Problem Set 3

Chapter 4

Monday, 11th of January 2016

Problem 1. As of today, in the English Isthmian League, Enfield Town have won 13 matches, drawn 4 and lost 11; Harrow Borough have won 10, drawn 6 and lost 10; and Hendon have won 8, drawn 7 and lost 15. In most association football leagues, including the Isthmian League, a team gets 3 points for a win, 1 point for a draw and 0 points for a loss. With a single matrix multiplication, calculate the number of points the above teams have.

Problem 2(a). The inverse of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by $\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. What can you spot that must be true for the inverse of a 2×2 matrix to exist? (*Hint: Focus on the scalar in front of the matrix*).

Problem 2(b). Suppose the condition you gave above is true. Show that the proposed matrix is indeed the inverse of a 2×2 matrix.

Problem 3(a). Consider the system of equations from Problem Set 2, question 4(a):

$$\begin{aligned} Q_d &= a - bP \\ Q_s &= -c + dP \end{aligned}$$

where Q_d represents quantity demanded, Q_s quantity supplied and P price. Letting $Q_s = Q_d$, write the above system in matrix form.

Problem 3(b). Using the formula for the inverse of a 2×2 matrix, find the equilibrium price and quantity. Make sure that this coincides with the result we found in Problem Set 2.

Problem 4. A famous anecdote of mathematician Carl Friedrich Gauss is that when he misbehaved in primary school, his teacher told him to add up all the numbers from 1 to 100 (inclusive). After just a few seconds, Gauss produced the correct answer: 5050. He did so by noticing that $1+100 = 101$, $2+99 = 101$, $3+98 = 101$ and so on, and that one can make fifty such pairs with the integers from 1 to 100. Thus, the total sum is $50 \times 101 = 5050$. Even though Gauss was precocious and overall a decent mathematician, the story is likely apocryphal. The following exercise will employ similar intuition and summation notation to find the result.

Problem 4(a). Notice that

$$\begin{aligned} 1+2+3+...+99+100 = & \\ &= (101-100)+(101-99)+...+(101-2)+(101-1) \\ &= (101-1)+(101-2)+...+(101-99)+(101-100) \end{aligned}$$

Write top and bottom sides of the above equality in summation notation (that is, using the symbol $\sum_{n=1}^{N}$). Expanding parentheses and knowing that $\sum_{n=1}^{N} x = Nx$, show that Gauss was right.

Problem 4(b). *Harder:* Can you find the general formula for the sum of all integers from 1 to N? Verify that this formula produces the desired result for the first 100 integers.

Problem 5(a). Let $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $v_3 = \begin{pmatrix} 11 \\ 7 \end{pmatrix}$. Find scalars λ_1 and λ_2 such that $\lambda_1 v_1 + \lambda_2 v_2 = v_3$ (*Suggestion: There are many ways to do this, but try using matrices and matrix inverses*).

Problem 5(b). Having answered the above, can the set of vectors v_1, v_2, v_3 be a basis?

Problem 6(a). Consider a table with x fair coins on it. All of the coins showing heads are placed to one side of the table and all of the coins showing tails are placed on the opposite side. You go to the heads side and upturn 1/4 of them so that they show tails. Without moving the upturned coins yet, you go to the tails side and flip all the coins on that side. Only after having done this, you reorganise the coins, partitioning the heads and tails to their respective side. What is the Markov transition matrix?

Problem 6(b). Suppose you start with all x coins heads up, and repeat the above process a few times. Notice that the numbers of coins showing heads and tails seems to be getting closer and closer to a certain proportion. Can you make an informed guess as to what the steady state is? Verify that this is indeed the steady state.

Problem 7. Linear algebra is useful for systems of linear equations. Can you cite an important concept in economics, often called a "law", that makes linear equations unlikely to accurately model many economic phenomena?