Special Elements Of The Bubble Algebra

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Abstract:

We provide a method for constructing central idempotents in the bubble algebra, and show how it can be split into direct sum of sub-algebras. We also give a description of the structure of its center.

Keywords: Temperley-Lieb algebra, multi-colour partition algebras and central idempotenets.

Introduction

The bubble algebra $\mathbb{T}_{n,m}(\delta_0, \dots, \delta_{m-1})$ was introduced by Grimm and Martin[3], and then its definition has generalized by using the definition of the partition algebra and as result we obtain the multi-colour partition algebra $\mathbb{P}_{n,m}(\delta_0, \dots, \delta_{m-1})$, see Hmaida[5].

The generic representations of the bubble algebra has been studied by Grimm and Martin[3], and they proved that it is semi-simple when none of parameters δ_i is a root of unity. Also Jegan[7] showed how certain idempotents in the bubble algebra could be use to simplify many problems on the algebra $\mathbb{T}_{n,m}(\delta_0, \dots, \delta_{m-1})$ as investigating the homomorphisms between the cell modules of the algebra, since it is a cellular algebra. Later, we, in [5], investigated the non-generic representations of the bubble algebra and the generic ones of the multi-colour partition algebra, and showed that the algebra $\mathbb{P}_{n,m}(\delta_0, \dots, \delta_{m-1})$ is non-semisimple over the complex field if and only if δ_i is a non-negative integer less than $2n - 1$ for some $j \in \mathbb{Z}_{m-1}$.

We generalized the technique that has been used in [5] and proved that the representations of any finite-dimensional cellular algebra with idempotents that satisfying specific conditions can be totally determined by the representations of its idempotent sub-algebras, for more details see Hmaida[6].

The representation theory of a unital algebra over a field with a splitting modular system is related to primitive central idempotents of its ordinary case. For example, a decomposition of the identity into a sum of primitive central idempotents gives the blocks of the algebra, see e.g. [1].

Our aim in this paper is to construct a family of central idempotents of the bubble algebra and give a simple description of its center.

Basic Definitions

Before discussing the multi-colour partition algebra we shall introduce the partition algebra $\mathbb{P}_n(\delta)$ and some of its subalgebra. Fix a natural number n, the algebra $\mathbb{P}_n(\delta)$ generated by the set, which is denoted by $\mathcal{P}_n$, of all partitions of the set $\underline{n} \cup \underline{n'} := \{1, \dots, n\} \cup \{1', \dots, n'\}$.

Each set partition can be represented by a graph: the graph is drawn in a rectangle with n nodes on the top and n nodes on the bottom, the vertices that are in the same part at the partition are represented as lines drawn inside the rectangle connecting these vertices. will often label our nodes starting with 1 on the left to n on the top edge and starting on the bottom edge with $1'$ on the left to n'. The diagram representing a partition is not unique, since there are different ways to drawing the edges. Two such diagrams are equivalent if they have the same connected components.

We define a multiplication on this basis as follows. If A, B are partition diagrams, the element AB is defined by stacking diagrams. In such a product of diagrams closed loops may appear, we remove each loop and replace it by δ. Figure 1 illustrates multiplication.

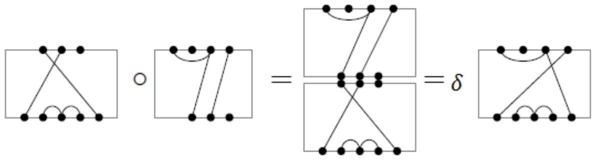


Figure 1

The Temperley-Lieb algebra is a sub-algebra of the algebra $\mathbb{P}_n(\delta)$, and the diagrams representing partitions that spanning the Temperley-Lieb algebra $TL_n(\delta)$ are planar (non-crossing) and their parts all have size two. For example see Figure 2.

Figure 2

The partition algebra was defined by Martin[8], and its representation theory has been investigated by many people, for example Halverson and Ram[4], Martin [8] and Martin and Woodcock [10]. Also, the representation theory of Temperley-Lieb algebra is well known, see e.g. Martin [9] , Ridout and Saint [11] and Westbury [12].

3. The Multi-Colour Partition algebras

For any positive integer m, let $\mathfrak{C}_0, \dots, \mathfrak{C}_{m-1}$ be different colours where none of them is white, and $\delta_0, \dots, \delta_{m-1}$ be scalars corresponding to these colours.

We construct basis elements of the multi-colour partition algebra $\mathbb{P}_{n,m}(\delta_0, \dots, \delta_{m-1})$ in similar way to the algebra $\mathbb{P}_n(\delta)$. Define the set $\Phi_{n,m} := \{(A_0, \dots, A_{m-1}) | \{A_0, \dots, A_{m-1}\} \in \mathcal{P}_n\}$ and the set $\mathcal{P}_{n,m}$ to be the union of sets $\prod_{i=0}^{m-1} \mathcal{P}_{A_i}$ where $(A_0, \dots, A_{m-1}) \in \Phi_{n,m}$ and $\mathcal{P}_{A_i}$ is the set of all set partitions of the set A_i.

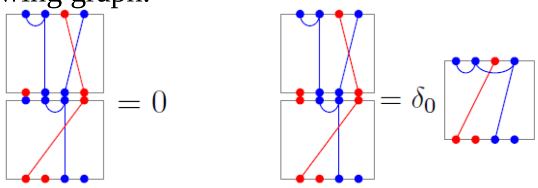
The element $d = (d_0, \dots, d_{m-1}) \in \prod_{i=0}^{m-1} \mathcal{P}_{A_i}$ is represented by the same diagram as the partition $\cup_{i=0}^{m-1} d_i$ after colouring it as follows. We use the colour $\mathfrak{C}_i$ to draw all the edges and the nodes in the partition d_i. A diagram represents an element in $\mathcal{P}_{n,m}$ is not unique. We say two diagrams are equivalent if they represent the same tuple of partitions. The term multi-colour partition diagram will be used to mean an equivalence class of a given diagram. For example, the diagrams in Figure 3 are equivalent.

Figure 3

We define the following sets for each element $d \in \prod_{i=0}^{m-1} \mathcal{P}_{A_i}$:

$$top(d) = \left(A_0 \cap \underline{n}, \dots, A_{m-1} \cap \underline{n}\right), \quad bot(d) = \left(A_0 \cap \underline{n}', \dots, A_{m-1} \cap \underline{n}'\right).$$

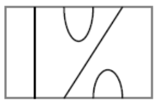
The composite of two diagrams is defined if the two diagrams have the same number of end points. In this case the composite is zero unless the colours match up precisely. If they do match up the composite is a multi-colour partition which is obtained by linking the diagrams together as for the partition algebra and replacing any $\mathfrak{C}_j$ loop appearing inside the diagram by the scalar δ_j times the rest of the diagram. See the following graph.



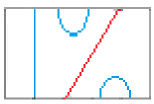
A planar multi-colour partition in the set $\mathcal{P}_{n,m}$ is a multi-colour partition represented by a diagram that does not have edge crossings in the same colour. The bubble algebra is generated by the set

$$\mathcal{T}_{n,m} = \{d \in \mathcal{P}_{n,m} \,|\, d \text{ is planar and all blocks of size } 2\}.$$

As all diagrams that represent multi-colour partitions are obtained from colouring diagrams in the monoid $\mathcal{P}_n$, such that all nodes in a part of a partition and their edges which connect them have the same colour. Thus we can determine a multi-colour diagram by knowing its top and its bottom and the uncoloured image of it. For example, let D be the diagram



One of the coloured copies of D is



which can be denoted by $D^{(1,1,1,0)}_{(1,0,1,1)}$, we use the elements of $\mathbb{Z}^n_{m-1}$ to represent the deferent colours. In this example 1 represents the blue and red is represented by zero. By using this notation, the identity of the algebras $\mathbb{P}_{n,m}(\delta_0, \dots, \delta_{m-1})$ and $\mathbb{T}_{n,m}(\delta_0, \dots, \delta_{m-1})$ is $\sum_{x \in \mathbb{Z}^n_{m-1}} 1_x$, where 1_x is the coloured image of $id \in \mathfrak{S}_n$, $\mathfrak{S}_n$ is the symmetric group on n letters, where the node i is only connected to i' with an $\mathfrak{C}_{x_i}$-edge.

The diagrams of shape id $id \in \mathfrak{S}_n$ are orthogonal idempotents, since

$$1_x 1_y = \begin{cases} 1_x, & if\ x = y, \\ 0, & otherwise, \end{cases} \tag{1}$$

for all $x, y \in \mathbb{Z}^n_{m-1}$. Thus we have a decomposition of the identity as a sum of orthogonal idempotents. Also, from the graphical visualization, it is evident that

$$1_x D_z^y = \begin{cases} D_z^y, & if\ x = y, \\ 0, & otherwise, \end{cases} \quad \text{and} \quad D_z^y 1_x = \begin{cases} D_z^y, & if\ x = z, \\ 0, & otherwise, \end{cases} \tag{2}$$

where $x, y, z \in \mathbb{Z}_{m-1}^n$ and $D_z^y \in \mathcal{P}_n$.

Theorem 3.1.4 in [7] shows that

$$1_x \mathbb{T}_{n,m}(\delta_0, \dots, \delta_{m-1}) 1_x \cong TL_{\lambda_0}(\delta_0) \otimes \cdots \otimes TL_{\lambda_{m-1}}(\delta_{m-1}), \tag{3}$$

and Theorem 2.34 in [5] similarly shows that

$$1_x \mathbb{P}_{n,m}(\delta_0, \dots, \delta_{m-1}) 1_x \cong \mathbb{P}_{\lambda_0}(\delta_0) \otimes \cdots \otimes \mathbb{P}_{\lambda_{m-1}}(\delta_{m-1}), \tag{4}$$

where $\lambda_i = \#\{x_j | x_j = i\}$, for all $x \in \mathbb{Z}_{m-1}^n$. The previous isomorphisms can be proved by using a map sending any tuple of diagrams in tensor product of algebras to the multi-colour partition diagram formed by drawing these diagrams in one frame one by one with put in consideration the distribution of colours in $1_x \mathbb{T}_{n,m}(\delta_0, \dots, \delta_{m-1}) 1_x$ using different colours such that the diagram from $TL_{\lambda_i}(\delta_i)$ is drawn in the colour $\mathfrak{C}_i$. Similarly for $1_x \mathbb{P}_{n,m}(\delta_0, \dots, \delta_{m-1}) 1_x$.

The elements 1_x and 1_y are conjugate if and only if $\lambda_i = \lambda'_i$ for each $0 \le i \le m-1$, where $\lambda_i = \#\{x_j | x_j = i\}$ and $\lambda'_i = \#\{y_j | y_j = i\}$, for more details see Theorem 2.33 in [5].

Central Idempotents in The Bubble Algebra

Let $\mathbb{T}_{n,2}{}^{+}(\delta_0, \delta_1)$ be the subspace of $\mathbb{T}_{n,2}(\delta_0, \delta_1)$ that is spanned by all the diagrams in $\mathcal{T}_{n,2}$ which have an even number of blue-nodes on the top face. Since making an arc needs two nodes on the same face, thus the number of blue-nodes on the bottom face of the diagrams in $\mathbb{T}_{n,2}{}^{+}(\delta_0, \delta_1)$ will be also an even number. Similarly, define $\mathbb{T}_{n,2}{}^{-}(\delta_0, \delta_1)$ to be the subspace of $\mathbb{T}_{n,2}(\delta_0, \delta_1)$ that is spanned by all the diagrams in $\mathcal{T}_{n,2}$ which have an odd number of blue-nodes on the top face.

Lemma 1. [5, Lemma 5.11]. For any $n \ge 1$, we have

$$\mathbb{T}_{n,2}(\delta_0, \delta_1) \cong \mathbb{T}_{n,2}{}^{+}(\delta_0, \delta_1) \oplus \mathbb{T}_{n,2}{}^{-}(\delta_0, \delta_1)\,,$$

as an algebra.

We can generalize the previous lemma, to get a decomposition of the bubble algebra with more colours. For example, in the case of three colours see the next lemma.

Lemma 2. For any $n \ge 1$, we have

$$\mathbb{T}_{n,3}(\delta_0, \delta_1, \delta_2) \cong \mathbb{T}_{n,3}{}^{++} \oplus \mathbb{T}_{n,3}{}^{+-} \oplus \mathbb{T}_{n,3}{}^{-+} \oplus \mathbb{T}_{n,3}{}^{--},$$

as an algebra, where $\mathbb{T}_{n,3}{}^{++}$ be the subspace of $\mathbb{T}_{n,3}(\delta_0, \delta_1, \delta_2)$, or simply $\mathbb{T}_{n,3}$, that is spanned by all the diagrams in $\mathcal{T}_{n,3}$ which have an even number of both blue-nodes and red-nodes on the top face. $\mathbb{T}_{n,3}{}^{+-}$ be the subspace of $\mathbb{T}_{n,3}$ that is spanned by all the diagrams in $\mathcal{T}_{n,3}$ which have an even number of blue-nodes and an odd number of red-nodes on the top face(conversely $\mathbb{T}_{n,3}{}^{-+}$). $\mathbb{T}_{n,3}{}^{--}$ be the subspace of $\mathbb{T}_{n,3}$ that is spanned by all the diagrams in $\mathcal{T}_{n,3}$ which have an odd number of both blue-nodes and red-nodes on the top face.

Proof. This come from the fact any diagram in $\mathcal{T}_{n,3}$ will be in ${\mathbb{T}_{n,3}}^{++}$ or ${\mathbb{T}_{n,3}}^{+-}$ or ${\mathbb{T}_{n,3}}^{-+}$ or ${\mathbb{T}_{n,3}}^{--}$. Furthermore, it is clear that their intersection is zero and the product of any two diagrams from different spaces always will be zero.
■

There are many techniques to computing the central elements and the idempotents in an algebra. In the case of the algebras $\mathbb{P}_{\mathrm{n,m}}$ and $\mathbb{T}_{\mathrm{n,m}}$, we will use the ones of the algebras $\mathbb{P}_n(\delta)$ and $TL_n(\delta)$ to construct the central idempotents of the algebras $\mathbb{P}_{n,m}$ and $\mathbb{T}_{n,m}$.

Let A be an algebra, we will the notation $Z(A)$ to donate the center of the algebra A.

Theorem 3. For any $n \geq 1$, we have

$$Z(\mathbb{T}_{n,m}) \subseteq \sum_{x\in\mathbb{Z}_{m-1}^n} Z(TL_{\lambda_{x,0}}(\delta_0)) \otimes \cdots \otimes Z(TL_{\lambda_{x,m-1}}(\delta_{m-1})),$$
$$Z(\mathbb{P}_{n,m}) \subseteq \sum_{x\in\mathbb{Z}_{m-1}^n} Z(\mathbb{P}_{\lambda_{x,0}}(\delta_0)) \otimes \cdots \otimes Z(\mathbb{P}_{\lambda_{x,m-1}}(\delta_{m-1})),$$

where $\lambda_{x,i} = \#\{x_j | x_j = i\}$.

Proof. Let $e = \sum_{\alpha\in\mathcal{P}_{n,m}} a_\alpha \alpha$ be an element in the center of the algebra $\mathbb{P}_{n,m}$. Now since $1_x\alpha = \alpha$, $\alpha 1_y = \alpha$ whenever $top(\alpha) = x$ and $bot(\alpha) = y$, otherwise they will be zero, therefore

$$top(\alpha) = bot(\alpha) \text{ whenever } a_\alpha \neq 0.$$

But then $\alpha \in 1_{top(\alpha)}\mathbb{P}_{n,m}1_{top(\alpha)}$.

Now if rewrite e as following $e = \sum_{x\in\mathbb{Z}_{m-1}^n} e_x := \sum_{x\in\mathbb{Z}_{m-1}^n}\sum_{\alpha\in\mathcal{P}_{n,m},top(\alpha)=x} a_\alpha\alpha$, thus $e_x \in 1_x\mathbb{P}_{n,m}1_x$ and since e in $Z(\mathbb{P}_{n,m})$ so $e_x \in Z(1_x\mathbb{P}_{n,m}1_x)$ and by using equation (4) we are done. Similarly for the first relation.
■

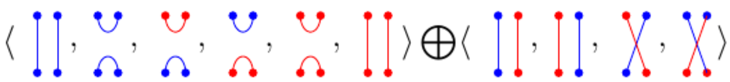
Now if we let $e = \sum_{x\in\mathbb{Z}_{m-1}^n} e_x = \sum_{x\in\mathbb{Z}_{m-1}^n}\sum_{\alpha\in\mathcal{P}_{n,m},top(\alpha)=x} a_\alpha\alpha$ be central idempotent in the algebra $\mathbb{P}_{n,m}$, hence

$$e^2 = \sum_{x\in\mathbb{Z}_{m-1}^n}(e_x)^2,$$

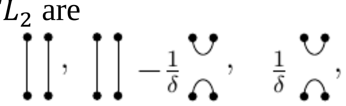
as the product on $\mathcal{P}_{n,m}$ will vanish, if the colours do not match. Hence e will be an idempotent as long as e_x is an idempotent in the algebra $1_x\mathbb{P}_{n,m}1_x$ for each x (similarly for $\mathbb{T}_{n,m}$). Hence to construct a central idempotent in $\mathbb{P}_{n,m}$, we start with pick up a central idempotent in the algebra $\mathbb{P}_{\lambda_0}(\delta_0)\otimes\cdots\otimes\mathbb{P}_{\lambda_{m-1}}(\delta_{m-1})$ for some x and check what are the rest of idempotents that satisfy $D_x^u e_x = e_u D_x^u$ where $u \neq x$.

5. Example

Let $n = 2$, from Lemma 1 the algebra $\mathbb{T}_{n,2}$ is isomorphic to

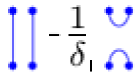


The central idempotents in TL_2 are

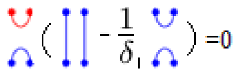


and the central idempotent in TL_1 is

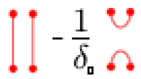
When we start with the element



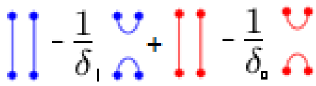
we get



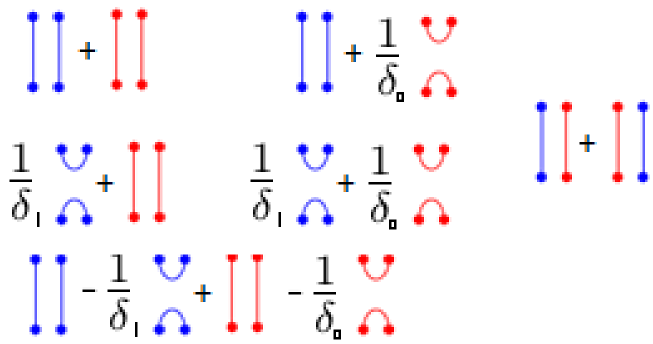
But the only element that satisfy the opposite is



so the next element is central idempotent of $\mathbb{T}_{n,2}$:



By following the same steps we find all the central idempotents of $\mathbb{T}_{n,2}$:



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