Questions

1. Let $\mathbf{p}$ and $\mathbf{q}$ be two vectors in $\Re^3$ and let $\mathbf{p}' = (2, 7, 4, 3)$ and $\mathbf{q}' = (1, 3, 5, 2)$ be a homogeneous coordinate representation of these vectors. Using the $\Re^3$ representation of these vectors, calculate $\mathbf{p}' \times \mathbf{q}'$ and $\mathbf{p}' \cdot \mathbf{q}'$.

2. We say that two matrices $\mathbf{A}$ and $\mathbf{B}$ commute if $\mathbf{A}\ \mathbf{B} = \mathbf{B}\ \mathbf{A}$. Matrices do not commute in general, but certain types of matrices do. Which of the following types of matrices commute? Assume matrices are 4×4.

   - scaling matrices $\mathbf{S}_1\mathbf{S}_2$
   - translation matrices $\mathbf{T}_1\mathbf{T}_2$
   - rotation matrices, $\mathbf{R}_1\mathbf{R}_2$
   - rotation and translation, $\mathbf{RT}$
   - scaling and translation, $\mathbf{ST}$
   - scaling and rotation, $\mathbf{SR}$

   To prove each one formally, you would need to write out a general form of each type of matrix and do the brute force calculation which wouldn't help you much in terms of how to think about it. Instead, try to visualize it. For cases of failure imagine a counterexample.

3. There are two ways to define a 3D dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$, namely the sum of $u_i v_i$ for $i = 1, ..3$, and $\mid \mathbf{u} \mid \mid \mathbf{v} \mid \cos(\theta)$. In lecture 2, I sketched out an argument for why these definitions are equivalent. Write out the same sketch for the 3D case.

4. Consider a 3×3 matrix that represents a rotation $R_z(\theta)$ about the z axis. What are the three eigenvalues and eigenvectors of this transformation?

   This question is beyond the scope of the course. It is only for those of you are comfortable with linear algebra and in particular, with complex numbers. Before you start turning the linear algebra crank to answer this question, ask yourself about the physical intuition here: an eigenvector just gets multiplied by a scalar (the eigenvalue). What might that mean in the case of a rotation?

Answers

1. Cross and dot products are not defined in a (4D) homogeneous coordinate representation. So we need to write these vectors in $\Re^3$.

$$\mathbf{p}' = (2,7,4,3) = (\frac{2}{3}, \frac{7}{3}, \frac{4}{3}, 1) \qquad \mathbf{q}' = (1,3,5,2) = (\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 1)$$

so we need to calculate the cross and dot products of

$$(\frac{2}{3}, \frac{7}{3}, \frac{4}{3}) \qquad (\frac{1}{2}, \frac{3}{2}, \frac{5}{2}).$$

Straightforward calculation yields

$$\mathbf{p}' \times \mathbf{q}' = (\frac{23}{6}, -1, -\frac{1}{6}) \qquad \mathbf{p}' \cdot \mathbf{q}' = \frac{43}{6}$$

2. Here I just list the answers.

- scaling matrices commute, $\mathbf{S}_1\mathbf{S}_2 = \mathbf{S}_2\mathbf{S}_1$
- translation matrices commute, $\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_2\mathbf{T}_1$
- rotation matrices typically do *not* commute, $\mathbf{R}_1\mathbf{R}_2 = \mathbf{R}_2\mathbf{R}_1$ (though they do commute always if rotation is about a common axis)
- rotation and translation typically do *not* commute, $\mathbf{RT} \neq \mathbf{TR}$
- scaling and translation typically do *not* commute, $\mathbf{ST} \neq \mathbf{TS}$
- scaling and rotation typically do *not* commute, $\mathbf{SR} \neq \mathbf{RS}$

For each non-trivial case, you would need to write the matrices in their general form, and then multiply them through. For example, $\mathbf{TR} \neq \mathbf{RT}$:

$$\begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & T_x \\ R_{21} & R_{22} & R_{23} & T_y \\ R_{31} & R_{32} & R_{33} & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

whereas

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & (R_{11}T_x + R_{12}T_y + R_{13}T_z) \\ R_{21} & R_{22} & R_{23} & (R_{21}T_x + R_{22}T_y + R_{23}T_z) \\ R_{31} & R_{32} & R_{33} & (R_{31}T_x + R_{32}T_y + R_{33}T_z) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. Suppose that the angle between $\mathbf{u}$ and $\mathbf{v}$ is γ. Consider a rotation matrix $\mathbf{R}$ that rotates $\mathbf{u}$ to $(|\mathbf{u}|, 0, 0)$ and that rotates $\mathbf{v}$ to $|\mathbf{v}|(\cos\gamma, \sin\gamma, 0)$. Then

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{R}\mathbf{u}) \cdot \mathbf{R}\mathbf{v} = \mathbf{u}^T \mathbf{R}^T \mathbf{R} \mathbf{v} \ = \ |\mathbf{u}| \ |\mathbf{v}| \cos\gamma.$$

4. Here I assume you remember how to compute the eigenvalues and eigenvectors of a matrix. Remember that you may end up with complex numbers when you do so.

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving for $\det(\mathbf{R}_z(\theta) - \lambda \mathbf{I}) = 0$ gives

$$(1-\lambda) \ ((\cos\theta - \lambda)^2 + \sin^2\theta) = 0$$

which gives three solutions: $\lambda = 1, \cos\theta \pm i \sin\theta$. The eigenvalue $\lambda = 1$ has an eigenvector $\hat{\mathbf{z}} = (0, 0, 1)$. That's the obvious one.

The two complex eigenvalues are more subtle. The magnitude of these eigenvalues is 1, which at least corresponds with the concept of rotation not changing length. But what are the corresponding eigenvectors. When you turn the linear algebra crank, you get that the eigenvectors for $\lambda = \cos\theta \pm i \sin\theta$ are $(\pm i, 1, 0)$. As stated, these eigenvectors are not of unit length but that doesn't matter: eigenvectors can be of any length.