Magnetic field of a bar magnet at an axial point (end-on position). Let NS be a bar magnet of length $2l$ and of pole strength q_m. Suppose the magnetic field is to be determined at a point P which lies on the axis of the magnet at a distance r from its centre, as shown in Fig. 5.13.

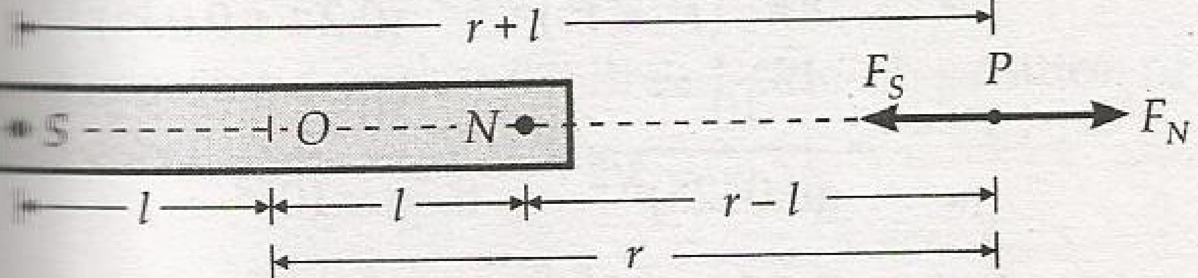

Fig. 5.13 Magnetic field of a bar magnet at an axial point.

Imagine a *unit north pole* placed at point P. Then from Coulomb's law of magnetic forces, the force exerted by the N-pole of strength q_m on unit north pole will be

$$F_N = \frac{\mu_0}{4\pi} \cdot \frac{q_m}{(r-l)^2}, \qquad \text{along } \overrightarrow{NP}$$

Similarly, the force exerted by S-pole on unit north pole is

$$F_S = \frac{\mu_0}{4\pi} \cdot \frac{q_m}{(r+l)^2}, \qquad \text{along } \overrightarrow{PS}$$

Therefore, the strength of the magnetic field $\vec{B}$ at point P is

B_{axial} = Force experienced by a unit north - pole at point P

$$= F_N - F_S = \frac{\mu_0 q_m}{4\pi}\left[\frac{1}{(r-l)^2} - \frac{1}{(r+l)^2}\right]$$

$$= \frac{\mu_0 q_m}{4\pi} \cdot \frac{4rl}{(r^2-l^2)^2}$$

But $q_m \cdot 2l = m$, is the magnetic dipole moment, so

$$B_{\text{axial}} = \frac{\mu_0}{4\pi} \cdot \frac{2mr}{(r^2 - l^2)^2}$$

For a short bar magnet, $l << r$, therefore, we have

$$B_{\text{axial}} = \frac{\mu_0}{4\pi} \cdot \frac{2m}{r^3}, \quad \text{along } \vec{NP} \qquad \text{—(1)}$$

Clearly, *the magnetic field at any axial point of magnetic dipole is in the same direction as that of its magnetic dipole moment i.e.,* from S-pole to N-pole, so we can write

$$\vec{B}_{\text{axial}} = \frac{\mu_0}{4\pi} \cdot \frac{2\vec{m}}{r^3}$$

$\frac{2\vec{m}k}{r^3}$

Magnetic field of a bar magnet at an equatorial point (broadside-on position). Consider a bar magnet NS of length $2l$ and of pole strength q_m. Suppose the magnetic field is to be determined at a point P lying on the equatorial line of the magnet NS at a distance r from its centre, as shown in Fig. 5.14.

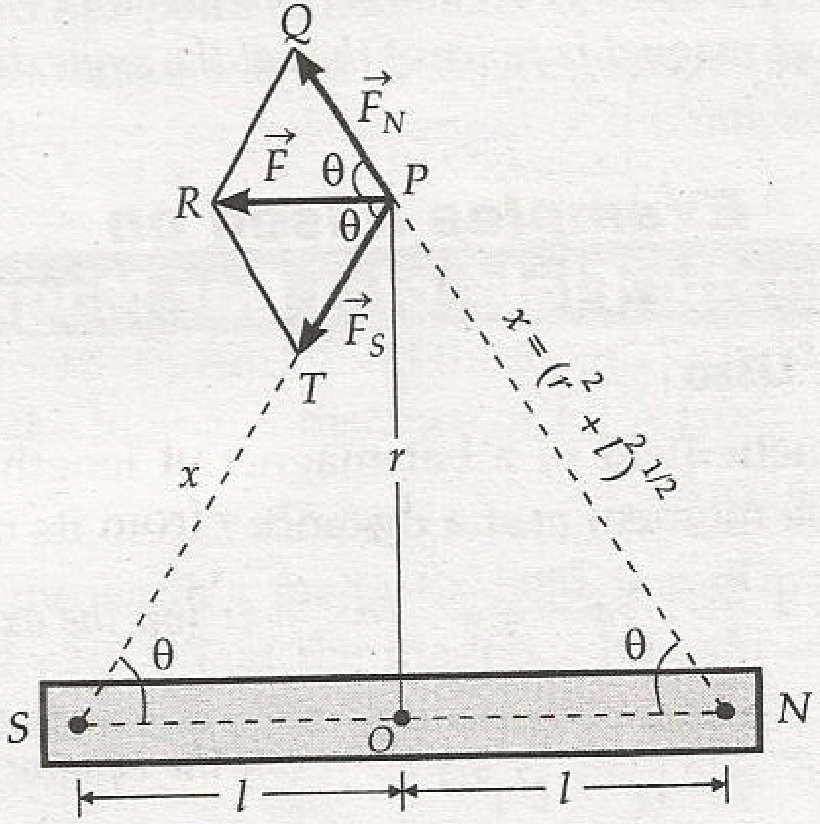

Fig. 5.14 Magnetic field of a bar magnet at an equatorial point.

Imagine a *unit north-pole* placed at point P. Then from Coulomb's law of magnetic forces, the force exerted by the N-pole of the magnet on unit north-pole is

$$F_N = \frac{\mu_0}{4\pi} \cdot \frac{q_m}{x^2}, \quad \text{along } NP$$

Similarly, the force exerted by the S-pole of the magnet on unit north-pole is

$$F_S = \frac{\mu_0}{4\pi} \cdot \frac{q_m}{x^2}, \quad \text{along } PS$$

As the magnitudes of F_N and F_S are equal, so their vertical components get cancelled while the horizontal components add up along PR.

Hence the magnetic field at the equatorial point P is

$$B_{equa} = \text{Net force on a unit } N\text{-pole placed at point } P$$

$$= F_N \cos\theta + F_S \cos\theta$$

$$= 2\, F_N \cos\theta \qquad [\because\ F_N = F_S]$$

$$= 2 \cdot \frac{\mu_0}{4\pi} \cdot \frac{q_m}{x^2} \cdot \frac{l}{x} \qquad \left[\because\ \cos\theta = \frac{l}{x}\right]$$

or $$B_{equa} = \frac{\mu_0}{4\pi} \cdot \frac{m}{(r^2 + l^2)^{3/2}} \qquad [\because\ x = (r^2 + l^2)^{1/2}]$$

where $m = q_m \cdot 2l$, is the magnetic dipole moment.

Again for a short magnet, $l << r$, so we have

$$B_{equa} = \frac{\mu_0}{4\pi} \cdot \frac{m}{r^3}, \text{ along } PR \qquad ...(2)$$

Clearly, *the magnetic field at any equatorial point of a magnetic dipole is in the direction opposite to that of its magnetic dipole moment i.e.*, from N-pole to S-pole. So we can write

$$\vec{B}_{equa} = -\frac{\mu_0\, \vec{m}}{4\pi r^3}$$

$-\frac{k\vec{m}}{r^3}$

On comparing equations (1) and (2), we note *that the magnetic field at a point at a certain distance on the axial line of a short magnet is twice of that at the same distance on its equatorial line.*

Torque on a magnetic dipole in a uniform magnetic field. Consider a bar magnet *NS* of length $2l$ placed in a uniform magnetic field $\vec{B}$. Let q_m be the pole strength of its each pole. Let the magnetic axis of the bar magnet make an angle θ with the field $\vec{B}$, as shown in Fig. 5.21(*a*).

Force on N-pole $= q_m B$; along $\vec{B}$

Force on S-pole $= q_m B$, opposite to $\vec{B}$

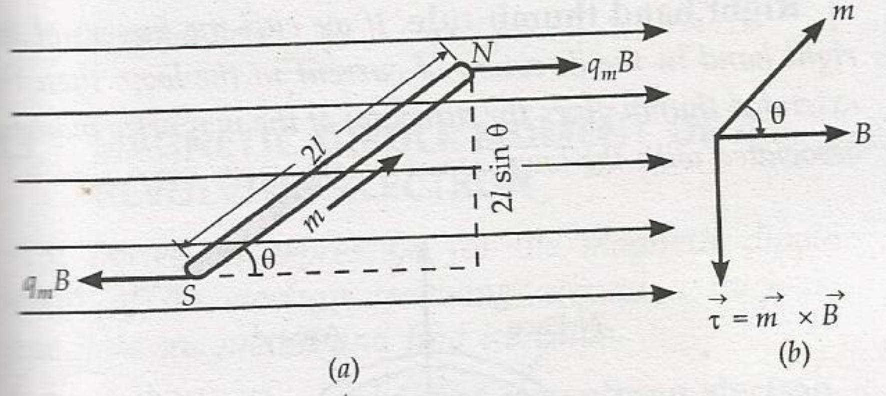

Fig. 5.21 (*a*) Torque on a bar magnet in a magnetic field. (*b*) Relation between the directions of $\vec{\tau}$, $\vec{m}$, $\vec{B}$.

The forces on the two poles are equal and opposite. They form a couple. Moment of couple or torque is given by

$$\tau = \text{Force} \times \text{perpendicular distance}$$

$$= q_m B \times 2l \sin\theta = (q_m \times 2l)\, B \sin\theta$$

or $$\tau = mB\sin\theta \qquad ...(1)$$

where $m = q_m \times 2l$, is the magnetic dipole moment of the bar magnet. In vector notation,

$$\vec{\tau} = \vec{m} \times \vec{B} \qquad ...(2)$$

The direction of the torque $\vec{\tau}$ is given by the right hand screw rule as indicated in Fig. 5.21(*b*). The effect of the torque $\vec{\tau}$ is to make the magnet align itself parallel to the field $\vec{B}$. That is why a freely suspended magnet aligns itself in the north-south direction because the earth has its own magnetic field which exerts a torque on the magnet tending it to align along the field.

of stable and unstable

Potential energy of a magnetic dipole. As shown in Fig. 5.21(*a*), when a magnetic dipole is placed in a uniform magnetic field $\vec{B}$ at angle θ with it, it experiences a torque

$$\tau = mB \sin \theta$$

This torque tends to align the dipole in the direction of $\vec{B}$.

If the dipole is rotated against the action of this torque, work has to be done. This work is stored as *potential energy* of the dipole.

The work done in turning the dipole through a small angle $d\theta$ is

$$dW = \tau \, d\theta = mB \sin \theta \, d\theta$$

If the dipole is rotated from an initial position $\theta = \theta_1$ to the final position $\theta = \theta_2$, then the total work done will be

$$W = \int dW = \int_{\theta_1}^{\theta_2} mB \sin\theta \, d\theta = mB[-\cos\theta]_{\theta_1}^{\theta_2}$$

$$= -mB(\cos\theta_2 - \cos\theta_1)$$

This work done is stored as the potential energy U of the dipole.

$$\therefore \quad U = -mB(\cos\theta_2 - \cos\theta_1)$$

The potential energy of the dipole is zero when $\vec{m} \perp \vec{B}$. So potential energy of the dipole in any orientation θ can be obtained by putting $\theta_1 = 90°$ and $\theta_2 = \theta$ in the above equation.

$$\therefore \quad U = -mB(\cos\theta - \cos 90°)$$

or $$U = -mB\cos\theta = -\vec{m} \cdot \vec{B}$$

Special Cases

1. When $\theta = 0°$, $U = -mB\cos 0° = -mB$

   Thus the potential energy of a dipole is *minimum* when $\vec{m}$ is *parallel* to $\vec{B}$. In this state, the magnetic dipole is in ***stable equilibrium***.
2. When $\theta = 90°$, $U = -mB\cos 90° = 0$.
3. When $\theta = 180°$, $U = -mB\cos 180° = +mB$.

Thus the potential energy of a dipole is *maximum* when $\vec{m}$ is *antiparallel* to $\vec{B}$. In this state, the magnetic dipole is in ***unstable equilibrium***.

Current loop as a magnetic dipole. We know that the magnetic field produced at a large distance r from the centre of a circular loop (of radius a) along its axis is given by

$$B = \frac{\mu_0 I a^2}{2 r^3}$$

or
$$B = \frac{\mu_0}{4\pi} \cdot \frac{2 I A}{r^3} \qquad ...(1)$$

where I is the current in the loop and $A = \pi a^2$ is its area. On the other hand, the electric field of an electric dipole at an axial point lying far away from it is given by

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2p}{r^3} \qquad ...(2)$$

where p is the electric dipole moment of the electric dipole.

On comparing equations (1) and (2), we note that both B and E have same distance dependence $\left(\frac{1}{r^3}\right)$. Moreover, they have same direction at any far away point, not just on the axis. This suggests that a circular current loop behaves as a magnetic dipole of magnetic moment,

$$m = IA$$

In vector notation,

$$\vec{m} = I\,\vec{A}$$

This result is valid for planar current loop of any shape. *Thus the magnetic dipole moment of any current loop is equal to the product of the current and its loop area.* Its direction is defined to be normal to the plane of the loop in the sense given by right hand thumb rule.

Magnetic dipole moment of a revolving electron. According to Bohr model of hydrogen-like atoms, negatively charged electron revolves around the positively charged nucleus. This uniform circular motion of the electron is equivalent to a current loop which possesses a magnetic dipole moment = IA. As shown in Fig. 5.23, consider an electron revolving anticlockwise around a nucleus in an orbit of radius r with speed v and time period T.

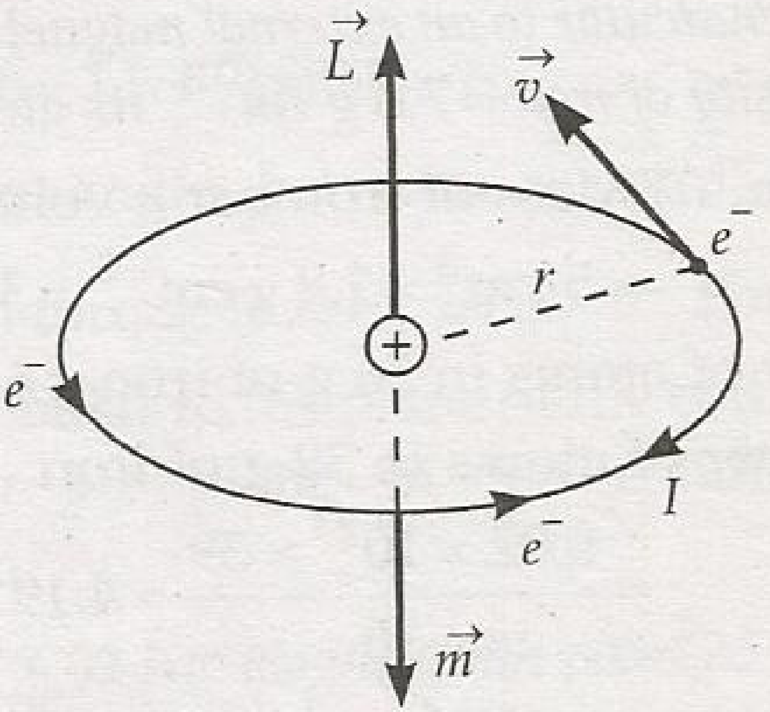

Fig. 5.23 Orbital magnetic moment of a revolving electron.

Equivalent current,

$$I = \frac{\text{Charge}}{\text{Time}} = \frac{e}{T} = \frac{e}{2\pi r / v} = \frac{ev}{2\pi r}$$

Area of the current loop, $A = \pi r^2$

Therefore, the *orbital magnetic moment* (magnetic moment due to orbital motion) of the electron is

$$\mu_l = IA = \frac{ev}{2\pi r} \cdot \pi r^2$$

or $$\mu_l = \frac{evr}{2} \qquad ...(1)$$

As the negatively charged electron is revolving anticlockwise, the associated current flows clockwise. According to right hand thumb rule, the direction of the magnetic dipole moment of the revolving electron will be perpendicular to the plane of its orbit and in the downward direction, as shown in Fig. 5.23

Also, the angular momentum of the electron due to its orbital motion is

$$l = m_e vr \qquad ...(2)$$

The direction of $\vec{l}$ is normal to the plane of the electron orbit and in the upward direction, as shown in Fig. 5.23.

Dividing equation (1) by (2), we get

$$\frac{\mu_l}{l} = \frac{evr/2}{m_e vr} = \frac{e}{2m_e}$$

The above ratio is a constant called ***gyromagnetic ratio***. Its value is 8.8×10^{10} C kg^{-1}. So

$$\mu_l = \frac{e}{2m_e} l$$

Vectorially,

$$\vec{\mu}_l = -\frac{e}{2m_e} \vec{l}$$

The negative sign shows that the direction of $\vec{l}$ is opposite to that of $\vec{\mu}_l$. According to *Bohr's quantisation condition*, the angular momentum of an electron in any permissible orbit is integral multiple of $h/2\pi$, where h is Planck's constant, *i.e.*,

$$l = \frac{nh}{2\pi}, \quad \text{where } n = 1, 2, 3, ...$$

$$\therefore \quad \mu_l = n\left(\frac{eh}{4\pi m_e}\right)$$

This equation gives orbital magnetic moment of an electron revolving in nth orbit.

Bohr magneton. *It is defined as the magnetic moment associated with an electron due to its orbital motion in the first orbit of hydrogen atom.* It is the minimum value of μ_l which can be obtained by putting $n = 1$ in the above equation. Thus Bohr magneton is given by

$$\mu_B = (\mu_l)_{\min} = \frac{eh}{4\pi m_e}$$

Putting the values of various constants, we get

$$\mu_B = \frac{1.6 \times 10^{-19}\ \text{C} \times 6.63 \times 10^{-34}\ \text{Js}}{4 \times 3.14 \times 9.11 \times 10^{-31}\ \text{kg}}$$

$$= \mathbf{9.27 \times 10^{-24}\ Am^2}.$$

Besides the orbital angular momentum $\vec{l}$, an electron has spin angular momentum $\vec{S}$ due to its spinning motion. *The magnetic moment possessed by an electron due to its spinning motion is called* ***intrinsic magnetic moment*** *or* ***spin magnetic moment***. It is given by

$$\vec{\mu}_s = -\frac{e}{m_e}\vec{S}$$

The total magnetic moment of the electron is the vector sum of these two momenta. It is given by

$$\vec{\mu} = \vec{\mu}_l + \vec{\mu}_s = -\frac{e}{2m_e}(\vec{l} + 2\vec{S})$$