Lecture 27

Balls and Bins

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1 Preliminaries

1. For any positive integer k

$$e^k = \frac{k^0}{0!} + \frac{k^1}{1!} + \frac{k^2}{2!} + ... + \frac{k^k}{k!} + ...\infty$$

$$e^k > \frac{k^k}{k!}$$

Hence,

$$\frac{e^k}{k^k} > \frac{1}{k!} \tag{1}$$

2. For any events A_1,A_2,A_3,.....,A_n,

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P\left(A_i\right) \tag{2}$$

Above inequality is also called probability union bound

3. **Claim :** For positive integers m and n,

$$m \geq \frac{3\log n}{\log\log n} \implies n\frac{e^m}{m^m} \leq \frac{1}{n} \tag{3}$$

Let $l(n) = \log n$, $l^2(n) = \log\log n$ and $l^3(n) = \log\log\log n$

$$n\frac{e^m}{m^m} \leq e^{l(n)}\left(\frac{e^{l^3(n)}}{e^{l^2(n)}}\right)^{\frac{3l(n)}{l^2 n}}$$

$$n\frac{e^m}{m^m} \leq e^{l(n)}\left(e^{l^3 n - l^2(n)}\right)^{\frac{3l(n)}{l^2(n)}}$$

Since $l^2(n) > 0$,

$$n\frac{e^m}{m^m} \leq e^{l(n)}\left(e^{\frac{3l(n)l^3 n}{l^2 n} - 3l(n)}\right)$$

$$n\frac{e^m}{m^m} \leq \left(e^{\frac{3l(n)l^3n}{l^2n} - 2l(n)}\right)$$

Since,

$$\frac{l^3(n)}{l^2(n)} \to 0$$

Therefore,

$$n\frac{e^m}{m^m} \leq e^{l(n)-2l(n)}$$

Therefore,

$$n\frac{e^m}{m^m} \leq e^{-\log n}$$

Therefore,

$$n\frac{e^m}{m^m} \leq \frac{1}{n}$$

4. For positive integers m and n,

$$\binom{n}{m}\left(\frac{1}{n}\right)^m = \frac{1}{m!}\frac{n!}{(n-m)!}\left(\frac{1}{n}\right)^m$$

$$\binom{n}{m}\left(\frac{1}{n}\right)^m = \frac{1}{m!}\left(\frac{n-m+1}{n}\frac{n-m+2}{n} - - - - - - - - - - - - - \frac{n}{n}\right)$$

Since,

$$\frac{n-m+1}{n}\frac{n-m+2}{n} - - - - - - - - - - - - - - \frac{n}{n} < 1$$

Therfore,

$$\binom{n}{m}\left(\frac{1}{n}\right)^m < \frac{1}{m!} \tag{4}$$

2 Maximum number of balls in bin

Consider a situation in which there are n balls and n bins. Each ball is thrown uniformly at random into one of the bin. Our aim is to find what is the probability that any bin will receive large number of balls. We start our analysis by finding the probability **P** that first bin receives at least m balls. There are total of $\binom{n}{m}$ ways in which m balls can be chosen out of n balls, basically our aim is to give at least m balls to first bins, hence we choose m balls which will for sure go into first bin and for rest of balls we just don't care, since our aim is at least m balls. Since there are $\binom{n}{m}$ combinations possible we name each combination

$A_1, A_2, A_3, ..., A\binom{n}{m}$. Let E_i be event that all the balls of A_i go into our first bin. Hence,

$$P = P\left(\bigcup_{i=1}^{\binom{n}{m}} E_i\right)$$

Therefore from equation 2,

$$P \leq \sum_{i=1}^{\binom{n}{m}} P\left(E_i\right) \tag{5}$$

Since,

$$P\left(E_i\right) = \left(\frac{1}{n}\right)^m \tag{6}$$

From equation 5 and 6,

$$P \leq \binom{n}{m}\left(\frac{1}{n}\right)^m \tag{7}$$

From equation 4 and 7,

$$P < \frac{1}{m!} \tag{8}$$

From equation 1 and 8,

$$P < \frac{e^m}{m^m} \tag{9}$$

From equation 3 and 9 if we take $m \geq \frac{3\log n}{\log\log n}$,

$$P < \frac{1}{n^2}$$

But this is probability for first bin only, for all n bins we need to multiply it by n. Hence our analysis reveals that with very small probability maximum number of balls in any bin is greater then $\frac{3\log n}{\log\log n}$, for say $n = 10^{10}$, with probability less than $\frac{1}{10^{10}}$ maximum number of balls in any bin is greater then 30.