

Integration by Parts comes right from the product rule which says:

$$f(x) \cdot g'(x) + f'(x) \cdot g(x) = (f(x) \cdot g(x))'$$

if we simply subtract $f'(x) \cdot g(x)$ from both sides we get:

$$f(x) \cdot g'(x) = (f(x) \cdot g(x))' - f'(x) \cdot g(x)$$

and if we integrate both sides:

$$\int (f(x) \cdot g'(x)) dx = \int (f(x) \cdot g(x))' dx - \int (f'(x) \cdot g(x)) dx$$

we can cancel the derivative and integral on the middle term:

$$\int (f(x) \cdot g'(x)) dx = f(x) \cdot g(x) - \int (f'(x) \cdot g(x)) dx$$

and if we just replace f and g with the more familiar u and v we get:

$$\int u \cdot \frac{dv}{dx} dx = u \cdot v - \int v \cdot \frac{du}{dx} dx$$

or upon doing some questionable work with the differentials (treating them like true fractions) we get the most popular way to state I.B.P.

$$\int u \cdot \frac{dv}{dx} dx = u \cdot v - \int v \cdot \frac{du}{dx} dx$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

Applying this to your problem of $\int e^x \cdot \cos(x) dx$ we need to firstly use the I.B.P. formula once:

let: $u = e^x$ and $dv = \cos(x)$

then: $du = e^x$ and $v = \sin(x)$ and

$$\int e^x \cdot \cos(x) dx = e^x \cdot \sin(x) - \int e^x \cdot \sin(x) dx$$

unfortunately, this hardly looks any better as we now have $\int e^x \cdot \sin(x) dx$ which is much the same as the original integral, but this is where a cool trick happens. Let's apply I.B.P. again:

let: $u = e^x$ and $dv = \sin(x)$

then: $du = e^x$ and $v = -\cos(x)$ and:

$$\int e^x \cdot \sin(x) dx = -e^x \cdot \cos(x) - \int e^x \cdot -\cos(x) dx$$

we can bring the negative out front:

$$\int e^x \cdot \sin(x) dx = -e^x \cdot \cos(x) + \int e^x \cdot \cos(x) dx$$

and we have encountered our original integral again. Watch what happens if we substitute this into the first equation we had:

this: $\int e^x \cdot \cos(x) dx = e^x \cdot \sin(x) - \int e^x \cdot \sin(x) dx$ becomes:

$$\int e^x \cdot \cos(x) dx = e^x \cdot \sin(x) - \left(-e^x \cdot \cos(x) + \int e^x \cdot \cos(x) dx \right)$$

we can then distribute the negative:

$$\int e^x \cdot \cos(x) dx = e^x \cdot \sin(x) + e^x \cdot \cos(x) - \int e^x \cdot \cos(x) dx$$

and we see we have the same integral on both sides. We can just add it to the l.h.s to get:

$$2 \cdot \int e^x \cdot \cos(x) dx = e^x \cdot \sin(x) + e^x \cdot \cos(x) \text{ or,}$$

$$\int e^x \cdot \cos(x) dx = \frac{e^x \cdot \sin(x) + e^x \cdot \cos(x)}{2} \text{ or,}$$

$$\int e^x \cdot \cos(x) dx = e^x \cdot \frac{\sin(x) + \cos(x)}{2} \text{ and don't forget, plus C}$$

$$\int e^x \cdot \cos(x) dx = e^x \cdot \frac{\sin(x) + \cos(x)}{2} + C$$