

Ques 1

There are 2 coins with us. First is the normal unbiased coin having tail on one side and head on the other. The other is a faulty coin having head on both the sides. We choose one of the coins uniformly at random and flip it. What is the probability that the tossed coin is the faulty one provided

(a) We witnessed a tail.

Since there is no 'tail' on the faulty coin,
the $\text{Pr}(\text{Tossed coin is faulty}) = 0$

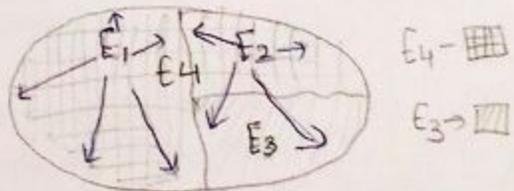
(b) We witnessed a head.

Consider the complete universe of events here

Given E_4 , we have to find the probability of E_1

$$\begin{aligned} P(E_1/E_4) &= \frac{P(E_1 \cap E_4)}{P(E_4)} \\ &= \frac{\text{Pr}(\text{we toss faulty coin and get head})}{\text{Pr}(\text{we get head})} \end{aligned}$$

$$= \frac{\text{Pr}(\text{we toss faulty coin}) \times \text{Pr}(\text{head in case of faulty})}{\text{Pr}(\text{we get head})}$$



E_1 = Event that faulty coin is chosen

E_2 = Event that fair coin is chosen

E_3 = Event of getting tail

E_4 = Event of getting head

$$= \frac{P_2(\text{we toss faulty coin}) \times P_2(\text{we get a head if we toss faulty})}{P_2(\text{we toss faulty coin and get a head}) + P_2(\text{we toss$$

$$\text{Fair coin and get a head})$$

$$= \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2}} \rightarrow \text{Since there is only head on the faulty coin.}$$

$$\frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2}$$

$$= \frac{\cancel{1}}{1 + \frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

Ques-2

Professor's intelligent software. Cheated student declared guilty honest with probability 0.8. Innocent is declared guilty with probability 0.0001. 10% of the students have actually cheated. Randomly picked student, software says he has cheated what is the probability that he has actually cheated?

Ans

| | Student cheated | Student has not cheated |
|-------------------------------|-----------------|-------------------------|
| Software gives correct result | | |
| Software gives false result | $P = 0.8$ | $P = 0.0001$ |

Let C be the event that the student has cheated and S be the event that the s/w says a student has cheated

$$P(C/S) = \frac{P(C \cap S)}{P(S)}$$

$$\# = \frac{P(C \cap S)}{P(C)P(S|C) + P(\bar{C})P(S|\bar{C})}$$

$$= \frac{\frac{10}{100} \times 0.2}{\frac{10}{100} \times 0.2 + \frac{90}{100} \times 0.0001}$$

$$= \frac{0.02}{0.02 + 0.0009}$$

$$= \frac{2000}{2009} = 0.995$$

Hence the software has a very high performance

$$P(X \text{ wins}) = 596$$

Ques 3 Analysis of matrix multiplication verification

Q We pick vectors from \mathbb{Z}_n^k , $k=3,4,5 \dots$

Let us understand first what happens in the binary case

$$AB=C \text{ or not}$$

$$D=(AB-C)$$

$D\vec{x}=0$ for wrong vectors when $AB \neq C$

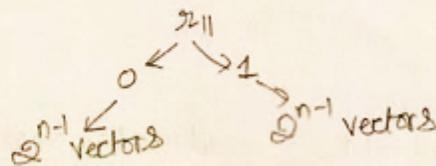
$$\text{Consider } [d_{11} \ d_{12} \ \dots \ d_{1n}] \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix} = 0$$

$D \neq 0 \Rightarrow \exists$ one entry in D which is non zero. Let it be d_{11}

$$\text{Now, } \exists x_{11}=0 \text{ and } D\vec{x}=0$$

\Rightarrow It is not possible $x_{11}=1$ and $D\vec{x}=0$

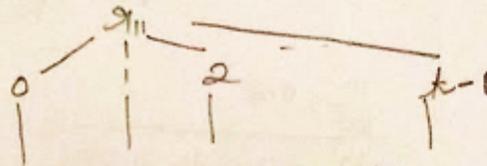
provided other elements of \vec{x} are fixed



x_{11} can take only one of the values 0 or 1. Hence other options are ruled out.

$$\Rightarrow P(D\vec{x}=0) \leq \frac{1}{2}$$

If we take \vec{x} from \mathbb{Z}_n^k



Only one of the branch can be pursued $\Rightarrow P(D\vec{x}=0) \leq \frac{1}{k}$

b) vector can chosen over \mathbb{Z}_n .

$$\lim_{k \rightarrow \infty} \frac{1}{k} = 0 \Rightarrow P(D\vec{x}=0) \text{ approaches almost zero}$$

So the algorithm always gives correct result.

4) Give examples of events A, B and C where

- All are mutually dependent

If we roll a die and $E[1]$ = Event that 1 appears,
 $E[3]$ = Event that 3 appears $E[5]$ = Event that 5 appears,
all of these are mutually dependent because only one
of these can appear on the front.

- A, B, C all are independent

If we roll three dice

$E[1]$ = Event of 1 appearing on first die

$E[3]$ = Event of 3 appearing on second die

$E[5]$ = Event of 5 appearing on third die

These events are independent, because what appears on
one die has nothing to do with what appears on other
die.

- A pair is independent, but together they are dependent

Assume a ~~new~~ technical die, which has the following property -
It has a screen, when you roll a die, there can be any combination
of the numbers from 1 to 6 that can appear. (A die a combination
of 6 dice)
~~the~~ The number of numbers which appear depends on
the screen capacity k .

If $k=1 \Rightarrow E[\text{a number appearing}]$ is totally dependent on
others.

If $k=6 \Rightarrow$ Complete independence

If $k=2$, then you see the independence between the
appearance of numbers, you can see
two events are independent, but the third
is dependent.

Another example - A barber shop having 2 chairs.

5) Partitioning of the class

a) Technique of flipping coin \approx Choosing one subset randomly

Flipping coin $P(\text{choosing a student}) = 1/2$

$$P(k \text{ student chosen}) = {}^n C_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k}$$
$$= \frac{{}^n C_k}{2^n}$$

Choosing subset

There are 2^n possible subsets

$$P(\text{picking a subset}) = \frac{1}{2^n}$$

Number of subsets having k elements $= {}^n C_k$

$$P(\text{picking } k \text{ students}) = P(\text{picking a subset having } k \text{ elements})$$
$$= \frac{{}^n C_k}{2^n}$$

Hence both methods are equal because their probability mass function is same.

b) Pick two subsets X & Y .

a) $P_X(X \subseteq Y)$

$= P_X(Y \text{ has same elements as } X, \text{ but in addition can have some more elements})$

Assume X has k elements x_1, x_2, \dots, x_k

$P_X(X \subseteq Y) = \left(\frac{1}{2}\right)^k \rightarrow Y$ also selects these

$$P_X(X \text{ has } k \text{ elements}) = {}^n C_k / 2^n$$

$$P_X(X \subseteq Y) = \sum_{k=0}^n P_X(X \text{ has } k \text{ elements}) \times P_X(X \subseteq Y)$$

$$= \sum_{k=0}^n \left(\frac{1}{2}\right)^k \times \frac{{}^n C_k}{2^n} = \frac{1}{2^n} \sum_{k=0}^n {}^n C_k \left(\frac{1}{2}\right)^k$$

$$= \frac{1}{2^n} \times \left(\frac{3}{2}\right)^n = \left(\frac{3}{4}\right)^n$$

using $(1+x)^n = \sum_{k=0}^n {}^n C_k x^k$

ii) $Y \subseteq X$

It is the same as previous = $\left(\frac{3}{4}\right)^n$

iii) X and Y are equal

Assume X has k elements x_1, x_2, \dots, x_k

$$Pr(Y=X) = \left(\frac{1}{2}\right)^k \times \left(\frac{1}{2}\right)^{n-k}$$

if X has k elements \downarrow for not selecting these k elements \rightarrow selecting the remaining elements

k ranges from 0 to n

$$Pr(Y=X) = \sum_{k=0}^n {}^n C_k \frac{1}{2^n} \times \frac{1}{2^n}$$
$$= \frac{1}{2^n \times 2^n} \sum_{k=0}^n {}^n C_k = \frac{1}{2^{2n}}$$

(iv) $X \cup Y = S$

Again if X has k elements, then Y has all elements which X does not have. Extra elements may or may not be present

If X has k elements x_1, x_2, \dots, x_k ,

$$Pr[X \cup Y = S] = \left(\frac{1}{2}\right)^{n-k}$$

$$Pr[X \cup Y = S] = \sum_{k=0}^n \frac{{}^n C_k}{2^n} \times \left(\frac{1}{2}\right)^{n-k}$$

$$= \frac{1}{2^n} \sum_{k=0}^n {}^n C_k \left(\frac{1}{2}\right)^{n-k}$$

$$= \frac{1}{2^n} \times \left(\frac{3}{2}\right)^n$$

$$= \left(\frac{3}{4}\right)^n$$

6. Consider tic-tac-toe
 a. Mark places uniformly at random.

$P(X \text{ won})$

Number of configurations = 2^9

Let us look at the number of configurations where X wins

If there are only 0, 1 or 2 'X' then X does not win

If there are 3, 4 or 9 'X', X always wins

10. ~~There~~

Number of win configurations (3 'X') = 8

Number of win configurations (4 'X') = $8 \times 6C_1 = 48$

Number of win configurations (5 'X') = $10(4) + 10(2) + 8(2) + 9(6) + 1 = 98$

Number of win configurations (6 'X') = 82

7 'X' = 36

8 'X' = 9

9 'X' = 1

$P(X \text{ wins}) = \frac{\text{Number of configurations where 'X' wins}}{\text{Total number of configurations}}$

$$= \frac{8 + 48 + 98 + 82 + 36 + 9 + 1}{512} = \frac{282}{512}$$

$P(\text{tie})$

Let us assume X wins and there we see in how many configurations

Y also wins

6X: 6 ways

5X: $6 \times 6 = 36$ ways

4X: 36 ways

3X: 6 ways

$$P(\text{tie}) = \frac{6 + 36 + 36 + 6}{512} = \frac{84}{512}$$

Q In the polynomial verification problem, if we choose the random number with replacement, how does the analysis change?

ans

Let $f(x)$ and $g(x)$ be the polynomials of degree d

$\phi(x) = f(x) - g(x)$ has maximum degree d

We choose r from $\{1, 2, \dots, 10^6 d\}$ and check if $\phi(x) = 0$ or not.

If $\phi(x) \neq 0$, the polynomials are definitely not equal

If $\phi(x) = 0$, they might/might not be equal

$\hookrightarrow P(\text{they are unequal}) = \text{Probability of picking one of the roots of } \phi(x)$

Assume we use k experiments with replacement

$$P(\text{wrong result}) = \frac{d}{10^6 d} \times \frac{(d-1)}{(10^6 d - 1)} \times \frac{(d-2)}{(10^6 d - 2)} \times \dots \times \frac{(d-k)}{(10^6 d - k)}$$

If $k > d$

$$\Rightarrow P(\text{wrong result}) = 0$$

also $\frac{d-1}{10^6 d - 1} > \frac{d}{10^6}$

$\Rightarrow P(\text{wrong result})$ increases also. So we get a better bound generally if we choose the numbers with replacement

(11) 10,000 people
70,000 - Like smiling 30,000 - Don't like smiling
Let $P(\text{wrong voting}) = \frac{1}{1000}$

$$E[\text{votes in favour}] = ?$$

There are 70,000 people in favour

$$P(\text{they voted in favour}) = \frac{999}{1000}$$

$$E[\text{votes in favour}] = 70,000 \times \frac{999}{1000} = 69930$$

There are 30,000 people in oppose

$$P(\text{they voted in favour}) = \frac{1}{1000}$$

$$E[\text{votes in favour}] = 30,000 \times \frac{1}{1000} = 30$$

$$E[\text{votes in favour}] = 69930 + 30 = 69960$$

(12) A gambler - $P(\text{win in a round}) = \frac{1}{3}$. Starts with zero. Show that the expected number of steps in which her net profit is positive can be upper bounded by an absolute constant.

Ans I am playing for n rounds.

In some rounds, I might have money in my hand. Let X be a random variable telling how many such rounds were there.

$$X_i = 1 \text{ if I have money at } i\text{th step} \\ = 0 \text{ else.}$$

$$E[X] = \sum_{i=1}^n E[X_i]$$

$$= \sum_{i=1}^n p(x_i)$$

... (1)

To have money in i th round, I should have won in $\geq \frac{i}{2}$ rounds.

Y = Number of rounds where I win.

What is $P[Y \geq i/2]$?

Chernoff Bound. $P[Y \geq (1+\delta)\mu] \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu$

$$\mu = (i) \times \frac{1}{3} = \frac{i}{3}$$

$$\delta = \frac{1}{2}$$

$$P[Y \geq i/2] \leq \left(\frac{e^{1/2}}{(1.5)^{1.5}}\right)^{i/2} = (0.8975)^{i/2}$$

$$P(\text{I have money at } i\text{th step}) \leq (0.8975)^{i/2}$$

$$E[X] \leq (0.8975)^{1/2} + (0.8975)^{2/2} + \dots + (0.8975)^{n/2}$$

$$= \frac{0.947 [1 - 0.8975^n]}{0.053}$$

$$= \frac{17.868 (1 - 0.8975^n)}{\quad} \leq 1$$

$$\leq \underline{\underline{17.868}} \text{ (upper bounded by an absolute constant)}$$

