Communicating Mathematics III

Theory and Applications of the Fibonacci p-numbers

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Abstract

The theory of the Fibonacci p-numbers, one of many generalisations of the Fibonacci numbers, is fascinating. An overview of this theory is given, before two recent and very contrasting applications are explored and analysed.

This piece of work is a result of my own work except where it forms an assessment based on group project work. In the case of a group project, the work has been prepared in collaboration with other members of the group. Material from the work of others not involved in the project has been acknowledged and quotations and paraphrases suitably indicated.

Contents

Introduction

The Fibonacci numbers are a widely acclaimed sequence of integers generated by a recursion relation, where the n^{th} term is the sum of the previous two terms. Although Indian mathematicians such as Pingala were already aware of their existence, the Fibonacci numbers are credited to Leonardo of Pisa, the Italian mathematician who introduced the western world to the Hindu-Arabic numerals. Leonardo, or Fibonacci as he is commonly known, discussed the Fibonacci numbers in his 1202 book *Liber Abaci* as the solution to a problem concerning a rabbit population. While the sequence may seem juvenile at first, mathematicians have been fascinated for centuries by its underlying beauty.

The Golden Ratio, the limit of the ratio of successive Fibonacci numbers, has itself been of particular interest over the years. There are numerous claims regarding appearances of the Golden Ratio in the world around us, many of which are disputed. Some suggest that geometric objects displaying the Golden Ratio are the most aesthetically pleasing and that the Golden Ratio appears in Egyptian pyramids, the Acropolis in Athens and Da Vinci's 'The Vetruvian Man' to name just a few. There have also been suggestions that the Golden Ratio can be used to predict changes in stock market prices and even the proposition that the Golden Ratio has special significance to the state of Illinois in the United States of America! [18] Many of these claims lack sufficient supporting evidence. There are however many proven applications of the Fibonacci numbers and the Golden Ratio in areas ranging from Numerical Analysis to Theoretical Physics, and Fibonacci number theory has often proved unexpectedly useful - the appearance of the Fibonacci numbers in the 'negative solution' to Hilbert's Tenth Problem is a notable example. [21]

This report focuses upon the theory and applications of the Fibonacci p-numbers, of which the Fibonacci numbers are a special case. The aim of the first chapter is to familiarise the reader with the core material which is required in subsequent chapters. In particular, the origins of the Fibonacci p-numbers and the Golden p-Proportion are discussed, and properties and identities, which appear analagously in later chapters, are proved. The second chapter introduces the Fibonacci (p, m)-numbers, a generalisation of the Fibonacci p-numbers. It is shown that there exists a closed form expression for the n^{th} Fibonacci (p, m)-number. Using this closed form expression, Fibonacci theory is extended from a discrete to a continuous domain in the third chapter. Some interesting curves and surfaces are derived and a potential application in physical cosmology is discussed. Fibonacci Coding Theory, a very contrasting application of the Fibonacci p-numbers, is analysed in the final chapter. Research of this topic is in its infancy and as such, the material seen here is the interleaving of published research and my own thoughts and analysis.

Chapter 1

The Generalised Golden Section

The primary aim of this chapter is to show the mathematical elegance of the Fibonacci p-numbers and their relation to the Golden p-Section, as well as providing a framework to build upon in later chapters. Properties and identities of these numbers, which appear analagously in subsequent chapters, are proved.

1.1 The Fibonacci p-numbers

Consider the following problem, adapted from [26]:

Newborn rabbit couples take p months to mature into adult couples and then produce one newborn couple per month from month $p+1$ onwards. All newborn couples consist of one male and one female, and rabbits are assumed to be immortal. If a newborn couple is put in an enclosed place in the first month, how many rabbit couples will there be in total in the n^{th} month?

We will look for a general solution to this problem.

Let b_m denote a rabbit couple that has been maturing for m months with $0 \leq m \leq p$. b_0 is a newborn couple and $A = b_p$ is an adult couple. Then the intermediate stages of maturing, dependent on the value of p, are:

$$\begin{array}{ccc} b_0 & \longrightarrow & b_1 \\ b_1 & \longrightarrow & b_2 \\ b_2 & \longrightarrow & b_3 \\ \vdots & & \vdots \\ b_{p-1} & \longrightarrow & A \end{array}$$

For the case $p = 0$, newborn rabbit couples immediately mature into adult couples and then produce babies one month later. Therefore the reproduction process is modelled by $A \to AA$ (an adult rabbit couple 'becomes' two different adult rabbit couples), and the rabbit population doubles each month. This is a trivial case with little application to the life sciences as organisms take at least a little time to mature. As a result we will consider cases where $p > 0$.

For example, when $p = 3$ the intermediate stages from a newborn to an adult couple are:

$$\begin{array}{ccc} b_0 & \longrightarrow & b_1 \\ b_1 & \longrightarrow & b_2 \\ b_2 & \longrightarrow & A \end{array}$$

n	Rabbit Couples					A	b_0	b_1	b_2	Total
1	b_0					0	1	0	0	1
2	b_1					0	0	1	0	1
3	b_2					0	0	0	1	1
4	A					1	0	0	0	1
5	$A\,b_0$					1	1	0	0	2
6	$A\,b_0\,b_1$					1	1	1	0	3
7	$A\,b_0\,b_1\,b_2$					1	1	1	1	4
8	$A\,b_0\,b_1\,b_2\,A$					2	1	1	1	5
9	$A\,b_0\,b_1\,b_2\,A$	$A\,b_0$				3	2	1	1	7
10	$A\,b_0\,b_1\,b_2\,A$	$A\,b_0$	$A\,b_0\,b_1$			4	3	2	1	10
11	$A\,b_0\,b_1\,b_2\,A$	$A\,b_0$	$A\,b_0\,b_1$	$A\,b_0\,b_1\,b_2$		5	4	3	2	14
12	$A\,b_0\,b_1\,b_2\,A$	$A\,b_0$	$A\,b_0\,b_1$	$A\,b_0\,b_1\,b_2$	$A\,b_0\,b_1\,b_2\,A$	7	5	4	3	19

Table 1.1: The 'rabbit problem' over the first 12 months when $p = 3$.

Table 1.1 shows the growth of the rabbit population over the first 12 months. The total number of rabbits is decomposed into newborn couples b_0, adult couples A, and the couples in the intermediate stages b_1 and b_2.

When $p = 3$, the total number of rabbit couples follows the sequence

$$1, 1, 1, 1, 2, 3, 4, 5, 7, 10, 14, 19 \ldots$$

Notice that this is the arithmetic sequence corresponding to the recursion relation

$$G(n) \;=\; G(n-1) + G(n-4) \qquad\qquad n \in \mathbb{Z},$$

with $G(1) = G(2) = G(3) = G(4) = 1$. In other words the n^{th} term of the sequence is found by summing the previous term with the term three previous to that, and the first four terms of the sequence are all equal to 1. This sequence can also be seen in the columns for A, b_0, b_1 and b_2.

Similarly, $p = 4$ gives the sequence

$$1, 1, 1, 1, 1, 2, 3, 4, 5, 6, 8, 11, 15, 20 \ldots$$

which is generated by the recursion relation

$$H(n) \;=\; H(n-1) + H(n-5) \qquad\qquad n \in \mathbb{Z},$$

with $H(1) = H(2) = H(3) = H(4) = H(5) = 1$.

Upon inspection of sequences for further values of p, we can conclude that in general, the total number of rabbit couples in month n is the n^{th} Fibonacci p-number [26]:

Definition. *(The Fibonacci p-numbers)*
For any integer $p \geq 0$ and $n \geq p+2$, the n^{th} Fibonacci p-number is given by the recursion relation

$$F_p(n) \;=\; F_p(n-1) + F_p(n-p-1) \qquad\qquad n \in \mathbb{Z}, \tag{1.1}$$

with the initial conditions $F_p(1) = F_p(2) = \cdots = F_p(p) = F_p(p+1) = 1$.

A visual representation of (1.1) below shows that the n^{th} Fibonacci p-number is the sum of the previous term and the term p previous to that:

The Fibonacci p-numbers also extend to negative values of n. By taking $n = p + 1$ in the recursion relation (1.1) we have

$$F_p(p+1) \;=\; F_p(p) + F_p(0).$$

As the initial conditions in the definition above give $F_p(p) = F_p(p+1) = 1$, then $F_p(0) = 0$.

As there are $p + 1$ initial conditions for (1.1), we can use the same method to find the first p Fibonacci p-numbers for $n \leq 0$:

$$F_p(0) \;=\; F_p(-1) \;=\; F_p(-2) \;=\; \ldots \;=\; F_p(-p+1) = 0 \tag{1.2}$$

These are in essence the initial conditions for the negative Fibonacci p-numbers.

Now, rearranging the recursion relation (1.1) gives $F_p(n - p - 1) = F_p(n) - F_p(n - 1)$. A shift by $p + 1$ terms gives the following equivalent recursion relation:

$$F_p(n) \;=\; F_p(n+p+1) - F_p(n+p), \qquad n \leq -p.$$

This recursion relation and the initial conditions (1.2) enable us to find that:

$$F_p(-p) \;=\; F_p(1) - F_p(0) \;=\; 1. \tag{1.3}$$

Continuing the sequence gives

$$F_p(-p-1) \;=\; F_p(-p-2) \;=\; \ldots \;=\; F_p(-2p+1) = 0, \tag{1.4}$$

and

$$F_p(-2p) \;=\; F_p(-p+1) - F_p(-p) \;=\; -1. \tag{1.5}$$

This continues for all negative values of n. We will not focus on Fibonacci p-numbers for negative n, but they will prove useful later. Table 1.2 gives the Fibonacci p-numbers for $p = 1, 2, 3, 4, 5$ and $-9 \leq n \leq 8$. [31]

n	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
$F_1(n)$	34	-21	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13	21
$F_2(n)$	2	0	-2	1	1	-1	0	1	0	0	1	1	1	2	3	4	6	9
$F_3(n)$	1	0	1	-1	0	0	1	0	0	0	1	1	1	1	2	3	4	5
$F_4(n)$	1	-1	0	0	0	1	0	0	0	0	1	1	1	1	1	2	3	4
$F_5(n)$	-1	0	0	0	1	0	0	0	0	0	1	1	1	1	1	1	2	3

Table 1.2: The extended Fibonacci p-numbers for $p = 1, 2, 3, 4, 5$. [31]

Given the highly idealised nature of the generalised rabbit problem it may be difficult to forsee its application on a wider scale in the life sciences. However note that for all cases where $p > 0$, the reproduction process modelled by

$$A \longrightarrow A\, b_0$$

is asymmetric, as the adult rabbit couple A 'becomes' two different couples A and b_0. This has similarities to the asymmetric binary cell division common in bacteria, insects and plants. In [24] Spears and Bicknell-Johnson analyse models of cell growth based on the Fibonacci 2- and 3-numbers and conclude that the models "provide rational bases for the occurrence of Fibonacci and other recursive phyllotaxis and patterning in biology".[26]

However the overriding importance of the generalised rabbit problem is that it produces the Fibonacci p-numbers which have numerous applications in many varied disciplines other than the life sciences. Notable cases are the discovery of a new measurement theory, a new theory of real numbers, an application in electrical engineering and a coding theory which is discussed in detail in Chapter 4 of this report.

1.2 Q_p matrices

The Fibonacci p-numbers can be represented in the following form:

$$\underbrace{\begin{pmatrix} F_p(n) \\ F_p(n-1) \\ F_p(n-2) \\ \vdots \\ F_p(n-p+1) \\ F_p(n-p) \end{pmatrix}}_{\mathbf{f_n}} = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}}_{Q_p^T} \underbrace{\begin{pmatrix} F_p(n-1) \\ F_p(n-2) \\ F_p(n-3) \\ \vdots \\ F_p(n-p) \\ F_p(n-p-1) \end{pmatrix}}_{\mathbf{f_{n-1}}} \tag{1.6}$$

where $\mathbf{f_n}$ and $\mathbf{f_{n-1}}$ are $(p+1) \times 1$ vectors and Q_p^T is the transpose of the Fibonacci Q_p matrix:

$$Q_p = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{1.7}$$

Q_p is a $(p+1) \times (p+1)$ matrix which contains a $p \times p$ identity matrix in its upper right corner. Regardless of the value of p, the first column always begins and ends with a 1, and has zeros elsewhere. The last row always begins with a 1 and has zeros in all other positions. [31]

To see that (1.6) does represent the Fibonacci p-numbers, apply matrix multiplication to give the recursion relation for the Fibonacci p-numbers (1.1) and the following trivial expressions:

$$\begin{aligned} F_p(n-1) &= F_p(n-1), \\ F_p(n-2) &= F_p(n-2), \\ &\vdots \\ F_p(n-p) &= F_p(n-p). \end{aligned}$$

The Fibonacci Q_p matrices for the cases $p = 0, 1, 2, 3$ are:

$$Q_0 = (1), \quad Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Perhaps the reason why the Q_p matrix is intriguing, and encourages mathematicians to explore its properties, is the direct relation between the Fibonacci p-numbers and the elements of the Q_p^n matrix. With the help of Table 1.2 we can see a pattern emerging when we consider the powers of the Q_1 matrix for example:

$$Q_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} F_1(2) & F_1(1) \\ F_1(1) & F_1(0) \end{pmatrix}, \quad Q_1^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} F_1(3) & F_1(2) \\ F_1(2) & F_1(1) \end{pmatrix},$$

$$Q_1^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} F_1(4) & F_1(3) \\ F_1(3) & F_1(2) \end{pmatrix}, \quad Q_1^4 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} F_1(5) & F_1(4) \\ F_1(4) & F_1(3) \end{pmatrix},$$

and the powers of the Q_2 matrix:

$$Q_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} F_2(2) & F_2(1) & F_2(0) \\ F_2(0) & F_2(-1) & F_2(-2) \\ F_2(1) & F_2(0) & F_2(-1) \end{pmatrix},$$

$$Q_2^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} F_2(3) & F_2(2) & F_2(1) \\ F_2(1) & F_2(0) & F_2(-1) \\ F_2(2) & F_2(1) & F_2(0) \end{pmatrix},$$

$$Q_2^3 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} F_2(4) & F_2(3) & F_2(2) \\ F_2(2) & F_2(1) & F_2(0) \\ F_2(3) & F_2(2) & F_2(1) \end{pmatrix},$$

$$Q_2^4 = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} F_2(5) & F_2(4) & F_2(3) \\ F_2(3) & F_2(2) & F_2(1) \\ F_2(4) & F_2(3) & F_2(2) \end{pmatrix}.$$

In general the n^{th} power of the Q_p matrix takes the following form [31]:

Theorem. *For any integer $p \geq 1$ and $n \in \mathbb{Z}$, the n^{th} power of the Fibonacci Q_p matrix is given by:*

$$Q_p^n = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix}, \qquad (1.8)$$

where $F_p(n)$ is the n^{th} Fibonacci p-number.

Proof. We will use induction to prove this theorem[1].

Firstly, using (1.1)–(1.5), we show that it is true for the simple case when $n = 1$:

$$Q_p^1 = \begin{pmatrix} F_p(2) & F_p(1) & \cdots & F_p(3-p) & F_p(2-p) \\ F_p(2-p) & F_p(1-p) & \cdots & F_p(3-2p) & F_p(2-2p) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_p(0) & F_p(-1) & \cdots & F_p(1-p) & F_p(-p) \\ F_p(1) & F_p(0) & \cdots & F_p(2-p) & F_p(1-p) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

This is the Q_p matrix which was defined earlier (1.7). Therefore the theorem holds true for the case $n = 1$.

Now assume that it also holds for Q_p^{n-1}:

$$Q_p^{n-1} = \begin{pmatrix} F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \\ F_p(n-p) & F_p(n-p-1) & \cdots & F_p(n-2p+1) & F_p(n-2p) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_p(n-2) & F_p(n-3) & \cdots & F_p(n-p-1) & F_p(n-p-2) \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \end{pmatrix}.$$

We now use that $Q_p^n = Q_p^{n-1} Q_p$:

$Q_p^{n-1} Q_p =$

$$\begin{pmatrix} F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \\ F_p(n-p) & F_p(n-p-1) & \cdots & F_p(n-2p+1) & F_p(n-2p) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_p(n-2) & F_p(n-3) & \cdots & F_p(n-p-1) & F_p(n-p-2) \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} F_p(n) + F_p(n-p) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p) + F_p(n-2p) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_p(n-2) + F_p(n-p-2) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) + F_p(n-1) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix}$$

$$= \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix}.$$

This is the Q_p^n matrix, hence the theorem is proved. □

[1]The original proof of this theorem can be found in [25]. Unfortunately I was unable to locate this source, so the proof shown here is my own.

Remark. Q_p^n is a $(p+1)\times(p+1)$ matrix where the first row is made from a decreasing Fibonacci p-sequence of length $p+1$, starting with the term $F_p(n+1)$:

$$\ldots, F_p(n+1), F_p(n), \ldots, F_p(n-p+2), F_p(n-p+1), \ldots\ .$$

The second row consists of the sequence above shifted forwards by p terms. All subsequent rows are found by shifting the sequence in the previous row backwards by a term.

The application of the Fibonacci p-numbers in Fibonacci Coding Theory, discussed later in this report, uses the Q_p^n matrix to a great extent and in particular requires the following identity involving its determinant. The proof is adapted from [31].

Theorem. *For any integer $p \geq 0$ and $n \in \mathbb{Z}$,*

$$\det Q_p^n \;=\; (-1)^{pn}. \tag{1.9}$$

Proof. Firstly we will consider the determinant of the Q_p matrix, aided by the following matrix theory from [15].

Let A be a square matrix and let a_{ij} denote the matrix element in the i^{th} row and j^{th} column of A. The cofactor A_{ij} of a_{ij} in the expansion of $\det A$ *is $(-1)^{i+j}$ times the determinant of the submatrix of A obtained by deleting the i^{th} row and j^{th} column of A.*

Consider the matrices:

$$Q_2 \;=\; \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_3 \;=\; \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Q_2 is a submatrix of Q_3 obtained by deleting row 4 and column 5 of Q_3. Using the matrix theory above, the cofactor of the element in the 4^{th} row and 5^{th} column of Q_3 is $(-1)^{4+5}$ times the determinant of Q_2. As the element in the 4^{th} row and 5^{th} column of Q_3 is 1 and there are zeros elsewhere in that row and column:

$$\det Q_3 \;=\; -\det Q_2.$$

By analogy we can prove that

$$\det Q_2 \;=\; -\det Q_1.$$

We can easily calculate that $\det Q_0 = 1$ and $\det Q_1 = -1$. Therefore we find that $\det Q_2 = 1$, $\det Q_3 = -1$ and so on. In general for integer $p \geq 0$:

$$\det Q_p \;=\; (-1)^p. \tag{1.10}$$

Now, using matrix theory [2] and (1.10):

$$\det Q_p^n \;=\; (\det Q_p)^n \;=\; (-1)^{pn}.$$

Hence, we have proved the Theorem. □

1.3 The Fibonacci numbers

We now look in more detail at a particular case of the Fibonacci p-numbers. The Fibonacci 1-numbers are generated if p, the number of months in which a newborn rabbit couple matures, is taken to be one. This was the original 'rabbit problem' discussed by Fibonacci in Liber Abaci. The problem solved in §1.1 is simply a generalisation of Fibonacci's rabbit problem where rabbits can take any number of months to mature. From now on we will drop the 1 and refer to these numbers by their more common name, the Fibonacci numbers, and denote them by $\{F(n)\}$.

So by setting $p = 1$ in the recursion relation (1.1) we obtain:

$$F(n) \;=\; F(n-1) + F(n-2), \qquad n \in \mathbb{Z} \tag{1.11}$$

with the initial conditions $F(1) = F(2) = 1$.

This recurrence relation produces the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

where each term for $n \geq 3$ is the sum of the previous two terms. Extending the Fibonacci numbers for $n \leq 0$, as in Table 1.2, reveals the interesting property [10]:

$$F(-n) = (-1)^{n+1} F(n). \tag{1.12}$$

There are many more identities and properties of the Fibonacci numbers, so many that one could probably devote an entire report to them. However the following identities appear frequently throughout the applications later in this report, and often in a generalised form. Therefore it is useful to see and be able to prove these identities for the basic case $p = 1$. For these proofs we will alter our notation for the Fibonacci numbers slightly so that $F(n)$ is written as F_n. This is simply for clarity.

Theorem (Cassini's identity)**.**

$$F_{n-1}F_{n+1} - F_n^2 \;=\; (-1)^n. \tag{1.13}$$

Proof. Earlier we proved that $\det Q_p^n = (-1)^{pn}$, (1.9). Note that when $p = 1$ this reduces to

$$\det Q_1^n = (-1)^n. \tag{1.14}$$

From (1.8) we can see that

$$Q_1^n \;=\; \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix},$$

and therefore

$$\det Q_1^n \;=\; F_{n-1}F_{n+1} - F_n^2.$$

Using (1.14), it follows that

$$F_{n-1}F_{n+1} - F_n^2 \;=\; (-1)^n.$$

□

Remark. Since Cassini's identity (1.13) is equivalent to $\det Q_1^n = (-1)^n$, which is a particular case of (1.9), we are lead to conclude that $\det Q_p^n = (-1)^{pn}$ is in fact a generalised form of the Cassini identity. [31]

The Cassini identity is a special case of Catalan's identity [22]:

Theorem (Catalan's identity)**.** *For positive integers n and r, with $n > r$:*

$$F_n^2 - F_{n+r}F_{n-r} \;=\; (-1)^{n-r}F_r^2. \tag{1.15}$$

To prove Catalan's identity, we require the following two lemmas[2]:

Lemma 1. *For positive integers a and b with $a > 1$, the following identity holds:*

$$F_{a+b} \;=\; F_aF_{b+1} + F_{a-1}F_b. \tag{1.16}$$

Proof. We will prove this lemma by induction on b.

For the simple case when $b = 1$:

$$\begin{aligned} F_{a+1} &= F_aF_2 + F_{a-1}F_1 \\ &= F_a + F_{a-1}, \end{aligned}$$

which is true by (1.11).

Now assume that the identity is true for $b < b_0$, where $b_0 \geq 2$ is an integer, and consider the case when $b = b_0$. Then

$$F_aF_{b_0+1} + F_{a-1}F_{b_0} \;=\; F_a(F_{b_0} + F_{b_0-1}) + F_{a-1}(F_{b_0-1} + F_{b_0-2}) \tag{1.17}$$

$$\;=\; (F_aF_{b_0} + F_{a-1}F_{b_0-1}) + (F_aF_{b_0-1} + F_{a-1}F_{b_0-2}). \tag{1.18}$$

Using the induction hypothesis (1.16), with $b = b_0 - 1$ and $b = b_0 - 2$, (1.18) is equivalent to:

$$\begin{aligned} F_aF_{b_0+1} + F_{a-1}F_{b_0} &= F_{a+b_0-1} + F_{a+b_0-2} \\ &= F_{a+b_0}. \end{aligned}$$

□

Lemma 2. *For all positive integers $t \geq 2$, the following identity holds:*

$$F_{t-1}^2 + F_tF_{t-1} - F_t^2 \;=\; (-1)^t. \tag{1.19}$$

Proof. Again, we will prove by induction.

For $t = 2$, we use (1.11) to show that

$$\begin{aligned} F_1^2 + F_2F_1 - F_2^2 &= 1^2 + 1 \cdot 1 - 1^2 \\ &= (-1)^2. \end{aligned}$$

Therefore the identity holds for $t = 2$.

Assume that the identity is true for all t satisfying $2 \leq t < t_0$, where $t_0 \in \mathbb{Z}$. Then using (1.11):

$$F_{t_0-1}^2 + F_{t_0}F_{t_0-1} - F_{t_0}^2 \;=\; F_{t_0-1}^2 + (F_{t_0-1} + F_{t_0-2})\,F_{t_0-1} - (F_{t_0-1} + F_{t_0-2})^2\,.$$

[2] The proofs given by [22] have errors. These proofs are my own corrected versions.

Expanding the squared bracket and simplifying gives:

$$\begin{aligned} F_{t_0-1}^2 + F_{t_0}F_{t_0-1} - F_{t_0}^2 &= F_{t_0-1}^2 - F_{t_0-1}F_{t_0-2} - F_{t_0-2}^2 \\ &= -\left(F_{t_0-2}^2 + F_{t_0-1}F_{t_0-2} - F_{t_0-1}^2\right). \end{aligned}$$

By the inductive hypothesis (1.19), with $t = t_0 - 1$:

$$\begin{aligned} -(F_{t_0-2}^2 + F_{t_0-1}F_{t_0-2} - F_{t_0-1}^2) &= (-1)(-1)^{t_0-1} \\ &= (-1)^{t_0}. \end{aligned}$$

□

Now we can prove Catalan's identity:

$$F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2.$$

Proof. Make the substitutions $x = n - r$ and $a = r$ to give:

$$F_{x+a}^2 - F_{x+2a}F_x = (-1)^x F_a^2.$$

Taking the left side and applying Lemma 1 gives

$$\begin{aligned} F_{x+a}^2 - F_{x+2a}F_x &= (F_xF_{a+1} + F_{x-1}F_a)^2 - (F_xF_{2a+1} + F_{x-1}F_{2a})F_x \\ &= F_x^2F_{a+1}^2 + 2F_xF_{a+1}F_{x-1}F_a + F_{x-1}^2F_a^2 - (F_xF_{a+1+a} + F_{x-1}F_{a+a})F_x. \end{aligned}$$

Applying Lemma 1 again results in

$$\begin{aligned} F_{x+a}^2 - F_{x+2a}F_x &= F_x^2F_{a+1}^2 + 2F_xF_{a+1}F_{x-1}F_a + F_{x-1}^2F_a^2 \\ &\quad - F_x\Big(F_x(F_{a+1}F_{a+1} + F_aF_a) + F_{x-1}(F_aF_{a+1} + F_{a-1}F_a)\Big). \end{aligned}$$

Simplifying gives

$$F_{x+a}^2 - F_{x+2a}F_x = F_xF_{x-1}F_a(F_{a+1} - F_{a-1}) + F_a^2(F_{x-1}^2 - F_x^2).$$

Using the recursion relation (1.11) we have that

$$F_a = F_{a+1} - F_{a-1}.$$

Therefore

$$\begin{aligned} F_{x+a}^2 - F_{x+2a}F_x &= F_xF_{x-1}F_a(F_a) + F_a^2(F_{x-1}^2 - F_x^2) \\ &= F_a^2(F_{x-1}^2 + F_xF_{x-1} - F_x^2). \end{aligned}$$

By Lemma 2,

$$F_{x+a}^2 - F_{x+2a}F_x = (-1)^x F_a^2.$$

□

1.4 The Golden p-Section

In *The Elements*, Euclid proposed the problem of the division of a line into *extreme and mean ratio.* In this section a generalisation of Euclid's problem is discussed, and its relation to the Fibonacci p-numbers becomes clear later in this chapter.

Definition. The point B divides a line AC in a *Golden p-Section* if

$$\frac{AB}{BC} = \left(\frac{AC}{AB}\right)^p \qquad p \in \mathbb{Z}_{\geq 0}. \tag{1.20}$$

In other words, the ratio of the longer segment to the shorter segment is equal to the p^{th} power of the ratio of the total length of the line to the longer segment.

There are infinite integer values for $p \geq 0$, so there are infinite ways to divide the line in a Golden p-Section (1.20). The line divisions for the first five values of p are shown in Figure 1.1.

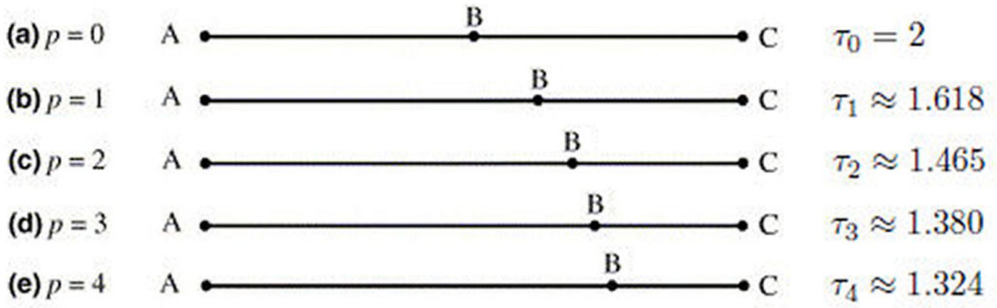


Figure 1.1: The Golden p-Sections for $p = 0, 1, 2, 3, 4$. [26]

Recall the generalised rabbit problem in §1.1, where p is the number of months a newborn rabbit couple takes to mature into an adult couple. For $p = 0$, the reproduction process is symmetric ($A \to AA$). But for $p > 0$, the reproduction process is asymmetric ($A \to A\,b_0$). The same occurs with the Golden p-section above - the line division is symmetric about B when $p = 0$ and asymmetric for $p > 0$.

We now wish to find the ratio $\frac{AC}{AB}$ which satisfies the Golden p-Section (1.20). Arbitrarily setting the length $AB = 1$ and the length $AC = x$ gives $BC = x - 1$. Substituting into (1.20) gives

$$\frac{1}{x-1} = x^p$$

which, when rearranged, is the *golden algebraic equation*

$$x^{p+1} - x^p - 1 = 0. \tag{1.21}$$

This is an equation of the $(p+1)^{th}$ degree and therefore has $p + 1$ roots which we shall denote $x_1, x_2, \ldots, x_p, x_{p+1}$. We can show that all of these roots are distinct. [16]

Theorem. *The golden algebraic equation of the Fibonacci p-numbers, $x^{p+1} - x^p - 1 = 0$, does not have multiple roots.*

Proof. We will prove this theorem by contradiction[3].

Let $f(z) = z^{p+1} - z^p - 1$ and suppose that α is a multiple root of $f(z) = 0$. Note that $f(z) \neq 0$ when $\alpha = 0$ and $\alpha = 1$, so they are not possible solutions.

Since α is a multiple root the following must hold:

$$f(\alpha) = \alpha^{p+1} - \alpha^p - 1 = 0,$$

$$f'(\alpha) = (p+1)\alpha^p - p\alpha^{p-1} = 0.$$

Rearranging and factorising these gives

$$f(\alpha) = \alpha^p(\alpha - 1) - 1 = 0, \tag{1.22}$$

$$f'(\alpha) = \alpha^{p-1}((p+1)\alpha - p) = 0. \tag{1.23}$$

We know that $\alpha = 0$ and $\alpha = 1$ are not solutions. Therefore $\alpha = \frac{p}{p+1}$ from (1.23).

Substituting $\alpha = \frac{p}{p+1}$ into (1.22) gives

$$\left(\frac{p}{p+1}\right)^p \left(\frac{p}{p+1} - 1\right) - 1 = 0$$

$$\iff \left(\frac{p}{p+1}\right)^p \left(\frac{p-(p+1)}{p+1}\right) = 1$$

$$\iff -\frac{1}{p+1}\left(\frac{p}{p+1}\right)^p = 1.$$

This implies that

$$\frac{1}{p+1}\left(\frac{p}{p+1}\right)^p = -1.$$

This is a contradiction as $\frac{1}{p+1}(\frac{p}{p+1})^p$ is positive for all p (recall that p is defined for integers greater than or equal to zero). Hence, the equation $f(z)$ does not have multiple roots. ☐

Here are some examples showing the roots of the algebraic equation found for the cases $p = 0, 1, 2$:

Example 1. *The golden algebraic equation of degree 1*
When $p = 0$, (1.21) reduces to $x - 1 - 1 = 0$, so $x_1 = 2$. This corresponds to case a) in Figure 1.1 above, where $\frac{AC}{AB} = 2$.

Example 2. *The golden algebraic equation of degree 2*
When $p = 1$ we have the equation $x^2 - x - 1 = 0$. This is well known as the Fibonacci quadratric equation which is derived from the original problem of dividing a line into *extreme and mean ratio* (see case b) in Figure 1.1. Using the quadratic formula $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ we find the two roots to be

$$x_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad x_2 = \frac{1-\sqrt{5}}{2}. \tag{1.24}$$

[3] The proof in [16] uses a lemma for which the proof is false. However the proof of the theorem itself does not actually require use of the lemma at all. Therefore the proof shown here is my adapted version.

The positive root x_1 is traditionally denoted by the greek letter τ (although in some texts it is ϕ) and is commonly called the *Golden Ratio*. The negative root x_2 is equal to $-\frac{1}{\tau}$ and the ratios $\frac{AC}{AB}$ and $\frac{AB}{BC}$ are both equal to τ. [10]

The Golden Ratio $\tau = 1.61803399\ldots$ is argued by some to be the "most irrational of irrational numbers" as when written in continued fraction[4] form it converges to a single value the slowest. In Mario Livio's words, "the Golden Ratio is farther away from being expressible as a fraction than any other irrational number". [20]

Example 3. *The golden algebraic equation of degree 3*
For $p = 2$, (1.21) reduces to

$$x^3 - x^2 - 1 = 0. \tag{1.25}$$

The three roots of a monic cubic equation of the form $x^3 + ax^2 + c = 0$ are found using the formulae (see Appendix A):

$$\begin{aligned} x_1 &= -\frac{1}{3}\left(a + \sqrt[3]{\frac{m+\sqrt{n}}{2}} + \sqrt[3]{\frac{m-\sqrt{n}}{2}}\right), \\ x_2 &= -\frac{1}{3}\left(a + \omega_2\sqrt[3]{\frac{m+\sqrt{n}}{2}} + \omega_1\sqrt[3]{\frac{m-\sqrt{n}}{2}}\right), \\ x_3 &= -\frac{1}{3}\left(a + \omega_1\sqrt[3]{\frac{m+\sqrt{n}}{2}} + \omega_2\sqrt[3]{\frac{m-\sqrt{n}}{2}}\right), \end{aligned} \tag{1.26}$$

where

$$\begin{aligned} m &= 2a^3 + 27c, & n &= m^2 - 4a^6, \\ \omega_1 &= -\frac{1}{2} + \frac{1}{2}\sqrt{3}i, & \omega_2 &= -\frac{1}{2} - \frac{1}{2}\sqrt{3}i. \end{aligned}$$

Since there are three possible values for each cube root $r = \sqrt[3]{\frac{m+\sqrt{n}}{2}}$ and $s = \sqrt[3]{\frac{m-\sqrt{n}}{2}}$ we must choose r and s such that $rs = a^2$.

So for our equation $x^3 - x^2 - 1 = 0$ we have $a = c = -1$. Using the formulas above we find that:

$$\begin{aligned} x_1 &= \frac{h^2 + 2h + 4}{6h} &&= 1.4655712319\ldots, \\ x_2 &= -\frac{h^2 - 4h + 4}{12h} + i\sqrt{3}\left(\frac{h}{12} - \frac{1}{3h}\right) &&= -0.233\cdots - (0.793\ldots)i, \\ x_3 &= -\frac{h^2 - 4h + 4}{12h} - i\sqrt{3}\left(\frac{h}{12} - \frac{1}{3h}\right) &&= -0.233\cdots + (0.793\ldots)i, \end{aligned}$$

[4] A continued fraction is of the form $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$ where $a_0, a_1, a_2, \ldots$ are positive integers, except perhaps for a_0. The Golden Ratio as a continued fraction is $\tau = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}$. [5]

where $h = \sqrt[3]{116 + 12\sqrt{93}}$. [28]

It is clear that the root x_1 is positive, real and irrational. x_2 and x_3 are complex conjugates of the form $z = a - bi$ and $z = a + bi$ respectively, and since h is an irrational number the real parts of x_2 and x_3 are irrational.

1.5 The Golden p-Proportion

Note that in all three of our examples there is exactly one root which is both positive and real, and without loss of generality we have let x_1 be this root. In fact it is true in general that of all the roots of the golden algebraic equation (1.21), only one is both positive and real for each value of p. [28]

This root can also be denoted by τ_p and is called the *Golden p-Proportion* as it corresponds to the ratio $\frac{AC}{AB}$ which satisfies the Golden p-Section above. The approximate values of τ_p for $p = 0, 1, 2, 3, 4$ are shown in Figure 1.1 above, and they suggest that τ_p monotonically decreases as p increases. We can see geometrically that $\tau_p \to 1$ as $p \to \infty$, since the length of the AB will get closer and closer to the length of AC as p becomes large.

§1.1 and §1.4 may initially seem unrelated. However the link between the Fibonacci p-numbers and the Golden p-Section becomes clear if we consider the limit of the ratio of two consecutive Fibonacci p-numbers as n tends to infinity [28]:

$$\lim_{n\to\infty} \frac{F_p(n)}{F_p(n-1)} \;=\; x. \tag{1.27}$$

We know from (1.1) that

$$\begin{aligned} \frac{F_p(n)}{F_p(n-1)} &= \frac{F_p(n-1) + F_p(n-p-1)}{F_p(n-1)} \\ &= 1 + \frac{F_p(n-p-1)}{F_p(n-1)}. \end{aligned}$$

Since

$$\frac{F_p(n-p-1)}{F_p(n-1)} \;\equiv\; \frac{F_p(n-2)\cdot F_p(n-3)\cdots F_p(n-p-1)}{F_p(n-1)\cdot F_p(n-2)\cdots F_p(n-p)},$$

we can express the ratio of two consecutive Fibonacci p-numbers in the form

$$\frac{F_p(n)}{F_p(n-1)} \;=\; 1 + \frac{F_p(n-2)}{F_p(n-1)} \cdot \frac{F_p(n-3)}{F_p(n-2)} \cdots \frac{F_p(n-p-1)}{F_p(n-p)}. \tag{1.28}$$

As a result of (1.27), we have that $\lim_{n\to\infty} \frac{F_p(n-1)}{F_p(n)} \;=\; \frac{1}{x}$. So as $n \to \infty$, (1.28) becomes

$$x \;=\; 1 + \frac{1}{x^p}.$$

Rearranging gives the algebraic equation $x^{p+1} - x^p - 1 = 0$ and we know that the positive root of this is τ_p. Therefore

$$\lim_{n\to\infty} \frac{F_p(n)}{F_p(n-1)} \;=\; \tau_p. \tag{1.29}$$

Example 4. *The Fibonacci numbers*

It was Johannes Kepler who first observed that the ratio of consecutive Fibonacci numbers oscillates about (see Figure 1.2) and tends to the Golden Ratio τ as $n \to \infty$.

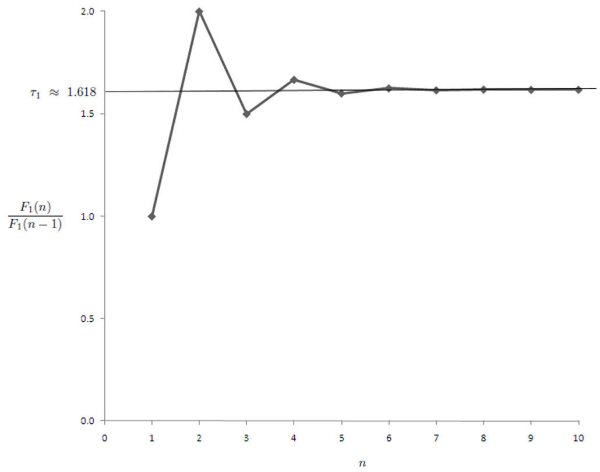


Figure 1.2: The oscillations of the ratio of consecutive Fibonacci numbers about the Golden Ratio τ.

In summary: we have seen that the Fibonacci p-numbers, a generalisation of the Fibonacci numbers, can be derived from a 'rabbit problem' where newborn rabbit couples take p months to mature. The limit of the ratio of consecutive Fibonacci p-numbers is the Golden p-Proportion τ_p. This corresponds to the value of the ratio $\frac{AC}{AB}$ which satisfies the Golden p-Section and is the only positive, real root of the distinct roots, $x_1, x_2, \ldots, x_{p+1}$, of the golden algebraic equation $x^{p+1} - x^p - 1 = 0$.

Chapter 2

A closed form expression for the m-extension of the Fibonacci p-numbers

There are many ways to generalise the Fibonacci numbers. We saw one of these generalisations in the previous chapter in the form of the Fibonacci p-numbers and we can generalise further by introducing the m-extension of the Fibonacci p-numbers. We will not study the basic theory and properties of these numbers in any great detail as much of it follows analogously from the topics discussed in the previous chapter. Instead, this chapter introduces their closed form expression, which is necessary knowledge for the more complex work in Chapter 3 on Fibonacci curves and surfaces.

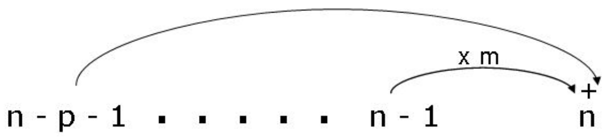
2.1 The m-extension of the Fibonacci p-numbers

Definition. For any integer $p \geq 0$ and real number $m > 0$, the n^{th} term of the m-extension of the Fibonacci p-numbers is defined by [17]:

$$F_{p,m}(n) = mF_{p,m}(n-1) + F_{p,m}(n-p-1) \qquad n \in \mathbb{Z}, \tag{2.1}$$

for $n \geq p+2$. The terms for $n = 1, 2, \ldots, p+1$ are given by the initial conditions $F_{p,m}(n) = m^{n-1}$.

This recursion relation has similarities to that of the Fibonacci p-numbers, which is produced by setting $m = 1$ in (2.1). However, a weighting of m is now given to the previous term. A visual representation of (2.1) is given below:



From now on for simplicity, we will refer to the m-extension of the Fibonacci p-numbers as the *Fibonacci (p, m)-numbers*.

Recall that earlier we found the Fibonacci p-numbers for $n \leq 0$ by rearranging the recursion relation for the Fibonacci p-numbers (1.1). The same method can be applied for the Fibonacci (p, m)-numbers, and we find that:

$$\begin{aligned}
F_{p,m}(0) &= F_{p,m}(-1) = \ldots = F_{p,m}(-p+1) = 0, \\
F_{p,m}(-p) &= 1, \\
F_{p,m}(-p-1) &= F_{p,m}(-p-2) = \ldots = F_{p,m}(-2p+1) = 0, \\
F_{p,m}(-2p) &= -m, \\
F_{p,m}(-2p-1) &= 1, \\
F_{p,m}(-2p-2) &= 0, \ \ldots .
\end{aligned}$$

We can continue the sequence for infinite values of $n \leq 0$. [17]

We saw earlier, when deriving the result (1.29), that the recursion relation for the Fibonacci p-numbers reduces to the golden algebraic equation (1.21) if we use (1.27). Similarly, if we make the assumption that the limit of the ratio of consecutive Fibonacci (p, m) numbers exists and equals a number x, then (2.1) reduces to the algebraic equation

$$x^{p+1} - mx^p - 1 = 0, \tag{2.2}$$

which has the roots $x_1, x_2, \ldots, x_{p+1}$. Note that, due to the factor of m in (2.1), these roots differ from the roots of the golden algebraic equation (1.21) discussed in the previous chapter.

It is clear that, as $x_1, x_2, \ldots, x_{p+1}$ solve (2.2), we can obtain the following identity which connects powers of the roots:

$$x_j^n = mx_j^{n-1} + x_j^{n-p-1} \tag{2.3}$$

where $n \in \mathbb{Z}$ and $j = 1, 2, \ldots, p+1$.

The positive root of the algebraic equation (2.2) is denoted by $\Phi_{p,m}$ and is called the *Golden (p, m)-Proportion*. Analagous to (1.29), $\Phi_{p,m}$ satisfies

$$\lim_{n\to\infty} \frac{F_{p,m}(n)}{F_{p,m}(n-1)} = \Phi_{p,m}. \tag{2.4}$$

For the case $m = 1$, $\Phi_{p,1}$ is equivalent to the Golden p-Proportion τ_p, discussed earlier in §1.5.

Let us consider a couple of particular cases of the Fibonacci (p, m)-numbers:

Example 5. *The Fibonacci $(1, m)$-numbers* [30]

Taking $p = 1$ in the recursion relation for the Fibonacci (p, m)-numbers (2.1) gives:

$$F_{1,m}(n) = mF_{1,m}(n-1) + F_{1,m}(n-2), \tag{2.5}$$

with two initial conditions $F_{1,m}(1) = 1$ and $F_1(2) = m$. This generates the Fibonacci $(1, m)$-numbers[1]:

$$\ldots, m^2 + 1, -m, 1, 0, 1, m, m^2 + 1, m^3 + 2m, \ldots$$

where the n^{th} term in the sequence is calculated by summing m of the previous term with the term previous to that. Letting $m = 1$ gives the Fibonacci numbers

$$\ldots, 2, -1, 1, 0, 1, 2, 3, 5, 8, 13, \ldots$$

[1] The Fibonacci $(1, m)$-numbers are referred to as the Fibonacci k-numbers in [11, 12, 13] and as the Fibonacci numbers of order m in [30].

which were discussed in §1.3 of Chapter 1.

The corresponding algebraic equation for the Fibonacci $(1, m)$-numbers is found by letting $p = 1$ in the equation (2.2):

$$x^2 - mx - 1 = 0, \tag{2.6}$$

and solving for x using the quadratic formula gives two real solutions:

$$x_1 = \frac{m + \sqrt{4 + m^2}}{2}, \quad x_2 = \frac{m - \sqrt{4 + m^2}}{2}. \tag{2.7}$$

The positive root x_1 corresponds to the Golden $(1, m)$-Proportion $\Phi_{1,m}$ and we can express the negative root in terms of $\Phi_{1,m}$ as follows:

$$\begin{aligned} x_2 &= \frac{(m - \sqrt{4 + m^2})}{2} \times \frac{m + \sqrt{4 + m^2}}{m + \sqrt{4 + m^2}} \\ &= -\frac{2}{m + \sqrt{4 + m^2}} \\ &= -\frac{1}{\Phi_{1,m}}. \end{aligned} \tag{2.8}$$

The roots of (2.6) are now denoted as

$$\Phi_{1,m} = \frac{m + \sqrt{4 + m^2}}{2} \quad \text{and} \quad -\frac{1}{\Phi_{1,m}} = \frac{m - \sqrt{4 + m^2}}{2}.$$

We can obtain identities through addition and subtraction of these two roots such as

$$\Phi_{1,m} - \frac{1}{\Phi_{1,m}} = m$$

and

$$\Phi_{1,m} + \frac{1}{\Phi_{1,m}} = \sqrt{4 + m^2}. \tag{2.9}$$

Example 6. *The Fibonacci $(2, m)$-numbers*

The recursion relation for the Fibonacci $(2, m)$-numbers is

$$F_{2,m}(n) = mF_{2,m}(n - 1) + F_{2,m}(n - 3),$$

by taking $p = 2$ in (2.1), and the initial conditions are $F_{2,m}(1) = 1$, $F_{2,m}(2) = m$ and $F_{2,m}(3) = m^2$. This produces the Fibonacci $(2, m)$-sequence

$$\ldots, -m, 0, 1, 0, 0, 1, m, m^2, m^3 + 1, m^4 + m, \ldots$$

By taking $p = 2$ in (2.2), the algebraic equation for the Fibonacci $(2, m)$-numbers is

$$x^3 - mx^2 - 1 = 0. \tag{2.10}$$

As with an earlier example (1.25), this is a monic cubic equation of the form $x^3 + ax^2 + c = 0$. Thus using the set of general formulas (1.26) with $a = -m$ and $c = 1$, we find that the cubic equation (2.10) has the roots [17]:

$$x_1 = \frac{h^2+2mh+4m^2}{6h}, \quad (2.11)$$

$$x_2 = -\frac{h^2-4mh+4m^2}{12h} + i\sqrt{3}\left(\frac{h}{12}-\frac{m^2}{3h}\right), \quad (2.12)$$

$$x_3 = -\frac{h^2-4mh+4m^2}{12h} - i\sqrt{3}\left(\frac{h}{12}-\frac{m^2}{3h}\right), \quad (2.13)$$

where $h = \sqrt[3]{108+8m^3+12\sqrt{81+12m^3}}$.

2.2 Binet formulas

As we have seen, the n^{th} Fibonacci (p, m)-number can be calculated using the recursion relation (2.1). This method requires us to calculate all preceding Fibonacci (p, m)-numbers in the sequence and, even when using a computer, it would become tiresome to compute for large values of n.

However, all sequences defined recurrently have a closed form expression. The closed form expression for the Fibonacci numbers was derived by Jacques Binet in 1843 and, although Leonhard Euler, Daniel Bernoulli and Abraham de Moivre were aware of it more than a century earlier, the name 'Binet formula' remained. [34]

The Binet formula for the Fibonacci numbers expresses the n^{th} Fibonacci number as the sum of powers of the roots (1.24) multiplied by constant coefficients. We wish to generalise this for the Fibonacci (p, m)-numbers (2.1) by the following theorem. The proof is adapted from [17].

Theorem. *For any $p \geq 1$ and $m \geq 1$ where p and m are integers, the n^{th} Fibonacci (p, m)-number can be represented in the form:*

$$F_{p,m}(n) = k_1(x_1)^n + k_2(x_2)^n + \cdots + k_{p+1}(x_{p+1})^n \quad (2.14)$$

where $x_1, x_2, \ldots, x_{p+1}$ are the roots of the algebraic equation (2.2) and $k_1, k_2, \ldots, k_{p+1}$ are constant coefficients which satisfy:

$$\begin{cases} F_{p,m}(0) & = k_1 + k_2 + \cdots + k_{p+1} = 0, \\ F_{p,m}(1) & = k_1x_1 + k_2x_2 + \cdots + k_{p+1}x_{p+1} = 1, \\ F_{p,m}(2) & = k_1(x_1)^2 + k_2(x_2)^2 + \cdots + k_{p+1}(x_{p+1})^2 = m, \\ \vdots & \vdots \qquad \vdots \qquad \vdots \\ F_{p,m}(p) & = k_1(x_1)^p + k_2(x_2)^p + \cdots + k_{p+1}(x_{p+1})^p = m^{p-1}, \\ F_{p,m}(p+1) & = k_1(x_1)^{p+1} + k_2(x_2)^{p+1} + \cdots + k_{p+1}(x_{p+1})^{p+1} = m^p. \end{cases} \quad (2.15)$$

Proof. First, recall that the initial conditions of the Fibonacci (p, m)-numbers (2.1) are given by $F_{p,m}(n) = m^{n-1}$ for $1 \leq n \leq p+1$. We have also found that $F_{p,m}(0) = 0$. Substituting $n = 0, \ldots, p+1$ into (2.14) gives the conditions (2.15) for the constant coefficients $k_1, k_2, \ldots, k_{p+1}$ above.

We now prove (2.14) by induction - the basic cases are given by (2.15) above. Now assume that the theorem holds for $n < n_0$, where $n_0 \geq p+2$ is an integer, and consider the case when $n = n_0$.

Using (2.3):

$$k_1(x_1)^{n_0} + k_2(x_2)^{n_0} + \cdots + k_{p+1}(x_{p+1})^{n_0} = k_1\Big(m(x_1)^{n_0-1} + (x_1)^{n_0-p-1}\Big) + k_2\Big(m(x_2)^{n_0-1} + (x_2)^{n_0-p-1}\Big)$$

$$+ \ \ldots \ + k_{p+1}\Big(m(x_{p+1})^{n_0-1} + (x_{p+1})^{n_0-p-1}\Big)$$

$$= m\ \Big(k_1(x_1)^{n_0-1} + k_2(x_2)^{n_0-1} + \cdots + k_{p+1}(x_{p+1})^{n_0-1}\Big)$$

$$+ \Big(k_1(x_1)^{n_0-p-1} + k_2(x_2)^{n_0-p-1} + \cdots + k_{p+1}(x_{p+1})^{n_0-p-1}\Big).$$

By the induction hypothesis:

$$k_1(x_1)^{n_0-1} + k_2(x_2)^{n_0-1} + \cdots + k_{p+1}(x_{p+1})^{n_0-1} \ = \ F_{p,m}(n_0-1),$$

and

$$k_1(x_1)^{n_0-p-1} + k_2(x_2)^{n_0-p-1} + \cdots + k_{p+1}(x_{p+1})^{n_0-p-1} \ = \ F_{p,m}(n_0-p-1).$$

Therefore

$$\begin{aligned} k_1(x_1)^{n_0} + k_2(x_2)^{n_0} + \cdots + k_{p+1}(x_{p+1})^{n_0} \ &= \ mF_{p,m}(n_0-1) + F_{p,m}(n_0-p-1) \\ &= \ F_{p,m}(n_0). \end{aligned}$$

□

Therefore to find Binet's formula for any given p, we must first find the roots of the corresponding algebraic equation (2.2) and then calculate the coefficients $k_1, k_2, \ldots, k_{p+1}$ using the conditions stated in the theorem. The following examples show the method for the Fibonacci $(1,m)$-numbers and the Fibonacci $(2,m)$-numbers.

Example 7. *Continuation of Example 6: The Fibonacci $(1,m)$-numbers.*

Earlier we found the roots of (2.6) to be

$$\Phi_{1,m} = \frac{m+\sqrt{4+m^2}}{2} \quad \text{and} \quad -\frac{1}{\Phi_{1,m}} = \frac{m-\sqrt{4+m^2}}{2}.$$

So, by the theorem above, Binet's formula for the Fibonacci $(1,m)$-numbers takes the form

$$F_{1,m}(n) \ = \ k_1\Phi_{1,m}^n + k_2\left(-\frac{1}{\Phi_{1,m}}\right)^n. \tag{2.16}$$

From the definition of the Fibonacci (p,m)-numbers (2.1) and subsequent calculations of (p,m) numbers for $n \le 0$, we know that $F_{1,m}(0) = 0$, $F_{1,m}(1) = 1$ and $F_{1,m}(2) = m$. Therefore k_1 and k_2 must satisfy

$$\begin{cases} F_{1,m}(0) \ = \ k_1 + k_2 \ = \ 0, \\ F_{1,m}(1) \ = \ k_1\Phi_{1,m} + k_2\left(-\dfrac{1}{\Phi_{1,m}}\right) \ = \ 1, \\ F_{1,m}(2) \ = \ k_1\Phi_{1,m}^2 + k_2\left(-\dfrac{1}{\Phi_{1,m}}\right)^2 \ = \ m. \end{cases} \tag{2.17}$$

Substituting $k_1 = -k_2$ into the second equation gives:

$$k_1\Phi_{1,m} + k_1\left(\frac{1}{\Phi_{1,m}}\right) \ = \ k_1\left(\Phi_{1,m} + \frac{1}{\Phi_{1,m}}\right) \ = \ 1.$$

Using (2.9) we find that

$$k_1 = \frac{1}{\sqrt{4+m^2}}, \quad \text{and} \quad k_2 = -k_1 = -\frac{1}{\sqrt{4+m^2}}.$$

The third equation in (2.17) can be used as to check if necessary.

We can now write (2.16) as

$$\begin{aligned} F_{1,m}(n) &= \frac{1}{\sqrt{4+m^2}}\Phi_{1,m}^n - \frac{1}{\sqrt{4+m^2}}\left(-\frac{1}{\Phi_{1,m}}\right)^n \\ &= \frac{1}{\sqrt{4+m^2}}\left[\left(\frac{m+\sqrt{4+m^2}}{2}\right)^n - \left(\frac{m-\sqrt{4+m^2}}{2}\right)^n\right]. \end{aligned} \tag{2.18}$$

This is Binet's formula for the Fibonacci $(1,m)$-numbers, also known as the Gazale formula after Midhat Gazale who first derived it. If we were to take $m=1$ we would obtain Binet's original formula for the n^{th} term of the Fibonacci sequence (1.11):

$$F(n) \;=\; \frac{1}{\sqrt{5}}\left(\tau^n - \left(\frac{1}{\tau}\right)^n\right), \tag{2.19}$$

where $\tau = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio. [30]

Example 8. *Continuation of Example 7: The Fibonacci $(2,m)$-numbers*
Binet's formula for the Fibonacci $(2,m)$-numbers is of the form:

$$F_{2,m}(n) \;=\; k_1(x_1)^n + k_2(x_2)^n + k_3(x_3)^n.$$

In Example 7 we found the roots of the algebraic equation (2.10) to be:

$$\begin{aligned} x_1 &= \frac{h^2+2mh+4m^2}{6h}, \\ x_2 &= -\frac{h^2-4mh+4m^2}{12h} + i\sqrt{3}\left(\frac{h}{12}-\frac{m^2}{3h}\right), \\ x_3 &= -\frac{h^2-4mh+4m^2}{12h} - i\sqrt{3}\left(\frac{h}{12}-\frac{m^2}{3h}\right), \end{aligned}$$

where $h = \sqrt[3]{108+8m^3+12\sqrt{81+12m^3}}$.

By (2.1), the initial conditions for the Fibonacci $(2,m)$-numbers are $F_{2,m}(1)=1$,$F_{2,m}(2)=m$ and $F_{2,m}(3)=m^2$, and we also know that $F_{2,m}(0)=0$. Therefore the coefficients k_1, k_2 and k_3 must satisfy:

$$\begin{cases} F_2(0) \;=\; k_1+k_2+k_3 \;=\; 0, \\ F_2(1) \;=\; k_1x_1+k_2x_2+k_3x_3 \;=\; 1, \\ F_2(2) \;=\; k_1(x_1)^2+k_2(x_2)^2+k_3(x_3)^2 \;=\; m, \\ F_2(3) \;=\; k_1(x_1)^3+k_2(x_2)^3+k_3(x_3)^3 \;=\; m^2. \end{cases}$$

As there are three unknowns we only require three of the these four equations, although the remaining equation is useful for checking purposes. The solution to this system of equations is

$$\begin{aligned} k_1 &= \frac{2h(h+2m)}{h^3+8m^3}, \\ k_2 &= \frac{h(-(h+2m)+i\sqrt{3}(h-2m))}{h^3+8m^3}, \\ k_3 &= \frac{h(-(h+2m)-i\sqrt{3}(h-2m))}{h^3+8m^3}, \end{aligned} \tag{2.20}$$

with $h = \sqrt[3]{108+8m^3+12\sqrt{81+12m^3}}$.

Therefore we have found that the n^{th} Fibonacci $(2, m)$ number can be calculated using the closed form expression:

$$\begin{aligned} F_{2,m}(n) &= \frac{2h(h+2m)}{h^3+8m^3}\left(\frac{h^2+2mh+4m^2}{6h}\right)^n \\ &+ \frac{h(-(h+2m)+i\sqrt{3}(h-2m))}{h^3+8m^3}\left(-\frac{h^2-4mh+4m^2}{12h}+i\sqrt{3}\left(\frac{h}{12}-\frac{m^2}{3h}\right)\right)^n \\ &+ \frac{h(-(h+2m)-i\sqrt{3}(h-2m))}{h^3+8m^3}\left(-\frac{h^2-4mh+4m^2}{12h}-i\sqrt{3}\left(\frac{h}{12}-\frac{m^2}{3h}\right)\right)^n, \end{aligned} \tag{2.21}$$

where $h = \sqrt[3]{108+8m^3+12\sqrt{81+12m^3}}$. [17]

As h is an irrational number, this formula is a combination of complex numbers with irrational real and imaginary parts. It is therefore intriguing that this reduces to an integer for every value of n.

In summary: The Fibonacci (p, m)-numbers are a generalisation of the Fibonacci p-numbers, and therefore possess many similar properties. It was proved that the generalised Binet formula allows the n^{th} Fibonacci (p, m)-number to be calculated without knowledge of previous terms in the sequence.

Chapter 3

Fibonacci on a continuous domain

The generalised Binet formula is now used to extend the Fibonacci (p, m)-numbers to a continuous domain. The properties and geometry of Fibonacci curves and surfaces are discussed and a potential application in physical cosmology is proposed.

3.1 Continuous functions for general p

Let us consider the general form of Binet's formula for the Fibonacci (p, m)-numbers:

$$F_{p,m}(n) \;=\; k_1(x_1)^n + k_2(x_2)^n + \cdots + k_{p+1}(x_{p+1})^n, \tag{3.1}$$

where $x_1, x_2, \ldots, x_{p+1}$ are the roots of the algebraic equation (2.2) and $k_1, k_2, \ldots, k_{p+1}$ are the coefficients satisfying the equations (2.15). The value of p determines the number of roots and coefficients, how many of them are real and how many are complex valued.

If p is even, there will be an odd number of roots and coefficients. Only x_1, the Golden (p, m)-Proportion, is real while the other p roots $x_2, \ldots, x_{p+1}$ can be grouped into $\frac{p}{2}$ pairs of complex conjugate roots, $(a_t - ib_t)$ and $(a_t + ib_t)$ where $t = 1, 2, \ldots, \frac{p}{2}$. Similarly, the coefficient k_1 is real valued and the other p coefficients can be grouped into the complex conjugate pairings $(c_t + id_t)$ and $(c_t - id_t)$.

If p is odd, there will be an even number of roots and coefficients. The roots x_1 and x_2 will be real. The other $p - 1$ roots will form $\frac{p-1}{2}$ pairs of complex conjugate roots, $(a_t - ib_t)$ and $(a_t + ib_t)$ where $t = 1, 2, \ldots, \frac{p-1}{2}$. And similarly the coefficients k_1 and k_2 will be real numbers and the other coefficients can be grouped into $\frac{p-1}{2}$ complex conjugate pairs $(c_t + id_t)$ and $(c_t - id_t)$.

Therefore Binet's formula (3.1) for all integer values of $p \geq 1$, can be represented as two separate formulas, one for even p and the other for odd p. [17]

For even p:

$$\begin{aligned} F_{p,m}(n) \;=\; & k_1(x_1)^n + (c_1 + id_1)(a_1 - ib_1)^n + (c_1 - id_1)(a_1 + ib_1)^n \\ & + \cdots + (c_{\frac{p}{2}} + id_{\frac{p}{2}})(a_{\frac{p}{2}} - ib_{\frac{p}{2}})^n + (c_{\frac{p}{2}} - id_{\frac{p}{2}})(a_{\frac{p}{2}} + ib_{\frac{p}{2}})^n, \end{aligned} \tag{3.2}$$

and for odd p:

$$\begin{aligned} F_{p,m}(n) \;=\; & k_1(x_1)^n + k_2(x_2)^n + (c_1 + id_1)(a_1 - ib_1)^n + (c_1 - id_1)(a_1 + ib_1)^n \\ & + \cdots + (c_{\frac{p-1}{2}} + id_{\frac{p-1}{2}})(a_{\frac{p-1}{2}} - ib_{\frac{p-1}{2}})^n + (c_{\frac{p-1}{2}} - id_{\frac{p-1}{2}})(a_{\frac{p-1}{2}} + ib_{\frac{p-1}{2}})^n. \end{aligned} \tag{3.3}$$

Both and contain expressions of the form[1]

$$(c+id)(a-ib)^n+(c-id)(a+ib)^n \;=\; c\big((a+ib)^n+(a-ib)^n\big)+id\big((a-ib)^n-(a+ib)^n\big). \quad (3.4)$$

Let's consider $(a \pm ib)^n$. We can convert this from cartesian to polar coordinates by using the change of variables:

$$a \;=\; r\cos\theta, \quad b \;=\; r\sin\theta, \quad r \;=\; \sqrt{a^2+b^2},$$

so that

$$\begin{aligned}(a \pm ib)^n &= (r\cos\theta \pm ir\sin\theta)^n \\ &= r^n(\cos\theta \pm i\sin\theta)^n. \end{aligned} \quad (3.5)$$

We can now apply De Moivre's formula:

$$(\cos\theta \pm i\sin\theta)^n \;=\; \cos(n\theta) \pm i\sin(n\theta)$$

to find that

$$\begin{aligned}(a+ib)^n+(a-ib)^n &= r^n\big(\cos(n\theta)+i\sin(n\theta)\big)+r^n\big(\cos(n\theta)-i\sin(n\theta)\big) \\ &= 2r^n\cos(n\theta)\end{aligned}$$

and

$$\begin{aligned}(a-ib)^n-(a+ib)^n &= r^n\big(\cos(n\theta)-i\sin(n\theta)\big)-r^n\big(\cos(n\theta)+i\sin(n\theta)\big) \\ &= -2ir^n\sin(n\theta).\end{aligned}$$

Therefore (3.4) can be written as:

$$\begin{aligned}(c+id)(a-ib)^n+(c-id)(a+ib)^n &= 2cr^n\cos(n\theta)+id\big(-2ir^n\sin(n\theta)\big) \\ &= 2r^n\big(c\cos(n\theta)+d\sin(n\theta)\big).\end{aligned} \quad (3.6)$$

If we use the change of variables

$$c \;=\; R\cos\gamma, \quad d \;=\; R\sin\gamma, \quad R \;=\; \sqrt{c^2+d^2},$$

then

$$(c+id)(a-ib)^n+(c-id)(a+ib)^n \;=\; 2r^n\big(R\cos\gamma\cos(n\theta)+R\sin\gamma\sin(n\theta)\big).$$

This allows us to use the trigonometric identity

$$\cos\alpha\cos\beta+\sin\alpha\sin\beta \;=\; \cos(\alpha-\beta),$$

which gives

$$\begin{aligned}(c+id)(a-ib)^n+(c-id)(a+ib)^n &= 2r^n\big(R\cos(n\theta-\gamma)\big) \\ &= 2(a^2+b^2)^{\frac{n}{2}}\sqrt{c^2+d^2}\cos(n\theta-\gamma).\end{aligned} \quad (3.7)$$

[1]From here I have completed the derivation of (3.7) as this is not explicity given in [17].

Therefore Binet's formula for even p (3.1) can be written as:

$$\begin{aligned} F_{p,m}(n) &= k_1(x_1)^n + 2(a_1^2+b_1^2)^{\frac{n}{2}}\sqrt{c_1^2+d_1^2}\cos(n\theta_1-\gamma_1) \\ &\quad + \cdots + 2(a_{\frac{p}{2}}^2+b_{\frac{p}{2}}^2)^{\frac{n}{2}}\sqrt{c_{\frac{p}{2}}^2+d_{\frac{p}{2}}^2}\cos(n\theta_{\frac{p}{2}}-\gamma_{\frac{p}{2}}) \\ &= k_1(x_1)^n + 2\sum_{t=1}^{\frac{p}{2}}(a_t^2+b_t^2)^{\frac{n}{2}}\sqrt{c_t^2+d_t^2}\cos(n\theta_t-\gamma_t). \end{aligned} \tag{3.8}$$

For odd p, (3.1) can be written as:

$$\begin{aligned} F_{p,m}(n) &= k_1(x_1)^n + k_2(x_2)^n\cos(\pi n) + 2(a_1^2+b_1^2)^{\frac{n}{2}}\sqrt{c_1^2+d_1^2}\cos(n\theta_1-\gamma_1) \\ &\quad + \cdots + 2(a_{\frac{p-1}{2}}^2+b_{\frac{p-1}{2}}^2)^{\frac{n}{2}}\sqrt{c_{\frac{p-1}{2}}^2+d_{\frac{p-1}{2}}^2}\cos(n\theta_{\frac{p-1}{2}}-\gamma_{\frac{p-1}{2}}) \\ &= k_1(x_1)^n + k_2(x_2)^n\cos(\pi n) + 2\sum_{t=1}^{\frac{p-1}{2}}(a_t^2+b_t^2)^{\frac{n}{2}}\sqrt{c_t^2+d_t^2}\cos(n\theta_t-\gamma_t), \end{aligned} \tag{3.9}$$

where $\theta = \arccos\frac{a}{\sqrt{a^2+b^2}}$ and $\gamma = \arccos\frac{c}{\sqrt{c^2+d^2}}$.

These formulas give the corresponding Fibonacci (p, m)-number for each value of n - the formula we use is dependent on whether p is even or odd. However calculating Fibonacci (p, m)-numbers was not the main purpose of deriving these formulas, as in most cases it would be simpler to use (3.1).

Instead, rewriting the general Binet formula as (3.8) and (3.9) allows for continuous functions $\{FF_{p,m}(x)\}$ to be obtained, one for even p and one for odd p, by replacing n by a continuous variable x.

For even p:

$$FF_{p,m}(x) = k_1(x_1)^x + 2\sum_{t=1}^{\frac{p}{2}}(a_t^2+b_t^2)^{\frac{x}{2}}\sqrt{c_t^2+d_t^2}\cos(x\theta_t-\gamma_t), \tag{3.10}$$

and for odd p:

$$FF_{p,m}(x) = k_1(x_1)^x + k_2(x_2)^x\cos(\pi x) + 2\sum_{t=1}^{\frac{p-1}{2}}(a_t^2+b_t^2)^{\frac{x}{2}}\sqrt{c_t^2+d_t^2}\cos(x\theta_t-\gamma_t). \tag{3.11}$$

Authors considering particular cases of these continuous functions in [12, 13, 26, 32] refer to them as Quasi-sine functions, due to some similarities to the shape of the sine function. Therefore we will call $FF_{p,m}(x)$ the *Quasi-sine Fibonacci (p, m)-function*. This function connects the co-ordinates $(n,F_{p,m}(n))$ by a smooth curve, effectively extending the Fibonacci (p, m)-numbers to a continuous domain x.

3.2 The Quasi-sine Fibonacci $(1, m)$-function

To show some of the properties of the Quasi-sine Fibonacci (p, m)-function we now focus on a particular case, the Quasi-sine Fibonacci $(1, m)$-function which is found by taking $p = 1$. We consider the curve's geometry, and some interesting identities which are very similar to identities and relations discussed in previous chapters. The analysis of these curves can be carried out in a similar manner

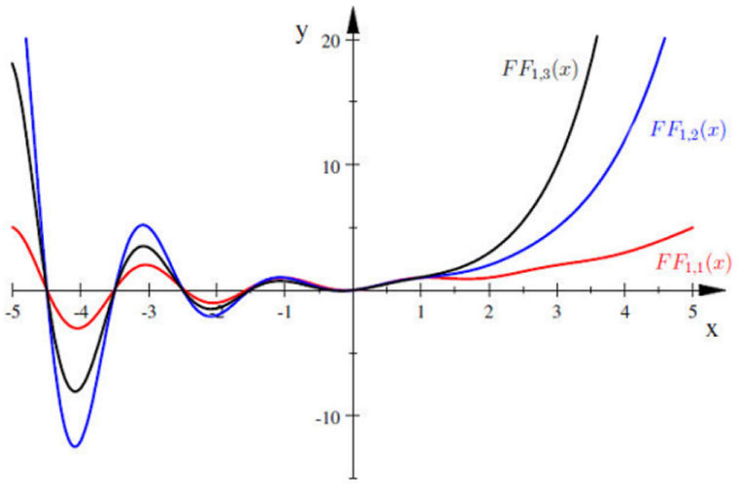


Figure 3.1: The Quasi-sine Fibonacci $(1, m)$-function for $m = 1, 2, 3$. [17]

with all cases of the Quasi-sine Fibonacci (p, m)-function.

In Example 7 we found the Binet formula for the Fibonacci $(1, m)$-numbers to be

$$F_{1,m}(n) = \frac{1}{\sqrt{4+m^2}}\left[\Phi_{1,m}^n - \left(-\frac{1}{\Phi_{1,m}}\right)^n\right] = \frac{\Phi_{1,m}^n - (-1)^n\Phi_{1,m}^{-n}}{\sqrt{4+m^2}},$$

where $\Phi_{1,m} = \frac{m+\sqrt{4+m^2}}{2}$.

By taking into consideration that $\cos(\pi n) = (-1)^n$ for $n \in \mathbb{Z}$ and replacing the discrete variable n by the continuous variable x we obtain the Quasi-sine Fibonacci $(1, m)$-function:

$$FF_{1,m}(x) = \frac{\Phi_{1,m}^x - \cos(\pi x)\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}. \quad (3.12)$$

The curves for $FF_{1,m}(x)$ where $m = 1, 2, 3$ are shown in Figure 3.1. Note that, for discrete values of x, the equation for the Quasi-sine Fibonacci $(1, m)$-function is equivalent to Binet's formula for the Fibonacci $(1, m)$-numbers. Therefore $FF_{1,m}(x)$ passes through the co-ordinates $(x, F_{1,m}(x))$ when x is an integer, and regardless of the value of m, $FF_{1,m}(x)$ will always pass through the origin.

Away from the origin, the shape of the curve differs dependent on whether x is positive or negative. For $x < 0$, the $\cos(\pi x)\Phi_{1,m}^{-x}$ term in (3.12) is more influential than the $\Phi_{1,m}^x$ term. This is why, for negative x, the curve oscillates about the x axis in a similar fashion to the sine function and the amplitude increases as x becomes more negative. Whereas for positive x, the $\Phi_{1,m}^x$ term becomes more influential and the $\cos(\pi x)\Phi_{1,m}^{-x}$ term less so. Hence the curve possesses a shape similar to that of the exponential function for $x > 0$.

However, dependent on the value of m, oscillations may still occur for $x > 0$. Looking closely at the curves in Figure 3.1, it is possible to see that $FF_{1,1}(x)$ does have one maximum (at $x \approx 1.09458$)

and one minimum (at $x \approx 1.67669$) for positive x, whereas the curves $FF_{1,2}(x)$ and $FF_{1,3}(x)$ do not possess any extreme points. This suggests that as the real number m increases, the curve oscillates less for positive x. In fact, Falcon and Plaza argued that there must exist a critical value m_0 whereby any curves for $m > m_0$ do not have an extreme points for $x > 0$. They found that $m_0 \approx 1.282974$. [12]

The Quasi-sine Fibonacci $(1, m)$-function possesses many identities analagous to those of the Fibonacci $(1, m)$-numbers and its special case, the Fibonacci numbers. The following identites are shown as they are analagous to some identities proved earlier.

Theorem (Recursion relation)**.**

$$FF_{1,m}(x) \;=\; m \cdot FF_{1,m}(x-1) + FF_{1,m}(x-2). \tag{3.13}$$

Proof. This proof is adapted from [12]. By the formula (3.12):

$$\begin{aligned} mFF_{1,m}(x-1) + FF_{1,m}(x-2) \;&=\; m\left(\frac{\Phi_{1,m}^{x-1} - \cos(\pi(x-1))\Phi_{1,m}^{-(x-1)}}{\sqrt{4+m^2}}\right) \\ &\quad + \frac{\Phi_{1,m}^{x-2} - \cos(\pi(x-2))\Phi_{1,m}^{-(x-2)}}{\sqrt{4+m^2}}. \end{aligned} \tag{3.14}$$

The cosine function is periodic with period 2π, and this means that for $r \in \mathbb{Z}$:

$$\cos(\theta \pm r\pi) \;=\; (-1)^r \cos(\theta). \tag{3.15}$$

Therefore

$$\cos(\pi(x-1)) \;=\; \cos(\pi x - \pi) \;=\; -\cos(\pi x)$$

and

$$\cos(\pi(x-2)) \;=\; \cos(\pi x - 2\pi) \;=\; \cos(\pi x).$$

Hence (3.14) becomes:

$$\begin{aligned} mFF_{1,m}(x-1) + FF_{1,m}(x-2) \;&=\; m\left(\frac{\Phi_{1,m}^{x-1} + \cos(\pi x)\Phi_{1,m}^{-x+1}}{\sqrt{4+m^2}}\right) \\ &\quad + \frac{\Phi_{1,m}^{x-2} - \cos(\pi x)\Phi_{1,m}^{-x+2}}{\sqrt{4+m^2}} \\ &=\; \frac{m\Phi_{1,m}^{x-1} + \Phi_{1,m}^{x-2} + \cos(\pi x)\Phi_{1,m}^{-x}(m\Phi_{1,m} - \Phi_{1,m}^2)}{\sqrt{4+m^2}}. \end{aligned}$$

As $\Phi_{1,m}$ is a root of the algebraic equation (2.2) when $p = 1$, we can use the identity (2.3) to show that

$$\Phi_{1,m}^x = m\Phi_{1,m}^{x-1} + \Phi_{1,m}^{x-2} \quad \text{and} \quad \Phi_{1,m}^2 = m\Phi_{1,m} + 1.$$

Therefore

$$\begin{aligned} mFF_{1,m}(x-1) + FF_{1,m}(x-2) \;&=\; \frac{\Phi_{1,m}^x + \cos(\pi x)\Phi_{1,m}^{-x}(m\Phi_{1,m} - m\Phi_{1,m} - 1)}{\sqrt{4+m^2}} \\ &=\; \frac{\Phi_{1,m}^x - \cos(\pi x)\Phi_{1,m}^{-x}}{\sqrt{4+m^2}} \\ &=\; FF_{1,m}(x). \end{aligned}$$

□

This recursion relation for the Quasi-sine Fibonacci $(1, m)$-function (3.13) is the continuous form of the recursion relation for the Fibonacci $(1, m)$-numbers (2.5). The latter is obtained by letting x take only integer values in (3.13) as we know that this gives the Fibonacci $(1, m)$-numbers.

Another property of the Quasi-sine Fibonacci $(1, m)$-function is given by the following theorem [12]:

Theorem (Asymptotic quotient)**.** *For $r \in \mathbb{Z}$,*

$$\lim_{x\to\infty} \frac{FF_{1,m}(x)}{FF_{1,m}(x-r)} \;=\; \Phi_{1,m}^{r}.$$

Proof. Firstly, it is clear that $\Phi_{1,m} = \frac{m+\sqrt{4+m^2}}{2} > 1$ as we defined $m > 0$. Then using (3.12):

$$\lim_{x\to\infty} \frac{FF_{1,m}(x)}{FF_{1,m}(x-r)} \;=\; \lim_{x\to\infty} \frac{\Phi_{1,m}^{x} - \cos(\pi x)\Phi_{1,m}^{-x}}{\Phi_{1,m}^{x-r} - \cos(\pi(x-r))\Phi_{1,m}^{-x+r}}.$$

As $\Phi_{1,m} \neq 0$, we can divide both the numerator and the denominator by $\Phi_{1,m}^{x-r}$:

$$\lim_{x\to\infty} \frac{FF_{1,m}(x)}{FF_{1,m}(x-r)} \;=\; \lim_{x\to\infty} \frac{\Phi_{1,m}^{r} - \cos(\pi x)\Phi_{1,m}^{-2x+r}}{1 - \cos(\pi(x-r))\Phi_{1,m}^{-2x+2r}}.$$

We know that $\Phi_{1,m} > 1$ so, as $x \to \infty$, $\Phi_{1,m}^{-2x+r}$ and $\Phi_{1,m}^{-2x+2r}$ will tend to zero giving

$$\lim_{x\to\infty} \frac{FF_{1,m}(x)}{FF_{1,m}(x-r)} \;=\; \Phi_{1,m}^{r}.$$

□

In §1.3 we proved Catalan's identity (1.15) for the Fibonacci numbers, a basic case of the Fibonacci (p, m)-numbers when $p = m = 1$. It is interesting to see that an analagous form of Catalan's identity holds for the Quasi-sine Fibonacci $(1, m)$-function. The following theorem is equivalent to that which is given in [12] but has been stated in a form which corresponds to Catalan's formula proved in §1.3. A step in the proof, which was omitted in [12], has also been included.

Theorem (Catalan's identity for the Quasi-sine Fibonacci $(1, m)$-function)**.** *If $r \in \mathbb{Z}$, then*

$$\big(FF_{1,m}(x)\big)^2 - FF_{1,m}(x+r)\cdot FF_{1,m}(x-r) \;=\; (-1)^r\cos(\pi x)\big(FF_{1,m}(r)\big)^2. \tag{3.16}$$

Proof.

$$\big(FF_{1,m}(x)\big)^2 - FF_{1,m}(x+r)\cdot FF_{1,m}(x-r) \;=$$

$$\left(\frac{\Phi_{1,m}^{x} - \cos(\pi x)\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right)^2 - \left(\frac{\Phi_{1,m}^{x+r} - \cos(\pi(x+r))\Phi_{1,m}^{-x-r}}{\sqrt{4+m^2}}\right)\left(\frac{\Phi_{1,m}^{x-r} - \cos(\pi(x-r))\Phi_{1,m}^{-x+r}}{\sqrt{4+m^2}}\right).$$

Using the property (3.15):

$$\big(FF_{1,m}(x)\big)^2 - FF_{1,m}(x+r)\cdot FF_{1,m}(x-r) \;=$$

$$\left(\frac{\Phi_{1,m}^{x} - \cos(\pi x)\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right)^2 - \left(\frac{\Phi_{1,m}^{x+r} + (-1)^{r+1}\cos(\pi x)\Phi_{1,m}^{-x-r}}{\sqrt{4+m^2}}\right)\left(\frac{\Phi_{1,m}^{x-r} + (-1)^{r+1}\cos(\pi(x))\Phi_{1,m}^{-x+r}}{\sqrt{4+m^2}}\right).$$

Then, expanding the brackets and simplifying the resulting expression gives

$$\big(FF_{1,m}(x)\big)^2 - FF_{1,m}(x+r)\cdot FF_{1,m}(x-r) \;=\; \frac{(-1)^r\cos(\pi x)\Phi_{1,m}^{2r} + (-1)^r\cos(\pi x)\Phi_{1,m}^{-2r} - 2\cos(\pi x)}{4+m^2}. \tag{3.17}$$

Now note that, as $\cos^2(\pi r) = 1 = (-1)^r\cos(\pi r)$, then

$$(-1)^r\cos(\pi x)\Phi_{1,m}^{-2r} \;\equiv\; (-1)^r\cos(\pi x)\cos^2(\pi r)\Phi_{1,m}^{-2r} \tag{3.18}$$

and

$$2\cos(\pi x) \;\equiv\; 2(-1)^r\cos(\pi x)\cos(\pi r). \tag{3.19}$$

Using (3.18) and (3.19), the equation (3.17) can then be written as

$$\big(FF_{1,m}(x)\big)^2 - FF_{1,m}(x+r)\cdot FF_{1,m}(x-r) \;=\; \frac{(-1)^r\cos(\pi x)\Phi_{1,m}^{2r} + (-1)^r\cos(\pi x)\cos^2(\pi r)\Phi_{1,m}^{-2r} - 2(-1)^r\cos(\pi x)\cos(\pi r)}{4+m^2}$$

Factorising this gives:

$$\begin{aligned}\big(FF_{1,m}(x)\big)^2 - FF_{1,m}(x+r)\cdot FF_{1,m}(x-r) \;&=\; (-1)^r\cos(\pi x)\left(\frac{\Phi_{1,m}^{2r} - 2\cos(\pi r) + \cos^2(\pi r)\Phi_{1,m}^{-2r}}{4+m^2}\right)\\ &=\; (-1)^r\cos(\pi x)\left(\frac{\Phi_{1,m}^{r} - \cos(\pi r)\Phi_{1,m}^{-r}}{\sqrt{4+m^2}}\right)^2\\ &=\; (-1)^r\cos(\pi x)\big(FF_{1,m}(r)\big)^2.\end{aligned}$$

□

Note that Cassini's identity is a special case of Catalan's identity above when $r = 1$.

3.3 The hyperbolic Fibonacci $(1, m)$-functions

We now discuss a pair of continuous functions which have connections to the Fibonacci $(1, m)$-numbers and the Golden $(1, m)$-Proportion, $\Phi_{1,m}$. These continuous functions are also geometrically related to the Quasi-sine Fibonacci $(1, m)$-function seen in the previous section. As with the Quasi-sine Fibonacci $(1, m)$-function, this pair of functions can be generalised for all p.

Binet's formula for the Fibonacci $(1, m)$-numbers:

$$F_{1,m}(n) \;=\; \frac{\Phi_{1,m}^{n} - (-1)^n\Phi_{1,m}^{-n}}{\sqrt{4+m^2}}$$

can be seperated into formulas for even and odd values of n:

$$F_{1,m}(n) \;=\; \begin{cases}\dfrac{\Phi_{1,m}^{n} - \Phi_{1,m}^{-n}}{\sqrt{4+m^2}}, & \text{if } n \text{ is even}\\[2ex] \dfrac{\Phi_{1,m}^{n} + \Phi_{1,m}^{-n}}{\sqrt{4+m^2}}, & \text{if } n \text{ is odd.}\end{cases}$$

By replacing the discrete variable n by the continuous variable x, we obtain the *hyperbolic Fibonacci sine* $(1,m)$*-function*:

$$sF_{1,m}(x) \;=\; \frac{\Phi_{1,m}^{x} - \Phi_{1,m}^{-x}}{\sqrt{4+m^2}}, \tag{3.20}$$

and the *hyperbolic Fibonacci cosine* $(1,m)$*-function*:

$$cF_{1,m}(x) \;=\; \frac{\Phi_{1,m}^{x} + \Phi_{1,m}^{-x}}{\sqrt{4+m^2}}, \tag{3.21}$$

where $\Phi_{1,m} = \frac{m+\sqrt{4+m^2}}{2}$ is the Golden $(1,m)$-Proportion. [30]

As these functions are derived from Binet's formula it is credible to suggest that identities will exist that are similar to those of the Fibonacci $(1,m)$-numbers and special cases such as the Fibonacci numbers. Two such identities are analagous to the recursion relation for the Fibonacci $(1,m)$-numbers:

Theorem (Recursion relations for the hyperbolic Fibonacci $(1,m)$-functions)**.**

$$cF_{1,m}(x) \;=\; m \cdot sF_{1,m}(x-1) + cF_{1,m}(x-2), \tag{3.22}$$

$$sF_{1,m}(x) \;=\; m \cdot cF_{1,m}(x-1) + sF_{1,m}(x-2). \tag{3.23}$$

Proof. We will prove the first recursion relation[2] (3.22). The proof of the second relation (3.23) is very similar.

$$\begin{aligned} m \cdot sF_{1,m}(x-1) + cF_{1,m}(x-2) \;&=\; m\left(\frac{\Phi_{1,m}^{x-1} - \Phi_{1,m}^{-x+1}}{\sqrt{4+m^2}}\right) + \frac{\Phi_{1,m}^{x-2} + \Phi_{1,m}^{-x+2}}{\sqrt{4+m^2}} \\ &=\; \frac{\Phi_{1,m}^{x-2}(m\Phi_{1,m}+1) + \Phi_{1,m}^{-x+2}(1-\Phi_{1,m}^{-1})}{\sqrt{4+m^2}}. \end{aligned}$$

To simplify this expression we can use that $\Phi_{1,m}$ is a root of the algebraic equation (2.2). Therefore, by substituting $x = \Phi_{1,m}$ we obtain $\Phi_{1,m}^2 = m\Phi_{1,m}+1$. Dividing this through by $\Phi_{1,m}^2$ and rearranging gives $\Phi_{1,m}^{-2} = 1 - \Phi_{1,m}^{-1}$. Hence we can write

$$\begin{aligned} m \cdot sF_{1,m}(x-1) + cF_{1,m}(x-2) \;&=\; \frac{\Phi_{1,m}^{x} + \Phi_{1,m}^{-x}}{\sqrt{4+m^2}} \\ &=\; cF_{1,m}(x). \end{aligned}$$

□

We saw Catalan's identity earlier for the Fibonacci numbers (1.15) and also in an analagous form for the Quasi-sine Fibonacci $(1,m)$-function (3.16). An analagous form of Catalan's identity also exists for the hyperbolic Fibonacci $(1,m)$-functions[3] [12]:

Theorem (Catalan's identity for the hyperbolic Fibonacci $(1,m)$-functions)**.** *If* $r \in \mathbb{Z}$*, then*

$$[cF_{1,m}(x)]^2 - cF_{1,m}(x+r) \cdot cF_{1,m}(x-r) \;=\; -[sF_{1,m}(r)]^2. \tag{3.24}$$

[2] This proof follows a similar method to a proof in [30].

[3] This identity has been written in a form which coincides with the conventions used previously in this report.

Proof.

$$\begin{aligned}
[cF_{1,m}(x)]^2 - cF_{1,m}(x+r)\cdot cF_{1,m}(x-r) &= \left(\frac{\Phi_{1,m}^{x}+\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right)^2 - \left(\frac{\Phi_{1,m}^{x+r}+\Phi_{1,m}^{-x-r}}{\sqrt{4+m^2}}\right)\left(\frac{\Phi_{1,m}^{x-r}+\Phi_{1,m}^{-x+r}}{\sqrt{4+m^2}}\right)\\
&= \frac{\Phi_{1,m}^{2x}+2+\Phi_{1,m}^{-2x}-(\Phi_{1,m}^{2x}+\Phi_{1,m}^{2r}+\Phi_{1,m}^{-2r}+\Phi_{1,m}^{-2x})}{4+m^2}\\
&= -\left(\frac{\Phi_{1,m}^{2r}-2+\Phi_{1,m}^{2r}}{4+m^2}\right)\\
&= -\left(\frac{\Phi_{1,m}^{r}-\Phi_{1m}^{-r}}{\sqrt{4+m^2}}\right)^2\\
&= -[sF_{1,m}(r)]^2.
\end{aligned}$$

□

The hyperbolic Fibonacci sine and cosine functions were named as such due to their similarities to the functions $\sinh x$ and $\cosh x$. These similarities arise because the former are obtained from the latter by a direct substitution of the constant terms - the term $\Phi_{1,m}$ is substituted for e, and the constant coefficient is altered. Figures 3.2(a) and 3.2(b) show the similarities in overall shape between the respective functions, although it also illustrates that the gradients of the hyperbolic Fibonacci functions are steeper. Scaling the hyperbolic Fibonacci functions and plotting $y = sF_{1,m}(2x)$ and $y = \sinh x$ together (Figure 3.2(c)), and $y = cF_{1,m}(2x)$ and $y = \cosh x$ together (Figure 3.2(d)), shows an even closer resemblence. The graphs in Figure 3.2 were drawn using the computer program *Maple*.

It is apparent from Figure 3.2 that both $sF_{1,m}(x)$ and $\sinh x$ are odd functions as both are symmetric with respect to the x axis. It is common knowledge that this is the case for $\sinh x$ but we can be sure that it is true for $sF_{1,m}(x)$ by the following:

$$sF_{1,m}(-x) = \frac{\Phi_{1,m}^{-x}-\Phi_{1,m}^{x}}{\sqrt{4+m^2}} = -\left(\frac{\Phi_{1,m}^{x}-\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right) = -sF_{1,m}(x). \tag{3.25}$$

$cF_{1,m}(x)$ and $\cosh x$ are both even functions and the former is shown explicitly by[4]:

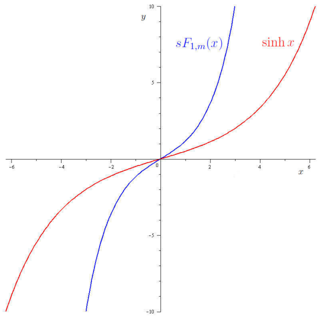
$$cF_{1,m}(-x) = \frac{\Phi_{1,m}^{-x}+\Phi_{1,m}^{x}}{\sqrt{4+m^2}} = \frac{\Phi_{1,m}^{x}+\Phi_{1,m}^{-x}}{\sqrt{4+m^2}} = cF_{1,m}(x). \tag{3.26}$$

The hyperbolic Fibonacci functions are not only similar to the hyperbolic sine and cosine functions in terms of their shape and symmetry but they also possess similar identities. The following are a short selection of such identities.

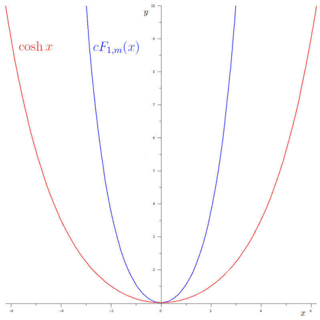
Theorem (Pythagorean theorem)**.** *The following identity is similar to* $\cosh^2 x - \sinh^2 x = 1$. *For* $m, x \in \mathbb{R}$,

$$\big(cF_{1,m}(x)\big)^2 - \big(sF_{1,m}(x)\big)^2 = \frac{4}{4+m^2}. \tag{3.27}$$

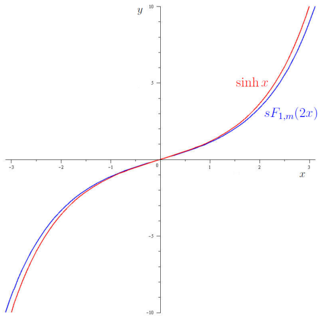
[4] I have corrected an error in the proof given in [30].



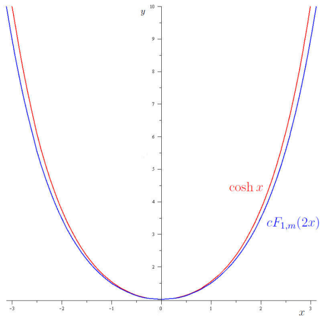
(a) $\sinh x$ and $sF_{1,m}(x)$.



(b) $\cosh x$ and $cF_{1,m}(x)$.



(c) $\sinh x$ and $sF_{1,m}(2x)$.



(d) $\cosh x$ and $cF_{1,m}(2x)$.

Figure 3.2: Comparisons between the hyperbolic sine and cosine functions, hyperbolic Fibonacci sine and cosine $(1, m)$-functions and scaled hyperbolic Fibonacci sine and cosine $(1, m)$-functions.

Proof. [30]

$$\begin{aligned}
\big(cF_{1,m}(x)\big)^2 - \big(sF_{1,m}(x)\big)^2 &= \left(\frac{\Phi_{1,m}^x + \Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right)^2 - \left(\frac{\Phi_{1,m}^x - \Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right)^2 \\
&= \frac{\Phi_{1,m}^{2x} + 2 + \Phi_{1,m}^{-2x} - \Phi_{1,m}^{2x} + 2 - \Phi_{1,m}^{-2x}}{4+m^2} \\
&= \frac{4}{4+m^2}.
\end{aligned}$$

□

Theorem (Sum and difference identities)**.** *For $x, y \in \mathbb{R}$,*

$$cF_{1,m}(x \pm y) = \frac{\sqrt{4+m^2}}{2}\Big(cF_{1,m}(x)cF_{1,m}(y) \pm sF_{1,m}(x)sF_{1,m}(y)\Big), \tag{3.28}$$

$$sF_{1,m}(x \pm y) = \frac{\sqrt{4+m^2}}{2}\Big(cF_{1,m}(x)sF_{1,m}(y) \pm sF_{1,m}(x)cF_{1,m}(y)\Big). \tag{3.29}$$

Proof. We will only prove the identity for $cF_{1,m}(x+y)$, as the proofs of the other three identites are very similar. The proof is adapted from [12].

By (3.20) and (3.21):

$$
\begin{aligned}
cF_{1,m}(x+y) &= \frac{\sqrt{4+m^2}}{2}\Big(cF_{1,m}(x)cF_{1,m}(y)+sF_{1,m}(x)sF_{1,m}(y)\Big) \\
&= \frac{\sqrt{4+m^2}}{2}\left(\frac{(\Phi^{x}_{1,m}+\Phi^{-x}_{1,m})(\Phi^{y}_{1,m}+\Phi^{-y}_{1,m})+(\Phi^{x}_{1,m}-\Phi^{-x}_{1,m})(\Phi^{y}_{1,m}-\Phi^{-y}_{1,m})}{4+m^2}\right) \\
&= \left(\frac{\Phi^{x+y}_{1,m}+\Phi^{x-y}_{1,m}+\Phi^{y-x}_{1,m}+\Phi^{-x-y}_{1,m}+\Phi^{x+y}_{1,m}-\Phi^{x-y}_{1,m}-\Phi^{y-x}_{1,m}+\Phi^{-x-y}_{1,m}}{2\sqrt{4+m^2}}\right) \\
&= \frac{\Phi^{x+y}_{1,m}+\Phi^{-x-y}_{1,m}}{\sqrt{4+m^2}}.
\end{aligned}
$$

By the definition of the hyperbolic Fibonacci cosine function (3.21), the proof is complete. □

Theorem (De Moivre's formula for the hyperbolic Fibonacci functions)**.** *For $m, x \in \mathbb{R}$ and $n \in \mathbb{Z}$ the following identity holds:*

$$
\Big(cF_{1,m}(x) \pm sF_{1,m}(x)\Big)^n = \left(\frac{2}{\sqrt{4+m^2}}\right)^{n-1}\Big(cF_{1m}(nx) \pm sF_{1,m}(nx)\Big). \tag{3.30}
$$

Proof. We will prove De Moivre's formula by considering three cases[5], and using identities and properties of the hyperbolic Fibonacci sine and cosine functions seen previously.

For $n \geq 1$ we use induction on n. The theorem clearly holds for the basic case $n = 1$. Assume that it also holds for a positive integer k:

$$
\Big(cF_{1,m}(x) \pm sF_{1,m}(x)\Big)^k = \left(\frac{2}{\sqrt{4+m^2}}\right)^{k-1}\Big(cF_{1m}(kx) \pm sF_{1,m}(kx)\Big)
$$

We now show that the theorem is true for $n = k+1$:

$$
\Big(cF_{1,m}(x) \pm sF_{1,m}(x)\Big)^{k+1} = \Big(cF_{1,m}(x) \pm sF_{1,m}(x)\Big)^k\Big(cF_{1,m}(x) \pm sF_{1,m}(x)\Big)
$$

By the induction hypothesis:

$$
\begin{aligned}
\Big(cF_{1,m}(x) \pm sF_{1,m}(x)\Big)^{k+1} &= \left(\frac{2}{\sqrt{4+m^2}}\right)^{k-1}\Big(cF_{1,m}(kx) \pm sF_{1,m}(kx)\Big)\Big(cF_{1,m}(x) \pm sF_{1,m}(x)\Big) \\
&= \left(\frac{2}{\sqrt{4+m^2}}\right)^{k-1}\Big(cF_{1,m}(kx)cF_{1,m}(x)+sF_{1,m}(kx)sF_{1,m}(x) \\
&\quad \pm\ cF_{1,m}(kx)sF_{1,m}(x) \pm sF_{1,m}(kx)cF_{1,m}(x)\Big) \\
&= \left(\frac{2}{\sqrt{4+m^2}}\right)^{k-1}\Bigg(cF_{1,m}(kx)cF_{1,m}(x)+sF_{1,m}(kx)sF_{1,m}(x) \\
&\quad \pm\ \Big(cF_{1,m}(kx)sF_{1,m}(x)+sF_{1,m}(kx)cF_{1,m}(x)\Big)\Bigg).
\end{aligned}
$$

[5]This theorem is stated but not proved in [30].

Using the sum and difference identities (3.28) and (3.29):

$$\begin{aligned}\Big(cF_{1,m}(x) \pm sF_{1,m}(x)\Big)^{k+1} &= \left(\frac{2}{\sqrt{4+m^2}}\right)^{k-1} \Big(\frac{2}{\sqrt{4+m^2}} cF_{1,m}(kx+x) \pm \frac{2}{\sqrt{4+m^2}} sF_{1,m}(kx+x)\Big) \\ &= \left(\frac{2}{\sqrt{4+m^2}}\right)^{k} \Big(cF_{1,m}((k+1)x) \pm sF_{1,m}((k+1)x)\Big).\end{aligned}$$

Therefore the theorem is true for $n = k+1$ if it is true for $n = k$. Hence, by induction it is true for all $n \geq 1$.

For the case $n = 0$ we can show the result to be correct:

$$\begin{aligned}\Big(cF_{1,m}(x) \pm sF_{1,m}(x)\Big)^{n} &= \left(\frac{2}{\sqrt{4+m^2}}\right)^{-1} \Big(cF_{1m}(0) \pm sF_{1,m}(0)\Big) \\ &= \frac{\sqrt{4+m^2}\big(\Phi_{1,m}^0 + \Phi_{1,m}^0 \pm (\Phi_{1,m}^0 - \Phi_{1,m}^0)\big)}{2\sqrt{4+m^2}} \\ &= 1.\end{aligned}$$

For $n < 0$, let m be a positive integer such that $n = -m$. Then

$$\begin{aligned}\big(cF_{1,m}(x) \pm sF_{1,m}(x)\big)^{n} &= \big(cF_{1,m}(x) \pm sF_{1,m}(x)\big)^{-m} \\ &= \frac{1}{\big(cF_{1,m}(x) \pm sF_{1,m}(x)\big)^{m}} \\ &= \left(\frac{2}{\sqrt{4+m^2}}\right)^{-(m-1)} \frac{1}{cF_{1,m}(mx) \pm sF_{1,m}(mx)}\end{aligned}$$

Multiplying the nominator and the denominator by $cF_{1,m}(mx) \mp sF_{1,m}(mx)$ gives

$$\begin{aligned}\big(cF_{1,m}(x) \pm sF_{1,m}(x)\big)^{n} &= \left(\frac{2}{\sqrt{4+m^2}}\right)^{-m+1} \frac{cF_{1,m}(mx) \mp sF_{1,m}(mx)}{\big(cF_{1,m}(mx) \pm sF_{1,m}(mx)\big)\big(cF_{1,m}(mx) \mp sF_{1,m}(mx)\big)} \\ &= \left(\frac{2}{\sqrt{4+m^2}}\right)^{-m+1} \frac{cF_{1,m}(mx) \mp sF_{1,m}(mx)}{\big(cF_{1,m}(x)\big)^2 - \big(sF_{1,m}(x)\big)^2}.\end{aligned}$$

By the Pythagorean theorem (3.27), we know that $\big(cF_{1,m}(x)\big)^2 - \big(sF_{1,m}(x)\big)^2 = \frac{4}{4+m^2}$.

Therefore

$$\begin{aligned}\big(cF_{1,m}(x) \pm sF_{1,m}(x)\big)^{n} &= \left(\frac{2}{\sqrt{4+m^2}}\right)^{-m+1} \left(\frac{4+m^2}{4}\right) \big(cF_{1,m}(mx) \mp sF_{1,m}(mx)\big) \\ &= \left(\frac{2}{\sqrt{4+m^2}}\right)^{-m-1} \big(cF_{1,m}(mx) \mp sF_{1,m}(mx)\big).\end{aligned}$$

Using that $sF_{1,m}(mx)$ is an odd function (3.25), and $cF_{1,m}(mx)$ is an even function (3.26):

$$\begin{aligned}\big(cF_{1,m}(x) \pm sF_{1,m}(x)\big)^n &= \left(\frac{2}{\sqrt{4+m^2}}\right)^{-m-1} \big(cF_{1,m}(-mx) \pm sF_{1,m}(-mx)\big) \\ &= \left(\frac{2}{\sqrt{4+m^2}}\right)^{n-1} \big(cF_{1,m}(nx) \pm sF_{1,m}(nx)\big).\end{aligned}$$

Hence, we have proved the theorem for all n. □

Moving away from the identities of the hyperbolic Fibonacci functions and back to the discussion of the geometry of these curves, we conclude this section by considering the relationship between $sF_{1,m}(x)$, $cF_{1,m}(x)$ and the Quasi-sine Fibonacci function $FF_{1,m}(x)$.

Let n be an integer on the continuous domain x. Then for $n = 2k$ with $k \in \mathbb{Z}$ (ie. even integers):

$$\begin{aligned}FF_{1,m}(2k) &= \frac{\Phi_{1,m}^{2k} - \cos(2\pi k)\Phi_{1,m}^{-2k}}{\sqrt{4+m^2}} \\ &= \frac{\Phi_{1,m}^{2k} - \Phi_{1,m}^{-2k}}{\sqrt{4+m^2}} \\ &= sF_{1,m}(2k) \;=\; F_{1,m}(2k).\end{aligned} \tag{3.31}$$

For $n = 2k+1$ (ie. odd integers):

$$\begin{aligned}FF_{1,m}(2k+1) &= \frac{\Phi_{1,m}^{2k+1} - \cos(2\pi k+\pi)\Phi_{1,m}^{-2k-1}}{\sqrt{4+m^2}} \\ &= \frac{\Phi_{1,m}^{2k+1} + \Phi_{1,m}^{-2k-1}}{\sqrt{4+m^2}} \\ &= cF_{1,m}(2k+1) \;=\; F_{1,m}(2k+1).\end{aligned} \tag{3.32}$$

Therefore the Quasi-sine Fibonacci $(1,m)$-function coincides with the hyperbolic Fibonacci sine $(1,m)$-function at the co-ordinates $(2k, F_{1,m}(2k))$, and the hyperbolic Fibonacci cosine $(1,m)$-function at $(2k+1, F_{1,m}(2k+1))$. Figure 3.3 shows that the hyperbolic Fibonacci functions are in fact the envelopes of the Quasi-sine Fibonacci function.

3.4 The Quasi-sine Fibonacci $(2,m)$-function

We have seen an example of the Quasi-sine (p,m)-Fibonacci function where p is odd, and the properties of this function and its envelopes, the hyperbolic Fibonacci sine and cosine functions. This section takes us through an example of the Quasi-sine (p,m)-Fibonacci function where p is even and utilises the general theory from §3.1.

The particular case we will derive is the Quasi-sine Fibonacci $(2,m)$-function. Recall that in Ex-

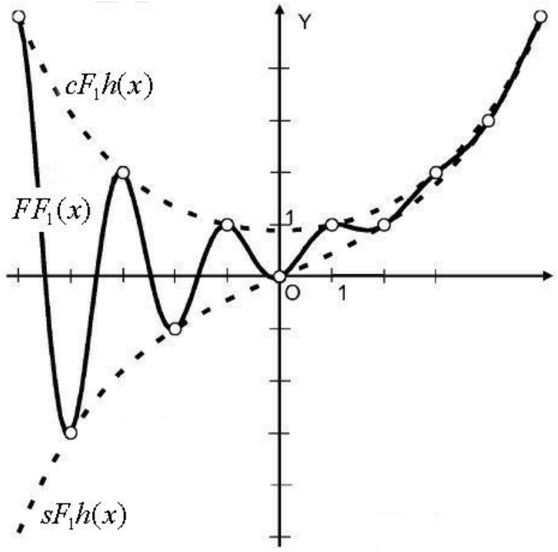


Figure 3.3: The Quasi-sine Fibonacci $(1, m)$-function and its envelopes. [32]

ample 8 we found the Binet formula for the Fibonacci $(2, m)$-numbers to be

$$F_{2,m}(n) = \frac{2h(h+2m)}{h^3+8m^3}\left(\frac{h^2+2mh+4m^2}{6h}\right)^n$$
$$+ \frac{h(-(h+2m)+i\sqrt{3}(h-2m))}{h^3+8m^3}\left(-\frac{h^2-4mh+4m^2}{12h}+i\sqrt{3}\left(\frac{h}{12}-\frac{m^2}{3h}\right)\right)^n$$
$$+ \frac{h(-(h+2m)-i\sqrt{3}(h-2m))}{h^3+8m^3}\left(-\frac{h^2-4mh+4m^2}{12h}-i\sqrt{3}\left(\frac{h}{12}-\frac{m^2}{3h}\right)\right)^n,$$

where $h = \sqrt[3]{108+8m^3+12\sqrt{81+12m^3}}$.

Notice that in line with the general theory discussed in §3.1, there are an odd number of roots and coefficients - three roots and three coefficients to be exact. The first root is the Golden $(2, m)$-Proportion while the other two roots are complex conjugates. The coefficients of the second and third roots are also complex conjugate.

Hence, the Binet formula above can be written in the form:

$$F_{2,m}(n) = \frac{2h(h+2m)}{h^3+8m^3}\left(\frac{h^2+2mh+4m^2}{6h}\right)^n + (c-id)(a+ib)^n + (c+id)(a-ib)^n,$$

where

$$a = -\frac{h^2-4mh+4m^2}{12h}, \qquad b = \sqrt{3}\left(\frac{h}{12}-\frac{m^2}{3h}\right),$$
$$c = \frac{-h(h+2m)}{h^3+8m^3}, \qquad d = \frac{h\sqrt{3}(h-2m)}{h^3+8m^3}.$$

We can then use the general formula (3.10) with $p = 2$ to rewrite Binet's formula as

$$F_{2,m}(n) = \frac{2h(h+2m)}{h^3+8m^3}\left(\frac{h^2+2mh+4m^2}{6h}\right)^n + 2\sqrt{c^2+d^2}(a^2+b^2)^{\frac{n}{2}}\cos(n\theta-\gamma),$$

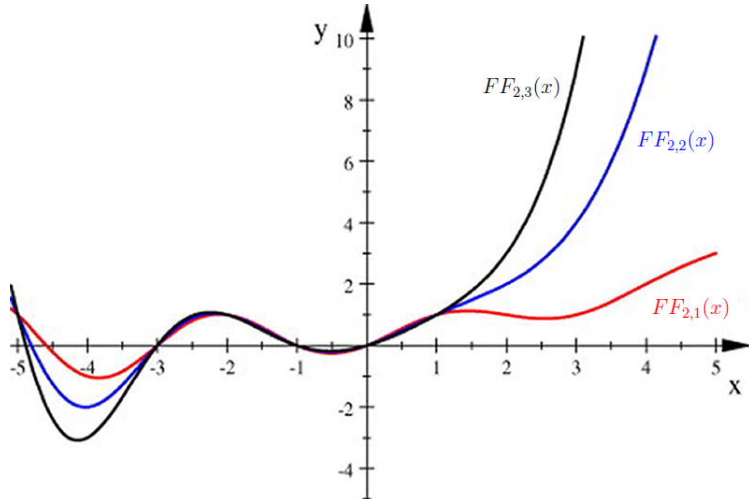


Figure 3.4: The Quasi-sine Fibonacci $(2, m)$-function for $m = 1, 2, 3$. [17]

where $\theta = \arccos \frac{a}{\sqrt{a^2+b^2}}$ and $\gamma = \frac{c}{\sqrt{c^2+d^2}}$.

Substituting the continuous variable x for n gives the Quasi-sine Fibonacci $(2, m)$-function [17]:

$$F_{2,m}(x) = \frac{2h(h+2m)}{h^3+8m^3}\left(\frac{h^2+2mh+4m^2}{6h}\right)^x + \left\{\frac{4h\sqrt{h^2-2hm+4m^2}}{h^3+8m^3}\left(\frac{(h-2m)\sqrt{h^2+2hm+4m^2}}{6h}\right)\right.$$
$$\left.\times\ \cos\left(x \arccos\left(\frac{2m-h}{2\sqrt{h^2+2hm+4m^2}}\right) - \arccos\left(-\frac{2m+h}{2\sqrt{h^2-2hm+4m^2}}\right)\right)\right\}. \quad (3.33)$$

The graph of this function is shown in Figure 3.4. We could apply similar analysis to this function as we did to the Quasi-sine Fibonacci $(1, m)$ and find analogous identities. We can also hypothesise that the hyperbolic Fibonacci sine and cosine $(2, m)$-functions will be the envelopes of this function for similar reasons to those discussed in §3.3. However we will not discuss the Quasi-sine Fibonacci $(2, m)$-function in any greater detail in this chapter.

In the remainder of this chapter we introduce some curves and surfaces lying in three-dimensional space. These are related to the Quasi-sine Fibonacci $(1, m)$-function and the hyperbolic Fibonacci $(1, m)$-functions. We then discuss a potential application within physical cosmology.

3.5 The three-dimensional Fibonacci $(1, m)$-spiral

The Quasi-sine Fibonacci $(1, m)$-function is the projection onto the plane XOY of the three-dimensional Fibonacci $(1, m)$-spiral. The spiral is produced by the translational movement of a point on an infinite rotating horn-shaped surface and is defined by the function [13]:

$$CFF_{1,m}(x) = \frac{\Phi_{1,m}^x - \cos(\pi x)\Phi_{1,m}^{-x}}{\sqrt{4+m^2}} + i\,\frac{\sin(\pi x)\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}. \quad (3.34)$$

When x takes integer values, the imaginary part of $CFF_{1,m}(x)$ vanishes, as $\sin(\pi x) = 0$. This leaves the Gazale formula (2.18), and therefore the Fibonacci $(1, m)$-numbers are produced at integer values of x.

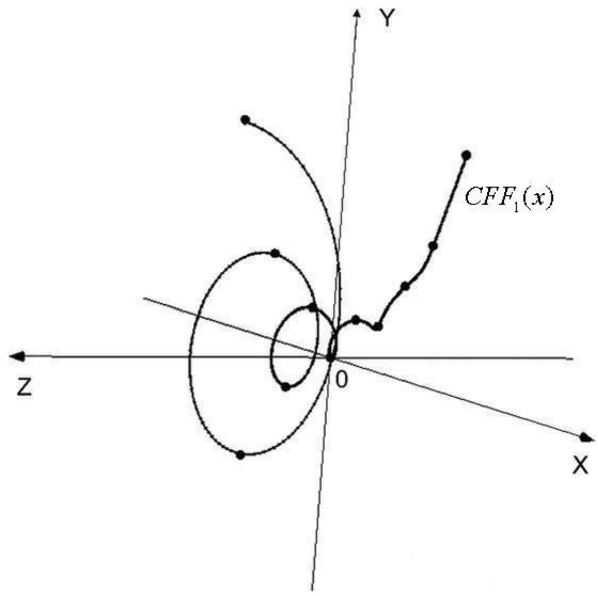


Figure 3.5: The three-dimensional Fibonacci $(1, m)$-spiral for $m = 1$. [27]

As with the Quasi-sine Fibonacci $(1, m)$-function and the hyperbolic Fibonacci $(1, m)$-functions, the three-dimensional Fibonacci $(1, m)$-spiral possesses identities analagous to the Fibonacci $(1, m)$-numbers. Two such identities are the recurrence relation [12]:

$$CFF_{1,m}(x) \;=\; m \cdot CFF_{1,m}(x-1) + CFF_{1,m}(x-2), \tag{3.35}$$

and Catalan's identity:

$$\big(CFF_{1,m}(x)\big)^2 - CFF_{1,m}(x+r) \cdot CFF_{1,m}(x-r) \;=\; (-1)^r \big(CFF_{1,m}(r)\big)^2. \tag{3.36}$$

We have already seen proofs of similar identities for the Quasi-sine Fibonacci $(1, m)$-function and the hyperbolic Fibonacci $(1, m)$-functions. Therefore the proofs of (3.35) and (3.36) will not be given here.

3.6 Metallic Shofars

If we consider the axis OY as the real axis and the axis OZ as the imaginary axis, then

$$y(x) \;=\; \mathrm{Re}\big(CFF_{1,m}(x)\big) \;=\; \frac{\Phi_{1,m}^{x} - \cos(\pi x)\Phi_{1,m}^{-x}}{\sqrt{4+m^2}} \tag{3.37}$$

and

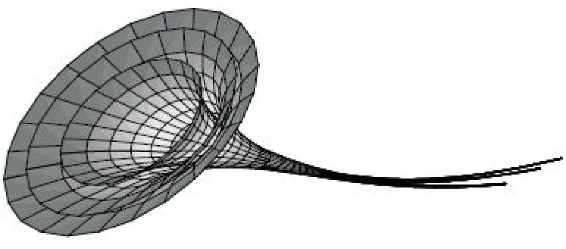
$$z(x) \;=\; \mathrm{Im}\big(CFF_{1,m}(x)\big) \;=\; \frac{\sin(\pi x)\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}. \tag{3.38}$$

Equation (3.37) can be rewritten as:

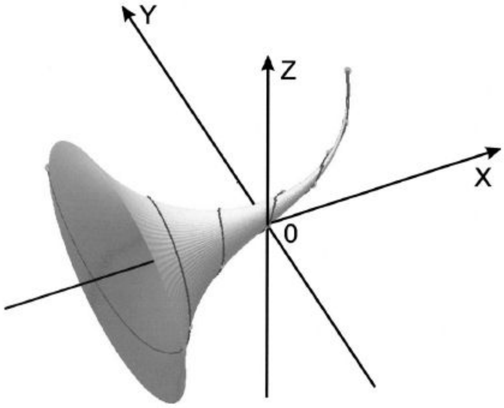
$$y(x) - \frac{\Phi_{1,m}^{x}}{\sqrt{4+m^2}} \;=\; -\frac{\cos(\pi x)\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}. \tag{3.39}$$

If we then square both sides of each of the equations (3.38) and (3.39) we obtain:

$$\left(y - \frac{\Phi_{1,m}^{x}}{\sqrt{4+m^2}}\right)^2 \;=\; \left(\frac{\cos(\pi x)\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right)^2$$



(a) The Metallic Shofars for $m = 1, 2, 3$. [14]



(b) The three-dimensional Fibonacci $(1, 1)$-spiral lying on the surface of the Golden Shofar, which is produced when $m = 1$. [32]

Figure 3.6

and

$$z^2 = \left(\frac{\sin(\pi x)\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right)^2.$$

Therefore

$$\left(y - \frac{\Phi_{1,m}^{x}}{\sqrt{4+m^2}}\right)^2 + z^2 = \left(\frac{\cos(\pi x)\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right)^2 + \left(\frac{\sin(\pi x)\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right)^2$$

$$= \left(\frac{\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right)^2 \left(\cos^2(\pi x) + \sin^2(\pi x)\right).$$

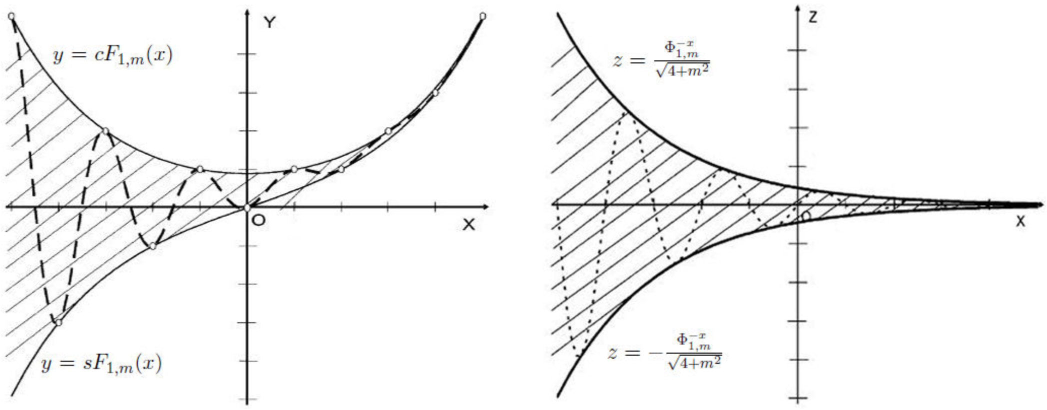
This can be simplified using the trigonometric identity $\sin^2(\theta) + \cos^2(\theta) = 1$ to give:

$$\left(y - \frac{\Phi_{1,m}^{x}}{\sqrt{4+m^2}}\right)^2 + z^2 = \left(\frac{\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right)^2. \tag{3.40}$$

This is the equation for a hyperbolic (negatively curved) surface of revolution called the *Metallic Shofar*[6] (Figure 3.6(a)) which possesses the shape of a horn or funnel with a bell and tail which extend infinitely. The three-dimensional Fibonacci $(1, m)$-spiral lies on its surface. Figure 3.6(b) shows this for $m = 1$. [13]

Rearranging (3.40) shows the relationship between the Metallic Shofar and the hyperbolic Fibonacci

[6] A shofar is a horn commonly found at Jewish religious occassions.



(a) Projection onto the plane XOY. [27] (b) Projection onto the plane XOZ. [27]

Figure 3.7

$(1, m)$-functions:

$$
\begin{aligned}
z^2 &= \left(\frac{\Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right)^2 - \left(y - \frac{\Phi_{1,m}^{x}}{\sqrt{4+m^2}}\right)^2 \\
&= \left(\frac{\Phi_{1,m}^{-x}}{\sqrt{4+m^2}} + \left(y - \frac{\Phi_{1,m}^{x}}{\sqrt{4+m^2}}\right)\right) \cdot \left(\frac{\Phi_{1,m}^{-x}}{\sqrt{4+m^2}} - \left(y - \frac{\Phi_{1,m}^{x}}{\sqrt{4+m^2}}\right)\right) \\
&= \left(y - \frac{\Phi_{1,m}^{x} - \Phi_{1,m}^{-x}}{\sqrt{4+m^2}}\right) \cdot \left(\frac{\Phi_{1,m}^{x} + \Phi_{1,m}^{-x}}{\sqrt{4+m^2}} - y\right) \\
z^2 &= (y - cF_{1,m}(x)) \cdot (sF_{1,m}(x) - y)
\end{aligned}
$$

Therefore we can see in Figure 3.7(a) that projecting the surface onto the plane XOY by taking $z = 0$ results in a region bounded by the curves $y = cF_{1,m}(x)$ and $y = sF_{1,m}(x)$. The interior curve is the Quasi-sine Fibonacci $(1, m)$-function, the projection of the three-dimensional Fibonacci $(1, m)$-spiral. Similarly, the projection of the Metallic Shofar in the plane XOZ (Figure 3.7(b)) gives a region bounded by the exponential curves $z = \pm(\Phi_{1,m}^{-x}/\sqrt{4+m^2})$. [13]

If we let x be a constant in equation (3.40), we obtain an equation of the form $(y - y_0)^2 + (z - z_0)^2 = r^2$. This is the equation of a circle centered at (y_0, z_0) with radius r. Therefore all intersections of the Metallic Shofar by planes parallel to the plane YOZ give a circle centered at $(y, z) = (\Phi_{1,m}^{x}/\sqrt{4+m^2}, 0)$ with radius $\Phi_{1,m}^{-x}/\sqrt{4+m^2}$.

It is interesting to note that the authors who derived the Metallic Shofar have proposed that it could model the spatial section of the universe. As the universe cannot be viewed from the outside, we rely on astronomical data and astrophysical discoveries to try to determine the accuracy of such hypotheses. According to the article [3], recordings of cosmic microwave background radiation collected by NASA's Wilkinson Microwave Anisotropy Probe do not rule out the possibility that the universe has the shape of an 'infinitely long horn'. The Metallic Shofar is certainly compliant with this description, but it does not seem to be considered as seriously as other horn topologies, such as the Picard horn. Furthermore, hyperbolic geometries such as the Metallic Shofar are not the only ones proposed. Some have also considered spherical geometries, such as the Poincaré Dodecahedral

Space[7], which have a positive curvature, and many believe the universe to be flat. [23]

Determining the topology of the universe is important in determining its future. A negatively curved universe, which includes one that has the shape of a Metallic Shofar, continues to expand forever. This expansion is accelerated by the presence of dark matter and ultimately, it would lead either to a universe too cold to sustain life (the 'Big Freeze') or the disintegration of matter into unbound elementary particles and radiation (the 'Big Rip'). [33] Therefore it is clear that further research into the shape of universe is required. It will be interesting to see whether the hypothesis that the universe has the shape of a Metallic Shofar survises further investigation.

In summary: The Quasi-sine Fibonacci (p, m)-function is derived from the general form of Binet's formula using polar coordinates and De Moivre's formula. It possesses many properties which are analagous to the Fibonacci (p, m)-numbers. The hyperbolic Fibonacci (p, m)-functions are the envelopes of the Quasi-sine Fibonacci (p, m)-function, and they possess many similar properties to the hyperbolic sine and cosine functions. The Quasi-sine Fibonacci function is the projection of the three-dimensional Fibonacci spiral. This spiral lies on the surface of the Metallic Shofar, a hyperbolic surface proposed as a possible model for the spatial section of our universe.

[7] A closed 3-manifold constructed by gluing pairs of opposite faces of a dodecahedron together.

Chapter 4

Fibonacci Coding Theory

The Fibonacci p-numbers can also be applied to a very different field - coding theory. In this chapter we introduce the Fibonacci coding theory and method before analysing its practicality.

4.1 An introduction to coding theory

The modern world relies heavily on the transmission of data over varying distance. Coding theory is a combination of pure mathematics, algebra and engineering, which addresses difficulties in the reliable and efficient transmission of digital data from an information source to an information sink.

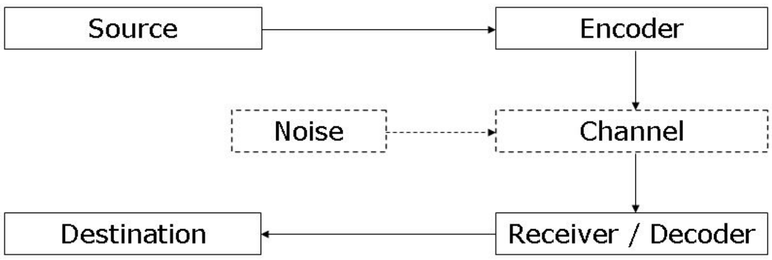


Figure 4.1: A basic model of the communication system. [7]

Figure 4.1 shows a communication system where a digital message is sent from a *source* to its *destination* through a *communication channel.* This data transmission can occur either in the space domain, where a message is transferred between two distinct locations; or the time domain, where some data is stored and retrieved at a later time. Coding of a message can generally be seperated into two distinct categories, *source coding* and *channel coding.* In practice either one or both types of coding may be used.

Source coding is a form of data compression where the aim is to increase efficiency within the network. There are two forms of source coding - *lossy* and *lossless.* Lossy source coding discards data to reduce both the transmission time through the channel and the memory space required to store the data. It cannot be reversed to obtain all of the original data. An example of this type of data compression is found with JPEG image files where the image quality is reduced if the file size is made smaller. Lossless source coding, such as Zip data compression on the internet, usually exploits statistical redundancy to make data more compact without a loss of data or quality. Data compressed by a lossless source code can be decompressed to its original form. [9]

A channel is a medium through which data can be transferred and *channel coding* can follow one of two methods depending on whether this channel is noisy or noiseless. A noisy channel is one that is unreliable as there is the possibility that 'noise' can cause errors to the transmitted message such that the output from the channel is not the same as the input. Scratches and dust on a CD causing a track to skip and meteor showers causing images of deep space sent from satellite stations to distort are a couple of examples. For the purpose of our application we are interested in noisy channels[1]. *Error-correcting codes* are used with the objective of detecting and correcting any errors caused within a noisy channel.

In the following chapter we will consider the transmission of a single message through the communication system. We will assume that source coding has already been applied to this message such that it is now represented by an ordered list of t symbols, $U = \{u_1, u_2, \ldots, u_t\}$. These symbols belong to an *alphabet* $\Lambda_r = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$, a finite set of r distinct symbols. It would be unwise to send a message through the channel in this form as there are no means by which the receiver on the other side of the channel can detect or correct any errors which occur to the symbols in the message due to noise. This is why channel coding is used. The message U is encoded using an error correcting code to produce a codeword, c. There is a one-to-one mapping between a message U and its coresponding codeword. In other words, no two messages U and V can both be encoded to the same codeword and vice versa.

The coding adds additional information which helps the receiver to detect and correct errors. This additional information is called *redundancy* and can also become erroneous due to noise in the channel. Therefore it does not fully guarantee that the receiver will be able to detect or correct any errors in a recieved code. Redundancy is proportional to the probability of a successful transmission but inversely proportional to efficiency within the network. Finding the least redundancy necessary to obtain a sufficiently high probability of successful transmission is one of the fundamental problems in coding theory.

The codeword is sent through a channel about which we must make some assumptions. Firstly, the channel is *discrete* which means that only finite alphabets are used; and secondly it is *memoryless* as a symbol error to one element of the codeword does not increase or decrease the probability of an error occuring elsewhere in the codeword. A further assumption is that errors do not change the number of elements in the codeword.

On the other side of the channel, an output is received that may or may not be equivalent to the codeword that was sent. The receiver uses the redundant information and possibly knowledge about properties of the error correcting code to detect where errors have occured. If all errors are corrected, this is called *complete decoding* and the corrected output is then decoded to obtain the original message. If there are detected errors which remain uncorrected, then we say that *incomplete decoding* has occurred. In this instance, retransmission may be requested if the channel is *two-way* and it is possible to do so. However when a channel is *one-way* - an example being from a deep space probe to Earth - retransmission is not possible and the message is lost.

[1]Coding theory differs for noiseless channels as the aim is to optimise the channel's usage.

4.2 The Fibonacci coding method

Let the *message matrix* M be a square $(p+1)\times(p+1)$ matrix where p is an integer greater than or equal to one. [31] Matrix elements within M are denoted[2] by $m_{i,j}$, where i corresponds to rows of M and j corresponds to columns of M, with $1 \leq i, j \leq (p+1)$:

$$M_{(p+1)\times(p+1)} = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,p+1} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_{p+1,1} & m_{p+1,2} & \cdots & m_{p+1,p+1} \end{pmatrix}.$$

Given a message $U = \{u_1, u_2, \ldots, u_t\}$, an ordered list of t message symbols, the first step in the Fibonacci coding method is to arrange the symbols into a message matrix M. This is done row by row from top left to bottom right, subject to the following conditions:

1. The ordering of the message symbols must be preserved;
2. Each matrix element $m_{i,j}$ must contain at least one message symbol.

The first condition is maintained so that the message U can be read directly from M at the destination. To be able to uphold the second condition, the parameter p must be chosen such that $(p+1)^2 \leq t$, to ensure that there are at least as many message symbols as matrix elements.

Example 9. Consider the message $\{1,3,5,7,9,2,4,6,8\}$. There are nine message elements so p must be chosen such that $(p+1)^2 \leq 9$. Therefore the message can be arranged either in a 2×2 or a 3×3 matrix. Two permitted arrangements are:

$$i)\ M_{2\times 2} = \begin{pmatrix} 135 & 79 \\ 24 & 68 \end{pmatrix}, \qquad ii)\ M_{3\times 3} = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 2 \\ 4 & 6 & 8 \end{pmatrix}.$$

However these two arrangements are not permitted as they both violate one of the conditions above:

$$iii)\ M'_{2\times 2} = \begin{pmatrix} 135 & 24 \\ 79 & 68 \end{pmatrix}, \qquad iv)\ M'_{3\times 3} = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 2 \\ 4 & 68 & \end{pmatrix}.$$

Note that if p is chosen such that $(p+1)^2 < t$, as is the case with $i)$ in Example 10, then at least one matrix element $m_{i,j}$ will hold more than one message element[3]. To ensure that there is no confusion at the destination as to whether, for example, 135 should be interpreted: as the integer *one hundred and thirty-five*; as the integers *one* and *thirty-five* placed next each other; as *thirteen* and *five* placed next to each other; or as three seperate integers *one*, *three* and *five*, we will assume that, after source coding, all the message symbols belong to the alphabet $\Lambda_{10} = \{0,1,2,3,4,5,6,7,8,9\}$. In practice a larger alphabet could be considered.

However in the intermediary stages between the source and destination, message symbols placed together in a single matrix element $m_{i,j}$ *are* treated as a single symbol. For example, during coding, decoding and error correction the element $m_{1,1}$ of $i)$ in Example 10 is interpreted as the integer *one hundred and thirty-five* and not as three seperate integers placed together. Therefore the message

[2] Note that much of the notation used within this chapter differs from that used in [4] and [31]. This is simply to aid the understanding of the coding theory.

[3] This is an example of the *pidgeonhole principal* which states that if y items are put into x pidgeonholes where $y > x$, then at least one pidgeonhole must contain more than one item.

symbols must be arranged in M such that 0 is never the first symbol in a 'block' of symbols in a matrix element $m_{i,j}$. This is so that, for example, 012 is not interpreted as the integer *twelve* and then decoded as 12.

Note that once the message symbols have been inserted into the message matrix M, it is possible that some matrix elements $m_{i,j}$ may only contain a single symbol which is 0. In order to derive an error correcting relation later in this chapter, we will require that all message matrix elements must be strictly positive. [31] Therefore an integer μ is added to each element $m_{i,j}$ to obtain a matrix $\hat{M}$ where all elements $\hat{m}_{i,j}$ are strictly positive:

$$\hat{M}_{(p+1)\times(p+1)} = \begin{pmatrix} \hat{m}_{1,1} & \hat{m}_{1,2} & \cdots & \hat{m}_{1,p+1} \\ \hat{m}_{2,1} & \hat{m}_{2,2} & \cdots & \hat{m}_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{m}_{p+1,1} & \hat{m}_{p+1,2} & \cdots & \hat{m}_{p+1,p+1} \end{pmatrix}. \tag{4.1}$$

Now recall that in §1.2, we represented the recursion relation for the Fibonacci p-numbers in matrix form (1.6), using the Fibonacci Q_p matrix (1.7). We then proved by induction that the $(p+1)\times(p+1)$ Q_p^n matrix is

$$Q_p^n = \begin{pmatrix} F_p(n+1) & F_p(n) & \cdots & F_p(n-p+2) & F_p(n-p+1) \\ F_p(n-p+1) & F_p(n-p) & \cdots & F_p(n-2p+2) & F_p(n-2p+1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ F_p(n-1) & F_p(n-2) & \cdots & F_p(n-p) & F_p(n-p-1) \\ F_p(n) & F_p(n-1) & \cdots & F_p(n-p+1) & F_p(n-p) \end{pmatrix}, \tag{4.2}$$

where $F_p(n)$ is the n^{th} Fibonacci p-number (1.1).

The Q_p^n matrix is used as the *encoding matrix* as its dimensions are equal to those of $\hat{M}$ and it contains properties which prove useful later. The value of n is chosen freely - the optimal value of n will be discussed later. The parameter p has already been determined when choosing the dimensions of the message matrix M.

To encode the matrix $\hat{M}$ we use the formula [31]:

$$\hat{M} \times Q_p^n = E \tag{4.3}$$

and obtain the *code matrix*[4]:

$$E = \begin{pmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,p+1} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{p+1,1} & e_{p+1,2} & \cdots & e_{p+1,p+1} \end{pmatrix}. \tag{4.4}$$

This code matrix is sent through the channel, together with the determinant of $\hat{M}$. In the channel, symbol errors may occur to one or more of the elements $e_{i,j}$ in the code matrix E. These errors can be represented in the form of an *error matrix*:

$$\Sigma = \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,p+1} \\ \sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p+1,1} & \sigma_{p+1,2} & \cdots & \sigma_{p+1,p+1} \end{pmatrix}. \tag{4.5}$$

[4]The code matrix is equivalent to the codeword discussed in §4.1. Codeword is the term conventionally used, as elements are usually represented in vector form.

Therefore, on the other side of the channel, the *received code matrix* E' is the addition of the code matrix E and the error matrix Σ:

$$E' = \begin{pmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,p+1} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{p+1,1} & e_{p+1,2} & \cdots & e_{p+1,p+1} \end{pmatrix} + \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,p+1} \\ \sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p+1,1} & \sigma_{p+1,2} & \cdots & \sigma_{p+1,p+1} \end{pmatrix}$$

$$= \begin{pmatrix} e'_{1,1} & e'_{1,2} & \cdots & e'_{1,p+1} \\ e'_{2,1} & e'_{2,2} & \cdots & e'_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ e'_{p+1,1} & e'_{p+1,2} & \cdots & e'_{p+1,p+1} \end{pmatrix}. \quad (4.6)$$

Also received is $(\det \hat{M})'$, which may or may not be equivalent to $\det \hat{M}$, depending on whether noise has caused an error. We use two checking relations to detect if errors are present in the matrix E'. If no errors are detected then the received code matrix E' is equivalent to the sent code matrix E and the error matrix Σ consists entirely of zero elements. The matrix E is then decoded to $\hat{M}$, using the inverse of the Q_p^n matrix, by the formula:

$$E \times Q_p^{-n} = \hat{M}. \quad (4.7)$$

However if errors are detected in the matrix E', then the two checking relations are used to try to correct the errors. If error correction is successful the matrix E is obtained. This is decoded to $\hat{M}$ as above and therefore complete decoding has occurred. Then the original message matrix M is obtained by subtracting the constant μ from $\hat{M}$ and the message is subsequently read from M at the destination. However, if the errors are detected but cannot be corrected, then incomplete decoding has occurred and retransmission is requested if the channel is two-way. [31]

4.3 Checking relations

We will now prove two checking relations which, as previously mentioned, are required for error detection and correction.

Theorem. *For $p, n \in \mathbb{Z}$ and $p \geq 1$:*

$$\det E = (-1)^{pn} \times \det \hat{M}. \quad (4.8)$$

Proof. This proof is taken from [31], although different notation is used.

Consider the formula for encoding the matrix $\hat{M}$:

$$E = \hat{M} \times Q_p^n.$$

Taking the determinant of both sides of the equation and using matrix theory[5] gives:

$$\det E = \det(\hat{M} \times Q_p^n)$$

$$= \det \hat{M} \times \det Q_p^n.$$

In §1.2, we proved that $\det Q_p^n = (-1)^{pn}$. Therefore

$$\det E = (-1)^{pn} \times \det \hat{M}.$$

□

[5] If A and B are two square matrices of the same dimension then $\det(AB) = \det(A) \times \det(B)$. [2]

Theorem. *For $p = 1$, and finite values of n, the following connections between elements of the code matrix E exist:*

$$\frac{e_{1,1}}{e_{1,2}} \approx \tau, \qquad \frac{e_{2,1}}{e_{2,2}} \approx \tau.$$

Proof. This proof is adapted from [31].

Consider the Fibonacci decoding formula $\hat{M} = E \times Q_p^{-n}$:

$$\begin{pmatrix} \hat{m}_{1,1} & \hat{m}_{1,2} \\ \hat{m}_{2,1} & \hat{m}_{2,2} \end{pmatrix} = \begin{pmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} \end{pmatrix} \times \frac{1}{\det Q_1^n} \begin{pmatrix} -F(n-1) & F(n) \\ F(n) & -F(n+1) \end{pmatrix}.$$

Applying the identity (1.9), this is equivalent to:

$$\begin{pmatrix} \hat{m}_{1,1} & \hat{m}_{1,2} \\ \hat{m}_{2,1} & \hat{m}_{2,2} \end{pmatrix} = \begin{pmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} \end{pmatrix} \times (-1)^{-n} \begin{pmatrix} -F(n-1) & F(n) \\ F(n) & -F(n+1) \end{pmatrix}.$$

Matrix multiplication results in the following four equations which give the matrix elements of $\hat{M}$ in terms of elements of E and the Fibonacci numbers:

$$\begin{aligned}
\hat{m}_{1,1} &= -\big((-1)^{-n}F(n-1)\big)e_{1,1} + \big((-1)^{-n}F(n)\big)e_{1,2}, \\
\hat{m}_{1,2} &= \big((-1)^{-n}F(n)\big)e_{1,1} - \big((-1)^{-n}F(n+1)\big)e_{1,2}, \\
\hat{m}_{2,1} &= -\big((-1)^{-n}F(n-1)\big)e_{2,1} + \big((-1)^{-n}F(n)\big)e_{2,2}, \\
\hat{m}_{2,2} &= \big((-1)^{-n}F(n)\big)e_{2,1} - \big((-1)^{-n}F(n+1)\big)e_{2,2}.
\end{aligned}$$

Recall that the elements of $\hat{M}$ are all strictly positive. Therefore

$$\begin{aligned}
-\big((-1)^{-n}F(n-1)\big)e_{1,1} + \big((-1)^{-n}F(n)\big)e_{1,2} &> 0, \\
\big((-1)^{-n}F(n)\big)e_{1,1} - \big((-1)^{-n}F(n+1)\big)e_{1,2} &> 0, \\
-\big((-1)^{-n}F(n-1)\big)e_{2,1} + \big((-1)^{-n}F(n)\big)e_{2,2} &> 0, \\
\big((-1)^{-n}F(n)\big)e_{2,1} - \big((-1)^{-n}F(n+1)\big)e_{2,2} &> 0.
\end{aligned}$$

Dividing both sides of each inequality by $(-1)^{-n}$ and rearranging we have that:

$$F(n)e_{1,2} > F(n-1)e_{1,1}, \tag{4.9}$$

$$F(n)e_{1,1} > F(n+1)e_{1,2}, \tag{4.10}$$

$$F(n)e_{2,2} > F(n-1)e_{2,1}, \tag{4.11}$$

$$F(n)e_{2,1} > F(n+1)e_{2,2}. \tag{4.12}$$

By rearranging to isolate $e_{1,1}$ in (4.9) and (4.10), we find that

$$\frac{F(n+1)}{F(n)}e_{1,2} < e_{1,1} < \frac{F(n)}{F(n-1)}e_{1,2},$$

which is equivalent to

$$\frac{F(n+1)}{F(n)} < \frac{e_{1,1}}{e_{1,2}} < \frac{F(n)}{F(n-1)}. \tag{4.13}$$

Similarly, from (4.11) and (4.12):

$$\frac{F(n+1)}{F(n)}e_{2,2} \;<\; e_{2,1} \;<\; \frac{F(n)}{F(n-1)}e_{2,2},$$

which is equivalent to

$$\frac{F(n+1)}{F(n)} \;<\; \frac{e_{2,1}}{e_{2,2}} \;<\; \frac{F(n)}{F(n-1)}. \tag{4.14}$$

Therefore both $\frac{e_{1,1}}{e_{1,2}}$ and $\frac{e_{2,1}}{e_{2,2}}$ are bounded above and below by the quotient of two consecutive Fibonacci numbers.

Recall from §1.5 that

$$\lim_{n\to\infty}\frac{F_p(n)}{F_p(n-1)} \;=\; \tau_p \;=\; \lim_{n\to\infty}\frac{F_p(n+1)}{F_p(n)},$$

where τ_p is the Golden p-Proportion - the positive, real root of the golden algebraic equation (1.21). In particular, recall from Example 4 that the Golden p-Proportion for the case when $p = 1$ is the Golden Ratio $\tau = \frac{1+\sqrt{5}}{2}$.

By the Squeeze Theorem[6], as n tends to infinity both $\frac{e_{1,1}}{e_{1,2}}$ and $\frac{e_{2,1}}{e_{2,2}}$ tend to the Golden Ratio τ. For finite values of n:

$$\frac{e_{1,1}}{e_{1,2}} \;\approx\; \tau, \qquad \frac{e_{2,1}}{e_{2,2}} \;\approx\; \tau.$$

□

This corollary and its proof are formed using material from [4]:

Corollary. *Let k be an integer such that $1 \leq k \leq p$ and $2 \leq j+k \leq p+1$. Then, for $p \geq 1$ and finite n:*

$$\frac{e_{i,j}}{e_{i,j+k}} \;\approx\; \tau_p^k. \tag{4.15}$$

Outline of proof. The proof for general p is very similar to the proof for $p = 1$. Begin with the formula $\hat{M} = E \times Q_p^{-n}$ and apply matrix multiplication to obtain equations for $\hat{m}_{i,j}$ in terms of the code elements of E and Fibonacci p-numbers. Use that all elements $\hat{m}_{i,j}$ are positive and obtain inequalities of the form:

$$\frac{F_p(n)}{F_p(n-k)} \;\lesseqgtr\; \frac{e_{i,j}}{e_{i,j+k}} \;\lesseqgtr\; \frac{F_p(n+1)}{F_p(n-k+1)}.$$

Applying (1.29) and the Squeeze Theorem gives that:

$$\lim_{n\to\infty}\frac{e_{i,j}}{e_{i,j+k}} \;=\; \tau_p^k, \tag{4.16}$$

and for finite n:

$$\frac{e_{i,j}}{e_{i,j+k}} \;\approx\; \tau_p^k.$$

□

[6]The Squeeze Theorem, otherwise known as the Pinching Theorem or the Sandwich Theorem, is as follows:

Suppose that $f(x) \leq g(x) \leq h(x)$ hold for all x in some open interval containing a, except possibly at a itself. a can be any real number or it can be $-\infty$ or $+\infty$. Suppose also that

$$\lim_{x\to a} f(x) \;=\; \lim_{x\to a} h(x) \;=\; L.$$

Then $\lim_{x\to a} g(x) \;=\; L$. [1]

Example 10. For $p = 2$, the relationships between the elements $e_{i,j}$ are:

$$\begin{aligned}
\frac{e_{1,1}}{e_{1,2}} &\approx \tau_2, & \frac{e_{1,2}}{e_{1,3}} &\approx \tau_2, & \frac{e_{1,1}}{e_{1,3}} &\approx \tau_2^2, \\
\frac{e_{2,1}}{e_{2,2}} &\approx \tau_2, & \frac{e_{2,2}}{e_{2,3}} &\approx \tau_2, & \frac{e_{2,1}}{e_{2,3}} &\approx \tau_2^2, \\
\frac{e_{3,1}}{e_{3,2}} &\approx \tau_2, & \frac{e_{3,2}}{e_{3,3}} &\approx \tau_2, & \frac{e_{3,1}}{e_{3,3}} &\approx \tau_2^2,
\end{aligned}$$

where $\tau_2 \approx 1.465$. The full derivation for $p = 2$ can be seen in [4].

Example 11. When $p = 99$, there are $10,000$ elements within the code matrix E so it would be tiresome to find the relationships between all of the elements. However using (4.15) we know that, for example:

$$\frac{e_{1,10}}{e_{1,100}} \approx \tau_{99}^{90}, \quad \text{and} \quad \frac{e_{100,10}}{e_{100,100}} \approx \tau_{99}^{90}.$$

4.4 Error detection and correction

In the communication system, the 'receiver' receives the matrix E' and $(\det \hat{M})'$. We now introduce the method of detecting and correcting errors which may be present in E'. This method requires the use of the checking relations (4.8) and (4.15).

Theorem. *If both* E' *and* $(\det \hat{M})'$ *do not contain errors, then:*

$$\det E' = (-1)^{pn} \times (\det \hat{M})'. \tag{4.17}$$

Proof. In the previous section we proved that

$$\det E = (-1)^{pn} \times \det \hat{M}.$$

If E' does not contain any errors then it is equivalent to the sent code matrix E, and therefore $\det E = \det E'$. Similarly, if the $(\det \hat{M})'$ does not contain an error, it is equivalent to $\det \hat{M}$ which was sent. Hence the theorem is proved. □

Remark. The converse does not hold.

To convince yourself of this, consider the following scenarios where $\det E' = (-1)^{pn} \times (\det \hat{M})'$ does not imply that E' and $(\det \hat{M})'$ are error-free:

i) There are two or more errors[7] present in E' such that the determinant of E' is equal to the determinant of E. $(\det \hat{M})'$ has no error and is therefore equivalent to $\det \hat{M}$.

For example, suppose the matrix $E = \left(\begin{smallmatrix} 89 & 55 \\ 144 & 89 \end{smallmatrix}\right)$ is sent and $E' = \left(\begin{smallmatrix} 89 & 60 \\ 132 & 89 \end{smallmatrix}\right)$ is received. Both have a determinant of one, meaning that the equality (4.17) will hold, even though E' has two errors.

ii) There are errors to both E' and $(\det \hat{M})'$ such that (4.17) still holds.

For example, suppose the matrix $E = \left(\begin{smallmatrix} 89 & 55 \\ 144 & 89 \end{smallmatrix}\right)$ is sent and $E' = \left(\begin{smallmatrix} 89 & 50 \\ 150 & 89 \end{smallmatrix}\right)$, which has a determinant of 421, is received. Also, $(\det \hat{M})' = (-1)^n \times 421$ is received which differs from $\det \hat{M}$ which was sent. Although both $\det E'$ and $(\det \hat{M})'$ have errors, the equality (4.17) still holds as $421 = (-1)^n \times (-1)^n \times 421$.

Therefore we may find that (4.17) is true, but we still cannot be sure that neither E' nor $(\det \hat{M})'$ possess errors. In this situation we must apply the following result:

[7] If there is only one error present then $\det E'$ cannot be equivalent to $\det E$.

Theorem. *If the matrix E' does not have errors, the following connections between its elements exist:*

$$\frac{e'_{i,j}}{e'_{i,j+k}} \approx \tau_p^k, \tag{4.18}$$

where $1 \leq k \leq p$ and $2 \leq j+k \leq p+1$.

Proof. If E' does not have errors, then it is equivalent to E and therefore the result follows from the identity (4.15). □

Therefore on receipt of E' and $(\det \hat{M})'$, if both (4.17) and (4.18) are satisfied then we conclude that there are no errors present and proceed to decode E'. However if either (4.17) or (4.18) are not satisfied, then we conclude that either: E' has at least one error; $(\det \hat{M})'$ differs from $\det \hat{M}$; or both have errors[8].

Consider again the scenarios discussed above. In each scenario the matrix E' satisfies (4.17) but does not (4.18), and therefore we conclude correctly that errors are present in either E', $(\det \hat{M})'$ or both.

If we do conclude that errors have occurred, we assume at first that an error does not lie in $(\det \hat{M})'$ and try to correct any errors in E'. We can propose later that $(\det \hat{M})'$ has an error if error correction on E' is unsuccessful.

If there are errors present in E', there could be between one and $(p+1)^2$ inclusive. We begin by assuming that there is one error present and try to solve the resulting problem. If we are unsuccessful, we then assume there are two errors present and so on.

There are $\binom{(p+1)^2}{1} = (p+1)^2$ ways in which a single error can occur in E' and so there are $(p+1)^2$ cases to consider. For each case, we denote the erroneous matrix element by $x_{i,j}$ and treat it as an unknown, and all other matrix elements $e'_{i,j}$ are considered equivalent to their corresponding $e_{i,j}$ and are therefore denoted as such. Hence the cases we consider are:

$$\begin{array}{ccc}
E' = \begin{pmatrix} x_{1,1} & e_{1,2} & \cdots & e_{1,p+1} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{p+1,1} & e_{p+1,2} & \cdots & e_{p+1,p+1} \end{pmatrix}, & \cdots & E' = \begin{pmatrix} e_{1,1} & e_{1,2} & \cdots & x_{1,p+1} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{p+1,1} & e_{p+1,2} & \cdots & e_{p+1,p+1} \end{pmatrix}, \\
\vdots & \ddots & \vdots \\
E' = \begin{pmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,p+1} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p+1,1} & e_{p+1,2} & \cdots & e_{p+1,p+1} \end{pmatrix}, & \cdots & E' = \begin{pmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,p+1} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{p+1,1} & e_{p+1,2} & \cdots & x_{p+1,p+1} \end{pmatrix}.
\end{array} \tag{4.19}$$

For each case, we use (4.17) to obtain an equation involving the variable $x_{i,j}$, whose value is dependent on the other elements in the matrix. If $x_{i,j}$ is found not to be an integer, then its corresponding case cannot be equivalent to the sent matrix E as all elements of E are integers.

[8]The authors of [4] and [31] do not mention the need for the second checking relation (4.18) when detecting errors. However, as dicussed, this checking relation is required.

The remaining $x_{i,j}$'s are re-entered into their corresponding matrix in (4.19). If one of these matrices is equivalent to the sent code matrix E, then (4.18) will hold for that matrix, but not for any other matrices. However if no matrix satisfies (4.18), then we conclude that there cannot be exactly one error in E'. [31]

Example 12. Let us consider the error correction method thus far for $p = 1$. Suppose that we receive the matrix

$$E' = \begin{pmatrix} e'_{1,1} & e'_{1,2} \\ e'_{2,1} & e'_{2,2} \end{pmatrix},$$

and $(\det \hat{M})'$ such that either (4.17) or (4.18) does not hold. We assume that $(\det \hat{M})'$ is not erroneous and consider the four possible ways in which one error could be present in E':

$$i)\ E' = \begin{pmatrix} x_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} \end{pmatrix}, \quad ii)\ E' = \begin{pmatrix} e_{1,1} & x_{1,2} \\ e_{2,1} & e_{2,2} \end{pmatrix},$$

$$iii)\ E' = \begin{pmatrix} e_{1,1} & e_{1,2} \\ x_{2,1} & e_{2,2} \end{pmatrix}, \quad iv)\ E' = \begin{pmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & x_{2,2} \end{pmatrix}.$$

Applying the checking relation (4.17) to each of these four cases gives the linear equations:

$$\begin{aligned} i)\ x_{1,1}e_{2,2} - e_{1,2}e_{2,1} &= (-1)^n \times (\det \hat{M})', \\ ii)\ e_{1,1}e_{2,2} - x_{1,2}e_{2,1} &= (-1)^n \times (\det \hat{M})', \\ iii)\ e_{1,1}e_{2,2} - e_{1,2}x_{2,1} &= (-1)^n \times (\det \hat{M})', \\ iv)\ e_{1,1}x_{2,2} - e_{1,2}e_{2,1} &= (-1)^n \times (\det \hat{M})'. \end{aligned}$$

Solving for $x_{i,j}$ in each case gives

$$\begin{aligned} i)\ x_{1,1} &= \frac{(-1)^n \times (\det \hat{M})' + e_{1,2}e_{2,1}}{e_{2,2}}, \\ ii)\ x_{1,2} &= \frac{(-1)^{n+1} \times (\det \hat{M})' + e_{1,1}e_{2,2}}{e_{2,1}}, \\ iii)\ x_{2,1} &= \frac{(-1)^{n+1} \times (\det \hat{M})' + e_{1,1}e_{2,2}}{e_{1,2}}, \\ iv)\ x_{2,2} &= \frac{(-1)^n \times (\det \hat{M})' + e_{1,2}e_{2,1}}{e_{1,1}}. \end{aligned}$$

We then check whether any of the cases, where $x_{i,j}$ is an integer, satisfy the checking relation (4.18). For example, suppose that we find $x_{1,1}$ to be an integer. We would choose this corrected value of $e'_{1,1}$ if:

$$\frac{x_{1,1}}{e_{1,2}} \approx \tau, \qquad \frac{e_{2,1}}{e_{2,2}} \approx \tau$$

If no case satisfies (4.18), then there is either more than one error in E', or $(\det \hat{M})'$ is erroneous. [31]

Returning to the error correction for general p: if either no $x_{i,j}$ is an integer or (4.18) does not hold for any of the $(p+1)^2$ cases, there is more than one error in E' or $(\det \hat{M})'$ recieved differs from $\det \hat{M}$. The next step would be to try to error correct as if there are exactly two errors present in E', and there are $\binom{(p+1)^2}{2}$ ways in which this could occur. Let $y_{i,j}$ denote an erroneous matrix element. All other elements $e'_{i,j}$ are deemed equivalent to their corresponding $e_{i,j}$ and are therefore denoted as such. The cases to consider are:

$$E' = \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & e_{1,p+1} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{p+1,1} & e_{p+1,2} & \cdots & e_{p+1,p+1} \end{pmatrix}, \quad \cdots \quad E' = \begin{pmatrix} e_{1,1} & \cdots & y_{1,p} & y_{1,p+1} \\ e_{2,1} & \cdots & e_{2,p} & e_{2,p+1} \\ \vdots & \ddots & \vdots & \vdots \\ e_{p+1,1} & \cdots & e_{p+1,p} & e_{p+1,p+1} \end{pmatrix},$$

$$\vdots \qquad\qquad \ddots \qquad\qquad \vdots$$

$$E' = \begin{pmatrix} e_{1,1} & e_{1,2} & \cdots & e_{1,p+1} \\ e_{2,1} & e_{2,2} & \cdots & e_{2,p+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{p+1,1} & y_{p+1,2} & \cdots & e_{p+1,p+1} \end{pmatrix}, \quad \cdots \quad E' = \begin{pmatrix} e_{1,1} & \cdots & e_{1,p} & e_{1,p+1} \\ e_{2,1} & \cdots & e_{2,p} & e_{2,p+1} \\ \vdots & \ddots & \vdots & \vdots \\ e_{p+1,1} & \cdots & y_{p+1,p} & y_{p+1,p+1} \end{pmatrix}.$$

Applying the checking relation (4.17) to each case gives $\binom{(p+1)^2}{2}$ equations involving two variables - some may be linear, others may be non-linear. Some equations may have a finite number of integer solutions while others may have an infinite number of integer solutions or no integer solutions at all. The solution which satisfies the second checking relation (4.18) must be chosen, if such a solution exists. Otherwise, we continue and try to correct for three errors. [31]

Example 13. *Continuation of Example* 12.

Suppose that in Example 12 we found no $x_{i,j}$ which satisfies (4.18). We now try to error correct as if there are two errors present. There are $\binom{4}{2} = 6$ ways in which two errors could occur in E':

$$i)\ E' = \begin{pmatrix} y_{1,1} & y_{1,2} \\ e_{2,1} & e_{2,2} \end{pmatrix}, \quad ii)\ E' = \begin{pmatrix} y_{1,1} & e_{1,2} \\ y_{2,1} & e_{2,2} \end{pmatrix}, \quad iii)\ E' = \begin{pmatrix} y_{1,1} & e_{1,2} \\ e_{2,1} & y_{2,2} \end{pmatrix},$$

$$iv)\ E' = \begin{pmatrix} e_{1,1} & y_{1,2} \\ y_{2,1} & e_{2,2} \end{pmatrix}, \quad v)\ E' = \begin{pmatrix} e_{1,1} & y_{1,2} \\ e_{2,1} & y_{2,2} \end{pmatrix}, \quad vi)\ E' = \begin{pmatrix} e_{1,1} & e_{1,2} \\ y_{2,1} & y_{2,2} \end{pmatrix}.$$

Let's consider the cases $i), ii), v)$ and $vi)$. Taking the determinant of E' and applying (4.17) to each of these cases, we obtain equations of the form $ax + by = c$. These are linear Diophantine equations[9] but with certain restrictions imposed upon a, b, x and y. As all the code elements are positive integers, this implies that: a,x and y are positive integers; b is a negative integer due to the minus sign in the determinant of E'; and c may be a positive or negative integer depending on the value of $(-1)^n \times (\det \hat{M})'$. A linear Diophantine equation possesses infinite integer solutions if c is a multiple of the greatest common divisor of a and b. These solutions can be found using the Extended Euclidean Algorithm. However if c is not a multiple of the greatest common divisor of a and b, then no integer solutions exist. [5]

For cases $iii)$ and $iv)$, taking the determinant of E' and applying (4.17) gives nonlinear equations of the form $xy = c$, where c is a constant integer and $x, y > 0$ are variables. These are equations for rectangular hyperbolas in the upper right quadrant where the horizontal and vertical axes are asymptotic. Whether there are integer solutions or not depends on whether c is positive or negative. If c is positive, there will exist a finite number of integer solutions but always at least the two solutions, $(x, y) = (1, c)$ and $(x, y) = (c, 1)$. Any other integer solutions lie on the curve between these two

[9] A linear diophantine equation is of the form $ax + by = c$ where a, b, c are integers and the variables x, y can only take integer values.

co-ordinates. However if $c \leq 0$, then no solutions exist for $x, y > 0$.

If, from all the cases $i) - iv)$, there is an integer solution which satisfies the checking relation (4.18) then this $y_{i,j}$ is the corrected value of its corresponding $e'_{i,j}$.

If we are not able to error correct for two errors we continue to error correct for three errors and so on, up to and including error correction for $(p+1)^2 - 1$ errors if necessary. However we are not able to error correct for $(p+1)^2$ errors, as all $(p+1)^2$ elements in the matrix E' would be treated as unknowns. In other words, there is a lack of necessary information. Therefore, if we are not able to correct for $(p+1)^2 - 1$ errors, we must propose that there are either $(p+1)^2$ errors in the matrix E', or $(\det \hat{M})'$ is incorrect, or both. Incomplete decoding will have occurred and retransmission would be requested if the channel is two-way. [31]

4.5 A numerical example

Algorithmic methods for a general theory can sometimes be difficult to digest without an example to follow. Therefore we now give an example of the Fibonacci coding method.

A message is to be sent through the channel from the source to the destination. After source coding, the message is in the form $U = \{0, 9, 1, 2, 1, 5\}$, an ordered list of $t = 6$ symbols from the alphabet $\Lambda_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We must choose the parameter $p \geq 1$ such that $(p+1)^2 < 6$. This condition only allows us to choose $p = 1$, so the message matrix M will be a 2×2 matrix. We arrange the message symbols into M according to the conditions stated earlier:

$$M = \begin{pmatrix} 0 & 9 \\ 12 & 15 \end{pmatrix}.$$

As $m_{1,1} = 0$, we must add an integer μ to each $m_{i,j}$ to obtain a matrix $\hat{M}$ where all $\hat{m}_{i,j} > 0$. Taking $\mu = 1$ we obtain

$$\hat{M} = \begin{pmatrix} 1 & 10 \\ 13 & 16 \end{pmatrix},$$

which has a determinant of -114.

We now select the encoding matrix Q_p^n given by (1.8). The parameter $p = 1$ but we can choose n freely. We will choose Q_1^{14}, a matrix containing the thirteenth, fourteenth and fifteen Fibonacci numbers:

$$Q_1^{14} = \begin{pmatrix} F(15) & F(14) \\ F(14) & F(13) \end{pmatrix} = \begin{pmatrix} 610 & 377 \\ 377 & 233 \end{pmatrix}.$$

We then encode $\hat{M}$ to obtain the code matrix E by using the formula $\hat{M} \times Q_1^{14} = E$:

$$\begin{pmatrix} 1 & 10 \\ 13 & 16 \end{pmatrix} \begin{pmatrix} 610 & 377 \\ 377 & 233 \end{pmatrix} = \begin{pmatrix} 4380 & 2707 \\ 13962 & 8629 \end{pmatrix}.$$

The matrix E and $\det \hat{M} = -114$ are sent through the channel.

Suppose that on the other side of the channel we receive

$$E' = \begin{pmatrix} 4385 & 2707 \\ 13962 & 8699 \end{pmatrix} \quad \text{and} \quad (\det \hat{M})' = -114.$$

It is unknown whether errors are present in either E' or $(\det \hat{M})'$ due to noise. Therefore, we need to check whether the checking relation $\det E' = (-1)^{pn} \times (\det \hat{M})'$ is satisfied. Firstly, we calculate

$$\begin{aligned} \det E' &= 4385 \times 8699 - 2707 \times 13962 \\ &= 349981. \end{aligned}$$

We then see that the checking relation does not hold:

$$349981 \neq (-1)^{14} \times (-114).$$

Therefore either between one and four errors have occurred to E', or $(\det \hat{M})'$ differs from $\det \hat{M}$. At first we assume that $(\det \hat{M})'$ is correct and proceed with error correction on E'.

There are $\binom{4}{1} = 4$ cases to consider when error correcting for one error:

$$i)\ E' = \begin{pmatrix} x_{1,1} & 2707 \\ 13962 & 8699 \end{pmatrix}, \quad ii)\ E' = \begin{pmatrix} 4385 & x_{1,2} \\ 13962 & 8699 \end{pmatrix},$$

$$iii)\ E' = \begin{pmatrix} 4385 & 2707 \\ x_{2,1} & 8699 \end{pmatrix}, \quad iv)\ E' = \begin{pmatrix} 4385 & 2707 \\ 13962 & x_{2,2} \end{pmatrix}.$$

The notation $x_{i,j}$ represents an erroneous matrix element and it's value is determined by the other elements in the matrix. Thus, suppose $x_{1,1}$ is where the error occurred and the remaining $e'_{i,j}$ are equivalent to $e_{i,j}$. Then we will be able to obtain the true value of $x_{1,1}$ by using the checking relation (4.8). However, in reality we do not know whether $x_{1,1}$ is actually where an error has occurred, so we must apply (4.17) to all the cases $i) - iv)$:

$$\begin{aligned} i)\ \ 8699x_{1,1} - 2707 \times 13962 &= (-1)^{14} \times (-144), \\ ii)\ \ 4385 \times 8699 - 13962x_{1,2} &= (-1)^{14} \times (-144), \\ iii)\ 4385 \times 8699 - 2707x_{2,1} &= (-1)^{14} \times (-144), \\ iv)\ \ 4385x_{2,2} - 2707 \times 13962 &= (-1)^{14} \times (-144). \end{aligned}$$

Solving these equations, we find that:

$$x_{1,1} \approx 4344.751, \quad x_{1,2} \approx 2732.077 \quad x_{2,1} \approx 14091.341, \quad x_{2,2} \approx 8619.154.$$

There are no integer solutions so we can conclude that there is either more than one error in E' or $\det \hat{M}$ is erroneous.

Hence we proceed to correct for two errors. There are $\binom{4}{2}$ cases to consider:

$$\begin{aligned} &i)\ E' = \begin{pmatrix} y_{1,1} & y_{1,2} \\ 13962 & 8699 \end{pmatrix}, \quad ii)\ E' = \begin{pmatrix} y_{1,1} & 2707 \\ y_{2,1} & 8699 \end{pmatrix}, \quad iii)\ E' = \begin{pmatrix} y_{1,1} & 2707 \\ 13962 & y_{2,2} \end{pmatrix}, \\ &iv)\ E' = \begin{pmatrix} 4385 & y_{1,2} \\ y_{2,1} & 8699 \end{pmatrix}, \quad v)\ E' = \begin{pmatrix} 4385 & y_{1,2} \\ 13962 & y_{2,2} \end{pmatrix}, \quad vi)\ E' = \begin{pmatrix} 4385 & 2707 \\ y_{2,1} & y_{2,2} \end{pmatrix}. \end{aligned} \tag{4.20}$$

We apply (4.17) to all the cases $i) - vi)$ to obtain the equations:

$$\begin{aligned} i)\;\; 8699y_{1,1} - 13962y_{1,2} &= (-1)^{14} \times (-114), \\ ii)\;\; 8699y_{1,1} - 2707y_{2,1} &= (-1)^{14} \times (-114), \\ iii)\;\; y_{1,1}y_{2,2} - 2707 \times 13962 &= (-1)^{14} \times (-114), \\ iv)\;\; 4385 \times 8699 - y_{1,2}y_{2,1} &= (-1)^{14} \times (-114), \\ v)\;\; 4385y_{2,2} - 13962y_{1,2} &= (-1)^{14} \times (-114), \\ vi)\;\; 4385y_{2,2} - 2707y_{2,1} &= (-1)^{14} \times (-114). \end{aligned}$$

Cases $i), ii), v)$ and $vi)$ are of the form $ax + by = c$ and, as we seek integer solutions, they are Linear Diophantine equations. We find that for all of these cases $\gcd(a, b) = 1$, thus c is a multiple of $\gcd(a, b)$. Therefore there are infinite solutions for cases $i), ii), v)$ and $vi)$. Using the Extended Euclidean Algorithm, we find that the solutions for $y_{i,j} > 0$ are[10]:

$$i)\;\; (y_{1,1} = 13710 + 13962q,\; y_{1,2} = 8542 + 8699q), \tag{4.21}$$

$$ii)\;\; (y_{1,1} = 2482 + 2707q,\; y_{2,1} = 7976 + 8699q), \tag{4.22}$$

$$v)\;\; (y_{1,2} = 3782 + 4385q,\; y_{2,2} = 12042 + 13962q), \tag{4.23}$$

$$vi)\;\; (y_{2,1} = 1317 + 4385q,\; y_{2,2} = 813 + 2707q), \tag{4.24}$$

where $q \in \mathbb{Z}_{\geq 0}$.

For cases $iii)$ and $iv)$ we must consider solutions which satisfy:

$$iii)\;\; y_{1,1}y_{2,2} = 37795020, \tag{4.25}$$

$$iv)\;\; y_{1,2}y_{2,1} = 38145229. \tag{4.26}$$

A particular solution obtained from (4.25) is $(y_{1,1}, y_{2,2}) = (4380, 8629)$. If we enter this solution back into its corresponding matrix $iii)$ in (4.20), we see that the second checking relation (4.18) holds:

$$\frac{4380}{2707} \approx 1.618027 \approx \tau, \qquad \frac{13962}{8629} \approx 1.618032 \approx \tau.$$

Therefore, through the error correction method above, we have corrected the received matrix E' to obtain the sent matrix E:

$$E = \begin{pmatrix} 4380 & 2707 \\ 13962 & 8629 \end{pmatrix}.$$

Subtraction of the sent matrix E from the received matrix E' gives the error matrix

$$\Sigma = \begin{pmatrix} 5 & 0 \\ 0 & 70 \end{pmatrix},$$

from which we can see the magnitude of the symbol errors and the positions where they occurred.

[10]The equations were solved with the aid of [19].

We then decode the code matrix E to obtain $\hat{M}$:

$$\begin{aligned} E \times Q_1^{-14} &= \begin{pmatrix} 4380 & 2707 \\ 13962 & 8629 \end{pmatrix} \begin{pmatrix} 233 & -377 \\ -377 & 610 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 10 \\ 13 & 16 \end{pmatrix} = \hat{M}. \end{aligned}$$

Finally we subtract $\mu = 1$ from all elements $\hat{m}_{i,j}$ to give the original message matrix

$$M = \begin{pmatrix} 0 & 9 \\ 12 & 15 \end{pmatrix},$$

from which the message can be read as $U = \{0, 9, 1, 2, 1, 5\}$. U has been transmitted succesfully through the channel from the source to the destination.

4.6 Theory versus practicality

A good coding theory is not only one which produces sound results on paper, but one which has few practical difficulties and can be applied successfully in the real world. Is Fibonacci coding a good coding theory? In this section we explore this using a combination of [4] and my own thoughts.

Error correction forms a major part of the Fibonacci coding method. It was mentioned previously that a maximum of $(p+1)^2$ errors can occur to a code matrix E containing $(p+1)^2$ elements. In theory the error correction method allows us to successfully correct errors in all scenarios where between one and $p^2 + 2p$ inclusive errors occur. For example, when $p = 1$ we can successfully error correct up to and including threefold errors, but cannot correct fourfold errors. As there are $\binom{4}{k}$ ways in which k errors can occur in a 2×2 code matrix E, the total number of possible error combinations for $p = 1$ is:

$$\binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 15.$$

We can successfully error correct for all these error combinations, except for the one case where there are fourfold errors present. Thus S_1, the probability of successful error correction for $p = 1$, is

$$S_1 = \frac{14}{15} \approx 0.933.$$

When $p = 2$, we can correct all error combinations except the one case of ninefold errors. Therefore

$$S_2 = \frac{510}{511} \approx 0.998.$$

In fact, the following result for general p is stated in [4]. We prove the result here.

Theorem. *For* $p \geq 1$*, the probability of successful error correction is*

$$S_p = \frac{2^{(p+1)^2} - 2}{2^{(p+1)^2} - 1}. \tag{4.27}$$

To enable us to prove this theorem, we require the following lemma:

Lemma. *For* $n, k \in \mathbb{Z}$*,*

$$\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1 \tag{4.28}$$

Proof of Lemma. Letting $x = y = 1$ in the binomial formula

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = (x+y)^n$$

gives the identity

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

This can be written as

$$\binom{n}{0} + \sum_{k=1}^{n} \binom{n}{k} = 2^n,$$

which is equivalent to

$$\sum_{k=1}^{n} \binom{n}{k} = 2^n - 1.$$

□

Proof of Theorem. Using the Lemma above, the total number of possible error combinations is

$$\begin{aligned} \binom{(p+1)^2}{1} + \binom{(p+1)^2}{2} + \cdots + \binom{(p+1)^2}{(p+1)^2} &= \sum_{k=1}^{(p+1)^2} \binom{(p+1)^2}{k} \\ &= 2^{(p+1)^2} - 1 \end{aligned} \quad (4.29)$$

For all values of p, we are able to correct all possible error combinations except where there are $(p+1)^2$ errors present. We are therefore able to correct $2^{(p+1)^2} - 2$ error combinations from a total of $2^{(p+1)^2} - 1$ possible error combinations. The probability of successful error correction, S_p, will equal the number of error combinations we are able to correct as a proportion of the total number of possible error combinations. Therefore

$$S_p = \frac{2^{(p+1)^2} - 2}{2^{(p+1)^2} - 1}.$$

□

Let the random variable X be the number of errors that occur to E. Note that, we have assumed that $P(X = 1) = P(X = 2) = \cdots = P(X = (p+1)^2)$. In other words, one error is as likely to occur as $(p+1)^2$ errors. However this is not the case in reality. Suppose that θ is the probability[11] that an error occurs to the matrix element $e_{i,j}$ and $1 - \theta$ is therefore the probability that an error does not occur to that element. Then X follows a binomial distribution with probability mass function:

$$P(X = k) = \binom{(p+1)^2}{k} \theta^k (1-\theta)^{(p+1)^2 - k}$$

for $k = 0, 1, 2, \ldots, (p+1)^2$. It is likely that $\theta < \frac{1}{2}$, in which case the probability that $(p+1)^2$ errors occur will be relatively small compared to the probabilities that other numbers of errors occur. Therefore, with $\theta < \frac{1}{2}$, a plausible statement to make is [8]:

$$S_p > \frac{2^{(p+1)^2} - 2}{2^{(p+1)^2} - 1}.$$

[11] As mentioned in the motivation to coding theory, we assume the channel is memoryless, meaning that a symbol error to one element does not increase or decrease the probability that a symbol error occurs to another element.

Note that $S_p \to 1$ as $p \to \infty$ and therefore, in theory we would like to choose p as large as is feasibly possible so that the probability of successful error correction is as close to one as possible. However there are several practical drawbacks to this. As p increases, the number of elements in the matrices increases quadratically and as a result the number of arithmetic operations $(+, -, \times, \div)$ performed during the method increases quickly - perhaps quadratically or even exponentially. An algorithm which grows exponentially is usually not a practical one. Therefore, if p is large and the checking relations (4.17) and (4.18) fail when we recieve E' and $(\det \hat{M})'$, would it be more efficient to ask for retransmission if the channel is two way, rather than potentially attempting to correct $p^2 + 2p$ errors?

Another practical issue with a large value of p is our ability to solve equations which contain a large number of variables. Linear equations in one variable, which we obtain when error correcting for one error, are simple to solve. However when we error correct for two errors or more we will encounter linear and non-linear equations with an increasing number of variables. Is it realistic to assume that we are able to solve equations with up to and including $p^2 + 2p$ variables?

This leads to a potential issue concerning our ability to find a particular solution which satisfies the relation (4.18). Using brute-force to try to find such a solution among a finite number of solutions is not ideal, but it is still a method which works. However if there are an infinite number of solutions to consider, a brute-force method is obviously not practical. It is not only time-consuming but potentially never-ending if there are actually no solutions which satisfy (4.18). A graphical approach is also impractical when there are many variables. Therefore, is there an efficient method which enables us to find the particular solution we seek when our equations involve many variables?

How do we decide how 'close' the approximation (4.18) is required to be for us to conclude that a particular solution is correct? The answer to this is related to our choice of n when selecting the encoding matrix Q_p^n. The larger the value of n, the more exact the approximation (4.18) due to (4.16). Therefore increasing n indirectly increases the accuracy of error correction. However, we also need to consider redundancy caused by n - additional information which is not necessary to be able to accurately error correct. [31] If $n = a$ is the smallest value of n such that error correction occurs with full accuracy, then choosing $n > a$ will add redundancy to the coding method and, as mentioned earlier, redundancy is inversely proportional to efficiency.

So, is Fibonacci coding a good coding theory? In theory, it *is* a good coding theory as the probability of successful error correction converges to one. However, there remain many issues regarding practicality, such as: the choices of p and n; our ability to solve equations which contain many variables; our ability to find a particular integer solution which satisfies (4.18), from an infinite number of solutions; and problems regarding the approximations (4.18), and the accuracy of the method. It is clear that more research is required before Fibonacci coding theory can be applied to real life scenarios. However, applying Fibonacci coding with a low value of p seems plausible as, even for the simplest case when $p = 1$, the ability of the Fibonacci coding method to successfully error correct "exceeds essentially all well-known correcting codes" [31].

Conclusion

Over the years, countless texts have been written about the Fibonacci numbers, the Golden Ratio and their applications. The aim of this report was to stay off the beaten track and introduce the reader to some lesser known applications related to the Fibonacci p-numbers. Much of the material discussed has only been published within the last five years and therefore is very much current research.

On the surface, the Fibonacci p-numbers may seem very elementary and questions may be asked as to why they have undergone so much research. After all, any mathematics which stems from a highly idealised rabbit problem may be deemed as elementary at first. However it becomes clear that the topic is in fact vast, and at times complex, especially when the theory for large values of p is considered. We saw the underlying beauty of the Fibonacci p-numbers in the first chapter through their relationship with the Golden p-Proportion, and it became apparent that they possess interesting properties and identities. In fact, given the huge number of properties and identities for the Fibonacci numbers, there may be many more for the Fibonacci p-numbers which are yet to be discovered.

Establishing a closed form expression for the n^{th} Fibonacci (p, m)-number, a generalisation of the n^{th} Fibonacci p-number, enabled Fibonacci theory to be extended from the discrete to the continuous domain. We derived functions for Fibonacci curves and surfaces which lie in two and three dimensional space and argued that, although it is possibly a far-fetched idea, the Metallic Shofar cannot be dismissed as a possible model for the spatial section of the universe. Little additional research of the Metallic Shofar and the Fibonacci curves is necessary, although it may be interesting if not particularly useful, to investigate the geometrical aspects such as Gaussian curvature and geodesics.

However it became apparent that much research is necessary if Fibonacci coding theory is to be put into practice. Many of the potential practical drawbacks are not mentioned in the published research available on this topic, as they tend to focus upon the method of Fibonacci coding and its success in theory. Therefore much of the analysis of the coding theory is my own, and I have left open questions which may be the subject of research in the future. I certainly feel some numerical analysis of the coding algorithm is vital, as well as collaborations with engineers and computer scientists, if this coding theory is to provide the safe tranmission of data in our communications networks in the future.

There are many more intiguing applications of the Fibonacci p-numbers, which unfortunately could not be covered in this report. If these are of interest, the works of the Ukrainian mathematician Alexey Stakhov are most certainly worth reading.

Appendix A

Formulae for the roots of the monic cubic equation

A monic polynomial is one in which the coefficient of the highest order term is 1. The roots of the monic cubic equation $x^3 + ax^2 + bx + c = 0$ are:

$$x_1 = -\frac{1}{3}\left(a + \sqrt[3]{\frac{m+\sqrt{n}}{2}} + \sqrt[3]{\frac{m-\sqrt{n}}{2}}\right)$$

$$x_2 = -\frac{1}{3}\left(a + \omega_2\sqrt[3]{\frac{m+\sqrt{n}}{2}} + \omega_1\sqrt[3]{\frac{m-\sqrt{n}}{2}}\right)$$

$$x_3 = -\frac{1}{3}\left(a + \omega_1\sqrt[3]{\frac{m+\sqrt{n}}{2}} + \omega_2\sqrt[3]{\frac{m-\sqrt{n}}{2}}\right)$$

where

$$m = 2a^3 - 9ab + 27c$$

$$k = a^2 - 3b$$

$$n = m^2 - 4k^3 = (2a^3 - 9ab + 27c)^2 - 4(a^2 - 3b)^3$$

$$\omega_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$$

$$\omega_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{3}i$$

Note that from $n = m^2 - 4k^3$:

$$k^3 = \frac{m^2 - n}{4}$$

$$\iff k^3 = \left(\frac{m+\sqrt{n}}{4}\right)\left(\frac{m-\sqrt{n}}{4}\right)$$

$$\iff k = \sqrt[3]{\frac{m+\sqrt{n}}{4}}\sqrt[3]{\frac{m-\sqrt{n}}{4}}$$

Therefore, since there are three possible values for each cube root $r = \sqrt[3]{\frac{m+\sqrt{n}}{2}}$ and $s = \sqrt[3]{\frac{m-\sqrt{n}}{2}}$ we must choose r and s such that $rs = k$. [6]

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