Yale University Department of Music

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Author(s): Jack Douthett and Peter Steinbach
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PARSIMONIOUS GRAPHS: A STUDY IN PARSIMONY, CONTEXTUAL TRANSFORMATIONS, AND MODES OF LIMITED TRANSPOSITION

Jack Douthett and Peter Steinbach

ABSTRACT

Connections between parsimonious structures and *modes of limited transposition* from three set classes are explored. A graph-theoretic approach proves useful in illustrating the symmetries inherent in parsimonious structures and modes of limited transposition. Four parsimonious graphs called *mode graphs* are constructed. Each mode graph consists of several components, and the vertices in each of these components represent triads or seventh chords embedded in a particular mode of limited transposition. Two parsimonious methods of *modulating* between modes of limited transposition are explored, one by *bridging* and the other by *coupling* components of mode graphs. Bridging techniques of modulation lead to two tori, one for triads and the other for seventh chords. In both tori, *contextual transformations* are evident in their structures, and the torus for triads is equivalent to the toroidal version of the *Oettingen/Riemann Tonnetz*. Coupling techniques of modulation lead to the graphs known as *Cube Dance* and *Power Towers*. Analytical implications of patterns of chord sequences embedded in parsimonious graphs are also discussed.

1. Introduction

Thanks to the 1993 SUNY-Buffalo Working Group, its 1997 sequel, and recent works by Klumpenhouwer (1994), Hyer (1995), Lewin (1996), Cohn (1996, 1997), and Clampitt (1997), new techniques are emerging that yield fresh insight into the complexities of voice leading, especially in the music of the late-nineteenth and early-twentieth centuries.[1] The resurrection of the theories of Hugo Riemann, initiated by Lewin (1982 and 1987, 175–80) and Hyer (1989), has played an important role in the development of these new techniques. *Neo-Riemannian Theory* puts forth a group-theoretic approach to Riemann's ideas, and contextual transformations that operate on consonant triads are fundamental to this theory. Three of these transformations, Parallel, Leittonwechsel, and Relative (the PLR family of contextual transformations), transform the modality of a consonant triad by inverting the triad about an axis that leaves two of its pitch classes fixed, and Cohn (1997) exploited this property to advance the concept of *parsimony* (law of the shortest way).

Parsimonious structures (e.g., *Tonnetze*) generally have high degrees of symmetry and are independent of tonal centers. As a result of the metamorphosis from the tonally centered eighteenth-century diatonicism to the tonal indeterminacy of the late nineteenth century, these symmetric structures have become important theoretical and analytical tools. With this new emphasis on symmetry, it is not surprising to discover that pitch-class collections from set classes 6-20, 8-28, and 9-12—pitch-class sets abundant in their degrees of symmetry—are associated with parsimony (for convenience, we will refer to these collections as *modes of limited transposition*[2]); Cohn discusses some of these relationships in his recent articles.

In this paper we take a graph-theoretic approach to the above and construct *parsimonious graphs* whose vertices represent either triads or seventh chords. In Section 2 the *relation definition* provides the needed tool to construct these graphs. Parsimonious graphs known as *mode graphs* are constructed in Section 3. These graphs consist of several components, and the vertices in each of these components represent chords embedded in a particular mode of limited transposition. In Section 4 *modulating* between modes of limited transposition by *bridging* parsimonious chords in different components of a mode graph will be investigated, and this leads to the construction of two tori. One of these tori is equivalent to the toroidal version of the *Oettingen/Riemann Tonnetz,* and hence, all the contextual transformations (Parallel, Leittonwechsel, and Relative) can be seen on this torus.[3] Moreover, these transformations have analogs apparent in the other torus, and these transformations are examined as well. A second method of modulating between modes of limited trans-

position via a *coupling chord* is discussed in Section 5, and the graphs known as *Cube Dance* and *Power Towers* illustrate this type of modulation. In the concluding section the definition of parsimony will be relaxed a bit, and the graphs resulting from the loosening of this definition will be investigated. Analytical implications of patterns of chord sequences embedded in parsimonious graphs are also discussed in this section.

2. The Relation Definition and Parsimony

The definition of *parsimony* is still evolving and, at present, not completely consistent. Richard Cohn (1997) restricts the definition of parsimony to trichords, and in the context of the usual modulo 12 system he confines his attention to the set class of consonant triads. Two triads are parsimonious if they have precisely two pitch classes in common. This results in the displacement of the remaining pitch class by interval class 1 or 2 (a half step or a whole step). Adrian Childs (1998) considers parsimonious seventh chords from the set class 4-27. In this case, two seventh chords are parsimonious if they have two pitch classes in common, and the remaining pitch classes can be paired so that the pitch classes in each pair differ by interval class 1. The differences between these approaches are obvious, but they are consistent in that common tones are required to remain fixed, while the other tones are allowed to move by no more than interval class 2. With this in mind we make the following definition:

Relation Definition: Let X and Y be two pcsets. We say X and Y are $P_{m,n}$-related (written $X\,P_{m,n}Y$) if there exists a set $\{x_k\}_{k=0}^{m+n-1}$ and a bijection τ: $X \rightarrow Y$ such that $X \backslash Y$ (the set of pcs in X that are not in Y) = $\{x_k\}_{k=0}^{m+n-1}$, $\tau(x) = x$ if $x \in X \cap Y$, and

$$\tau(x_k) = \begin{cases} x_k \pm 1 (\text{mod} 12) \text{ if } 0 \leq k \leq m - 1 \\ x_k \pm 2 (\text{mod} 12) \text{ if } m \leq k \leq m + n - 1. \end{cases}$$

To put this definition in musical terms, when X and Y are conceived as chords, the tones common to both X and Y remain fixed, and of the remaining tones (there will be $m + n$ of them), m of them move by a half step and n by a whole step.[4] Figure 1 depicts four different scenarios for consonant triads and how they relate in accordance with this definition. Note also that this definition implies that $P_{m,n}$-related chords have the same cardinality (since τ is bijective), but they need not be in the same set class.[5]

This relation definition leads to a somewhat flexible interpretation of parsimony. For example, for Cohn, *parsimonious chords* are consonant triads that are either $P_{1,0}$- or $P_{0,1}$-related, while Childs defines a parsimonious pair as seventh chords in set class 4-27 that are $P_{2,0}$-related. This

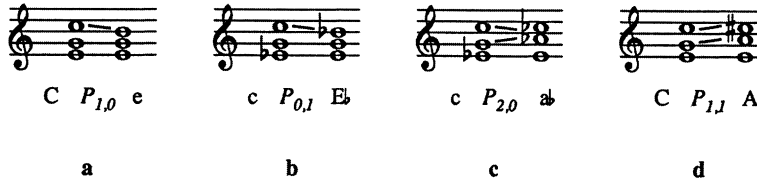


Figure 1. $P_{m,n}$-related Consonant Triads

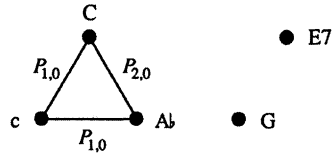


Figure 2. Graph of $\mathcal{F}$ induced by $P_{1,0}$ and $P_{2,0}$

flexibility is convenient in that it permits one to explore parsimonious structures of varying types and degrees, depending on particular objectives.[6] In Sections 3, 4, and 5 our definition of *parsimonious chords* will be confined to chords that are either $P_{1,0}$- or $P_{0,1}$-related. We will employ these two types of parsimony ($P_{1,0}$-parsimony and $P_{0,1}$-parsimony) separately or together, depending on our objectives. In the last section we will admit $P_{2,0}$-related chords into our definition of parsimony.

Parsimonious graphs are graphs whose vertices represent members of a given family of chords and whose vertex adjacency is governed by some parsimony-related rule. For example, consider the family $\mathcal{F}$ = {C,A♭,G,c,E7}, and suppose the rule governing adjacency on this family is that two vertices are connected by an edge if their corresponding chords are $P_{1,0}$- or $P_{2,0}$-related. Then since C $P_{2,0}$ A♭ $P_{1,0}$ c $P_{1,0}$ C and G and E7 are not $P_{1,0}$- or $P_{2,0}$-related to any chord in $\mathcal{F}$, *the graph of* $\mathcal{F}$ *induced by* $P_{1,0}$ *and* $P_{2,0}$ consists of a triangle whose vertices represent C, A♭, and c and two isolated vertices representing G and E7 (Figure 2). We will also consider the union of parsimonious graphs to be a parsimonious graph.

Although we will focus on graphs that are essentially independent of diatonic influence, it should be noted that some parsimonious graphs are strongly influenced by diatonicism. Most notably, the graph of set class 7-35 induced by $P_{1,0}$ is the *cycle of fifths* and belongs to a family of *generalized cycles of fifths* referred to by Cohn as *unidirectional P-cycles*.[7]

3. Mode graphs

We begin by constructing four basic parsimonious graphs. Since each of these graphs is separated into components whose vertices represent chords embedded in a particular *mode* of limited transposition, these graphs will be referred to as *mode graphs.* The modes of limited transposition associated with these graphs are the *hexatonic* (from set class 6-20), *octatonic* (8-28), and *enneatonic* (9-12) sets. The prefixes to the names assigned to these mode graphs indicate the associated modes of limited transposition. The parsimony that induces the first two mode graphs (one for triads and the other for seventh chords) is restricted to $P_{1,0}$-parsimony, and the construction of the second parallels that of the first. Both types of parsimony are needed to induce the last two graphs, and these graphs will parallel in construction as well.

HexaCycles

Let $\mathcal{T}_1$ be the family of all trichords whose step-interval sequences are permutations of 3, 4, and 5. Whence, $\mathcal{T}_1$ is the set class of all consonant triads. We call the graph of $\mathcal{T}_1$ induced by $P_{1,0}$ *HexaCycles,* and its components are *hexatonic cycles* for triads (Figure 3). As discussed by Cohn (1996) the consonant triads in each of these cycles coincide with those embedded in the hexatonic set listed below the cycle.

OctaTowers

To parallel this construction for seventh chords, let $\mathcal{S}_1$ be the family of all tetrachords whose step-interval sequences are permutations of 2, 3, 3, and 4. It follows that $\mathcal{S}_1$ is the family of all dominant, half-diminished, and minor seventh chords, and the graph of $\mathcal{S}_1$ induced by $P_{1,0}$, which we call *OctaTowers,* consists of three *octatonic tower* components (Figure 4). Each tower consists of four quadrilaterals joined in a circuit. In this

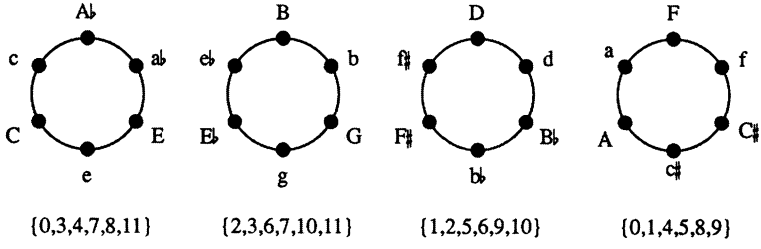


Figure 3. HexaCycles

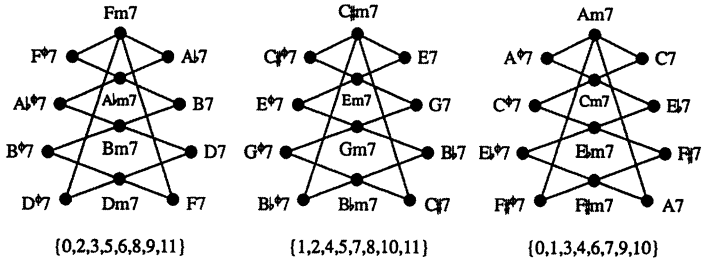


Figure 4. OctaTowers

case, the seventh chords in each octatonic tower coincide with those embedded in the octatonic set listed below the tower.

OctaCycles

To construct this graph, it is necessary to partition $\mathcal{T}_1$ into three subfamilies, $\mathcal{T}_{1,1}$, $\mathcal{T}_{1,2}$, and $\mathcal{T}_{1,3}$, such that each subfamily consists of the consonant triads embedded in a particular octatonic set. The graph *OctaCycles* is the union of the graphs of $\mathcal{T}_{1,k}$ (k=1,2,3) induced by $P_{1,0}$ and $P_{0,1}$ (Figure 5). These cyclic components are called *octatonic cycles,* and the triads in each cycle can be found embedded in the octatonic set listed below the cycle.

EnneaCycles

In this case $\mathcal{S}_1$ must be subdivided into four subfamilies, $\mathcal{S}_{1,1}$, $\mathcal{S}_{1,2}$, $\mathcal{S}_{1,3}$, and $\mathcal{S}_{1,4}$, such that each subfamily consists of the seventh chords embedded in a particular enneatonic set. It is easily verified that these subfamilies partition $\mathcal{S}_1$. The graph *EnneaCycles* is the union of the graphs of $S_{1,k}$ (k=1,2,3,4) induced by $P_{1,0}$ and $P_{0,1}$ (Figure 6). These cyclic components are called *enneatonic cycles,* and the enneatonic set in which the seventh chords in each of these cycles are embedded is listed under the cycle.

4. The tori

One method of *modulating* between modes of limited transposition is to *bridge* components of a mode graph. This is done by joining parsimonious chords in separate components with an edge. This transforms the above mode graphs into two tori, one for triads and the other for seventh chords.

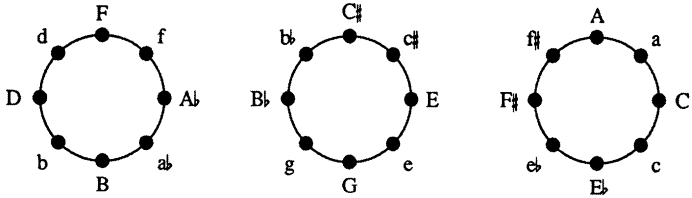
The Chicken-Wire Torus

The torus (graph) of $\mathcal{T}_1$ induced by $P_{1,0}$ and $P_{0,1}$ is called the *Chicken-Wire Torus* because of its hexagonal faces (Figure 7a). To realize this

torus, bend the upper and lower edges around so they meet, thus making an open-ended cylinder. Then pull the left and right ends of this cylinder around to form a donut, but before joining the ends, twist one end of the cylinder one-third of a revolution so that the corresponding triads on each end line up (Figure 7b).[8] This twist plays an important role in the group symmetries of this graph, to be discussed at the end of this section.

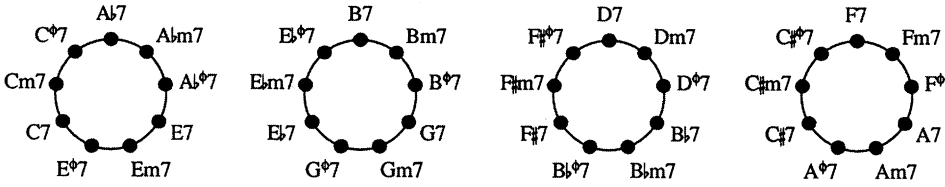
This graph is the *geometric dual* of the *Tonnetz;* that is, the Chicken-Wire Torus can also be constructed by mapping each face in the *Tonnetz* to a vertex and joining vertices with an edge if their corresponding faces in the *Tonnetz* share an edge. It can also be verified that the *Tonnetz* is the dual of the Chicken-Wire Torus. Thus, each torus can be obtained from the other, and hence, the Chicken-Wire Torus and the *Tonnetz* must contain equivalent information. The Chicken-Wire Torus, however, provides a different perspective.

In the Chicken-Wire Torus, the edges represent transformations from the PLR family of contextual transformations. The solid edges represent the P (Parallel) transformation and connect triads (represented by vertices) that are $P_{1,0}$-related and have the same root; the dotted edges represent the L (Leittonwechsel) transformation and connect triads that are $P_{1,0}$-related and have different roots; and the dashed edges represent the



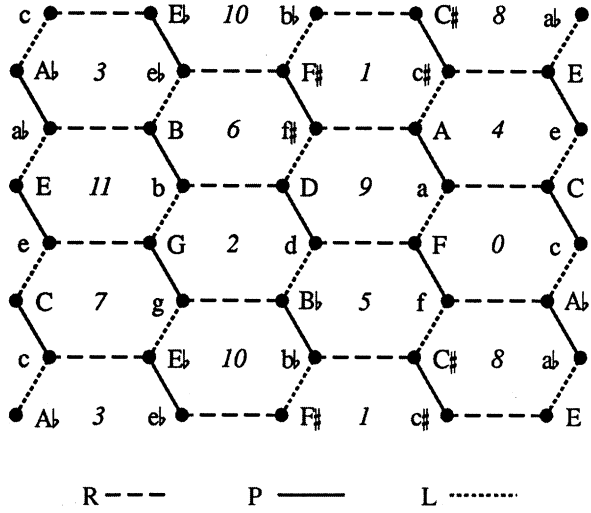
{0,2,3,5,6,8,9,11} {1,2,4,5,7,8,10,11} {0,1,3,4,6,7,9,10}

Figure 5. OctaCycles



{0,2,3,4,6,7,8,10,11} {1,2,3,5,6,7,9,10,11} {0,1,2,4,5,6,8,9,10} {0,1,3,4,5,7,8,9,11}

Figure 6. EnneaCycles



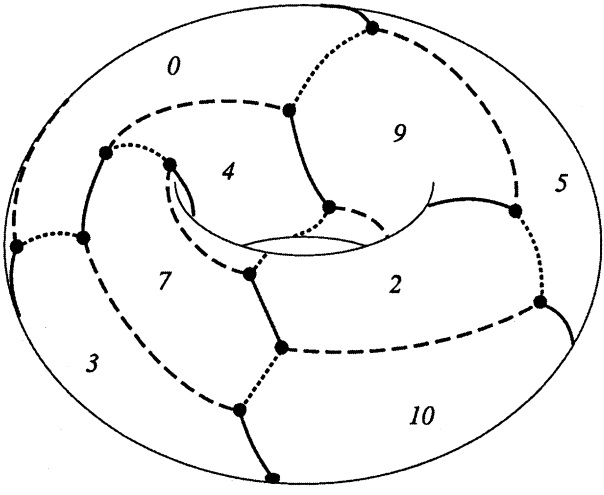


Figure 7. Chicken-wire Torus

R (Relative) transformation and connect triads that are $P_{0,1}$-related. The cycles of alternating dotted and solid edges are hexatonic cycles (the components of HexaCycles), and since the dotted and solid edges represent the L and P transformations, respectively, these are the cycles induced by the binary-generated chain of transformations <PL>. The dashed edges

are *hexatonic bridges*, since they connect parsimonious triads embedded in different hexatonic sets. Note that the transformation represented by the bridges is the Relative transformation; thus, the triads incident to each hexatonic bridge are $P_{0,1}$-related.

The three 8-cycles (cycles of length 8) of alternating dashed and solid edges are the octatonic cycles (the components of OctaCycles) and are induced by the binary-generated chain of transformations <PR>. Since the dotted edges connect parsimonious triads in separate octatonic cycles, they are the *octatonic bridges* (for triads). It follows that the triads incident to an octatonic bridge (Leittonwechsel) are $P_{1,0}$-related.

There are two other generated chains of transformations to be noted. The 24-cycle of alternating dotted and dashed edges (a *hamiltonian cycle,* since it passes through each vertex in the Chicken-Wire Torus exactly once) is induced by the binary-generated chain <LR>. This cycle is the union of all the hexatonic and octatonic bridges. As such, the triads in this cycle move quickly through the hexatonic and octatonic sets, resting only briefly in any one set.

The last chain to be discussed here is the ternary-generated chain <LPR>. This chain induces the twelve hexagonal faces (6-cycles) in the Chicken-Wire Torus. The triads in each of these cycles have a single pitch class in common, and this pitch class becomes the name of the hexagonal face.

Note that if the hexatonic bridges (Relative) are cut from the Chicken-Wire Torus then the torus reduces to HexaCycles, and if the octatonic bridges (Leittonwechsel) are cut, the torus reduces to OctaCycles. Cutting P reduces the torus to the 24-cycle induced by <LR> (the union of the hexatonic and octatonic bridges). As can be seen from the above, the Chicken-Wire Torus provides a convenient picture that shows how consonant triads interact within the web of modes of limited transposition and parsimonious voice leading.

The Towers Torus

The *Towers Torus* is the torus of $\mathcal{S}_1$ induced by $P_{1,0}$ and $P_{0,1}$ shown in Figure 8. Its name comes from the fact that the circuits of quadrilaterals (square faces) hinged by minor seventh chords are the octatonic towers in Figure 4. To realize this torus, bend the upper and lower edges around so they meet, thus making an open-ended cylinder. Then pull the left and right ends of this cylinder around to form a donut. But before joining the ends, twist one end of the cylinder one-fourth of a revolution so that the corresponding chords on each end line up. As in the case of the Chicken-Wire Torus, this twist is important to the group symmetries of the Towers Torus, which will be discussed at the end of this section.

There are Parallel*, Leittonwechsel*, and Relative* contextual transformations for seventh chords that are analogous to the unstarred transformations for triads. In the Chicken-Wire Torus the transformation P

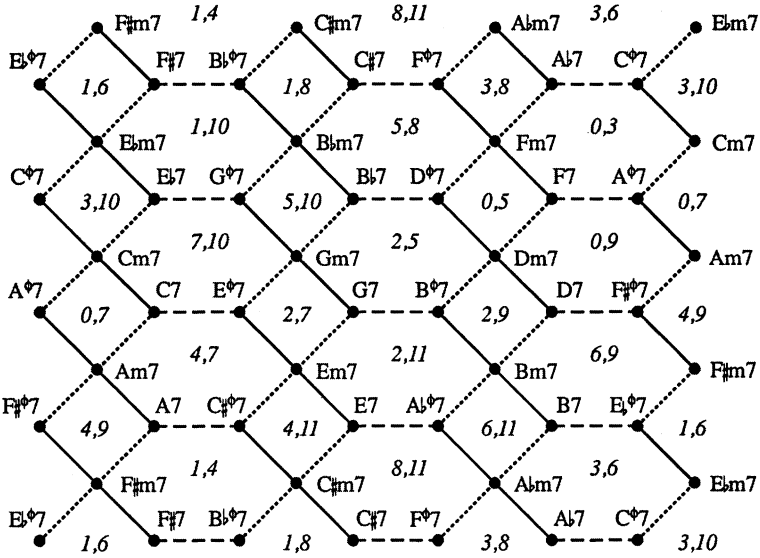


Figure 8. Towers Torus

(Parallel) exchanges triads that have the same root (see solid edges in Figure 7). In the case of seventh chords there are two Parallel* transformations. The transformation P_1^* exchanges the half-diminished and minor seventh chords that have the same root, and P_2^* exchanges the minor and dominant-seventh chords with the same root (see solid edges in Figure 8). In the Chicken-Wire Torus the L (Leittonwechsel) transformation exchanges root-distinct $P_{1,0}$-related triads (see dotted edges in Figure 7). For seventh chords there are two Leittonwechsel* transformations; L_1^* exchanges root-distinct $P_{1,0}$-related half-diminished and minor seventh chords, and L_2^* exchanges root-distinct $P_{1,0}$-related dominant and minor seventh chords (see dotted edges in Figure 8). Finally, the Relative transformation R exchanges triads that are $P_{0,1}$-related (dashed edges in Figure 7) and the Relative* transformation R^* exchanges two seventh-chords that are $P_{0,1}$-related (see dashed edges in Figure 8).

Whereas the Towers Torus gives no information about the hexatonic sets, it does provide new insights into the seventh chord interaction between the octatonic and enneatonic sets. In fact, the roles that the octatonic and enneatonic sets play in the Towers Torus are analogous to those of the hexatonic and octatonic sets in the Chicken-Wire Torus. The dashed edges (Relative*) are *octatonic bridges* (for seventh chords) and join $P_{0,1}$-related seventh chords that are in separate octatonic towers (Figure 4). The four 9-cycles of dashed and solid edges are the enneatonic cycles (Figure 6) and the dotted edges (Leittonwechsel*) are *enneatonic bridges*. Thus, seventh chords incident to an enneatonic bridge are $P_{1,0}$-related.

Although more complex, many of the chains of transformations for seventh chords also parallel the chains for triads. The two binary-generated chains $\langle P_1^* L_1^* \rangle$ and $\langle P_2^* L_2^* \rangle$ parallel the chain <PL> in that all induce cycles of pairwise $P_{1,0}$-related chords embedded in modes of limited transposition. These starred chains split (but do not partition) the octatonic towers into two subfamilies of seventh chords. The cycles induced by $\langle P_1^* L_1^* \rangle$ partition the subfamily of half-diminished and minor seventh chords into three 8-cycles of pairwise $P_{1,0}$-related seventh chords. The seventh chords in each cycle are those embedded in a particular octatonic set. The cycles induced by $\langle P_2^* L_2^* \rangle$ partition the subfamily of dominant and minor seventh chords into three 8-cycles of pairwise $P_{1,0}$-related seventh chords similarly embedded in the octatonic sets. These cycles can be observed in the Towers Torus by following the cycles of alternating dotted and solid edges in Figure 8.

The ternary-generated chain $\langle P_1^* P_2^* R^* \rangle$ is analogous to the <PR> chain. The cycles induced by this chain partition Towers Torus into the four enneatonic cycles, similar to the way that the cycles induced by the <PR> chain partition the Chicken-Wire Torus into the three octatonic cycles.

Paralleling the cycle of triads induced by <LR>, the two 18-cycles induced by the ternary-generated chain of transformations $\langle L_1^* L_2^* R^* \rangle$ are the union of all the octatonic and enneatonic bridges. Whence, the seventh chords in these cycles pass quickly through the octatonic and enneatonic sets, settling only briefly in any particular set.

There are two other ternary-generated chains, $\langle L_1^* P_2^* R^* \rangle$ and $\langle P_1^* L_2^* R^* \rangle$. The cycles induced by these chains are 36-cycles and appear to have no particularly relevant analog in the Chicken-Wire Torus. Although they are similar to the cycle of triads induced by the <LR> chain in that they are hamiltonian cycles, they lack the more important property of being the union of the octatonic and enneatonic bridges.

There are also cycles of interest induced by longer chains of transformations. For example, the seventh chords in each 4-cycle face and 6-cycle face (the squares and hexagons in the Towers Torus) share the pair of pitch classes that name the face of the corresponding square or hexa-

gon. The corresponding chains of transformations generating these cycles are $<P_1^*L_1^*P_2^*L_2^*>$ and $<P_1^*L_1^*R^*P_2^*L_2^*R^*>$. In the sense that these cycles define the names of the faces, they are analogous to the cycles induced by the <PLR> chain of transformations in the Chicken-Wire Torus. There are also forty-eight 10-cycles in which the seventh chords share a single pitch class.

Also paralleling the Chicken-Wire Torus, if the octatonic bridges (Relative*) are cut from the Towers Torus the resultant graph is Octa-Towers, and if the enneatonic bridges (Leittonwechsel*) are cut this torus reduces to EnneaCycles. Whence, just as the Chicken-Wire Torus provides a convenient way to observe the interaction of parsimonious triads and their relationship to the hexatonic and octatonic sets, so does the Towers Torus present a convenient way of observing the interaction of parsimonious seventh chords and their relationships to the octatonic and enneatonic sets.

Symmetries of the Tori

Symmetries of a graph are adjacency-preserving vertex permutations, and the symmetries of the Chicken-Wire Torus (Figure 7), the Towers Torus (Figure 8), and the *Tonnetz* are all governed by the chromatic 12-cycle of pitch classes. It is well known that the group of symmetries of this 12-cycle is the dihedral group D_{12} and consists of 24 permutations in three different ways:

(1) *12 rotations of the cycle are transpositions by fixed intervals.* On the Chicken-Wire Torus these are even translations (major-to-major) of the lattice along the <LR> 24-chain. The motion is similar on the *Tonnetz*, since this *Tonnetz* is the geometric dual of the Chicken-Wire Torus (the groups of symmetries of geometric duals are isomorphic). On the Towers Torus of seventh chords, this transposition is achieved by a horizontal translation (m7-to-m7) of the lattice.

(2) *6 reflections of the chromatic cycle occur on axes passing through and fixing two opposite (tritone) pitch classes.* These are half-turns of the Chicken-Wire Torus about the center of a hexagonal face or, equivalently, about the midpoint of an R edge, exchanging major and minor triads.[9] On the Towers Torus these are half-turns about a m7 vertex or about the midpoint of an R^* edge, mapping m7-to-m7 (self-inverse) and exchanging dominant and half-diminished seventh chords.

(3) *6 reflections of the chromatic cycle occur on axes passing through opposite edges of the cycle, fixing no pitch classes.* These are half-turns of the Chicken-Wire Torus about the midpoint of an L edge or, equivalently, a P edge, again exchanging major and minor tri-

ads. On the Towers Torus these are the half-turns about the center of a hexagonal face or a square face, with maps as in (2).

All 24 symmetries preserve the Parallel, Leittonwechsel, and Relative edge types of the Chicken-Wire Torus. On the Towers Torus, (2) and (3) exchange the Leittonwechsel* edge types (L_1^* and L_2^*) and exchange the Parallel* edge types. Because these lattices are twisted to form the tori, no other symmetries exist. At a glance the Chicken-Wire Torus suggests a possible reflection of the lattice about a vertical or horizontal axis, but (because of the twist) this reflection produces immediate contradictions such as the exchange of <PR> 8-chains for <LR> 24-chain. Without this twist the symmetries would total 144. Similar problems are encountered with reflections about vertical or horizontal axes of the Towers Torus. A left glove turned inside out is no longer a left glove and cannot map automorphically to its original state.

5. Cube Dance and Power Towers

Another method of parsimonious modulation between modes of limited transposition is to introduce a chord that belongs to none of the components in a given mode graph but is $P_{1,0}$-related to chords in two or more of its components. Thus, one passes from one component to the next through an intermediate *coupling chord.* The following graphs, *Cube Dance* and *Power Towers,* illustrate this technique.[10]

Cube Dance

We start with HexaCycles (Figure 3) and look for trichords with the greatest potential to couple hexatonic cycles via $P_{1,0}$-relations. One class of trichords that stands out in this search is the set class of augmented triads, since altering any pitch of an augmented triad up or down by a half-step produces a consonant triad. Whence, each augmented triad is $P_{1,0}$-related to six consonant triads. By allowing the augmented triads to interact with the hexatonic cycles via $P_{1,0}$-parsimony, each of these cycles is transformed into a cube as illustrated in Figure 9a. In this figure, the cycles with bold edges are hexatonic cycles. Note that each augmented triad is common to precisely two cubes. The union of these cubes generates the graph known as *Cube Dance* (Figure 9b).[11] Equivalently, if $\mathcal{A}$ is the family (set class) of all augmented triads and $\mathcal{T}_2 = \mathcal{T}_1 \cup \mathcal{A}$ then *Cube Dance* is the graph of $\mathcal{T}_2$ induced by $P_{1,0}$.

The numbers inclosed in squares that surround this graph are the *sum classes* of the three triads of like modality adjacent to the numbers. Sum classes are discussed in detail by Cohn (1998).

From this graph it is easy to see how the augmented triads serve as the *couplings* between hexatonic cycles and function nicely as a way of mod-

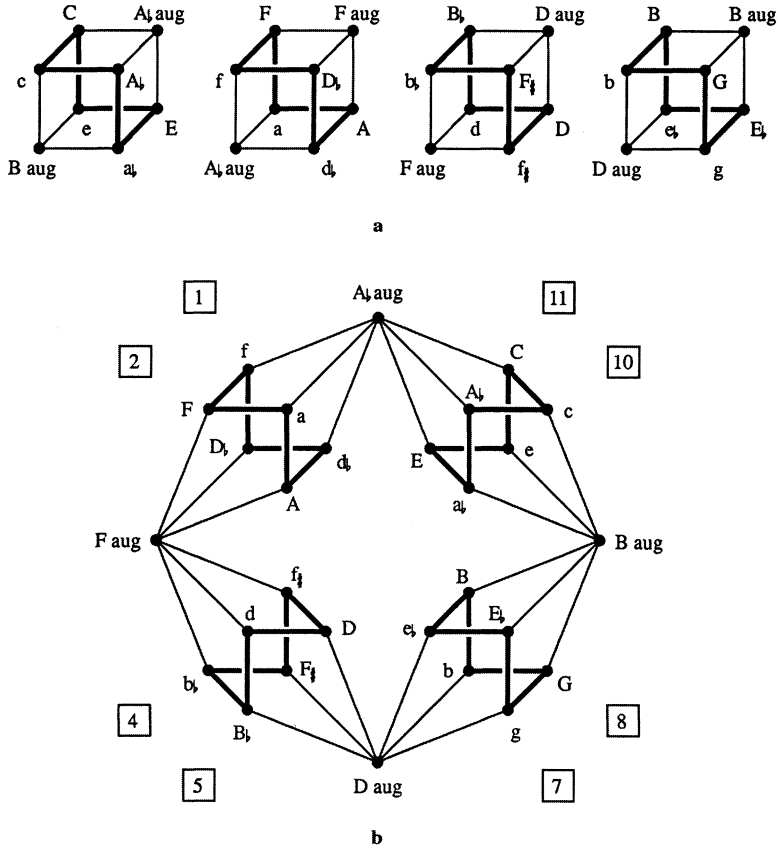


Figure 9. Cube Dance

ulating between hexatonic sets. In fact, the augmented triad coupling two hexatonic cycles can also be found embedded in both associated hexatonic sets (a common trichord). Any triad in a given hexatonic cycle can move through a coupling augmented triad to a neighboring hexatonic cycle in three ways or back to a triad within its own cycle in three ways. If the transition is to a neighboring hexatonic cycle then the coupled consonant triads are related by inversion, and if the transition is within a single cycle, they are related by transposition.

Power Towers

Power Towers is the seventh-chord analog of Cube Dance. We start with OctaTowers (Figure 4) and look for tetrachordal couplings for the octatonic towers. For the same kind of reasons that augmented triads served well as coupling trichords for hexatonic cycles, diminished seventh chords are ideal coupling tetrachords for octatonic towers. If $\mathcal{D}$ is the family of all diminished seventh chords and $\mathcal{S}_2 = \mathcal{S}_1 \cup \mathcal{D}$, then the graph of $\mathcal{S}_2$ induced by $P_{1,0}$ is called *Power Towers* (Figure 10).[12]

As in the case of Cube Dance, the numbers inclosed in squares that surround this graph are the *sum classes* of the transpositionally equivalent seventh chords adjacent to the numbers and are discussed by Cohn (1998).

This graph illustrates how diminished seventh chords function as couplings between octatonic towers, thus providing a way to modulate between octatonic sets. Similar to the augmented triad in Cube Dance, the diminished seventh chord coupling two octatonic towers is embedded in both associated octatonic sets. Diminished seventh chords are only adjacent to dominant and half-diminished seventh chords, and if the coupled seventh chords are in different towers they are related by inversion, and if they are within the same tower they are related by transposition.

Although augmented triads and diminished seventh chords as coupling chords offer maximum connectivity between components of the mode graphs, diminished triads, major seventh chords, and many other trichords and tetrachords can also serve as coupling chords.

6. Applications and extensions

Cohn's recent articles on maximally smooth cycles and hexatonic sets (1996) and on neo-Riemannian operators (1997) include many interesting analyses of passages by Beethoven, Brahms, Franck, Liszt, Mahler, Schubert, and Wagner. In the latter article, Cohn plots several of the triadic sequences found in the works of these composers on the *Tonnetz*, which, as discussed in the fourth section, is equivalent to the Chicken-Wire Torus. If a sequence of triads plotted on either of these graphs produces a simple pattern then there is an implied musical structure. When plotted on the Chicken-Wire Torus the following three sequences (originally plotted in Cohn 1997 on the *Tonnetz*) are simple illustrations of this.

The first example is a sequence from the first movement of Brahms's *Concerto for Violin and Cello.* The sequence is (A♭, g♯, E, e, C, c, A♭, g♯). It is clear from the Chicken-Wire Torus (Figure 7) that this sequence follows one of the zigzag patterns of alternating dotted and solid edges around the small diameter of the torus. This pattern implies that the sequence is a hexatonic cycle, and the triads in this sequence are those

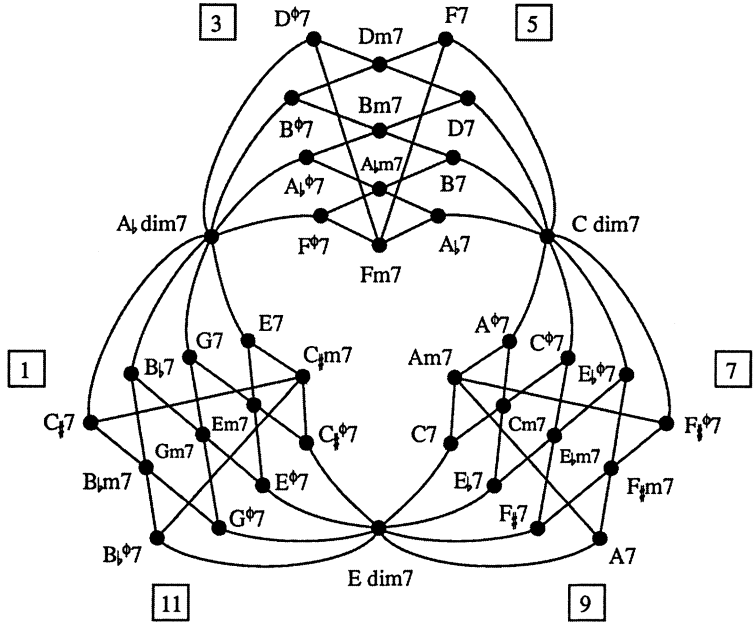


Figure 10. Power Towers

embedded in the hexatonic set listed below the corresponding cycle in Figure 3.

Another example comes from the Andante of the overture to Schubert's *Die Zauberharfe,* where the sequence (c, E♭, e♭, G♭, f♯ A, a, C, c) can be found. Placing this sequence on the Chicken-Wire Torus it is easy to see that it follows the alternating solid and dashed edges around the large diameter of the torus. It follows that the triads in this sequence are those embedded in the octatonic set listed below the corresponding octatonic cycle in Figure 5.

Cohn (1991, 1992, and 1997) has also observed a remarkable sequence of 19 triads in the second movement of Beethoven's *Ninth Symphony* (mm. 143–76) generated by the binary chain <LR>; the sequence is (C, a, F, d, B♭, g, E♭, c, A♭, f, D♭, b♭, G♭, e♭, B, g♯, E, c♯, A). From a diatonic standpoint, this sequence passes quickly through the keys, without establishing any particular tonality. But what can be said of its relationship with hexatonic and octatonic sets? Recalling that the <LR>-generated cycle (the alternating dotted and dashed edges on the Chicken-Wire Torus) is the union of all the hexatonic and octatonic bridges (Section 4),

this sequence passes even more rapidly through the hexatonic and octatonic sets, resting for no more than two consecutive triads in any one set. This sequence is notable in that there is no more efficient way of avoiding modes of limited transposition.

As of yet a search for analytical examples of $P_{1,0}$- and $P_{0,1}$-related seventh chords has not been made. However, Lewin (1996) and Childs (1998) have given us insight into the works of Chopin, Scriabin, Wagner, and Stravinsky by exploiting $P_{2,0}$-parsimony for seventh chords. For example, Lewin (1996) applies $P_{2,0}$-parsimony to the famous Wagnerian sequence (F∅7, E7, A♭∅7, G7, D∅7, B7), which appears at the opening of the Prelude from *Tristan*. The first seventh chord in this sequence is $P_{2,0}$-related to all the others in the sequence, and many of the other pairs are $P_{2,0}$-related as well. Without going into depth here (see Lewin [1996, 207–209] for details), Lewin constructs a sequence of *local* graphs (graphs that involve only the seventh chords in the sequence) based on $P_{2,0}$-parsimony. These graphs reveal the presence of a high degree of symmetry at every stage of this sequence except at the last stage, which is the only graph that includes the B7 chord. Lewin observes that at this stage the high degree of symmetry is disrupted in preparation for a tonal cadence.

The (*global*) graph of the family (set class) of all dominant- and half-diminished-seventh chords induced by $P_{2,0}$ reveals a surprising connection between the seventh chords and octatonic sets in this example. Childs's (1998) categorization of all possible $P_{2,0}$-related pairs of half-diminished and dominant seventh chords is useful in the construction of this graph. His categorization reveals that the set class 4-27 can be partitioned into six subfamilies, each consisting of four pairwise $P_{2,0}$-related seventh chords equivalent under transposition. With some work the graph of the set class 4-27 induced by $P_{2,0}$ can be realized as a (twisted) torus whose cross-sections consist of the six subfamilies mentioned above (Figure 11a).[13] We call this graph *Pipeline*. For clarity, only four of the twelve edges connecting each pair of cross-sections are shown in Figure 11a, and the tube section and panel in Figure 11b show how the other eight connecting edges join the pair.[14] The seventh chords in each cross-section can be found embedded in the octatonic set listed below them.

If a non-musical observer were asked to use Pipeline to trace the *Tristan* sequence and determine a pattern, the observer might note the following:

> The path oscillates between an even-numbered cross-section (cross-section 2) and the odd-numbered cross-section to its *right* (cross-section 3), except at the last seventh chord, which appears in the cross-section to the *left* of the initial even-numbered cross-section (cross-section 1).

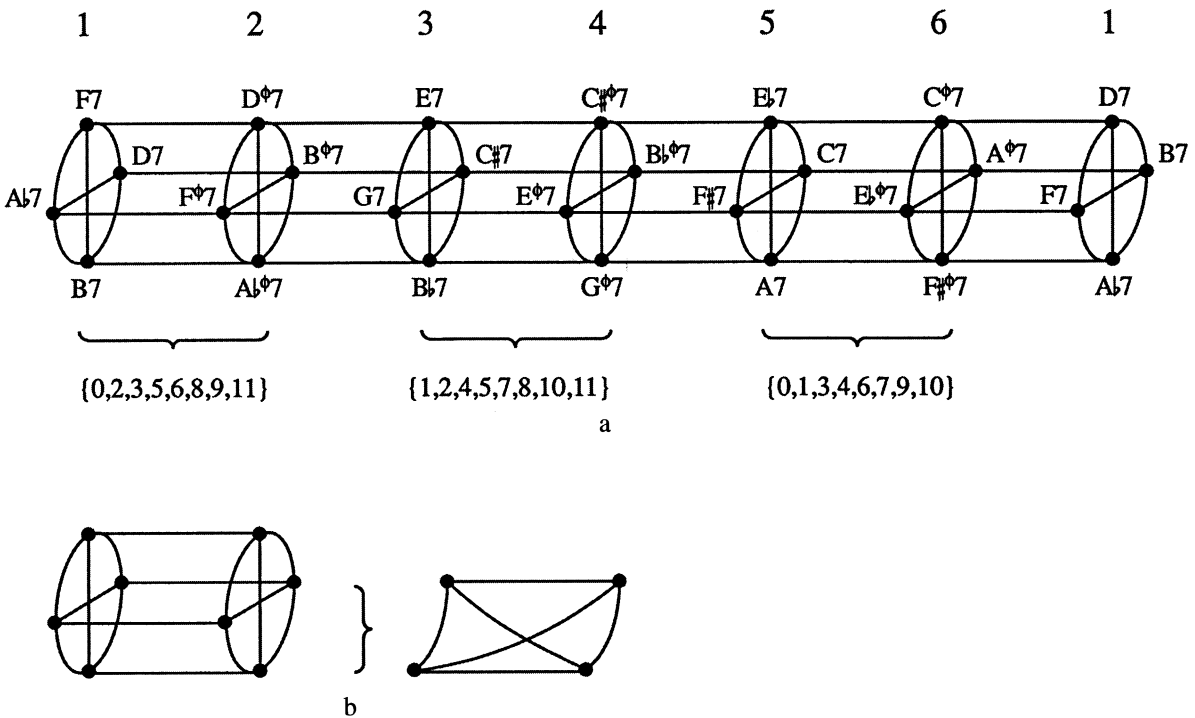


Figure 11. Pipeline

On the surface, this observation might seem to have little or no musical significance. However, because of the parsimonious organization manifested in Pipeline, the above non-musical observation is equivalent to the following analysis:

> The first five seventh chords of this sequence continually modulate between a pair of octatonic sets by *bridging* back and forth between half-diminished seventh chords in one octatonic set (cross-section 2) and dominant seventh chords in another (cross-section 3). In each of the first five seventh chords, precisely three of the pitch classes are in the intersection of the two associated octatonic sets; thus, only one pitch class determines in which octatonic set each seventh chord is embedded. Suddenly, at the last seventh chord in this sequence there are three departures from this pattern: the last seventh chord is embedded in the same octatonic set as its immediate predecessor; the last dominant seventh chord is embedded in an octatonic set that has previously been reserved as the source for only half-diminished seventh chords; and the last seventh chord contains three pitch classes not in the intersection of the two associated octatonic sets (these three departures are all a result of the last dominant seventh chord being in the first cross-section). These departures from the previous pattern suggest a change in direction, and the sequence prepares for a tonal cadence.[15]

This analysis holds any time the pattern observed by the non-musician appears in Pipeline.

At this point it is uncertain whether this graph-theoretic interpretation is equivalent to or in addition to the sequential symmetry observed by Lewin. In support of the former, Lewin (1996, 209) points out that similar sequential symmetries can be found in other Wagnerian passages, including one from the opening of the Immolation Scene with the sequence (A7, G∅7, C7, B♭∅7, E♭7, C♯∅7, C♯7). It is easy to see from Figure 11a that this sequence also follows the same pattern as that observed by the non-musician. Thus, the analysis given for the first *Tristan* sequence applies to this one as well.[16]

$P_{2,0}$-parsimony also induces some interesting triadic structures. For example, we call the graph of $\mathcal{T}_2$ induced by $P_{2,0}$ *Weitzmann's Waltz* (Figure 12). The four hexagonal structures consisting of consonant triads are the graph-theoretic representations of octahedra, and the consonant triads in each are those that make a *Weitzmann region* (sum-class corner) discussed by Cohn (2000, 1998).[17] Although each consonant triad can be found embedded in two enneatonic sets, there is precisely one enneatonic set in which *all six* consonant triads of a given octahedron are embedded, and these enneatonic sets are listed alongside their corresponding octahedra in Figure 12. Modulating between enneatonic sets is accomplished by *coupling* pairs of octahedra with augmented triads that are $P_{2,0}$-related

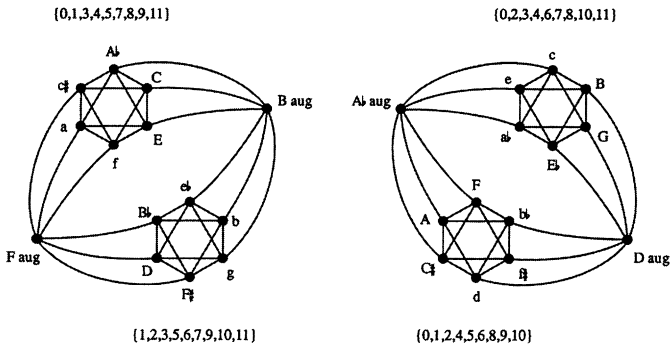


Figure 12. Weitzmann's Waltz

to consonant triads in the octahedra being coupled. As in the case of Cube Dance and Power Towers, if the consonant triads being coupled are in the same octahedron, the triads are related by transposition, and if they are in neighboring octahedra they are related by inversion. Also note that the tonic-dominant relationship for minor keys is a $P_{2,0}$-parsimonious one (e.g., a $P_{2,0}$E). It follows that, although Weitzmann's Waltz is clearly not a diatonic structure, there is at least a hint of diatonicism in this graph.

From a theoretical point of view, parsimonious graphs provide a visual representation of group actions on families of chords. This can prove useful to those who have a limited background in group theory as well as those whose background is more extensive. For those with a limited background, these graphs offer a convenient way of *seeing* (literally) parsimonious organization within families of chords, thus providing an intuitive sense of the complexities of group actions on these families. For those with a more extensive background, parsimonious graphs can provide a way of discovering relationships between group structures and families of chords. Such discoveries can lead to an understanding of how seemingly disparate structures relate to one another, as suggested by our discussion of the tori in Section 4 and in endnotes 9, 13, and 16.

The visual aspects of these graphs can also prove useful in analysis. Simple patterns observed in these graphs imply a musical structure involving an interaction between parsimony and modes of limited transposition, and this structure can be grasped by an understanding of the graph's construction. In some cases (as in the given examples of sequences from Wagner's *Tristan*) these musical structures might otherwise be difficult to observe. This suggests that parsimonious graphs might prove useful in revealing other musical structures previously hidden in the complexities of late nineteenth- and early twentieth-century voice-leading.

NOTES

1. Cohn (1996, 9–11) illustrates very convincingly the shortcomings of a traditional approach, given the tonal indeterminacy of much late nineteenth-century music.
2. We call collections from these set classes *modes of limited transposition* because of their limited number of distinct transpositions. Although there is no intended connection between our modes of limited transposition and those of Messiaen, it should be noted that collections from set classes 8-28 and 9-12 are, in fact, Messiaen's second and third modes of limited transposition.
3. The original *Oettingen/Riemann Tonnetz* was constructed to accommodate just intonation and, hence, the lattice filled two-dimensional Euclidean space. As discussed in Cohn 1997, in pitch-class space the *Oettingen/Riemann Tonnetz* can be realized as a lattice on the surface of a torus.
4. This notation is a generalization of the notations employed in Lewin 1996 and Childs 1998. What Lewin refers to as DOUTH*m* and Childs as P_m is $P_{m,0}$. Thus, $P_{m,n}$-relations take into account whole-step motion as well as half-step motion.
5. Callender (1998) defines parsimony in terms of sequences of pitch-class sets. His definition permits pitch-class *splitting* (e.g., E goes to E♭ and F) and *fusing* (E♭ and F go to E), thus allowing parsimonious comparisons of pitch-class sets with different cardinalities. As Callender illustrates in his analyses of passages from works by Scriabin, the ability to make such comparisons offers an important flexibility not possible with the *relation definition* we present in Section 2. Although not needed for our objectives in this paper, it is also possible to extend the *relation definition* to accommodate pitch-class splitting and fusing; simply allow τ to be a *surjective* map from the larger set onto the smaller set when the pitch-class sets differ in cardinality. The benefits and/or problems created by this extension have not yet been explored.
6. With respect to *parsimonious degree,* one could make an argument, for example, that $P_{0,1}$-parsimony and $P_{2,0}$-parsimony are *of the same degree,* since each has a total half-step displacement of 2; $P_{0,1}$-parsimony displaces one tone by a whole step while $P_{2,0}$-parsimony displaces two tones by a half step each.
7. One of the major topics of discussion at the 1993 meeting of the *SUNY-Buffalo Working Group* was *P-cycles* (also known as *Cohn cycles*). These cycles are components of graphs of set classes induced by $P_{1,0}$. In the general setting these cycles are not restricted to the usual modulo-12 system, and few set classes in any system are capable of generating graphs with cyclic components. Because of the pitch-class motion between adjacent sets around these cycles, Cohn divides these cycles into two categories; *unidirectional P-cycles* and *toggling P-cycles.* A *unidirectional P-cycle* is a type of *generalized cycle of fifths* in that the induced graph is a *single* cycle of "closely-related keys," linking these cycles to the theory of *maximally even sets* developed in Clough and Douthett 1991. Clampitt (1997, 139) has shown that the graph of a set class induced by $P_{1,0}$ is a unidirectional *P*-cycle if and only if the set class consists of *maximally even sets* such that the sizes of these sets and the cardinality of the chromatic universe in which they are embedded are coprime. If the induced graph consists of more than one cyclic component then the components of the graph are *toggling P-cycles.* The components of the graph we call *HexaCycles* (Figure 3), discussed in Section 3, are toggling

P-cycles. In his article on Cohn functions, Lewin (1996) has constructed an algorithm that, when restricted to $P_{1,0}$-related sets, generates *all* set classes whose induced graphs consist of *P*-cycles.

8. The Chicken-Wire Torus also appears in the toroidal structures generated by the psychological test data on neo-Riemannian transformations discussed in Krumhansl 1998.
9. For example, half-turns about the centers of the *tritone* faces 1 or 7 or about the midpoints of the R edges incident to d and F, or a♭ and B, leave these faces and edges fixed (see Figure 7a). The other faces map by name in the same way the pitch classes map on the 12-cycle when reflected about the (1-7)-axis.
10. Our intent in this section is to illustrate how a graph-theoretic approach can be useful in observing and understanding how modes of limited transposition, parsimony, and coupling modulations function together. Other properties inherent in these graphs and in other related structures have been investigated in Cohn 1994, 2000, and 1998.
11. The name *Cube Dance* was first suggested by Carol Krumhansl at the 1993 meeting of the *SUNY-Buffalo Working Group.*
12. As originally presented by Douthett at the 1993 meeting of the *SUNY-Buffalo Working Group,* French-sixth chords were also included in this graph. With the inclusion of both French-sixth and diminished seventh chords, each of the three octatonic towers is transformed into a 4-cube (4-dimensional cube), and *Power Towers* becomes a *4-Cube Trio* (three 4-cubes joined at diminished seventh chords to form a circuit).
13. To put this in terms of Childs's *C transformations*, each family of four seventh-chords that make up a cross-section in Pipeline is an orbit of the set class 4-27 under the group of C transformations. To put this in terms of the Parallel*, Leittonwechsel*, and Relative* transformations, these families are the collection of orbits of the set of half-diminished seventh chords under the group generated by $P_1^*L_1^*$ $(=\{(P_1^*L_1^*)^k\}_{k=0}^3)$ and the orbits of the set of dominant seventh chords under the group generated by $P_2^*L_2^*$. Whence, these transformations partition these families into "half-diminished" and "dominant" orbits. By employing the work of Cohn (1998) there is yet another way to characterize these families. They are the six *sum classes* of set class 4-27. Precisely how these three characterizations relate to one another has not yet been worked out.
14. Note that not every pair of consecutive seventh chords in the *Tristan* sequence consists of $P_{2,0}$-related seventh chords; one pair contains $P_{0,1}$-related seventh chords, and those in another pair are $P_{4,0}$-related. Technically, the adjacency rule should include $P_{0,1}$- and $P_{4,0}$-related pairs as well. However, these inclusions would unnecessarily complicate Pipeline by increasing the number of edges between adjacent cross-sections, and it would have no effect on our observations and analysis that follow in the text.
15. The curious reader might find it interesting to compare our analysis with that of Boretz (1972, 159–217).
16. It can be seen from Cohn's (1998) work on *sum classes* that there is another observation equivalent to that made by the non-musician and the analysis following that observation. Considering only the seventh chords in set class 4-27, the sequences mentioned oscillate between seventh chords whose total pitch-class sums (modulo

12) are 3 (or 7 or 11) and seventh chords whose sums are 1 (or 5 or 9, respectively) until the last seventh chord, whose total pitch-class sum is 5 (or 9 or 1, respectively).

17. When introduced by Douthett at the 1993 meeting of the *SUNY-Buffalo Working Group,* this graph was called the *Anniversary Waltz.* After observing the connection between the octahedra in this graph and Weitzmann regions we have decided to change its name to *Weitzmann's Waltz.*