INTRODUCTION TO QUANTUM INFORMATION

29 August 2022, VCQ Summer School

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TOPICS COVERED:

- Quantum systems
- Quantum maps and operations
- Device independent quantum information
- CHSH inequality
- Max value of CHSH in quantum theory
- Max value of CHSH in other theories
- Calendar Problem
- Geometry of correlations.

QUANTUM SYSTEMS

A quantum system is described on a Hilbert space $\mathcal{H}$.
A density operator on this space is a positive, semi-definite, self-adjoint operator that is normalised.

$$\rho = \sum_i p_i |i\rangle\langle i| \quad , \quad \sum_i p_i = 1 .$$

PURE SEPARABLE

$$|\psi_{AB}\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$$

MIXED SEPARABLE

$$\rho_{AB} = \sum_i p_i \, \rho_A^i \otimes \rho_B^i$$

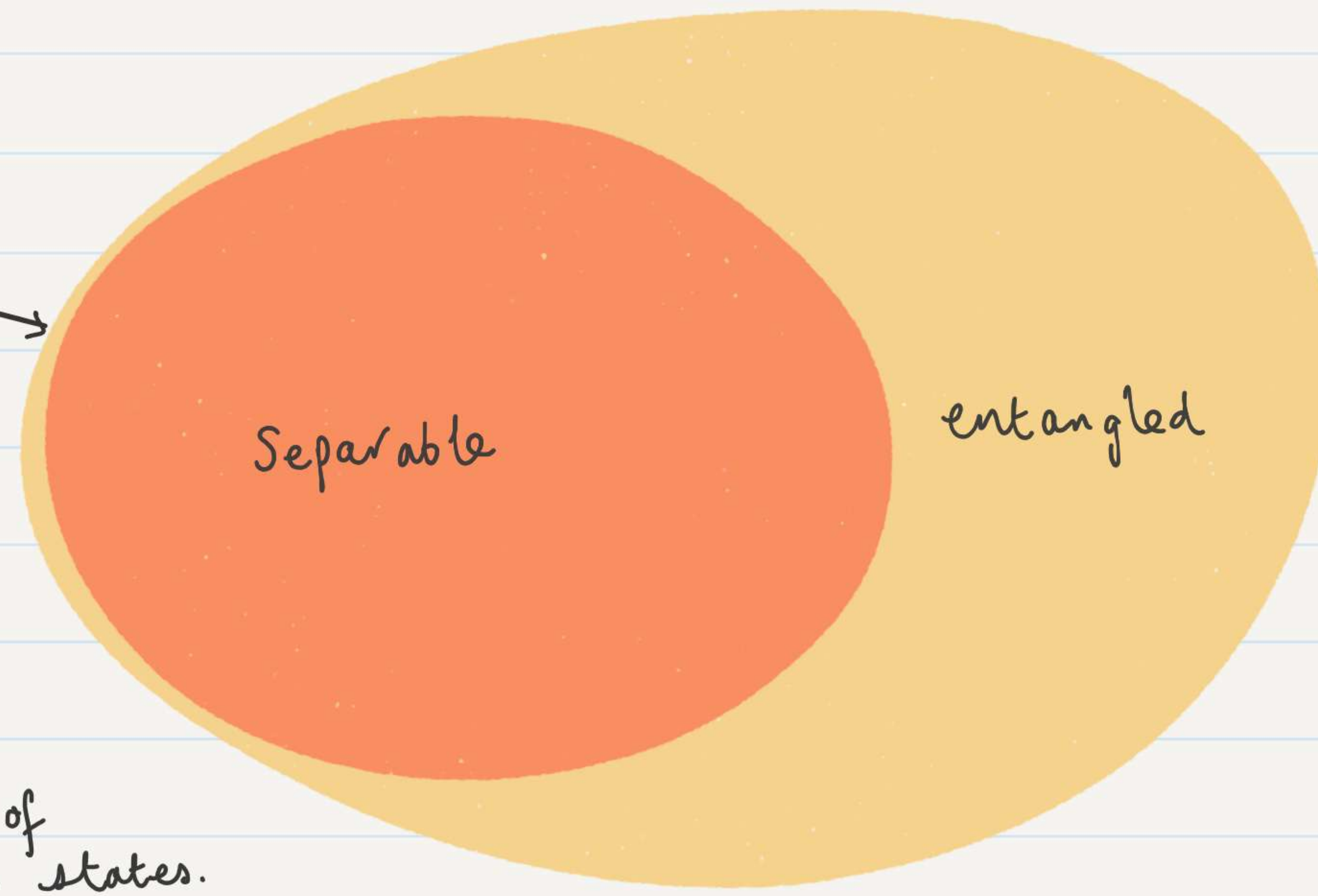
if a state cannot be decomposed like this, it is said to be entangled.
Entanglement is hard to detect. Why? Decomposition. In general finding a separable decomposition is NP-HARD.

(Gurrits and Barnum 2004/2008).

CONVEXITY IN QUANTUM THEORY

Quantum states form a convex set $p\rho_1 + (1-p)\rho_2 \in \mathcal{L}(\mathcal{H})$

Convex set that intersects the boundary



- all pure states lie on the boundary, but <u>not</u> all states on the boundary are pure
- separable pure states are of measure zero in the set of all states.

PURE SEPARABLE $\rho_{AB} = |\phi_A\rangle\langle\phi|_A \otimes |\phi_B\rangle\langle\phi_B|$

ENTANGLED

MIXED SEPARABLE $\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i$

$\rho_{AB} \neq \sum_i p_i \rho_A^i \otimes \rho_B^i$

QUANTUM MAPS

$\Lambda_B : \mathcal{L}(\mathcal{H}_{B1}) \longrightarrow \mathcal{L}(\mathcal{H}_{B2})$ Bob's quantum map.

POSITIVE

if $\Lambda_A(\rho) \geqslant 0$ for all $\rho \geqslant 0$ $\rho \in \mathcal{L}(\mathcal{H}_{A1})$ "it preserves positivity"

K-POSITIVE

if $\mathcal{I}_k \otimes \Lambda_B(\rho_{AB}) \geqslant 0$ for all $\rho_{AB} \geqslant 0$ $\rho_{AB} \in \mathcal{L}(\mathcal{H}_{A1} \otimes \mathcal{H}_{B1})$

identity map $\mathcal{L}(\mathcal{H}_k) \to \mathcal{L}(\mathcal{H}_k)$

COMPLETELY POSITIVE

if $\mathcal{I}_k \otimes \Lambda_B(\rho_{AB}) \geqslant 0$ for all k

k-positive for all k.

CHECK

it is enough to check if $\mathcal{I}_n \otimes \Lambda_B(\rho_{AB}) \geqslant 0$ where $n = \dim(B1)$ $\forall \rho_{AB}$

enough to check states on Alice same as Bob.

EVEN MORE...

it is enough to check positivity on a single state $|\phi^+\rangle = \sum_i \frac{1}{\sqrt{n}} |i\rangle_{A1} |i\rangle_{B1}$

Λ_B is completely positive iff $\mathcal{I}_n \otimes \Lambda_B(|\phi^+\rangle\langle\phi^+|) \geqslant 0$

KRAUS DECOMPOSITION

$$\mathcal{M}_a(\rho) = \sum_\mu K_a^\mu \rho K_a^{\mu\dagger} \qquad \sum_\mu K_a^\mu K_a^{\mu\dagger} = E_a$$

$E_a \geqslant 0$ is the POVM element. $\sum_a E_a = \mathbb{1}$ $\forall a.$ $E_a^2 = E_a$ projector.

THE BUILDING BLOCKS OF QUANTUM OPERATIONS

Quantum operations can be composed of <u>three</u> basic operations.

1. Adding a system $\rho \longrightarrow \rho \otimes \sigma$

2. Tracing out a subsystem $\rho_{AB} \longrightarrow \rho_A = tr_B[\rho_{AB}]$

3. Unitary operations $\rho \longrightarrow U \rho U^{\dagger}$

$$\Lambda_A(\rho_A) = \rho_A'$$
$$= tr_B[U_{AB}(\rho_{AB})U^{\dagger}_{AB}]$$

Steinspring dilation, Church of the larger Hilbert space.

Examples of quantum maps.

$\Lambda(\rho) = U \rho U^{\dagger}$ unitary

$\Lambda(\rho) = \rho^T$ transposition $\longrightarrow$ positive, 1-positive, <u>not</u> 2-positive

$\Lambda(\rho) = \frac{\mathbb{1}}{2}$ trace and replace

$\Lambda(\rho) =$ " measure an reprepare

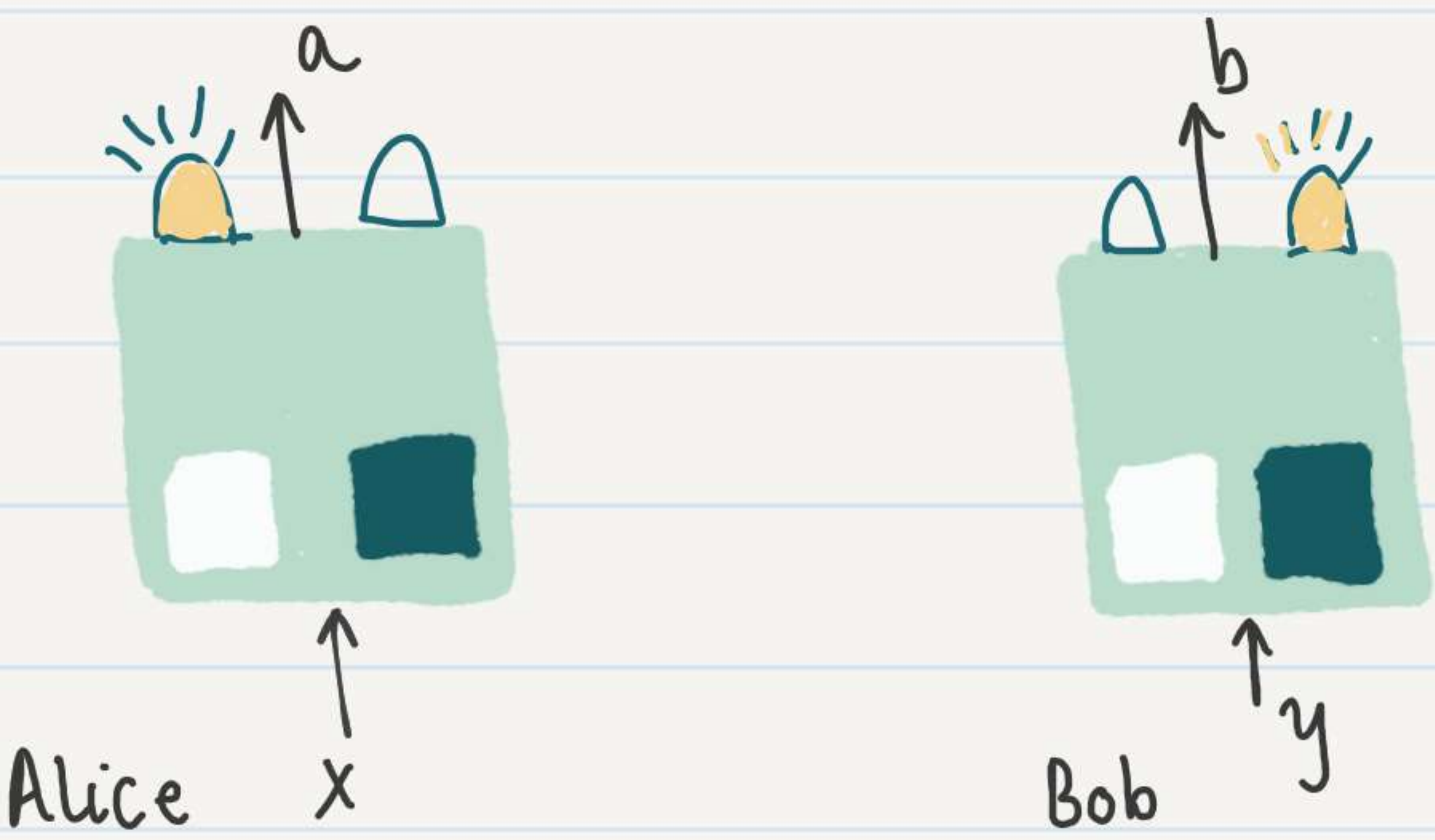
DOES QUANTUM THEORY PREDICT
DIFFERENT EXPERIMENTAL RESULTS
AND IS IT POSSIBLE TO DETECT THEM?
IF SO, IS IT POSSIBLE TO HARNESS
THESE DIFFERENCES TO DO SOMETHING
USEFUL? (see xiao's TALK).

DOES QUANTUM THEORY PREDICT DIFFERENT EXPERIMENTAL RESULTS AND IS IT POSSIBLE TO DETECT THEM? IF SO, IS IT POSSIBLE TO HARNESS THESE DIFFERENCES TO DO SOMETHING USEFUL? (see xiao's TALK).

→ BELL INEQUALITIES

BOXES : A DEVICE INDEPENDENT FORMALISM

Why? No assumptions on the devices in lab. Anyone anywhere can do the exp. provided that they collect the statistics.



TWO PARTY BLACK BOX

"untrusted"

x, y are classical inputs,
a, b are classical outputs.
In the setup Alice and Bob input x and y into their boxes. The boxes 'respond' by producing a and b.

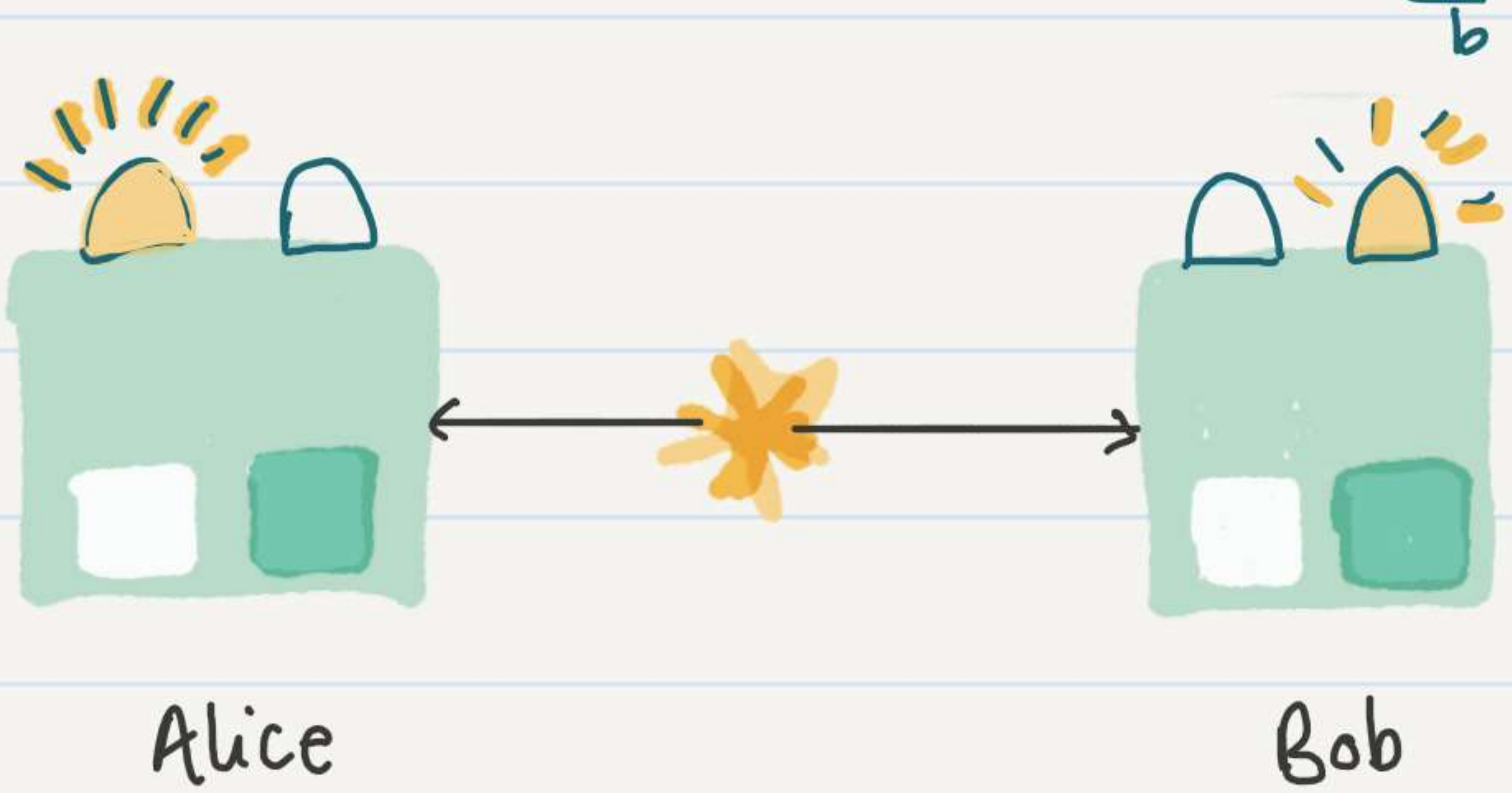
$\{p(ab|xy)\}$ joint conditional probability distribution.

NO-SIGNALLING RULE

"no communication", A and B cannot exchange messages and 'signals' don't show up in their correlations.

$$\sum_a p(ab|xy) = p(b|xy) = p(b|x'y) = p(b|y) \quad \forall x, x'$$

$$\sum_b p(ab|xy) = p(a|xy) = p(a|xy') = p(a|x) \quad \forall y, y'$$



In quantum theory we have an explanation for what happens in the box.

CHSH INEQUALITY

TASK: maximise prob. of success

RULES no-signalling

CHSH (Clauser, Horny, Shimony, Holt)... can be thought of as a game whose objective it is to maximise the probability of success. In each round of the experiment A and B win if they meet any of the following conditions:

x y	$a\,b$		$a\,b$
0 0	00	or	11
0 1	00	or	11
1 0	00	or	11
1 1	01	or	10

WINNING CHSH CORELLATIONS $a \oplus b = x \cdot y$

in CHSH we assume that x, y are randomly distributed and uncorrelated
$p(x,y) = p(x)\,p(y) = \frac{1}{4} \quad \forall x, y.$

We are interested in 3 kinds of "black box". A general one that obeys the no-signalling rule, a quantum one, and a classical one we want to compare

Alice Bob

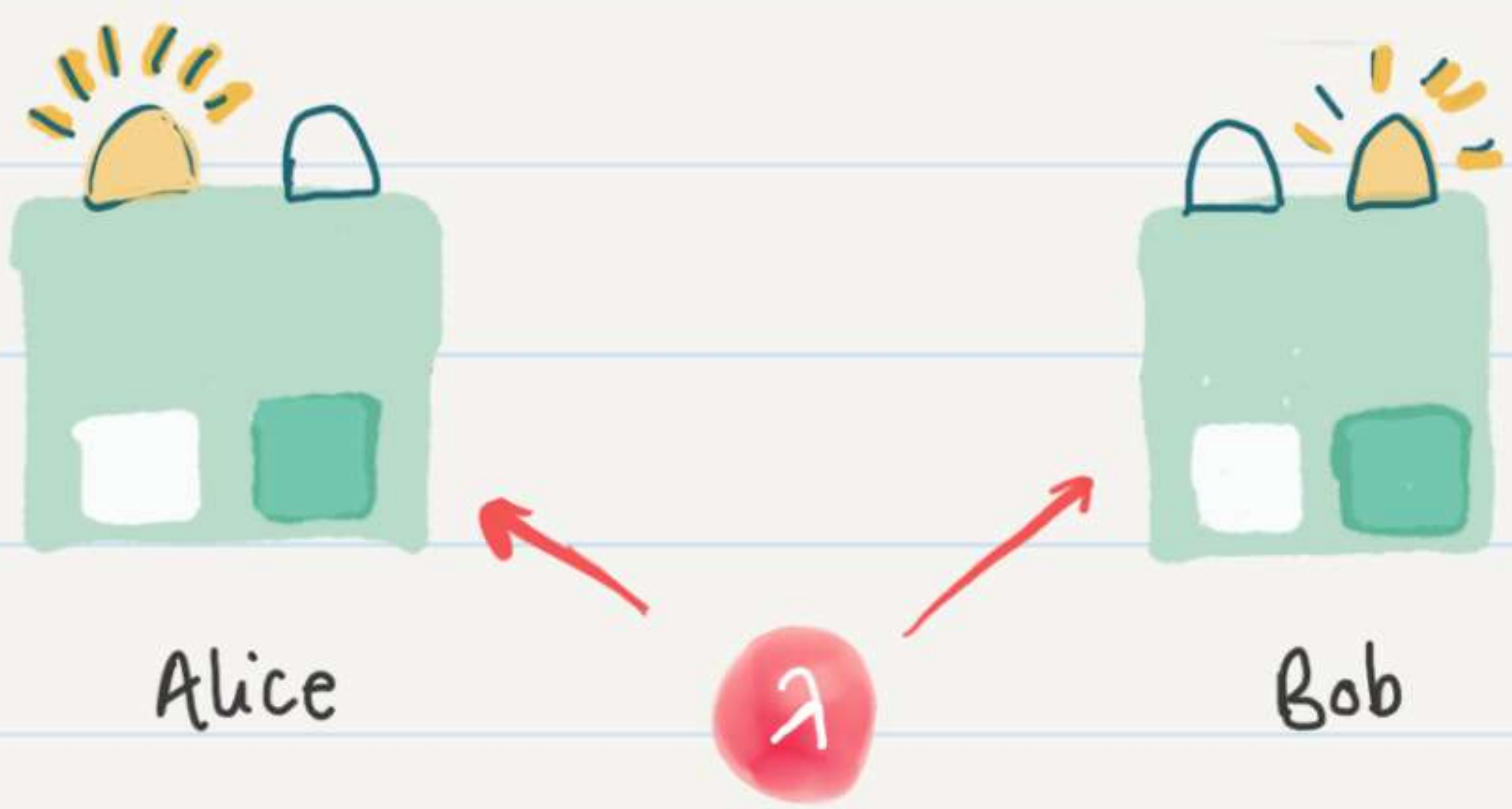
In quantum theory the statistics are given by the Born rule

$p(ab|xy) = \mathrm{tr}\left[M_{a|x} \otimes M_{b|y}\, \rho_{AB}\right]$

$\{M_{a|x}\}$ are CP maps

$\{M_x\}$ CPTP

The correlations of a and b are local iff they can be written $p(ab|xy) = \int d\lambda\, q(\lambda)\, p(a|x,\lambda)\, p(b|y,\lambda)$



Alice Bob

λ = local hidden variable that can be thought of as a ϑ of instructions.

DETERMINISTIC DISTRIBUTIONS

A local distribution can always be rewritten as a convex combination of deterministic distributions.

A deterministic distribution (response) by Alice are the functions:

$$D(a|x,\lambda=0) = \begin{cases} 1 & a=0 \\ 0 & \text{otherwise} \end{cases}, \quad D(a|x,\lambda=1) = \begin{cases} 1 & a = x \oplus 1 \\ 0 & \text{otherwise} \end{cases}, \ldots$$

i.e. those functions for which she always gives the same response. Response may be a function of the input. Local probability tables:

$a \backslash x$	0	1
0	1	1
1	0	0

the element $p(0|1,\lambda=0)$

$a \backslash x$	0	1
0	0	1
1	1	0

$\lambda = 1$

from the examples above.

λ enumerates the strategy. If Alice has m inputs $x \in \{0, \ldots, m-1\}$ and d outputs $a \in \{0, \ldots, d-1\}$ then the number of strategies is $|\lambda| = d^m$. 4 ways to fill in the table

Any local distribution can be writ-en as a convex combination of deterministic strategies.

$$
\begin{aligned}
p_L(ab|xy) &= \int d\lambda\, q(\lambda)\, p(a|x,\lambda)\, p(b|y,\lambda) \\
&= \int d\lambda\, q(\lambda) \sum_s D(a|x,s)\, p(s|\lambda) \sum_t D(b|y,t)\, p(t|\lambda) \\
&= \sum_s \sum_t D(a|x,s)\, D(b|y,t) \int d\lambda\, q(\lambda)\, p(s|\lambda)\, p(t|\lambda) \\
&= \sum_{s,t} D(a|x,s)\, D(b|y,t)\, \tilde{p}(s,t) \\
&= \sum_r D(a|x,r)\, D(b|y,r)\, \tilde{p}(r)
\end{aligned}
$$

DETERMINISTIC BOX

		$x=0$		$x=1$	
		0	1	0	1
$y=0$	0	0	1	1	0
	1	0	0	0	0
$y=1$	0	0	1	1	0
	1	0	0	0	0

A bipartite deterministic box in the no-signalling scenario has the properties: each box must add up to one due to normalisation $\sum_{a,b} p(ab|xy) = 1 \;\; \forall x,y$.

for no signalling

$p(a|x)$ = add up column-wise in box

$p(b|y)$ = add up row-wise in box

16 ways to fill the table.

		$x=0$		$x=1$		
		0	1	0	1	
$y=0$	0	0	1	1	0	$p_B(0\|0)=1$
	1	0	0	0	0	$p_B(1\|0)=0$
$y=1$	0	0	1	1	0	$p_B(0\|1)=0$
	1	0	0	0	0	$p_B(1\|1)=0$
		$p_A(0\|0) = 0$	$p_A(1\|0) = 1$	$p_A(0\|1) = 1$	$p_A(1\|1) = 0$	

MAX VALUE OF CHSH IN QUANTUM THEORY

We want to know how classical, quantum and other resources compare in the CHSH game. In terms of correllators CHSH can pe written as.

Winning correlations

$$a \oplus b = x \cdot y$$

$$C = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$$

this is the expectation value of Alice's measurement 0 and Bob's measurement zero.

$pA + (1-p)B \leq \max\{A, B\}$
$A, B \geq 0$. let $A > B$.

$$C_Q = \langle \psi | A_0 \otimes B_0 | \psi \rangle + \langle \psi | A_0 \otimes B_1 | \psi \rangle + \langle \psi | A_1 \otimes B_0 | \psi \rangle - \langle \psi | A_1 \otimes B_1 | \psi \rangle$$

since $a, b, x, y \in \{0, 1\}$ and $A_0 = \Pi^0_{A0} - \Pi^1_{A0}$ $B_0 = \Pi^0_{B0} - \Pi^1_{B0}$ and $\Pi^0_{A0} + \Pi^1_{A0} = \mathbb{1}$ ...etc

$\Pi^1_{A0} = \mathbb{1} - \Pi^0_{A0}$.

$$C_Q = \langle \psi | (2\Pi^0_{A0} - \mathbb{1})(2\Pi^0_{B0} - \mathbb{1}) | \psi \rangle + \langle \psi | (2\Pi^0_{A0} - \mathbb{1}) \cdots$$

$$= 4p(00|00) - 2p_A(0|0) - 2p_B(0|0) + 1 + \cdots$$

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to find the max. attainable value in quantum theory means we must find the state $|\psi\rangle$ and operators A_0, B_0, A_2, B_1 that maximise the expression. This is completely general in the sense that the optimisation in theory goes over states of arbitrary dimension.

$$C_Q \leq \| (A_0 \otimes B_0) + (A_0 \otimes B_1) + (A_1 \otimes B_0) + (A_1 \otimes B_1) |\psi\rangle \|$$
$$\leq \| (A_0 \otimes (B_0 + B_1)) |\psi\rangle \| + \| (A_1 \otimes (B_0 - B_1)) |\psi\rangle \|$$

note that $\|A_i\| \leq 1$ $\quad \|B_i\| \leq 1$ $\quad$ let $|\Phi_b\rangle = (\mathbb{1} + B_b)|\psi\rangle$ and $\| |\Phi_b\rangle \| \leq 1$

$$C_Q \leq \| |\Phi_0\rangle + |\Phi_1\rangle \| + \| |\Phi_0\rangle - |\Phi_1\rangle \|$$
$$\leq \sqrt{2 + 2 \operatorname{Re} \langle \phi_0 | \phi_1 \rangle} + \sqrt{2 - 2 \operatorname{Re} \langle \phi_0 | \phi_1 \rangle}$$
$$= \sqrt{2 + 2x} + \sqrt{2 - 2x}$$

max at $x = 0 \implies C_{quant} \leq 2\sqrt{2}$

MAX VALUE OF CHSH IN QUANTUM THEORY

$$C_Q \leq 2\sqrt{2}$$

This bound still does not tell us what <u>state</u> and measurements acheive this value. It turns out <u>it is enough</u> to have a singlet

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

and $A_0 = \sigma_z$, $A_1 = \sigma_x$

$B_0 = \frac{\sigma_x + \sigma_z}{\sqrt{2}}$ $B_1 = \frac{\sigma_z - \sigma_x}{\sqrt{2}}$.

It is not hard to verify that for classical resources the bound is 2

$$C = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle$$

$$\text{"} C_{classical} = 1 + 1 + 1 - 1 \text{"}$$

A classical strategy is such that the values of A_0, A_1, B_0, B_1 are preassigned beforehand (deterministic) or a probabilistic strategy eg. 'output 1 1/4 of the time and -1 3/4 of the time).

	Classical strategy 1.	
$(A_0)(B_0)$	$(1)\cdot(1)$	$= 1$
$(A_0)(B_1)$	$(1)\cdot(1)$	$= 1$
$(A_1)(B_0)$	$(-1)\cdot(1)$	$= -1$
$-(A_1)(B_1)$	$(-1)\cdot(1)$.	$= -1$
		2

here the outputs have been changed from 0,1 to $\{-1,1\}$ to avoid the pathology resulting from multiplying by zero. $\{0,1\}$ are just <u>labels</u>

$$C_{CL} \leq 2$$

MAX VALUE OF CHSH IN OTHER THEORIES

IS QUANTUM THEORY SPECIAL BECAUSE IT OBEYS SPECIAL RELATIVITY

NO.

POPESCU ROHRLICH 1994.

There exist correlations (i.e. probability distributions) that are different to the ones produced by quantum theory that also obey special relativity (i.e no signalling).

$$P_{PR}(ab|xy) = \begin{cases} \frac{1}{2} & a \oplus b = x \cdot y \\ 0 & \text{otherwise} \end{cases}$$

		x = 0: 0	x = 0: 1	x = 1: 0	x = 1: 1
y = 0	0	½	0	½	0
y = 0	1	0	½	0	½
y = 1	0	½	0	0	½
y = 1	1	0	½	½	0

$p(01|11) = \frac{1}{2}$

$\Rightarrow \sum_{a,b} p(ab|xy) = 1 \quad \forall x,y$ normalisation

$\Rightarrow p(a|x,y) = p(a|x,\bar{y}) \quad \forall a,x,y,\bar{y}$ no-signalling row-wise

$p(b|x,y) = p(b|\bar{x},y)$ column wise

- PR correlations are very strong. Amongst other things they trivialise communication complexity (Brassard, Buhlmann, Linden ... 2006). For an example, see the "Calendar Problem" due to Van Dam.
- In general there are 8 ways to correctly fill in the table

$$P_{PR}(ab|xy) = \begin{cases} \frac{1}{2} & a \oplus b = x \cdot y \oplus \alpha x \oplus \beta y \oplus \delta \\ 0 & \text{otherwise} \end{cases}$$

$a, b, x, y, \alpha, \beta, \delta \in \{0, 1\}.$

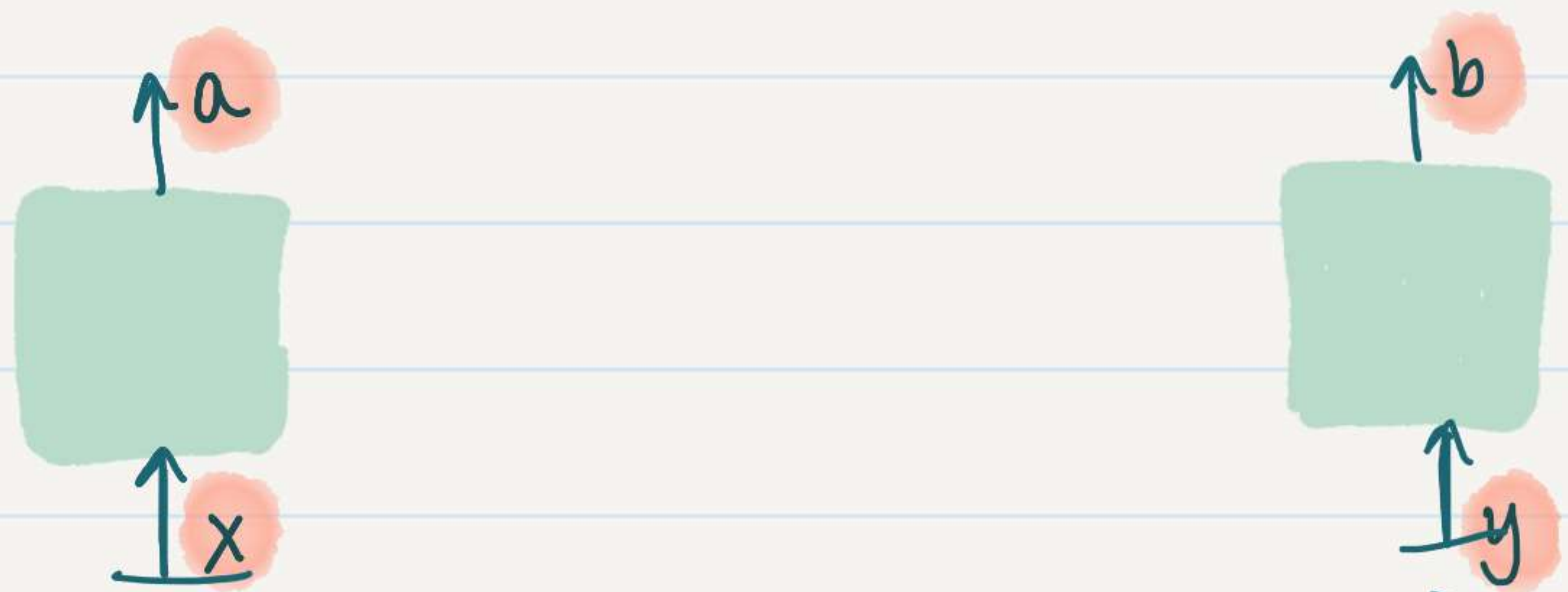
CALENDAR PROBLEM

Busy = red = 0
Free = blank = 1

Question : are the number of days that we are both free even or odd?

0		0	0	sum $a \oplus b$
0	$\oplus$	1	1	
1		1	0	
0		0	0	

PR $a \oplus b = x \cdot y$



Alice adds up all her a's and sends to Bob.

0	0
1	0
1	1
0	1

PR corrrellations acheive the algebraic maximum of CHSH, which is 4.

We can decompose each operator into projectors

$A_0 := \Pi^0_{A_0} - \Pi^1_{A_0}$ $\quad A_1 := \Pi^0_{A_1} - \Pi^1_{A_1}$, $\quad B_i := \Pi^0_{B_i} - \Pi_{B_i}^1$ in order to rewrite the CHSH expression in terms of probabilities. Since $\Pi^0_{A_0} + \Pi^1_{A_0} = \mathbb{1}$, then $A_0 = 2\Pi^0_{A_0} - \mathbb{1}$, and the same for the others.

$$C = \langle (2\Pi^0_{A_0} - \mathbb{1})(2\Pi^0_{B_0} - \mathbb{1})\rangle + \langle A_0 B_1 \rangle + \cdots$$
$$= 4p(00|00) - 2p_A(0|0) - 2p_B(0|0) + 1 + \langle A_0 B_1 \rangle + \cdots$$

substitute for the PR probabilities and one arrives at the conclusion.

	expectation value of CHSH	success probability
classical	$\leq \lvert 2 \rvert$	$\frac{3}{4} = 0.75$
quantum	$\leq \lvert 2\sqrt{2} \rvert$	$\frac{2+\sqrt{2}}{4} \simeq 0.85$
PR	$\lvert 4 \rvert$	1

GEOMETRY OF CORRELATIONS

QUANT. THEORY

Quantum mechanics is convex in the states and in the probabilities in the probabilities. $p(ab|xy) = q \cdot p'(ab|xy)_{\rho'} + (1-q)\, p''(ab|xy)_{\rho''} \in Q$

CONVEX SETS

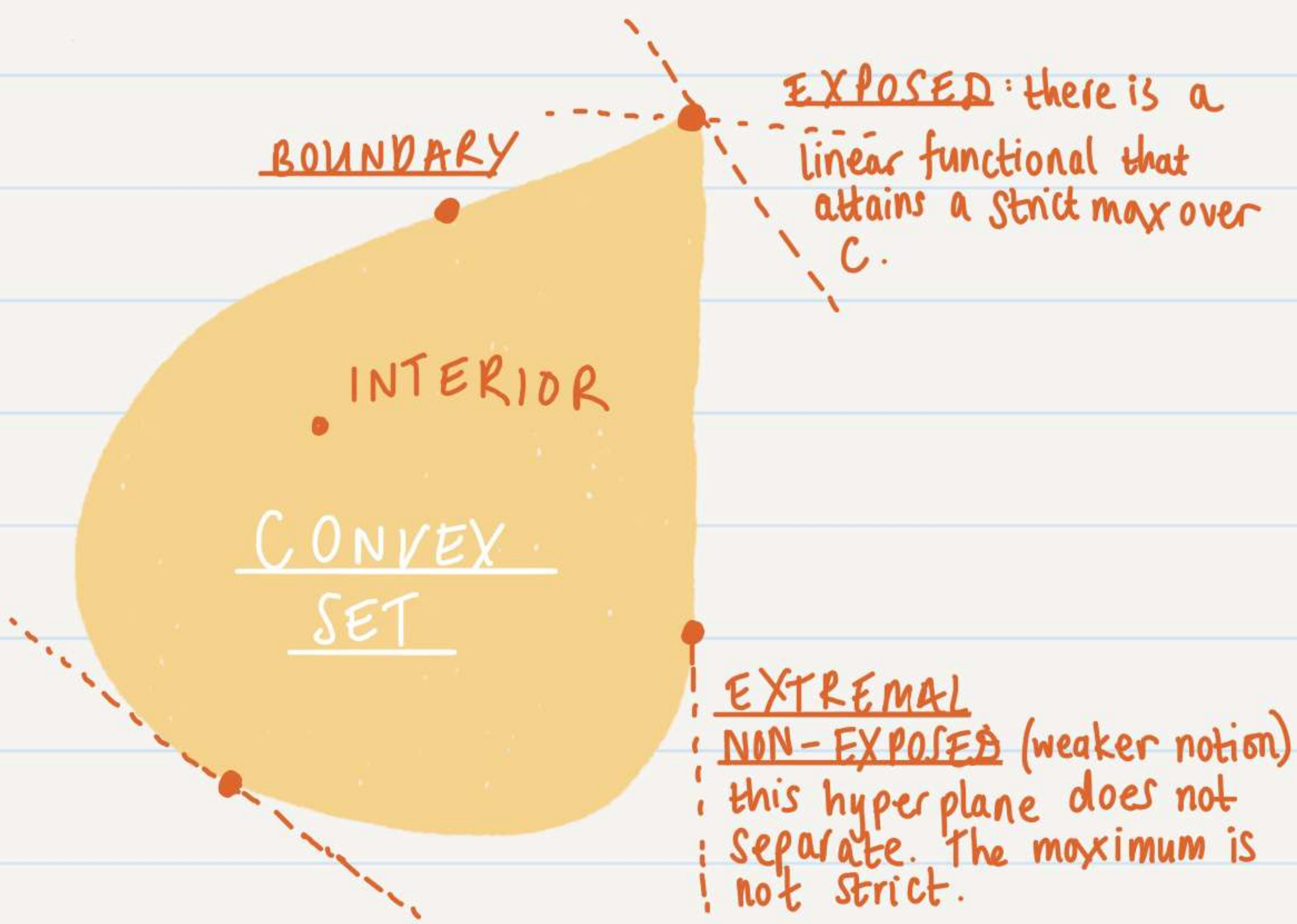
A set C which lives in a vector space $\mathbb{R}^n$ is called a convex set if the line segment joining any pair of point in C lies in C

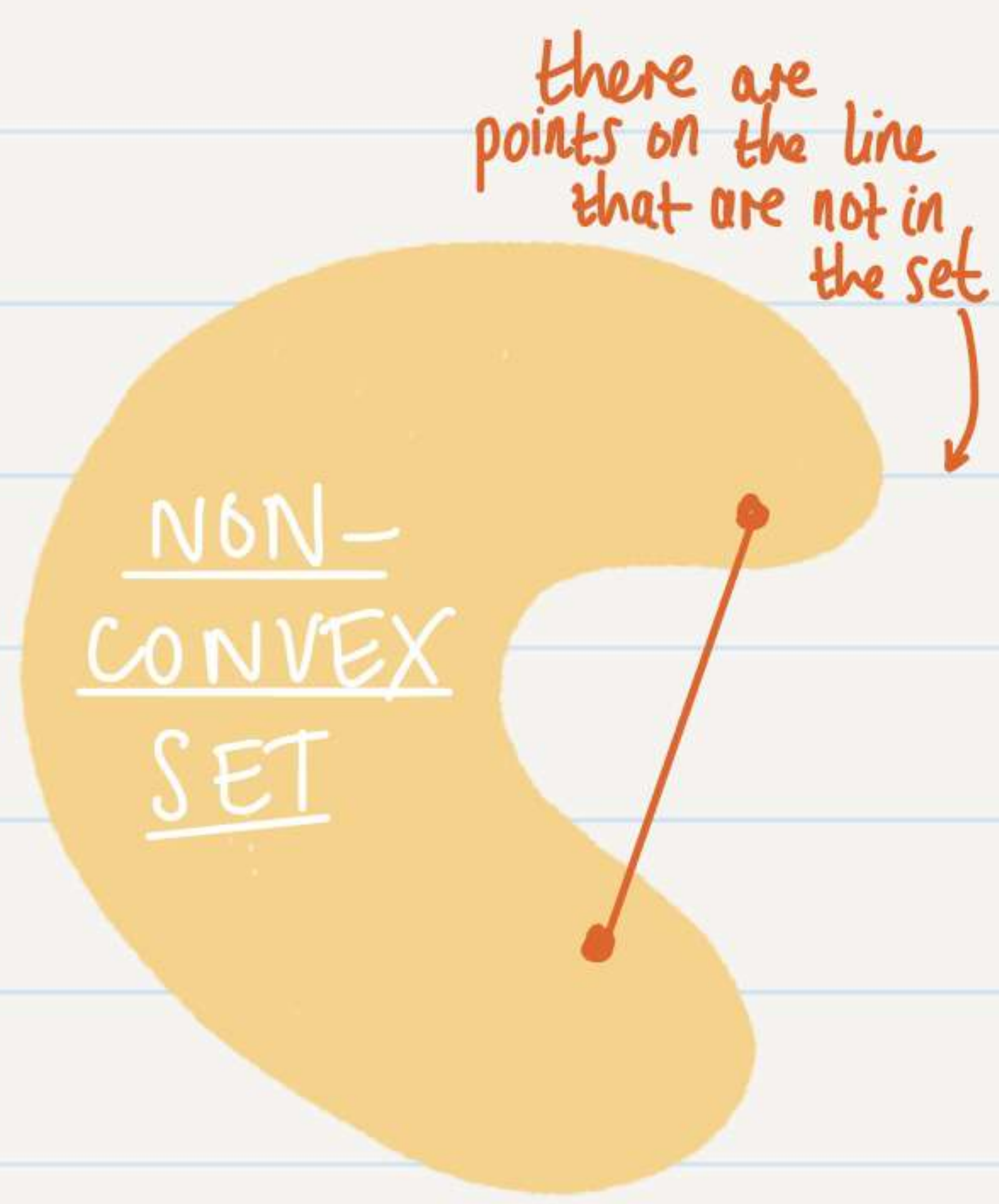
$$p\, c_1 + (1-p)\, c_2 \in C \qquad c_1, c_2 \in C \quad p \in [0,1].$$

HYPERPLANE

For $\vec{a} \in \mathbb{R}^n, b \in \mathbb{R}$, a hyperplane is defined as the set of vectors satisfying the condition

$$H = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{a} = b\}.$$





EXTREMAL

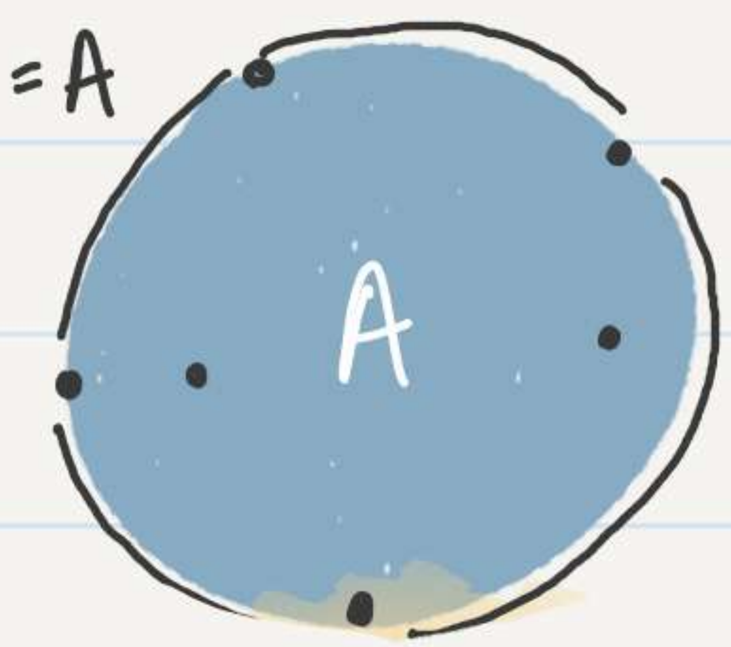
Cannot be written as a non-trivial convex combination of $c \in C$.
$c = \lambda c_1 + (1-\lambda) c_2 \Rightarrow \lambda = 0$ or 1 or $c_1 = c_2 = c$.

CONVEX HULL

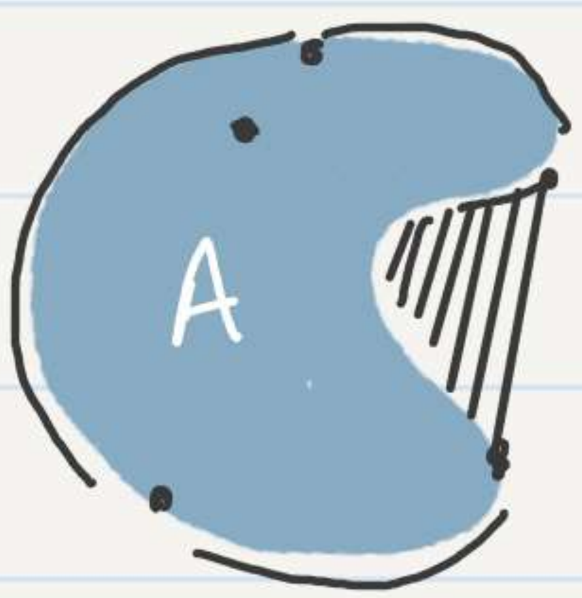
for a set $A \subseteq \mathbb{R}^n$, the convex hull conv A is the set of all convex combination of points in A

$$\text{conv} A = \left\{ \lambda_1 a_1 + \dots + \lambda_m a_m \,\middle|\, m \in \mathbb{N}, \lambda_i \geq 0, \sum_i \lambda_i = 1, a_i \in A \right\}$$

if A is convex, then
conv = A



if A is not convex, then conv $A \neq A$



MINKOWSKI THM

"density matrices can be decomposed into pure states".

$\rho = \sum_i p_i |i\rangle\langle i|$
$= \sum_i p_i \Pi_i$

every compact convex set is the convex hull of its extreme points (=vertices)
example: state space of a quantum n-level system.

$C_n := \{ \rho \in \mathbb{C}^{n \times n} \mid tr[\rho] = 1, \rho \geq 0 \}$. COMPACT, CONVEX

$\partial_{ext} C_n = \{ |\psi\rangle\langle\psi| \mid |\psi\rangle \in \mathbb{C}^n, \|\psi\| = 1 \}$ PURE STATES.

denotes set of extremal points.

CARATHEODORY'S THM

If A is an r-dim set in $\mathbb{R}^n$, then a can be expressed as a convex combination of $r+1$ or fewer points in A.

POLYTOPE

The convex hull of a finite set of points in $\mathbb{R}^n$

FROM BOXES TO POLYTOPES

We put the 16 probabilities $\{p(ab|xy)\}$ associated with the (2,2,2) scenario (2 parties, 2 inputs, 2 outputs) into a vector

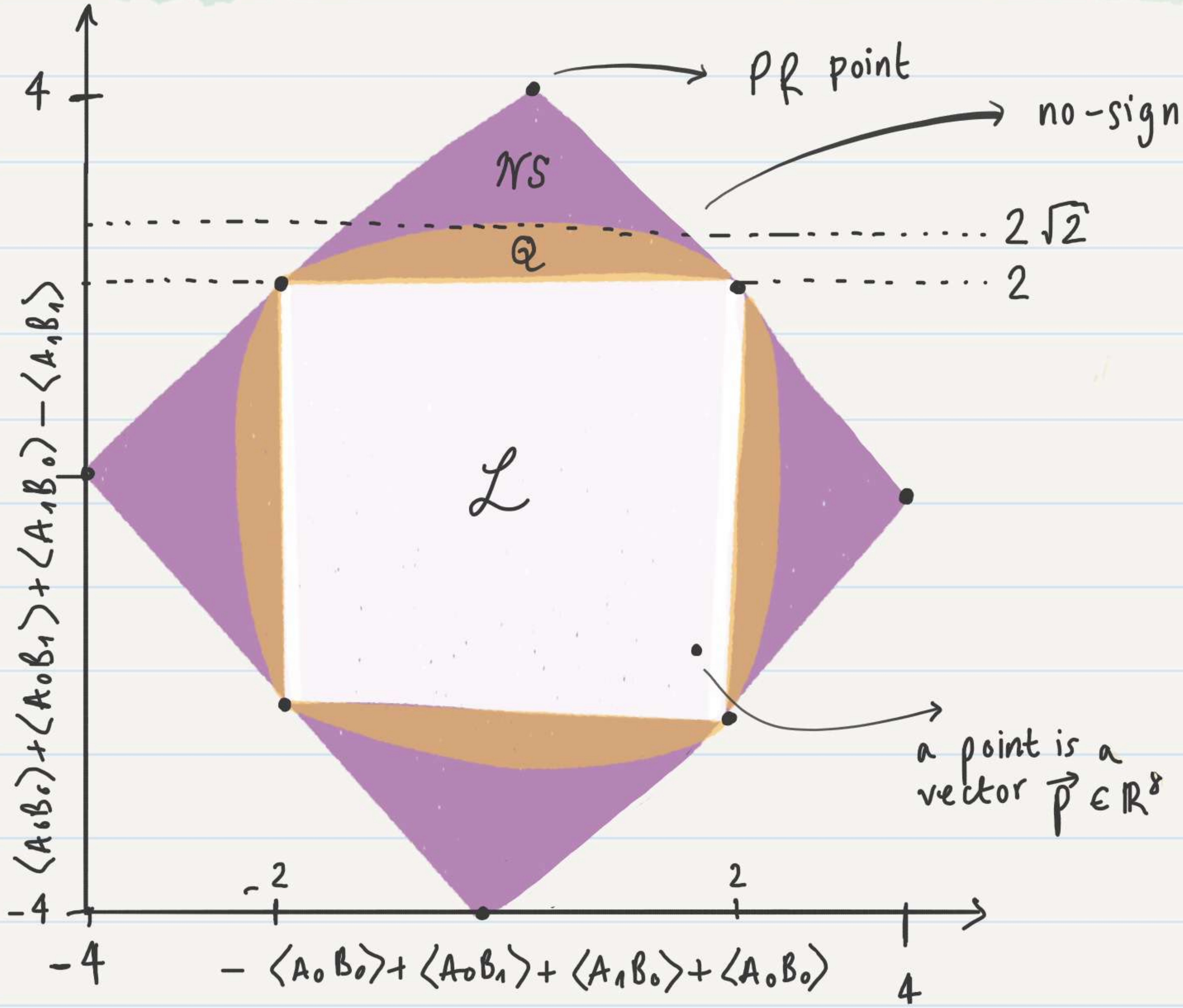
$$\vec{p} = \left(p(00|00), p(01|00), p(10|00), \ldots \quad \ldots, p(11|11)\right)$$

Each element of the vector can be thought of as a variable. We want to know how many independent variables there are, so we count the number of constraints.

CONSTRAINT	Nº OF CONSTRAINTS
1. positivity $p(ab\|xy) \geq 0 \quad \forall x,y,a,b.$	16
2. normalisation $\sum_{ab} p(ab\|xy) = 1 \quad \forall x,y$	-4
3. no-signalling $\sum_a p(ab\|xy) = p(b\|xy) := p(b\|y) \quad \forall b,y,x$	-2
$\sum_b p(ab\|xy) = p(a\|xy) := p(a\|x) \quad \forall a,x,y$	-2
	8

GEOMETRY OF CORRELATIONS

$\mathcal{L} \subset Q \subset NS$



2D SLICE THROUGH THE SET OF NO-SIGNALLING CORRELATIONS