# Lecture - 15

Wednesday, 31 August 2016 (14:25 - 15:15)

Randomized Approximation Algorithm for Max 3-sat

**Definition 1.** *A Boolean formula is said to be satisfiable if there exists some assignment of the involved variables that makes the formula evaluate to TRUE.*

**Definition 2.** *Given a Boolean expression of the form $C_1 \wedge C_2 \wedge \dots C_k$, where each clause $C_i$ is of length 3, over a set of variables $X = \{x_1, x_2, \dots x_n\}$, the **3-SAT** problem questions the satisfiability of the expression. Each clause is given by the disjunction of three literals.*

If one had to determine the solution of the 3-SAT problem through brute force, she would have to evaluate the expression over $2^n$ possible truth assignments of all the $n$ variables. In this lecture, we saw the *Maximum 3-Satisfiability Problem* which converts an instance of 3-SAT to an optimization problem and determines a truth assignment that maximizes the satisfiability of a maximum number of clauses. We will look at a randomized algorithm that indeed satisfies a sufficiently large fraction of the clauses.

**Randomized Algorithm:** We consider each variable $x_1, x_2, \dots x_n$ and assign each of them 0 or 1 independently and uniformly at random. We then check how many clauses of the given expression are satisfied.

The question we addressed is to determine the expected number clauses that would be satisfied given such a random assignment. Thus, we model the given problem as follows.

- We begin by tossing a fair coin $n$ times to determine the truth values of the $2n$ associated literals $(x_1, \overline{x_1}, x_2, \overline{x_n}, \dots x_1, \overline{x_n})$.
- Since we want the expected number of clauses, we assume that the Boolean expression is constructed such that each literal in each clause is picked uniformly at random from the set of all $2n$ literals.
- We can observe that, independent of the truth values assigned to the $n$ variables, the set of all $2n$ literals would be equally assigned a 0 or a 1!!

Let $Z$ be a random variable that denotes the number of clauses that end up being satisfied. What we wish to determine is $E(Z)$.

$$Z = Z_1 + Z_2 + \dots Z_k \tag{1}$$

where, each $Z_i$ is an indicator random variable such that $Z_i = 1$ if the $i^{th}$ clause is satisfied, 0 otherwise and we know that we have $k$ clauses $C_1, C_2, \dots C_k$. Therefore, we have that the probability of a clause not being satisfied when each literal is picked uniformly at random equal to probability that all the picked literals are assigned to be FALSE :

$$\begin{aligned} Pr(Z_i \neq 1) &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{8} \end{aligned}$$

Therefore, the probability that a clause is satisfied is equal to :

$$Pr(Z_i = 1) = 1 - \frac{1}{8} = \frac{7}{8}$$

Therefore, from linearity of expectation in equation 1 we can say that :

$$\begin{aligned} E(Z) &= E(Z_1) + E(Z_2) + \cdots + E(Z_k) \\ &= \frac{7k}{8} \end{aligned}$$

Thus, we can conclude that such a random assignment of truth values to variable leads to $\frac{7}{8}^{th}$ of the total clauses to be satisfied on an average. We go one step ahead and analyze the probability that a random assignment satisfies at least $\frac{7}{8}^{th}$ of the total clauses. The aim is to see if we can provide an upper bound on this probability.

Let the probability that a random assignment satisfies at least $\frac{7}{8}^{th}$ of the total clauses be $\geq p$. Let $p_j$ denote the probability that precisely $j$ clauses are satisfied, $\forall j, 0 \leq j \leq k$. We are interested in determining

$$p = \sum_{j \geq 7k/8} p_j$$

Thus, by the definition of expectation we have:

$$\begin{aligned} E(Z) &= \sum_{j=0}^{k} j p_j \\ \frac{7k}{8} &= \sum_{j=0}^{k} j p_j \\ \frac{7k}{8} &= \sum_{j < 7k/8} j p_j + \sum_{j \geq 7k/8} j p_j \end{aligned}$$

Now, let $k'$ be the largest integer that is strictly lesser than $\frac{7k}{8}$. We also note that

$$1 - p = \sum_{j < 7k/8} p_j$$

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$$\begin{aligned} \frac{7k}{8} &\leq \sum_{j < 7k/8} k' p_j + \sum_{j \geq 7k/8} k p_j \\ &= k'(1-p) + kp \\ \frac{7k}{8} &\leq k' + kp \end{aligned}$$

Since we know that k' is a natural number strictly lesser than $\frac{7}{8}^{th}$ of k, we can conclude that $\frac{7k}{8} - k' \geq \frac{1}{8}$

$$\therefore p \geq \frac{\frac{7k}{8} - k'}{k} \geq \frac{1}{8k}$$

Therefore, we see that the probability of success in satisfying at least $\frac{7}{8}^{th}$ of the clauses is bounded below by $\frac{1}{8k}$. Hence, a repeated trail of at most $8k$ times guarantees such an assignment.