DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING, IIT ROPAR

Tutorial-2

CSL-471, Probability and Computing

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1 INSTRUCTIONS

1. The tutorial will be discussed in the class on September 1, 15:20 - 16:10 and 17:10- 18:00 hrs.

2. In case of any doubt or clarification wrt any question, please open a thread on the google group csl471@iitrpr.ac.in and initiate a discussion.

2 QUESTIONS

1. (Rating : **) There are two coins with us. First is the normal unbiased coin having tail on one side and head on the other, the probability of both- head or tail being equal to appear on flipping. The other is a coin having head on both the sides, so always head appears on flipping it. We call this coin the faulty coin. We choose one of these coins uniformly at random and flip it. What is the probability that the tossed coin is the faulty one, provided :

    - We witness a tail on flipping this coin ?
    - We witness a head on flipping this coin ?

2. (Rating : **) A professor comes up with an intelligent software to detect whether a student has cheated in an exam or not, based on the student's handwriting in the answer script. There is a small amount of false positives and false negatives associated with the software. If a student cheats, the software might declare the student honest with a probability of 0.8. If a student has not cheated, the software might hold the student guilty with a probability of 0.0001. 10% of the students in the class actually cheated in

the mid semester exams. The professor picks an answer script uniformly at random and his software says that the student has cheated. What is the probability that the student has actually cheated in the exam?

3. (Rating : **) We discussed matrix multiplication verification in the class. In the discussed algorithm, we picked 100 vectors uniformly at random from $\mathbb{Z}_2^n$ where $\mathbb{Z}_2 = \{0,1\}$. Discuss the same analysis for the following cases:
    - We pick vectors from $\mathbb{Z}_k^n$ where $k = 3,4,5,......$
    - We pick vectors from $\mathbb{Z}$ where $\mathbb{Z}$ is the set of integers.

4. (Rating : **) We know the meaning of mutually independent and dependent events. Give examples of three events A,B, and C where:
    - A,B and C, all are mutually dependent
    - A,B and C, all are independent.
    - Pick any two of them, they are independent. But as soon as you pick the third one, they become dependent. i.e. A pair is independent but all three of them together are dependent.

5. (Rating : **) A new faculty joins a school and the school administration decides to shift some of the students from the existing class to the class of the newly joined teacher.
    - Let the class be represented by a set of students, say $S = \{s_1, s_2, s_3, ..., s_n\}$. Some students have to be selected out of this set to be taught by the newly joined teacher in the school. We know that there are 2^n ways of selecting some students out of all students, i.e. number of subsets of the set S. The following is what the school administration does for it- A coin is flipped for each student in the class, if head turns up, the student is shifted to the new class, else not. Prove that this technique used for shifting the students is same as choosing one of the 2^n possible subsets of the set S uniformly at random. We call it activity 1.
    - Assume we perform activity 1 two times and obtain two sets of students X and Y. What is the probability that (i) X is a subset of Y (ii) Y is a subset of X. (iii) X and Y are equal. (iv) $X \cup Y = S$

6. (Rating : **) Consider the tic-tac-toe game.
    - Mark the places uniformly at random with O and X. What is the probability that X has won? What is the probability of a tie?
      Note: The number of O and X need not be equal. We are marking each place uniformly at random.
    - Consider the full fledged tic-tac-toe game, where O and X make moves alternately but uniformly at random in the empty spaces. Now, what is the probability that X wins?

7. (Rating : ***)Given a binary string of length n, what is the expected length of the longest streak of 0s that one can see?

8. (Rating : **) In the polynomial verification problem we have done in class, we chose a random number from the range $\{1, 2, ..., 10^6 d\}$. The variable was chosen with replacement. Obtain the error bound for the problem if this random variable is chosen without replacement.

9. (Rating : ***) Explain how the polynomial verification algorithm discussed in the class can be used to test if there is a perfect matching in the given bipartite graph.
Hint : Determinant of adjacency matrix of a bipartite graph having a perfect matching is non zero.

10. (Rating : ***) Given two functions, $F : \{1, ..., n-1\} \rightarrow \{1, ..., m-1\}$. Given that for $0 \leq x, y \leq n-1$, $F((x+y) mod\ n) = (F(x)+F(y)) mod\ m$. The values of F are stored in a database. One of the administrator of the database mistakenly changes 1/3 of the entries in the database (database holding the values of F).
Describe a simple randomised algorithm, which when given z determines $F(z)$ correctly with a probability atleast 1/2.
How can you improve the performance of your method, if you are allowed to use your initial algorithm 5 times?

11. (Rating : **) If you are going to Milan, you better smile. It is illegal not to smile in Milan and can cost you a fine of $100. Assume a hypothetical situation where the citizens of Milan were allowed to vote for this rule, to smile, or not to smile. We assume all the 100,000 citizens participated in the voting. As it happens, the people in Milan love smiling and 70,000 people are in support of the smiling law and 30,000 are in opposition.
On the voting day, there seemed to be a slight confusion among the citizens because of the layout of the ballot. Hence, each voter with a probability of 1/1000 voted for the wrong choice (and ended up voting for what she did not want). What is the expected number of votes in the favour of the "SMILE PLEASE" rule?

12. (Rating : **) Assume a gambler playing the following game. At the beginning, her net profit is zero. She played for n rounds. In every round, the net profit increases by 1 with a probability of 1/3 and decreases by 1 with a probability of 2/3. Show that the expected number of steps in which her net profit is positive can be upper bounded by an absolute constant, independent of the value of n.