A characterisation of the measures for which the L^p spaces are Hilbert spaces.

November 30, 2020

In the following, $\mathbb{K}$ will stand for either $\mathbb{R}$ or $\mathbb{C}$ and a.e. will stand for "almost everywhere".

Let $(X, \mathfrak{M}, \mu)$ be a measure space, $p \in [1, +\infty)\backslash\{2\}$. The following are equivalent:

i) $L^p(\mu)$ is a Hilbert space

ii) If $A, B \in \mathfrak{M}$ have finite measure and $A \cap B = \emptyset$, then either $\mu(A) = 0$ or $\mu(B) = 0$

iii) $\dim L^p(\mu) \leq 1$

Proof

"$i) \implies ii)$": We will prove that the negation of $ii)$ implies the negation of $i)$, i.e. that, if $\exists A, B \in \mathfrak{M}$ such that $A \cap B \neq \emptyset$ and both A and B have strictly positive finite measure, then $L^p(\mu)$ is not a Hilbert space.

In order to prove this, we will use the parallelogram law, which states that a norm $||\cdot||$, defined on a vector space X, is induced by an inner product $\iff$ $\forall x, y \in X : ||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$.

Let $\mathbf{1}_A,\ \mathbf{1}_B$ be the indicator functions of A and B respectively, which are elements of $L^p(\mu)$ because A and B have finite measure. Since A and B are disjoint, $|\mathbf{1}_A + \mathbf{1}_B| = |\mathbf{1}_A - \mathbf{1}_B| = \mathbf{1}_{A\cup B}$, so

$$||\mathbf{1}_A + \mathbf{1}_B||_p^2 + ||\mathbf{1}_A - \mathbf{1}_B||^2 = 2||\mathbf{1}_{A\cup B}||_p^2 = 2\left(\int_X \mathbf{1}_{A\cup B} \mathrm{d}\mu\right)^{\frac{2}{p}} = 2(\mu(A) + \mu(B))^{\frac{2}{p}}$$

Analogously,

$$2(||\mathbf{1}_A||_p^2 + ||\mathbf{1}_B||_p^2) = 2(\mu(A)^{\frac{2}{p}} + \mu(B)^{\frac{2}{p}})$$

Therefore, $||\mathbf{1}_A + \mathbf{1}_B||_p^2 + ||\mathbf{1}_A - \mathbf{1}_B||^2 = 2(||\mathbf{1}_A||_p^2 + ||\mathbf{1}_B||_p^2)$ if and only if $(\mu(A) + \mu(B))^{\frac{2}{p}} = (\mu(A)^{\frac{2}{p}} + \mu(B)^{\frac{2}{p}})$, which, dividing by the left hand side of the equation, is equivalent to

$$\frac{\mu(A)^{\frac{2}{p}} + \mu(B)^{\frac{2}{p}}}{(\mu(A) + \mu(B))^{\frac{2}{p}}} = \left(\frac{\mu(A)}{\mu(A) + \mu(B)}\right)^{\frac{2}{p}} + \left(\frac{\mu(B)}{\mu(A) + \mu(B)}\right)^{\frac{2}{p}} = 1$$

If we let $t = \frac{\mu(A)}{\mu(A)+\mu(B)}$, then $0 < t < 1$ and $\frac{\mu(B)}{\mu(A)+\mu(B)} = 1 - t$. Thus, the parallelogram law for $\mathbf{1}_A$, and $\mathbf{1}_B$ is equivalent to $t^{\frac{2}{p}} + (1-t)^{\frac{2}{p}} = 1$.

However, if $p < 2$, then $\frac{2}{p} > 1$, so $t^{\frac{2}{p}} < t$ and $(1-t)^{\frac{2}{p}} < 1-t$, which implies $t^{\frac{2}{p}} + (1-t)^{\frac{2}{p}} > 1$. If $p > 2$, then $\frac{2}{p} < 1$, so $t^{\frac{2}{p}} > t$ and $(1-t)^{\frac{2}{p}} > 1-t$, which implies $t^{\frac{2}{p}} + (1-t)^{\frac{2}{p}} > 1$.

"$ii) \implies iii)$": Let $S = \{s : X \to \mathbb{K} : s \text{ is simple and measurable}\}$ and let $S_f = \{s \in S : \mu(\{x \in X : s(x) \neq 0\}) < +\infty\}$, where we identify two functions if they are equal a.e. It is a well known fact that S_f is dense in $L^p(\mu)$ (see Theorem 3.13 from Rudin's "Real and Complex Analysis").

Let $s \in S_f$. We can write $s = \sum_{j=1}^{n} \alpha_j \mathbf{1}_{A_j}$, where $\alpha_1, ..., \alpha_n \in \mathbb{K}\backslash\{0\}$ and $A_1, ..., A_n \in \mathfrak{M}$ are mutually disjoint and have finite measure. By hypothesis, there is at most one index $k \in \{1, .., n\}$ such that $\mu(A_k) > 0$. Therefore, $j \neq k \implies \mu(A_j) = 0 \implies \mathbf{1}_{A_j} = 0$ a.e. and thus $s = \alpha_k \mathbf{1}_{A_k}$ a.e. If $S_f = \{0\}$, then $\{0\}$ is dense in $L^p(\mu)$ and therefore $L^p(\mu) = \{0\}$. If this is not the case, let $s \in S_f\backslash\{0\}$. By the previous argument, $s = \alpha \mathbf{1}_A$ a.e. for some $A \in \mathfrak{M}$, $\alpha \in \mathbb{K}\backslash\{0\}$, with $\mu(A) < +\infty$. If $t \in S_f$, there exist $B \in \mathfrak{M}$, $\beta \in \mathbb{K}\backslash\{0\}$ such that $t = \beta \mathbf{1}_B$ a.e. and $\mu(B) < +\infty$. Now, A and $B\backslash A$ are disjoint and have finite measure, so one of them has measure 0, but A can't have measure zero because otherwise $\mathbf{1}_A = 0$ a.e., so $\mu(B\backslash A) = 0$. If $\mu(A \cap B) = 0$, then, since $B = (B\backslash A) \cup (A \cap B)$, $\mu(B) = 0$, so $t = 0$ a.e. If on the other hand $\mu(A \cap B) > 0$, then, since $A = (A\backslash B) \cup (A \cap B)$, we have that $\mu(A\backslash B) = 0$, therefore

$$\mathbf{1}_A = \mathbf{1}_B = \mathbf{1}_{A \cap B} \text{ a.e.} \implies t = \beta \mathbf{1}_B = \frac{\beta}{\alpha} \alpha \mathbf{1}_A = \frac{\beta}{\alpha} s \text{ a.e.}$$

Hence, S_f is one dimensional, so it is complete and therefore closed in $L^p(\mu)$. But S_f is also dense in $L^p(\mu)$, so $L^p(\mu) = S_f$ and is therefore one dimensional.

"$ii) \implies iii)$": It follows from the easy fact that every norm on a one dimensional space is induced by an inner product.