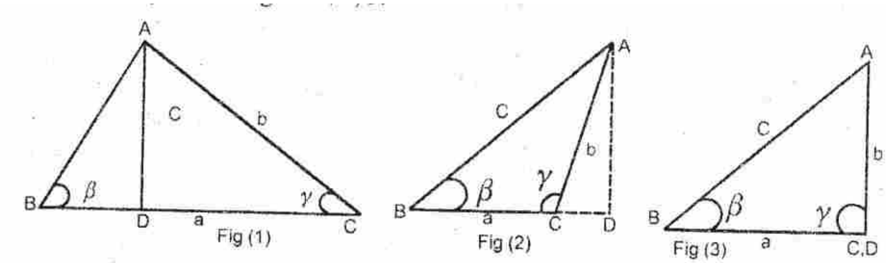


Fig. 6.15

Draw AD $\perp$ BC or BC produced.

Then from ΔABC (for all figures)

$$\frac{AD}{AB} = \text{Sin}\beta \qquad \therefore\ AD = c\text{Sin}\beta \ldots\ldots\ldots\ldots\ldots.. \quad (1)$$

If γ is acute in figure (1) $\frac{AD}{AC} = \text{Sin}\ \gamma \qquad \Rightarrow AD = b\ \text{Sin}\ \gamma$

If γ is obtuse in figure (2) $\frac{AD}{AC} = \text{Sin}\ (180 - \gamma) = \text{Sin}\ \gamma$

$$\Rightarrow AD = b\ \text{Sin}\ \gamma$$

If γ is right in figure (3) $\frac{AD}{AC} = 1 = \text{Sin}\ 90^o = \text{Sin}\ \gamma$

$$AD = b\ \text{Sin}\ \gamma$$

In each case we have

$$AD = b\ \text{Sin}\ \gamma \ldots\ldots\ldots\ldots. \quad (2)$$

From (1) & (2), we have

It can similarly be proved that:

$$\frac{a}{\text{Sin}\alpha} = \frac{b}{\text{Sin}\beta}, \text{ Similarly, } \frac{a}{\text{Sin}\alpha} = \frac{c}{\text{Sin}\ \gamma}$$

Hence ,

$$\frac{a}{\text{Sin}\alpha} = \frac{b}{\text{Sin}\beta} = \frac{c}{\text{Sin}\gamma}$$

This is known as law of sines.

Note: we use sine formula when

i. one side and two angles are given

ii. two sides and the angle opposite one of them are given

Example 1:

In any ΔABC

$a = 12$, $b = 7$, $\alpha = 40^o$ Find β

Solution:

By law of sines $\dfrac{a}{\text{Sin}\alpha} = \dfrac{b}{\text{Sin}\beta} = \dfrac{c}{\text{Sin}\gamma}$

$$\Rightarrow \frac{a}{\text{Sin}\alpha} = \frac{b}{\text{Sin}\beta} = \frac{12}{\text{Sin}40^o} = \frac{7}{\text{Sin}\beta}$$

$$\Rightarrow \text{Sin}\beta = \frac{7\text{Sin}40^o}{12} = \frac{7(0.6429)}{12}$$

$$\text{Sin }\beta = 0.3750$$

$$\Rightarrow \beta = \text{Sin}^{-1}(.3750) \qquad \Rightarrow \beta = 22^o1'$$

Example 2:

In any ΔABC, $b = 24$, $c = 16$

Find the ratio of $\text{Sin}\,\beta$ to $\text{Sin}\,\gamma$

Solution:

By law of sines

$$\frac{a}{\text{Sin}\alpha} = \frac{b}{\text{Sin}\beta} = \frac{c}{\text{Sin}\gamma}$$

$$\Rightarrow \frac{b}{\text{Sin}\beta} = \frac{c}{\text{Sin}\gamma} \Rightarrow \frac{\text{Sin}\beta}{\text{Sin}\gamma} = \frac{b}{c} = \frac{24}{16} = \frac{3}{2}$$

Example 3:

A town B is 15 km due North of a town A. The road from A to B runs North 27^o, East to G, then North 34^o, West to B. Find the distance by road from town A to B.

Solution:

Given that: $c = 15$ km $\alpha = 24^o$, $\beta = 34^o$

We have to find

Distance from A to B by road.

Since $\alpha + \beta + \gamma = 180^o \quad \Rightarrow \quad 27^o + 34^o + \gamma = 180^o$

$$\gamma = 119^o$$

By law of sines:

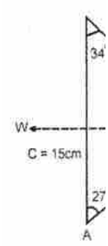


Fig. 7.16

$$\frac{a}{\text{Sin}\alpha} = \frac{b}{\text{Sin}\beta} = \frac{c}{\text{Sin}\gamma}$$

$$\Rightarrow \quad \frac{a}{\text{Sin}\alpha} = \frac{c}{\text{Sin}\gamma} \quad \Rightarrow \quad a = \frac{\text{Sin}\alpha}{\text{Sin}\gamma}$$

$$a = \frac{15\text{Sin}27^o}{\text{Sin}119^o} = \frac{15(0.4539)}{0.8746} = 7.78$$

$$\text{Also } \frac{b}{\text{Sin}\beta} = \frac{c}{\text{Sin}\gamma} \quad \Rightarrow \quad \frac{b}{\text{Sin}34^o} = \frac{15}{\text{Sin}119^o}$$

$$b = \frac{15\text{Sin}34^o}{\text{Sin}119^o} = \frac{15(0.5592)}{0.8746} = 9.59$$

Thus distance from A to B by road:

= b + a = 9.59 + 7.78 = 17.37km

Exercise7.3

In any triangle ABC if:

Q1. $a = 10$ $b = 15$ $\beta = 50^o$ Find α

Q2. $a = 20$ $c = 32$ $\gamma = 70^o$ Find α

Q3. $a = 3$ $b = 7$ $\beta = 85^o$ Find α

Q4. $a = 5$ $c = 6$ $\alpha = 45^o$ Find γ

Q5. $a = 20\sqrt{3}$ $\alpha = 75^o$ $\gamma = 60^o$ Find c

Q6. $a = 211.3$ $\beta = 48^o 16'$ $\gamma = 71^o\ 38'$ Find b

Q7. $a = 18$ $\alpha = 47^o$ $\beta = 102^o$ Find c

Q8. $a = 475$ $\beta = 72^o 15'$ $\gamma = 43^o\ 30'$ Find b

Q9. $a = 82$ $\beta = 57^o$ $\gamma = 78^o$ Find a

Q10. $\alpha = 60^o$ $\beta = 45^o$ Find the ratio of b to c

Q11. Two shore batteries at A and B, 840 meters apart are firing at a target C. The measure of angle ABC is 80^o and the measure of angle BAC is 70^o. Find the measures of distance AC and BC.

Answers 7.3

1. $\alpha = 30^o\ 42'\ 37''$ 2. $\alpha = 35^o\ 37'\ 58''$

3. $\alpha = 25^o\ 16'\ 24''$ 4. $\gamma = 58^o\ 3'$

5. $c = 31.06$ 6. $b = 181.89$
7. $c = 12.68$ 8. $b = 449.22$
9. $a = 69.13$ 10. 0.7319
11. 1578.68, 1654.46m

7.5 The Law of Cosines:

This law states that "the square of any sides of a triangle is equal to the sum of the squares of the other two sides minus twice their product times the cosine of their included angle. That is

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = c^2 + a^2 - 2ac\cos\beta$$
and $$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Proof:

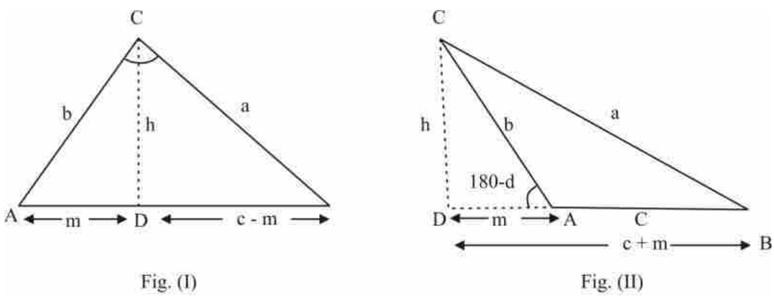


Fig. 6.17

Let β be an acute angle of ΔABC, draw $CD \perp AB$

Let $AD = m$ and $CD = h$

In right triangle BCD, we have

$$(BC)^2 = (BD)^2 + (CD)^2$$
$$a^2 = (BD)^2 + h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

(i) If α is an acute angle, then from (i)

In right triangle ACD,

$$\text{Sin}\,\alpha = \frac{h}{b} \quad \Rightarrow h = b\,\text{Sin}\,\alpha$$

and $$\text{Cos}\,\alpha = \frac{m}{b} \quad \Rightarrow m = b\,\text{Cos}\,\alpha$$

So, $\boxed{BD = c - m = c - b\cos\alpha}$

Putting the values of h and BD in equation (1)

$$a^2 = (c - b\cos\alpha)^2 + (b\sin\alpha)^2$$
$$= c^2 - 2bc\cos\alpha + b^2\cos^2\alpha + b^2\sin^2\alpha$$
$$= c^2 - 2bc\cos\alpha + b^2(\cos^2\alpha + \sin^2\alpha)$$

$= c^2 - 2bc\cos\alpha + b^2$

$\boxed{a^2 = b^2 + c2 - 2bc\cos\alpha}$

(ii) If α in an obtuse angle, then from fig (ii)

In right triangle ACD,

$$\text{Sin}(180 - \alpha) = \frac{h}{b}$$

$$\text{Sin}\,\alpha = \frac{h}{b} \quad \Rightarrow h = b\sin\alpha$$

and $\cos(180 - \alpha) = \dfrac{m}{b}$

$$-\cos\alpha = \frac{m}{b} \quad \Rightarrow m = -b\cos\alpha$$

So, $\boxed{BD = c + m = c - b\cos\alpha}$

Putting the values of h and BD in equation (1)

$$a^2 = (c - b\cos\alpha)^2 + (b\sin\alpha)^2$$

we get, $a^2 = b^2 + c^2 - 2bc\cos\alpha$

Similarly we obtain

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

and $c^2 = b^2 + c^2 - 2bc\cos\alpha$

Also when three sides are given, we find

$\text{Cos}\,\alpha = \dfrac{b^2+c^2-a^2}{2bc}$, $\text{Cos}\,\beta = \dfrac{a^2+c^2-b^2}{2ac}$ and

$\text{Cos}\,\gamma = \dfrac{a^2+b^2-c^2}{2ab}$

Note: we use the cosine formula, when

(i) Two sides and their included angle are given.

(ii) When the three sides are given.

Example 1: In any by using the law of cosines

$a = 7$, $c = 9$, $\beta = 112^0$ Find b

Solution: By law of cosines

$b^2 = a^2 + c^2 - 2ac\cos\beta$

$b^2 = (7)^2 + (9)^2 - 2(7)(9)\text{ Cos }112^o$

$= 49 + 81 - 126(.3746)$

$b^2 = 130 + 47.20 = 177.2$

$b = 13.31$

Example 2: Two man start walking at the same time from a cross road, both walking at 4 km/hour. The roads make an angle of measure 80^o with each other. How far apart will they be at the end of the two hours?

Solution: Let, A be the point of starting of two man V = 4 km/hour

Distance traveled by two men after 2 hours = vt

= 4 x 2 = 8km

Thus, we have to find BC = a = ?

By law of cosine:

$a^2 = b^2 + c^2 - 2bc \cos\alpha$

$a^2 = (8)^2 + (8)^2 - 2(8)(8) \text{ Cos } 80^o = 128 - 128\ (0.1736)$

$a^2 = 105.77 \quad \Rightarrow \quad a = 10.28\text{km}$

Thus, two men will apart 10.28 km after two hours.

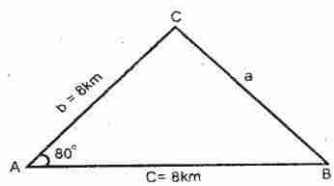


Fig. 6.18

Fig. 7.18

Exercise 7.4

In any triangle ABC by using the law of cosines:

1. a = 56 c = 30 $\beta = 35^o$ Find b
2. b = 25 c = 37 $\alpha = 65^o$ Find a
3. b = 5 c = 8 $\alpha = 60^o$ Find a
4. a = 212 c = 135 $\beta = 37^o\ 15^/$ Find b
5. a = 16 b = 17 $\gamma = 25^o$ Find c
6. a = 44 b = 55 $\gamma = 114^o$ Find c
7. a = 13 b = 10 c = 17 Find α and β
8. Three villages P, Q and R are connected by straight roads. Measure PQ is 6 km and the measure QR is 9km. The measure of the angle between PQ and QR is 120^o. Find the distance between P and R.
9. Two points A and B are at distance 55 and 32 meters respectively from a point P. The measure of angle between AP and BP is 37^o. Find the distance between B and A.
10. Find the cosine of the smallest measure of an angle of a triangle with 12, 13 and 14 meters as the measures of its sides.

Answers 7.4

1. b = 35.83 2. a = 34.83 3. a =7
4. b = 132.652 5. c = 7.21 6. c = 83.24
7. $\alpha = 49^o\ 40^/\ 47^{//}$ $\beta = 35^o\ 54^/\ 30^{//}$
8. 13.08km 9. 35.18m 10. 52^o 37'

7.6 Solution of Oblique Triangles:

Definition:

The triangle in which have no right angle is called oblique triangle.

A triangle has six elements (i.e. three sides and three angles) if any three of a triangle are given, provided at least one of them is a side, the remaining three can be found by using the formula discussed in previous articles i.e. law of sines and law of cosines.

There are four important cases to solve oblique triangle.

Case I: Measure of one side and the measures of two angles.

Case II: Measure of two sides and the measures of the angle included by them.

Case III: When two sides and the angle opposite to one of them is given.

Case IV: Measure of the three sides.

Example 1:

Solve the ABC with given data.

$a = 850, \qquad \alpha = 65^o, \qquad \beta = 40^o$

Solution:

Given that:

$a = 850, \qquad \alpha = 65^o, \qquad \beta = 40^o$

$b = ? \qquad c = ? \qquad \gamma = ?$

Since, $\alpha + \beta + \gamma = 180^o$

$65^o + 40^o + \gamma = 180 \ \gamma \qquad \Rightarrow \ = 75^o$

By law of sines to find b:

$$\frac{a}{\text{Sin } \alpha} = \frac{b}{\text{Sin } \beta}$$

$$\frac{850}{\text{Sin } 65^o} = \frac{b}{\text{Sin } 40^o} \Rightarrow \quad b = \frac{850 \text{ Sin } 40^o}{\text{Sin } 65^o}$$

$$b = \frac{850(0.64.28)}{0.9063} = 602.85$$

To find c, by law of Sines

$$\frac{b}{\text{Sin } \beta} = \frac{a}{\text{Sin } \alpha} \quad \Rightarrow c = \frac{b \text{ Sin } \gamma}{\text{Sin } \beta}$$

$$c = \frac{602.85 \text{ Sin } 75^o}{\text{Sin } 40^o} = \frac{602.85(0.9659)}{0.6428}$$

$= \quad 905.90$

Example2:

Solve the triangle with given data:

$a = 45 \qquad b = 34 \qquad \gamma = 52^o$

Solution:

Given a = 45 b = 34 $\gamma = 52^o$

c = ? β = ? γ = ?

To find c, we use law of cosines

$c^2 = a^2 + b^2 - 2ab \text{ Cos } \gamma$

$c^2 = (45)^2 + (34)^2 - 2(45)(34) \text{ Cos } 52^o$

$c^2 = 2025 + 1156 - (3060)(0.6157)$

$c^2 = 1297 \quad \Rightarrow \quad \boxed{c = 36.01}$

To find α, we use law of sines

$$\frac{a}{\text{Sin}\alpha} = \frac{b}{\text{Sin}\gamma} \Rightarrow \frac{45}{\text{Sin}\alpha} = \frac{36.01}{\text{Sin}52^o}$$

$$\text{Sin}\alpha = \frac{45 \text{ Sin}52^o}{36.01} = \frac{45(0.7880)}{36.01} = 0.9847$$

$\alpha = \text{Sin}^{-1}(0.9847) = 97^o\ 58^/\ 39^{//}$

To find β, we use $\alpha + \beta + \gamma = 180^o$

$79^o\ 58^/\ 39^{//} + \beta + 52^o = 180^o$

$\beta = 48^o\ 01^/\ 20^{//}$

Exercise 7.5

Solve the triangle ABC with given data.

Q1. c = 4 $\alpha = 70^o$ $\gamma = 42^o$

Q2. a = 464 $\beta = 102^o$ $\gamma = 23^o$

Q3. b = 85 $\beta = 57^o15^/$ $\gamma = 78^o18^/$

Q4. b = 56.8 $\alpha = 79^o31^/$ $\beta = 44^o24^/$

Q5. b = 34.57 $\alpha = 62^o11^/$ $\beta = 63^o22^/$

Q 6. Find the angle of largest measure in the triangle ABC where:

(i) a = 224 b = 380 c = 340

(ii) a = 374 b = 514 c = 425

Q7. solve the triangle ABC where:

(i) a = 74 b = 52 c = 47

(ii) a = 7 b = 9 c = 7

(iii) a = 2.3 b = 1.5 c = 2.7

Answers 7.4

Q1.	$a = 5.62$	$b = 5.54$	$\beta = 68^o$	
Q2.	$\alpha = 55^o$	$b = 454$	$c = 221.31$	
Q3.	$\alpha = 44^o27'$	$a = 70.78$	$c = 98.97$	
Q4.	$a = 79.82$	$c = 67.37$	$\gamma = 56^o\,0'$	
Q5.	$a = 34.20$	$c = 31.47$	$\gamma = 54^o\,2'$	
Q6.	(i) $81^o\,55'\,57''$	(ii) $79^o\,47'\,53''$		
Q7.	(i)	$\alpha = 96^o37'$	$\beta = 44^o16'$	$\gamma = 39^o\,07'$
	(ii)	$\alpha = 50^o$	$\beta = 80^o$	$\gamma = 50^o$
	(iii)	$\alpha = 58^o21'$	$\beta = 33^o45'$	$\gamma = 87^o\,55'$

Summary

1. Right Triangle:

A triangle which has one angle given equal to a right angle.

2. Oblique Triangle:

The triangle in which have no right angle is called oblique triangle.

3. Law of Sines

In any ΔABC, the measures of the sides are proportional to the sines of the opposite angles.

i.e. $\dfrac{a}{\text{Sin}\alpha} = \dfrac{b}{\text{Sin}\beta} = \dfrac{c}{\text{Sin}\gamma}$

4. Law of Cosines

(i) $a^2 = b^2 + c^2 - 2bc \text{ Cos } \alpha$

(ii) $b^2 = a^2 + c^2 - 2ac \text{ Cos } \beta$

(iii) $c^2 = a^2 + b^2 - 2ab \text{ Cos } \gamma$

(iv) $\text{Cos } \alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$

(v) $\text{Cos } \beta = \dfrac{a^2 + c^2 - b^2}{2ac}$

(vi) $\text{Cos } \gamma = \dfrac{a^2 + b^2 - c^2}{2ab}$

Angle of Elevation:

The angle AOP which the ray from an observer's eye at O to an object P at O to an object P at a higher level, makes with horizontal ray OA through O is called the angle of elevation.

Angle of Depression:

The angle AOP which the ray from an observer's eye at O to an object at P at a lower level makes with the horizontal ray OA through O is called the angle of depression.

Short Questions

Write the short answers of the following:

Q.1: Define the law of sine.

Q.2: Define the Laws of cosines

Q.3: In right triangle ABC, $\gamma = 90^o$, $a = 5$, $c = 13$ then find the value of angle α .

Q.4: Given that $\gamma = 90^o$, $\alpha = 35^{o,}$ $a = 5$, find angle β

Q.5: In right triangle ABC $b = 6$, $\alpha=35^o$, $\gamma =90^o$, Find side 'a'

Q.6: Given that $\alpha = 30^o$, $\gamma = 135^o$, and $c = 10$, find a

Q.7: In any triangle ABC, if $a =20$, $c = 32$ and $c =70^o$, Find A.

Q.8: In any triangle ABC if $a = 9$, $b = 5$, and $\gamma = 32^{o}$, Find c.

Q.9: The sides of a triangle are 16, 20 and 33 meters respectively. Find its greatest angle.

Q.10: Define angle of elevation and depression.

Q.11: A string of a flying kite is 200 meters long, and its angle of elevation is 60^o. Find the height of the kite above the ground taking the string to be fully stretched.

Q.12: A minaret stands on the horizontal ground. A man on the ground, 100 m from the minaret, Find the angle of elevation of the top of the minaret to be 60^o. Find its height.

Q.13: The shadow of Qutab Minar is 81m long when the measure of the angel of elevation of the sun is $41^o\ 31'$. Find the height of the Qutab Minar.

Q.14: In any triangle ABC in which

$b = 45, c =34$, $\alpha = 52^o$, find a

Q.15: In any triangle ABC is which

$a = 16$, $b = 17$, $\gamma = 25^o$ find c

Q.16: In any triangle ABC in which

$a = 5$, $c =6$, $\alpha = 45^o$ Find $\sin \gamma$

Q.17: $b =25$, $c =37$ $a =65^o$ find a

Q.18: a = 16, b = 17, $\gamma = 25^o$ find c

Q.19: a =3, b = 7, $\beta = 85^o$, Find α .

Answers

3. $22^o\ 37'$. 4. $\beta = 55^o$ 5. a = 4.2 6. a = 7.07

7. A $= 35^o\ 77'\ 58''$ 8. c = 5.48 9. $\gamma = 132^o\ 34'$ 11. h = 173.2 m

12. h = 173.20 m 13. h = 71.66 m 14. a= 36.04 15. c = 7.21

16. $\gamma = 58^o\ 3'$ 17. a = 34.82 18 c= 7.21 19 $\alpha = 25^o 14'\ 14''$

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. Law of sines is:

(a) $\dfrac{a}{\sin B} = \dfrac{b}{\sin A} = \dfrac{c}{\sin c}$ (b) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

(c) $\dfrac{a}{\sin B} = \dfrac{b}{\sin A} = \dfrac{c}{\sin C}$ (d) $\dfrac{a}{\sin B} = \dfrac{b}{\sin C} = \dfrac{a}{\sin A}$

__2. In a triangle ABC $\angle A = 70^o$, $\angle B = 60^o$, then $\angle C$ is:

(a) 30^o (b) 40^o (c) 50^o (d) 60^o

__3. When angle of elevation is viewed by an observer, the object is:

(a) Above (b) Below

(c) At the same level (d) None of these

__4. If $b = 2$, $A = 30^o$, $B = 45^o$, then a is equal to:

(a) 2 (b) $\sqrt{2}$ (c) $\dfrac{\sqrt{3}}{2}$ (d) $\dfrac{2}{\sqrt{3}}$

__5. If $a = 2$, $b = 2$, $A = 30^o$, then B^o is:

(a) 45^o (b) 30^o (c) 60^o (d) 90^o

__6. If in a triangle ABC, the sides b, c and angle A are given, then the side a is:

(a) $a^2 = b^2 + c^2 + 2bc \cos A$ (b) $a^2 = b^2 - c^2 - 2ab \cos A$

(c) $a^2 = b^2 + c^2 - 2bc \cos A$ (d) $a^2 = b^2 - c^2 + 2ab \cos A$

__7. In a triangle ABC, the law of cosine is:

(a) $\text{Cos } A = \dfrac{b^2 + c^2 + a^2}{2bc}$ (b) $\text{Cos } A = \dfrac{b^2 + c^2 - a^2}{2bc}$

(c) $\text{Cos } A = \dfrac{b^2 + c^2 + a^2}{2ab}$ (d) $\text{Cos } A = \dfrac{b^2 + c^2 - a^2}{2ac}$

__8. If in a triangle ABC, $b = 2$, $c = 2$, $A = 60^o$, then side a is:

(a) 2 (b) 3 (c) 4 (d) 5

__9. If in a triangle ABC, $a = 1$, $b = \sqrt{2}$, $C = 60^o$, then side c is:

(a) $\sqrt{2}$ (b) 2 (c) 1 (d) 3

__10. If in a triangle ABC, $b = 2$, $c = 3$, $a = 1$, then cos A is:

(a) 1 (b) 2 (c) 3 (d) 4

__11. If in a triangle ABC, $a = 3$, $b = 4$, $c = 2$, then cos C is:

(a) $\dfrac{1}{2}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{7}{8}$ (d) 3

__12. If $b \sin C = c \sin B$, then, $a \sin B$ is equal to:

(a) c sin A (b) b sin c (c) b sin A (d) b sin B

__13. In a right triangle if one angle is 30^o, then the other will be:
(a) 45^o (b) 50^o (c) 60^o (d) 75^o

__14. In a right triangle if one angle is 45^o, then the other will be:
(a) 45^o (b) 50^o (c) 60^o (d) 75^o

__15. If $B = 90^o$, $b = 2$, $A = 30^o$, then side a is:
(a) 4 (b) 3 (c) 2 (d) 1

__16. If $c = 90^o$, $a = 1$, $c = 2$, then angle A is:
(a) 90^o (b) 60^o (c) 45^o (d) 30^o

__17. If $c = 90^o$, $b = 1$, $c = \sqrt{2}$, then side a is:
(a) 1 (b) 2 (c) $\sqrt{2}$ (d) 3

__18. If $c = 90^o$, $b = 1$, $c = \sqrt{2}$, then angle A is:
(a) 15^o (b) 30^o (c) 45^o (d) 60^o

__19. The distance of a man from the foot of a tower, 100m high if the angle of elevation of its top as observed by the man is 30^o is:
(a) 50m (b) 100m (c) 150m (d) 200m

__20. A pilot at a distance of 50m, measure the angle of depression of a tower 30^o, how far is the plane from the tower:
(a) 50m (b) 25m (c) 20m (d) 10m

Answers

Q1:

1.	b	2.	c	3.	a	4.	d	5.	b
6.	c	7.	b	8.	a	9.	c	10.	a
11.	c	12.	c	13.	c	14.	a	15.	d
16.	d	17.	a	18.	c	19.	d	20.	b

Chapter 8
Vectors and Scalars

8.1 Introduction:

In this chapter we shall use the ideas of the plane to develop a new mathematical concept, vector. If you have studied physics, you have encountered this concept in that part of physics concerned with forces and equilibrium.

Physicists were responsible for first conceiving the idea of a vector, but the mathematical concept of vectors has become important in its own right and has extremely wide application, not only in the sciences but in mathematics as well.

8.2 Scalars and Vectors:

A quantity which is completely specified by a certain number associated with a suitable unit without any mention of direction in space is known as scalar. Examples of scalar are time, mass, length, volume, density, temperature, energy, distance, speed etc. The number describing the quantity of a particular scalar is known as its magnitude. The scalars are added subtracted, multiplied and divided by the usual arithmetical laws.

A quantity which is completely described only when both their magnitude and direction are specified is known as vector. Examples of vector are force, velocity, acceleration, displacement, torque, momentum, gravitational force, electric and magnetic intensities etc. A vector is represented by a Roman letter in bold face and its magnitude, by the same letter in italics. Thus **V** means vector and V is magnitude.

8.3 Vector Representations:

A vector quantity is represented by a straight line segment, say $\overrightarrow{PQ}$. The arrow head indicate the direction from P to Q. The length of the Vector represents its magnitude. Sometimes the vectors are represented by single letter such as V or $\vec{V}$. The magnitude of a vector is denoted by |V| or by just **V**, where | $\vec{V}$ | means modulus of $\vec{V}$ which is a positive value

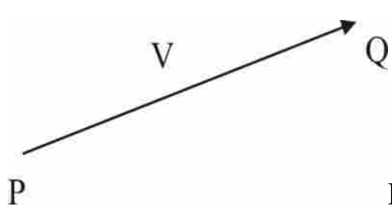


Fig. 1

8.4 Types of Vectors:

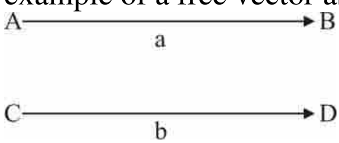
1. Unit Vector:

A vector whose magnitude is unity i.e., 1 and direction along the given vector is called a unit Vector. If $\vec{a}$ is a vector then a unit vector in the direction of $\vec{a}$, denoted by $\hat{a}$ (read as a cap), is given as,

$$\hat{a} = \frac{\vec{a}}{|a|} \quad \text{or} \quad \vec{a} = |a|\,\hat{a}$$

2. Free Vector:

A vector whose position is not fixed in space. Thus, the line of action of a free vector can be shifted parallel to itself. Displacement is an example of a free vector as shown in figure 1:



3. Localized or Bounded Vectors:

A vector which cannot be shifted parallel to itself, i.e., whose line of action is fixed is called a localized or bounded vector. Force and momentum are examples of localized vectors.

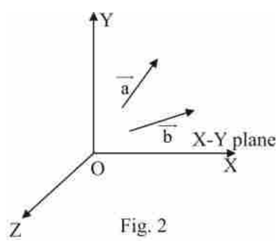


Fig. 2

4. Coplanar Vectors:

The vectors which lies in the same plane are called coplanar vectors, as shown in Fig. 2.

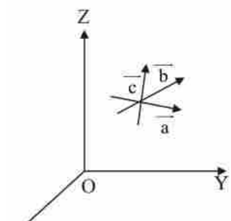


Fig. 3

5. Concurrent Vectors:

The vectors which pass through the common point are called concurrent vectors. In the figure no.3 vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are called concurrent as they pass through the same point.

6. Negative of a Vector:

The vector which has the same magnitude as the vector $\vec{a}$ but opposite in direction to $\vec{a}$ is called the negative to $\vec{a}$. It is represented by $-\vec{a}$. Thus of $\overrightarrow{AB} = -\vec{a}$ then $\overrightarrow{BA} = -\vec{a}$

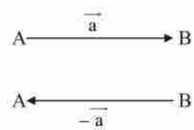


Fig. 4

7. Null or Zero Vector:

It is a vector whose magnitude is zero. We denote the null vector by O. The direction of a zero vector is arbitrary.

The vectors other than zero vectors are proper vectors or non-zero vectors.

8. Equal Vectors:

Two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are said to be equal if they have the same magnitude and direction. If $\overrightarrow{a}$ and $\overrightarrow{b}$ are equal vectors then $\overrightarrow{a} = \overrightarrow{b}$

9. Parallel and Collinear Vectors:

The vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel if for any real number n,

$\overrightarrow{a} = n\,\overrightarrow{b}$. If

(i) $n > 0$ then the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ have the same direction.

(ii) $n < 0$ then $\overrightarrow{a}$ and $\overrightarrow{b}$ have opposite directions.

Now, we can also define collinear vectors which lie along the same straight line or having their directions parallel to one another.

10. Like and Unlike Vectors:

The vectors having same direction are called like vectors and those having opposite directions are called unlike vectors.

11. Position Vectors (PV):

If vector $\overrightarrow{OA}$ is used to specify the position of a point A relative to another point O. This $\overrightarrow{OA}$ is called the position vector of A referred to O as origin. In the figure 4 $\overrightarrow{a} = \overrightarrow{OA}$ and $\overrightarrow{OB} = \overrightarrow{b}$ are the position vector (P.V) of A and B respectively. The vector $\overrightarrow{AB}$ is determined as follows:

By the head and tail rules,

$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

Or $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{b} - \overrightarrow{a}$

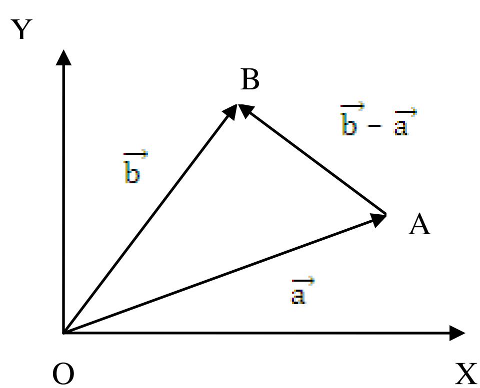


Fig. 5

8.5 Addition and Subtraction of Vectors:

1. Addition of Vectors:

Suppose $\vec{a}$ and $\vec{b}$ are any two vectors. Choose point A so that $\vec{a} = \overrightarrow{OA}$ and choose point C so that $\vec{b} = \overrightarrow{AC}$. The sum, $\vec{a} + \vec{b}$ of $\vec{a}$ and $\vec{b}$ is the vector is the vector $\overrightarrow{OC}$. Thus the sum of two vectors $\vec{a}$ and $\vec{b}$ is performed by the Triangle Law of addition.

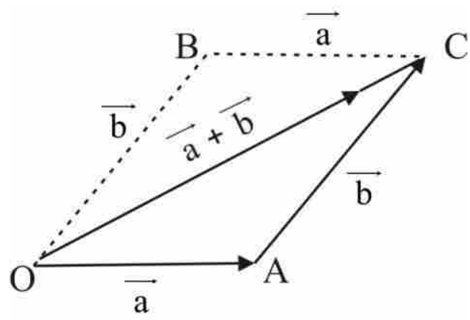


Fig. 6

2. Subtraction of Vectors:

If a vector $\vec{b}$ is to be subtracted from a vector $\vec{a}$, the difference vector $\vec{a} - \vec{b}$ can be obtained by adding vectors $\vec{a}$ and $-\vec{b}$.

The vector $-\vec{b}$ is a vector which is equal and parallel to that of vector but its arrow-head points in opposite direction. Now the vectors $\vec{a}$ and $-\vec{b}$ can be added by the head-to-tail rule. Thus the line $\overrightarrow{AC}$ represents, in magnitude and direction, the vector $\vec{a} - \vec{b}$.

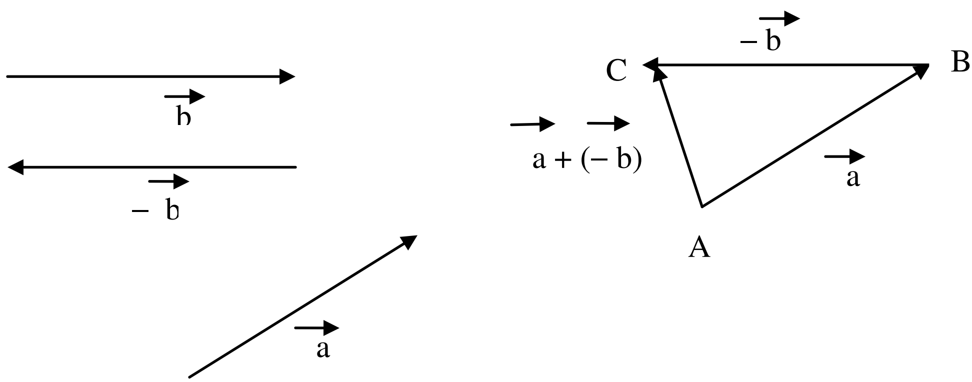


Fig . 7

Properties of Vector Addition:

i. Vector addition is commutative

i.e., $\vec{a} + \vec{b} = \vec{a} + \vec{b}$ where $\vec{a}$ and $\vec{b}$ are any two vectors.

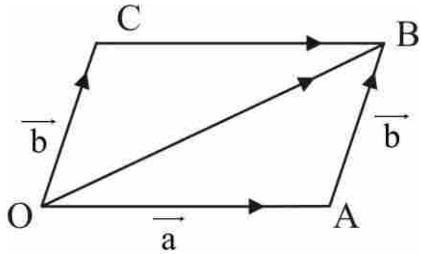
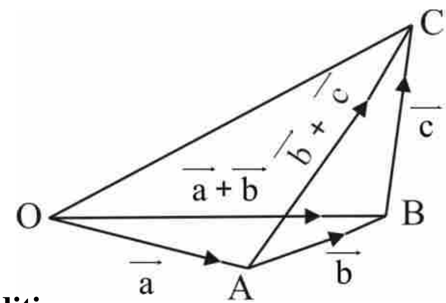


Fig. 8

(ii) Vectors Addition is Associative:

i.e. $\left(\vec{a} + \vec{b}\right) + \vec{c} = \vec{a} + \left(\vec{b} + \vec{c}\right)$

where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are any three vectors.



(iii) $\vec{O}$ is the identity in vectors addition:

Fig.9

For every vector $\vec{a}$

$\vec{a} + \vec{O} = \vec{a}$

Where $\vec{O}$ is the zero vector.

Remarks: Non-parallel vectors are not added or subtracted by the ordinary algebraic Laws because their resultant depends upon their directions as well.

8.6 Multiplication of a Vector by a Scalar:

If $\vec{a}$ is any vectors and K is a scalar, then $K\vec{a} = \vec{a}K$ is a vector with magnitude $|K|.|a|$ i.e., $|K|$ times the magnitude of $\vec{a}$ and whose direction is that of vector $\vec{a}$ or opposite to vector $\vec{a}$ according as K is positive or negative resp. In particular $\vec{a}$ and $-\vec{a}$ are opposite vectors.

Properties of Multiplication of Vectors by Scalars:

1. The scalar multiplication of a vectors satisfies

$$m(n\vec{a}) = (mn)\vec{a} = n(m\vec{a})$$

2. The scalar multiplication of a vector satisfies the distributive laws

i.e., $(m + n)\vec{a} = m\vec{a} + n\vec{a}$

and $\quad m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

Where m and n are scalars and $\vec{a}$ and $\vec{b}$ are vectors.

8.7 The Unit Vectors i, j, k (orthogonal system of unit Vectors):

Let us consider three mutually perpendicular straight lines OX, OY and OZ. These three mutually perpendicular lines determine uniquely the position of a point. Hence these lines may be taken as the co-ordinates axes with O as the origin.

We shall use i, j and k to denote the Unit Vectors along OX, OY and OX respectively.

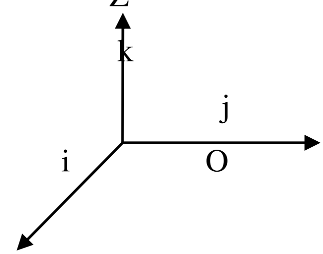


Fig. 9a

8.8 Representation of a Vector in the Form of Unit Vectors i, j and k.

Let us consider a vector $\vec{r} = \overrightarrow{OP}$ as shown in fig. 11. Then x i, y j and z k are vectors directed along the axes,

$$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} = \overrightarrow{OA} + \overrightarrow{OB} \qquad \text{because}$$

and $\quad \overrightarrow{OQ} = xi + yi$

Because $\quad \overrightarrow{QP} = zk$

$$\overrightarrow{OP} = \overrightarrow{OQ} + \overrightarrow{QP}$$

and $\quad \vec{r} = \overrightarrow{OP} = xi + yj + zk$

Here the real numbers x, y and z are the components of Vector $\vec{r}$ or the co-ordinates of point P in the direction of OX, OY and OZ respectively. The vectors xi, yj and zk are called the resolved parts of the vector $\vec{r}$ in the direction of the Unit vectors i, j and k respectively.

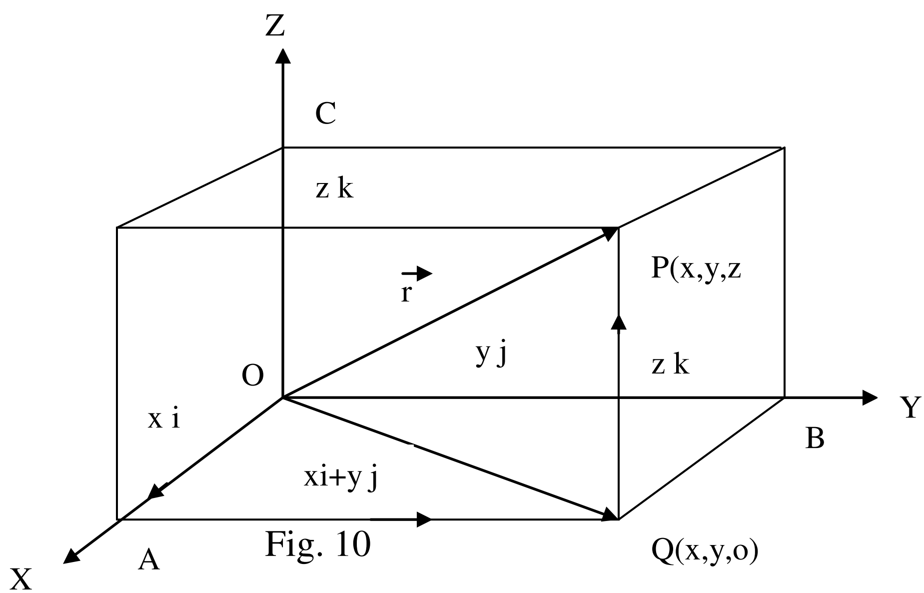


Fig. 10

8.9 Components of a Vector when the Tail is not at the Origin:

Consider a vector $\vec{r} = \overrightarrow{PQ}$ whose tail is at the point $P(x_1, y_1, z_1)$ and the head at the point $Q(x_2, y_2, z_2)$. Draw perpendiculars PP' and QQ' on x-aixs.

$P'Q' = x_2 - x_1$ = x-component of $\vec{r}$

Now draw perpendiculars PP^o and QQ^o on y-axis.

Then $P^oQ^o = y_2 - y_1$ = y-component of $\vec{r}$

Similarly $z_2 - z_1$ = z-component of $\vec{r}$

Hence the vector $\vec{r}$ can be written as,

$\vec{r} = \overrightarrow{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

Or, $\vec{r} = \overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$

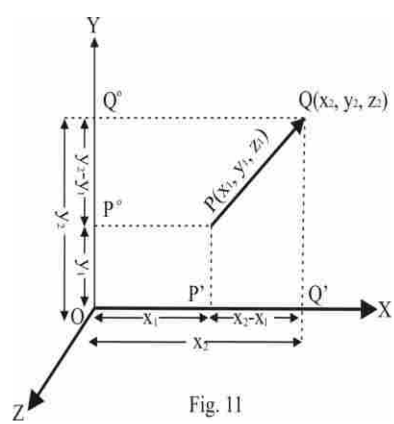


Fig. 11

8.10 Magnitude or Modulus of a Vector:

Suppose x, y and z are the magnitude of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ as shown in fig. 10.

In the right triangle OAQ, by Pythagorean Theorem

$$OQ^2 = x^2 + y^2$$

Also in the right triangle OQP, we have

$$OP^2 = OQ^2 + QP^2$$
$$OP^2 = x^2 + y^2 + z^2$$

Or $\quad |\overrightarrow{r}| = |OP| = \sqrt{x^2 + y^2 + z^2}$

Thus if $\quad \overrightarrow{r} = \overrightarrow{PQ} = xi + yj + zk$

Then , its magnitude is

$$|\overrightarrow{r}| = \sqrt{x^2 + y^2 + z^2}$$

If $\quad \overrightarrow{r} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

Then $\quad |\overrightarrow{r}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example 1:

If $P_1 = P(7, 4, -1)$ and $P_2 = P(3, -5, 4)$, what are the components of P_1P_2? Express $\overrightarrow{P_1\ P_2}$ in terms of i, j and k.

Solution:

x-component of $P_1\ P_2 = x_2 - x_1 = 3 - 7 = -4$

y-component of $P_1\ P_2 = y_2 - y_1 = -5 - 4 = -9$

and z-component of $P_1\ P_2 = z_2 - z_1 = 4 - (-1) = 5$

also $\quad \overrightarrow{P_1\ P_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

$$\overrightarrow{P_1\ P_2} = -4i - 9j + 5k$$

Example 2: Find the magnitude of the vector

$$|\bar{u}| = \frac{3}{5}i - \frac{2}{5}j + \frac{2\sqrt{3}}{5}k$$

Solution:

$$|\bar{u}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{2}{5}\right)^2 + \left(\frac{2\sqrt{3}}{5}\right)^2}$$

$$= \sqrt{\frac{9}{25} + \frac{4}{25} + \frac{12}{25}} = \sqrt{\frac{25}{25}}$$

$$|\bar{u}| = 1$$

Note: Two vectors are equal if and only if the corresponding components of these vectors are equal relative to the same co-ordinate system.

Example 3:

Find real numbers x, y and z such that

$xi + 2yj - zk + 3i - j = 4i + 3k$

Solution:

Since $(x + 3)i + (2y - 1)j + (-z)k = 4i + 3k$

Comparing both sides,we get

$x + 3 = 4, \; 2y - 1 = 0, \quad -z = 3$

$x = 1, \quad y = \frac{1}{2}, \quad z = -3$

Note 2: If $\overrightarrow{r_1} = x_1 i + y_1 j + z_1 k$

$\overrightarrow{r_2} = x_2 i + y_2 j + z_2 k$

Then the sum vector =

$$\overrightarrow{r_1} + \overrightarrow{r_2} = (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$$

Or $$\overrightarrow{r_1} + \overrightarrow{r_2} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

Example 4:

$\overrightarrow{a} = 3i - 2j + 5k$ and $\overrightarrow{b} = -2i - j + k$.

Find $2\overrightarrow{a} - 3\overrightarrow{b}$ and also its unit vector.

Solution:

$$\begin{aligned} 2\overrightarrow{a} - 3\overrightarrow{b} &= 2(3i - 2j + 5k) - 3(-2i - j + k) \\ &= 6i - 4j + 10k + 6i + 3j - 3k \\ &= 12i - j + 7k \end{aligned}$$

If we denote $2\overrightarrow{a} - 3\overrightarrow{b} = \overrightarrow{c}$, then $\overrightarrow{c} = 12i - j + 7k$

and $|\vec{c}| = \sqrt{12^2 + (-1)^2 + 7^2} = \sqrt{144 + 1 + 49} = \sqrt{194}$

Therefore, $$\hat{c} = \frac{\overrightarrow{c}}{|c|} = \frac{12i - j + 7k}{\sqrt{194}}$$

$$\hat{c} = \frac{12}{\sqrt{194}} i - \frac{1}{\sqrt{194}} j + \frac{1}{\sqrt{194}} k$$

Note 3: Two vectors $\vec{r_1} = x_1 i + y_1 j + z_1 k$ and $\vec{r_2} = x_2 i + y_2 j + z_2 k$ are parallel if and only if $\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$.

8.11 Direction Cosines:

Let us consider that the vector $\vec{r} = \overrightarrow{OP}$ which makes angles α, β and γ with the coordinate axes OX, OY and OZ respectively. Then Cos α, Cos β and Cos γ are called the direction cosines of the vector $\overrightarrow{OP}$. They are usually denoted by l, m and n respectively.

If $\overrightarrow{OP} = \vec{r} = xi + yj + zk$, then x, y and z are defined as the direction ratios of the vector $\vec{r}$ and $\vec{r} = \sqrt{x^2 + y^2 + z^2}$. Since the angles A, B and C are right angles (by the fig. 11), so in the right triangles.

OAP, OBP and OCP the direction cosines of $\vec{r}$ can be written as,

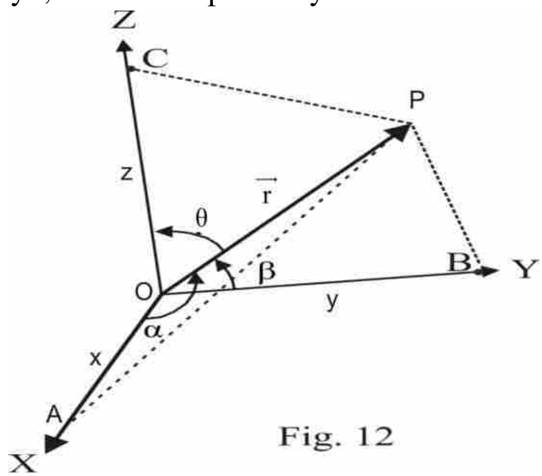


Fig. 12

$$l = \cos\alpha = \frac{x}{|r|} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$m = \cos\beta = \frac{y}{|r|} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

and $$n = \cos\gamma = \frac{z}{|r|} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Note 1: Since the unit vector $\hat{r} = \frac{\vec{r}}{|r|} = \frac{xi + yj + zk}{|r|}$

$$\hat{r} = \frac{x}{|r|}i + \frac{y}{|r|}j + \frac{z}{|r|}k$$

$$\hat{r} = \text{Cos }\alpha\text{ i} + \text{Cos }\beta\text{ j} + \text{Cos }\gamma\text{ k}$$

Or $$\hat{r} = li + mj + nk$$

Therefore the co-efficient of i, j and k in the unit vector are the direction cosines of a vector.

Note 2: $$l^2 + m^2 + n^2 = \frac{x^2}{|r|^2} + \frac{y^2}{|r|^2} + \frac{z^2}{|r|^2}$$

$$=\frac{x^2+y^2+z^2}{|r|^2}=\frac{x^2+y^2+z^2}{x^2+y^2+z^2}=1$$

Example 5:

Find the magnitude and direction cosines of the vectors $3i + 7j - 4k$, $i - 5j - 8k$ and $6i - 2j + 12k$.

Solution:

Let
$$\overrightarrow{a} = 3i + 7j - 4k$$
$$\overrightarrow{b} = i - 5j - 8k$$
$$\overrightarrow{c} = 6i - 2j + 12k$$

Now
$$\hat{a} = \frac{\overrightarrow{a}}{|a|} = \frac{3i + 7j - 4k}{\sqrt{74}}$$
$$= \frac{3}{\sqrt{74}}i + \frac{7}{\sqrt{74}}j - \frac{4}{\sqrt{74}}k$$

So the direction cosines of $\overrightarrow{a}$ are: $\frac{3}{\sqrt{74}}, \frac{7}{\sqrt{74}}, -\frac{4}{\sqrt{74}}$

Similarly the direction cosines of $\overrightarrow{b}$ are: $\frac{1}{\sqrt{90}}, -\frac{5}{\sqrt{90}}, -\frac{8}{\sqrt{90}}$

and the direction cosines of $\overrightarrow{c}$ are: $\frac{6}{\sqrt{184}}, -\frac{2}{\sqrt{184}}, \frac{12}{\sqrt{184}}$

Exercise8.1

Q.1 If $\overrightarrow{a} = 3i - j - 4k$, $b = -2i + 4j - 3k$ and $\overrightarrow{c} = i + 2j - k$.
Find unit vector parallel to $3\overrightarrow{a} - 2\overrightarrow{b} + 4\overrightarrow{c}$.

Q.2 Find the vector whose magnitude is 5 and which is in the direction of the vector $4i - 3j + k$.

Q.3 For what value of m, the vector $4i + 2j - 3k$ and $mi - j + \sqrt{3}k$ have same magnitude?

Q.4 Given the points $A = (1, 2, -1)$, $B = (-3, 1, 2)$ and $C = (0, -4, 3)$
(i) find $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{AC}$ (ii) Show that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

Q.5 Find the lengths of the sides of a triangle, whose vertices are $A = (2, 4, -1)$, $B = (4, 5, 1)$, $C = (3, 6, -3)$ and show that the triangle is right angled.

Q.6 If vectors $3i + j - k$ and $\lambda i - 4j + 4k$ are parallel, find the value of λ.

Q.7 Show that the vectors $4i - 6j + 9k$ and $-6i + 9j - \frac{27}{2}k$ are parallel.

Q.8 Find real numbers x, y and z such that

(a) $7xi + (y - 3)j + 6k = 10i + 8j - 3zk$

(b) $(x + 4)i + (y - 5)j + (z - 1)k = 0$

Q.9 Given the vectors $\overrightarrow{a} = 3i - 2j + 4k$ and $\overrightarrow{b} = 2i + j + 3k$ find the magnitude and direction cosines of

(i) $\overrightarrow{a} - \overrightarrow{b}$ (ii) $3\overrightarrow{a} - 2\overrightarrow{b}$

Q.10 If the position vector of $\overrightarrow{A}$ and $\overrightarrow{B}$ $5i - 2j + 4k$ and $i + 3j + 7k$ respectively, find the magnitude and direction cosines of $\overrightarrow{AB}$.

Answers 8.1

Q.1 $\frac{1}{\sqrt{398}}(17i - 3j - 10k)$ Q.2 $\frac{5}{\sqrt{26}}(4i - 3j + k)$

Q.3 ± 5 Q.4 $(-4, -1, 3), (3, -5, 1), (-1, -6, 4)$

Q.5 $AB = AC = 3$, $BC = 3\sqrt{2}$ Q.6 $\lambda = -12$

Q.8 (a) $x = \frac{10}{7}$ $y = 11$ $z = -2$ (b) $x = -4$, $y = 5$, $z = 1$

Q.9 (a) $\sqrt{11}; \frac{1}{\sqrt{11}}\ \frac{-3}{\sqrt{11}}\ \frac{1}{\sqrt{11}}$ (b) $5\sqrt{5}; \frac{1}{\sqrt{5}}\ \frac{-8}{5\sqrt{5}}\ \frac{6}{5\sqrt{5}}$

Q.10 $\sqrt{50}; \frac{-4}{\sqrt{50}}, \frac{5}{\sqrt{50}}, \frac{3}{\sqrt{50}}$

8.12 Product of Vectors:

1. Scalar Product of two Vectors:

If $\overrightarrow{a}$ and $\overrightarrow{b}$ are non-zero vectors, and θ is the angle between them, then the scalar product of $\overrightarrow{a}$ and $\overrightarrow{b}$ is denoted by $\overrightarrow{a} \cdot \overrightarrow{b}$ and read as $\overrightarrow{a}$ dot $\overrightarrow{b}$. It is defined by the relation

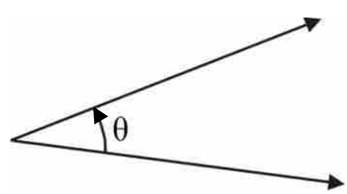


Fig.14

$$\overrightarrow{a} \cdot \overrightarrow{b} = |a|\ |b| \cos\theta \qquad \ldots\ldots\ldots\ldots \quad (1)$$

If either $\overrightarrow{a}$ or $\overrightarrow{b}$ is the zero vector, then $\overrightarrow{a} \cdot \overrightarrow{b} = 0$

Remarks:

i. The scalar product of two vectors is also called the dot product because the "." used to indicate this kind of multiplication. Sometimes it is also called the inner product.

ii. The scalar product of two non-zero vectors is zero if and only if they are at right angles to each other. For $\vec{a} \cdot \vec{b} = 0$ implies that Cos $\theta = 0$, which is the condition of perpendicularity of two vectors.

Deductions:

From the definition (1) we deduct the following:

i. If $\vec{a}$ and $\vec{b}$ have the same direction, then

$\theta = 0^o \Rightarrow \text{Cos } 0^o = 1$

$\vec{a} \cdot \vec{b} = |\vec{a}|\ |\vec{b}|$

ii. If $\vec{a}$ and $\vec{b}$ have opposite directions, then

$\theta = \pi \quad \Rightarrow \cos \pi = -1$

$\therefore \quad \vec{a} \cdot \vec{b} = -|\vec{a}|\ |\vec{b}|$

Iii $\vec{a} \cdot \vec{b}$ will be positive if $0 \le \theta < \frac{\pi}{2}$

and negative if, $\frac{\pi}{2} < \theta \le \pi$

iv The dot product of $\vec{a}$ and $\vec{b}$ is equal to the product of magnitude of $\vec{a}$ and the projection of $\vec{b}$ on $\vec{a}$. This illustrate the geometrical meaning of $\vec{a} \cdot \vec{b}$. In the fig. 15 $|\vec{b}|$ Cos θ is the projection of $\vec{b}$ on $\vec{a}$.

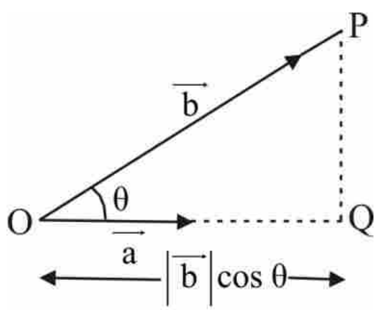


Fig. 15

v From the equation (1)

$\vec{b} \cdot \vec{a} = |\vec{b}|\ |\vec{a}|$ Cos θ

$= |\vec{a}|\ |\vec{b}|$ Cos θ

$\vec{b} \cdot \vec{a} = \vec{a} \cdot \vec{b}$

Hence the dot product is commutative.

Corollary 1:

If $\overrightarrow{a}$ be a vector, then the scalar product $\overrightarrow{a} \cdot \overrightarrow{a}$ can be expressed with the help of equation (1) as follows:

$$\overrightarrow{a} \cdot \overrightarrow{a} = |\overrightarrow{a}||\overrightarrow{a}| \text{Cos } 0^o = |\overrightarrow{a}|^2$$

Or $$|\overrightarrow{a}| = = \sqrt{\vec{a} \cdot \vec{a}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

This relation gives us the magnitude of a vector in terms of dot product.

Corollary 2:

If i, j and k are the unit vectors in the directions of X−, Y− and Z− axes, then from eq. (2)

$$i^2 = i.i = |i|\ |i| \text{ Cos}0^o$$
$$i^2 = 1$$

so $i^2 = j^2 = k^2 = 1$

and $i \cdot j = j \cdot i = 0$ Because $\text{Cos}90^o = 0$

$$i \cdot k = k \cdot i = 0$$
$$k \cdot i = i \cdot k = 0$$

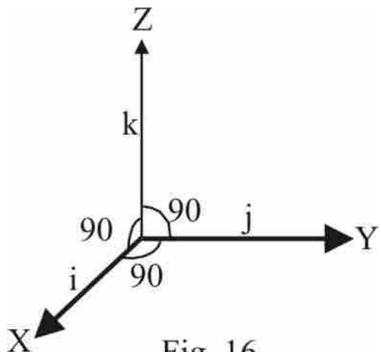


Fig. 16

Corollary 3:

(Analytical expression of $\vec{a} \cdot \vec{b}$)

Scalar product of two vectors in terms of their rectangular components.

For the two vectors

$$\overrightarrow{a} = a_1i + a_2j + a_3k$$

and $$\overrightarrow{b} = b_1i + b_2j + b_3k$$

the dot product is given as,

$$\overrightarrow{a} \cdot \overrightarrow{b} = (a_1i + a_2j + a_3k) \cdot (b_1i + b_2j + b_3k)$$
$$= a_1b_1 + a_2b_2 + a_3b_3 \text{ as } i^2 = j^2 = k^2 = 1$$
$$\text{and } i \cdot j = j \cdot k = k \cdot i = 0$$

Also $\overrightarrow{a}$ and $\overrightarrow{b}$ are perpendicular if and only if $a_1b_1 + a_2b_2 + a_3b_3 = 0$

Example 6:

If $\overrightarrow{a} = 3i + 4j - k$, $\overrightarrow{b} = -2i + 3j + k$ find $\overrightarrow{a} \cdot \overrightarrow{b}$

Solution:

$$\overrightarrow{a} \cdot \overrightarrow{b} = (3i + 4j - k) \cdot (-2i + 3j + k)$$
$$= -6 + 12 - 1$$
$$= 5$$

Example 7:

For what values of λ, the vectors 2i – j + 2k and 3i + 2λj are perpendicular?

Solution:

Let $\vec{a} = 2i - j + 2k$ and $\vec{b} = 3i + 2\lambda j$

Since $\vec{a}$ and $\vec{b}$ are perpendicular,

So $\vec{a} \,.\, \vec{b} = 0$

$(2i - j + 2k) \,.\, (3i + 2\lambda j) = 0$

$6 - 2\lambda = 0$

Or $\lambda = 3$

Example 8:

Find the angle between the vectors $\vec{a}$ and $\vec{b}$, where

$\vec{a} = i + 2j - k$ and $\vec{b} = -i + j - 2k$.

Solution:

As $\vec{a} \,.\, \vec{b} = |\vec{a}||\vec{b}| \text{ Cos } \theta$

Therefore $\text{Cos } \theta = \dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}|.|\vec{b}|}$, $\vec{a}\,.\,\vec{b} = -1 + 2 + 2 = 3$

$|\vec{a}| = \sqrt{1+4+1} = \sqrt{6}$, $|\vec{b}| = \sqrt{1+1+4} = \sqrt{6}$

$\text{Cos } \theta = \dfrac{3}{\sqrt{6}\sqrt{6}}$

$\text{Cos } \theta = \dfrac{3}{6} = \dfrac{1}{2}$

$\theta = \text{Cos}^{-1}\dfrac{1}{2} = 60^{o}$

Example 9:

Consider the points A, B, C, D where coordinates are respectively (1, 1, 0), (−1, 1, 0), (1, −1, 0), (0, −1, 1). Find the direction cosines of AC and BD and calculate the angle between them.

Solution:

Now we have A(1, 1, 0), B(−1, 1, 0), C(1, −1, 0), D(0, −1, 1)

$\vec{a} = \overline{AC} = (1-1)i + (-1-1)j + (0-0)k = -2j$

$\therefore$ Unit vector along AC = $\dfrac{\overrightarrow{AC}}{|AC|} = \dfrac{-2j}{2} = -j$

$\therefore$ The direction cosines of AC are 0, −1, 0

Now

$$\vec{b} = \overrightarrow{BD} = (0+1)i + (-1-1)j + (1-0)k = i - 2j + k$$

Unit vector along BD = $\dfrac{\overrightarrow{BD}}{|BD|} = \dfrac{i-2j+k}{\sqrt{1+4+1}} = \dfrac{i-2j+k}{\sqrt{6}}$

$\therefore$ The direction cosines of BD are:

$$\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$$

Let, θ be the angle between AC and BD then:

$$\text{Cos } \theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{\overrightarrow{AC}\cdot\overrightarrow{BD}}{|\overrightarrow{AC}|\,|\overrightarrow{BD}|}$$

$$= \frac{(-2j).(i-2j+k)}{(2)\sqrt{6}}$$

$$= \frac{(-2)(-2)}{(2)\sqrt{6}} = \frac{2}{\sqrt{6}}$$

$$\theta = \text{Cos}^{-1}\left(\frac{2}{\sqrt{6}}\right)$$

Example 10:

Show that if $|a + b| = |a - b|$ then a and b are perpendicular.

Solution:

We have $|a + b| = |a - b|$

$\therefore$ $|a + b|^2 = |a - b|^2$ taking square.

$a^2 + b^2 + 2a.b = a^2 + b^2 - 2a.b$

$4a.b = 0$ or $a.b = 0$

Hence $\bar{a}$ and $\bar{b}$ are perpendicular.

2. Vector Product:

If $\vec{a}$ and $\vec{b}$ are non-zero vectors and θ is the angle between $\vec{a}$ and $\vec{b}$, then the vector product of $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$, is the vector $\vec{c}$ which is perpendicular to the plane determined by $\vec{a}$ and $\vec{b}$. It is defined by the relation,

$$\vec{c} = \vec{a} \times \vec{b} = \left(|\vec{a}||\vec{b}|\text{Sin } \theta\right)n$$

Where $|\vec{a}||\vec{b}|\text{Sin}\,\theta$ is the magnitude of $\vec{c}$ and n is the Unit Vector in the direction of $\vec{c}$. The direction of $\vec{c}$ is determined by the right hand rule.

The vector product is also called the 'cross product' or 'Outer product' of the vectors.

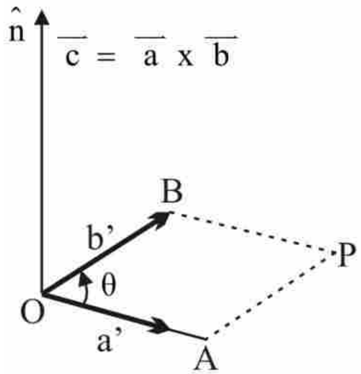


Fig. 17

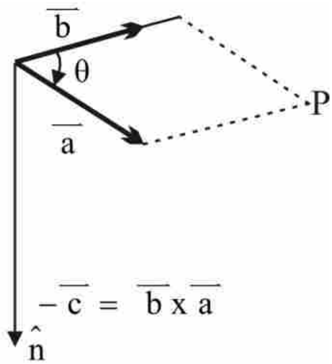


Fig. 18

Remarks:

If we consider $\vec{b} \times \vec{a}$, then $\vec{b} \times \vec{a}$ would be a vector which is opposite in the direction to $\vec{a} \times \vec{b}$.

Hence $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

Which gives that $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$ in general

Hence the vector product is not commutative.

Deductions:

The following results may be derived from the definition.

i. The vector product of two non-zero vectors is zero if $\vec{a}$ and $\vec{b}$ are parallel, the angle between $\vec{a}$ and $\vec{b}$ is zero. Sin $0^o = 0$, Hence $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = 0$.

For a x b = 0 implies that $\text{Sin}\,\theta = 0$ which is the condition of parallelism of two vectors. In particular $\vec{a} \times \vec{a} = 0$. Hence for the unit vectors i, j and k,

$$i \times i = j \times j = k \times k = 0$$

ii. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, then $\vec{a} \times \vec{b}$ is a vector whose magnitude is $|\vec{a}||\vec{b}|$ and whose direction is such that the vectors a, b, a x b form a right-handed system of three mutually perpendicular

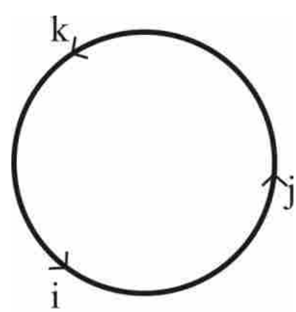


Fig. 19

vectors. In particular i x j = (1) (1) Sin 90^o k (k being perpendicular to i and j) = k

Similarly j x i = −k, i x k = −j, k x j = −i

Hence the cross product of two consecutive unit vectors is the third unit vector with the plus or minus sign according as the order of the product is anti-clockwise or clockwise respectively.

iii. Since $\left|\overrightarrow{a} \times \overrightarrow{b}\right| = \left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right| \text{Sin}\,\theta$ ...... (2)

Which is the area of the parallelogram whose two adjacent sides are $\left|\overrightarrow{a}\right|$ and $\left|\overrightarrow{b}\right|$.

Hence, area of parallelogram OABC = $\left|\overrightarrow{a} \times \overrightarrow{b}\right|$

and area of triangle OAB = $\frac{1}{2}\left|\overrightarrow{a} \times \overrightarrow{b}\right|$

If th vertices of a parallelogram are given, then

area of parallelogram OABC = $\left|\overrightarrow{OA} \times \overrightarrow{OB}\right|$

and, area of triangle OAB = $\frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OB}\right|$

iv. If n is the unit vector in the directions of $\overrightarrow{c} = \overrightarrow{a} \times \overrightarrow{b}$ then

$$n = \frac{\overrightarrow{c}}{|c|} = \frac{\overrightarrow{a} \times \overrightarrow{b}}{|a \times b|}$$

or $$n = \frac{\overrightarrow{a} \times \overrightarrow{b}}{|a||b| \text{Sin}\,\theta}$$

from equation (2) we also find.

$$\text{Sin}\,\theta = \frac{\left|\overrightarrow{a} \times \overrightarrow{b}\right|}{|a||b|}$$

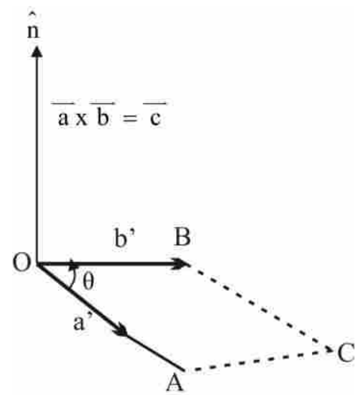


Fig. 20

8.13 Rectangular form of $\overrightarrow{a} \times \overrightarrow{b}$
(Analytical expression of $\overrightarrow{a} \times \overrightarrow{b}$)

If $\overrightarrow{a} = a_1i + a_2j + a_3k$

and $\overrightarrow{b} = b_1i + b_2j + b_3k$

then $\overrightarrow{a} \times \overrightarrow{b} = (a_1i + a_2j + a_3k) \times (b_1i + b_2j + b_3k)$

$= (a_1b_2k - a_1b_3j - a_2b_1k + a_2b_3i + a_3b_1j - a_3b_2i)$

$= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$

This result can be expressed in determinant form as

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 11:

If $\overrightarrow{a} = 2i + 3j + 4k$ $\overrightarrow{b} = I - j + k$, Find

(i) $\overrightarrow{a} \times \overrightarrow{b}$

(ii) Sine of the angle between these vectors.

(iii) Unit vector perpendicular to each vector.

Solution:

(i) $$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\overrightarrow{a} \times \overrightarrow{b} = i(3+4) - j(2-4) + k(-2-3)$$
$$= 7i + 2j - 5k$$

(ii) $$\text{Sin } \theta = \frac{\left|\overrightarrow{a} \times \overrightarrow{b}\right|}{\left|\overrightarrow{a}\right|\left|\overrightarrow{b}\right|} = \frac{\sqrt{7^2 + 2^2 + (-5)^2}}{\sqrt{2^2 + 3^2 + 4^2}.\sqrt{1^2 + (-1)^2 + 1^2}}$$

$$= \frac{\sqrt{78}}{\sqrt{29}\sqrt{3}}$$

$$\text{Sin } \theta = \sqrt{\frac{26}{29}}$$

(iii) If $\hat{n}$ is the unit vector perpendicular to $\overrightarrow{a}$ and $\overrightarrow{b}$ then

$$\hat{n} = \frac{\overrightarrow{a} \times \overrightarrow{b}}{\left|\overrightarrow{a} \times \overrightarrow{b}\right|} = \frac{7i + 2j - 5k}{\sqrt{78}}$$

Example 12:

$\overrightarrow{a} = 3i + 2k$, $\overrightarrow{b} = 4i + 4j - 2k$

$\overrightarrow{c} = i - 2j + 3k$, $\overrightarrow{d} = 2i - j + 5k$

Compute $(\overrightarrow{d} \times \overrightarrow{c}).(\overrightarrow{a} - \overrightarrow{b})$

Solution:

$$\overrightarrow{d} \times \overrightarrow{c} = \begin{vmatrix} i & j & k \\ 2 & -1 & 5 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= i(-3 + 10) - j(6 - 5) + k(-4 + 1)$$
$$= 7i - j - 3k$$

Also $\overrightarrow{a} - \overrightarrow{b} = -i - 4j + 4k$

Hence $(\overrightarrow{d} \times \overrightarrow{c}).(\overrightarrow{a} - \overrightarrow{b}) = (7i - j - 3k).(-i - 4j + 4k)$

$= -7 + 4 - 12$

$= -15$

Example 13:

Find the area of the parallelogram with adjacent sides,

$\overrightarrow{a} = i - j + k$, and $\overrightarrow{b} = 2j - 3k$

Solution:

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 0 & 2 & -3 \end{vmatrix}$$

$= i(3 - 2) - j(-3 - 0) + k(2 + 0)$

$= i + 3j + 2k$

Area of parallelogram $= |\overrightarrow{a} \times \overrightarrow{b}| = \sqrt{1 + 9 + 4}$

$= \sqrt{14}$ square unit.

Example 14:

Find the area of the triangle whose vertices are

A(0, 0, 0), B(1, 1, 1) and C(0, 2, 3)

Solution:

Since $\overrightarrow{AB} = (1 - 0, 1 - 0, 1 - 0)$

$= (1, 1, 1)$

and $\overrightarrow{AC} = (0 - 0, 2 - 0, 3 - 0)$

$\overrightarrow{AC} = (0, 2, 3)$

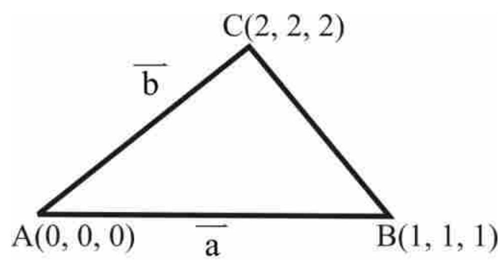


Fig. 21

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 2 & 3 \end{vmatrix}$$

$= i(3 - 2) - j(3 - 0) + k(2 - 0)$

$= i - 3j + 2k$

Area of the triangle ABC $= \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}\sqrt{1^2 + (-3)^2 + 2^2}$

$= \frac{\sqrt{14}}{2}$ square unit

Example 15:

Prove by the use of cross-product that the points

A(5, 2, −3), B(6, 1, 4), C(−2, −3, 6) and D(−3, −2, −1) are the vertices of a parallelogram.

Solution:

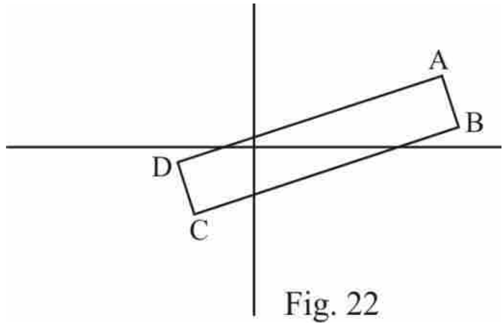


Fig. 22

Since $\overrightarrow{AB} = (1, -1, 7)$

$\overrightarrow{DC} = (+1, -1, +7)$

$\overrightarrow{BC} = (-8, -4, 2)$

and $\overrightarrow{AD} = (-8, -4, 2)$

$$\overrightarrow{AB} \times \overrightarrow{DC} = \begin{vmatrix} i & j & k \\ 1 & -1 & 7 \\ 1 & -1 & 7 \end{vmatrix}$$

$$= i(-7-7) - j(+7-7) + k(1-1)$$

$\overrightarrow{AB} \times \overrightarrow{DC} = 0$, so, $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are parallel.

Also $$\overrightarrow{BC} \times \overrightarrow{AD} = \begin{vmatrix} i & j & k \\ -8 & -4 & 2 \\ -8 & -4 & 2 \end{vmatrix}$$

$$= i(0) - j(0) + k(0)$$

$\overrightarrow{BC} \times \overrightarrow{AD} = 0$, so, $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are parallel.

Hence the given points are the vertices of a parallelogram.

Exercise 8.2

Q.1 Find $\overrightarrow{a} . \overrightarrow{b}$ and $\overrightarrow{a} \times \overrightarrow{b}$

(i) $\overrightarrow{a} = 2i + 3j + 4k$ $\quad \overrightarrow{b} = i - j + k$

(ii) $\overrightarrow{a} = i + j + k$ $\quad \overrightarrow{b} = -5i + 2j - 3k$

(iii) $\overrightarrow{a} = -i - j - k$ $\quad \overrightarrow{b} = 2i + j$

Q.2 Show that the vectors $3i - j + 7k$ and $-6i + 3j + 3k$ are at right angle to each other.

Q.3 Find the cosine of the angle between the vectors:

(i) $\overrightarrow{a} = 2i - 8j + 3k$ $\quad \overrightarrow{b} = 4j + 3k$

(ii) $\overrightarrow{a} = i + 2j - k$ $\quad \overrightarrow{b} = -j - 2k$

(iii) $\overrightarrow{a} = 4i + 2j - k$ $\quad \overrightarrow{b} = 2i + 4j - k$

Q.4 If $\overrightarrow{a} = 3i + j - k$, $\overrightarrow{b} = 2i - j + k$ and $\overrightarrow{c} = 5i + 3k$, find $(2\overrightarrow{a} + \overrightarrow{b}).\overrightarrow{c}$.

Q.5 What is the cosine of the angle between $\overline{P_1P_2}$ and $\overline{P_3P_4}$

If $P_1(2,1,3)$, $P_2(-4, 4, 5)$, $P_3(0, 7, 0)$ and $P_4(-3, 4, -2)$?

Q.6 If $\vec{a} = [a_1, a_2, a_3]$ and $\vec{b} = [b_1, b_2, b_3]$, prove that:

$$\vec{a}.\vec{b} = \frac{1}{2}\left[\left|\vec{a} + \vec{b}\right|^2 - \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2\right]$$

Q.7 Find $(\vec{a} + \vec{b}).(\vec{a} - \vec{b})$ if $\vec{a} = i + 2j + 3k$ and $\vec{b} = 2i - j + k$.

Q.8 Prove that for every pair of vectors $\vec{a}$ and $\vec{b}$

$$(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = \left|\vec{a}\right|^2 - \left|\vec{b}\right|^2$$

Q.9 Find x so that $\vec{a}$ and $\vec{b}$ are perpendicular,

(i) $\vec{a} = 2i + 4j - 7k$ and $\vec{b} = 2i + 6j + xk$

(ii) $\vec{a} = xi - 2j + 5k$ and $\vec{b} = 2i - j + 3k$

Q.10 If $\vec{a} = 2i - 3j + 4k$ and $\vec{b} = 2j + 4k$

Find the component or projection of $\vec{a}$ along $\vec{b}$.

Q.11 Under what condition does the relation $(\vec{a}.\vec{b})^2 = \vec{a}^2\,\vec{b}^2$ hold for two vectors $\vec{a}$ and $\vec{b}$.

Q.12 If the vectors $3i + j - k$ and $\lambda\, i - 4\, j + 4\, k$ are parallel, find value of λ.

Q.13 If $\vec{a} = i - 2j + k$, $\vec{b} = i + 2j - 4k$, $\vec{c} = 2i - 3j + k$ Evaluate:

(i) $(\vec{a} \times \vec{b}).(\vec{a} \times \vec{c})$ (ii) $(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})$

Q.14 If $\vec{a} = i + 3j - 7k$ and $\vec{b} = 5i - 2j + 4k$. Find:

(i) $\vec{a}.\vec{b}$ (ii) $\vec{a} \times \vec{b}$

(iii) Direction cosines of $\vec{a} \times \vec{b}$

Q.15 Prove that for the vectors $\vec{a}$ and $\vec{b}$

(i) $\left|\vec{a} \times \vec{b}\right|^2 + \left|\vec{a}.\vec{b}\right|^2 = \left|\vec{a}\right|^2 \left|\vec{b}\right|^2$

(ii) $(\vec{a} - \vec{b}) \times (\vec{a} \times \vec{b}) = 2(\vec{a} \times \vec{b})$

Q.16 Prove that for vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$[\vec{a} \times (\vec{b} + \vec{c})] + [\vec{b} \times (\vec{c} + \vec{a})] + [\vec{c} \times (\vec{a} + \vec{b})] = 0$$

Q.17 Find a vector perpendicular to both the lines, $\overrightarrow{AB}$ and $\overrightarrow{CD}$

where A is (0, 2, 4), B is (3, –1, 2), C is (2, 0, 1) and D is (4, 2, 0).

Q.18 Find $\left|(\overrightarrow{a} \times \overrightarrow{b}) \times c\right|$ if $\overrightarrow{a} = i - 2j - 3k$, $\overrightarrow{b} = 2i + j - k$, $\overrightarrow{c} = i + 3j - 2k$.

Q.19 Find the sine of the angle and the unit vector perpendicular to each:

(i) $\overrightarrow{a} = i + j + k$ and $\overrightarrow{b} = 2i + 3j - k$

(ii) $\overrightarrow{a} = 2i - j + k$ and $\overrightarrow{b} = 3i + 4j - k$

Q.20 Given $\overrightarrow{a} = 2i - j$ and $\overrightarrow{b} = j + k$, if $\left|\overrightarrow{c}\right| = 12$ and $\overrightarrow{c}$ is perpendicular to both $\overrightarrow{a}$ and $\overrightarrow{b}$, write the component form of $\overrightarrow{c}$.

Q.21 Using cross product, find the area of each triangle whose vertices have the following co-ordinates:

(i) (0, 0, 0),(1, 1, 1),(0, 0, 3)

(ii) (2, 0, 0), (0, 2, 0), (0, 0, 2)

(iii) (1, -1, 1),(2, 2, 2), (4, -2, 1)

Q.22 Find the area of parallelogram determined by the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ $\overrightarrow{a} = i + 2j + 3k$ and $\overrightarrow{b} = -3i - 2j + k$.

Answers 8.2

Q.1 **(i)** 3; $7i - 2j - 5k$ **(ii)** $-6, -5i - 2i + 7k$ **(iii)** $-3;\ -i - 2j + k$

Q.3 **(i)** $-\dfrac{33}{5\sqrt{77}}$ **(ii)** 0 **(iii)** $\dfrac{17}{21}$ **Q.4** 37

Q.5 $\dfrac{5}{7\sqrt{22}}$ **Q.7** 8 **Q.9(i)**4 **(ii)** $-\dfrac{17}{2}$ **Q.10** $\sqrt{5}$

Q.11 [0, 11] **Q.12** $\lambda = -12$ **Q.13** (i) 15 (ii)$i - 2j + k$

Q.14 **(i)** -29 **(ii)** $-2i - 39j - 17k$ **(iii)** $-\dfrac{2}{\sqrt{1814}}, \dfrac{-39}{\sqrt{1814}}, \dfrac{-17}{\sqrt{1814}}$

Q.17 $7i - j + 12k$ **Q.18** $5\sqrt{26}$

Q.19 **(i)** $\sqrt{\dfrac{13}{21}}, \dfrac{-4i + 3j + k}{\sqrt{26}}$ **(ii)** $\sqrt{\dfrac{155}{156}}, \dfrac{-3i + 5j + 11k}{\sqrt{155}}$

Q.20 $-4i - 8j + 8k$

Q.21 **(i)** $\dfrac{3\sqrt{2}}{2}$ sq. unit. **(ii)** $2\sqrt{3}$ sq. unit.

(iii) $\dfrac{\sqrt{110}}{2}$ sq. unit. Q.22 $\sqrt{180}$ sq. unit

Summary

A vector is a quantity which has magnitude as well as direction while scalar is a quantity which has only magnitude. Vector is denoted as $\overrightarrow{AB}$ or $\overrightarrow{OP}$.

1. If P (x, y, z) be a point in space, then the position vector of P relative to 0 = $\overrightarrow{OP}$.
2. Unit coordinator vectors x, j, k are taken as unit vector s along axis $\overrightarrow{OP}$ = xi + yj + zk.
3. Magnitude of a vector. i.e. $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$
4. Unit vector of a (non-zero vector), then $\hat{a} = \frac{\overrightarrow{a}}{|a|}$
5. Direction cosines of $\overrightarrow{OP}$ = xi + yj + zk then,

$$\cos\alpha = \frac{x}{|\overrightarrow{OP}|}.\ \text{Cos}\beta = \frac{y}{|\overrightarrow{OP}|}.\text{Cos}\gamma = \frac{z}{|\overrightarrow{OP}|}$$

Scalar product:

The scalar product of two vector $\overrightarrow{a}$ and $\overrightarrow{b}$ is defined as $\overrightarrow{a} \cdot \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}|\text{Cos}\,\theta$

1. If a.b = 0, vectors are perpendicular.
2. i.j = j.k = i.k = zero while i.i = j.j = k.k = 1
3. $\overrightarrow{a} \cdot \overrightarrow{b} = (a_1i + a_2j + a_3k).(b_1i + b_2j + b_3k) = a_1b_1 + a_2b_2 + a_3b_3$

Vector product:

The vector or cross product of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ denoted $\overrightarrow{a} \times \overrightarrow{b}$ and is defined as: $\overrightarrow{a} \times \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}|\text{Sin}\,\theta\ n$, $\text{Sin}\,\theta = \frac{|a \times b|}{|a||b|}$

1. $n = \frac{\overrightarrow{a} \times \overrightarrow{b}}{|a \times b|}$ unit vector.
2. $\overrightarrow{a} \times \overrightarrow{b} = 0$. $\overrightarrow{a}$ and $\overrightarrow{b}$ are parallel or collinear.
3. i x j = j x j = k x k = 0 and i x j = k.j x k = i, k x i = j
4. $\overrightarrow{a} \times \overrightarrow{b} = -\overrightarrow{b} \times \overrightarrow{a}$
5. $\overrightarrow{a} \times \overrightarrow{b} = (a_1i + a_2j + a_3k) \times (b_1i + b_2j + b_3k)$

$$= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Short Questions

Write the short answers of the following:

Q.1: What is scalar? Give examples.
Q.2: What is a vector? Give example.
Q.3: What is unit vector?
Q.4: Find the formula for magnitude of the vector $\vec{r} = xi + yj + zk$

Q.5: Find the magnitude of vector $-2i - 4j + 3k$

Q.6: What are parallel vectors?
Q.7: Find α, so that $|\alpha i + (\alpha + 1)j + 2k| = 3$
Q.8: If cosα , cosβ , cosγ are direction cosines of a vector

$\vec{r} = xi + yj + zk$, then show that $\cos^2\alpha + \cos^2\beta + co^2\gamma = 1$

Q.9: Find the unit vector along vector $4i - 3j - 5k$.

Q.10: Find the unit vector parallel to the sum of the vectors

$\bar{a} = [2, 4, -5]$, $\vec{b} = [1, 2, 3]$

Q.11: Given the vectors, $\bar{a} = 3i - 2j + k$, $\bar{b} = 2i - 4j - 3k$

$\bar{c} = -i + 2j + 2k$, Find a +b + c

Q.12: Given the vectors $\bar{a} = 3i + j - k$ and $\bar{b} = 2i + j - k$,find magnitude

of $3\bar{a} - \bar{b}$

Q.13: Find a vector whose magnitude is 2 and is parallel to $5i + 3j + 2k$.

Q.14: Define scalar product of two vectors.

Q.15 Find $\bar{a} . \bar{b}$ if $\bar{a} = i + 2j + 2k$, $\bar{b} = 3i - 2j - 4k$

Q.16: Find (a +b) . (a – b) if

$\bar{a} = 2i + 2j + 3k$, $\bar{b} = 2i - j + k$
Q.17: Define Vector product.

Q.18: If $\bar{a} = 2i + 3j + 4k$, $\bar{b} = i - j + k$

Find $|\vec{a} \times \vec{b}|$

Q.19: Find the area of parallelogram with adjacent sides,

$\bar{a} = 7i - j + k$ and $\bar{b} = 2j - 3k$

Q.20: For what value of λ, the vectors $2i - j + 2k$ and $3i + 2\lambda j$ are perpendicular.

Q.21: Under what conditions does the relation $\bar{a} \cdot \bar{b} = |\bar{a}|\,|\bar{b}|$ hold?

Q.22: Find scalars x and y such that $x(i + 2j) + y(3i + 4j) = 7$

Q.23: Prove that if $\bar{a} = i + 3j - 2k$ and $\bar{b} = i - j - k$, then $\bar{a}$ and $\bar{b}$ are perpendicular to each other.

Answers

5.	$\sqrt{29}$	7.	$1, -2$	9.	$\dfrac{4i - 3j - 5k}{5\sqrt{2}}$
10.	$\dfrac{3i + 6j - 2k}{7}$	11.	$4i - 4j + 0k$	12.	$\sqrt{54}$
13.	$\dfrac{10i + 6j + 4k}{\sqrt{38}}$	15.	-9	16.	8
18.	$\sqrt{78}$	19.	$\sqrt{14}$ sq. unit	20.	$\lambda = 3$
21	$\theta = 0^\circ$	22.	$x = \dfrac{-1}{2}$, $y = 5/2$		

Objective Type Question

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

__1. Magnitude of the vector 2i – 2j – k is:
(a) 4 (b) 3 (c) 2 (d) 1

__2. Unit vector of i + j + k is:
(a) i + j + k (b) $\frac{1}{3}$(i + j + k)
(c) $\frac{1}{\sqrt{3}}$(i + j + k) (d) $\frac{1}{2}$ (i + j + k)

__3. Unit vector of i – 2j – 2k is:
(a) i – 2j – 2k (b) $\frac{1}{3}$(i – 2j – 2k)
(c) $\frac{1}{\sqrt{3}}$(i – 2j – 2k) (d) $\frac{1}{2}$ (i – 2j – 2k)

__4. If $\hat{i}$, $\hat{j}$ and k are orthogonal unit vectors, then j x i is:
(a) k (b) −k (c) 1 (d) −1

__5. The magnitude of a vector $\hat{i}-3\hat{j}+5k$ is:
(a) 3 (b) 25 (c) 35 (d) $\sqrt{35}$

__6. In l, m and n are direction cosine of a vector, then:
(a) $l^2 - m^2 - n^2 = 1$ (b) $l^2 - m^2 + n^2 = 1$
(c) $l^2 + m^2 - n^2 = 1$ (d) $l^2 + m^2 + n^2 = 1$

__7. If θ is the angle between the vector $\vec{a}$ and $\vec{b}$, then cos θ is:
(a) $\vec{a} \cdot \vec{b}$ (b) $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$
(c) $\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$ (d) $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$

__8. If $\vec{a} = a_1j + a_2j + a_3k$, $\vec{b} = b_1i + b_2j + b_3k$, then $\vec{a} \cdot \vec{b}$ is:
(a) $a_1b_1j + a_2b_2j + a_3b_3k$ (b) $a_1b_1 + a_2b_2 + a_3b_3$
(c) $a_1b_2j + a_2b_3j + a_3b_1k$ (d) None of these

__9. $\vec{a} \cdot \vec{b} = 0$ implies that $\vec{a}$ and $\vec{b}$ are:
(a) Perpendicular (b) Parallel

(c) Non-parallel (d) Oblique

__10. If $\vec{a} = i + j + k$ and $\vec{b} = -i - j - mk$ are perndicular then m will be equal to:
(a) 1 (b) -2 (c) ± 1 (d) ± 3

__11. $\vec{a} \,.\, \vec{b}$ is a:
(a) Vector quantity (b) Scalar quantity
(c) Unity (d) None of these

__12. $\vec{a} \,.\, \vec{a}$ is equal to:
(a) 1 (b) a^2 (c) $|\vec{a}|$ (d) None of these

__13. If $\vec{a} = 2i - 3j + k$ and $\vec{b} = -i + 2j + 7k$ then $\vec{a} \,.\, \vec{b}$ is equal to:
(a) -1 (b) -2 (c) -3 (d) -4

__14. If $\vec{a} \times \vec{b} = 0$ then $\vec{a}$ and $\vec{b}$ is:
(a) Non-parallel (b) Parallel
(c) Perpendicular (d) None of these

__15. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is:
(a) $|\vec{a}||\vec{b}|\cos\theta$ (b) $|\vec{a}||\vec{b}|\sin\theta$
(c) $|\vec{a}||\vec{b}|\sin\theta\ \hat{n}$ (d) $|\vec{a}||\vec{b}|\cos\theta\ \hat{n}$

__16. If $\hat{n}$ is the unit vector in the direction of $\vec{a} \times \vec{b}$, then $\hat{n}$ is:
(a) $\dfrac{\vec{a} \times \vec{b}}{|\vec{a}||\vec{b}|}$ (b) $\dfrac{\vec{a} \,.\, \vec{b}}{|\vec{a}||\vec{b}|}$ (c) $\dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|\sin\theta}$ (d) $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

__17. $|\vec{a} \times \vec{b}|$ is area of the figure called:
(a) Triangle (b) Rectangle (c) Parallelogram (d) Sector

__18. $\vec{a} \times \vec{b}$ is equal to:
(a) $-\vec{b} \times \vec{a}$ (b) $\vec{b} \times \vec{a}$ (c) $|\vec{a} \times \vec{b}|$ (d) $|\vec{b} \times \vec{a}|$

__19. If $\vec{a}$ and $\vec{b}$ are collinear vectors, then:
(a) $\vec{a} \times \vec{b} = 0$ (b) $\vec{a} \,.\, \vec{b} = 0$
(c) $\vec{a} - \vec{b} = 0$ (d) $\vec{a} + \vec{b} = 0$

__20. $|\vec{a} \times \vec{b}|$ is a:
(a) Vector quantity (b) Scalar quantity
(c) Unity (d) None of these

Answers

1.	b	2.	c	3.	b	4.	b	5.	d
6.	d	7.	b	8.	b	9.	a	10.	c
11.	b	12.	b	13.	a	14.	b	15.	c
16.	d	17.	c	18.	a	19.	a	20.	b

Chapter 9
Matrices and Determinants

9.1 Introduction:

In many economic analysis, variables are assumed to be related by sets of linear equations. Matrix algebra provides a clear and concise notation for the formulation and solution of such problems, many of which would be complicated in conventional algebraic notation. The concept of determinant and is based on that of matrix. Hence we shall first explain a matrix.

9.2 Matrix:

A set of mn numbers (real or complex), arranged in a rectangular formation (array or table) having m rows and n columns and enclosed by a square bracket [ ] is called m×n matrix (read "m by n matrix") .

An m×n matrix is expressed as

$$A=\begin{bmatrix} a_{11} & a_{12} & ----a_{1n} \\ a_{21} & a_{22} & ----a_{2n} \\ --- & ---- & ------ \\ --- & ---- & ------ \\ a_{m1} & a_{m2} & ----a_{mn} \end{bmatrix}$$

The letters a_{ij} stand for real numbers. Note that a_{ij} is the element in the ith row and jth column of the matrix .Thus the matrix A is sometimes denoted by simplified form as (a_{ij}) or by $\{a_{ij}\}$ i.e., $A = (a_{ij})$

Matrices are usually denoted by capital letters A, B, C etc and its elements by small letters a, b, c etc.

Order of a Matrix:

The order or dimension of a matrix is the ordered pair having as first component the number of rows and as second component the number of columns in the matrix. If there are 3 rows and 2 columns in a matrix, then its order is written as (3, 2) or (3 x 2) read as three by two. In general if m are rows and n are columns of a matrix, then its order is (m x n).

Examples:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

are matrices of orders (2 x 3), (3 x 1) and (4 x 4) respectively.

9.3 Some types of matrices:

1. Row Matrix and Column Matrix:

A matrix consisting of a single row is called a **row matrix or a row vector**, whereas a matrix having single column is called a **column matrix or a column vector.**

2. Null or Zero Matrix:

A matrix in which each element is '0' is called a Null or Zero matrix. Zero matrices are generally denoted by the symbol O. This distinguishes zero matrix from the real number 0.

For example $O = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a zero matrix of order 2 x 4.

The matrix O_{mxn} has the property that for every matrix A_{mxn},

$A + O = O + A = A$

3. Square matrix:

A matrix A having same numbers of rows and columns is called a square matrix. A matrix A of order m x n can be written as A_{mxn}. If m = n, then the matrix is said to be a square matrix. A square matrix of order n x n, is simply written as A_n.

Thus $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ are square matrix of order 2 and 3

Main or Principal (leading)Diagonal:

The principal diagonal of a square matrix is the ordered set of elements a_{ij}, where i = j, extending from the upper left-hand corner to the lower right-hand corner of the matrix. Thus, the principal diagonal contains elements a_{11}, a_{22}, a_{33} etc.

For example, the principal diagonal of

$$\begin{bmatrix} 1 & 3 & -1 \\ 5 & 2 & 3 \\ 6 & 4 & 0 \end{bmatrix}$$

consists of elements 1, 2 and 0, in that order.

Particular cases of a square matrix:

(a)Diagonal matrix:

A square matrix in which all elements are zero except those in the main or principal diagonal is called a diagonal matrix. Some elements of the principal diagonal may be zero but not all.

For example $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are diagonal matrices.

In general $A = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ --- & --- & --- & --- \\ --- & --- & --- & --- \\ a_{n1} & a_{n2} & --- & a_{nn} \end{bmatrix} = (a_{ij})_{nxn}$

is a diagonal matrix if and only if

$a_{ij} = 0$ for $i \neq j$

$a_{ij} \neq 0$ for at least one $i = j$

(b) Scalar Matrix:

A diagonal matrix in which all the diagonal elements are same, is called a scalar matrix i.e.

Thus

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ are scalar matrices

(c) Identity Matrix or Unit matrix:

A scalar matrix in which each diagonal element is 1(unity) is called a unit matrix. An identity matrix of order n is denoted by I_n.

Thus $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are the identity matrices of order 2 and 3 .

In general, $A = \begin{bmatrix} a_{11} & a_{12} & ----a_{1n} \\ a_{21} & a_{22} & ----a_{2n} \\ --- & ---- & ------- \\ --- & ---- & ------- \\ a_{m1} & a_{m2} & ----a_{mn} \end{bmatrix} = [a_{ij}]_{mxn}$

is an identity matrix if and only if

$a_{ij} = 0$ for $i \neq j$ and $a_{ij} = 1$ for $i = j$

Note: If a matrix A and identity matrix I are comformable for multiplication, then I has the property that

AI = IA = A i.e., I is the identity matrix for multiplication.

4. Equal Matrices:

Two matrices A and B are said to be equal if and only if they have the same order and each element of matrix A is equal to the corresponding element of matrix B i.e for each i, j, $a_{ij} = b_{ij}$

Thus $A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{4}{2} & 2-1 \\ \sqrt{9} & 0 \end{bmatrix}$

then A = B because the order of matrices A and B is same and $a_{ij} = b_{ij}$ for every i , j.

Example 1: Find the values of x , y , z and a which satisfy the matrix equation

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4a-6 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

Solution : By the definition of equality of matrices, we have

$x + 3 = 0$ ……………………………..(1)

$2y + x = -7$ ……………………………(2)

$z - 1 = 3$ ……………………………(3)

$4a - 6 = 2a$ ……………………………(4)

From (1) $x = -3$

Put the value of x in (2) , we get $y = -2$
From (3) $z = 4$
From (4) $a = 3$

5. The Negative of a Matrix:

The negative of the matrix A_{mxn}, denoted by $-A_{mxn}$, is the matrix formed by replacing each element in the matrix A_{mxn} with its additive inverse. For example,

If $$A_{3x2} = \begin{bmatrix} 3 & -1 \\ 2 & -2 \\ -4 & 5 \end{bmatrix}$$

Then $$-A_{3x2} = \begin{bmatrix} -3 & 1 \\ -2 & 2 \\ 4 & -5 \end{bmatrix}$$

for every matrix A_{mxn}, the matrix $-A_{mxn}$ has the property that

$$A + (-A) = (-A) + A = 0$$

i.e., $(-A)$ is the additive inverse of A.

The sum $B_{m-n} + (-A_{mxn})$ is called the difference of B_{mxn} and A_{mxn} and is denoted by $B_{mxn} - A_{mxn}$.

9.4 Operations on matrices:

(a) Multiplication of a Matrix by a Scalar:

If A is a matrix and k is a scalar (constant), then kA is a matrix whose elements are the elements of A , each multiplied by k

For example, if $A = \begin{bmatrix} 4 & -3 \\ 8 & -2 \\ -1 & 0 \end{bmatrix}$ then for a scalar k,

$$kA = \begin{bmatrix} 4k & -3k \\ 8k & -2k \\ -k & 0 \end{bmatrix}$$

Also, $3\begin{bmatrix} 5 & -8 & 4 \\ 0 & 3 & -5 \\ 3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 15 & -24 & 12 \\ 0 & 9 & -15 \\ 9 & -3 & 12 \end{bmatrix}$

(b) Addition and subtraction of Matrices:

If A and B are two matrices of same order m×n then their sum A + B is defined as C, m×n matrix such that each element of C is the sum of the corresponding elements of A and B .

for example

If $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 0 \end{bmatrix}$

Then $C = A + B = \begin{bmatrix} 3+1 & 1+0 & 2+2 \\ 2-1 & 1+3 & 4+0 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 4 \\ 1 & 4 & 4 \end{bmatrix}$

Similarly, the difference A – B of the two matrices A and B is a matrix each element of which is obtained by subtracting the elements of B from the corresponding elements of A

Thus if $A = \begin{bmatrix} 6 & 2 \\ 7 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 1 \\ 3 & 4 \end{bmatrix}$

then $A - B = \begin{bmatrix} 6 & 2 \\ 7 & -5 \end{bmatrix} - \begin{bmatrix} 8 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 6-8 & 2-1 \\ 7-3 & -5-4 \end{bmatrix}$

$= \begin{bmatrix} -2 & 1 \\ 4 & -9 \end{bmatrix}$

If A, B and C are the matrices of the same order mxn

then $A + B = B + A$

and $(A + B) + C = A + (B + C)$ i.e., the addition of matrices is commutative and Associative respectively.

Note: The sum or difference of two matrices of different order is not defined.

(c) Product of Matrices:

Two matrices A and B are said to be conformable for the product AB if the number of columns of A is equal to the number of rows of B. Then the product matrix AB has the same number of rows as A and the same number of columns as B.

Thus the product of the matrices A_{mxp} and B_{pxn} is the matrix $(AB)_{mxn}$. The elements of AB are determined as follows:

The element C_{ij} in the ith row and jth column of $(AB)_{mxn}$ is found by $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots\ldots\ldots + a_{in}b_{nj}$

for example, consider the matrices

$$A_{2x2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B_{2x2} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Since the number of columns of A is equal to the number of rows of B ,the product AB is defined and is given as

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

Thus c_{11} is obtained by multiplying the elements of the first row of A i.e., a_{11} , a_{12} by the corresponding elements of the first column of B i.e., b_{11} , b_{21} and adding the product.

Similarly , c_{12} is obtained by multiplying the elements of the first row of A i.e., a_{11} , a_{12} by the corresponding elements of the second column of B i.e., b_{12} , b_{22} and adding the product. Similarly for c_{21} , c_{22} .

Note :

1 . Multiplication of matrices is not commutative i.e., AB $\neq$ BA in general.
2 . For matrices A and B if AB = BA then A and B commute to each other
3 . A matrix A can be multiplied by itself if and only if it is a square matrix.The product A.A in such cases is written as A^2.
Similarly we may define higher powers of a square matrix i.e., $A . A^2 = A^3$, $A^2. A^2 = A^4$
4. In the product AB, A is said to be pre multiple of B and B is said to be post multiple of A.

Example 1: **If A =** $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ **and B =** $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ **Find AB and BA.**

Solution:

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 1+2 \\ -2+3 & -1+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2-1 & 4+3 \\ 1-1 & 2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 7 \\ 0 & 5 \end{bmatrix}$$

This example shows very clearly that multiplication of matrices in general, is not commutative i.e., AB $\neq$ BA.

Example 2: If

Example 2: If A = $\begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ **and B =** $\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$**, find AB**

Solution:

Since A is a (2 x 3) matrix and B is a (3 x 2) matrix, they are conformable for multiplication. We have

$$AB = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3+2+6 & -3+1+2 \\ 1+0+3 & -1+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 \\ 4 & 0 \end{bmatrix}$$

Remark:

If A, B and C are the matrices of order (m x p), (p x q) and (q x n) respectively, then

i. (AB)C = A(BC) i.e., Associative law holds.

ii. $\left.\begin{array}{l} C(A+B) = CA + CB \\ \text{and } (A+B)C = AC + BC \end{array}\right\}$ i.e distributive laws holds.

Note: that if a matrix A and identity matrix I are conformable for multiplication, then I has the property that

AI = IA = A i.e, I is the identity matrix for multiplication.

Exercise 9.1

Q.No. 1 Write the following matrices in tabular form:

i. A = $[a_{ij}]$, where i = 1, 2, 3 and j = 1, 2, 3, 4
ii. B = $[b_{ij}]$, where i = 1 and j = 1, 2, 3, 4
iii. C = $[c_{jk}]$, where j = 1, 2, 3 and k = 1

Q.No.2 Write each sum as a single matrix:

i. $\begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 3 & 0 \\ -2 & 1 & 0 \end{bmatrix}$

ii. $\begin{bmatrix} 1 & 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 1 & 3 \end{bmatrix}$

iii. $\begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}$

iv. $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 6 & 2 \\ 1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

v. $2\begin{bmatrix} 6 & 1 \\ 0 & -3 \\ -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 4 & 2 \\ 0 & 1 \\ -5 & -1 \end{bmatrix}$

Q.3 Show that $\begin{bmatrix} b_{11}-a_{11} & b_{12}-a_{12} \\ b_{21}-a_{21} & b_{22}-a_{22} \end{bmatrix}$ is a solution of the matrix equation X + A = B, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$.

Q.4 Solve each of the following matrix equations:

i. $X + \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix}$

ii. $X + \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} -4 & -8 \\ -2 & 0 \end{bmatrix}$

iii. $3X + \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 4 & -1 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 1 \\ -1 & -2 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

iv. $X + 2I = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$

Q.5 Write each product as a single matrix:

i. $\begin{bmatrix} 3 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$

ii. $[3 \quad -2 \quad 2]\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

iii. $\begin{bmatrix} 2 & -2 & -1 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 & 5 \\ -1 & -1 & 3 \\ -1 & -2 & 4 \end{bmatrix}$

iv. $\begin{bmatrix} -1 & -2 & 5 \\ -1 & -1 & 3 \\ -1 & -2 & 4 \end{bmatrix}\begin{bmatrix} 2 & -2 & -1 \\ 1 & 1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$

Q.6 If $A = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 2 \\ 4 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, find $A^2 + BC$.

Q.7 Show that if $A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, then

(a) $(A + B)(A + B) \neq A^2 + 2AB + B^2$

(b) $(A + B)(A - B) \neq A^2 - B^2$

Q.8 Show that:

(i) $\begin{bmatrix} -1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 5 & -1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -a + 2b + 3c \\ 2a + b \\ 3a + 5b - c \end{bmatrix}$

(ii) $\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ +\sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & +\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q.9 If $A = \begin{bmatrix} 2 & -2\sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix}$

Show that A and B commute.

Answers 9.1

Q.1(i) $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

(ii) $\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \end{bmatrix}$

(iii) $\begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix}$

Q.2 (i) $\begin{bmatrix} 8 & 4 & 4 \\ 1 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & 6 & 9 \end{bmatrix}$

(iii) $\begin{bmatrix} 10 \\ 3 \\ -3 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 3 & 4 \\ -1 & 6 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

(v) $\begin{bmatrix} 0 & 4 & 0 \\ -9 & 13 & 7 \end{bmatrix}$

Q.4 (i) $\begin{bmatrix} 2 & 2 \\ -5 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 1 & -\frac{1}{3} \\ -1 & -1 & -1 \\ -\frac{4}{3} & \frac{2}{3} & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$

Q.5 (i) $\begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Q.6 $\begin{bmatrix} 6 & 17 \\ 8 & 9 \end{bmatrix}$

9.5 Determinants:

The Determinant of a Matrix:

The determinant of a matrix is a scalar (number), obtained from the elements of a matrix by specified, operations, which is characteristic of the matrix. The determinants are defined only for square matrices. It is denoted by det A or |A| for a square matrix A.

The determinant of the (2 x 2) matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by det A = |A| = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$$= a_{11}\,a_{22} - a_{12}\,a_{21}$$

Example 3: If $A = \begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix}$ find |A|

Solution:

$$|A| = \begin{vmatrix} 3 & 1 \\ -2 & 3 \end{vmatrix} = 9 - (-2) = 9 + 2 = 11$$

The determinant of the (3 x 3) matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{ denoted by } |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is given as, det A = |A|

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Note: Each determinant in the sum (In the R.H.S) is the determinant of a submatrix of A obtained by deleting a particular row and column of A.

These determinants are called minors. We take the sign + or −, according to $(-1)^{i+j} a_{ij}$

Where i and j represent row and column.

9.6 Minor and Cofactor of Element:

The minor M_{ij} of the element a_{ij} in a given determinant is the determinant of order (n – 1 x n – 1) obtained by deleting the ith row and jth column of A_{nxn}.

For example in the determinant

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The minor of the element a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

The minor of the element a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

The minor of the element a_{13} is $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ and so on.

The scalars $C_{ij} = (-1)^{i+j} M_{ij}$ are called the cofactor of the element a_{ij} of the matrix A.

Note: The value of the determinant in equation (1) can also be found by its minor elements or cofactors, as

$a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$ Or $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

Hence the det A is the sum of the elements of any row or column multiplied by their corresponding cofactors.

The value of the determinant can be found by expanding it from any row or column.

Example 4: If $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{bmatrix}$

find det A by expansion about (a) the first row (b) the first column.

Solution (a)

$$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= 3\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & -2 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix}$$

$= 3(4 + 6) - 2(0 + 2) + 1\,(0 - 1)$

$= 30 - 4 - 1$

$|A| = 25$

(b) $|A| = 3\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix}$

$= 3\,(4 + 6) + 1\,(-4 - 1)$

$= 30 - 5$

$|A| = 25$

9.7 Properties of the Determinant:

The following properties of determinants are frequently useful in their evaluation:

1. Interchanging the corresponding rows and columns of a determinant does not change its value (i.e., $|A| = |A'|$). For example, consider a determinant

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$ ... (2)

Now again consider

$$|B| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Expand it by first column

$|B| = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$

which is same as equation (2)

so $|B| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

or $|B| = |A|$

2. If two rows or two columns of a determinant are interchanged, the sign of the determinant is changed but its absolute value is unchanged.

For example if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Consider the determinant,

$$|B| = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

expand by second row,

$$|B| = -a_1(b_2c_3 - b_3c_2) + b_1(a_2c_3 - a_3c_2) - c_1(a_2b_3 - a_3b_2)$$
$$= -(a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2))$$

The term in the bracket is same as the equation (2)

So $$|B| = -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Or $|B| = -|A|$

3. If every element of a row or column of a determinant is zero, the value of the determinant is zero. For example

$$|A| = \begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
$$= 0(b_2c_3 - b_3c_2) - 0(a_2c_3 - a_3c_2) + 0(a_2b_3 - a_3b_3)$$
$$|A| = 0$$

4. If two rows or columns of a determinant are identical, the value of the determinant is zero. For example, if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$
$$= a_1(b_1c_3 - b_3c_1) - b_1(a_1c_3 - a_3c_1) + c_1(a_1b_3 - a_3b_1)$$
$$= a_1b_1c_3 - a_1b_3c_1 - a_1b_1c_3 + a_3b_1c_1 + a_1b_3c_1 - a_3b_1c_1$$
$$|A| = 0$$

5. If every element of a row or column of a determinant is multiplied by the same constant K, the value of the determinant is multiplied by that constant. For example if,

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Consider a determinant, $|B| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$|B| = ka_1(b_2c_3 - b_3c_3) - kb_1(a_2c_3 - a_3c_2) + kc_1(a_2b_3 - a_3b_2)$
$= k(a_1(b_2c_3 - b_3c_3) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2))$

So $|B| = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Or $|B| = K|A|$

6. The value of a determinant is not changed if each element of any row or of any column is added to (or subtracted from) a constant multiple of the corresponding element of another row or column. For example, if

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Consider a matrix,

$$|B| = \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$= (a_1+ka_2)(b_2c_3-b_3c_2)-(b_1+kb_2)(a_2c_3-a_3c_2)+(c_1+kc_2)(a_2b_3-a_3b_2)$
$= [a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$
$= [ka_2(b_2c_3 - b_3c_2) - kb_2(a_2c_3 - a_3c_2) + kc_2(a_2b_3 - a_3b_2)]$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k(0) \text{ because row 1st and 2nd are identical}$$

$|B| = |A|$

7. The determinant of a diagonal matrix is equal to the product of its diagonal elements. For example, if

$$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$= 2(-15 - 0) - (0 - 0) + 0(0 - 0)$

$= 30$, which is the product of diagonal elements.

i.e., $2(-5)3 = -30$

8. The determinant of the product of two matrices is equal to the product of the determinants of the two matrices, that is $|AB| = |A||B|$. for example, if

$$A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad B = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

$$\text{Then } AB = \begin{vmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{vmatrix}$$

$$\begin{aligned} |AB| &= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) \\ &\quad - (a_{11}b_{12} + a_{12}b_{22} - a_{11}b_{22})(a_{21}b_{11} + a_{22}b_{21}) \\ &= a_{11}b_{11}\, a_{21}b_{12} + a_{11}b_{11}\, a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} \\ &\quad + a_{12}b_{21}\, a_{22}b_{22} - a_{11}b_{12}\, a_{21}b_{11} - a_{11}b_{12}\, a_{22}b_{21} \\ &\quad - a_{12}b_{22}\, a_{21}b_{11} - a_{12}b_{22}\, a_{22}b_{21} \end{aligned}$$

$$\begin{aligned} |AB| &= a_{11}b_{11}\, a_{22}b_{22} + a_{12}b_{21}\, a_{21}b_{12} - a_{11}b_{12}\, a_{22}b_{21} \\ &\quad - a_{12}b_{22}\, a_{21}b_{11} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (A) \end{aligned}$$

and $|A| = a_{11}a_{22} - a_{12}a_{21}$

$|B| = b_{11}b_{22} - b_{12}b_{21}$

$$\begin{aligned} |A|\,|B| &= a_{11}b_{11}\, a_{22}\, b_{22} + a_{12}b_{21}\, a_{21}\, b_{12} - a_{11}b_{12}\, a_{22}\, b_{21} \\ &\quad - a_{12}b_{22}\, a_{21}\, b_{11} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (B) \end{aligned}$$

R.H.S of equations (A) and (B) are equal, so

$|AB| = |A|\,|B|$

9. The determinant in which each element in any row, or column, consists of two terms, then the determinant can be expressed as the sum of two other determinants

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

Expand by first column.

Proof:

L.H.S $= (a_1+\alpha_1)(b_2c_3-b_3c_2)-(a_2+\alpha_2)(b_1c_3-b_3c_1)+(a_3+\alpha_3)(b_1c_2-b_2c_1)$

$= [(a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)]$

$+ [(\alpha_1(b_2c_3-b_3c_2)- \alpha_2(b_1c_2-b_3c_1)+ \alpha_3(b_1c_2-b_2c_1)]$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

= R.H.S

Similarly

$$\begin{vmatrix} \alpha_1+a_1 & b_1+\beta_1 & c_1 \\ \alpha_2+a_2 & b_2+\beta_2 & c_2 \\ \alpha_3+a_3 & b_3+\beta_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \beta_1 & c_1 \\ a_2 & \beta_2 & c_2 \\ a_3 & \beta_3 & c_3 \end{vmatrix}$$

$$+ \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & c_1 \\ \alpha_2 & \beta_2 & c_2 \\ \alpha_3 & \beta_3 & c_3 \end{vmatrix}$$

And, $$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2+\alpha_2 & b_2+\beta_2 & c_2+\gamma_2 \\ a_3+\alpha_3 & b_3+\beta_3 & c_3+\gamma_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \text{sum of six determinant} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Also $$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & \beta_2 & \gamma_2 \\ a_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Example 5: Verify that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Solution:

Multiply row first, second and third by a, b and c respectively, in the L.H.S., then

$$\text{L.H.S} = \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

Take abc common from 3rd column

$$= \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

Interchange column first and third

$$= -\begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix}$$

Again interchange column second and third

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$= \text{R.H.S}$

Example 6: Show that

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b - c)\,(c - a)\,(a - b)$$

Solution:

$$\text{L.H.S} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

subtracting row first from second and third row

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

from row second and third taking (b – a) and (c – a) common.

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$

expand from first column

$$= (b-a)(c-a)(c+a-b-a)$$
$$= (b-a)(c-a)(c-b)$$

Or L.H.S $= (b-c)(c-a)(a-b)(-1)(-1)$

$$= (b-c)(c-a)(a-b) = \text{R.H.S}$$

Example 7: Without expansion, show that

$$\begin{vmatrix} 6 & 1 & 3 & 2 \\ -2 & 0 & 1 & 4 \\ 3 & 6 & 1 & 2 \\ -4 & 0 & 2 & 8 \end{vmatrix} = 0$$

Solution:

In the L.H.S Taking 2 common from fourth row, so

$$\text{L.H.S} = 2\begin{vmatrix} 6 & 1 & 3 & 2 \\ -2 & 0 & 1 & 4 \\ 3 & 6 & 1 & 2 \\ -2 & 0 & 1 & 4 \end{vmatrix}$$

Since rows 2nd and 3rd are identical, so

$$= 2(0) = 0$$

L.H.S = R.H.S

9.8 Solution of Linear Equations by Determinants: (Cramer's Rule)

Consider a system of linear equations in two variables x and y,

$$a_1x + b_1y = c_1 \quad (1)$$

$$a_2x + b_2y = c_2 \quad (2)$$

Multiply equation (1) by b_2 and equation (2) by b_1 and subtracting, we get

$$x(a_1b_2 - a_2b_1) = b_2c_1 - b_1c_2$$

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \quad (3)$$

Again multiply eq. (1) by a_2 and eq. (2) by a_1 and subtracting, we get

$$y(a_2b_1 - a_1b_2) = a_2c_1 - a_1c_2$$

$$y = \frac{a_2c_1 - a_1c_2}{a_2b_1 - a_1b_2}$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \quad (4)$$

Note that x and y from equations (3) and (4) has the same denominator $a_1b_2 - a_2b_1$. So the system of equations (1) and (2) has solution only when $a_1b_2 - a_2b_1 \neq 0$.

The solutions for x and y of the system of equations (1) and (2) can be written directly in terms of determinants without any algebraic operations, as

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \text{ and } y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

This result is called Cramer's Rule.

Here $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = |A|$ is the determinant of the coefficient of x and y in equations (1) and (2)

If $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = |A|$

and $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = |A|$

Then $x = \dfrac{|A_x|}{|A|}$ and $y = \dfrac{|A_y|}{|A|}$

Solution for a system of Linear Equations in Three Variables:

Consider the linear equations:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Hence the determinant of coefficients is

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ if } |A| \neq 0$$

Then by Cramer's Rule the value of variables is:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{|A|} = \frac{|A_x|}{|A|}$$

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{|A|} = \frac{|A_y|}{|A|}$$

and $$z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{|A|} = \frac{|A_z|}{|A|}$$

Example 8: Use Cramer's rule to solve the system

$$-4x + 2y - 9z = 2$$
$$3x + 4y + z = 5$$
$$x - 3y + 2z = 8$$

Solution:

Here the determinant of the coefficients is:

$$|A| = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 4 & 1 \\ 1 & -3 & 2 \end{vmatrix}$$

$$= -4(8 + 3) - 2(6 - 1) - 9(-9 - 4)$$

$$= -44 - 10 + 117$$

$$|A| = 63$$

for $|A_x|$, replacing the first column of $|A|$ with the corresponding constants 2, 5 and 8, we have

$$|A_x| = \begin{vmatrix} 2 & 2 & -9 \\ 5 & 4 & 1 \\ 8 & -3 & 2 \end{vmatrix}$$

$$= 2(11) - 2(2) - 9(-47) = 22 - 4 + 423$$

$$\boxed{|A_x| = 441}$$

Similarly,

$$|A_y| = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 5 & 1 \\ 1 & 8 & 2 \end{vmatrix}$$

$$= -4(2) - 2(5) - 9(19)$$

$$= -8 - 10 - 171$$

$$\boxed{|A_y| = -189}$$

and

$$|A_z| = \begin{vmatrix} -4 & 2 & 2 \\ 3 & 4 & 5 \\ 1 & -3 & 8 \end{vmatrix}$$

$$= -4(47) - 2(19) + 2(-13)$$

$$= -188 - 38 - 26$$

$$\boxed{|A_z| = -252}$$

Hence $x = \dfrac{|A_x|}{|A|} = \dfrac{441}{63} = 7$

$$y = \frac{|A_y|}{|A|} = \frac{-189}{63} = -3$$

$$z = \frac{|A_z|}{|A|} = \frac{-252}{63} = -4$$

So the solution set of the system is $\{(7, -3, -4)\}$

Exercise 9.2

Q.1 Expand the determinants

(i) $\begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \\ -2 & 1 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & b & 1 \\ a & b & 1 \\ 1 & 1 & 1 \end{vmatrix}$

(iii) $\begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}$

Q.2 Without expansion, verify that

(i) $\begin{vmatrix} -2 & 1 & 0 \\ 3 & 4 & 1 \\ -4 & 2 & 0 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -5 & 0 \end{vmatrix}$

(iii) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$ (iv) $\begin{vmatrix} bc & ca & ab \\ a^3 & b^3 & c^3 \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix} = 0$

(v) $\begin{vmatrix} x+1 & x+2 & x+3 \\ x+4 & x+5 & x+6 \\ x+7 & x+8 & x+9 \end{vmatrix} = 0$

(vi) $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} e & b & h \\ d & a & g \\ f & c & k \end{vmatrix}$

Q.3 Show that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1x+d_1 & c_2x+d_2 & c_3x+d_3 \end{vmatrix} = x \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

Q.4 Show that

(i) $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix} = 0$ (ii) $\begin{vmatrix} a & b & c \\ a & a+b & a+b+c \\ a & 2a+b & 3a+2b+c \end{vmatrix} = a^3$

(iii) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ b+c & c+a & a+b \end{vmatrix} = (b-c)(c-a)(a-b)$

Q.5 Show that:

(i) $\begin{vmatrix} \ell & a & a \\ a & \ell & a \\ a & a & \ell \end{vmatrix} = (2a+\ell)(\ell-a)^2$

(ii) $\begin{vmatrix} a+\ell & a & a \\ a & a+\ell & a \\ a & a & a+\ell \end{vmatrix} = \ell^2(3a+\ell)$

Q.6 prove that:

(i) $\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$

(ii) $\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2 (a + b + c + \lambda)$

(iii) $\begin{vmatrix} \text{Sin}\,\alpha & \text{Cos}\,\alpha & 0 \\ -\text{Sin}\,\beta & \text{Cos}\,\beta & \text{Sin}\,\gamma \\ \text{Cos}\,\beta & \text{Sin}\,\beta & \text{Cos}\,\gamma \end{vmatrix} = \text{Sin}\,(\alpha+\beta+\gamma)$

Q.7 Find values of x if

(i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$ (ii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

Q.8 Use Cramer's rule to solve the following system of equations.

(i) $x - y = 2$, $x + 4y = 5$ (ii) $3x - 4y = -2$, $x + y = 6$

(iii) $x - 2y + z = -1$, $3x + y - 2z = 4$, $y - z = 1$ (iv) $2x + 2y + z = 1$, $x - y + 6z = 21$, $3x + 2y - z = -4$

(v) $x + y + z = 0$, $2x - y - 4z = 15$, $x - 2y - z = 7$ (vi) $x - 2y - 2z = 3$, $2x - 4y + 4z = 1$, $3x - 3y - 3z = 4$

Answers 9.2

Q.1 (i) -41 (ii) 0 (iii) x^3

Q.7 (i) $x = -2, 3$ (ii) $x = 3, 4$

Q.8 (i) $\left\{\left(\frac{13}{5}, \frac{3}{5}\right)\right\}$ (ii) $\left\{\left(\frac{22}{7}, \frac{20}{7}\right)\right\}$

(iii) $\{(1, 1, 0)\}$ (iv) $\{(1, -2, 3)\}$

(v) $\{(3, -1, -2)$ (vi) $\left\{\left(-\frac{1}{3}, -\frac{25}{24}, -\frac{5}{8}\right)\right\}$

9.9 Special Matrices:

1. Transpose of a Matrix

If $A = [a_{ij}]$ is mxn matrix, then the matrix of order n x m obtained by interchanging the rows and columns of A is called the transpose of A. It is denoted A^t or A'.

Example if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $A^t = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

2. Symmetric Matrix:

A square matrix A is called symmetric if $A = A^t$
for example if

$$A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, \quad \text{then} \quad A^t = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = A$$

Thus A is symmetric

3. Skew Symmetric:

A square matrix A is called skew symmetric if $A = -A^t$

for example if $B = \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$,then

$$B^t = \begin{bmatrix} 0 & 4 & -1 \\ -4 & 0 & 3 \\ 1 & -3 & 0 \end{bmatrix} = (-1)\begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

$$B^t = -B$$

Thus matrix B is skew symmetric.

4. Singular and Non-singular Matrices:

A square matrix A is called singular if $|A| = 0$ and is non-singular if $|A| \neq 0$, for example if

$$A = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}, \text{ then } |A| = 0, \text{ Hence A is singular}$$

and if $A = \begin{bmatrix} 3 & 1 & 6 \\ -1 & 3 & 2 \\ 1 & 0 & 0 \end{bmatrix}$, then $|A| \neq 0$,

Hence A is non-singular.

Example: Find k If $A = \begin{bmatrix} k-2 & 1 \\ 5 & k+2 \end{bmatrix}$ is singular

Solution: Since A is singular so $\begin{vmatrix} k-2 & 1 \\ 5 & k+2 \end{vmatrix} = 0$

$(k-2)(k+2) - 5 = 0$

$k^2 - 4 - 5 = 0$

$k^2 - 9 = 0 \Rightarrow \quad K = \pm 3$

5. Adjoint of a Matrix:

Let $A = (a_{ij})$ be a square matrix of order n x n and (c_{ij}) is a matrix obtained by replacing each element a_{ij} by its corresponding cofactor c_{ij} then $(c_{ij})^t$ is called the adjoint of A. It is written as adj. A.

For example , if

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Cofactor of A are:

$A_{11} = 5$, $A_{12} = -2$, $A_{13} = +1$

$A_{21} = -1$, $A_{22} = 2$, $A_{23} = -1$

$A_{31} = 3$, $A_{32} = -2$, $A_{33} = 3$

Matrix of cofactors is

$$C = \begin{bmatrix} 5 & -2 & +1 \\ -1 & 2 & -1 \\ 3 & -2 & 3 \end{bmatrix}$$

$$C^t = \begin{bmatrix} 5 & -1 & 3 \\ -2 & 2 & -2 \\ +1 & -1 & 3 \end{bmatrix}$$

Hence adj A = C^t $= \begin{bmatrix} 5 & -1 & 3 \\ -2 & 2 & -2 \\ +1 & -1 & 3 \end{bmatrix}$

Note: Adjoint of a 2×2 Matrix:

The adjoint of matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by adjA is defined as

adjA = $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

6. Inverse of a Matrix:

If A is a non-singular square matrix , then $A^{-1} = \dfrac{\text{adj } A}{|A|}$

For example if matrix A = $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

Then adj A = $\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

$|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$

Hence $A^{-1} = \dfrac{\text{adj } A}{|A|} = \dfrac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

Alternately:

For a non singular matrix A of order (n x n) if there exist another matrix B of order (n x n) Such that their product is the identity matrix I of order (n x n) i.e., AB = BA = I

Then B is said to be the inverse (or reciprocal) of A and is written as B = A^{-1}

Example 9: If A = $\begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}$ and B = $\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$ then show that AB = BA = I and therefore, B = A^{-1}

Solution:

$$AB = \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $BA = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence AB = BA = I

and therefore $B = A^{-1} = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$

Example 10: Find the inverse, if it exists, of the matrix.

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

Solution:

$|A| = 0 + 2(-2 + 3) - 3(-2 + 3) = 2 - 3$

$|A| = -1$, Hence solution exists.

Cofactor of A are:

$A_{11} = 0$, $A_{12} = 1$, $A_{13} = 1$

$A_{21} = 2$, $A_{22} = -3$, $A_{23} = 2$

$A_{31} = 3$, $A_{32} = -3$, $A_{33} = 2$

Matrix of transpose of the cofactors is

$$\text{adj } A = C' = \begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

So

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-1} \begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

9.11 Solution of Linear Equations by Matrices:

Consider the linear system:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \text{------} + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \text{------} + a_{2n}x_n = b_2 \\ \quad | \qquad\quad | \qquad\qquad\qquad | \\ a_{n1}x_1 + a_{n2}x_2 + \text{------} + a_{nn}x_n = b_n \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

It can be written as the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \text{------} & a_{1n} \\ a_{21} & a_{22} & \text{------} & a_{2n} \\ | & | & & | \\ a_{n1} & a_{n2} & \text{------} & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ | \\ b_n \end{bmatrix}$$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \text{------} & a_{1n} \\ a_{21} & a_{22} & \text{------} & a_{2n} \\ | & | & & | \\ a_{n1} & a_{n2} & \text{------} & a_{nn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix}$

$B = \begin{bmatrix} b_1 \\ b_2 \\ | \\ b_n \end{bmatrix}$

Then latter equation can be written as,

$$AX = B$$

If $B \neq 0$, then (1) is called non-homogenous system of linear equations and if $B = 0$, it is called a system of homogenous linear equations.

If now $B \neq 0$ and A is non-singular then A^{-1} exists.

Multiply both sides of $AX = B$ on the left by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$

$$1X = A^{-1}B$$

$$\text{Or} \qquad X = A^{-1}B$$

Where $A^{-1}B$ is an n x 1 column matrix. Since X and $A^{-1}B$ are equal, each element in X is equal to the corresponding element in $A^{-1}B$. These elements of X constitute the solution of the given linear equations.

If A is a singular matrix, then of course it has no inverse, and either the system has no solution or the solution is not unique.

Example 11: Use matrices to find the solution set of

$$x + y - 2z = 3$$
$$3x - y + z = 5$$
$$3x + 3y - 6z = 9$$

Solution:

Let $A = \begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ 3 & 3 & -6 \end{bmatrix}$

Since $|A| = 3 + 21 - 24 = 0$

Hence the solution of the given linear equations does not exists.

Example 12: Use matrices to find the solution set of

$$4x + 8y + z = -6$$
$$2x - 3y + 2z = 0$$
$$x + 7y - 3z = -8$$

Solution:

Let $A = \begin{bmatrix} 4 & 8 & 1 \\ 2 & -3 & 2 \\ 1 & 7 & -3 \end{bmatrix}$

Since $|A| = -32 + 48 + 17 = 61$

So A^{-1} exists.

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix}$$

Now since,

$X = A^{-1} B$, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix} \begin{bmatrix} -6 \\ 0 \\ -8 \end{bmatrix}$$

$$= \frac{1}{61}\begin{bmatrix} 30+152 \\ -48+48 \\ -102+224 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

Hence Solution set: $\{(x, y, z)\} = \{(-2, 0, 2)\}$

Exercise 9.3

Q.1 Which of the following matrices are singular or non-singular.

(i) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ 3 & 3 & -6 \end{bmatrix}$

Q.2 Which of the following matrices are symmetric and skew-symmetric

(i) $\begin{bmatrix} 2 & 6 & 7 \\ 6 & -2 & 3 \\ 7 & 3 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 3 & -5 \\ -3 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

Q.3 Find K such that the following matrices are singular

(i) $\begin{vmatrix} K & 6 \\ 4 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 2 & -1 \\ -3 & 4 & K \\ -4 & 2 & 6 \end{vmatrix}$ (iii) $\begin{vmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ k & 3 & -6 \end{vmatrix}$

Q.4 Find the inverse if it exists, of the following matrices

(i) $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 0 & 2 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{bmatrix}$

Q.5 Find the solution set of the following system by means of matrices:

(i) $2x - 3y = -1$, $x + 4y = 5$ (ii) $x + y = 2$, $2x - z = 1$, $2y - 3z = -1$ (iii) $x - 2y + z = -1$, $3x + y - 2z = 4$, $y - z = 1$

(iv) $-4x + 2y - 9z = 2$, $3x + 4y + z = 5$, $x - 3y + 2z = 8$ (v) $x + y - 2z = 3$, $3x - y + z = 0$, $3x + 3y - 6z = 8$

Answers 9.3

Q.1 (i) Non-singular (ii) Singular
(iii) Singular

Q.2 (i) Symmetric (ii) Skew-symmetric
(iii) Symmetric

Q.3 (i) 8 (ii) 5 (iii) 3

Q.4 (i) $\begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & \frac{1}{-7} \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{4}{5} & -\frac{1}{5} & -\frac{4}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{7}{10} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$

(iv) A^{-1} does not exist.

Q.5 (i) {(1, 1)} (ii) {(1, 1, 1)} (iii) {(1, 1, 0)}
(iv) {(7, -3, -4)} (v) no solution

Summary

1. If $A = [a_{ij}]$, $A = [b_{ij}]$ of order m x n. Then $A + B = [a_{ij} + b_{ij}]$ is also m x n order.
2. The product AB of two matrices A and B is conformable for multiplication if No of columns in A = No. of rows in B.
3. If $A = [a_{ij}]$ is m x n matrix, then the n x m matrix obtained by interchanging the rows and columns of A is called the transpose of A. It is denoted by A^t.
4. Symmetric Matrix:
   A square matrix A is symmetric if $A^t = A$.
5. If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ Then,

   (i) adj $A = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$, a_{ij} are the co-factor elements.

   And inverse of A is:

(ii) $A^{-1} = \dfrac{\text{adj } A}{|A|}$

6. A square matrix A is singular if $|A| = 0$.

Short Questions

Write the short answers of the following:

Q.1: Define row and column vectors.
Q.2: Define identity matrix.
Q.3: Define symmetric matrix.

Q.4: Define diagonal matrix.

Q.5: Define scalar matrix.

Q.6: Define rectangular matrix.

Q.7: Show that $A = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$ is singular matrix

Q.8: Show that $A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & -3 \\ 3 & -3 & 6 \end{vmatrix}$ is symmetric

Q.9: Show that $\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = b^3 + a^3$

Q.10: Evaluate $\begin{vmatrix} 1 & 2 & -2 \\ -1 & 1 & -3 \\ 2 & 4 & -1 \end{vmatrix}$

Q.11: Without expansion show that $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

Q.12: Find x and y if $\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix}$

Q.13: Find x and y if $\begin{bmatrix} x+3 & 1 \\ -3 & 3y-4 \end{bmatrix} = \begin{bmatrix} y & 1 \\ -3 & 2x \end{bmatrix}$

Q.14: If $A = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{vmatrix}$ and $\begin{vmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{vmatrix}$, find A – B

Q.15: Find inverse of $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

Q.16: If A is non-singular, then show that $(A^{-1})^{-1} = A$

Q.17: If A is any square matrix then show that AA^t is symmetric.

Q.18: Find K if $A = \begin{bmatrix} 4 & k & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{bmatrix}$ is singular matrix

Q.19: Define the minor of an element of a matrix.

Q.20: Define a co-factor of an element of a matrix.

Q.21: Without expansion verify that $\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \gamma+\alpha & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix} = 0$

Q.22: What are the minor and cofactor of 3 in matrix.

$$\begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}$$

Q.23: What are the minor and cofactor of 4 in matrix.

$$\begin{pmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{pmatrix}$$

Q.24: If $\begin{vmatrix} k-2 & 1 \\ 5 & k+2 \end{vmatrix} = 0$, Then find k.

Q.25: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, Then find A + B

Q.26: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, Then find A – B

Q.27: If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, Then find AB

Q.28: If $\begin{vmatrix} 2 & 3 \\ 4 & k \end{vmatrix}$ is singular , Then find k.

Q.29: Find A^{-1} if $A = \begin{vmatrix} 5 & 3 \\ 1 & 1 \end{vmatrix}$

Answers

Q10. 9 Q12. $x = -1$, $y = 2$ Q13. $x = -5$, $y = -2$

Q14. $\begin{vmatrix} -1 & -2 & 3 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{vmatrix}$ Q15. D^{-1} does not exist Q18. $k = 3$

Q22. $M_{11} = 16$, $C_{11} = 16$ Q23. $M_{32} = 26$, $C_{32} = -26$

Q24. $K = \pm 3$ Q25. $\begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$ Q26 $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

Q27. $\begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix}$ Q28. $k = 6$ Q29. $\frac{1}{2}\begin{bmatrix} 1 & -3 \\ -1 & 5 \end{bmatrix}$

,

Objective type Exercise

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

___1. The order of the matrix $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ is

(a) 2 x 1 (b) 2 x 2 (c) 3 x 1 (d) 1 x 3

___2. The order of the matrix [1 2 3] is

(a) 1 x 3 (b) 3 x 1 (c) 3 x 3 (d) 2 x 3

___3. The matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called

(a) Identity (b) scalar (c) diagonal (d) Null

___4. Two matrices A and B are conformable for multiplication if

(a) No of columns in A = No of rows in B

(b) No of columns in A = No of columns in B

(c) No of rows in A = No of rows in B

(d) None of these

___5. If the order of the matrix A is pxq and order of B is qxr, then order of AB will be:

(a) pxq (b) qxp (c) pxr (d) rxp

___6. In an identity matrix all the diagonal elements are:

(a) zero (b) 2 (c) 1 (d) none of these

___7. The value of determinant $\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ is:

(a) 6 (b) – 6 (c) 1 (d) 0

___8. If two rows of a determinant are identical then its value is

(a) 1 (b) zero (c) – 1 (d) None of these

___9. If A = $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$ is a matrix, then Cofactor of 4 is

(a) – 2 (b) 2 (c) 3 (d) 4

___10. If all the elements of a row or a column are zero, then value of the determinant is:

(a) 1 (b) 2 (c) zero (d) None of these

___11. Value of m for which matrix $\begin{bmatrix} 2 & 3 \\ 6 & m \end{bmatrix}$ is singular.

(a) 6 (b) 3 (c) 8 (d) 9

___12. If $[a_{ij}]$ and $[b_{ij}]$ are of the same order and $a_{ij} = b_{ij}$ then the matrix will be

(a) Singular (b) Null (c) unequal (d) equal

13. Matrix $[a_{ij}]_{mxn}$ is a row matrix if:

(a) $i = 1$ (b) $j = 1$ (c) $m = 1$ (d) $n = 1$

14. Matrix $[c_{ij}]_{mxn}$ is a rectangular if:

(a) $i \neq j$ (b) $i = j$ (c) $m = n$ (d) $m - n \neq 0$

15. If $A = [a_{ij}]_{mxn}$ is a scalar matrix if :

(a) $a_{ij} = 0 \quad \forall\, i \neq j$ (b) $a_{ij} = k \;\forall\, i = j$

(c) $a_{ij} = k \quad \forall\, i \neq j$ (d) (a) and (b)

16. Matrix $A = [a_{ij}]_{mxn}$ is an edentity matrix if :

(a) $\forall\, i = j$, $a_{ij} = 0$ (b) $\forall\, i = j$, $a_{ij} = 1$

(c) $\forall\, i \neq j$, $a_{ij} = 0$ (d) both (b) and (c)

17. Which matrix can be tectangular mayrix ?

(a) Diagonal (b) Identity (c) Scalar (d) None

18. If $A = [a_{ij}]_{mxn}$ then order kA is:

(a) m x n (b) km x kn (c) km x n (d) m x kn

19. $(A - B)^2 = A^2 - 2AB + B^2$, if and only if :

(a) $A + B = 0$ (b) $AB - BA = 0$ (c) $A^2 + B^2 = 0$ (d) (a) and c

20. If A and B ARE symmetric , then AB =

(a) BA (b) $A^t B^t$ (c) $B^t A^t$ (d) (a) and (c)

Answers

Q.1 (1) c (2) a (3) d (4) a (5) c (6) c
(7) a (8) b (9) a (10) c (11) d (12) d
(13) c (14) d (15) d (16) d (17) d (18) a
(19) b (20) d

Chapter 10
Area of triangles (Plane figures)

10.1 Introduction to Mensuration:

This is a branch of Applied Mathematics which deals with the calculation of length of Lines, areas and volumes of different figures.

Scope

It is widely applied in various branches of engineering. A chemical engineer has to find the volume or capacity of a plant, A civil engineer has to find the areas and volumes in embarkments, canal digging or dam works. An electrical engineer has to depend upon this branch of mathematics while calculating resistance or capacity of a conductor. In the same way a draftsman, a designer or an electrical supervisor very often uses mensuration in his work.

If we dismantle any complicated machine or its parts we will find the machine has been separated into different simple plane or solid figures like rings, cylinders, squares and prisms.

In this part of the text our aim is to enable the student to find areas, volume, and circumferences of different figures, so that when as a technician, he has to estimate the cost of material or to design a machine part, he may be able to understand the calculations needed to determine weight, strength and cost.

10.2 Plane Figures:

Plane figures are those figures which occupy an area with only two dimensions, room floors, grassy plots and tin sheets are examples of plane figure.

While the solid figures are those which occupy space with three dimensions, Shafts, Fly wheels, bolts, wooden boxes, and coal tar drum are examples of solids.

10.3 Triangle:

A triangle is a plane figure bounded by three straight lines. The straight lines AB, BC, CA which bound triangle ABC are called its sides. The side BC may be regarded as the base and AD as the height.

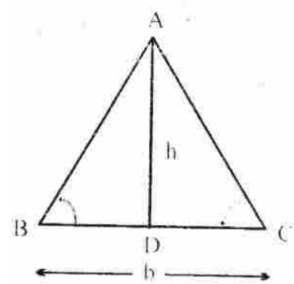


Fig. 10.1

Kinds of Triangles

There are six types of triangles three of them are classified according to their sides and the remaining three are according to their angles.

(a) Triangles classified to their angles:

(1) Right angled triangles

If one angle of a triangle is a right angle (90^o) then it is called a right angled triangle the side opposite to right side is called its by hypotenuse and remaining other two side an base and altitude.

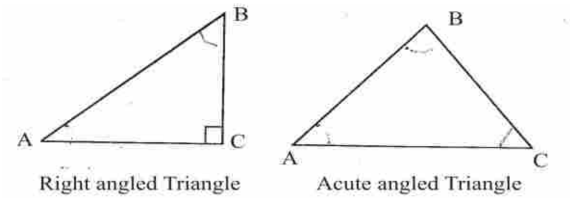


Fig. 10.2

(2) Acute Angled Triangles

If all the three angles of an triangle are acute (less then 90^o) then the triangle is called acute angled.

(3) Obtuse Angled Triangles

If one angle of a triangle is obtuse (greater than 90^o) the triangle is called obtuse angled.

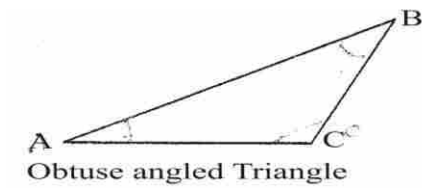


Fig. 10.3

(b) Triangle Classified according to their sides:

(1) Scalene Triangle

A triangle in which all sides are of different lengths is called scalene triangle.

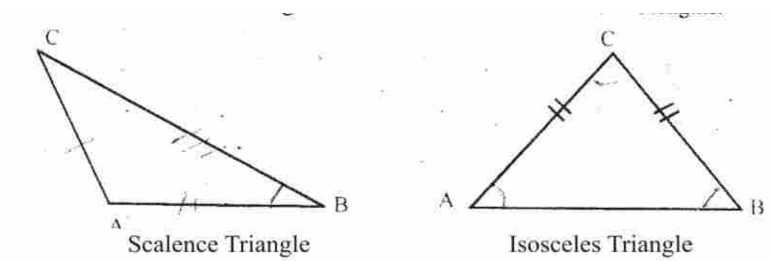


Fig. 10.4

(2) Isosceles Triangle

If two sides of triangle are equal, the triangle is called isosceles.

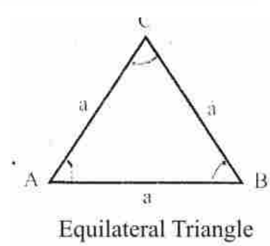


Fig. 10.5

(3) Equilateral Triangle

If all the three sides of a triangle are equal in lengths, the triangle is called Equilateral.

Perimeter:

The perimeter of a closed plane figure is the total distance around the edges of the figure.

Perimeter of a triangle with sides a, b & c:
Perimeter of the triangle = (a + b + c) units

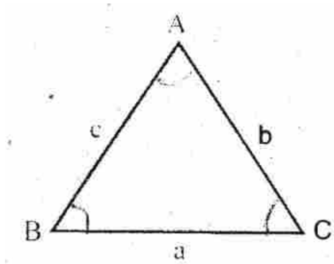


Fig. 10.6

10.4 Area of Triangles:

There are so many methods to find the area of triangle. We shall discuss them one by one.

(a) Area of a triangle in terms of its Height (altitude) and base:

Case I: When the triangle is Right angled:

Let ABC be a right angled triangle whose angle B is right angle. Side BC is the height (altitude) 'h' and side AB is the base 'b'.

Invert the same triangle in its new position ADC as shown in figure 10.7.

Area of ABC $= \frac{1}{2}$ area of rectangle BCDA.

$$= \frac{1}{2}(AB)(BC)$$

$$= \frac{1}{2}bh$$

Area $= \frac{1}{2}$(base) (height)

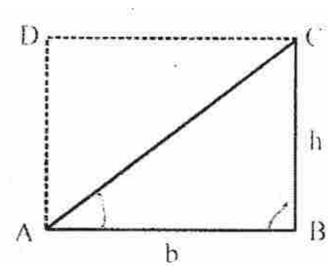


Fig. 10.7

Case II: When Triangle is acute-angled:

Let ABC be a triangle with its base 'b' and height h. When CD is perpendicular some to the base AB.

The area of ΔABC = Area of ΔADC + Area of ΔBDC

$$= \frac{1}{2}(AD)(CD) + \frac{1}{2}(DB)(CD)$$

$$= \frac{1}{2}(AD + DB)CD$$

$$= \frac{1}{2}(AB)(CD)$$

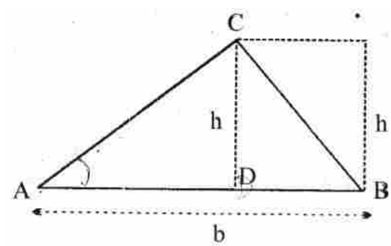


Fig. 10.8

$$= \frac{1}{2} b \times h$$

Hence Area $= \frac{1}{2}$(base x height)

Case III: When Triangle is obtuse-angled

Let ABC be triangle whose obtuse angle is B . Also draw CD perpendicular to AB produced.

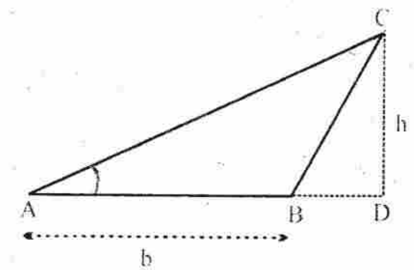


Fig. 10.9

Then,

Area of ΔABC = Area of ΔADC – Area of ΔBDC

$$= \frac{1}{2}(AD)(CD) - \frac{1}{2}(BD)(CD)$$

$$= \frac{1}{2}(AD - BD)CD$$

$$= \frac{1}{2}(AB)(CD) = \frac{1}{2}(b)(h)$$

Area $= \frac{1}{2}$(Base x height)

Note : Hence from case I, II and III it is concluded that

Area of a triangle whose base and height is given $= \frac{1}{2}$(Base x height)

Example 1:

Find the area of a triangle whose base is 12 cm. and hypotenuse is 20cm.

Soluyion :

Let ABC be a right triangle

Base = AB = 12cm.
Hieght = BC = h = ?
Hypotinuse = 20 cm.

By pythagoruse Theorem

$$AB^2 + BC^2 = AC^2$$
$$BC^2 = AC^2 - AB^2$$
$$h^2 = 20^2 - 12^2$$
$$= 400 - 144$$
$$= 256$$
$$h = 16 \text{ cm.}$$

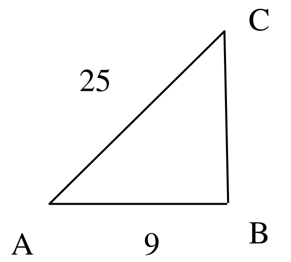


Fig. 10.10

Now , Area of triangle ABC $= \frac{1}{2}$(Base x height)

$= \frac{1}{2}(12 \times 16)$

$=$ 96 sq.cm

Example 2:

The discharge through a triangular notch is 270 cu cm/sec. Find the maximum depth of water if the velocity of water is 10 cm/sec and the width of water surface is 16cm.

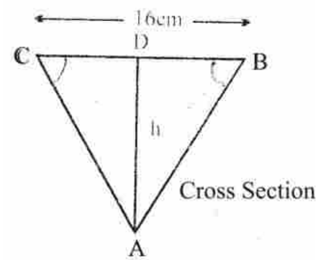


Fig. 10.11

Solution:

Given that

Discharge = 720 cu. cm/sec

Velocity of water = 10 cm/sec

Width of water = 16 cm.

Let 'h' be the depth of water

Area of cross-section = Area of Δ ABC

$= \frac{1}{2}(BC)(AD) = \frac{1}{2}(16)(h) = 8h$

Discharge = (Area of cross-section) velocity

720 = 8h(10)

h = 9cm

(b) Area of the Triangle when two adjacent sides and their included angle is given:

Let, ABC be a triangle with two sides b, c and included angel A are given. Draw CP perpendicular to AB.

Area of Δ ABC $= \frac{1}{2}$ base $\times$ height

$$= \frac{1}{2}\ AB \times CP$$

Since $\dfrac{CP}{AC} = \text{Sin } A$

$CP = AC \text{ Sin } A$

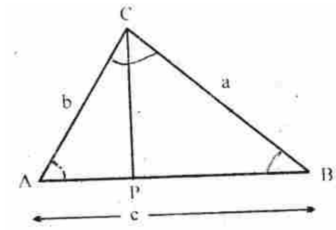


Fig. 10.13

Fig.10.1.12

Area of Δ ABC $= \dfrac{1}{2}\ AB \times CP$

$$= \frac{1}{2}\ AB \times AC\ \text{ Sin } A$$

$$= \frac{1}{2}\ b \times c \text{ Sin } A$$

$$A = \frac{1}{2} bc \text{ Sin } A = \frac{1}{2} ac \text{ Sin } B = \frac{1}{2} ab \text{ Sin } C$$

i.e Half of the product of two sides with sine of the angle between them.

Example 2:

Find the area of a triangle whose two adjacent sides are 17.5cm and 25.7cm respectively, and their included angle is 57^o.

Solution:

Let two adjacent sides be a = 17.5 cm

b = 25.7 cm and included angle $\theta = 57^o$

Area of Δ ABC $= \dfrac{1}{2} ab \text{ Sin } \theta$

$$= \frac{1}{2}(17.5)(25.7) \text{ Sin } 57^o$$

$$= (224.88) \text{ x } (.84) = 188.89 \text{ sq. cm}$$

(c) Area of an equilateral triangle:

A triangle in which all the sides are equal, all the angles are also equal that is 60 degree. If each side of the triangle is ‘a’.

Draw AP perpendicular to BC.

Then, $BP = PC = \dfrac{a}{2}$

Since, $|AB|^2 = |BP|^2 + |AP|^2$

$|AP|^2 = |AB|^2 - |BP|^2$

$$= a^2 - \left(\frac{a}{2}\right)^2$$

$$= a^2 - \frac{a^2}{4}$$

$$= \frac{3a^2}{4}$$

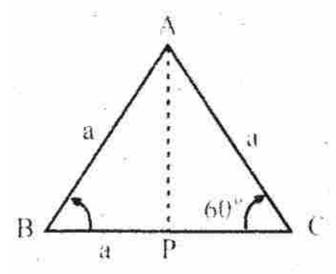


Fig. 10.14

$$AP = \frac{\sqrt{3}}{2}a$$

$$\text{Area of triangle} = \frac{1}{2} \text{ x (base) (height)}$$

$$= \frac{1}{2}a \times \frac{\sqrt{3}}{2}a$$

$$\boxed{A = \frac{\sqrt{3}}{4}a^2}$$

Example 3:

A triangle blank of equal sides is to punch in a copper plate, the area of the blank should be 24 sq. cm find the side.

Solution:

Area of triangular blank of equal sides = 24 sq. cm

Area of triangle of equal sides = $\frac{\sqrt{3}}{4}a^2$

$$24 = \frac{\sqrt{3}}{4}a^2$$

$$\sqrt{3}a^2 = 96$$

$$a^2 = 50.40$$

$a = 7.4$ cm each side

(d) Area of Triangle when all sides are given:

Let ABC be a triangle whose sides are a, b and c respectively, then

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}\ , \quad \text{where} \quad s = \frac{a+b+c}{2}$$

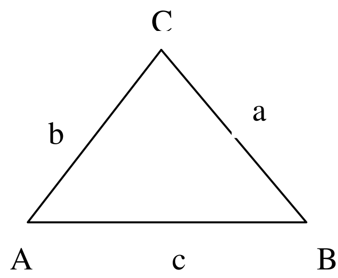


Fig.10.14

Which is called **Hero's Formul**

Corollary : For equilateral triangle ,

$$a = b = c$$

$$s = \frac{a + a + a}{2} = \frac{3a}{2}$$

$$s - a = s - b = s - c = \frac{3a}{2} - a = \frac{a}{2}$$

$$\text{Area of equilateral triangle} = \sqrt{\frac{3a}{2}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)}$$

$$A = \frac{\sqrt{3}}{2}a^2$$

Example 4: Find the area of a triangle whose sides are 51, 37 and 20cm respectively.

Solution : Here, a =51 cm, b =37 cm c = 20cm

$$s = \frac{a + b + c}{2} = \frac{51 + 37 + 20}{2} = 54$$

$$\text{Area by Hero's Formula} = \sqrt{S(s-a)(s-b)(s-c)}$$

$$= \sqrt{54\,(3)\,(17)\,(34)} = 306 \text{ sq.unit}$$

Example 5:

Find the area of a triangle whose sides are 51, 37 and 20 cm respectively.

Solution:

Let a = 51cm, b = 37cm, and c = 20cm

Be the given sides triangle

$$S = \frac{a + b + c}{2} = \frac{51+37+20}{2} = \frac{108}{2} = 54$$

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{54(54-51)(54-37)(54-20)} = \sqrt{54(3)(17)(34)}$$

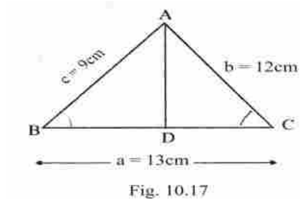
$$= 306 \text{ sq. cm}$$

Example 6:

The sides of a triangle are 13, 12 and 9cm respectively. Find the distance of the longest side from the opposite vertex.

Solution:

Let
Given sides be
a = 13cm, b = 12cm, c = 9cm



$$S = \frac{a+b+c}{2} = \frac{13+12+9}{2} = \frac{34}{2} = 17$$

Fig.10.15

$$\text{Area of } \Delta\, ABC = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{17(17-13)(17-12)(17-9)} = \sqrt{17(4)(5)(8)}$$

$$\text{Area} = \sqrt{2720} = 52.15 \text{ sq. cm.}$$

Let h = distance of the longest side from opposite vertex

$$\text{Also Area of } \Delta\, ABC = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(a)(h)$$

$$52.15 = \frac{1}{2}(\text{longest sides})(\text{required distance})$$

$$52.15 = \frac{1}{2}(13)(h)$$

$$h = \frac{2x52.15}{13} = 8.02\text{cm}$$

Exercise 10

Q.1. The hypotenuse of a right triangle is 10 cm and its height is twice of its base. Find the area of the triangle.

Q.2. From a point within an Equilateral triangle perpendicular are drawn to the three sides are 6, 7 and 8 cm respectively. Find the area of triangle.

Q.3. The sides of triangular pond are 242, 1212 and 1450m. Find the total amount of antiseptic medicine needed for spraying when one gallon of the medicine is sufficient for 100 square meter of water surface of the pond.

Q.4. The sides of a triangle are 21, 20 and 13cm respectively, find the area of the triangles into which it is divided by the perpendicular upon the longest side from the opposite angular point.

Q.5. From a point within a triangle, it is found that the three sides subtend equal angles. From this point, three line are drawn to meet the opposite edges. If these lines measure 5, 6 and 7cm respectively. Find the area of the triangle.

Q.6. The sides of a triangular lawn are proportional to the numbers 5, 12 and 13. The cost of fencing it at the rate of Rs 2 per meter is Rs 120. Find the sides. Also find the cost of turfing the lawn at 25 paisa per square meter.

Q.7. Find the area of the triangle whose sides are in the ratio 9 : 40 : 41 and whose perimeter in 180 meters.

Q.8. The corresponding bases of two triangles, giving equal altitude, are 8cm and 10cm. the area of the smaller is 108 sq.cm. Find the area of the larger triangle and altitude of each.

Answers 10

Q1. 19.98 sq.cm **Q2.** 254.60 sq. cm **Q3.** 290.40 gallons
Q4. 8.32cm, 24.96cm **Q5.** 46.33 sq. cm
Q6. 10m, 24m, 26m, Rs. 30 **Q.7** 720sq.cm **Q.8.** 135sq.cm, 27cm

Summary

(1) **Plane Figures**

Plane figures are those figures which occupy an area with only two dimensions e.g. room floor, grassy plots, tin sheets.

(2) **Triangle**

A triangle is plane figure bounded by three straight lines.

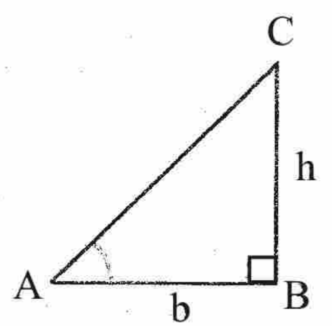


Fig. 10.18

Kinds of Triangles

(i) Obtuse angled triangle (in which one angle >90^o)
(ii)Right angled triangle (in which one angle in 90^o)
(iii)Acute angled triangle (in which one angle less than 90^o)
(iv)Iosceles triangle (two side equal)
(v) Equilateral triangle (three sides equal)
(vi)Scalene triangle (all sides are different)

(3)Area of Triangle in terms of its height (altitude) and base

Area = $\frac{1}{2}$(base) height

$$\text{Area} = \frac{1}{2}bh$$

(4) Area of an equilateral triangle with side 'a' is

$$\text{Area} = \frac{\sqrt{3}}{4}a^2$$

(5) Area of triangle when two adjacent sides and their included angle is given.

$$\text{Area} = \frac{1}{2}bc\sin\theta$$

Which is also called Snell's Formula.

(6) Area of a triangle when all sides are given

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

When $S = \dfrac{a+b+c}{2}$

It is called Hero's Formula

Perimeter of triangle = (a + b + c) units

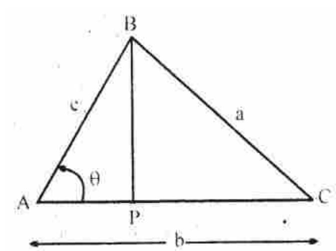


Fig. 10.19
Fig.10.17

Short Questions

Q.1: Define plane figure

Q.2: Define a triangle

Q.3: Define isosceles triangle.

Q.4: Define equilateral triangle.

Q.5: Write the area of an equilateral triangle with side ‘a’.

Q.6: Find the cost of turfing a triangular lawn at the rate of Rs.5 per sq.m, if its one side 20 m; and perpendicular on it from the opposite vertex is 30m.

Q.7: Find the area of equilateral triangle with side 4 m.

Q.8: Find the area of right triangle if base and altitude are 20m. and 10m respectively

Q.9: Find the area of a triangle whose two adjacent sides are 16 cm and 12 cm and their included angle is 30^o.

Q.10: What is the side of the equilateral triangle whose area is $9\sqrt{3}$ sq.cm.

Q11: Find the area of triangle with sides 5, 4 and 3 meters respectively.

Answers

Q6.	Rs.1500	Q7.	$4\sqrt{3}$ = sq.m.	Q8.	100 sq. m.
Q9.	48 sq. cm.	Q10.	6 cm.	Q11.	5.48 sq. m.

Objective type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

___1. Plane figures are those figures which occupy an area with only
(a) Two dimensions (b) Three dimensions
(c) 4-dimensions (d) None of these

___2. A right triangle has one angle is
(a) 30^o (b) 90^o (c) 60^o (d) 45^o

___3. Each angle of an acute triangle is
(a) Less than 90^o (b) greater than 90^o
(c) Equilate 90^o (d) None of these

___4. An obtuse triangle has one angle greater than
(a) 60^o (b) 30^o (c) 90^o (d) None of these

___5. In an equilateral triangle, each angle is
(a) 30^o (b) 45^o (c) 60^o (d) None of these

___6. Area of an equilateral triangle with side 'x' is
(a) $\frac{\sqrt{3}}{4}x^2$ (b) $\frac{\sqrt{3}}{2}x^2$ (c) $\frac{2}{\sqrt{3}}x$ (d) None of these

___7. Area of right triangle if base = 4cm, height = 6cm
(a) 24 sq. cm (b) 12 sq. cm
(c) 6 sq. cm (d) 10 sq. cm

___8. If a = 4cm, side of equilateral triangle, then area of equilateral triangle is
(a) $\frac{\sqrt{3}}{4}$ sq. cm. (b) $\frac{\sqrt{3}}{2}$ sq. cm.
(c) $4\sqrt{3}$ sq. cm. (d) $\sqrt{3}$ sq. cm.

___9. If a = 2cm, b = 3cm , c = 5cm sides of triangle, then perimeter of triangle is
(a) 8cm (b) 6cm (c) 10cm (d) 30cm

___10. If a = 4cm, b = 2cm are adjacent sides of triangle and $\theta = 30^o$ is the included angle then area is
(a) 2 sq. cm (b) 4 sq. cm (c) 8 sq. cm (d) 12 sq. cm

___11. Number of sides equal in isosceles triangle are
(a) 1 (b) 2 (c) 3 (d) 4

Answer

Q.1	(1)	a	(2)	b	(3)	a	(4)	c
	(5)	c	(6)	a	(7)	b	(8)	c
	(9)	c	(10)	a	(11)	b		

Chapter 11
Area of Quadrilateral

11.1 Quadrilateral

A plane figure bounded by four sides is known as a quadrilateral. The straight line joining the opposite corners is called its diagonal. The diagonal divides the quadrilateral in to two triangles.

Types of Quadrilateral

The following types of quadrilateral are

(1) Square (2) Rectangle (3) Parallelogram
(4) Rhombus (5) Trapezoid (6) Cyclic quadrilateral

11.2 Area of Quadrilateral

1. Square

A square is a plane figure of four sides in which all sides are equal, the opposite sides are parallel and diagonals are also equal. The angle between the adjacent sides is a right angle.

Let ABCD be a square whose each side has length equal to 'a' and AC is a diagonal which divides the square ABCD into equal right triangles, named ΔABC & ΔACD. Therefore,

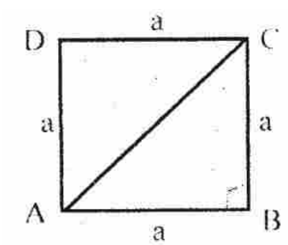


Fig. 11.1

Area of the square ABCD = Area of ΔABC + Area of ΔACD

$$= \frac{1}{2}(AB)(BC) + \frac{1}{2}(AD)(DC) = \frac{1}{2}(a)(a) + \frac{1}{2}(a)(a)$$

$$= \frac{1}{2}a^2 + \frac{1}{2}a^2 = a^2$$

Area of square = (Side)^2

Length of square = $\sqrt{\text{Area of square}}$

i.e. side = $\sqrt{\text{Area}}$

Perimeter of square = 4a

Example 1:

A square lawn whose area is 2.5 sq. km has to be enclosed within iron railing. Find the length of the railing and its cost at Rs 10.50 per meter.

Solution:

Given that

Area of square lawn = 2.5 sq. km

$= 2.5 \times (1000)^2$ sq. m

A $= 2500000$ sq. m

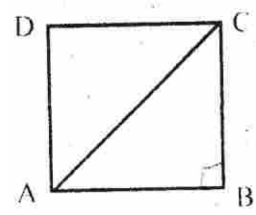


Fig. 11.2

One side of lawn = $\sqrt{2500000} = 1581$m

Perimeter of square lawn = 4 x 1581= 6324m

Cost for 1meter = 10.50 rupees

Cost for 6324 meters = 10.50 x 6324 = 66402 rupees

Example 2:

The diagonal of a square plate is 19 cm. Find the length of the plate and its area.

Solution:

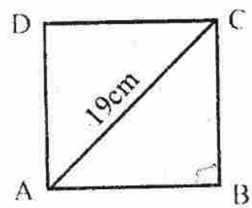


Fig. 11.3

Let ABC be a square plate.

Diagonal = AC = 19cm

Let a = length of square plate in right ΔABC

$(AB)^2 + (BC)^2 = (AC)^2 \Rightarrow a^2 + a^2 = (19)^2 \Rightarrow 2a^2 = 361$

$a^2 = 180.5 \Rightarrow a = 13.43$ cm

2. Rectangle

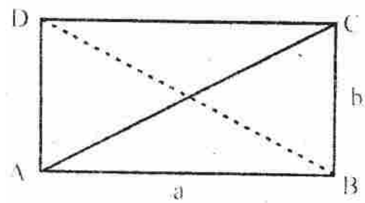


Fig. 11.4

A rectangle is a four sided figure whose opposite sides are parallel and equal in length, diagonals are equal and angles between adjacent sides are right angles.

Also diagonals of rectangle bisect each other

Let ABCD be rectangle having side AB = a and BC = b and the diagonal AC divides the rectangle into two right triangles. ΔABC and ΔADC.

Area of rectangle ABCD = Area of ΔABC + Area of ΔADC.

$$= \frac{1}{2}(AB)(BC) + \frac{1}{2}(DC)(AD) = \frac{1}{2}ab + \frac{1}{2}ab$$

Area of rectangle ABCD = ab

$\therefore$ Area = length x width

Note: Length = $\dfrac{\text{Area}}{\text{Width}}$ Width = $\dfrac{\text{Area}}{\text{length}}$

Perimeter of rectangle ABCD = AB + BC + CD + AD

$= 2a + 2b = 2(a + b)$

Perimeter = 2 (length + breadth)

Example 3:

The area of a rectangle is 20 sq. cm and one of its sides is 4cm long. Find the breadth and the perimeter of the rectangle.

Solution: Given that

Area of rectangle = 20 sq. cm
One of its side i.e. length = 4cm
$\therefore$ Breadth of rectangle = ?
Since area = length x breadth
20 = 4 x breadth
Also perimeter of rectangle = 2(length + breadth)
= 2(4 + 5) = 18cm

Example 4:

A rectangle field is 13m long and 10 m wide it has a cement path 3.5m wide around it. What is the area of the cement path?

Solution:

Given that
Width of cement path = 3.5cm
Length and width of rectangular field are 13m & 10m respectively.
Therefore
Area of field = (13 x 10)m^2 = 130m^2

Length of outer rectangle = 13 + 3.5 + 3.5 = 20m
& width of outer rectangle = 10 + 3.5 + 3.5 = 17m
$\therefore$ Area of outer rectangle = (20 x 17)m^2 = 340m^2
Hence, Area of cement path = Area of outer rectangular – Area of inner rectangular field
= 340m^2 – 130m^2 = 210m^2

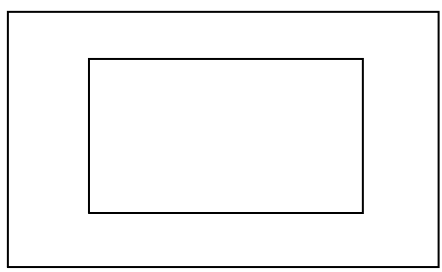


Fig.11.5

3. Parallelogram

A parallelogram is a quadrilateral whose opposite sides are equal in length and parallel, however, its diagonals are unequal and bisect each other.

Area of Parallelogram

Area of a parallelogram can be calculated in two ways

(i) When the base and height are given

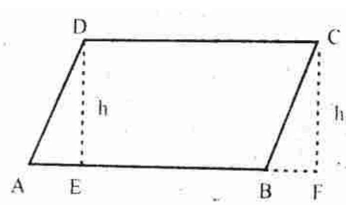


Fig. 11.6

Let ABCD be a parallelogram whose base AB=a and Height DE=h.

$\therefore$ Area of parallelogram ABCD

= Area of rectangle DEFC

= (Length)(breadth) = (EF)(DE)

= (DC)(DE) $\therefore$ EF = CD

= (AB)(DE) & CD = AB

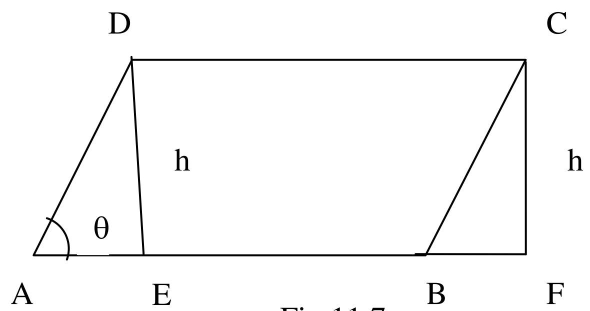
= a h

Area of parallelogram = (Base)(height)

(ii) When the two adjacent sides and included angle given

Let ABC be the parallelogram with AB = b, AD = c

As adjacent sides are b, c and θ be the include angle. Draw DE $\perp$r on AB from vertex D.



Fig,11.7

Thus Δ AED becomes a right triangle.

Area of parallelogram ABCD = (AB)(DE) -------- (1)

In right Δ AED, $\sin\theta = \dfrac{DE}{AD}$ $\Rightarrow$ DE = AD $\sin\theta$

DE = C $\sin\theta$

From (1)

$\therefore$ Area of llgm (parallelogram) ABCD = bc $\sin\theta$

Area of llgm (product of adjacent sides) $\sin\theta$

Example 5:

Find the area of a parallelogram whose base is 24cm and height 13 cm respectively.

Solution:

Here, b = 24cm, h = 13 cm

Area parallelogram = b x h = 24 x 13 = 312 sq. cm

Example 6:

Find the area of a parallelogram whose two adjacent sides of which are 70 cm and 80cm and their included angle 60^o.

Solution:

Here, b = 80, c = 70, $\theta = 60^o$

Area $= bc \operatorname{Sin}\theta$
$= 80 \times 70 \times \operatorname{Sin}60^o$
$= 5600\ (.866) = 4849$ sq. cm

4. Rhombus

A rhombus is a quadrilateral having all sides equal with unequal diagonal, which bisect each other. If a square is pressed from two opposite corners the rhombus is formed.

Area of Rhombus

(a) If side a and the included angle θ is given

Let ABCD be a rhombus and length of each side be 'a' and θ be the included angle.

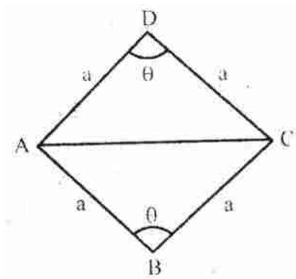


Fig. 11.9

Since diagonal AC divides the rhombus into two equal triangles ABC and ACD.

Therefore

Area of rhombus ABCD $= 2(\text{area of } \Delta ABC)$

$$= 2\left(\frac{1}{2} a\, a.\ \operatorname{Sin}\theta\right) = a^2 \operatorname{Sin}\theta$$

Area = $(\text{one side})^2 \operatorname{Sin}\theta$

Note : When diagonal of Rhombus d_1 and d_2 are given the side 'a' can be calculated as

$$a = \sqrt{\left(\frac{d_1}{2}\right)^2 + \left(\frac{d_2}{2}\right)^2}$$

(b) Area of rhombus, when two diagonals are given

Let AC = d_1 and BD = d_2 be the two diagonals.

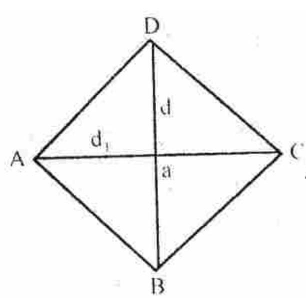


Fig.11.9

Fig. 11.10

Since diagonals of rhombus divide into four equal triangles.
Therefore,

$$\text{Area of rhombus} = 4(\text{area of one triangle})$$

$$= 4\left(\frac{1}{2} \text{ x } \frac{AC}{2} \text{ x } \frac{BD}{2}\right)$$

$$= \frac{AC \text{ x } BD}{2} \Rightarrow A = \frac{d_1 \text{ x } d_2}{2}$$

$$\text{Area of rhombus} = \frac{1}{2}(\text{product of two diagonals})$$

Example 7:

The length of each side of a rhombus is 120cm and two of its opposite angles are each 60^o find the area.

Solution:

Here each side a = 120cm

Angle, $\theta = 60^o$

Area $= a \text{ x } a \text{ x } \text{Sin}\theta$

$= 120 \text{ x } 120 \text{ x } \text{Sin}60$

$= 14400 \text{ x } 0.866$

$= 12470.4$ sq. cm.

Example 8:

The diagonals of a rhombus are 56 and 33 cm respectively. Find the length of side and the area of the rhombus.

Solution:

Let, ABCD is a rhombus with diagonal BD = 56cm = d_1

AC = 33 cm = d_2

Since diagonals of rhombus intersect at right angle triangle

∴ ABO in right angle triangle

Where $OB = \frac{BD}{2} = \frac{56}{2}$

OB = 28

$OA = \frac{AC}{2} = \frac{33}{2} = 16.5$

∴ $|AB|^2 = |AO|^2 + |BO|^2$

$= (16.5)^2 + (28)^2$

AB = 32.5 cm

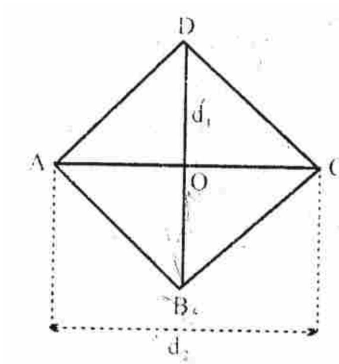


Fig. 11.11

$$\text{Area of Rhombus} = \frac{d_1 \text{ x } d_2}{2}$$

$$= \frac{56 \text{ x } 33}{2} = 924 \text{ sq. cm}$$

5. Trapezoid or Trapezium

A trapezoid is a quadrilateral which has two sides parallel and other two are unparallel. The parallel sides of trapezoid are called base. The altitude is the perpendicular distance between these bases.

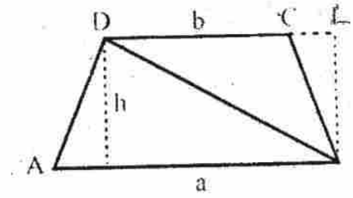


Fig. 11.12

Let, ABCD be a trapezoid whose sides AB and CD are parallel, AD and BC and unparallel sides.

Let AB = a, CD = b, and DP = h

BL= h, Since the diagonal BD divide the trapezoid into two triangles ABD & BCD.

Therefore,

$$\text{Area of Trapezium} = \text{Area } \Delta ABD + \text{Area } \Delta BCD$$

$$= \frac{1}{2} AB \times DP + \frac{1}{2} \times DC \times BL$$

$$= \frac{1}{2} ah + \frac{1}{2} bh$$

$$A = \frac{1}{2}(a + b)h$$

$$\text{Area of Trapezium} = \frac{\text{Sum of parallel sides}}{2} \times \text{height}$$

Example 9:

Find the area of a trapezium whose parallel sides are 57 cm and 85 cm and perpendicular distance between them is 4cm.

Solution:

Here, a = 85cm
b = 57cm
h = 4cm

$$\text{Area of Trapezium} = \frac{a + b}{2} \times h$$

$$= \frac{85+57}{2} \times 4 = 142 \times 2$$

$$A = 284 \text{ sq. cm}$$

Area of any quadrilateral

It may be found as the sum of two triangles formed by joining one of the diagonals of the quadrilateral.

OR

Area of any quadrilateral can be calculated by dividing it into two triangles.

Example 10:

Find the area of the quadrilateral ABCD in which the sides AB, BC, CD, DA and diagonal AC are 25, 60, 52, 39 and 65cm. respectively.

Solution:

Area Δ ABC $= \sqrt{S(S-a)(S-b)(S-c)}$

Where, 2S $= a + b + c$

$2S = 25 + 60 + 65$

$S = 75$

Area= $\sqrt{75(75-25)(75-60)(75-65)} = 750$ Sq. cm

Area = Δ ACD = $\sqrt{S(S-a)(S-b)(S-c)}$

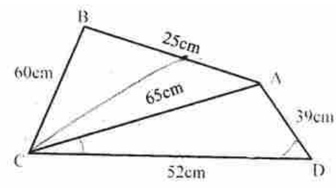


Fig. 11.15

Where,

$2S = a + b + c$

$2S = 52 + 39 + 65$

$S = 78$

A = $\sqrt{78(78-52)(78-39)(78-65)}$

A = 1014 Sq. cm

Area of quadrilateral ABCD = Δ ABC + Δ ACD

= 750 + 1014 = 1764 Sq. cm

Area of Cyclic Quadrilateral

A four sided figure which is in a circle is called a cyclic quadrilateral. Its area is given by the formula.

A = $\sqrt{(S-a)(S-b)(S-c)(S-d)}$

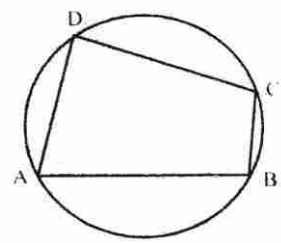


Fig. 11.16

Where,

$2S = a + b + c + d$

and a, b , c, d are sides

Example 11:

In a circular grassy plot a quadrilateral shape with its corners touching the boundary of the plot is to be paved with bricks. Find the area of the pavement in square meter if sides of the quadrilateral are 36, 77, 75, and 40 meter respectively.

Solution:

Since this is cyclic quadrilateral, so, a = 36, m. b = 77m, c = 75m, d= 40m

$\therefore$ Area = $\sqrt{(S-a)(S-b)(S-c)(S-d)}$

Where

$$S = \frac{a+b+c+d}{2} = \frac{36+77+75+40}{2} = 114$$

$$\text{Area} = \sqrt{(114-36)(114-77)(114-75)(114-40)}$$
$$= \sqrt{78 \text{ x } 37 \text{ x } 39 \text{ x } 74}$$
$$= 2886 \text{ Sq. m}$$

Exercise 11

Q1.. A lawn is in the shape of a rectangle of length 60m and width 40m outside the lawn there is a foot path of uniform width 1m, boarding the lawn, Find area of the path.

Q2. A track round the inside of a rectangular grassy plot 40m by 30m occupies 600 sq. m show that the width of the track is 5m.

Q3. A wire rectangle is pressed at the corners to form a parallelogram. The included angle is reduced to 60^o. Find the reduction in area if the original size of the rectangle is 16 x 12cm.

Q4. The perimeter of a rhombus is 146cm and one of its diagonal is 55cm. Find the other diagonal and area.

Q5. The diagonals of a rhombus are 80 cm and 60 cm respectively. Find the area and length of each side.

Q6. The difference between two parallel sides of a trapezoid is 8 cm. The perpendicular distance between them is 24m and the area of the trapezoid is 312 square meter. Find the two parallel sides.

Q7. The altitude of a triangle is 15cm and its base is 40cm find the area of trapezoid formed by a line parallel to the base of the triangle and 6cm from the vertex.

Q8. A swimming pool is in the form of an isosceles trapezoid. The length of its parallel banks are 72m and 45m and the perpendicular distance between them is 60m. Find the area of its water surface. Also find out the number of tiles required to line its bottom of each tile is 1.5m x 1.5m.

Q9. In a quadrilateral the diagonal is 125cm and the two perpendiculars on it from the other two angles are 19cm and 25cm respectively, find the area.

Q10. Two triangles are cut from a 14ft. square piece of sheet metal one has a base of 2.5ft. an altitude of 14ft. and the other has a base of 5ft. and an altitude of 7.5ft. Find area of sheet metal left in the piece.

Answers 11

Q1. 2604 sq.m	Q3. 26 sq.cm	Q4. 48 cm; 1320 sq. cm
Q5. 2400 sq.cm, 50cm	Q6. 17 m, 9m	Q7. 252sq.cm
Q8. 3510sq.m,1560	Q9. 2750sq.cm	Q10. 107.75 sq.ft.

Summary

1. Area of square with side 'a' = a^2 perimeter of square = 4a.
   Length of square = $\sqrt{\text{area of square}}$
2. Area of rectangle with length a and breadth b is ab.
   Perimeter of rectangle = 2(a + b)
3. **(i) Area of parallelogram when base and height are given**
   Area of ||gm = base x height
   (ii) Area of parallelogram when two adjacent sides and included angle are given
   If b and c are the adjacent sides and θ is the included angle.
   Then
   Area of ||gm (parallelogram) = bcsinθ
   i.e. Area of ||gm = (Product of adjacent sides) Sinθ
4. **Area of Rhombus**
   (i) If side a and the included angle θ
   Then
   Area of rhombus = $a^2 \sin\theta$
   (ii) If d_1 and d_2 are the two diagonals of rhombus
   Then
   Area of rhombus = $\dfrac{d_1 \text{ x } d_2}{2}$
5. **Area of Trapezoid**
   If a and b are parallel sides and h is the perpendicular distance between these parallel sides.
   Then
   Area of trapezoid = $\dfrac{\text{Sum of parallel sides}}{2}$ x height
   $$\text{Area} = \left(\frac{a+b}{2}\right) \text{x h}$$
6. **Area of any quadrilateral**
   Area of any quadrilateral can be calculated by dividing it into two triangles.
   i.e. if ABCD is any quadrilateral
   Then
   Area of quadrilateral ABCD = Area of ABC + Area of ΔACD
7. **Area of Cyclic quadrilateral**
   A = $\sqrt{(S-a)(S-b)(S-c)(S-d)}$
   Where S = $\dfrac{a+b+c+d}{2}$ and, a, b, c, d sides

Short Questions

Q.1: Define a quadrilateral.

Q.2: Write the area and perimeter of a square of sides 'a'.

Q.3: If the perimeter of a square is 40 cm. Find the area of the square.

Q.4: The area of a rectangle is 20 sq. cm and one of its sides is 4 cm long. Find the breadth and the perimeter of the rectangle.

Q.5: A wire rectangle of original size 2 by 3 cm is pressed to form a parallelogram. The included angle is reduced to 30^o, find the reduction is area.

Q.6: Find the base of a parallelogram whose area is 256 sq. cm. and height 32 cm.

Q.7: Define a rhombus.

Q.8: Write the area of Rhombus of side 'a' and the included angle 'θ' is given and when two diagonals are given.

Q.9: The diagonals of a rhombus area 40 m and 30 m. Find its area.

Q.10: The perimeter of a Rhombus is 140 cm and one of the opposite angles is 30^o, find the area.

Q.11: The diagonals of a Rhombus are 6 and 8 cm respectively. Find the length of the side of Rhombus.

Q.12: Find the area of Trapezoid whose parallel sides are 20 cm and 30 cm and perpendicular distance between them is 4 cm.

Q.13: Define a cyclic quadrilateral and write its area.

Q.14: The sides of a cyclic quadrilateral are 75, 55, 140 and 40 m find its area.

Answers

Q3.	100 sq.cm.	**Q4**.	5 cm. 18 cm.	**Q5.**	3 sq.cm.
Q6.	8 cm	**Q9.**	600 sq.cm	**Q10.**	612.5 sq.cm.
Q11.	5 cm	**Q12.**	100 sq. cm.	**Q14.**	3714.8 sq.m.

Objective Type Exercise

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

___1. Area of a square having side 4cm is equal to
(a) 30 sq. cm (b) 16 sq. cm
(c) 8 sq. cm (d) 20 sq. cm

___2. Perimeter of square having side 5 cm is
(a) 20 cm (b) 10 cm (c) 15cm (d) 25cm

___3. Each angle of a square is
(a) 30^o (b) 60^o (c) 90^o (d) 180^o

___4. Area of rectangle having sides 8 cm and 5 cm is
(a) 13 sq. cm (b) 40 sq. cm (c) 45 sq. cm (d) 30 sq. cm

___5. Area of parallelogram having 'a' and b as adjacent sides and θ is the included angles is
(a) $ab\text{Cos}\,\theta$ (b) $\frac{1}{2}ab\sin\theta$
(c) $ab\sin\theta$ (d) $a\sin\theta$

___6. Area of parallelogram having base 2cm and height 5cm is
(a) 20 sq. cm. (c) 30 sq. cm.
(c) 15 sq. cm. (d) 10 sq. cm.

___7. Area of rhombus of side 20 cm and included angle 30^o is
(a) 100 sq. cm (b) 200 sq. cm.
(c) 400 sq. cm (d) 50 sq. cm

___8. If diagonals of rhombus are 6 cm and 5 cm, then area is
(a) 15 sq. cm (b) 30 sq. cm
(c) 10 sq. cm (d) 11sq. cm

___9. If a = 4cm and b = 8cm are parallel sides and h = 5 cm distance between them, then area of trapezoid is
(a) 12 sq. cm (b) 40 sq. cm
(c) 30 sq. cm (d) 20 sq. cm

___10. A four sided figure whose two sides are parallel and two sides non-parallel is called
(a) Rectangle (b) Parallelogram
(c) Trapezium (d) Rhombus

___11. The length of the diagonal of a square of side 'x' is
(a) $\sqrt{2}.x$ sq. unit (b) $2\sqrt{2x}$ sq. cm
(c) $4\sqrt{2x}$ (d) $\sqrt{2x}$ sq. cm

___12. The area of a rhombus 24 sq. cm and one diagonal is 6 cm. Then other diagonal will be
(a) 8cm (b) 10cm (c) 4cm (d) 36cm

__.13 If perimeter of a square is 40cm, then side of square is
(a) 4cm (b) 20cm (c) 10cm (d) 5cm

Answers

(1)	b	(2)	a	(3)	c	(4)	b	
	(5)	c	(6)	d	(7)	b	(8)	a
	(9)	c	(10)	c	(11)	a	(12)	a
	(13)	c						

Chapter 12
Area of Polygons

12.1 Polygon

A plane figure bounded by a number of straight lines is called polygon. A polygon is said to be regular when all its sides and angles are equal.

A five sided polygon is called a pentagon.
A six sided polygon is called a hexagon.
A seven sided polygon is called a heptagon.
A eight sided polygon is called an octagon.
A nine sided polygon is called a nonagon.
A ten sided polygon is called a decagon.
A twelve sided polygon is called a dodecagon.

Inscribed Polygon

If a circle passes through the corners of a polygons then this polygon is called inscribed polygon and the circle is called the circumscribed circle. If the inscribed polygon is a regular polygon the centre of the circle is also the centre of the polygon. In the figure 'o' is the centre of the polygon and OF is the radius of the circumscribed circle and is denoted by 'R'.

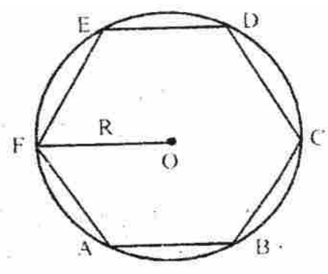


Fig. 12.1

Circumscribed Polygon:

A polygon which is drawn outside a circle so that the sides touch circumferences of the circle. In this case circle is called inscribed circle. If the polygon is regular then the centre of the circle is also the centre of the polygon. The radius of the inscribed circle is denoted by 'r' in the figure OG = r

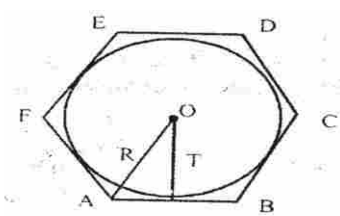


Fig. 12.2

Interior Angle:

The interior angle of a regular

Polygon of n sides = $\dfrac{2n-4}{n} \text{ x } 90^o$

For example for a hexagon, n = 6

Its interior angle $= \dfrac{2 \text{ x } 6 - 4}{6} \text{ x } 90$

$= \dfrac{12-4}{6} \text{ x } 90$

$= \dfrac{8}{6} \text{ x } 90^o = 120^o$

Angle at the centre opposite to each side of a regular.

Polygon $= \dfrac{360^o}{n}$

12.2 Method for Finding Area of Regular Polygon

A regular polygon can be divided into equal isosceles triangles by joining all the corners to the centre. Obviously, the number of triangle is the same as the number of sides of the polygon, then the area of the polygon is equal to the area of any one triangle multiplied by the number of triangle.

(i) Area of the Regular Polygon of "n" sides when length of a side is given:

Let, AB = a be the length of the side of regular polygon of n sides, "o" be the centre of the polygon, we formed "n" similar triangles.

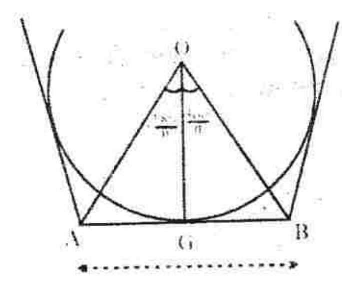


Fig. 12.3

Let, Δ AOB be one of the "n" triangles.

Then <AOB = $\dfrac{360^o}{n}$, <AOG = $\dfrac{180^o}{n}$

Area of the regular polygon = n x area of Δ AOB

$= n \text{ x } \dfrac{AB \text{ x } OG}{2}$

$$= n \times \frac{a}{2} \times OG$$

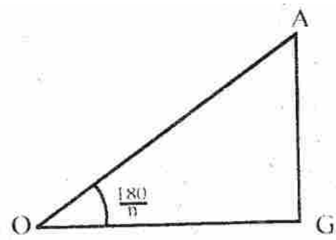


Fig. 12.4

Now, in right triangle AOG

$$\frac{OG}{AG} = \text{Cot}\,\frac{180^o}{n}$$

$$OG = AG\ \text{Cot}\ \frac{180^o}{n}$$

$$= \frac{a}{2}\text{Cot}\ \frac{180^o}{n}$$

Area of polygon $= n \times \frac{a}{2} \times \frac{a}{2}\ \text{Cot}\ \frac{180^o}{n}$

$$= \frac{na^2}{4}\ \text{Cot}\frac{180^o}{n}\text{Sq. unit}$$

Perimeter of polygon = na

Example 1:

Find the cost of carpeting an octagonal floor with sides measuring 12 meter if the carpet costs Rs. 12/- per square meter.

Solution:

n = 8, a = 12m

Area $= \frac{na^2}{4}\text{Cot}\frac{180^o}{8}$

$$= \frac{8x(12)^2}{4}\text{Cot}\frac{180^o}{8}$$

$$= \frac{8x144}{4}\text{Cot } 22.5$$

= 288 x 2.41

A =695.29 Sq. m

Cost of carpeting = 695.29 x 12 = Rs. 8343.48

(ii) Area of a regular polygon of "n" side when the radius of the inscribed circle "r" is given

Area of polygon = n x area of AOB

$$= n \times \frac{AB \times OG}{2}$$

AB = ?, OG = r

AB = 2AG

$$\frac{AG}{OG} = \tan\frac{180^o}{n}$$

$$AG = OG\tan\frac{180^o}{n}$$

$$\frac{AB}{2} = r\tan\frac{180^o}{n}$$

$$AB = 2r\tan\frac{180^o}{n}$$

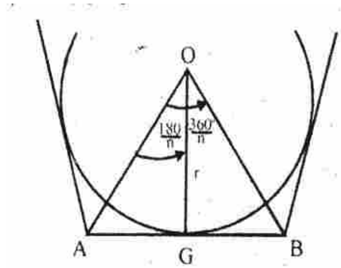


Fig. 12.5

Area of polygon $= n \text{ x } AB \text{ x } \dfrac{OG}{2}$

$$= n \text{ x } 2r\tan\frac{180^o}{n} \text{ x } \frac{r}{2}$$

$$= nr^2\tan\frac{180^o}{n} \text{ Sq. unit}$$

Perimeter $= 2nr\tan\dfrac{180^o}{n}$

Example 2:

A regular pentagon is circumscribed about circle with a radius of 20 cm. Find the area of the pentagon.

Solution:

Radius r = 20cm

n = 5 (pentagon)

Area of circumscribed polygon

$$= nr^2\tan\frac{180^o}{n}$$

$$= 5(20)^2\tan\frac{180^o}{5}$$

$= 5 \text{ x } 400 \text{ x } \tan 36^o$

= 2000 x (.726)

= 1453 sq. cm

(iii)Area of Regular Polygon of "n" sides when the radius of the circumscribed circle R is given

Let, OA = R, the radius of the circumscribed circle

Area of the polygon = n area of ΔAOB

$$= n \times \frac{OA \times OB}{2} \text{Sin } \frac{360^o}{n}$$

$$= n \times \frac{R \times R}{2} \text{Sin } \frac{360^o}{n}$$

$$\text{Area} = \frac{nR^2}{2} \text{Sin } \frac{360^o}{n} \text{ sq. unit}$$

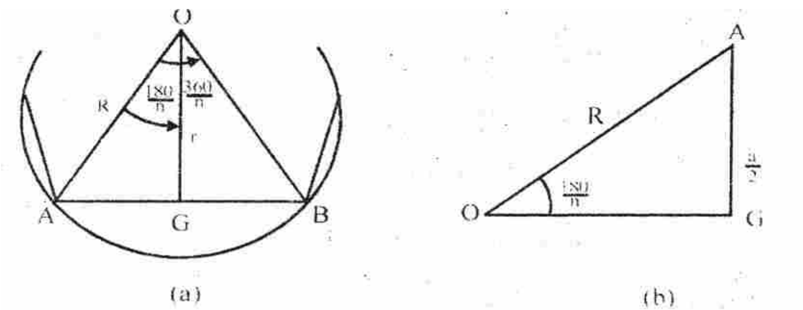


Fig. 12.6

Perimeter = n x side
= n x AB

But $\frac{AG}{OA} = \text{Sin } \frac{180^o}{n}$

$$AG = OA \text{ Sin } \frac{180^o}{n}$$

AB = 2AG

$$\frac{AB}{2} = OA \text{ Sin } \frac{180^o}{n}$$

$$AB = 2 \text{ OA Sin } \frac{180^o}{n}$$

Perimeter = n x AB

$$= n \times 2 \text{ OA Sin } \frac{180^o}{n}$$

$$= n \times 2 \times R \text{ Sin } \frac{180^o}{n}$$

$$= 2n \text{ R Sin } \frac{180^o}{n}$$

$$A = \frac{nR^2}{2} \text{Sin } \frac{360^o}{n}$$

$$P = 2n \text{ R Sin } \frac{180^o}{n}$$

Example 3:

A regular polygon of 10 sides is inscribed in a circle whose radius is 30 cm. Find the area and the perimeter of the polygon.

Solution:

We have,

R = 30cm, n = 10

$$\text{Area} = \frac{nR^2}{2} \text{Sin } \frac{360^o}{n}$$

$$= \frac{10 \text{ x } 30 \text{ x } 30}{2} \text{Sin } \frac{360^o}{10}$$

$= 4500 \text{ x Sin } 36^o$

$= 4500 \text{ x } (.5078) = 2650$ Sq. cm ………. (1)

$$\text{Perimeter} = 2nR \text{ Sin } \frac{180^o}{n}$$

$$= 2 \text{ x } 10 \text{ x } 30 \text{ x Sin } \frac{180^o}{n}$$

$= 600 \text{ x Sin } 18^o$

$= 600 \text{ x } (.31) = 186$cm …………. (2)

Example 4:

Compare the areas of an equilateral triangle, a square and a regular hexagon of equal perimeter.

Solution:

Let, perimeter of each figure = x

$\therefore$ One side of equilateral triangle = $\frac{x}{3}$

One side of square = $\frac{x}{4}$

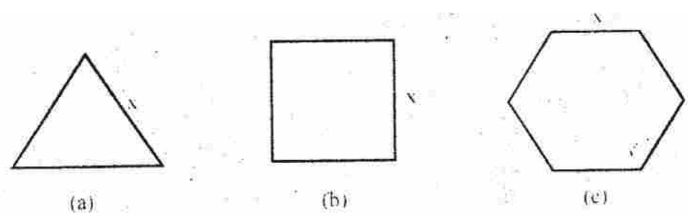


Fig. 12.7

One side of regular hexagon = $\frac{x}{6}$

Area of an equilateral triangle

$$= \frac{\sqrt{3}}{4}a^2$$

$$= \frac{\sqrt{3}}{4}\left(\frac{x}{3}\right)^2$$

$$= \frac{\sqrt{3}x^2}{36} \quad \ldots\ldots\ldots\ldots \quad \text{(i)}$$

Area of a square $= a^2$

$$= \left(\frac{x}{3}\right)^2 = \frac{x^2}{16} \quad \ldots\ldots\ldots\ldots \quad \text{(ii)}$$

Area of regular hexagon $= \frac{na^2}{4}\text{Cot}\,\frac{180^o}{n}$

$$= \frac{6(x/6)^2}{4}\text{Cot}\,\frac{180^o}{6}$$

$$= \frac{6}{4}\text{x}\,\frac{x^2}{36}\text{Cot}\,30^o$$

$$= \frac{x^2}{24}(\sqrt{3}) \ldots\ldots\ldots\ldots \quad \text{(iii)}$$

Comparison of area is

$$\frac{\sqrt{3}}{36}x^2 : \frac{x^2}{16} : \frac{\sqrt{3}}{24}x^2$$

Multiplying by $\frac{4}{x^2}$

$$\frac{\sqrt{3}}{9} : \frac{1}{1} : \frac{\sqrt{3}}{6}$$

Multiplying by 36

$$4\sqrt{3} : 9 : 6\sqrt{3}$$

Example 5:

A grassy plot has the shape of a regular hexagon each side 100m. Within the plot and along its sides a foot path is made 4m wide all around. Find the area of the grassy plot left within.

Solution:

For inner hexagon

Area of the regular polygon

When side is given = $\frac{na^2}{4}\text{Cot }\frac{180^o}{n}$

To find the length (side) CD, draw perpendicular CE and DF on AB.

In right angle triangle ACE

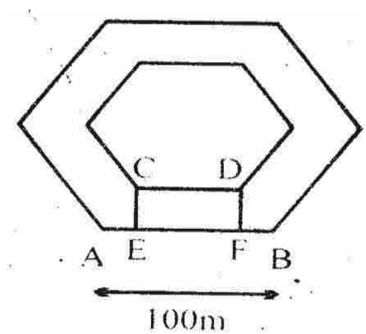


Fig. 12.8

$\frac{CE}{AE} = \tan 60^o$

AE $= \frac{CE}{\tan 60^o}$

$= \frac{4}{1.723}$

AE = 2.30 = FB

FB = 2.30 similarly

Also, CD = EF = AB – AE – FB

= 100 – 2.30 – 2.30

CD = 100 – 4.60 = 95.40m

Now,

n = 6, a = 95.40m

$A = \frac{na^2}{4}\text{Cot }\frac{180^o}{6}$

$= \frac{6(95.4)^2}{4} \times 1.723$

A = 23644.81 Sq. meter

Exercise 12

Q1. The area of a regular octagonal room is 51 sq. m Find the length of its side.

Q2. A regular decagon is inscribed in a circle the radius of which is 10cm. Find the area of the decagon.

Q3. Find the cross-sectional area of the largest hexagonal shank that can be cut from a round bar of copper 2cm in diameter.

Q4. Regular polygons of 15 sides are inscribed in and circumscribed about a circle whose radius is 12cm show that the difference of their areas is nearly 20 square cm.

Q5 A regular octagon circumscribes a circle of 2cm radius. Find the area of the octagon.

Q6. The distance between the corners of a hexagonal nut is 2.28 cm. Find the distance between the jaws of the wrench needed to fit this nut.

Q7. Find the area of a regular hexagon whose perimeter is 48cm. What is radius of its inscribed circle.

Q8. Find the area of a regular hexagon whose perimeter is 48cm, find also the perimeter and area of circumscribed circle of hexagon.

Q9. What is the length of the side and area of the largest hexagon that can be cut from 8cm. round bar.

Answers 12

Q1.	3.25	Q2.	293.8 sq. cm	Q3.	2.59 sq. cm
Q5.	13.3 sq.cm	Q6.	1.97 cm		

Q7. A = 166.72 sq. cm , r = 6.93cm

Q8. 166.72 sq. cm, 50.28 cm, 201.14 sq. cm. Q9. 41.569sq.cm

Summary

1. The interior angle of a regular

   Polygon of n sides = $\frac{2n-4}{n} \text{ x } 90^o$

2. Area of the regular polygon of n sides when length of a side 'a' is give $A = \frac{na^2}{n} \text{Cot} \frac{180^o}{n}$ Sq. Unit

   Perimeter = na

3. Area of a regular polygon of 'n' side when the radius of the inscribed aide 'r' is given

   $A = nr^2 \tan \frac{180^o}{n}$ sq. unit

   $P = 2nr \tan \frac{180^o}{n}$

4. Area of regular polygon of 'n' sides when the radius of the circumscribed circle 'R' is give

   $A = \frac{nR^2}{2} \text{Sin} \frac{360^o}{n}$ Sq. unit $\qquad P = 2nR \text{ Sin } \frac{180^o}{n}$

Short Questions

Q.1: Define a polygons.

Q.2: Define a regular polygon**.**

Q.3: Define inscribed polygon (circumscribed circle).

Q.4: Define circumscribed polygon (inscribed circle)

Q.5: Write the formula to find the angle of a regular polygon of n sides.

Q.6: Find the interior angle of hexagon.

Q.7: Write the area of the regular polygon of n sides when the length of a side is given.

Q.8: Write the formula of area of a regular polygon of 'n' sides when the radius of inscribed circle 'r' is given.

Q.9: Write the area of regular polygon of 'n' sides when the radius of the circumscribed circle 'R' is given.

Q.10: The perimeter of a regular hexagon is 12 cm, find its area

Q.11: Find the area of regular hexagon circumscribed about a circle of radius 2 cm.

Answers

Q5. The angle of a regular polygon of n sides is $\frac{2n -4}{n} \times 90^o$.

Q6. 120^o **Q10.** 10.38 sq. cm. **Q11.** 6.94 sq. cm.

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

___1. A seven sided figure is called
(a) Square (b) octagon (c) heptagon (d) pentagon

___2. If a is the side of polygon of n sides, then its area is
(a) $\frac{na^2}{4}\text{Cot}\,\frac{180^o}{n}$ (b) $\frac{na^2}{3}\;\frac{180^o}{n}$
(c) $\frac{na^2}{2}\text{Sin}\frac{360^o}{n}$ (d) None of these

___3. Area of a regular polygon of n sides whose radius of circumscribed circle is 'R' is
(a) $\frac{nR^2}{2}\text{Sin}\frac{360^o}{n}$ (b) $\frac{nR^2}{n}\text{Sin}\;\frac{360^o}{n}$
(c) $\frac{R^2}{2}\text{Cot}\;\frac{180^o}{n}$ (d) None of these

___4. Decagon have number of sides
(a) 8 (b) 9 (c) 10 (d) 12

_5. Area of a regular hexagon of side 'a' is
(a) $\frac{\sqrt{r}}{4}$ (b) $\frac{2}{\sqrt{3}}a^2$ (c) $\frac{3\sqrt{3}}{4}a^2$ (d) $2\sqrt{3a^2}$

___6. The perimeter of a regular hexagon is 12 cm, its area is
(a) 10.932 sq. cm (b) 10.392 cm^2
(c) 10.239 sq^2 (d) 10.329 cm^2

___7. Each angle of an octagon is
(a) 60^o (b) 120^o (c) 135^o (d) 150^o

___8. Area of regular octagon of side 'a' is
(a) 4.2884 a^2 (b) 4.24884 a^2
(c) 4.8284 a^2 (d) 4.1884 a^2

___9. A regular polygon having infinite number of sides is called
(a) Octagon (b) hexagon
(c) circle (d) heptagon

___10. Area of regular hexagon circumscribed about a circle of radius 2 cm is
(a) $\frac{24}{\sqrt{3}}$ (b) $\frac{20}{\sqrt{3}}$ (c) $\frac{36}{\sqrt{3}}$ (d) $\frac{2}{\sqrt{3}}$

Answers

Q.1 (1) c (2) a (3) a (4) c (5) c
(6) b (7) c (8) c (9) c (10) a

Chapter 13
Area of Circle

13.1 Circle

A circle is a plane figure bounded by one line which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to another.

The point C is called the center of the circle.

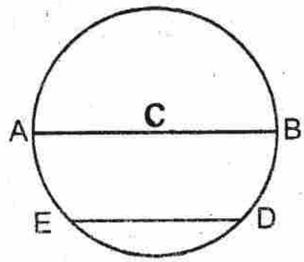


Fig. 13.1

Radius

A radius of a circle is a straight line drawn from the center to the circumference.

Diameter

A diameter of a circle is a straight line drawn through the center and terminated both ways by the circumference.

In figure AC is a radius, and AB a diameter.

Circumference of a circle = πd or $2\pi r$

Chord

A chord of a circle is a straight line joining two points on the circumference.

ED is chord.

13.2 Area of the Circle:

If r is the radius of the circle, d is the diameter of the circle.

Then Area of circle = πr^2 or $\frac{\pi}{4}d$

Hence r = $\frac{\sqrt{A}}{\pi}$

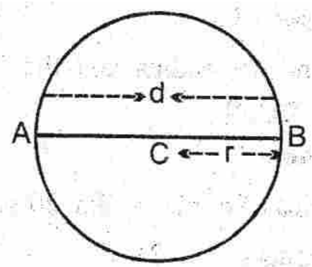


Fig. 13.2

13.3 Concentric Circles:

Concentric circles are such as have the same center.

Area of the Annulus (Ring)

Area between two concentric circles is known as annulus, for example, area of a washer, the area of cross-section of a concrete pipe. Area of the annulus = Area of outer circle – area of inner circle

$$= \frac{\pi}{4}D^2 - \frac{4}{\pi}d^2$$

$$= \frac{\pi}{4}(D^2 - d^2)$$

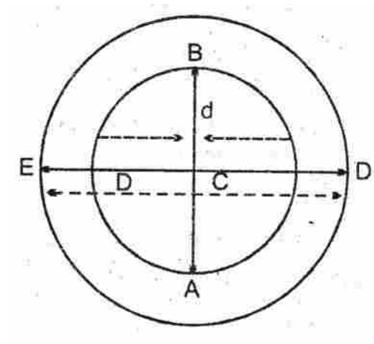


Fig. 13.3

Where D is the diameter of outer circle and d is the diameter of inner circle.

If R and r denote the radius of the outer and inner circles respectively.

Then ,Area of ring = $\pi R^2 - \pi r^2$

$= \pi(R^2 - r^2)$ square units

Example 1:

Find the radius and the perimeter of a circle the area of which is 9.3129 sq. cm.

Solution:

Area of circle = 9.3129 sq. cm.
Radius r = ?
Area of circle = πr^2
$9.3129 = (3.14)r^2$
$r^2 = \dfrac{9.3129}{3.14}$
$r^2 = 2.96$
r = 1.72 cm
Perimeter of the circle = $2\pi r$
= 2(3.14)(1.72) = 10.80cm.

Example 2:

A path 14 m wide surrounds a circular lawn whose diameter is 120 m. find the area of the path.

Solution:

Diameter of inner circle = 120m
Radius of inner circle = r = 60m
Radius of outer circle = R = 60 + 14 = 74m
Area of path = $\pi (R^2 - r^2)$

$$= \frac{22}{7}(74^2 - 60^2) = \frac{22}{7}(1876)$$

= 5896 sq. m

Example 3:

A hollow shaft with 5 m internal diameter is to have the same cross-sectional area as the solid shaft of 11m diameter. Find the external diameter of the hollow shaft.

Solution:

Let D = diameter of solid shaft = 11m

Area of the solid shaft = $\frac{\pi}{4}(11)^2 = \frac{121}{4}\pi$

Let, d = Internal diameter of hollow shaft, d = 5m

Let, D = External diameter of hollow shaft = ?

Area of annulus $= \frac{\pi}{4}(D^2 - d^2)$

$$= \frac{\pi}{4}[(D^2 - (5)^2] = \frac{\pi}{4}(D^2 - 25)$$

But, Area of annulus = Area of solid shaft

$$\frac{\pi}{4}(d^2 - 25) = \frac{121}{4}\pi$$

$\Rightarrow$ $D^2 - 25 = 121$

$D^2 = 146$

D = 12.1m which is external diameter

13.4 Sector of a Circle:

A sector of a circle is a figure bounded by two radii and the arc intercepted between them. The angle contained by the two radii is the angle of the sector.

In figure <PCQ is called the angle of sector PCQ

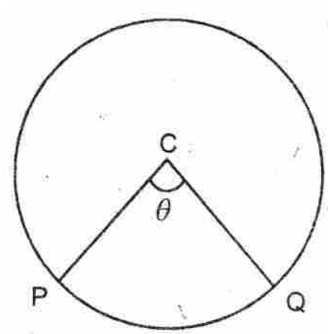


Fig. 13.4

1. Area of sector

When angle is given in degree.

Area of circle for angle $1^o = \frac{\pi r^2}{360}$

Hence, Area of sector for angle $N^o = \frac{\pi r^2}{360}$ x N

Length of arc = $l = \frac{2\pi r}{360}$. N

2. If angle <POQ is given in radian say θ radian.

Area of circle for angle 2π rad = πr^2

Area of circle for angle 1 rad = $\frac{\pi r^2}{2\pi} = \frac{1}{2}r^2$

Hence,

Area of sector for angle θ rad = $\frac{1}{2}r^2\theta$ --------(1)

3. Area of sector when arc ℓ and the radius of the circle r are given.

Since, Area of sector $= \frac{1}{2}r^2\theta$

$= \frac{1}{2}r^2\left(\frac{\ell}{r}\right)$ (because $\theta = \ell/r$)

A $= \frac{1}{2}rl$

Example 4:

Find the area of the sector of the circle whose radius is 4cm and length of the arc is 9cm.

Solution:

Let, AOB be the sector of the circle in which

OA = OB = r = 4cm

AB = l = 9cm

Area of the sector $= \frac{1}{2}lr$

$= \frac{1}{2} \times 9 \times 4$

= 18 sq. cm

Example 5:

Find the area of the sector of the circle when the radius of the circle is 15cm and the angle at the center is 60^o.

Solution:

Since, r = 15cm, and angle $\theta = 60^o$

Area of the sector $= \frac{\pi r^2}{360^o} \times 60^o$

$= \frac{22}{7}(15)^2 \times \frac{1}{6} = 117.8$ sq. cm

Example 6:

Find the expense of paving a circular court 60cm in diameter at Rs. 3.37 per square cm. If a space is left in the center for a fountain in the shape of a hexagonal each side of which is one cm.

Solution:

Area of the circle = $\frac{\pi}{4}d^2$, but d = 60cm

Area = $\frac{3.14}{4}$ x 60 x 60

= 2828.5714 sq. cm

Area of the hexagon = $\frac{na^2}{4}\text{Cot}\ \frac{180^o}{n}$

But n = 4, a = 1cm

Area of the hexagon = $\frac{6 \text{ x } (1)^2}{4}\text{Cot}\ \frac{180^o}{6}$

= $\frac{3}{2}\text{Cot}\ 60^o$

Area = 2.5980 sq. cm

Area of the plot which is to be paved:

= Area of the circle – area of the hexagon

= 2828.5714 – 2.5981

= 2826 sq. cm

∴ Expense = 2826 x 3.37

= 9523.5 rupees

13.5 Area of Segment:

A segment is a portion of circle which is cut off by a straight line not passing through the centre. The straight line AB is called the chord of the circle.

The segment smaller than a semicircle is called a minor segment and a segment greater than a semi-circle is called a major segment.

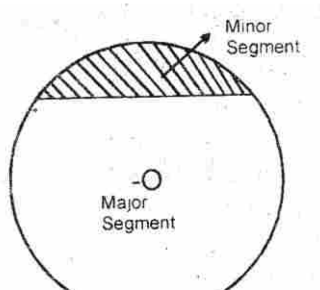


Fig. 13.6

Area of segment = Area of sector AOB $\pm$ area of ΔAOB

= $\frac{1}{2}r^2\theta \pm \frac{1}{2}r^2\text{Sin}\theta$

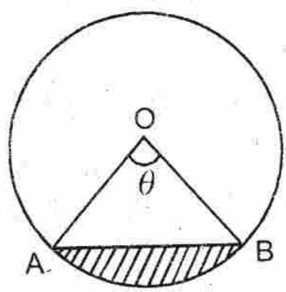


Fig. 13.7

For major segment, positive sign is taken and for minor segment, negative sign is used.

Area of the segment in terms of Height and Length of the Chord of the Segment:

If "h" is the maximum height and "c" is the length of the chord of the segment, then area of the segment is given by.

Area = $\dfrac{h}{6c}(3h^2+4c^2)$ ……………. (1)

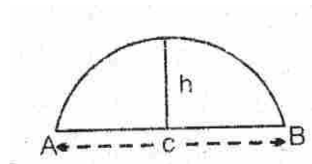


Fig. 13.8

Length of Chord and Maximum height of arc:

Let ACD is an arc of a circle with center 'O' and radius 'r' and ADB is the chord of length c and CD is maximum height 'h' of the segment.

(a) If 'h' and 'r' are given then 'c' can be calculated.

In right ΔOAD, by Pythagoras theorem,

$(OA)^2 = (AD)^2 + (OD)^2$

$(AD)^2 = (OA)^2 - (OD)^2$

$(AD)^2 = r^2 - (r-h)^2 = r^2 - r^2 + 2rh - h^2$

$AD = \sqrt{2hr - h^2}$

$\dfrac{C}{2} = \sqrt{2hr - h^2}$ because $AD = \dfrac{1}{2}(ADB)$

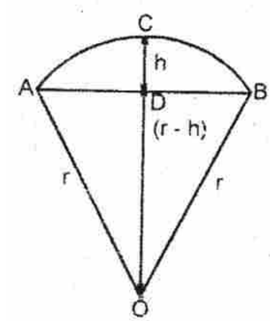


Fig. 13.9

$C = 2\sqrt{2hr - h^2}$ ……………. (i)

(b) If 'r' and 'c' are given, then 'h' can be calculated.

From (i)

Squaring (i) both sides

$C^2 = 4(2hr - h^2) \Rightarrow \quad c^2 = 8hr \Rightarrow 4h^2$

Or $4h^2 - 8hr + c^2 = 0$

Which is the quadratic equation in 'h'

$a = 4, \; b = -8r, \; c = c^2$

$$h = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{8r \pm \sqrt{64r^2 - 16c^2}}{8}$$

$$h = \frac{8r \pm 8\sqrt{r^2 - \dfrac{c^2}{4}}}{8} = r \pm \sqrt{r^2 - \left(\frac{c}{2}\right)^2}$$

+ve sign is used for major segment and – ve sign to be taken for minor segment.

Example 7:

Find the area of a segment the chord of which 8cm with a height of 2cm.

Solution:

Since, h = 2cm, c = chord of segment = 8cm

Area of segment $= \frac{h}{6c}(3h^2 + 4c^2)$

$= \frac{2}{6(8)}(3(2)^2 + 4(8)^2) = 11.16$ sq. cm

Example 8:

The span of a circular arch of 90^o is 120cm. Find the area of the segment.

Solution

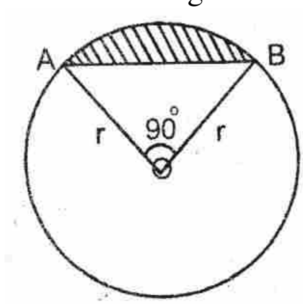


Fig. 13.10

Let O be the centre and r be the radius of the circle. Span AB = 120cm. In right angle ΔAOB,

$(OA)^2 + (OB)^2 = (AB)^2$, $r^2 + r^2 = (120)^2$

$\Rightarrow$ $2r^2 = 14400$

$r^2 = 7200$ …… (i) $\Rightarrow$ r = 84.85 cm

Now area of ΔAOB $= \frac{1}{2}(OB)(OA) = \frac{1}{2}(r)\ (r)$

$= \frac{1}{2}r^2$

Area of ΔAOB $= \frac{1}{2}(7200)$ by (i)

= 3600 sq. cm.

Area of sector $= \frac{\pi r^2}{360^o} \text{ x } N^o$

$= \frac{3.142 \text{ x } 7200}{360^o} \text{ x } 90^o = 5656$ sq. cm

Area of segment = Area of sector – Area of ΔAOB

= 5656 – 3600 = 2056 sq. cm

Example 9:

The chord of an arc is 5cm and the diameter of the circle is 7cm. Find the height of the arc.

Solution:

Here c = 5cm

d = 7cm

r = 3.5cm

$$\text{height of arc} = r \pm \sqrt{r^2 - \left(\frac{c}{2}\right)^2}$$

$$h = 3.5 \pm \sqrt{(3.5)^2 - \left(\frac{5}{2}\right)^2}$$

$$h = 3.5 \pm \sqrt{12.25 - 6.25}$$

$h = 3.5 \pm 2.45$

$h = 3.5 \pm 2.45$, $h = 3.5 - 2.45$

$h = 5.59$ cm, $h = 105$cm

Example 10:

Find the chord of arc whose height is 24 cm, in a circle of radius 15 cm.

Solution:

Here, $h = 24$cm

$r = 15$cm

Chord of arc $= 2\sqrt{2hr - h^2}$

$C = 2\sqrt{2(24)(15) - (24)^2}$

$C = 2\sqrt{720 - 576} = 2\sqrt{144} = 2(12)$

$C = 24$cm

13.6 Ellipse:

An ellipse is defined as the locus of a point which moves such that the sum of its distance from two fixed points remains constant.

The fixed points are the foci of the ellipse.

i.e. |PF| + |PF'| = constant

In figure, F and F′ are the two foci and D is the centre of the ellipse. AA' is the major axis and BB' is the minor axis, OA and OB are the semi-axes. Also 2a is the length of major axis and 2b is the length of minor axis.

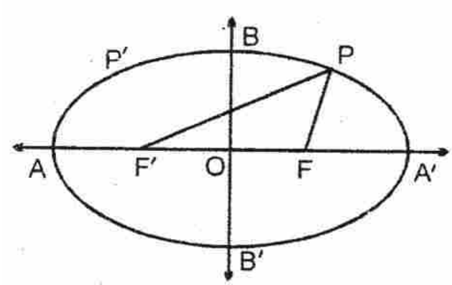


Fig. 13.11

Area of an ellipse

Area of an ellipse $= \pi ab$

Where a = semi-major axis

b = semi-minor axis

and perimeter of an ellipse $= \pi(a + b)$

Example 11:

It is desired to lay out a plot in the form of an ellipse. The area is 23100 sq. cm. The axes are in the ratio 3:2. Find the length of the fence required for this plot.

Solution:

Given area of plot in the form of ellipse = 23100 sq. cm.

Since axes are in the ratio 3:2

$\therefore \quad a = \dfrac{3}{2}b$

Area of plot $= \pi ab = \left(\dfrac{22}{7}\right) x \dfrac{3}{2} b \text{ x } b$

$23100 = 4.71b^2$

$b^2 = \dfrac{23100}{4.71} = 4904.46$

b = 70.03cm

$a = \dfrac{3}{2}(70.30) = 105\text{cm}$

Perimeter of plot = π (a + b)

$= \dfrac{22}{7}(70 + 105) \quad = 550 \text{ cm}$

Example 12:

An elliptical pipe has a major axis of 16cm and minor axis of 10cm. Find the diameter of a circular pipe that has the same area of cross-section.

Solution:

Major axis = 2a = 16cm

a = 8cm

Minor axis = 2b = 10cm

b = 5cm

A = π ab

A = (3.142)(8)(5) = 125.64 sq. cm.

Area of circle = πr^2 but D = 2r

Area of circle = $\pi \dfrac{D^2}{4}$

By given condition

Area of circle = Area of ellipse

$\pi \dfrac{D^2}{4} = 125.64$

$$D^2 = \frac{125.64 \text{ x } 4}{3.14} = 159.97$$

D = 12.64 cm

Exercise 13

Q.1 The area of a semi-circle is 130 sq. cm. Find its total perimeter. Hint $P = \pi r + 2r$.

Q.2 A road 10m wide is to be made around a circular plot of 75m diameter. Find the cost of the ground needed for the road at Rs 4.00 per square meter.

Q.3 The areas of two concentric circles are 1386 sq. cm and 1886.5 sq. cm respectively. Find the width of the ring.

Q.4 The area of a circle is 154 sq. cm. Find the length of the side of the inscribed squares.

Q.5 A circular arc has a base of 4cm and maximum height 1.6cm. Find radius, length of arc and area of segment.

Q.6 The height of an arc is 7cm and its chord 42cm. Find the diameter of the circle.

Q.7 In a circle of diameter 25cm, the chord of an arc in 10cm, find its height.

Q.8 The radius of a circle is 33.5cm. Find the area of a sector enclosed by two radii and an arc 133.74 cm in length.

Q.9 The inner diameter of a circular building is 54m and the base of the wall occupies a space of 352 sq. m. Find the thickness of the wall.

Q.10 The axis of an ellipse are 40 cm and 60 cm. Find its perimeter and area.

Q.11 The sides of the triangle are 8, 21 and 25 m. find the radius of the circle whose area is equal to the area of triangle.

Q.12 The area of a sector is 76π sq.cm and angle of the sector is 70^0. Find radius of the circle.

Answers 13

Q1. 46.8cm Q2. 10684 rupees Q3. 23.2 sq. cm

Q4. 9.89 cm Q5. 4.771 sq. cm; 2.05cm; 2.77cm

Q6. 70cm Q7. 1.04 cm Q8. 2240.14 sq.cm

Q9. 2m Q10. 157.1cm; 1885.20sq.cm

Q11. 4.99m Q12. 19.77cm.

Summary

1. Area of circle A = πr^2
2. Perimeter or circumferences of circle = $2\pi r$
3. Area of Annulus (ring)

   $A = \pi(R^2 - r^2)$

   Where R = radius of outer circle , r = radius of inner circle
4. (a) Area of the sector if angle is given in degree

   Area of the sector = $\dfrac{\pi r^2}{360^o} \text{ x } N^o$

   Length of the arc = $\dfrac{2\pi r}{360^o} \text{ x } N^o$

   (b) If the angle in radian, say θ radians,

   then Area of sector = $\dfrac{1}{2} r^2 \theta$

   (c) If l is the length of an arc and r, radius of the circle, then area of sector.

   $A = \dfrac{1}{2} lr$
5. Area of segment = Area of sector AOB $\pm \dfrac{1}{2} r^2 \sin\theta$

   +ve sign is taken for major axis
   -ve sign is taken for major axis
6. Area of the segment in terms of Height and Length of the chord of the segment

   Area = $\dfrac{h}{6c}(3h^2 + 4c^2)$

   Where h = maximum height, c = length of the chord.

Short Questions

Q.1: Define a circle.

Q.2: Define diameter of a circle

Q.3: Define chord of a circle.

Q.4: What is the area and circumference of circle .

Q.5: Find the radius of a circle the area of which is 9.3129 sq. cm.

Q.6: What are concentric circle.

Q.7: Define area of the Annulus (Ring).

Q.8: A path 14cmwide, surrounds a circular lawn whose diameter is 360 cm. Find the area of the path.

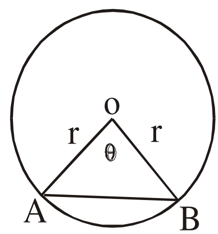
Q.9: Define a sector of the circle.

Q.10: Write the area of the sector.

Q.11: The minute hand of a clock is 12 cm long. Find the area which is described on the clock face between 6 A.M.to6.20A.M.

Q.12: Define a segment.

Q.13: Write the formula of Area of the minor segment and major segment when angle ‘θ’ and radius ‘r’ are given.



Q.14: Write the area of the segment in terms of Height and length of the chord of the segment.

Q.15: Find the area of a segment the chord of which 8 cm with a height of 2 cm.

Q.16: The area of a semi-circle is 130 sq. cm. Find its total perimeter.

Answers

Q5. r= 1.72 cm. **Q8**. 16456 sq.cm **Q11.** 150.7 sq.cm.

Q13. Area of minor segment $= \frac{1}{2} r^2 (\theta - \sin\theta)$

Area of a major segment $= \frac{1}{2} r^2 (2\pi - \theta + \sin\theta)$

Q15. 11.16 sq. cm. **Q16.** 46.7 cm.

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

___1. Area of a circle whose radius is 'a' cm is:

(a) πr^2 (b) πa^2 (c) πa (d) $\frac{\pi}{2}a^2$

__2. Circumference of a circle whose radius is $\frac{1}{2}$ cm is equal to

(a) 2π (b) $2\pi r$ (c) $\frac{\pi r}{2}$ (d) $\frac{r}{2}$

___3. Area of a sector of 60^o in a circle of radius 6 cm is:

(a) 6π (b) $6\pi^2$ (c) 36π (d) 3π

___4. The space enclosed between two concentric circles is called

(a) cone (b) ellipse (c) annulus (d) none of these

___5. Arc of circle with diameter 'd' is:

(a) $\frac{\pi}{2}r^2$ (b) $\frac{\pi}{2}d^2$ (c) $\frac{\pi}{4}d^2$ (d) None of these

___6. If R and r denote the radii of the outer and inner circles, then Area of annulus (ring) is:

(a) $\pi(R^2 - r^2)$ (b) $\frac{\pi}{2}(R^2 - r^2)$ (c) $\pi(R^2 + r^2)$

(d) $(R^2 - r^2)$

___7. If 2a and 2b are the major and minor axis of the ellipse, then area of ellipse is:

(a) ab (b) πab (c) $\frac{\pi}{2}ab$ (d) $\pi^2 ab$

___8. If 2a and 2b are the major and minor axis of the ellipse, the circumference of the ellipse is:

(a) $\pi(a + b)$ (b) $\pi(a - b)$

(c) $\pi(a + b)^2$ (d) $\pi^2(a - b)$

___9. If the area of circle is , then radius r is:

(a) 4 (b) 2 (c) 16 (d) 8

___10. If x and y the major and minor axis of the ellipse then area of ellipse is:

(a) $\frac{\pi xy}{2}$ (b) $\frac{\pi xy}{4}$ (c) πxy (d) $\pi^2 xy$

___11. The circumference of a pulley is $\frac{440}{7}$ cm, its diameter is:

(a) 20cm (b) 40cm (c) 80cm (d) 10cm

Answers

Q.1	(1)	b	(2)	a	(3)	a	(4)	c
	(5)	c	(6)	a	(7)	b	(8)	a
	(9)	a	(10)	b	(11)	a		

Chapter 14
Area of Irregular Plane Figures

In the previous chapters, areas of different regular figures have been calculated. In this chapter we find the areas of figures which are not regular and uniform in shape, such figures are called irregular. For example, a plot of land may be partly bounded by a winding river. In this case the exact area cannot be calculated, only the approximation of the area can be made. Some of the rules devised for finding the areas of irregular figures are given below.

14.1 Mid Ordinate Rule:

According to the rule, the base line AB of irregular figure ABCD is divided into a number of equal strips each of breadth s.

At the mid point of each strip a vertical line drawn is called the mid ordinate.

Let, a, b, c, d, e, f be the lengths of these and ordinates.

The area of each strip is approximately equal to s x mid ordinate.

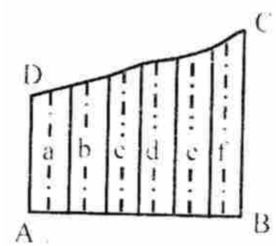


Fig. 14.1

Approximate area of the figure ABCD

= S(a + b + c + d + e + f)

Where S = breadth of each strip.

If the figure is divided into a different of strips, then area of the figure = S(a + b + c + d + ………………)

Example 1:

The mid-ordinates of a water plane are 15m apart and their lengths are 1.9, 6.6, 11, 5, 17, 4, 19.4, 20.8, 18.8, 15.8, 10.6 and 2.6 respectively. Find the area of the plane.

Solution:

By mid-ordinate rule

Area = S(A + b + c + d + ……..) where S is breadth of each strips and a, b, c, d, ………….. are lengths of mid-ordinates.

Now, S = 15, and the length of mid-ordinates are 1.9, 6.6, 11, 14.5, 17.4, 19.4, 20.5, 20.8, 18.8, 15.8, 10.6 and 2.6 m.

Area = 15(1.9 + 6.6 + 11 + 14.5 + 17.4 + 19.4 + 20.5)

= 15 x 159.9 sq. m = 2398.5 sq. m

14.2 Trapezoidal Rule:

The trapezoidal rule is another method for finding the area of irregular plane figures. Let ABCD be a irregular figure. To find the area of figure ABCD, divide the figure into a number of strip of equal width. Each strip is treated as trapezium. So the area of the figure ABCD is the sum of the areas of all these trapeziums. Let s be the width of each strip and a, b, c, d, …… be the ordinates of the sides of the strips.

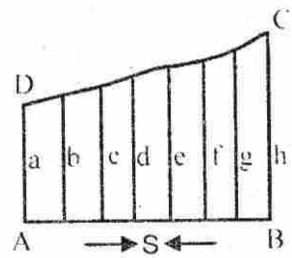


Fig. 14.2

Thus,

$$\text{Area} = S[\frac{\text{Sum of first and last ordinate}}{2} + (\text{Sum of remaining ordinate})]$$

$$\text{Area} = S[\frac{a+b}{2} = (a + d + e + f + g)]$$

Example 2

Find the area of cross-section of river along a lone where the depths at equal interval of 10m are noted 0, 7, 11, 15, 5, 0m respectively.

Solution:

Since the number of ordinates are 0, 7, 11, 15, 5, 0 we apply the trapezoidal rule.

$$\text{Area} = S[\frac{\text{Sum of first and last ordinate}}{2} + (\text{Sum of remaining ordinate})]$$

$$= 10[\frac{0+0}{2} + 7 + 11 + 15 + 5]$$

$$= 10 \times 38 = 380 \text{ sq. m}$$

14.3 Simpson's Rule:

The most important rule in practice is the Simpson's rule because of its simplicity and accuracy.

For the application of Simpson's rule the figure is divided into an even number of strip of equal width, thus giving an odd number of ordinates.

Area = $\dfrac{\text{width of each strip}}{3}$ [Sum of first and last ordinates + 2(Sum of odd ordinates + 4(Sum of even ordinates)]

A = $\dfrac{S}{3}$[A + 2D + 4E]

Where, A = sum of first and last ordinates
D = sum of odd ordinates
E = sum of even ordinates
S = width of each strip

So, this rule is applicable only where there will be an even number of strips or odd numbers of ordinates.

Example 3:

Find the area of the field, whose ordinates are 0, 20, 22.5, 33.5, 45, 42, 33.5, 25.5 and 0 meter respectively. The width of each strip is 14m. Find the approximately cost of purchasing the field at a cost of Rs 5,000/per m.

Solution:

The number of ordinates are odd therefore, we shall apply Simpson's rule.

Here, S = 14,
A = 0 + 0 =0
D = 22.5 + 45 + 33.5 = 101
E = 20 + 33.5 + 42 + 25.5 = 121

$$\text{Area} = \frac{S}{3}[A + 2D + 4E]$$
$$= \frac{14}{3}[0 + 2(101) + 4(121)]$$
$$= \frac{14}{3}[202 + 484]$$
$$= \frac{14}{3} \text{ x } 686 = 3201.33 \text{ sq. m}$$

Cost of purchasing the field = 3201.33 x 5000
= 160066 Rupees

Example 4:

Calculate the total area of a field with a base of 50m and ordinates 4, 6, 5, 7, 7, 6, 6.5, 7.5, 8, 8 and 9m respectively.

Solution:

Here base = 50m and the number of ordinates = 11

And strips = 10 so $S = \dfrac{50}{10} = 5$

A = 4 = 9 = 13
D = 5 + 7 + 6.5 + 8 = 26.5
E = 6 + 7 + 6 + 7.5 + 8 = 34.5

$$A = \frac{S}{3}[A + 2D + 4E]$$

$$= \frac{5}{3}[13 + 2(26.5) + 4(34.5)] = \frac{5}{3}(204) = 340 \text{ sq. m}$$

Example 5:

Find the area of the curve $y = x^2$ between the values x = 1 and x = 7.

Solution:

For values of x = 1, 2, 3 ……. 7, the corresponding values of y are in the given table.

X	1	2	3	4	5	6	7
Y	1	4	9	16	25	36	49

Ordinates are 1, 4, 9, 16, 25, 36, 49

$$\text{Area} = \frac{S}{3}[A + 2D + 4E]$$

When S = length of strip = 1

$$A = \frac{1}{3}[(1 + 49) + 2(9 + 25) + 4(4 + 16 + 36)]$$

$$A = \frac{1}{3}[50 + 2(34) + 4(56)] = \frac{1}{3}[50 + 68 + 224]$$

$$= \frac{1}{3}(342) = 114 \text{ sq. units}$$

Example 6:

Calculate the total area of the figure shown below: using Simpson rule. All the given dimensions are in cm

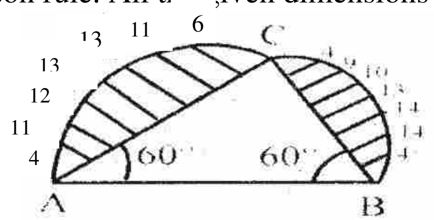


Fig. 14.3

Solution: The given figure is divided into three parts

$$A_1 = \text{Area of an equilateral } \Delta ABC = \frac{\sqrt{3}}{4}a^2$$

$$= \frac{\sqrt{3}}{4}(64)^2 = 1773.67 \text{ sq. cm}$$

Using Simpson rule

$A_2 =$ Area of the left side of ΔABC

$$= \frac{S}{3}[A + 2D + 4E]$$

$$= \frac{8}{3}[(0 + 0) + 2(11 + 13 + 11) + 4(4 + 12 + 13 + 6)]$$

$$A_2 = \frac{8}{3}[0 + 70 + 140] = \frac{8}{3}[210] = 560\text{sq. cm.}$$

Area of the right side of ΔABC

$$A_3 = \frac{8}{3}[(0 + 0)+2(7 + 13 + 14\ ) + 4(4 + 10 + 14 + 4)]$$

$$= \frac{8}{3}[0 + 68 + 4(37)] + \frac{8}{3}[68 + 148] = \frac{8}{3}(216)$$

$A_3 = 576$ sq. cm

Area of given figure $A_1 + A_2 + A_3$

$= (1773.67 + 560 + 576)$sq. cm $= 2909.67$ sq. cm

Example 7 :

Using Simpson's rule calculate the total area of the plot of land shown below.

Solution:

Consider a right angle ΔABC by Phthagoras theorem

$(AB)^2 = (AC)^2 + (BC)^2$

$(AB)^2 = (30)^2 + (40)^2$

$(AB)^2 = 900 + 1600 = 2500$

$AB = 50$m

Since, base of figure = 50m and number of strips = 8

$$\therefore \quad S = \frac{50}{8} = 6.25\text{m}$$

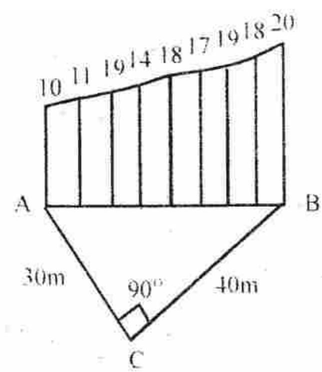


Fig. 14.4

By Simpson's Rule

$$A = \frac{S}{3}[A + 2D + 4E]$$

A = 10 + 20 = 30
D = 19 + 18 + 19 =56,
E = 11 + 14 + 17 + 18 = 60

$$A = \frac{6.25}{3}[30 + 112 + 240] = 795.8 \text{ sq. m}$$

Exercise 14

Q.1 Calculate the total area of a field with a base of 48m length and ordinates of 2, 7, 5, 7.5, 8.5, 6.5, 6.5, 3, 5.5, 8, 11.5, 10.5 and 9 respectively.

Q.2 An irregular figure has ordinates 3.5, 4.75, 5.25, 7.5, 8.25, 14.75, 6, 9.5, 4cm respectively. Length of the base of irregular figure is 20cm find the area.

Q.3 Find the area of an irregular plane figure which is divided into 6 strips of equal width of 10m each and with mid ordinates of 22, 25, 31, 36 and 39 meters respectively.

Q.4 Find the area of an irregular plane figure whose ordinates are 20, 23, 28, 32, 34, 37, and 40m respectively and the width of each trip is 7 meter.

Q.5 Find the area of an irregular plane figure whose ordinates are 7.75, 10.70, 11.20, 9.70, 7.75, 6.80, 6.30, 6.80 and 2.00 respectively. The width of each strip is 8.25m

Q.6 Find area of an irregular figure by Simpson's Rule if the ordinates are 9. 11, 13, 12, 10, 13, 15, 17, 14, 12, 7 meters and base = 73 meters.

Q.7 Following ordinates of equal intervals are down in a plot of base 1200 meters. Find the area 50, 60, 80, 90, 30, 50, 60, 80, 70, 90, 100, 120, 130.

Q.8 Sketch the graph of the function $y = 6x - x^2$. Determine the total area in the first quadrant between the curve and the y – axis by Simpson's Rule.

Answers 14

Q1.	340 sq. m	Q2.	160.41 sq. m	Q3.	1860 sq. m
Q4.	1288 sq. m	Q5.	539.68 m	Q6.	924.6 sq. m
Q7.	94000 sq.m	Q8.	36 sq.unit		

Summary

1 Area of irregular figure by mid ordinate rule
$A = S(a + b + c + d \ldots\ldots\ldots)$
When S = breadth of each strip

2 Trapezoidal Rule

$$\text{Area} = S[\frac{\text{Sum of first and last ordinate}}{2} + (\text{Sum of remaining ordinate})]$$

$$\text{Area} = S[\frac{a + g}{2} = (b + c + d + e + f)]$$

3 Simpson's rule $A = \frac{S}{3}[A + 2D + 4E]$

Where, A = sum of 1st + last ordinate
D = sum of odd ordinates
E = sum of even ordinates
S = width of each strip

Short Questions

Q.1: Write the Simpsons Rule.

Q.2: Find the area of cross-section of river along a line where the depths at equal interval of 10 m are noted 0, 7, 11, 15, 0 meter respectively.

Q.3: Using Simpson's rule, find the area of an irregular figure whose ordinates are 20, 23, 23, 32, 34, 37 and 40 cm respectively and the width of each strip is 7 cm.

Q.4: If base of a field 50 m and number of ordinates are 11, then find breadth of strip.

Q.5: Find the area of the curve $y = x^2$, between, the values $x = 1$ and $x = 7$.

Q.6: Define Irregular figure.

Answers

Q2.	366.3 sq.m.	Q3.	1253sq. cm.	Q4.	5m
Q5	114 sq.units.				

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

___1. In Simpson's rule, the number of ordinates are
(a) odd (b) even
(c) in fraction (d) None of these

___2. S, stands for, in calculating irregular figures
(a) breadth of strip (b) height of strip
(c) volume of strip (d) None of these

___3. Area of an irregular figure = $\frac{S}{3}[A + 2D + 4E]$ is
(a) mid-ordinate rule (b) Trapezoidal rule
(c) Simpson's rule (d) None of these

___4. In Simpson's Rule D, stands for
(a) Sum of 1st + last ordinates (b) Sum of odd ordinates
(c) Sum of even ordinates (d) None of these

___5. In Simpson's Rule so calculating an irregular figure, number of strips should be
(a) even (b) odd (c) double (d) Half

___6. Exact area of an irregular figure is calculated by
(a) Mid-ordinates rule (b) Simpson's
(c) Trapezoidal rule (d) None of these

__7. More accurate area of an irregular figure can be calculated by
(a) Mid ordinate rule (b) Simpson's Rule
(c) Trapezoidal Rule (d) Pythagoras theorem

___8. In Simpson's rule, E. stand for
(a) Sum of odd ordinates (b) Sum of even ordinates
(c) Sum of 1st and last ordinate (d) None of these

___9. If base of field 50m and number of ordinates are 11, then breadth of trips is equal to
(a) 5 (b) 10 (c) 15 (d) 4.54

___10. Simpson's Rule is
(a) $\frac{S}{2}[A + 2D + 4E]$ (b) $\frac{S}{3}[A - 2D + 4E]$
(c) $\frac{S}{3}[A + 2D + 4E]$ (d) $\frac{S}{3}[A + 4D + 2E]$

Answers

Q.1	(1)	a	(2)	a	(3)	c	(4)	b	(5)	b
	(6)	b	(7)	b	(8)	b	(9)	a	(10)	c

Chapter 15
Mensuration of Solid

15.1 Solid:

It is a body occupying a portion of three-dimensional space and therefore bounded by a closed surface which may be curved (e.g., sphere), curved and planer (e.g., cylinder) or planer (e.g., cube or prism).

15.2 Mensuration of Prisms:

Prism:

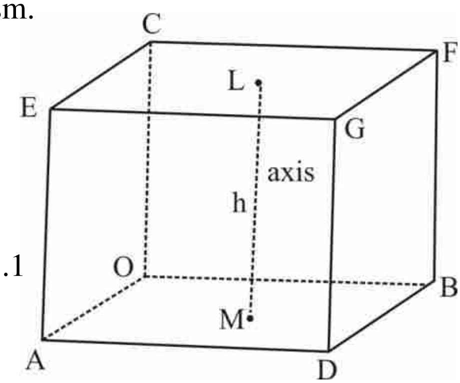
A solid bounded by congruent parallel bases or ends and the side faces (called the lateral faces) are the parallelograms, formed by joining the corresponding vertices of the bases. It is called a right prism if the lateral are rectangles. Otherwise an oblique prism. A common side of the two lateral faces is called a lateral edge.

Prism are named according to the shape of ends. A prism with a square base, a rectangular base, a hexagonal base, and a parallelogram base, is called a square prism, a rectangular prism, a hexagonal prism and a parallelepiped respectively.

Altitude:

The altitude of a prism is the vertical distance from the centre of the top to the base of the prism.

Fig. 15.1



Axis:

The axis of a prism is the distance between the centre of the top to the centre of the base.

In a right prism the altitude, the axis and the latera edge are the same lengths. In the Fig. 1 the lateral faces are OAEC, BDGF, OBCF dan ADEG. The bases are OABD and ECFG. LM = h is the axis of the prism, where L and M are the centres of the bases.

15.3 Surface Area of a Prims:

Lateral surface area of a prism is the sum of areas of the lateral faces. From Fig. 1.

Lateral surface area = Area of (OAEC + ADEG + BDGF + OBCF)

= (OA)h + (AD)h + (DB)h + (OB)h

= (OA + AD + DB + BO)h

= Perimter of the base x height of the prism (1)

Total surface area = Lateral surface area + Area of the bases (2)

Note: that, Total surface, surface area, total surface area and the surface of any figure represent the same meanings.

15.4 Volume of a Prism:

A sold occupies an amount of space called its volume. Certain solids have an internal volume or cubic capacity. (The term capacity, when not associated with cubic, is usually reserved for the volume of liquids or materials which pour, and special sets of units e.g.; gallon, litre, are used). The volume of solid is measured as the total number of unit cubes that it contains. If the solid is a Prism the volume can be computed directly from the formula,

$$V = l \cdot b \cdot h$$

Where, l, b and h denoted the length, breadth and height of the prism respectively. Also l . b denotes the area A of the base of the prism. Then

Volume of the Prims = Ah

= Area of the base x height of the prism (4)

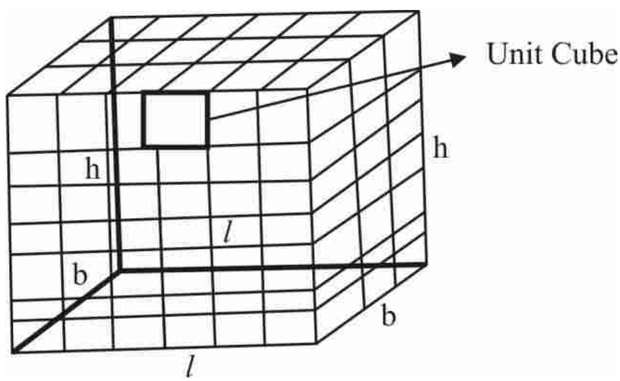


Fig. 15.2

l = 6 units b = 3 units and h = 5 units

The total number of unit cubes

= l b h = 6 x 3 x 5 = 90

So volume of prism = 90 cubic unit

Weight of solid = volume of solid x density of solid (density means weight of unit volume)

15.5 Types of Prism:

1. Rectangular Prism:

If the base of prism is a rectangle, it is called a rectangular prism.

Consider a rectangular prism with length a, breadth b and height c (Fig. 15.3)

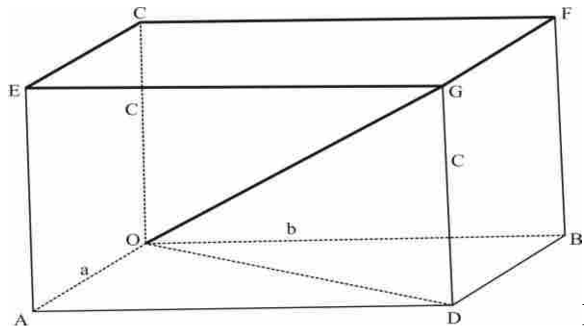


Fig. 15.3

(i) Volume of rectangular prism = abc cu. Unit

(ii) Lateral surface area = area of four lateral faces
= 2 ac + 2bc sq. unit
= (2a + 2b)c = Perimetr of base x height

(iii) Total surface area = Area of six faces
= 2ab + 2ac + 2ca
= 2 (ab + bc + ca) sq. unit

(iv) Length of the diagonal OG

In the light triangle ODG, by Pythagorean theorem,

$$OG^2 = OD^2 + DG^2 \ldots\ldots\ldots\ldots\ldots\ldots.. (1)$$
$$= OD^2 + C^2 \quad \ldots\ldots\ldots\ldots \quad (I)$$

Also in the right triangle OAD,

$$OD^2 = OA^2 + AD^2$$
$$= a^2 + b^2$$

Put OD^2 in equation (1)

(the line joining the opposite corners of the rectangular prism is called its diagonal).

$$OG^2 = a^2 + b^2 + c^2$$

Or $$|OG| = \sqrt{a^2 + b^2 + c^2}$$

2. Cube: A cube is a right prism will all sides equals. Let ‘a’ be the side of the cube Fig. 4

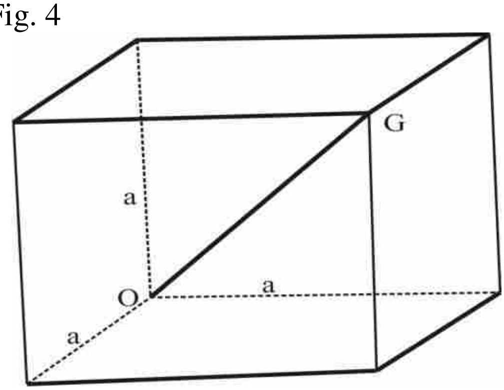


Fig. 15.4

(i) Volume of the cube $= a \,.\, a.\, a$

$= a^3$ cu . unit

(ii) Lateral surface area = Area of four lateral faces

$= 2a \,.\, a + 2a \,.\, a$

$= 4a^2$ sq. unit

(iii) Total surface area = Area of six faces

$= 6a^2$ sq. unit

(iv) The length of the diagonal

$$|OG| = \sqrt{a^2 + a^2 + a^2} = a\sqrt{3}$$

Example 1:

Find the volume, total surface, diagonal and weight of rectangular block of wood 7.5 cm long, 8.7 cm wide and 12 cm deep, 1 cu. cm = 0.7 gm.

Solution:

Let a = 7.5 cm, b = 8.7 cm, and c = 12cm

Then (i) Volume = abc = 7.5 x 8.7 x 12 = 783.00 cu. cm.

(ii) Total surface = 2 (ab + bc + ca)

= 2 (65.25 + 104.4 + 90)

= 519.3 sq. cm.

(iii) diagonal $= \sqrt{a^2 + b^2 + c^2} = \sqrt{7.5^2 + 8.7^2 + 12^2}$

$= \sqrt{275.94} = 16.6$ cm

(iv) Weight = Volume x density

= 783 x 0.7 = 548.1 gms

Example 2:

The side of a triangular prism are 25, 51 and 52 cm. and height is 60 cm. Find the side of a cube of equivalent volume.

Solution:

Volume of a prism = Area of base x height

$= \sqrt{S(s-a)(s-b)(s-c)}$ x h

When a = 25cm, b = 51cm, c = 52cm, h = 60cm

$$S = \frac{a+b+c}{2} = \frac{25+51+52}{2} = 64$$

S – a = 64–25 = 39, S – b = 64–51 = 13, S – c = 64–52 = 12

Volume of prism $= \sqrt{64(39)(13)(12)}$ x 60 = 37440 cu. cm

Volume of a cube of side l cm = l^3 cu. cm.

$l^3 = 37440$

$l = (37440)^{1/3} = 33.45$ cu. cm.

Example 3:

The volume of the cube is 95 cu. cm. Find the surface area and the edge of the cube.

Solution:

Volume of cube = 95 cu. cm.

Let 'a' be the side of cube, then

Volume = a^3

$a^3 = 95$

$a = (95)^{1/3} = 4.56$ cm

Surface area $= 6a^2 = 6(4.56)^2 = 124.92$ sq. cm

Example 4:

Find the number of bricks used in a wall 100 ft long, 10 ft high and one and half brick in thickness. The size of each is $9''$x $4\frac{1}{2}''$ x $3''$

Solution:

a = Length of wall = 100 ft = 1200 inches

b = Breadth of wall $= \left(9 + \frac{9}{2}\right)$ = 13.5 inches

h = Height of wall = 10ft = 120 inches

∴ Volume of the wall = abh = 1200 x 13.5 x 120 = 1944000 cu. in

Note: Length of brick = a = 9 inches

Breadth of brick $= b = \frac{9}{2}$ = 4.5 inches

Height of wall = h = 3 inches

Volume of the brick = abh $= 9 \text{ x } 9.5 \text{ x } 3 = \frac{243}{2}$ cu. in.

$$\text{Number of bricks} = \frac{\text{Volume of wall}}{\text{Volume of brick}} = \frac{1944000}{\frac{243}{2}} = 16000$$

Example 5:

Find the edge of a cube whose volume is equal to that of a rectangular bar measuring 126 cm long, 4 cm wide and 2 cm thick.

Solution:

Given that a = length of bar = 126 cm

b = breadth of bar = 4 cm

h = 2 cm

Volume of rectangular bar = abh

= 126 x 4 x 2 = 1008 cu. cm

Let 'a' be the edge of a cube

The volume of cube = a^3 cu. cm.

Since Volume of the cube = Volume of bar

$$a^3 = 1008 \Rightarrow a = (1008)^{1/3} = 10.002 \text{ cm}$$

3. Polygonal Prism:

If the base of prism is a polygon, the prism is called polygonal prism.

a. Volume of the polygonal prism when base is a regular polygon of n sides and h is th height = Area of the base x height.

i. $V = \frac{n\,a^2}{4} \cot\frac{180^o}{n} xh$, when side a is given.

ii. $V = \frac{n\,R^2}{2} \text{ Sin}\frac{360^o}{n} \text{x h}$, when radius R of circumscribed circle is given.

iii. $V = nr^2 \tan\frac{180^o}{n} \text{x h}$, when radius r of inscribed circle is given.

b. Lateral surface area = Perimeter of the base x height
= n a x height, a is the side of the base

c. Total surface area = Lateral surface area + Area of bases

Example 6:

A hexagonal prism has its base inscribed about a circle of radius 2cm and which has a height of 10cm is cast into a cube. Find the size of the cube.

Solution:

Hence n = 6, R = 2cm, h = 10cm

Let 'a' be the side of the cube

Volume of the hexagonal prism = $n\,R^2 \sin\frac{360^o}{n} \text{x h}$

$= 6 \text{ x } 4 \sin 60^o \text{ x } 10$

$= 240 \text{ x } 0.866$

$= 207.84$ cu. Cm.

Volume of the cube = Volume of the hexagonal prism

$a^3 = 207..84$

$a = 5.923$ cm

Note :

Weight of solid = Volume × Density

15.6 Frustum of a Prism:

When a solid is cut by a plane parallel to its base (or perpendicular to its axis), the section of the solid is called cross-section. If however, the plane section is not parallel to the bases, the portion of the prism between the plane section and the base is called a frustum.

In Fig. 5 ABCDEIGH represents a Frustum of a prism whose cutting plane EFGH is inclined an angle θ to the horizontal. In this case the Frustum can be taken as a prism whose base ABEF which is a Trapezium and height BC.

i. Volume of the Frustum:

Volume of the Frustum = Area of the Trapezium ABEF x BC

$$= \left(\frac{AE + BF}{2} \text{ x AB}\right) \text{x BC}$$

$$= \frac{h_1 + h_2}{2} \text{ x (AB + BC)}$$

Volume = average height x area of the base

Or Volume = average height x area of the cross-section

ii. Area of the Section EFGH:

Since the cutting plane is inclined an angle θ to the horigontal from Fig. 5.

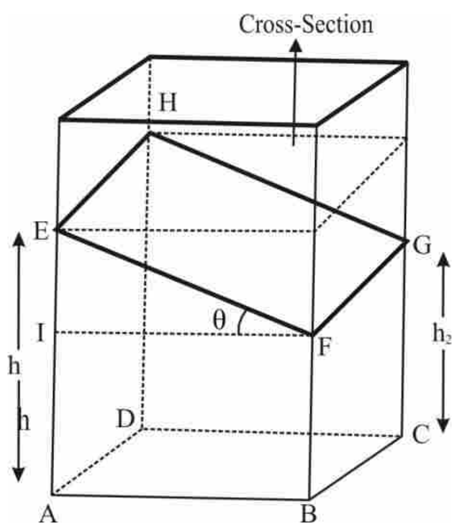


Fig. 5

$$\frac{FI}{EF} = \text{Cos } \theta$$

Or $$\frac{AB}{EF} = \text{Cos } \theta, \text{ as FI = AB}$$

Or $\dfrac{AB}{EF} \times \dfrac{BC}{FG} = \text{Cos}\ \theta$, as BC = FG

Or $\dfrac{\text{Area of the base}}{\text{Area of the Section EFGH}} = \text{Cos}\ \theta$

Or Area of the Section EFGH = $\dfrac{\text{Area of the base}}{\text{Cos}\ \theta}$

iii. Lateral surface area = Area of rectangle ADEH + BCFG + 2 (Area of Trapezium ABEF)

Or = Perimeter of the base x average height

iv. Total surface area = Area of the base + Area of the Section EFGH + Lateral surface area

Example 7:

The area of the cross-section of a prism is 50 sq. m. What is the area of the section making an angle of 60^o with the plane of the cross-section?

Solution:

Here $\theta = 60^o$, area of cross-section = 50 sq. m

By the formula

$$\text{Area of the section} = \frac{\text{Area of the base}}{\text{Cos}\ \theta}$$

$$= \frac{\text{Area of the cross-section}}{\text{Cos}\ \theta}$$

$$= \frac{50}{\text{Cos}\ 60^o} = \frac{50}{0.5} = 100 \text{ sq. m}$$

Example 8:

A hexagonal right prism whose base is inscribed in a circle of radius 2m, is cut by a plane inclined at an angle 45^o to the horizontal. Find the volume of the frustum and the area of the section when the height of the frustum are 8m and 6m respectively.

Solution:

$$\text{Area of base} = \frac{n\,R^2}{2}\sin\frac{360^o}{n}, \quad n = 6,\ R = 2$$

$$\text{Area of base} = \frac{6(2)^2}{2}\sin\frac{36^o}{6} = 12 \text{ Sin } 60^o = 12 \times .866$$

$$= 10.4 \text{ sq. m}$$

$$\text{Volume of the frustum} = \left(\frac{h_1 + h_2}{2}\right) \times (\text{area of base})$$

$$= \frac{8+6}{2} \text{ x } 10.4 = 73 \text{ cu. m}$$

Example 9:

The frustum of a prism has a transverse section that is a regular hexagon of side 35 cm. The average length of the edge is 1.2m. Find the volume and total surface area.

Solution: Given that, side of base of prism = 35 cm = 0.35m

$$\text{Area} = \frac{n\,a^2}{4}\cot\frac{180^o}{n} = \frac{6(.35)^2}{4}\cot\frac{180^o}{6}$$

$$= \frac{3}{2}(.1225)\cot 30^o$$

$$= 0.18375\ (1.7321) = 0.3183 \text{ sq. m.}$$

Average height = 1.2m

Volume of frustum = (average height)(area)
= 1.2(0.3183) cu. m
= 0.38 cu. m

Since there are six equal strips (rectangles) in lateral surface of the prism

Lateral surface = 6(area of strip)
= 6(.35x1.2) = 2.52 sq. m

Total surface area = L.S + 2 (area of base)
= 2.52 + (0.3183)2 = 3.1566 sq. m

Exercise 15

Q1. The length, width and height of a rectangular prism are 6, 4 and 3 meters respectively. Find the volume, the surface area and the length of the diagonal.

Q2. Find the cost of painting the outside of a rectangular box whose length is 64cm, breadth is 54cm and height is 51cm, at the rate of 37 paisa per sq. m.

Q3. A 10cm cube of cast iron is melted and cast into a hexagonal prism with a height of 12cm. Find the side of the base of prism.

Q4. The base of a right prism is a trapezium whose parallel sides are 17cm and 13cm, the distance between them being 18cm. If the height of the prism is 1m, find the volume.

Q5. What weight of water will fill a vessel in the form of a prism whose base is a regular hexagon of side 2ft. the height of the vessel being 6ft.

Q6. The sides of a triangular prism are 17, 25 and 28cm respectively. The volume of the prism is 4200 cu. cm what is its height?

Q7. Find the area of the whole surface of a right triangular prism whose height a 36m and the sides of whose base are 51, 37 and 20m.

Q8. The frustum of a prism has a transverse section that is a regular hexagon of side 30cm, the average length of the edge is 1.5m. Find the volume and total surface area.

Q9. How many cubic cm of wood are required to build a box without a lid (cover) if the wood is 1cm. thick and if the internal length, breadth and depth of the box are 30cm, 15cm and 25cm, respectively?

Q10. A pentagonal prism which has its base circumscribed about a circle of radius 1cm and which has height of 8cm is casted into cube, find the size of the cube.

Answers 15

Q1. 72 cu. m, 108 sq. m, 7.81 m **Q2.** Rs. 7010.76 **Q3.** 5.66m
Q4. 27000 cu. cm **Q5.** 3897 lbs; 62.35cu. ft **Q6.** 20cm
Q7. 4500sq. m **Q8** . 0.35 cu. m; 3.168 sq. m **Q9.** 2224 cu. m
Q10 3.07cm

Summary

Lateral surface area = Perimeter of the base x height of the prism
Total surface area = Lateral surface area + Area of the bases
Volume of prism = Length x breadth x height i.e. V = l b h
Also volume of prism = Area of the base x height of the prism
Total surface area = area of sin faces of rectangular prism
Length of diagonal of rectangular Prism = $\sqrt{a^2 + b^2 + c^2}$
Volume of cube = Area of base x height
Total surface area of cube = area of six faces = $6a^2$
Weight = Volume x density i.e. W = V . D

Volume of Hexagonal Prism = $\dfrac{n\,a^2}{4}\cot\dfrac{180^o}{n}\times h$

Volume of frustum $= \left(\dfrac{h_1 + h_2}{2}\right)$ area of base

Area of section $= \dfrac{\text{Area of the cross-section}}{\text{Cos}\,\theta}$

Short Questions

Q.1: Find the area of the whole surface of a right triangular prism whose height is 36 cm and sides of whose base 51,37 and 20 cm, respectively.

Q.2: The base of a right prism is an equilateral in triangle with a side of 4 cm and its heights is 25 cm. find its volume.

Q.3: Three cubes of metal whose edges are 3, 4 and 5 cm respectively, are melted without any loss of metal into a single-cube. Find (i) edge of the new cube (ii) surface area of the new cube.

Q.4: The inside measurement of a room are 8.5, 6.4 and 4.5 m height. How many men should sleep in the room, if each man is allowed 13.6 cu.m. of air?

Q.5: Find the volume of wood used in making an open wooden rectangular box 2cm thick , given that its internal dimension are 54 cm long, 46cm wide and 18 cm deep.

Q.6: A brick measures 18 cm by 9 cm by 6 cm. find the number of bricks that will be needed to built a wall 4.5 cm wide, 18 cm thick and 3.6 cm high.

Q.7: How many match box each 80 mm by 75 mm by 18mm, can be packed into a box 72 cm by 45 cm, 60 cm internally?

Q.8: Find the volume and weight of rectangular block of wood 7.5cm long, 8.7 cm wide and 12cm deep (1 cu.cm = 0.7 gm).

Q.9: The volume of the cube is 95 cu.cm. Find the surface area and the edge of the cube.

Q.10: Find the edge of a cube whose volume is equal to that of a rectangular bar measuring 126 cm long, 4 cm wide and 2 cm thick.

Q.11: The area of the cross-section of a prism is 50 sq. m. What is the area of the section making an angle of 60^o with the plane of the cross-section?

Q.12: A rectangle cuboid 9 cm long and 7 cm wide given that the volume of the cuboid is 315 cm^3. Find (a) the height of the cuboid (b) its total surface area.

Q.13: An open rectangular tank of length 16 cm, width 9 cm and height 13 cm contains water up to a height of 8 cm. calculate (a) Volume in liters (b) Total surface area of the tank.

Q.14: Find surface area of cube of volume 64 cm^3?

Q.15: The dimension of a marriage hall are 100 m, 50 m, and 18 m respectively find volume of hall.

Q.16: Define polygon prism.

Q.17: Write the formula which used to find volume, total surface area of rectangle prism.

Q.18: Write the formula which is used to find volume, lateral surface area and total surface are of polygon prism.

Answers

Q1. 4500 sq. m **Q2.** 173.2 cu.cm

Q3 (i)Edge = 6m (ii) Surface area = 216 sq m **Q4.** 18 men

Q5. 10488 cm^3 **Q6.** 3000 bricks **Q7.** 1800

Q8. (i)Volume = 783 cu.cm (ii) Weight= 548.1 gms

Q9. a = 4.56 cm ; 124.92 sq. cm **Q10.** 10.002 cm

Q11. 100 sq. m Q12. a) 5 cm (b) 286 cm^2

Q13. (a) 115 (b) 544 $cm^2 2\ cm^3$ Q14. 96 cm^2 Q15. 90, 000 cu.m

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

1. The cube is a right prism with
   a) square base b) rectangular base c) triangular base d) None
2. Volume of cube with side 2m is
   a) 4 b) 8 c) 16 d) 2
3. The length of the diagonal of cube if edge of cube 'a'
   a) $\sqrt{3}$ a b) a^3 c) 3a d) $3a^2$
4. Total surface area of the cube is
   a) a^2 b) $3a^2$ c) $6a^2$ d) $4a^2$
5. If volume of a cube is 27, then side of cube
   a) 3 b) 9 c) $\sqrt{3}$ d) $\sqrt{27}$
6. A prism with a polygon base is known as
   a) circular prism b) cubic prismc) polygonal prism d) None of these
7. Volume of hexagonal prism with height 'h' and side of the henagon 'a' is
   a) $\frac{3\sqrt{3}}{2}a^2h$ b) $\frac{1}{2}a^2h$ c) a^2h d) $3\sqrt{3}\ a^2h$
8. Volume of rectangular prism with dimensions are 6m, 4m and 2m respectively.
   a) 48 cu. m b) 12 cu. m c) 24 cu. m d) 36 cu. m
9. The portion of the prism between the plane section and its base is called
   a) ring b) annulus c) frustum d) None of these

Answers

Q.1

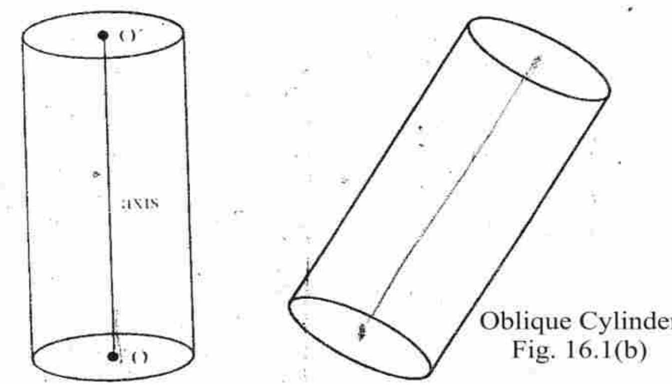
1. a	2. b	3. a	4. c	5. a
6. c	7. a	8. a	9. c	

Chapter 16
Mensuration of Cylinder

16.1 Cylinder:

A solid surface generated by a line moving parallel to a fixed line, while its end describes a closed figure in a plane is called a cylinder. A cylinder is the limiting case of prism.

If the generating line is perpendicular to the base, the cylinder is called as Right cylinder, otherwise oblique. The line joining the centres of the bases is called the axis of the cylinder.



Right Cyliner
Fig. 16.1(a)

Oblique Cylinder
Fig. 16.1(b)

16.2 Volume and Surface area of Different Kinds of Cylinders:

1. Right Circular Cylinder:

If the base of a right cylinder is a circle, it is called a right circular cylinder. In the right circular cylinder the axis is perpendicular to the base.

If r is the radius of the base, h is the height and d is the diameter of the base, then since cylinder is the limiting form of prism, therefore volume and surface area of the cylinder is also calculated by the same formula of prism.

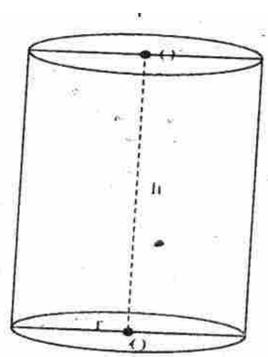


Fig. 16.2

(i) Volume of the cylinder = Area of the base x height

$= \pi r^2 h$

$= \frac{\pi d}{4} h$

(ii) Lateral surface area $=$ Perimeter of the base x height

$= 2\pi r h$

$= \pi d h$

(iii) Total surface area $=$ Lateral surface area + Area of bases

$= 2\pi r h + 2\pi r^2$

$= 2\pi r(h + r)$

Example 1:

The curved surface of a cylinder is 1000 sq. m. and the diameter of the base is 20m. Find the volume and height of the cylinder.

Solution:

Here d = 20m, r = 10m

Since Lateral surface $= 2\pi r h$

$1000 = 2\pi \times 10h$

$h = 15.9$ m

Volume of the cylinder $= \pi r^2 h$

$= \pi \times 100 \times 15.9$

$= 5000$ cu. m.

Example 2:

The radius of a right circular cylinder is 25 cm and its height is 15cm. Find its volume, lateral surface and the whole surface area.

Solution:

Here r = 25cm, h = 15cm

Volume $= \pi r^2 h = \pi \times 625 \times 15$

$= 29452.43$ cu. m.

Lateral surface area $= 2\pi r h$

$= \pi \times 40 \times 15 = 2356.19$ sq. sm.

Total surface area $=$ Lateral surface area + Area of bases

$= 2\pi r h + 2\pi r^2 h$

$= 2\pi r(r + h) = \pi \times 50(25 + 15)$

$= 6238.185$ sq. cm

2. Hollow Circular Cylinder:

The examples of hollow cylinders are pipes and bearing pushes, etc.

If R is the outer radius and r is the inner radius of the cylinder, then

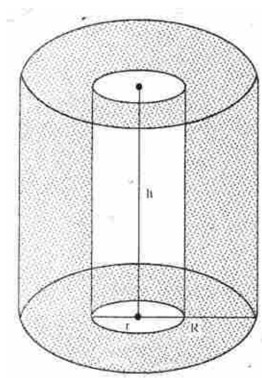


Fig. 16.3

(i) Volume (solid portion) = Volume of external cylinder – volume of internal cylinder

$= \pi R^2 h - \pi r^2 h$

$= \pi(R^2 - r^2)h$

(ii) Lateral surface area = External surface area + Internal surface area

$= 2\pi R h + 2\pi r h$

$= 2\pi(R + r)h$

(iii) Total surface area = Lateral area + Areas of solid bases

$= 2\pi(R + r)h + 2\pi(R^2 - r^2)$

Example 3:

Find the weight, lateral surface area and total surface area of iron pipe whose interior and exterior diameters measure 15cm and 17cm respectively, and length 10m; one cubic cm of iron weighting 0.8gm.

Solution:

Here d = 15cm r = 7.5cm

D = 17cm R = 8.5cm

h = 10cm = 1000cm

Volume = $\pi(R^2 - r^2)h$ $= \pi(72.25 - 64.75)1000$

= 2346.19 cu. cm.

Weight = Volume x density = 23561.9 x 0.8

= 18849.52gms

Length surface area $= 2\pi(R + r)h$

$= 2\pi(8.5 + 7.5)1000$

$= 2\pi$ x 16 x 1000

= 100530.96 sq. cm.

Total surface area

$= 2\pi(R + r)h + 2\pi(R^2 - r^2)$

= 100530.96 + 47.12

= 100578.08 sq. cm.

Example 4:

A well is to be dug 5m inside diameter and 36m in depth; find the quantity of earth to be excavated and the quantity of brick work required for a lining of 25 cm in thickness.

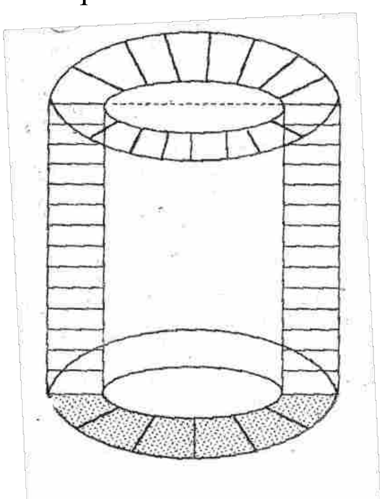


Fig. 16.4

Solution:

h = 36m

Inside diameter = $d = 5m$
$r = 2.5m$
Thickness of brick wok = 25cm
= .25m
Outside diameter = D = 5 + .50
= 5.5
R = 2.75m

Quantity of Earth to be excavated
= External volume of well
$= \pi R^2 h$
$= \pi \times 7.5625 \times 36$
= 855.3 cu. m.

Brick work = External volume – Internal volume
$= \pi R^2 h - \pi r^2 h$
= 855.3 – 706.9
= 148.44 cu. m

3. Elliptic Cylinder

If the bases of a cylinder are ellipses, it is called on "Elliptic cylinder."

If a and b are the semi-major and semi-minor axes and h is the height, then

(i) Volume = Area of the base x height
$= \pi a b h$

(ii) Lateral surface area = Perimeter of the base x height
$= \pi (a + b)h$

(iii) Total surface area = Lateral surface area + area of bases
$= \pi (a + h)h + 2\pi ab$

Example 5:

In making the pattern of a special container to hold 9 quarter. It is decided to have the bottom an ellipse with axes of 20 cm and 15 cm. Find the height of the container.

Solution:

Here 1 quarter = $\frac{1}{80}$ ton = 12.5kg = 28 Lbs

a = 20cm, b = 15cm

Volume = 9 quarters = 9 x 12.5 = 112.5 kg
= 112500 cu. cm.

Volume of the container $= \pi a b h$
$112500 = \pi \times 20 \times 15 \times h$
$h = 119.36$ cm

16.3 Frustum of a Right Circular Cylinder:

When a right circular cylinder is cut by a plane parallel to its base (or perpendicular to its axis) the section of the cylinder is called cross-section, which is a circle. If, however, the plane section is not parallel to the bases i.e., it is oblique, the portion of the cylinder between the plane section and the base is called Frustum of the right circular cylinder. This cutting section is an ellipse.

In the Fig.16.5 ACDE represents a frustum of the cylinder whose cutting plane AC is inclined an angle θ to the horizontal.

If r is the radius of the base and h_a is the average height of the Frustums, then

(i) Volume of the Frustum of circular cylinder

= Area of base x average height

$= \pi r^2 h_a$

(ii) Lateral surface area = Perimeter of the base x average height

$= 2\pi r h_a$

(iii) Total surface area = Area of the base + Area of the ellipse + Lateral surface area

For the ellipse, $AB = AC \cos\theta$

$$AB = \frac{AB}{\cos\theta} = \frac{2r}{\cos\theta}$$

So, the semi-mirror axis $= r$

and the semi-major axis $= \dfrac{AB}{2} = \dfrac{r}{\cos\theta}$

Hence area of the ellipse $= \pi a b = \dfrac{\pi r^2}{\cos\theta}$

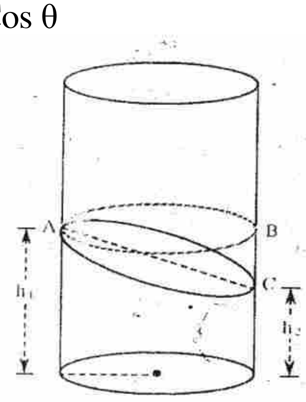


Fig. 16.5

Example 5:
A swimming pool is in the form of a cylinder of radius 10m. The depth of the water varies uniformly from 3m at one and end on 6m at the other end. Find how long it will take a pipe to fill it, if the diameter of the pipe is 10cm and the water in it runs at a uniform rate of 4m per second.

Solution: Radius of pool = 10m

$$h_a = \frac{3+6}{2} = 4.5\text{m}$$

Volume of the pool in the form of Frustum = $\pi r^2 h_a$

= π x 100 x 4.5 = 1413.72 cu. m.

Diameter of pipe = 10cm
Radius = 5cm = 0.05m
Length of pipe = 4m
Volume of pipe = $\pi r^2 x\ \ell = \pi$ x 0.0025 x 4 = 0.0314 cu. m

The quantity of water which the pipe can supply in one second.
= 0.0314 cu. m

So 0.0314 cu. m of pool fill in time = 1 sec

and 1413.72 cu. m of pool fill in time = $\dfrac{1413.72}{0.0314}$ = 45000.1 seconds
= 12.5 hours

Exercise 16

Q1. The diameter of a right circular cylinder is 38cm and its length is 28cm. Find its volume, lateral surface and total surface.

Q2. The whole surface of a right circular cylinder is 10 sq.m and the height is three times the radius of the base. Find the radius of the base.

Q3. A rectangular piece of iron sheet 1000 sq. cm in area is bent to form a cylinder 31.89cm in diameter. Find the height and volume of this cylinder.

Q4. Find the weight of 360m of lead pipe with inside diameter of 1.09cm and outer diameter of 3.09cm. Assume that lead has a density of 11.85 g/cu. cm.

Q5. Find the weight of iron in a pipe whose interior and exterior diameter measure 10cm and 11cm respectively and length 120cm (one cu. cm of iron weighing 0.26 lbs).

Q6. A right circular cylinder of radius 3 m and height 7m is forged into a hexagonal prism of side 1m. Find the height of the prism.

Q7. A cylindrical water drum has a base of radius 1.2m and its height is 3m. How many liters of water will it hold? (1 liter = 1000 cu. cm)

Q8. The rain that falls on the roof 25m kg 30m is conducted to a cylinder 10m in diameter. How great fall of rain would it take to fill the cylinder to a depth of 6m?

Q9. The volume of a cylinder is 252π cu.cm. and height 7 cm. Find its curved surface and total surface area.

Q10. A hollow shaft with 5 cm. internal diameter diameter is to have same cross- sectional area as a solid shaft of 12cm. diameter. Find the external diameter of shaft.

Q11. A boiler contain 400 tubes, each 5m long 10cm. external diameter. Find the heating surface of these tubes.

Q.12: 10 cylinderical pillars of a building have to be cleaned. If the diameter of each pillar is 50 cm and the height 4 cm, what will be the cost of cleaning these at the rate of 50 paisa per sq. m?

Q.13: A circular metal sheet 30 cm in diameter and 0.25 cm thick is melted and then recast into a cylindrical bar of diameter 5 cm. find the length of bar.

Answers 16

Q1.	31759 cu. cm, 3343 sq. cm, 5611 sq. cm			**Q2.**	0.63 m
Q3.	7958 cu. cm	**Q4.**	2798 **kg**	**Q5.**	514.8 lbs
Q6.	76.43m	**Q7.**	13571.68 liters	**Q8.**	0.628m
Q9.	264sq.cm; 490.29sq.cm.	**Q10.**	13cm.	**Q11**.	62.3 sq.m
Q12.	Rs. 31.40	**Q13.**	9 cm.		

Summary

1. Volume of cylinder = Area of base x height $= \pi r^2 h$
2. Curved surface area (lateral surface) = Perimeter x height
   i.e. L.S = $2\pi r h$
3. Total surface area = Lateral surface + area of two ends
   $= 2\pi r h + 2(\pi r^2) = 2\pi r(h + r)$
4. Volume of hollow cylinder = $\pi R^2 h - \pi r^2 h$
   i.e. V = $\pi h(R^2 - r^2)$
5. Lateral surface $= \pi (R + r)h$
6. Total surface $= 2\pi (R + r)h + 2\pi (R^2 - r^2)$
7. Weight = volume x density
   i.e. W = vd
8. Elliptic cylinder (A cylinder with a base which is an ellipse)
   (i) volume = V = $\pi ab h$
   (ii) S = Curved surface area (lateral surface) = $\pi (a + b)h$
   (iii) Total surface area = $\pi (a + b)h + 2\pi ab$

Short Questions

Q.1: Define cylinder.

Q.2: Find the cost of digging a well 3m in diameter and 24 m in depth at the rate of Rs.10 per cu.m.

Q.3: Define Hollow Circular Cylinder.

Q.4: Define Elliptic cylinder.

Q.5: The cylinder of an air compressor is required to have a working volume of 5 cu.m. if the radius is 5/6 m, what must be the stroke?

Q.6: In a hollow cylinder, the cylinder, the circles of cross-section are concentric. If the internal diameters of these circles be 2.2 cm and 3.8 cm respectively and the height be 6.5 cm, find the volume of hollow interior.

Q.7: Write the formula of volume of cylinder if radius is given.

Q.8: Write the formula of volume of hollow circular cylinder.

Q.9: Write the formula of total surface area of cylinder.

Q.10: Write formula of volume of elliptic cylinder and total area.

Q.11: The diameter of the base of a right circular cylinder is 14 cm and its height is 10 cm. find the volume and surface area of solid cylinder.

Q.12: If water flows through a 56 mm diameter pipe at a rate of 3 m/s, what volume of water, in liters, is discharge per minutes.

Q.13: Find the diameters of the cylinder given the following.

Volume 704 cm^3, height 14 cm

Q.14: Find height of cylinder if Volume 528 cm^3, diameter 4 cm

Answers

Q2.	RS. 1696.46	Q5.	2.29 m	Q6.	49 cu.cm.
Q7.	$V = \pi r^2 h$.	Q8.	$V = \pi (R^2 - r^2) h$	Q9.	$S = 2 \pi r h + 2 \pi r^2$
Q10.	$V = \pi a b h$, $S = \pi(a + b) h + 2 \pi a b$			**Q11.**	748 cm^2
Q12.	443.52 liters	Q13.	8 cm	Q14.	42 cm

Objective Type Exercise

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

___1. Volume of circular cylinder of height 'h' and radius r is

(a) πr^2h (2) $2\pi r h$
(3) $2\pi r h^2$ (4) $2\pi r^2h$

___2. Lateral surface area of right circular cylinder is

(a) πr^2 (b) $\pi r h$ (c) $2\pi r h$

___3. Volume of hollow cylinder if R and r are external and internal radii respectively is

(a) $\pi(R - r)$ (b) $2\pi(R^2 - r^2)$
(c) $\pi(R^2 - r^2)h$ (d) $\pi(R - r)h$

___4. Volume of an elliptic cylinder is

(a) πab (b) $\pi ab h$
(c) $\pi(a + b) h$ (d) $a b h$

___5. If 20 and 10 are the major and minor axis respectively, then volume of an elliptic cylinder of height 5m

(a) 1000π (b) 250π (c) 200π (d) 100π

___6. Volume of right circular cylinder of height 10cm and diameter is 4cm

(a) 40π (b) 160π (c) 80π (d) 4π

Answers

Q.1	(1)	a	(2)	c	(3)	c
	(4)	b	(5)	b	(6)	a

Chapter 17
Mensuration of Pyramid

17.1 Pyramid:

A pyramid is a solid whose base is a plane polygon and sides are triangles that meet in a common vertex. The triangular sides are called lateral faces. The common vertex is also called Apex.

A pyramid is named according to the shape of its base. If the base is a triangle, square, hexagon etc. the pyramid is called as a triangular pyramid, a square pyramid, a hexagonal pyramid etc. respectively.

Altitude (or height):

The altitude of a pyramid is the perpendicular distance from the vertex to the base.

Axis:

The axis of a pyramid is the distance from the vertex to the centre of the base.

17.2 Right or Regular Pyramid:

A pyramid whose base is a regular polygon and congruent isosceles triangles as lateral faces.

In a regular pyramid the axis is perpendicular to the base. Thus in a regular pyramid th axis and the altitude are identical.

Slant Height:

The slant height of a regular pyramid is the length of the median through the apex of any lateral face. In the Fig 17.1 OG is the slant height. It is denoted by ℓ.

Fig. 17.1

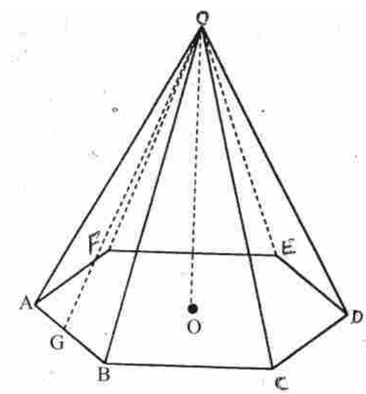


Fig. 17.1

Lateral edge:

It is the common side where the two, faces meet. In the OA is the lateral edge.

17.3 The surface area and Volume of a Regular Pyramid:

If a is the side of the polygon base, h is the height and ℓ is the slant height of a regular pyramid, then

(i) Lateral surface area = Sum of the triangular sides forming the pyramid, which are all equal in areas
= n (area of one triangle of base a and slant height ℓ)

$$= n\left(\frac{1}{2}a\,\ell\right)$$

$$= \frac{1}{2}(n\,a) \times \ell$$

$$= \frac{1}{2}\text{Perimeter of the base x slant height}$$

(ii) Total surface area = Lateral surface area + area of the base

(iii) Volume:

The volume of a pyramid may be easily derived from the volume of a cube. By joining the centre O of the cube with all vertices, six equal pyramids are formed. The base of each pyramid is one of the faces of the cube. Hence the volume of each pyramid is one-sixth of the volume of the cube. The height of each pyramid is

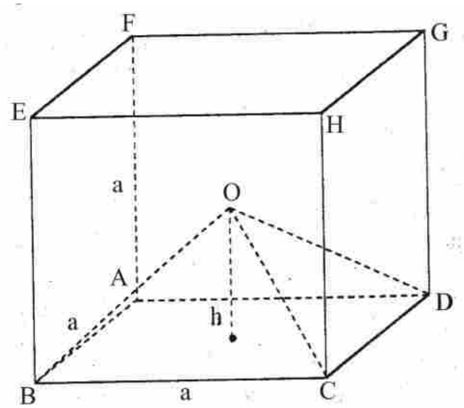


Fig. 17.2

$$h = \frac{a}{2}$$

Volume of each pyramid = $\frac{1}{6}$ the volume of the cube

$$= \frac{1}{6}a^3$$

$$= \frac{1}{3}a^2.\frac{a}{2}$$

$$= \frac{1}{3}a^2h$$

Volume of the pyramid = $\frac{1}{3}$ area of the base x height

Example 1:

Find the volume, the lateral surface area and the total surface area of the square pyramid of perpendicular height 9.41cm and the length of the side of base 2.92cm.

Solution:

Here h = 9.41cm, a = 2.92cm

(i) Volume $= \frac{1}{3}$ area of the base x height

$= \frac{1}{3}a^2h$

$= \frac{1}{3}8.526 \times 9.41$

$= 26.74$ cu. cm

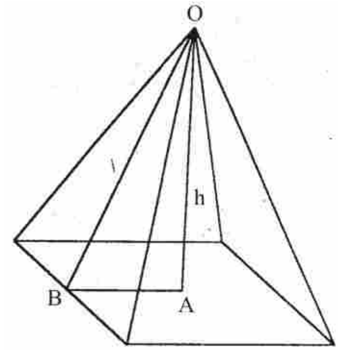


Fig. 17.3

(ii) For the lateral surface area we first calculate the slant height ℓ. In the right triangle OAB, OA = h = 9.41 cm, $AB = \frac{a}{2} = 1.45\,\text{cm}$

By Pythagorean theorem, $\ell^2 = h^2 + AB^2$

$= 85.55 + 2.13$

$\ell^2 = 90.68$

$\ell = 9.52$

Now lateral surface area $= \frac{1}{2}$ perimeter of the base x ℓ

$= \frac{1}{2}4a \times \ell$

$= 2 \times 2.92 \times 9.52$

$= 55.60$ sq. cm.

(iii) Total surface area= Lateral surface area + Area of the base

$= 55.60 + (2.92)^2$

$= 55.60 + 8.53$

$= 64.13$ sq. cm.

Example 2:

A right pyramid 10m high has a square base of which the diagonal is 10m. Find its slant surface.

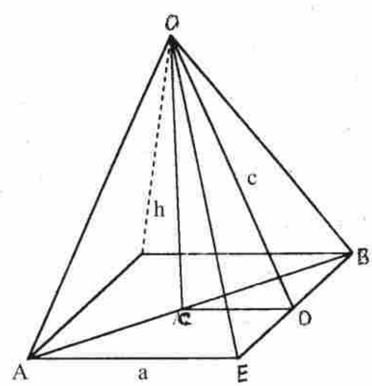


Fig. 17.4

Solution:

Here h = 10m

AB = d = 10m

$BC = \frac{d}{2} = 5\text{m}$

In the right triangle BCD,

$CD = BD$

So $BC^2 = CD^2 + BD^2$

$25 = 2\,CD^2$

Or $CD = \dfrac{5}{\sqrt{2}}$ m

Side of the base

$$a = BE = 2\,BD = 2 \cdot \frac{5}{\sqrt{2}}$$

$$= 5\sqrt{2}$$

Now, in the right triangle OCD,

$$OD^2 = OC^2 + CD^2$$

$$\ell^2 = 100 + \frac{25}{2} = 112.50$$

$$\ell = 10.6\text{m}$$

The slant surface area $= \dfrac{1}{2}$ perimeter of the base x ℓ

$$= \frac{1}{2} \times 4a \times \ell$$

$$= 2 \times 5\sqrt{2} \times 10.6$$

$$= 150 \text{ sq. m.}$$

Example 3:

The base of a right pyramid is a regular hexagon of side 4m and its slant surfaces are inclined to the horizontal at an angle of 30^o. Find the volume.

Solution:

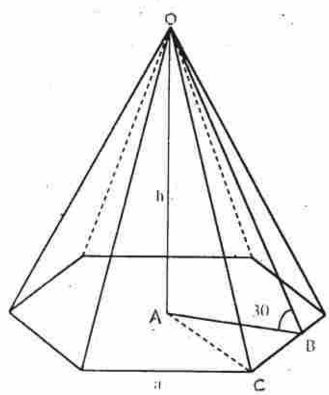


Fig. 17.5

Here, $a = 4$m

$\theta = \angle B = 30^o$

Area of the base $= \dfrac{n\,a^2}{4} \cot \dfrac{180}{n}$

$$= \frac{6 \text{ x } 4^2}{4} \cot 60^o$$

$$= 24\frac{1}{\sqrt{3}} = 13.86 \text{ sq. m}$$

In the right triangle ABC,

$$BC = \frac{a}{2} = 2\text{m}$$

Angle $C = 60^o$

So, $\tan 60^o = \frac{AB}{CB}$ OR $AB^2 = AC^2 - BC^2$

$AB = 2\sqrt{3}$ m

$AB = 2\sqrt{3}$

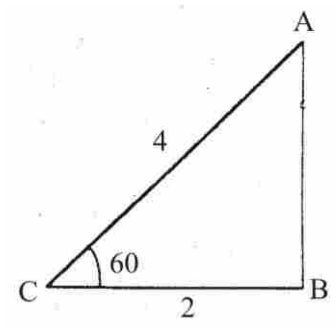


Fig. 17.6

Now, in the right triangle OAB

$\tan 30^o = \frac{h}{AB}$

$= 2\sqrt{3} \text{ x } \frac{1}{\sqrt{3}}$

$h = 2m$

Volume = $\frac{1}{3}$ area of the base x height

$= \frac{1}{3} \text{ x } 13.86 \text{ x } 2$

= 9.25 cu. m

Example 4:

The area of the base of a hexagonal pyramid is $54\sqrt{3}$ sq. m. and the area of one of its face is $9\sqrt{6}$ sq. m. Find the volume of the pyramid.

Solution:

Here, area of the base $= 54\sqrt{3}$ sq. m

Area of one side face $= 9\sqrt{6}$ sq. m

Volume $= \frac{1}{3}$ area of the base x h

$= \frac{1}{3}(54\sqrt{3}) \text{ x h}$

To find h, we have to find ℓ and AB.

Area of the hexagon $= \frac{n\,a^2}{4} \cot \frac{180}{n}$

$$54\sqrt{3} = \frac{6 \text{ x } a^2}{4} \cot 30^o$$

$$54\sqrt{3} = \frac{6}{4} \text{ x } \sqrt{3}\ a^2$$

$a^2 = 36$

$a = 6m$

Area of one triangle, say, OCD = $\frac{1}{2}$ a ℓ

$$9\sqrt{6} = \frac{1}{2} \text{ x } 6 \text{ x } \ell$$

$$\ell = 3\sqrt{6} \text{ m}$$

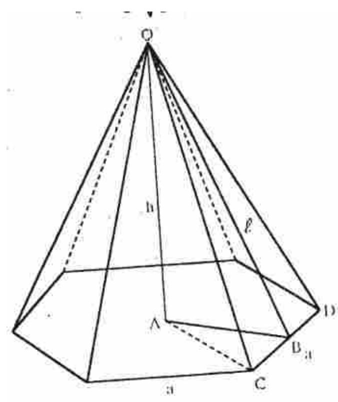


Fig. 17.7

In the right triangle ABC,

AC = 6m, BC = 3m

So $AB^2 = AC^2 - BC^2$

$= 36 - 9 = 27$

$AB = 3\sqrt{3}$ m

Now, in the right triangle OAB

$$OA^2 = OB^2 - AB^2$$

$$h^2 = \ell^2 - AB^2$$

$$= 54 - 27 = 27$$

$$h = 3\sqrt{3}$$

Therefore, volume = $\frac{1}{3}(54\sqrt{3}) \text{ x } 3\sqrt{3} = 162$ cu. m.

Exercise 17(A)

Q.1: Find the volume of a pyramid whose base is an equilateral triangle of side 3m and height 4m.

Q.2: Find the volume of a right pyramid whose base is a regular hexagon each side of which is 10m and height 50m.

Q.3: A regular hexagonal pyramid has the perimeter of its base 12cm and its altitude is 15cm. Find its volume.

Q.4; A pyramid with a base which is an equilateral triangle each side of which is 3m and has a volume of 120 cu. m find its height.

Q.5: A pyramid on a square base has every edge 100m long. Find the edge of a cube of equal volume.

Q.6: The faces of a pyramid on a square base are equilateral triangles. If each side of the base is 10m. Find the volume and the whole surface of the pyramid.

Q.7: Find the whole surface of a pyramid whose base is an equilateral triangle of side 3m and its slant height is 6m.

Q.8: The slant edge of a right regular hexagonal pyramid is 65 cm and the height is 56cm. Find the area of the base.

Q.9: Find the slant surface of a right pyramid whose height is 65m and whose base is a regular hexagon of side $48\sqrt{3}$ m.

Q.10: The sides of the base of a square pyramid are each 12.5cm and height of the pyramid is 8.5cm. Find its volume and lateral surface.

Q.11: Find volume of a square pyramid whose every edge is 100cm long.

Answers 17(A)

Q1.	$3\sqrt{3}$ cu.m	**Q2.**	4330.127 cu. m	**Q3.**	51.96 cu. m
Q4.	92.376 m	**Q5.**	61.77m	**Q6.**	235.70 cu. m; 273.2 sq. m
Q7.	30.897 sq. m	**Q8.**	3772.296 sq. cm	**Q9.**	$13968\sqrt{3}$ sq. m
Q10.	1328.125 cu. cm, 263.76 sq. cm			**Q11.**	235702.26 cu.cm

17.4 Frustum of a Pyramid:

When a pyramid is cut through by a plane parallel to its base, the portion of the pyramid between the cutting plane and the base is called a frustum of the pyramid. Each of the side face of the frustum of the pyramid is a trapezium.

Slant height:

The distance between the mid points of the sides of base and top. It is denoted by ℓ.

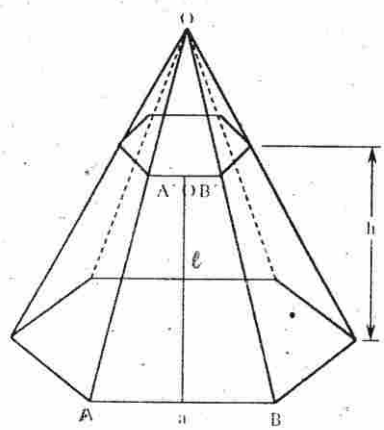


Fig. 17.8

17.5 Volume and surface area of Frustum of a Regular Pyramid:

Let, A_1 by the area of the base, and A_2 be the area of the top, a is the side of the base and b is the side of the top, ℓ is the slant height and h is the height of the frustum of a pyramid, then

(i) Volume of the frustum of a pyramid

$$= \frac{h}{3}(A_1 + A_2 + \sqrt{A_1A_2})$$

(ii) Lateral surface area = Sum of the areas of all the trapezium faces, which are equals

= n (area of one trapezium, say, ABA'B')

$$= n\left(\frac{a+b}{2} x\ell\right) \quad = \frac{1}{2}(na + nb) \text{ x } \ell$$

$$= \frac{1}{2}\text{sum of the perimeters of the base and top x slant height}$$

(iii) Total surface area= Lateral surface area + area of the base and the top

Example 5 :

A frustum of a pyramid has rectangular ends, the sides of the base being 20m and 32m. If the area of the top face is 700 sq. m. and the height of the frustum is 50m; find its volume.

Solution:

Here $A_1 = 20 \text{ x } 32 = 640$ sq. m , $A_2 = 700$ sq. m

$$\text{Volume} = \frac{h}{3}[A_1 + A_2 + \sqrt{A_1A_2}]$$

$$= \frac{50}{3}[640 + 700 + \sqrt{640 \text{ x } 700}]$$

$$= \frac{50}{3}[1340 + 669.33] \qquad = \frac{50}{3}(2009.33)$$

$$= 33488.80 \text{ cu. m.}$$

Example 6 :

A square pyramid 12m high is cut 8m from the vertex to form a frustum of a pyramid with a volume of 190 cu. m. Find the side of the base of the frustum of a pyramid.

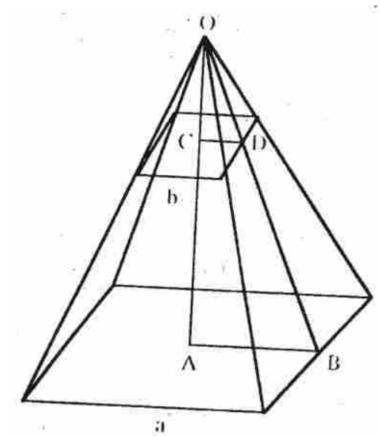


Fig. 17.9

Solution:

Here, volume of frustum of a pyramid = 190 cu. m

Height of pyramid = h = 12m

Height of the frustum of a pyramid = h_1 = 4m

OC = 8m

If 'a' and 'b' are the sides of base and top, then since the right triangles OAB and OCD' are similar.

So $$\frac{AB}{CD} = \frac{OA}{OC}$$

$$\frac{\frac{1}{2}a}{\frac{1}{2}b} = \frac{12}{8}$$

OR $$\frac{a}{b} = \frac{3}{2}$$

OR $$b = \frac{2a}{3}$$

Now, Volume of the frustum of a pyramid

$$= \frac{h_1}{3}[A_1 + A_2 + \sqrt{A_1A_2}]$$

$$190 = \frac{4}{3}[a^2 + b^2 + ab]$$

$$190 = \frac{4}{3}[a^2 + \frac{4a^2}{9} + \frac{2a^2}{3}]$$

$$190 = \frac{4}{3}[\frac{19a^2}{9}]$$

$$a^2 = \frac{190 \text{ x } 27}{76} = 67.5$$

$$a = 8.22\text{m}$$

Exercise 17(B)

Q.1 Find the volume and the total surface area of a frustum of a pyramid; the end being square of sides 8.6m and 4.8m respectively and the thickness of the frustum of a pyramid is 5m.

Q.2 Find the lateral surface area and volume of frustum of a square pyramid. The sides of the base and top are 6m and 4m respectively and the slant height is 8m.

Q.3 Find the net area of material required to make half dozen lamp shades each shaded as a hollow frustum of a square pyramid, having top and bottom sides of 10cm and 18cm respectively, and vertical height 16m.

Q.4 The sides of the top and bottom ends of a frustum of a square pyramid are 6m and 15m respectively. Its height being 30m. it is capped at the top by a square pyramid 12m from the base to the apex. Find the number of cu. m in the frustum of a square pyramid and in the cape.

Q.5 Find the cost of canvas, at the rate of Rs. 5 per square meter, required to make a tent in the form of a frustum of a square pyramid. The sides of the base and top are 6m and 4m respectively and the height is 8m, taking no account of waste.

Q.6 A square pyramid 15cm height and side of the base 12cm is cut by a plane parallel to the base and 9cm from the base. Find the ratio of the values of the two parts thus formed.

Q.7 What is the lateral area of a regular pyramid whose base is a square 12cm. on a side and whose slant height is 10cm? If a plane is passed parallel to the base and 4cm. from the vertex, what is the lateral area of the frustum?

Answers 17(B)

Q1. 229.63 cu. m ; 239.88 sq. m **Q2.** 201.15 cu. m ; 160.00 sq. m
Q3. 5541.45 sq. cm
Q4. Volume of frustum = V_1 = 2106 cu. m.
Volume of cape = V_2 = 144 cu. m
Number of cubic m = $V_1 + V_2$ = 2250 cu. m.
Q5. Rs. 806.23 **Q6.** 1 : 14.6 **Q7.** 240 sq. cm ; 180 sq. cm.

Summary

1. Lateral surface area of regular pyramid
$= \frac{1}{2}$(perimeter of the base) x slant height
2. Total surface area of regular pyramid
= Lateral surface area + area of the base
3. Volume of pyramid $= \frac{1}{3}$(area of the base) x height
4. Volume of the frustum of a pyramid $= \frac{h}{3}[A_1 + A_2 + \sqrt{A_1A_2}]$
5. Lateral surface area of frustum of a pyramid
$= \frac{1}{2}$(sum of the perimeters of base and top) x slant height
i.e. $\frac{1}{2}(P_1 + P_2) \times \ell$
6. Total surface area of frustum of a pyramid
= lateral surface + area of the base and top

Short Questions

Write the short answers of the following.

Q.1: Define pyramids.

Q.2: Find the volume of a pyramid whose base is an equilateral triangle of side 1m and whose height is 4 m.

Q.3: Find the whole surface of a pyramid whose base is an equilateral triangle of side 3 m and its slant height is 6m.

Q.4: Find the volume of a pyramid with a square base of side 10 cm. and height 15 cm.

Q.5: Find the volume of a pentagonal based pyramid whose area of base is 15 sq.cm and height is 15 cm.

Q.6: A square pyramid has a volume of 60 cu.cm and the side of the base is 6 cm. Find height of the pyramid.

Q.7: Find the volume of a square pyramid if the side of the base is 3cm. and perpendicular height is 10 cm.

Q.8: The height of pyramid with square base is 12 cm. and its volume is 100 cu.cm. Find length of side of square base

Answers

Q2.	0.58 cu.cm	Q3.	30.897 sq.m	Q4.	500 cu.cm.
Q5.	75 cu.cm	Q6.	5 cm.	Q7.	30 cu.cm.
Q8.	5cm				

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

___1. A solid figure whose base is a plane polygon and sides an triangles that meet in a common vertex is known as
(a) pyramid (b) cube
(c) frustum of a pyramid (d) None of these

___2. If the base of pyramid is hexagon, the pyramid is called
(a) triangular pyramid (b) square pyramid
(c) hexagonal pyramid (d) pentagonal pyramid

___3. If the base of pyramid is square, the pyramid is called
(a) square pyramid (b) hexagonal pyramid
(c) triangular pyramid (d) Rectangular pyramid

___4. If area of base of pyramid is 'A' and height 'h' then volume of pyramid
(a) $\frac{1}{3}Ah$ (b) $\frac{1}{2}Ah$ (c) $\frac{1}{6}Ah$ (d) Ah

___5. Volume of a pyramid whose area of base $6a^2$ and height 'h' is
(a) $\frac{1}{3}a^2h$ (b) $2a^2h$ (c) $3a^2h$ (d) a^2h

___6. Lateral surface area of regular pyramid if perimeter of base is P and slant height 'ℓ' is
(a) $P\ell$ (b) $\frac{1}{3}P\ell$ (c) $\frac{1}{2}P\ell$ (d) $\frac{1}{6}P\ell$

___7. The length of median through vertex (apex) of any lateral surface of a regular pyramid is
(a) length of diagonal (b) slant height
(c) height (d) axis

___8. Volume of frustum of pyramid is
(a) $\frac{h}{3}[A_1 + A_2 + \sqrt{A_1A_2}]$ (b) $\frac{h}{2}[A_1 + A_2 + \sqrt{A_1A_2}]$
(c) $\frac{h}{3}[A_1 + A_2 + \sqrt{A_1A_2}]$ (d) $h[A_1 + A_2 + \sqrt{A_1A_2}]$

___9. Each of the side face of frustum of the pyramid is a
(a) triangle (b) rectangle
(c) trapezium (d) square

___10. If P_1 and P_2 are perimeters of base and top of frustum of pyramid respectively then lateral surface area is

(a) $\frac{1}{2}P_1P_2\ell$ (b) $\frac{1}{2}(P_1+P_2)\ell$

(c) $\frac{1}{2}(P_1+P_2)h$ (d) $\frac{1}{3}(P_1+P_2)\ell$

Answers

Q.1	1.	a	2.	c	3.	a	4.	a
	5.	b	6.	c	7.	b	8.	a
	9.	c	10.	b				

Chapter 18
Mensuration of Cone

18.1 Cone:

A cone is a solid generated by a line, one end of which is fixed and the other end describes a closed curve in a plane. The fixed point is called the vertex or apex.

Axis:

The axis of the cone is the distance from the apex to the centre of the base.

Altitude:

It is the perpendicular distance from the apex to the base of the cone. It is also called height of the cone.

Circular Cone:

A cone whose base is a circle and whose lateral surface tapers uniformly to the apex, is called a circular cone.

Right Circular Cone:

It is a cone whose base is a circle and whose axis is perpendicular to the base.

A right circular cone can also be described when a right triangle is rotated about one of its side containing the right triangle.

Slant Height:

In a right cone it is the distance from the vertex to the circumference of the base. It is denoted by ℓ.

Note: A cone may also be defined as, when the number of sides of the base of a pyramid is increased indefinitely and the magnitude of each side diminished the surface of the pyramid tends to become the surface of a cone. Hence a cone is a limiting case of a pyramid.

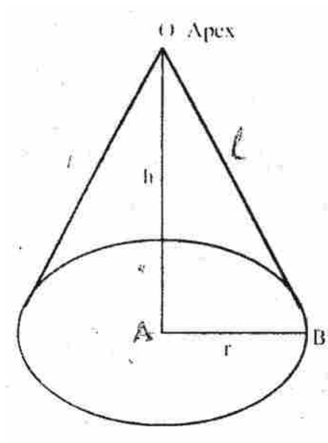


Fig. 18.1

18.2 Volume and Surface area of a Right Circular Cone:

If r is the radius of the base, h is the height and ℓ is the slant height, then since the cone is the limiting case of a pyramid, therefore volume and surface area of the cone is calculated by the same formula of pyramid. i.e.,

(i) Volume of the cone $= \frac{1}{3}$ area of the base x height

$$= \frac{1}{3}\pi r^2 h$$

(ii) Lateral surface of a Cone:

It is the curved surface area. If a hollow cone is made a cut along a straight line from the vertex to the circumference of the base, the cone is opened out and a sector of a circle with radius ℓ is produced. Since, the circumference of the base of the cone is $2\pi r$, therefore the arc length of the sector of the circle is $2\pi r$. Thus,

Lateral surface area of a cone = area of the Sector OAB

$= \frac{1}{2}$radius x arc length

$= \frac{1}{2}\ell \times 2\pi r$

$= \pi r \ell$

Where

ℓ = slant height

$\ell = \sqrt{r^2 + h^2}$

Or Lateral surface =

$\frac{1}{2}$Perimeter of the base x slant height

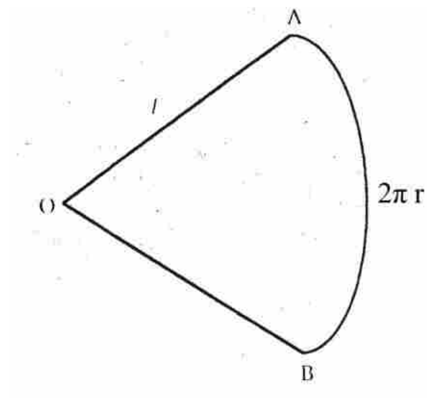


Fig. 18.2

$= \frac{1}{2} 2\pi r \times \ell$

$= \pi r \ell$

(ii) Total surface area = Lateral surface area + area of the base

$= \pi r \ell + \pi r^2$

$= \pi r(\ell + r)$

Example 1:

Find the volume and the total surface area of a cone of radius 6.6cm and height of 12.5cm.

Solution:

Here 4 = 6.6cm, h = 12.5cm

Volume $= \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi(6.6)^2 \times 12.5$

= 570.199 cu. cm

Since $\ell^2 = h^2 + r^2$

$= 12.5^2 + 6.6^2$

$= 156.25 + 43.56$

$\ell^2 = 199.81$

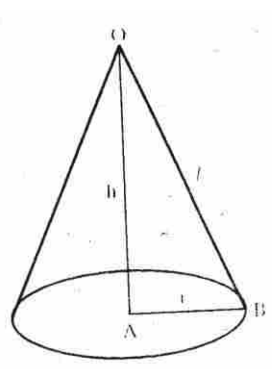


Fig. 18.3

$\ell = 14.14\text{cm}$

Total surface area $= \pi r(\ell + r)$

Total surface area $= \pi\, 6.6(14.14 + 6.60)$

$= \pi\, 6.6(20.74)$

$= 430.03$ sq. cm.

Example 2:

A pyramid on a regular hexagonal base is trimmed just enough to reduce it to a cone. Show that $\frac{1}{10}$ of the original volume of the pyramid is removed.

Solution:

Volume of the pyramid = $\frac{1}{3}$ area of the base x height

$$V_p = \frac{1}{3} n\, r^2 \tan\frac{180}{n} \times h$$

$$= \frac{1}{3} 6 \tan 30^o\, r^2 h$$

$$V_p = 2\frac{1}{\sqrt{3}} r^2 h$$

$$r^2 = \frac{\sqrt{3}}{2\,h} V_p$$

Volume of the cone =

$$V_c = \frac{1}{3} \text{area of the base x height}$$

$$= \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi \frac{\sqrt{3}}{2\,h} h V_p$$

$$= \frac{\sqrt{3}\,\pi}{6} V_p$$

$$V_c = 0.9\, V_p$$

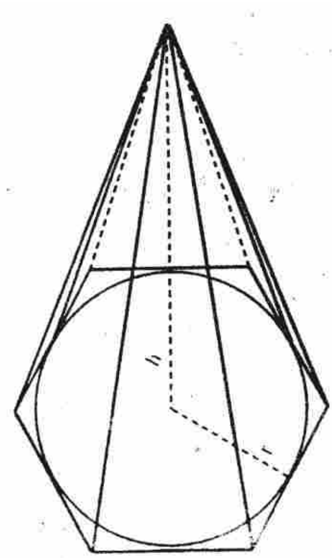


Fig. 18.4

Volume removed by trimming $= V_p - V_c$

$= V_p - 0.9V_p$

$= (1 - 0.9)V_p$

$= 0.1V_p$

$= \frac{1}{10} V_p$

$$= \frac{1}{10} \text{ volume of the pyramid}$$

Example 3:

Find what length of canvas $\frac{3}{4}$ m. wide is required to make a conical tent 8m in diameter and 3m high.

Solution:

Here, d = 8m, r = 4m, h = 3m

Lateral surface of the conical tent = $\pi r \ell$

$= \pi r\sqrt{r^2 + h^2}$ where $\ell = \sqrt{r^2 + h^2}$

$= \pi \text{ x } 4\sqrt{16+9}$

$= 20\pi$

$= 62.83$ sq. m

Now, area of the tent = 62.83

Width $= \frac{3}{4}$ m

Length $= \ell = ?$

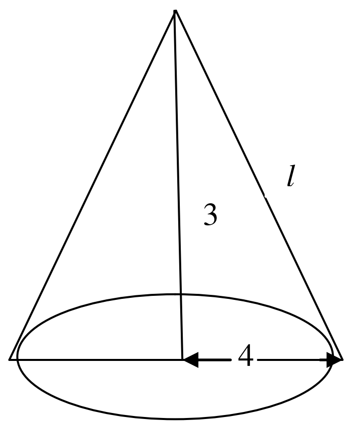


Fig.18.5

Width x length = area

$\frac{3}{4}\ell = 62.83$

$\ell = \frac{62.83 \text{ x } 4}{3}$

$\ell = 83.78$ m.

Exercise 18(A)

Q.1 The radius of the base of a right circular cone is 6m and the slant height is 6.5m, find the volume and the lateral surface area.

Q.2 The generating line of a right circular cone is inclined at an angle of 60^o to the horizontal. If the height of the cone be 15cm find its lateral surface and volume.

Q.3 The slant height of a cone is 25cm, and the area of its curved surface is 550 sq. cm. find its volume.

Q.4 Find the cost of the canvas for 50 conical tents, the height of each being 45cm and the diameter of the base 36cm at the rate of Rs. 9.40 sq. m.

Q.5 Find the cost of canvas 3/5m wide at the rate of Rs. 5 per meter, required to make a conical tent, 5m diameter and 2m high, taking no account of waste.

Q.6 Find the cost of painting at the rate of Rs. 6.25 per sq. cm. a conical spire 60cm in circumference at the base and 105cm in slant height.

Q.7 Find the curve surface area, the whole surface area, the volume and the vertical angle of a cone whose radius of the base is 45cm and height is 48cm.

Q.8 A right angled triangle of which the sides are 3cm and 4cm in length is made to turn round the longer side. Find the volume of the cone thus formed.

Q.9 A right angled triangle of which the sides are 65, 60 and 25cm in length is made to turn round its hypotenuse. Find the volume of the double cone thus formed.

Q.10 The area of the whole surface of a cone is 64 sq. m. and the slant height is 5 times the radius of the base. Find the radius of the base.

Q.11 Find the volume of the largest right circular cone that can be cut out of a cube whose edge is 2.8m.

Q.12 A cylindrical column 4m wide and 6m high is surmounted (cover) by a cone of the same width and 3m height. Find the area of the sheet metal required to cover its lateral surface.

Q.13 The vertical height of a conical tent is 42 dm and diameter of its base is 5.4m. Find the number of persons it can accommodate if each person is to allowed 2916 cu. dm of space.

Answers

Q.1 94.286 cu.m., 122.52 sq. m.

Q.2 471.2 sq. cm., 1178.03 cu. cm

Q.3 1232 cu. cm. Q.4 Rs. 128.8

Q.5 Rs. 209.54 Q.6 Rs. 19687.50

Q.7	9301.55 sq. cm ,15663.27 sq. cm, 37.699 cu. cm, $86^{o}\ 18'$		
Q.8	37.699 cu. cm,	Q.9	36283.97 cu. cm
Q.10	1.84m	Q.11	5.747 cu. m
Q.12	98.05 sq. m	Q.13	11

18.3 Frustum of Cone:

When a cone is cut through by a plane parallel to its base, the portion of the cone between the cutting plane and the base called a frustum of the cone. The frustum of a cone is the limiting form of the frustum of a pyramid.

18.4 Volume and Surface area of the Frustum of a Cone:

Let R and r are radii of the base and top respectively, h is the thickness or height and ℓ is the slant height of the frustum of a right cone. Since the frustum of a cone is the limiting form of the frustum of a pyramid. Therefore volume and surface area of the frustum of a cone is calculated by the same formula of the frustum of a pyramid.

(i) Volume of the frustum of a cone = $\frac{h}{3}(A_1+A_2+\sqrt{A_1A_2})$

$$= \frac{h}{3}(\pi R^2 + \pi r^2 + \sqrt{(\pi R^2)(\pi r^2)})$$

$$= \frac{h}{3}(\pi R^2 + \pi r^2 + \pi\, R\, r)$$

$$= \frac{\pi\, h}{3}(R^2 + r^2 + R\, r)$$

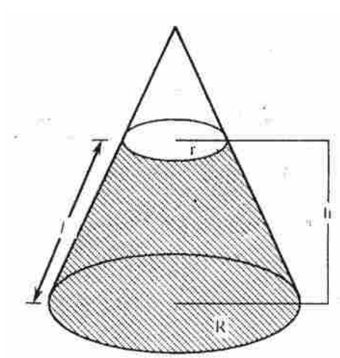


Fig. 18.5

(ii) Lateral surface area
= curved surface area

$$= \frac{1}{2}(\text{sum of the perimeters of base and top}) \times \text{slant height}$$

$$= \frac{1}{2}(2\pi R + 2\pi\, r)\ell$$

$$= \pi(R + r)\ell$$

(iii) Total surface area = Lateral surface area
+ area of the base and top

$$= \pi(R + r)\ell + \pi\, R^2 + \pi\, r^2$$

Note: If r, R and h are given, then $\ell^2 = h^2 + (R - r)^2$

Example 5:

Find the curved and total surface area and the volume of the frustum of a cone whose top and bottom diameters are 6m and 10m and the height is 12m.

Solution:

Here $d = 6m$ $r = 3m$

$D = 10m$ $R = 5m$

$h = 12m$

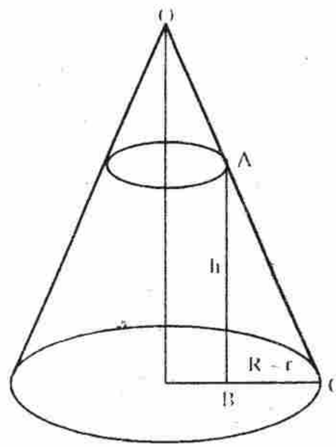


Fig. 18.6

$$\text{Volume} = \frac{\pi h}{3}(R^2 + r^2 + R r)$$

$$= \frac{\pi \text{ x } 12}{3}(25 + 9 + 15)$$

$$= \pi \, 4(49)$$

$$= 615.75 \text{ cu. cm}$$

$$\text{Lateral surface} = \pi \, \ell(R + r)$$

$$= \pi \, 12.166(8)$$

$$= 305.75 \text{ sq. m.}$$

In the right triangle ABC

$$\ell^2 = h^2 + (R - r)^2$$

$$= 144 + 4$$

$$\ell^2 = 148$$

$$\ell = 12.166m$$

Total surface area = Lateral surface area + area of the base and top

$$= 305.75 + \pi R^2 + \pi r^2$$

$$= 305.75 + 75.54 + 28.27$$

$$= 412.56 \text{ sq. m}$$

Example 6:

A material handling bucket is in the shape of the frustum of a right circular cone as shown in Fig. 7. Find the volume and total surface area of the bucket.

Solution:

Here $D = 15cm \Rightarrow R = 7.5cm$

$d = 10cm \Rightarrow r = 5cm$

$h = 17cm$

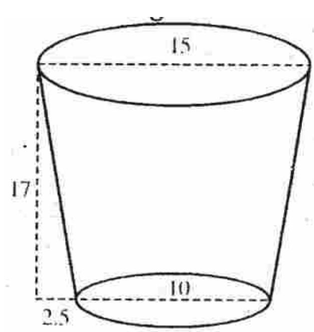


Fig. 18.7

$$\ell = \sqrt{17^2 + 2.5^2} = \sqrt{289 + 6.25}$$

$$= 17.18$$

$$\text{Now, Volume} = \frac{\pi h}{3}(R^2 + r^2 + R r)$$

$$= \pi \frac{17}{3}(56.25 + 37.5)$$

$$= \pi\frac{17}{3} \text{ x } 118.5 = 2114.03 \text{ cu. cm}$$

Lateral surface $= \pi \ell(R + r)$

$= \pi\ 17.18\ (12.5)$

$= 674.657$ sq. cm.

Total surface area $= 674.657$ + area of base and top

$= 674.657 + \pi R^2 + \pi r^2$

$= 674.657 + \pi\ (56.25 + 25)$

$= 674.657 + 255.25$

$= 929.9$ sq. cm.

Exercise 18(B)

Q.1 Find the curved surface area and volume of Frustum of a cone. The top and bottom diameters are 6m and 10m and the slant height 12.2m.

Q.2 A loud speaker diaphragm is in the form of a Frustum of a cone. If the base diameters are 40cm and 10cm and the vertical height is 35cm, find the curved surface area needed to cover the speaker in square meters.

Q.3 A cone 15cm high and of base diameter 12cm is cut by a plane parallel to the base and 9cm. from the base. Find the ratio of the volumes of the two parts thus formed.

Q.4 A cone 12cm high is cut 8cm from the vertex to form a frustum with a volume of 190 cu. cm. Find the radius of the base of cone.

Q.5 Find the cost of canvas at the rate of Rs. 5 per square meter required to make a tent in the form of a frustum of a cone. The diameters of the base and top are 6m and 4m respectively and the height is 8m, taking no account of waste.

Q.6 The diameters of the top and bottom of a frustum of a cone are 6m and 15m respectively. Its height being 30m. It is capped at the top by a cone 12m from the base to the apex. Find the number of cu. m in the frustum of a cone and in the cape.

Q.7 A right circular cone of altitude 16 inch is cut by a plane parallel to the base and 7 inch from the vertex. If the areas of the bases of the frustum thus formed are 49 sq. in and 81 sq. in., what is the volume of the frustum?

Q.8 A bucket is 11 inch in diameter at the bottom and 13 inch on top. If the slant height is 12 in., (a) What is the number of gallons of water it will hold (1 gallon 231 cu. in.); (b) How many square feet of material are used to make this bucket.

Answers 18(B)

Q.1	305.75 sq. m, 617.497 cu. m		
Q.2	0.3589 sq. m	Q.3	1:14.625
Q.4	6.64cm	Q.5	Rs. 633.21
Q.6	1767.15 cu. m.		
Q.7	579 cu. in		
Q.8	(a) 1.87 gallons	(b)	3.8 sq. ft.

Summary

1. Volume of cone = $\frac{1}{3}$(area of base) x altitude = $\frac{1}{3}\pi r^2 h$
2. Lateral surface of cone = $\pi r \ell$, where ℓ = slant height
   $\ell = \sqrt{r^2 + h^2}$
3. Total surface area of cone = Lateral surface area + area of the base
   $= \pi r \ell + \pi r^2$
   $= \pi r (\ell + r)$
4. Volume of the frustum of a cone = $\frac{h}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)$
   Where A_1 = area of base, A_2 = Area of top
   Or Volume of the frustum of a cone = $\frac{\pi h}{3}(R^2 + r^2 + Rr)$
5. Lateral surface area = curved surface area of frustum of cone
   $= \frac{1}{2}$(sum of the perimeters of base and top) slant height
   $= \frac{1}{2}(2\pi R + 2\pi r)\ell$
   $= \pi (R + r)\ell$
6. Total surface area of frustum of a cone = Lateral surface area + area of the base and top
   $= \pi(R+r)\ell + \pi R^2 + \pi r^2$

Short Questions

Write the short answers of the following.

Q.1 Define cones.

Q.2 Write the formula for Volume of a cone and total surface area of a cone.

Q.3 The circumference of base of a 9m high conical tent is 44 cm. find the volume of the air contained in it.

Q.4 The vertical height of a conical tent is 42 dm and the diameter of its base is 5.4 m. Find the number of persons it can accommodate if each person is to be allowed 2916 cu.dm. of space.

Q.5 A right triangle of which the sides are 3 cm and 4 cm length is made to turn around the longer side. Find the volume of the cone thus formed.

Q.6 Find the cost of painting @ Rs.7.5 per sq. cm a conical spire 64

Q.7 Find the volume of the largest cone that can be cut out ofa cube whose edge is 3 cm.

Q.8 Write formula of total curved surface area of cone and slant height of cone.

Answers

Q3.	462 cu.m.	Q4.	11 persons	Q5.	37.699 cu. cm
Q6.	Rs.25930	Q7.	7.069 cu. cm		

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

___1. Volume of a cone of height 'h' and base radius 'r' is

(a) $\frac{1}{3}\pi r^2 h$ (b) $\frac{1}{3}\pi r h$

(c) $\pi r^2 h$ (d) $\frac{1}{2}\pi r^2 h$

___2. Volume of cone of radius of base 3cm and heights 12cm is

(a) 108π cu. cm (b) 36π cu. cm

(c) 12π cu.cm. (d) 54 cu. cm

___3. The diameter of base of cone is 6cm and the height is 4 cm then slant height '' is

(a) 5cm (b) $\sqrt{52}$ cm

(c) $\sqrt{7}$ cm (d) $\sqrt{5}$ cm

___4. The curved surface of a right circular cone of radius 3cm and slant height is 6cm

(a) 54π (b) 9π (c) 18π (d) 6cm

___5. The line joining the vertex of a cone to the boundary line of the base is called

(a) height (b) slant height (c) length (d) perimeter

___6. Curved surface of a right cone of height 'R' and base radius 'r' is

(a) $\pi\sqrt{h^2+r^2}$ (b) $\frac{\pi r}{3}$ (c) $\frac{\pi}{3}\sqrt{h+r}$ (d) $\pi r\sqrt{h^2+r^2}$

___7. Volume of frustum of a cone with R and r as its end radii is

(a) $\frac{\pi h}{3}(R^2 + r^2 + Rr)$ (b) $\frac{\pi h}{3}(R^2+r^2+\pi r)$

(c) $\pi h(R^2+r^2+Rr)$ (d) $\frac{\pi h}{3}(R + r + Rr)$

Answers

Q.1	1.	a	2.	b	3.	a	4.	c
	5.	b	6.	d	7.	a		

Chapter 19
Mensuration of Sphere

19.1 Sphere:

A sphere is a solid bounded by a closed surface every point of which is equidistant from a fixed point called the centre. Most familiar examples of a sphere are baseball, tennis ball, bowling, and so forth.

Terms such as radius, diameter, chord, and so forth, as applied to the sphere are defined in the same sense as for the circle.

Thus, a radius of a sphere is a straight line segment connecting its centre with any point on the sphere. Obviously, all radii of the same sphere are equal.

Diameter of the sphere is a straight line drawn from the surface and after passing through the centre ending at the surface.

The sphere may also be considered as generated by the complete rotation of a semicircle about a diameter.

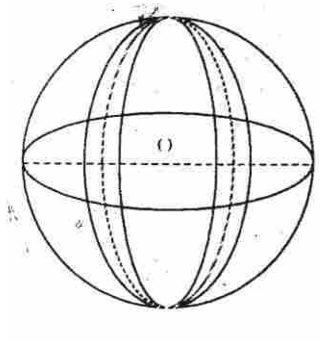


Fig. 19.1

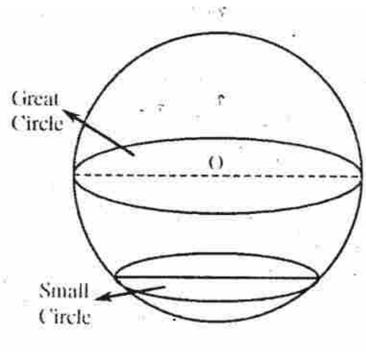


Fig. 19.2

Great and Small Circles:

Every section made by a plane passed through a sphere is a circle. If the plane passes through the centre of a sphere, the plane section is a great circle; otherwise, the section is a small circle (Fig. 2). Clearly any plane through the centre of the sphere contains a diameter. Hence all great circles of a sphere are equals have for their common centre, the centre of the sphere and have for their radius, the radius of the sphere.

Hemi-Sphere:

A great circle bisects the surface of a sphere. One of the two equal parts into which the sphere is divided by a great circle is called a hemi-sphere.

19.2 Surface Area and Volume of a Sphere:

If r is the radius and d is the diameter of a great circle, then

(i) Surface area of a sphere = 4 times the area of its great circle

$$= 4\pi r^2$$
$$= \pi d^2$$

(ii) Volume of a sphere $= \frac{4}{3}\pi r^3 = \frac{\pi}{6} d^3$

(iii) For a spherical shell if R and r are outer and inner radii respectively, then the volume of a shell is

$$= \frac{4}{3} \pi(R^3 - r^3) = \frac{\pi}{6}(D^3 - d^3)$$

Example 1:

The diameter of a sphere is 13.5m. Find its surface area and volume.

Solution:

Here $d = 13.5m$

Surface area $= 4\pi r^2 = \pi d^2$

$$= \pi(13.5)^2$$
$$= 572.56 \text{ sq. m.}$$

Volume of sphere $= \frac{4}{3}\pi r^3 = \frac{\pi}{6} d^3$

$$= \frac{\pi}{6}(13.5)^3 = 1288.25 \text{ cu. m.}$$

Example 2:

Two spheres each a 10m diameter are melted down and recast into a cone with a height equal to the radius of its base, Find the height of the cone.

Solution:

Here $d = 10m$

Radius of cone = height of the cone

$r = h$

Volume of sphere $= \frac{\pi}{6} d^3$

$$= \frac{\pi}{6} 10^3$$
$$= \frac{\pi}{6} \times 1000 = 523.599 \text{ cu. m}$$

Volume of two sphere= 1047.2 cu. m.

Volume of the cone = volume of two spheres

$$\frac{1}{3}\pi r^3 h = 1047.2$$

$$\frac{1}{3}\pi h^3 = 1047.2$$

$$h^3 = \frac{3 \text{ x } 1047.2}{\pi}$$

$$= 1000$$

$$h = 10m$$

Example 3:

How many leaden ball of a $\frac{1}{4}$cm. in diameter can be cost out of metal of a ball 3cm in diameter supposing no waste.

Solution:

Here diameter of leaden ball $= \frac{1}{4}\text{cm.} = d_1$

Diameter of metal ball $= 3\text{cm} = d_2$

Volume of leaden ball $= \frac{\pi}{6} d_1^3$

$$= \frac{\pi}{6}\left(\frac{1}{4}\right)^3$$

$$= 0.0082 \text{ cu. cm}$$

Volume of metal ball $= \frac{\pi}{6} d_2^3$

$$= \frac{\pi}{6}(3)^3$$

$$= 14.137 \text{ cu. cm}$$

Number of leaden ball $= \frac{14.137}{0.0082} = 1728.00$

Exercise 19(A)

Q.1 How many square meter of copper will be required to cover a hemispherical dome 30m in diameter?

Q.2 A lead bar of length 12cm, width 6cm and thickness 3cm is melted down and made in four equal spherical bullets. Find the radius of each bullet.

Q.3 A sphere of diameter 22cm is charged with 157 coulomb of electricity. Find the surface density of electricity. (Hint: surface density = coulomb. Change/sq. cm.)

Q.4 A circular disc of lead 3cm in thickness and 12cm diameter, is wholly converted into shots of radius 0.5cm. Find the number of shots.

Q.5 The radii of the internal and external surfaces of a hollow spherical shell are 3m and 5m respectively. If the same amount of material were formed into cube what would be the length of the edge of the cube?

Q.6 A spherical cannon ball, 6cm in diameter is melted and cast into a conical mould the base of which is 12cm. in diameter. Find the height of the cone.

Q.7 A solid cylinder of brass is 10cm. in diameter and 3m long. How many spherical balls each 2cm. in radius, can be made from it?

Q.8 A hundred gross billiard balls $2\frac{1}{2}$ in. in diameter are to be painted. Assuming the average covering capacity of white paint to be 500 sq. in. per gallon, one coat, how many gallons are required to paint the hundred gross billiard balls? (1 gallon = 231 cu. in)

Q.9 How many cu. ft. of gas are necessary to inflate a spherical balloon to a diameter of 60in?

Q.10 The average weight of a cu. ft. of copper is 555 Lbs. What is the weight of 8 solid spheres of this metal having a diameter of 12in?

Q.11 Find the relation between the volumes and the surface areas of the cylinder, sphere and cone, when their heights and diameters are equal.

Q.12 A storage tank, in the form of a cylinder with hemisphere ends, 15m. long overall and 2m. in diameter. Calculate the weight of water, in liters, contained when the tank is one-third full.

Q.13 A cost iron sphere of 8cm. in diameter is coated with a layer of lead 7cm thick. Density of lead is 11.85 gm/cu.cm . Find the total weight of the lead.

Answers 19(A)

Q.1	1413.72 sq. m	Q.2	2.34cm.	Q.3	0.1033 coulomb/sq. cm.
Q.4	648	Q.5	7.43 m	Q.6	3 cm
Q.7	703 Balls	Q.8	3.93 gallons	Q.9	65.45 cu. ft.
Q.10	2324.78 Lbs.	Q.11	3:2:1 ; $1:1:\frac{\sqrt{5}}{4}$	Q.12	17104.23 liters
Q.13	62915.6 gms.				

19.3 Zone (Frustum) of a Sphere:

The portion of the surface of a sphere included between two parallel planes, which intersect the sphere, is called a zone. The distance between the two planes is called height or thickness of the zone.

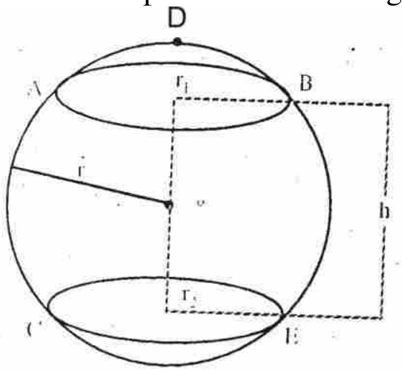


Fig. 19.3

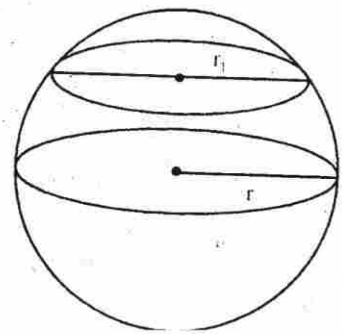


Fig. 19.4

19.4 Volume and Surface Area of the Zone:

Let r_1 and r_2 are the radii of the small circles respectively, r is the radius of the great circle and h is the height of the zone, then

(i) The volume of the zone of a sphere may be found by taking the difference between segment CDE and ADB (Fig. 3), that is

$$V = \frac{\pi h}{6}(h^2 + 3r_1^2 + 3r_2^2)$$

(ii) Surface area of the zone = Circumference of the great circle x height of the zone

$$= 2\pi r \times h$$

(iii) Total surface area of the zone $= 2\pi r h + \pi r_1^2 + \pi r_2^2$

(iv) For a special segment of one base, the radius of the lower base r_2 is equal to zero. Therefore,

$$V = \frac{\pi h}{6}(h^2 + 3r_1^2)$$

In this case, the total surface area of the segment

$$2\pi r h + \pi r_1^2$$

19.5 Spherical Segment or Cap of a Sphere:

If a plane cuts the sphere into two portions then each portion is known as a segment. The smaller portion is known as minor segment and the larger portion is known as major segment. Bowl is a spherical segment.

19.6 Volume and Surface Area of Spherical Segment:

If r_1 and r are the radius of the segment and sphere respectively and h is the height of the segment, then

(i) Volume of the segment of one base

$$= \frac{\pi h}{6}(h^2 + 3r_1^2)$$

$$= \frac{\pi h}{6}(3r - h)$$

(ii) Volume of the segment of two bases

$$= \frac{\pi h}{6}(h^2 + 3r_1^2 + 3r_2^2)$$

(iii) Surface area = Perimeter of the sphere x height of the zone

$$= 2\pi r h$$

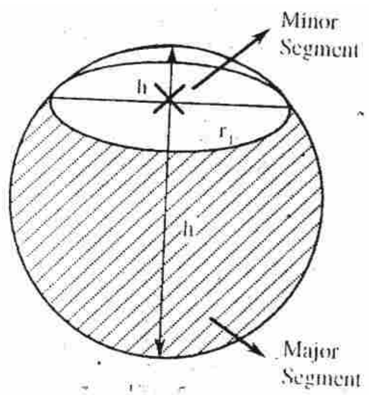


Fig. 19.5

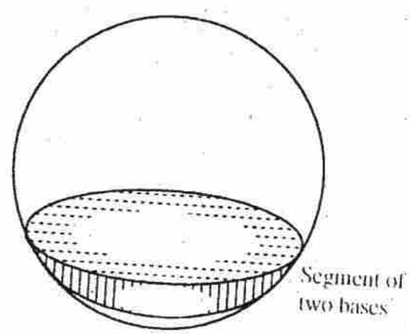


Fig. 19.6

19.7 Sector of Sphere:

A sector of a sphere is the solid subtended at the centre of the sphere by a segment cap (Fig. 7).

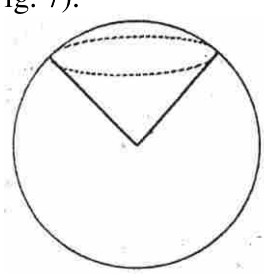


Fig. 19.7

Example 4:

A sphere is cut by two parallel planes. The radius of the upper circle is 7cm and the lower circle is 20cm. Both circles are on the same side of the sphere. The thickness of the zone is 9cm. Find the volume and the surface area of the zone.

Solution:

Here $r_1 = 7cm$ $r_2 = 20cm$, $h = 9cm$

Volume of the zone =

$$\frac{\pi h}{6}(h^2 + 3r_1^2 + 3r_2^2)$$

$$= \frac{\pi \text{ x } 9}{6}\{(9)^2 + 3(7)^2 + 3(20)^2\}$$

$$= \frac{\pi \text{ x } 3}{2}(81 + 147 + 1200)$$

$$= \pi \text{ x } \frac{3}{2}\text{x}1428$$

$= 6729.29$ cu. cm.

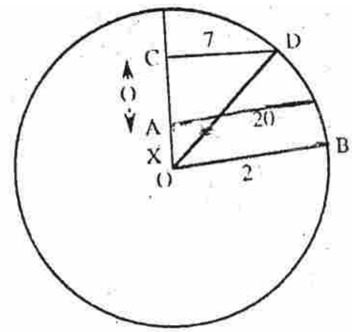


Fig. 19.8

In the right triangles OAB and OCD,

$$OB^2 = OA^2 + AB^2$$
$$= x^2 + (20)^2$$

and

$$OD^2 = OC^2 + CD^2$$
$$= (x + 9)^2 + (7)^2$$

Since $OB = OD$

Or $OB^2 = OD^2$

Or $x^2 + 20^2 = (x + 9)^2 + 7^2$

$$x^2 + 400 = x^2 + 18x + 81 + 49$$
$$400 = 18x + 130$$
$$x = 15cm$$

Now, from (1) $OB^2 = 15^2 + 20^2$

$$OB^2 = 625$$
$$OB = r = 25cm$$

So, area of the zone $= 2\pi r h$

$$= 2\pi \text{ x } 25 \text{ x } 9$$
$$= 1413.72 \text{ sq. cm.}$$

Example 5:

What proportion of the volume of a sphere 20cm. in diameter is contained between two parallel planes distant 6cm from the centre and on opposite side of it?

Solution:

Here $r_1 = 6cm$

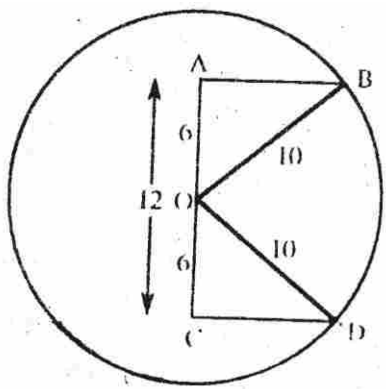


Fig. 19.9

$r_2 = 6cm$
$h = 12cm$
$d = 20cm$

Volume of the sphere $= \dfrac{\pi}{6}d^3$

$= \dfrac{\pi}{6}(20)^3 \qquad = 4188.79$ cu. cm

Volume of the zone $= \dfrac{\pi h}{6}(h^2 + 3r_1^2 + 3r_2^2)$

$= \dfrac{\pi \text{ x } 12}{6}(12^2 + 6 + 6^2)$

$= \pi \text{ x } 2(144 + 216) = 2261.95$ cu. cm.

Proportion of the volume of the sphere between the zone

$= \dfrac{2261.95}{4188.79} = 0.54$ or 54%

Exercise 19(B)

Q.1 The sphere of radius 8cm is cut by two parallel planes, one passing 2cm. from the centre and the other 7cm. from the centre. Find the area of the zone and volume of the segment between two planes if both are on the same side of the centre.

Q.2 Find the volume of the zone and the total surface area of the zone of a sphere of radius 8cm, and the radius of the smaller end is 6cm. The thickness of the zone is 12cm.

Q.3 What is the volume of a spherical segment of a sphere of one base if the altitude of the segment is 12cm. and radius of the sphere is 3cm.

Q.4 What percentage of the volume of a sphere 16cm. in diameter is contained between two parallel planes distant 4cm and 6cm. from the centre and on opposite side of it.

Q.5 Find the nearest gallons the quantity of water contains in a bowl whose shape is a segment of sphere. The depth of bowl is 7in. and radius of top is 11 in. (1 gallon = 231 cu. cm.)

Q.6 The core for a cast iron piece has the shape of a spherical segment of two bases. The diameters of the upper and lower bases are 2ft. and 6 ft. respectively, and the distance between the bases is 3ft. If the average weight of a cu. ft. of core is 100 Lbs. find the weight of the core.

Q.7 The bronze base of a statuette has the shape of a spherical segment of one base. The metal has a uniform thickness of 1/8in.; the diameter of the inner sphere is $4\frac{1}{2}$ in.; and the altitude of the segment formed by the inner shell is 2 in. find the weight of the bronze base. (The average weight of a cu. ft. bronze is 529 Lbs.)

Answers 19(B)

Q.1	251.33 sq. cm. 654.9 cu. cm	Q.2	775.97 sq. cm., 1943.33 cu. cm		
Q.3	0.134 cu. m.	Q.4	80.08%	Q.5	6.5 gallons
Q.6	6126.11 Lbs.	Q.7	1.15 Lbs.		

Summary

If r is the radius and d is the diameter of a great circle, then

1. S = Surface area of a sphere $= 4\pi r^2$
   Or $= \pi d^2$
2. Volume of sphere $= \frac{4}{3}\pi r^3$ or $\frac{\pi}{6} d^3$
3. For a spherical shell if R and r are the outer and inner radii respectively, then
   Volume of the shell $= \frac{4}{3}\pi(R^3 - r^3)$ or $\frac{\pi}{6}(D^3 - d^3)$

Zone of sphere

If the two parallel planes cut the sphere, then the portion of the sphere between the parallel planes is called the zone of the sphere.

(i) Volume of the zone = V = $\frac{\pi h}{6}(h^2+3r_1^2+3r_2^2)$ where r_1, r_2 are the radii of circles of base and top of zone.

(ii) Surface area of the zone = circumference of the great circle x height of the zone
i.e $2\pi r h$

(iii) Total surface area of the zone = $2\pi r h+\pi r_1^2+\pi r_2^2$

(iv) For a special segment of one base, the radius of the lower base $r_2=0$

$\therefore$ Volume of the segment = V = $\frac{\pi h}{6}(h^2+3r_1^2)$

In this case,

Total surface area of the segment = $2\pi r h+\pi r_1^2$

Short Questions

Q.1: Define sphere.
Q.2: Define Spherical shell
Q.3: A solid cylinder of glass, the radius of whose base in 9 cm and height 12 cm is melted and turned into a sphere. Find the radius of the sphere so formed.
Q.4: Find the thickness of a shell whose inner diameter measures 7 cm if it weight half as which as a solid ball of same diameter.
Q.5: How many square meter of copper will be required to cover a hemi-spherical done 30 m diameter?
Q.6: A spherical common ball, 6 cm in diameter is melted and cost into a conical mould the base of which is 12 cm in diameter. Find the height of the cone.
Q.7: How many cu.ft of gas are necessary to inflate a spherical ball to a diameter of 60 inch?
Q.8: A cost iron sphere of 8 cm in diameter is coated with a layer of lead 7 cm thick. Find the total weight of lead.
Q.9: A lead bar of length 10 cm, width 5 cm and thickness 4 cm is melted down and made 5 equal spherical bullets. Find radius of each bullet.
Q.10: A sphere of diameter 22cm is charged with 157 coulombs of electricity. Find the surface density of electricity.
(Hint: Surface density = $\frac{\text{charge}}{\text{Surface area}}$)
Q.11: How many lead balls, each of radius 1 cm can be made from a sphere whose radius is 8 cm?
Q.12: Write the formula for surface area of Segment of a sphere.
Q.13: Find the volume of a segment of a sphere whose height is $4\frac{1}{2}$ cm and diameter of whose base is 8 cm.
Q.14: The area of cross-section of a prism is 52 sq.m. What is the weight of the frustum of the prism of the smallest length is 10cm and the greatest length is 24.3 cm? Density of material 0.29 Lb/cu.cm.
Q.15: Write the formula of volume of sphere and hemi-sphere.

Answers

Q3.	r = 9	**Q4.**	0.50 cm.	**Q5.**	1413.72 sq. m.
Q6.	3 cm	**Q7.**	65.45 cu. ft	**Q8.**	5307.33 cu. cm
Q9.	2.12 cm	**Q10.**	0.1032 col/ sq.cm	**Q11.**	512 balls.
Q13.	160.8 cu. cm	**Q14.**	258.62 Lbs		

Objective Type Questions

Q.1 Each questions has four possible answers. Choose the correct answer and encircle it.

___1. The surface area of a sphere of radius 'r' is

(a) $4\pi\, r^3$ (b) $4\pi\, r^2$ (c) $\pi\, r^2$ (d) $\frac{4}{3}\pi\, r^3$

___2. The volume of a sphere of diameter D is

(a) $\frac{4}{3\pi}D^3$ (b) $4\pi D^2$ (c) $\frac{\pi}{6}D^3$ (d) $\frac{\pi}{4}D^2$

___3. If R and r are the external and internal radii of spherical shell respectively, then its volume is

(a) $\frac{4}{3}\pi\,(R^3 - r^3)$ (b) $\frac{4}{3}\pi\,(R^2 - r^2)$

(c) $4\pi\,(R^3 - r^3)$ (d) $\frac{\pi}{3}(R^3 - r^3)$

___4. If a plane cuts a sphere into two unequal positions then each portion is called

(a) circle (b) diameter (c) hemi-sphere (d) segment

___5. If two parallel planes cut the sphere by separating two segments, the portion between the planes is called

(a) zone (b) volume (c) surface (d) hemisphere

___6. Volume of hemi-sphere is

(a) $\frac{2}{3}\pi\, r^3$ (b) $\frac{4}{3}\pi\, r^3$ (c) $\frac{1}{2}\pi\, r^3$ (d) $3\pi\, r^2$

___7. Area of cloth to cover tennis ball of radius 3.5cm is

(a) 154 sq. cm (b) 77 sq. cm
(c) 308 sq. cm (d) 231 sq.cm

___8. Volume of sphere of radius '3' cm is

(a) 108π cu. cm (b) 36π cu. cm
(c) 12π cu. cm (d) 45π cu. cm

Answers

Q.1	(1)	b	(2)	c	(3)	a	(4)	d
	(5)	d	(6)	a	(7)	a	(8)	b

Linear and Area Measure:

Linear Measure:

12 inches (in)	=	1 foot (ft.)
3 feet	=	1 yard (yd)
36 inches	=	1 yard
220 yards	=	1 Furlong (Fr.)
8 Furlong	=	1 Mile (mi)
1760 yards	=	1 mile
1 meter (m)	=	1000 centimeter (cm.)
10 millimeters (mm)	=	1 cm.
10 cms	=	1 decimeter (dm.)
10 dm	=	1 m
10 m	=	1 Dekameter (DM.)
10 Dms.	=	1 Hectometer (HM.)
1 inch	=	2.54cm
1 ft.	=	0.3048m
1 yd	=	0.8144 m
1 mile	=	1.6093 Km
1cm	=	0.3937 in.
1 m	=	3.2808 ft.
1 m	=	1.0936 yd
1 Km	=	0.6214 mile
Area Measure:		
1 sq. ft.	=	144 sq. in.
1 sq. yd.	=	9 sq. ft.
1 sq. yd.	=	1296 sq. in
1 sq. mi	=	640 acres
1 acre	=	4840 sq. yd.
1 sq. cm	=	100 sq. m.m.
1 sq. m	=	10000 sq. cm.
Capacity:		
1 gallon	=	4.5461 litres
1 gallon	=	10 Lbs
1 cu. ft. of water	=	62.3 Lbs
1 litre	=	1000 cu. cm
1 gallon	=	231 cu. inch.