The Surface Integral: Theory and Application

Raja Bashir Elrogayeg
Fatma Altuhami Noah zaggout

Abstract:

In this study, we give a quick insight into surface integral and its application. We evaluate different cases of surface integrals. These cases include, the coordinate planes, the circular cylinder zones and spheres zones.
Highlighting various types of the surface integral, we provide different shapes that explain the idea.
Key words: integral surface, outward normal, stokes's theorem.

Introduction:

Surface integral is an integration over a surface, and it is a generalization for the linear integration in two - dimensional or more. Surface integration helps in the description of the normal phenomenon and arises in the Stokes's theorem and divergence theorem.
In this research we study the surface integration and discuss many aspects related with.

The Aim:

Our interest is to look at various cases of surface integrals and to explore different types of surface integrals that occur.

Definition [5]: If $\vec{F}$ is a continuous vector field defined on an oriented surface S with unite normal vector $\vec{n}$, then the surface integral of $\vec{F}$ over S is

$$\iint_S \vec{F}.dS = \iint_S \vec{F}.\vec{n}\,dS$$

This integral is also called the flux of F across S.

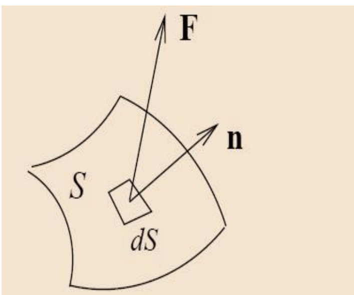


figure (1)

Types of surface integrals :

Consider a surface S in three – dimensional space. Let dS represents the area of a little bit of the surface and let $\vec{n}$ be the unite normal to the surface:
(i) If the surface is the boundary of a three – dimensional region, then $\vec{n}$ is outward normal, inward normal, for example the unite sphere in figure(2) and figure(3) .

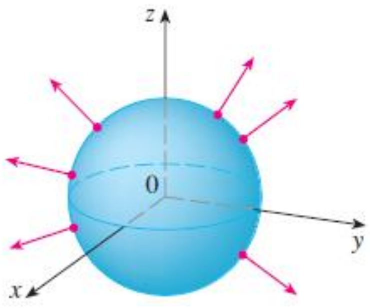


figure (2)

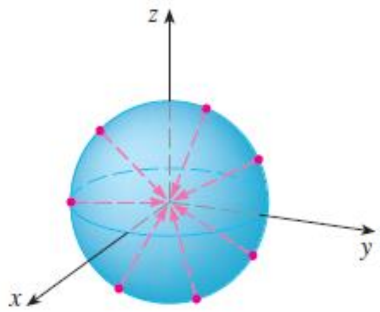
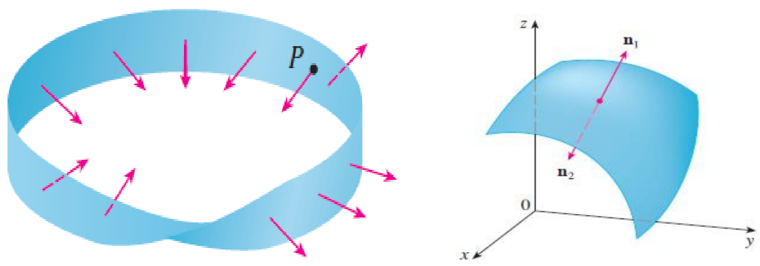


figure (3)

(ii) If S is an open surface, then either of the two normal vectors $\vec{n}_1$ or $-\vec{n}_1$ can be used, for example Möbius strip in figure (4). (See [1], and [2]).



figure(4)

The formulas of the surface integral [5]:
If the vector field is given by $\vec{F} = \langle P, Q, R \rangle$, then formulas of the surface integral are:
(i)- The regular formula:
Let the surface is given by $z = g(x, y)$, and the orientation is the “outward” orientation. the surface integral $\vec{F}$ of over S is,

$$\iint_S \vec{F}.\vec{n}\, dS = \iint_D \left(-P f_x - Q f_y + R\right) dA$$

where

$$dS = \left|\vec{\nabla} f\right| dA \quad , \quad \vec{n} = \frac{\vec{\nabla} f}{\left|\vec{\nabla} f\right|} \quad , \quad f(x,y,z) = z - g(x,y)$$

If we'd need “inward” orientation then we need to change the signs on the normal vector.

(ii)- The parametric formula:

Let the surface S is given parametrically by $\vec{r}(u,v)$, the surface integral is

$$\iint_S \vec{F}.\vec{n}\, dS = \iint_D \vec{F}.\left(\vec{r}_u \times \vec{r}_v\right) dA$$

where

$$dS = \left|\vec{r}_u \times \vec{r}_v\right| dA \quad , \quad \vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\left|\vec{r}_u \times \vec{r}_v\right|}$$

Evaluate surface integrals [2]:

The main part in finding the value of surface integral is where the surface S is found, accor-ding to the coordinates plane.

Case1- S is parallel to a coordinate plane:

If the surface S has the equation $z = c$, then the integration region D will be sit below or above S in the in the xy plane, where $\vec{n} = \vec{k}$, $dS = dxdy$, thus,

$$\iint_S \vec{F}.\vec{n}\, dS = \iint_D \vec{F}.\vec{k}\, dxdy$$

Similarly, if the region D sits in yz plane for the surface $x = c$, and in xz plane for the surface $y = c$ respectively, then

$$\iint_S \vec{F}.\vec{n}\, dS = \iint_D \vec{F}.\vec{i}\, dydz ,$$

$$\iint_S \vec{F}.\vec{n}\, dS = \iint_D \vec{F}.\vec{j}\, dxdz$$

Case2- S projects onto the coordinate plane:

(i) S projects onto xy plane:

In this case, each coordinate pair (x, y) corresponds to at most one point on the surface S which has the equation $f(x, y, z) = c$,(figure 5), and taking $\vec{n}$ to be the normal with positive $\vec{k}$ component, we have the formulas

$$dS = \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \vec{k}|} dxdy, \qquad \vec{n}dS = \frac{\vec{\nabla} f}{\vec{\nabla} f \cdot \vec{k}} dxdy$$

and then

$$\boxed{\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot \frac{\vec{\nabla} f}{\vec{\nabla} f \cdot \vec{k}} dxdy}$$

(ii) S projects onto yz plane:

In this case, $\vec{n}$ is taken to be the normal with positive $\vec{i}$ component:

$$dS = \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \vec{i}|} dydz, \qquad \vec{n}dS = \frac{\vec{\nabla} f}{\vec{\nabla} f \cdot \vec{i}} dydz$$

then

$$\boxed{\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot \frac{\vec{\nabla} f}{\vec{\nabla} f \cdot \vec{i}} dydz}$$

(iii) S projects onto xz plane:

Here, $\vec{n}$ is the normal with positive $\vec{j}$ component

$$dS = \frac{|\vec{\nabla} f|}{|\vec{\nabla} f \cdot \vec{j}|} dxdz, \qquad \vec{n}dS = \frac{\vec{\nabla} f}{\vec{\nabla} f \cdot \vec{j}} dxdz$$

thus

$$\boxed{\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot \frac{\vec{\nabla} f}{\vec{\nabla} f \cdot \vec{j}} dxdz}$$

(iv) S is of the form $z = f(r)$:

Here we can project onto xy plane, and use the polar coordinates:

$$dS = \sqrt{\left(f'(r)\right)^2 + 1}\, r\, dr\, d\theta,$$

$$\vec{n}\, dS = \langle -f'(r)\cos\theta, -f'(r)\sin\theta, 1\rangle\, r dr d\theta$$

and then

$$\iint_S F.n\, dS = \int_{\theta_1}^{\theta_2}\int_0^{r_1} F\left(r\cos\theta, r\sin\theta, f(r)\right).\langle -f'(r)\cos\theta, -f'(r)\sin\theta, 1\rangle\, r dr d\theta$$

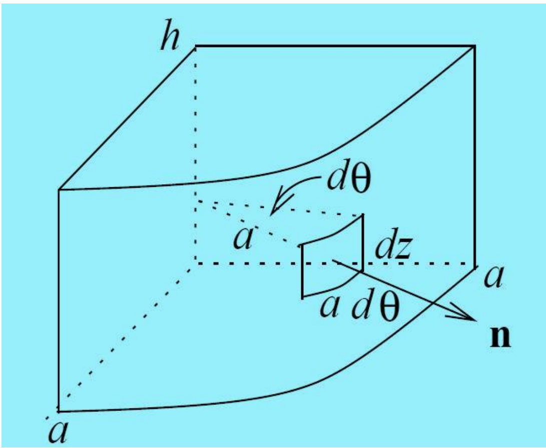
Case3- S is a portion of a circular cylinder $r = a$:

Used the cylindrical coordinates in figure (6), the vector $\langle x, y, 0\rangle$ is an outward normal to get $\vec{n}$ to formulate:

$$dS = r\, dz\, d\theta, \qquad \vec{n} = \frac{\langle x, y, 0\rangle}{r} = \frac{\langle r\cos\theta, r\sin\theta, 0\rangle}{r}$$

so

$$\iint_S F.n\, dS = \int_{\theta_1}^{\theta_2}\int_{z_1}^{z_2} F\left(r\cos\theta, r\sin\theta, z\right).\langle \cos\theta, \sin\theta, 0\rangle\, r dz d\theta$$



Figure(6)

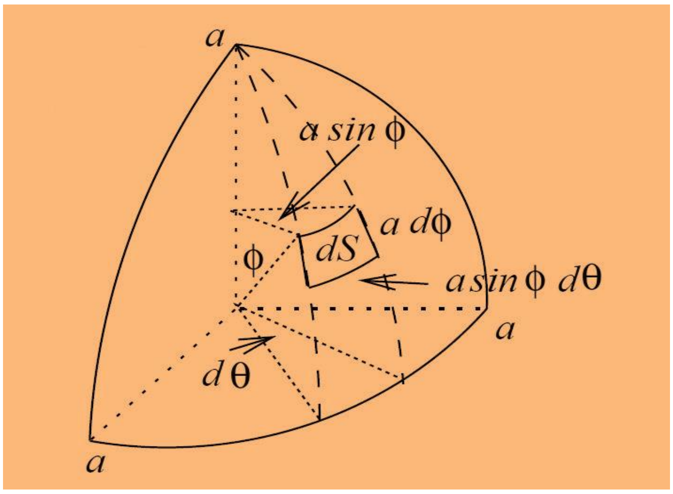
Case4- S is a portion of a sphere $\rho = const$:

If the surface S has equation $\rho = a > 0$ and the vector $\vec{r}$ is an outward normal, we have to use spherical coordinates as figure(7):

$$\vec{n}=\frac{\vec{r}}{\rho},\qquad dS=\rho^2\sin\varphi\, d\varphi\, d\theta$$

then

$$\iint_S F.n\,dS=\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}(F.\vec{r})(\theta,\varphi)\,\rho\sin\varphi\, d\varphi\, d\theta$$



Figure(7)

Example1: Let be the cylinder of radius 3 and height 5 given by $x^2+y^2=9$ and $0\le z\le 5$. Let $\vec{F}$ be the vector field $\vec{F}(x,y,z)=\langle 2x,2y,2z\rangle$. Find the surface integral of F over S .

Solution: From figure(8), the vector field and unite normal vector point outward.

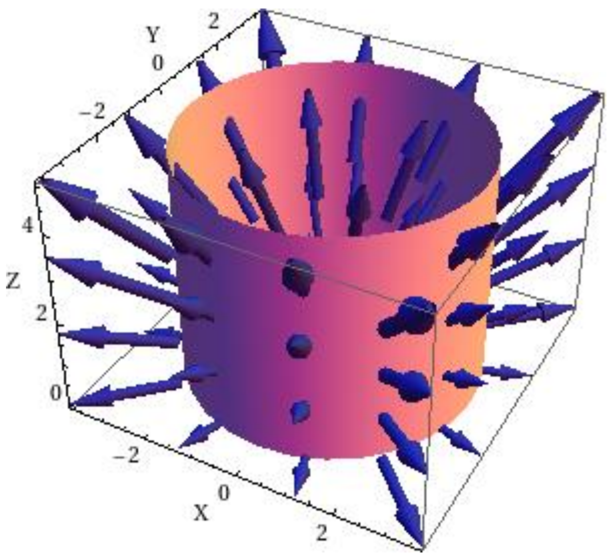


figure (8)

By the cylindrical coordinates S has equation

$$S: \vec{r}(\theta, z) = \langle 3\cos\theta, 3\sin\theta, z \rangle, \qquad for\ 0 \le \theta \le 2\pi, \quad 0 \le z \le 5.$$

and case4, we get

$$\vec{n} = \langle \cos\theta, \sin\theta, 0 \rangle$$

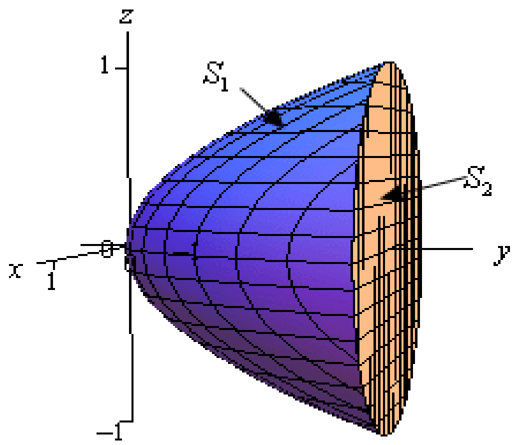
$$\iint_S \vec{F}.\vec{n}\,dS = \int_{\theta_1}^{\theta_2}\int_{z_1}^{z_2} \vec{F}(r\cos\theta, r\sin\theta, z).\langle \cos\theta, \sin\theta, 0\rangle r dz d\theta$$

$$= \int_0^{2\pi}\int_0^5 3\langle 6\cos\theta, 6\sin\theta, 2z\rangle.\langle \cos\theta, \sin\theta, 0\rangle dz\, d\theta$$

$$= 180\pi$$

Example 2: Evaluate $\iint_S F.dS$ where $F = \langle 0, y, -z \rangle$ and S is the surface given by the paraboloid $y = x^2 + y^2$, $0 \le y \le 1$ and the disk $x^2 + z^2 \le 1$ at $y = 1$. Assume that S has positive orientation.

Solution: From figure (9), the surface S is composed of two surfaces, we will denote the paraboloid by S_1 and the disc by S_2. Also in order for unit normal vectors on the paraboloid to point away from the region they will all need to point generally in the negative y direction. On the other hand, unit normal vectors on the disk will need to point in the positive y direction in order to point away from the region.



Figure(8)

So

$$\iint_S \vec{F}.dS = \iint_{S_1} \vec{F}.dS + \iint_{S_2} \vec{F}.\mathrm{d}S$$

S_1: The paraboloid

D being the projection of S_1 onto the xz plane, namely, the disc $x^2+z^2\le 1$, we define

$$f(x,y,z)=y-g(x,y)=y-x^2-z^2$$
$$\vec{\nabla} f=\langle -2x,1,-2z\rangle$$

by case 2 (iii)

$$\vec{n}\,dS=\frac{\vec{\nabla} f}{\vec{\nabla} f\,.\vec{j}}dx\,dz=\langle 2x,-1,2z\rangle dx\,dz$$

$$\iint_{S_1}\vec{F}.dS=\iint_{S_1}\vec{F}.\vec{n}\,dS$$

$$=\iint_{D}\langle 0,y,-z\rangle.\langle 2x,-1,2z\rangle dx\,dz$$

$$=\iint_{D}\left(-y-2z^2\right)dxdz=\iint_{D}\left(-x^2-3z^2\right)dxdz$$

by polar coordinates for this reign $x=r\cos\theta$, $z=r\sin\theta$, where $0\le\theta\le 2\pi$, $0\le r\le 1$, we get

$$\iint_{S_1}F.dS=-\int_0^{2\pi}\int_0^1\left(r^2\cos^2\theta+3r^2\sin^2\theta\right)rdrd\theta$$
$$=-\pi$$

S_2: The cap of the paraboloid, the disk $x^2+z^2\le 1$ is the portion of the plane $y=1$ that is in front of the disk of radius 1 in the xz plane, the unit normal vector of S_2 is $\vec{n}=\vec{j}$. So by case 1 and case-2 (ii) , we get

$$\iint_{S_2}\vec{F}.dS=\iint_{S_2}\vec{F}.\vec{n}\,dS=\iint_{S_2}ydS$$
$$=\iint_{D}dA=\int_0^{2\pi}\int_0^1 r\,dr\,d\theta$$
$$=\pi$$

then

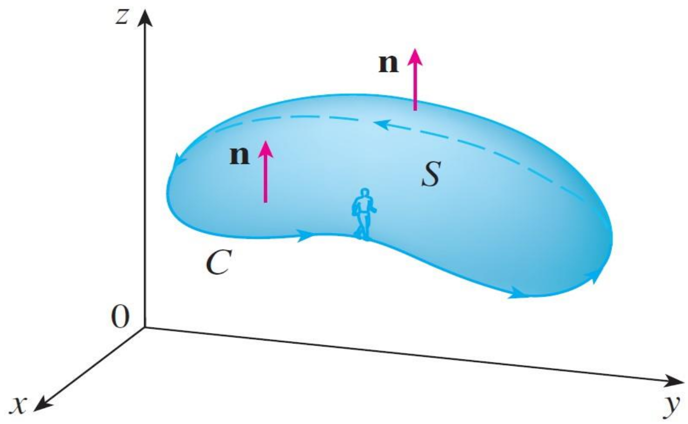
$$\iint_{S}\vec{F}.dS=\iint_{S_1}\vec{F}.dS+\iint_{S_2}\vec{F}.\text{dS}=-\pi+\pi=0$$

Stokes's theorem:

Stokes theorem is one of the important applications of the surface integral .
Stokes' theorem states that,

$$\int_C \vec{F}.ds = \iint_S curl\ \vec{F}.\vec{n}\ dS = \iint_S curl\ \vec{F}.\,dS$$

(Recall that a surface integral of a vector field is the integral of the component of the vector field perpendicular to the surface). We see that the integral on the tight is the surface integral of the vector field $curl\ F$ .Stokes theorem says that the surface integral of $curl\ F$ over a surface S (i.e., $\iint_S curl\ F.\,dS$) is the circulation of F around the boundary of the surface (i.e., $\int_C F.ds$ where $C = \partial S$). (For more details see [1] and [4])



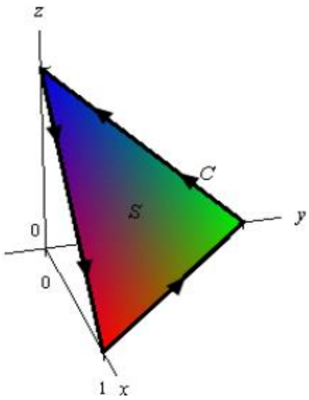
Figure(9)

Example3:

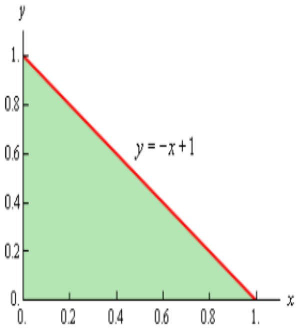
Use Stokes' Theorem to evaluate $\int_C \vec{F}.ds$ where $\vec{F} = \langle z^2, y^2, x \rangle$ and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ with counter-clockwise rotation.

Solution:

First, we will sketch the surface S figure(10-a), and region D in figure(10-b).



(a)



(b)

Figure(10)

Now, what we have is the boundary curve for the surface that we have need to use in the surface integral. However, as noted above all we need is any surface that has this as its boundary curve. So, let's use the following plane with upwards orientation for the surface.

Since the plane is oriented upwards this induces the positive direction on *C* as shown. The equation of this plane is

$$x + y + z = 1 \quad \Rightarrow \quad f\,(x, y, z) = x + y + z - 1$$

and

$$curl\ \vec{F} = \langle 0, 2z - 1, 0 \rangle$$

so by stokes's theorem

$$\int_C \vec{F}.ds = \iint_S curl\ \vec{F}.d\vec{S} = \iint_S curl\ \vec{F}.n\,dS$$

$$= \iint_D curl\ \vec{F}.\frac{\vec{\nabla} f}{\vec{\nabla} f\ .\vec{k}}\,dA$$

$$= \iint_D \langle 0, 2z - 1, 0 \rangle.\langle 1,1,1 \rangle dA$$

$$= \int_0^1 \int_0^{1-x} (1 - 2x - 2y)\,dydx$$

$$= -\frac{1}{6}$$

References:

1- Dawkins . P, Calculus III- Notes (Math 3435), Paul's Online Math Notes, ©2003- 2017.
2- Ladder .G, Notes on Surface Integrals, UNL Math , 2009.
3- Mattuck. A, Poonen. B, Shifrin. T, and LeDuc. S, V9. Surface Integrals, 18.02 Notes and Exercises, c M.I.T. 2010-2014.
4- Nykamp. D.Q, The Idea Behind Stokes' Theorem, *Math Insight*, part 22, 2374(2011). [5]- Stewart . J, Calculus, Brooks/Cole, Cengage Learning, Seventh Edition, © 2012, 2008.