1, (a) $\vec{E}^i = \hat{x} E_0 e^{-j\beta z}$

$$\vec{J}_{eq} = j\omega(\epsilon_1 - \epsilon_0)(\hat{x} E_0 e^{-j\beta z'})$$

$$\simeq \hat{x} j\omega(\epsilon_1 - \epsilon_0) E_0 (1 - j\beta z')$$

$$\vec{f} = \hat{x} j\omega(\epsilon_1 - \epsilon_0) E_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{a} (1 - j\beta z') e^{j\beta(\hat{r}\cdot\vec{r}')} r'^2 \sin\theta' dr' d\theta' d\phi'$$

$$e^{j\beta(\hat{r}\cdot\vec{r}')} = e^{j\beta r'(\hat{r}\cdot\hat{r}')}$$

$$\simeq 1 + j\beta r'(\hat{r}\cdot\hat{r}')$$

$$\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$$

Thus

$$e^{j\beta(\hat{r}\cdot\hat{r}')} \Rightarrow 1 + \sin\theta\sin\theta'\cos\phi\cos\phi' + \sin\theta\sin\theta'\sin\phi\sin\phi' + \cos\theta\cos\theta'$$

$$\equiv G$$

$$z' = r'\cos\theta'$$

Thus

$$\vec{f} = \hat{x} j\omega(\epsilon_1 - \epsilon_0) E_0 \int_0^{2\pi}\int_0^{\pi} (1 - j\beta r'\cos\theta') G r'^2 \sin\theta' dr' d\theta' d\phi'$$

but

$$\int_0^{2\pi} \cos\phi' d\phi' = \int_0^{2\pi} \sin\phi' d\phi' = 0$$

$$\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta = -\frac{1}{3}\cos^3\theta \Big|_0^{\pi} = \frac{2}{3}$$

$$\Rightarrow \int_0^{\pi} \cos\theta' \sin\theta' d\theta' = -\frac{1}{2}\cos^2\theta' \Big|_0^{\pi} = 0$$

Thus

$$\vec{f} = \hat{x} j\omega(\epsilon_1 - \epsilon_0) E_0 \int_0^{2\pi}\int_0^{\pi}\int_0^{a} (1 - j\beta r'\cos\theta')(1 + \cos\theta\cos\theta') r'^2 \sin\theta' dr' d\theta' d\phi'$$

$$= \hat{x}\, j\omega(\epsilon_1-\epsilon_0)E_0\int_0^{2\pi}\int_0^{\pi}\int_0^{a}(1-j\beta r'\cos\theta\cos^2\theta')\,r'^2\sin\theta'$$

$$\vec{f} = \hat{x}\, j\omega(\epsilon_1-\epsilon_0)E_0\left[4\pi\int_0^a r'^2\,dr' - j\beta\, 2\pi\cos\theta\int_0^{\pi}\int_0^{a} r'^3\cos^2\theta'\sin\theta'\,dr'\,d\theta'\right]$$

$$= \hat{x}\, j\omega(\epsilon_1-\epsilon_0)E_0\left[\frac{4}{3}\pi a^3 - j\beta\frac{\pi a^4}{3}\cos\theta\right]$$

$$f_\theta = j\omega(\epsilon_1-\epsilon_0)E_0\left[\frac{4}{3}\pi a^3 - j\beta\frac{\pi a^4}{3}\cos\theta\right]\cos\theta\cos\phi$$

$$f_\phi = -j\omega(\epsilon_1-\epsilon_0)E_0\left[\frac{4}{3}\pi a^3 - j\beta\frac{\pi a^4}{3}\cos\theta\right]\sin\phi$$

$$\vec{E}^s = \frac{\beta\eta}{4\pi r}e^{-j\beta r}\omega(\epsilon_1-\epsilon_0)E_0\left[\frac{4}{3}\pi a^3 - j\beta\frac{\pi a^4}{3}\cos\theta\right]\left(\hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi\right)$$

b) $$V = -E_0 r\sin\theta\cos\phi + \frac{\epsilon_1-\epsilon_0}{\epsilon_1+2\epsilon_0}a^3 r^{-2}\sin\theta\cos\phi\, E_0$$

$$= -E_0 x + E_0\frac{\epsilon_1-\epsilon_0}{\epsilon_1+2\epsilon_0}\frac{a^3}{r^2}\sin\theta\cos\phi$$

$$\vec{E} = -\nabla V$$

$$= E_0\hat{x} - \nabla\psi$$

$$\psi = E_0\frac{\epsilon_1-\epsilon_0}{\epsilon_1+2\epsilon_0}\frac{a^3}{r^2}\sin\theta\cos\phi$$

$$\nabla\psi = \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$= E_0\frac{\epsilon_1-\epsilon_0}{\epsilon_1+2\epsilon_0}a^3\left(\hat{r}\frac{-2}{r^3}\sin\theta\cos\phi + \hat{\theta}\frac{1}{r^3}\cos\theta\cos\phi - \hat{\phi}\frac{1}{r^3}\sin\phi\right)$$

$$\vec{E} = E_0\hat{x} + E_0\frac{\epsilon_1-\epsilon_0}{\epsilon_1+2\epsilon_0}\frac{a^3}{r^3}\left(2\sin\theta\cos\phi\,\hat{r} - \cos\theta\cos\phi\,\hat{\theta} + \sin\phi\,\hat{\phi}\right)$$

$$= E_0\hat{x} + E_0\frac{\epsilon_1-\epsilon_0}{\epsilon_1+2\epsilon_0}\frac{a^3}{r^3}\Big(\hat{x}(2\sin^2\theta\cos^2\phi - \cos^2\theta\cos^2\phi - \sin^2\phi)$$
$$+\hat{y}(2\sin^2\theta\cos\phi\sin\phi - \cos^2\theta\cos\phi\sin\phi + \sin\phi\cos\phi) + \hat{z}\,3\cos\theta\sin\theta\cos\phi\Big)$$

but $2\sin^2\theta - \cos^2\theta = 2\sin^2\theta - (1-\sin^2\theta) = 3\sin^2\theta - 1$

$$= E_0\hat{x} + E_0\frac{\epsilon_1-\epsilon_0}{\epsilon_1+2\epsilon_0}\frac{a^3}{r^3}\left(\hat{x}(3\sin^2\theta\cos^2\phi - 1) + \hat{y}\,3\sin^2\theta\cos\phi\sin\phi + \hat{z}\,3\cos\theta\sin\theta\cos\phi\right)$$

Let $k = E_0\frac{\epsilon_1-\epsilon_0}{\epsilon_1+2\epsilon_0}a^3$

Thus,

$$\vec{f} = j\omega(\epsilon_1-\epsilon_0)\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\Big[\hat{x}\Big(E_0+\frac{k}{r'^3}(3\sin^2\theta'\cos^2\phi'-1)\Big)+\hat{y}\frac{k}{r'^3}3\sin^2\theta'\cos\phi'\sin\phi'$$

$$+\hat{z}\frac{k}{r'^3}3\cos\theta'\sin\theta'\cos\phi'\Big]G\; r'^2\sin\theta'\,dr'd\theta'd\phi'$$

from part (a)

Let us break into components

$$f_x = j\omega(\epsilon_1-\epsilon_0)\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\Big[E_0+j\beta r'E_0(\sin\theta\cos\phi\sin\theta'\cos\phi'+\sin\theta\sin\phi\sin\theta'\sin\phi'$$

$$+\cos\theta'\cos\theta)+j\beta\frac{k}{r'^2}(3\sin^2\theta'\cos^2\phi'-1)(\sin\theta\cos\phi\sin\theta'\cos\phi'+\sin\theta\sin\phi\sin\theta'\sin\phi'$$

$$\cos\theta'\cos\theta)+\frac{k}{r'^3}(3\sin^2\theta'\cos^2\phi'-1)\Big]\,r'^2\sin\theta'\,dr'd\theta'd\phi'$$

but

$$\int_0^{2\pi}\cos\phi'\,d\phi' = \int_0^{2\pi}\sin\phi'\,d\phi' = 0$$

$$\int_0^{\pi}\cos\theta'\sin\theta'\,d\theta' = 0$$

$$\int_0^{2\pi}\cos^2\phi'\sin\phi'\,d\phi' = 0$$

$$\int_0^{2\pi}\cos^3\phi'\,d\phi' = \int_0^{2\pi}\cos\phi'\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\phi'\right)d\phi' = 0$$

We are left with

$$f_x = j\omega(\epsilon_1-\epsilon_0)\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\Big[E_0 r'^2\sin\theta' + j\beta k\,3\sin^3\theta'\cos\theta'\cos\theta\,\cos^2\phi'$$

$$\frac{k}{r'}(3\sin^3\theta'\cos^2\phi'-\sin\theta')\Big]dr'd\theta'd\phi'\Big]$$

$$\int_0^{\pi}\sin^3\theta'\cos\theta' = \frac{1}{4}\sin^4\theta'\Big|_0^{\pi} = 0$$

$$\int_0^{\pi}\sin^3\theta'\,d\theta' = \int_0^{\pi}\tfrac{1}{4}(3\sin\theta'-\sin 3\theta')\,d\theta'$$

$$= \left(-\frac{3}{4}\cos\theta'+\frac{1}{12}\cos 3\theta'\right)_0^{\pi}$$

$$= \frac{3}{2}-\frac{1}{6} = \frac{4}{3}$$

$$\int_0^{2\pi}\cos^2\phi' = \pi$$

Thus

$$f_x = j\omega(\epsilon_1 - \epsilon_0)\left[\frac{4}{3}\pi a^3 E_0 + \frac{k}{r'}\int_0^a \underbrace{(4\pi - 4\pi)}_{0}\, dr'\right]$$

$$= j\omega(\epsilon_1 - \epsilon_0)\frac{4}{3}\pi a^3 E_0$$

Let us consider the y-component

$$f_y = j\omega(\epsilon_1 - \epsilon_0)\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{k}{r'} 3\sin^3\theta' \cos\phi' \sin\phi' \Big(1 + j\beta r'(\sin\theta\cos\phi\sin\theta'\cos\phi'$$
$$+ \sin\theta\sin\phi\sin\theta'\sin\phi' + \cos\theta'\cos\theta)\Big)\, dr'\, d\theta'\, d\phi'$$

$$\int_0^{2\pi} \cos^2\phi' \sin\phi'\, d\phi' = 0 \qquad \int_0^{2\pi} \cos\phi' \sin\phi'\, d\phi' = 0$$

$$\int_0^{2\pi} \sin^2\phi' \cos\phi'\, d\phi' = 0$$

Thus

$$f_y = 0$$

Let us consider z-component

$$f_z = j\omega(\epsilon_1 - \epsilon_0)\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \frac{k}{r'}\, 3\cos\theta' \sin^2\theta' \cos\phi' \Big(1 + j\beta r'(\sin\theta\cos\phi\sin\theta'\cos\phi'$$
$$+ \sin\theta\sin\phi\sin\theta'\sin\phi' + \cos\theta'\cos\theta)\Big)\, dr'\, d\theta'\, d\phi'$$

$$\int_0^{2\pi} \cos\phi'\, d\phi' = 0$$

$$\int_0^{2\pi} \cos\phi' \sin\phi' = 0$$

$$\int_0^{\pi} \sin^3\theta' \cos\theta' = \sin^4\theta' \Big|_0^{\pi} = 0$$

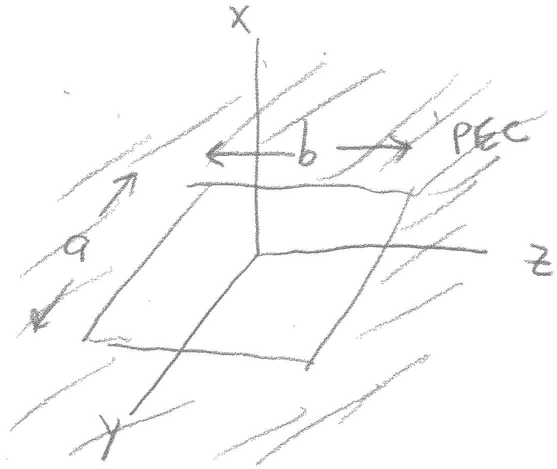
Thus

$$f_z = 0$$

We are left with $\vec{f} = j\omega(\epsilon_1 - \epsilon_0)\frac{4}{3}\pi a^3 E_0 \hat{x}$ which is the same as the first term in part (a)

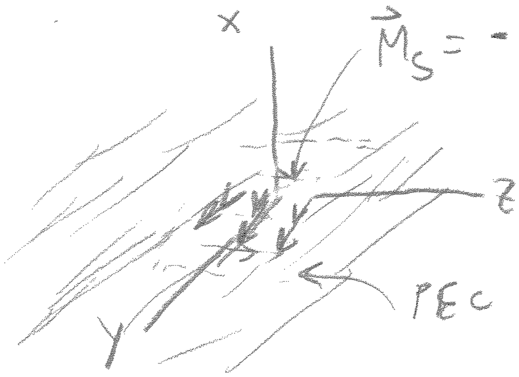
2

7.28



$$\vec{E}_a = \hat{z} E_o \cos\left(\frac{\pi y'}{a}\right)$$

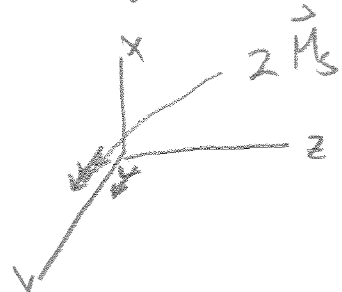
$\Downarrow$



$$\vec{M}_s = -\hat{x} \times \vec{E}_a = -\hat{x} \times \hat{z} E_o \cos\left(\frac{\pi y}{a}\right)$$

$$= \hat{y} E_o \cos\left(\frac{\pi y}{a}\right)$$

$\Downarrow$



$$2\vec{M}_s = \hat{y}\, 2E_o \cos\left(\frac{\pi y}{a}\right)$$

$$\vec{H} = \frac{-j\beta e^{-j\beta r}}{\eta 4\pi r}\left(\hat{\theta} g_\theta + \hat{\phi} g_\phi\right)$$

$$\vec{E} = \frac{j\beta e^{-j\beta r}}{4\pi r}\left(\hat{\theta} g_\phi - \hat{\phi} g_\theta\right)$$

$$\vec{g} = \iint 2\vec{M}_s e^{j\beta(\hat{r}\cdot\vec{r}')} ds'$$

$$\hat{y} = \sin\theta \sin\phi\, \hat{r} + \cos\theta \sin\phi\, \hat{\theta} + \cos\phi\, \hat{\phi}$$

$$\vec{g} = \hat{y}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}} 2E_0 \cos\left(\frac{\pi y'}{a}\right) e^{j\beta(\hat{r}\cdot\vec{r}')} dy'dz'$$

$$\hat{r}\cdot\vec{r}' = (\hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta)\cdot(y'\hat{y} + z'\hat{z})$$

$$= y'\sin\theta\sin\phi + z'\cos\theta$$

$$\Rightarrow \vec{g} = \hat{y}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{a}{2}}^{\frac{a}{2}} 2E_0 \cos\left(\frac{\pi y'}{a}\right) e^{j\beta y'\sin\theta\sin\phi}\, e^{j\beta z'\cos\theta} dy'dz'$$

$$= \hat{y}\,4E_0\left[\frac{\sin\left(\frac{\beta b}{2}\cos\theta\right)}{\beta\cos\theta}\right]\int_{-\frac{a}{2}}^{\frac{a}{2}}\cos\left(\frac{\pi y'}{a}\right) e^{j\beta y'\sin\theta\sin\phi}\, dy'$$

but

$$\int_{-\frac{a}{2}}^{\frac{a}{2}}\cos\left(\frac{\pi y'}{a}\right) e^{j\beta y'\sin\theta\sin\phi}\, dy' = \frac{1}{2}\int_{-\frac{a}{2}}^{\frac{a}{2}} e^{jy'\left(\beta\sin\theta\sin\phi + \frac{\pi}{a}\right)} + e^{jy'\left(\beta\sin\theta\sin\phi - \frac{\pi}{a}\right)}\, dy'$$

$$= \frac{1}{2}\left[\frac{e^{jy'\left(\beta\sin\theta\sin\phi + \frac{\pi}{a}\right)}}{j\left(\beta\sin\theta\sin\phi + \frac{\pi}{a}\right)} + \frac{e^{jy'\left(\beta\sin\theta\sin\phi - \frac{\pi}{a}\right)}}{j\left(\beta\sin\theta\sin\phi - \frac{\pi}{a}\right)}\right]_{-\frac{a}{2}}^{\frac{a}{2}}$$

Let $\alpha = \frac{\beta a}{2}\sin\theta\sin\phi$

Then

$$= \frac{1}{2}\left[\frac{e^{j\alpha}e^{j\frac{\pi}{2}} - e^{-j\alpha}e^{-j\frac{\pi}{2}}}{j\left(\beta\sin\theta\sin\phi + \frac{\pi}{a}\right)} + \frac{e^{j\alpha}e^{-j\frac{\pi}{2}} - e^{-j\alpha}e^{j\frac{\pi}{2}}}{j\left(\beta\sin\theta\sin\phi - \frac{\pi}{a}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{je^{j\alpha} + je^{-j\alpha}}{j\left(\beta\sin\theta\sin\phi + \frac{\pi}{a}\right)} + \frac{-je^{j\alpha} - je^{-j\alpha}}{j\left(\beta\sin\theta\sin\phi - \frac{\pi}{a}\right)}\right]$$

$$= \frac{\cos\alpha}{\beta\sin\theta\sin\phi + \frac{\pi}{a}} - \frac{\cos\alpha}{\beta\sin\theta\sin\phi - \frac{\pi}{a}}$$

$$= \cos\alpha \left[\frac{\beta \sin\theta \sin\phi - \frac{\pi}{a} - \beta \sin\theta \sin\phi - \frac{\pi}{a}}{(\beta \sin\theta \sin\phi)^2 - \left(\frac{\pi}{a}\right)^2} \right]$$

$$= \frac{-\frac{\pi a}{2}\cos\alpha}{\left(\frac{\beta a}{2}\sin\theta\sin\phi\right)^2 - \left(\frac{\pi}{2}\right)^2} = -\frac{\pi a}{2} \frac{\cos\alpha}{\alpha^2 - \left(\frac{\pi}{2}\right)^2}$$

Thus

$$\vec{g} = -\hat{y}\, 2\pi a E_0 \left[\frac{\sin\left(\frac{\beta b}{2}\cos\theta\right)}{\beta\cos\theta} \right] \left[\frac{\cos\alpha}{\alpha^2 - \left(\frac{\pi}{2}\right)^2} \right]$$

$$= -\hat{y}\, \pi a b E_0 \left[\frac{\sin\delta}{\delta} \right] \left[\frac{\cos\alpha}{\alpha^2 - \left(\frac{\pi}{2}\right)^2} \right] \qquad \delta = \frac{\beta b \cos\theta}{2}$$

$$g_\theta = -\pi a b E_0 \cos\theta \sin\phi \, \frac{\sin\delta}{\delta} \, \frac{\cos\alpha}{\alpha^2 - \left(\frac{\pi}{2}\right)^2}$$

$$g_\phi = -\pi a b E_0 \cos\phi \, \frac{\sin\delta}{\delta} \, \frac{\cos\alpha}{\alpha^2 - \left(\frac{\pi}{2}\right)^2}$$