NSW
EDUCATION
STANDARDS
AUTHORITY

2017 | HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2–5)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6–16)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the gradient of the line $2x + 3y + 4 = 0$?

A. $-\frac{2}{3}$

B. $\frac{2}{3}$

C. $-\frac{3}{2}$

D. $\frac{3}{2}$

2 Which expression is equal to $3x^2 - x - 2$?

A. $(3x - 1)(x + 2)$

B. $(3x + 1)(x - 2)$

C. $(3x - 2)(x + 1)$

D. $(3x + 2)(x - 1)$

3 What is the derivative of e^{x^2}?

A. $x^2 e^{x^2 - 1}$

B. $2xe^{2x}$

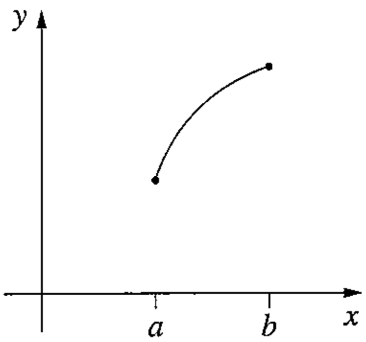
C. $2xe^{x^2}$

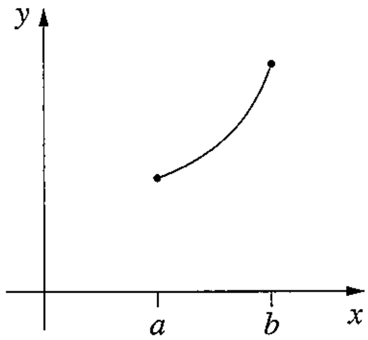
D. $2e^{x^2}$

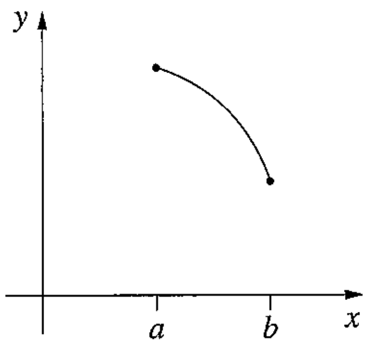
4 The function $f(x)$ is defined for $a \le x \le b$.

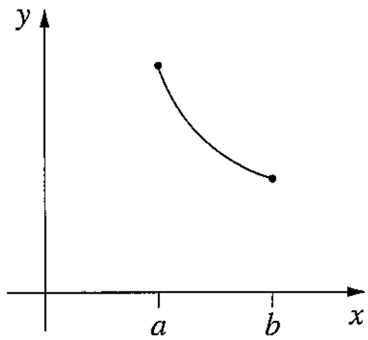
On this interval, $f'(x) > 0$ and $f''(x) < 0$.

Which graph best represents $y = f(x)$?

A. 

B. 

C. 

D. 

5 It is given that $\ln a = \ln b - \ln c$, where $a, b, c > 0$.

Which statement is true?

A. $a = b - c$

B. $a = \dfrac{b}{c}$

C. $\ln a = \dfrac{b}{c}$

D. $\ln a = \dfrac{\ln b}{\ln c}$

6 The point P moves so that it is always equidistant from two fixed points, A and B.

What best describes the locus of P?

A. A point

B. A circle

C. A parabola

D. A straight line

7 Which expression is equivalent to $\tan\theta + \cot\theta$?

A. $\operatorname{cosec}\theta + \sec\theta$

B. $\sec\theta \operatorname{cosec}\theta$

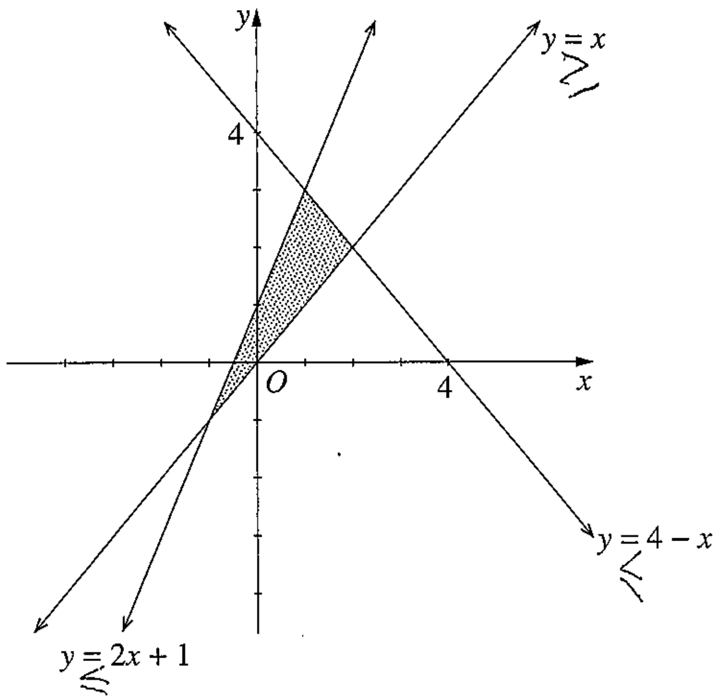
C. 2

D. 1

$\frac{\sin}{\cos} + \frac{\cos}{\sin}$

$\frac{1}{\cos\sin}$

8 The region enclosed by $y = 4 - x$, $y = x$ and $y = 2x + 1$ is shaded in the diagram below.



Which of the following defines the shaded region?

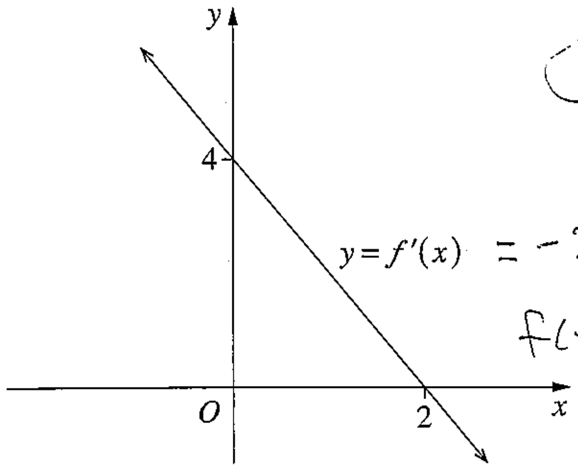
A. $y \leq 2x + 1, \quad y \leq 4 - x, \quad y \geq x$

B. $y \geq 2x + 1, \quad y \leq 4 - x, \quad y \geq x$

C. $y \leq 2x + 1, \quad y \geq 4 - x, \quad y \geq x$

D. $y \geq 2x + 1, \quad y \geq 4 - x, \quad y \geq x$

9 The graph of $y = f'(x)$ is shown.



$= -2x + 4$

$f(x) = -x^2 + 4x + c$

$-(2)^2 + 8 + c = 12$

$c = 8$

The curve $y = f(x)$ has a maximum value of 12.

What is the equation of the curve $y = f(x)$?

A. $y = x^2 - 4x + 12$

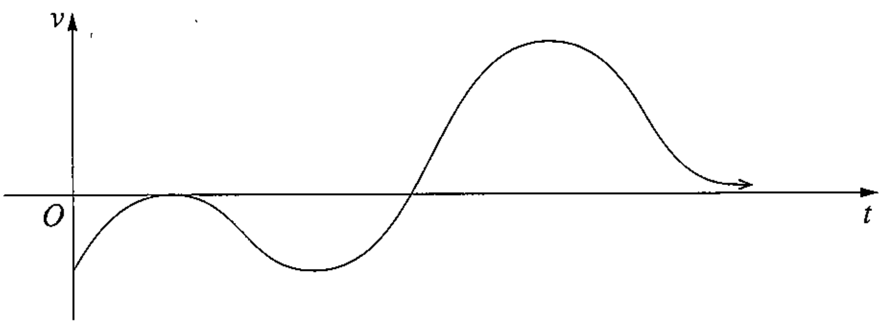
B. $y = 4 + 4x - x^2$

C. $y = 8 + 4x - x^2$

D. $y = x^2 - 4x + 16$

10 A particle is moving along a straight line.

The graph shows the velocity, v, of the particle for time $t \geq 0$.



How many times does the particle change direction?

A. 1

B. 2

C. 3

D. 4

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Rationalise the denominator of $\frac{2}{\sqrt{5}-1}$. $\frac{2(\sqrt5+1)}{(\sqrt5-1)(\sqrt5+1)} = \frac{2(\sqrt5+1)}{4} = \frac{\sqrt5+1}{2}$ **2**

(b) Find $\int (2x+1)^4 \, dx$. $= \int \frac{u^4}{2} du = \frac{u^5}{10} + C = \frac{(2x+1)^5}{10} + C$ **1**

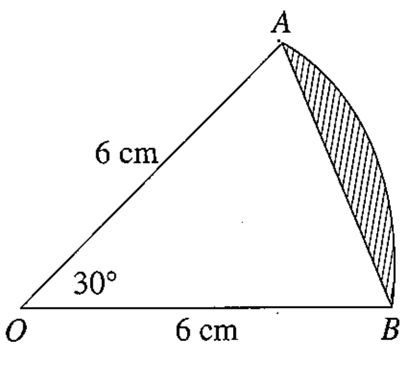
$u = 2x+1$
$du = 2dx$

(c) Differentiate $\frac{\sin x}{x}$. $\frac{x\cos x - \sin x}{x^2}$ **2**

(d) Differentiate $x^3 \ln x$. $3x^2 \ln x + x^2$ **2**

$= x^3(\ln x^3 + 1)$

(e) In the diagram, OAB is a sector of the circle with centre O and radius 6 cm, where $\angle AOB = 30°$.



NOT TO SCALE

(i) Find the exact value of the area of the triangle OAB. $\frac{1}{2}\times 6^2 \sin 30 = 9\text{cm}^2$ **1**

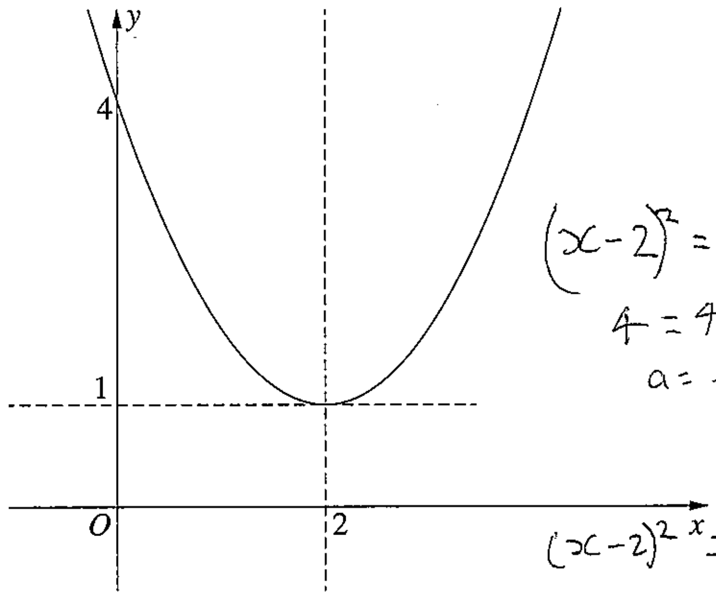
(ii) Find the exact value of the area of the shaded segment. $\frac{30}{360}\times\pi\times 6^2 - 9$ **1**

$= (3\pi - 9)\text{cm}^2$

Question 11 continues on page 7

Question 11 (continued)

(f) Determine the equation of the parabola shown. Write your answer in the form $(x-h)^2 = 4a(y-k)$. **2**



$(x-2)^2 = 4a(y-1)$

$4 = 4a(3)$

$a = \frac{1}{3}$

$(x-2)^2 = \frac{4}{3}(y-1)$

(g) Solve $|3x-1| = 2$. **2**

$3x - 1 = \pm 2$

$3x = 1 \pm 2$

$x = \frac{1 \pm 2}{3}$

$x = 1$ or $\frac{-1}{3}$

(h) Find the domain of the function $f(x) = \sqrt{3-x}$. **2**

$3 - x \geq 0$

$x \leq 3$

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) Find the equation of the tangent to the curve $y = x^2 + 4x - 7$ at the point $(1, -2)$. **2**

$y' = 2x + 4$

$m = 6$

$y + 2 = 6(x - 1)$

$y + 2 = 6x - 6$

$6x - y - 8 = 0$

(b) The diagram shows the region bounded by $y = \sqrt{16 - 4x^2}$ and the x-axis. **3**

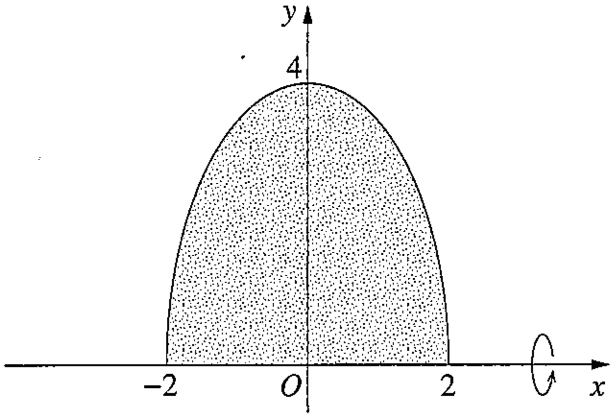
$2\pi\int_0^2 (16 - 4x^2)\,dx$

$= 2\pi\left[16x - \frac{4x^3}{3}\right]_0^2$

$= 2\pi\left[\left(16(2) - \frac{4(2)^3}{3}\right) - 0\right]$

$= 2\pi\left(32 - \frac{32}{3}\right)$

$= \frac{128\pi}{3}\ u^3$



The region is rotated about the x-axis to form a solid.

Find the exact volume of the solid formed.

(c) In an arithmetic series, the fifth term is 200 and the sum of the first four terms is 1200. **3**

Find the value of the tenth term.

$T_5 = 200$

$S_4 = 1200$

$T_{10} = ?$

Question 12 continues on page 9

$a + 4d = 200$

$\frac{4}{2}(2a + (4-1)d) = 1200$

$4a + 6d = 1200$

$2a + 3d = 600$

$2a + 8d = 400$

$5d = -200$

$d = -40$

$a = 200 + 160$

$= 360$

$T_{10} = a + 9d$

$= 360 - 9 \times 40$

$= 0$

Question 12 (continued)

i) $\dfrac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

(d) The points $A(-4, 0)$ and $B(1, 5)$ lie on the line $y = x + 4$.

The length of AB is $5\sqrt{2}$.

The points $C(0, -2)$ and $D(3, 1)$ lie on the line $x - y - 2 = 0$.

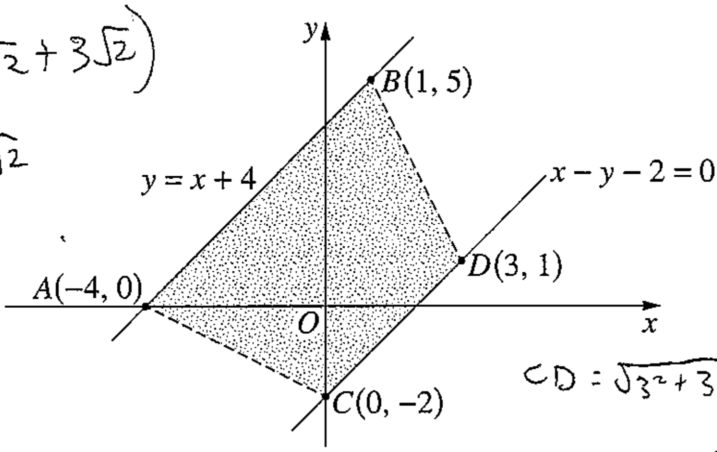
The points A, B, D, C form a trapezium as shown.

$\dfrac{|-4-0-2|}{\sqrt{1^2+(-1)^2}} = \dfrac{6}{\sqrt{2}} = 3\sqrt{2}$

ii) $A = \dfrac{3\sqrt{2}}{2}\left(5\sqrt{2}+3\sqrt{2}\right)$

$= \dfrac{3\sqrt{2}}{2} \times 8\sqrt{2}$

$= 24u^2$



NOT TO SCALE

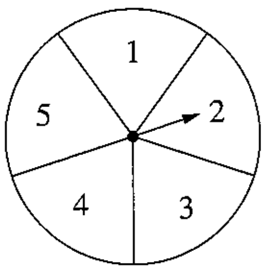
$CD = \sqrt{3^2+3^2} = 3\sqrt{2}$

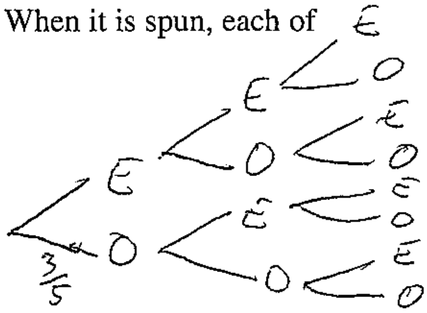
x_1, y_1

(i) Find the perpendicular distance from point $A(-4, 0)$ to the line $x - y - 2 = 0$. **1**

(ii) Calculate the area of the trapezium. **2**

(e) A spinner is marked with the numbers 1, 2, 3, 4 and 5. When it is spun, each of the five numbers is equally likely to occur.





The spinner is spun three times.

(i) What is the probability that an even number occurs on the first spin? **1** $\frac{2}{5}$

$\frac{98}{125}$ (ii) What is the probability that an even number occurs on at least one of the three spins? **1** $1-\left(\frac{3}{5}\right)^3$

$\frac{18}{125}$ (iii) What is the probability that an even number occurs on the first spin and odd numbers occur on the second and third spins? **1** $\left(\frac{2}{5}\right)\left(\frac{3}{5}\right)^2$

(iv) What is the probability that an even number occurs on exactly one of the three spins? **1**

$3 \times \frac{18}{125} = \frac{54}{125}$

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) Using the cosine rule, ~~find~~ the value of x in the following diagram. **3**

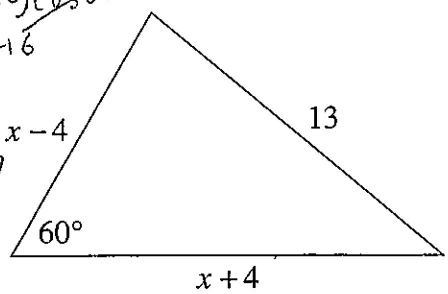
$169 = (x-4)^2 + (x+4)^2 - \cancel{2}(x^2-16)\cancel{\cos 60}$

$69 = x^2 - 8x + 16 + x^2 + 8x + 16 - x^2 + 16$

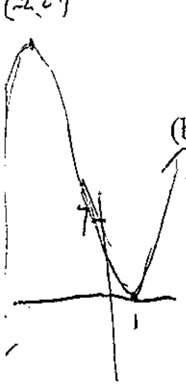
$21 = x^2$

$x = \pm 11$ but $x > 4$

$x = 11$



NOT TO SCALE



(b) Consider the curve $y = 2x^3 + 3x^2 - 12x + 7$. $y' = 6x^2 + 6x - 12$ $= 6(x^2 + x - 2)$ $= 6(x+2)(x-1)$ $y'' = 12x + 6$ $12(-2)+6 < 0$ $12(1)+6 > 0$

(i) Find the stationary points of the curve and determine their nature. **4**

max $\rightarrow (-2, 27)$ $(1, 0) \leftarrow$ min

(ii) Sketch the curve, labelling the stationary points. **2**

(iii) Hence, or otherwise, find the values of x for which $\dfrac{dy}{dx}$ is positive. **1**

$x < -2$, $x > 1$

(c) By letting $m = t^{\frac{1}{3}}$, or otherwise, solve $t^{\frac{2}{3}} + t^{\frac{1}{3}} - 6 = 0$. **2**

$m^2 + m - 6 = 0$

$(m+3)(m-2) = 0$

$m = -3, 2$

$t^{\frac{1}{3}} = -3, 2$

$t = -27, 8$

(d) The rate at which water flows into a tank is given by **3**

$$\frac{dV}{dt} = \frac{2t}{1+t^2},$$

where V is the volume of water in the tank in litres and t is the time in seconds.

Initially the tank is empty.

Find the exact amount of water in the tank after 10 seconds.

$V = \int \frac{2t}{1+t^2}\,dt$

$V = \ln|1+t^2| + C$

$0 = \ln 1 + C$

$C = 0$

$V = \ln|1+10^2|$

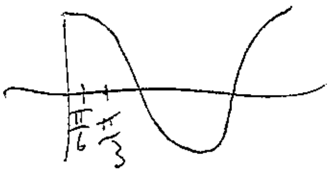
$= \ln 101$ u^3

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) Sketch the curve $y = 4 + 3\sin 2x$ for $0 \le x \le 2\pi$. **3**

(b) (i) Find the exact value of $\int_0^{\frac{\pi}{3}} \cos x \, dx$. $= \sin x \Big]_0^{\pi/3} = \frac{\sqrt{3}}{2}$ **1**

(ii) Using Simpson's rule with one application, find an approximation to the integral **2**



$$\int_0^{\frac{\pi}{3}} \cos x \, dx, \quad \frac{\frac{\pi}{3}}{6}\left[1 + 4\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\right] = \frac{\pi}{18}\left(\frac{3}{2} + 2\sqrt{3}\right)$$

$$= \frac{\pi}{36}\left(3 + 4\sqrt{3}\right)$$

leaving your answer in terms of π and $\sqrt{3}$.

(iii) Using parts (i) and (ii), show that **1**

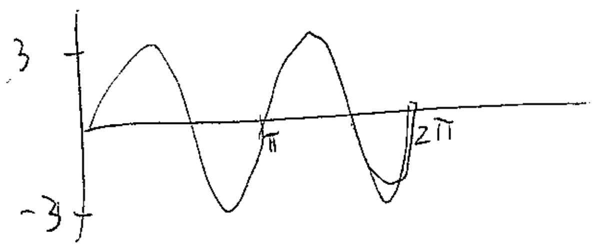
$$\pi \approx \frac{18\sqrt{3}}{3 + 4\sqrt{3}}.$$

$$\frac{\pi}{36}\left(3 + 4\sqrt{3}\right) \approx \frac{\sqrt{3}}{2}$$

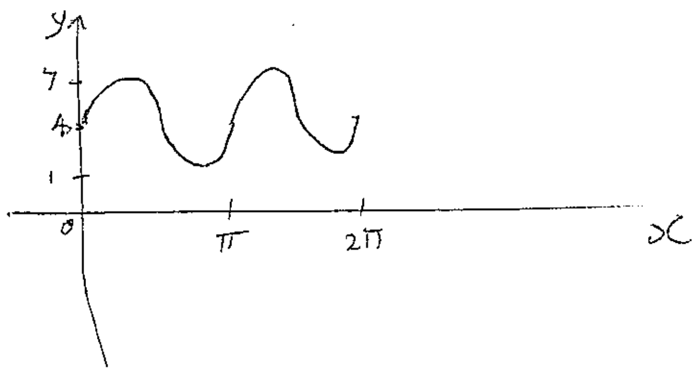
$$\pi\left(3 + 4\sqrt{3}\right) \approx 18\sqrt{3}$$

$$\pi \approx \frac{18\sqrt{3}}{3 + 4\sqrt{3}}$$

Question 14 continues on page 12



a)



Question 14 (continued)

(c) Carbon-14 is a radioactive substance that decays over time. The amount of carbon-14 present in a kangaroo bone is given by

$$C(t) = Ae^{kt},$$

where A and k are constants, and t is the number of years since the kangaroo died.

(i) Show that $C(t)$ satisfies $\frac{dC}{dt} = kC$. **1**

$\frac{dC}{dt} = kAe^{kt}$

$= kC$

(ii) After 5730 years, half of the original amount of carbon-14 is present. **2**

Show that the value of k, correct to 2 significant figures, is -0.00012.

$\frac{A}{2} = Ae^{k\times5730}$

$e^{5730k} = \frac{1}{2}$

$5730k = \ln(\frac{1}{2})$

$k = \frac{\ln(\frac{1}{2})}{5730}$

$= -0.00012096809\ldots$

$= -0.00012$ (2sf)

(iii) The amount of carbon-14 now present in a kangaroo bone is 90% of the original amount. **2**

Find the number of years since the kangaroo died. Give your answer correct to 2 significant figures.

$0.9A = Ae^{kt}$

$e^{-0.00012t} = 0.9$

$-0.00012t = \ln 0.9$

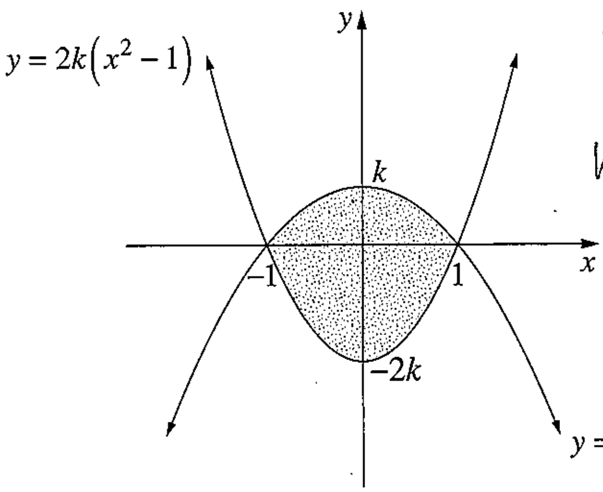
$t = \frac{\ln 0.9}{-0.00012}$

$= 878$ yrs

$= 880$ y (2sf)

(d) The shaded region shown is enclosed by two parabolas, each with x-intercepts at $x = -1$ and $x = 1$. **3**

The parabolas have equations $y = 2k(x^2 - 1)$ and $y = k(1 - x^2)$, where $k > 0$.



$2\int_0^1 [k(1-x^2) - 2k(x^2-1)]\,dx = 8$

$k\int_0^1 (1 - x^2 - 2x^2 + 2)\,dx = 4$

$k\int_0^1 (3 - 3x^2)\,dx = 4$

$k\int_0^1 (1 - x^2)\,dx = \frac{4}{3}$

$k\left[x - \frac{x^3}{3}\right]_0^1 = \frac{4}{3}$

Given that the area of the shaded region is 8, find the value of k.

$k\left[1 - \frac{1}{3} - 0\right] = \frac{4}{3}$

$\frac{2k}{3} = \frac{4}{3}$

$k = 2$

End of Question 14

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) The triangle ABC is isosceles with $AB = AC$ and the size of $\angle BAC$ is $x°$. Points D and E are chosen so that $\triangle ABC$, $\triangle ACD$ and $\triangle ADE$ are congruent, as shown in the diagram. **3**

Construct $AF \perp ED$

In $\triangle AEF$ and $\triangle ADF$

AF is common

$AD = AE$ (given)

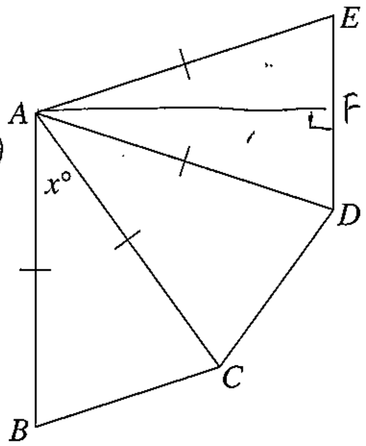
$\angle EFA = \angle DFA = 90$ (by construction)

$\therefore \triangle AEF \equiv \triangle ADF$ (RHS)

Hence $\angle EAF = \angle DAF$ (CACT)

$\therefore \angle FAD = \frac{x}{2}$

Also $\angle DAC = x$ (CACT)



$\angle FAB = 90$

$\therefore 2.5x = 90°$

$x = 36°$

Find the value of x for which AB is parallel to ED, giving reasons.

(b) Anita opens a savings account. At the start of each month she deposits $\$X$ into the savings account. At the end of each month, after interest is added into the savings account, the bank withdraws \$2500 from the savings account as a loan repayment. Let M_n be the amount in the savings account after the n^{th} withdrawal.

The savings account pays interest of 4.2% per annum compounded monthly. $\frac{4.2\%}{12} = 0.35\%$ per month $= 0.0035$

(i) Show that after the second withdrawal the amount in the savings account is given by **2**

$$M_2 = X\left(1.0035^2 + 1.0035\right) - 2500\left(1.0035 + 1\right).$$

(ii) Find the value of X so that the amount in the savings account is \$80 000 after the last withdrawal of the fourth year. 48 months **3**

Month 1 $\quad X + 0.0035X - 2500$

$= X(1.0035) - 2500$

Question 15 continues on page 14

Month 2 $\quad = \left(X(1.0035) - 2500 + X\right) \times 1.0035 - 2500$

$= X(1.0035^2) - 2500 \times 1.0035 + 1.0035X - 2500$

$= X(1.0035^2 + 1.0035) - 2500(1.0035 + 1)$

$X(1.0035^{48} + \dots + 1.0035) - 2500(1.0035^{47} + 1.0035^{46} + \dots + 1) = 80000$

Month 48: $X\left(\frac{1.0035^{49} - 1.0035}{0.0035}\right) - 2500\left(\frac{1.0035^{48} - 1}{0.0035}\right) = 80000$

$X = \$4019.42$

Question 15 (continued)

(c) Two particles move along the x-axis.

When $t = 0$, particle P_1 is at the origin and moving with velocity 3.

For $t \geq 0$, particle P_1 has acceleration given by $a_1 = 6t + e^{-t}$.

(i) Show that the velocity of particle P_1 is given by $v_1 = 3t^2 + 4 - e^{-t}$. **2**

When $t = 0$, particle P_2 is also at the origin.

For $t \geq 0$, particle P_2 has velocity given by $v_2 = 6t + 1 - e^{-t}$.

(ii) When do the two particles have the same velocity? **2**

(iii) Show that the two particles do not meet for $t > 0$. **3**

End of Question 15

i) $a_1 = 6t + e^{-t}$

$V_1 = \int 6t + e^{-t}\,dt$

$= \frac{6t^2}{2} - e^{-t} + c$

$3 = 3(0)^2 - e^{-0} + c$

$c = 4$

$V_1 = 3t^2 - e^{-t} + 4$

$= 3t^2 + 4 - e^{-t}$

ii) $6t + 1 - e^{-t} = 3t^2 + 4 - e^{-t}$

$3t^2 - 6t + 3 = 0$

$3(t^2 - 2t + 1) = 0$

$3(t-1)^2 = 0$

$t = 1$

iii) $x_1 = \int 3t^2 + 4 - e^{-t}\,dt$

$= t^3 + 4t + e^{-t} + c$

$0 = 0^3 + 0 + e^0 + c$

$c = -1$

$\cancel{x_1 = 3t^2 + x}$

$x_1 = t^3 + 4t + e^{-t} - 1$

$x_2 = \int 6t + 1 - e^{-t}\,dt$

$= 3t^2 + t + e^{-t} + c$

$0 = 3(0)^2 + 0 + e^0 + c$

$c = -1$

$x_2 = 3t^2 + t + e^{-t} - 1$

$t^3 + 4t + e^{-t} - 1 = 3t^2 + t + e^{-t} - 1$

$t^3 - 3t^2 + 3t = 0$

$t(t^2 - 3t + 3) = 0$

$b^2 - 4ac = (-3)^2 - 4(1)(3)$

$= 9 - 12$

$= -3 < 0$

$t = 0$ is only real root. Hence $x_1 = x_2$ at $t = 0$ but nowhere else

Question 16 (15 marks) Use the Question 16 Writing Booklet.

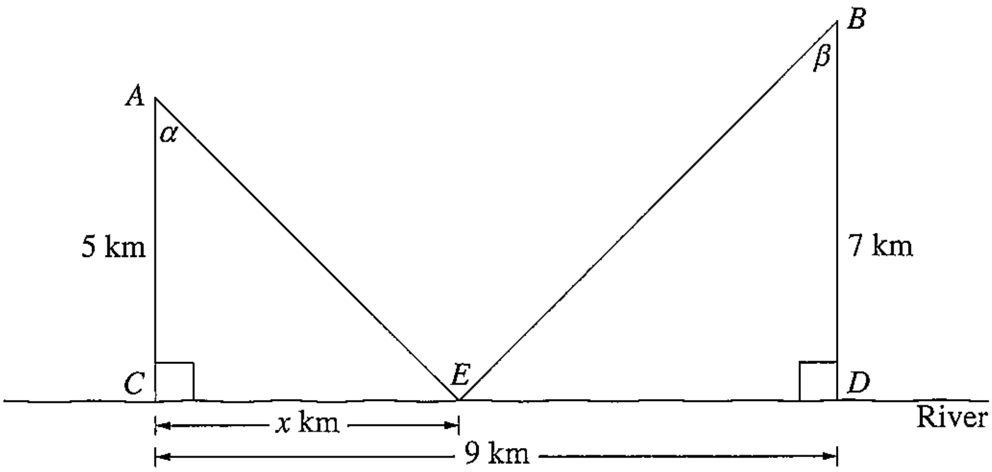
(a) John's home is at point A and his school is at point B. A straight river runs nearby.

The point on the river closest to A is point C, which is 5 km from A.

The point on the river closest to B is point D, which is 7 km from B.

The distance from C to D is 9 km.

To get some exercise, John cycles from home directly to point E on the river, x km from C, before cycling directly to school at B, as shown in the diagram.



The total distance John cycles from home to school is L km.

(i) Show that $L = \sqrt{x^2+25} + \sqrt{49+(9-x)^2}$. **1**

(ii) Show that if $\frac{dL}{dx} = 0$, then $\sin\alpha = \sin\beta$. **3**

(iii) Find the value of x that makes $\sin\alpha = \sin\beta$. **2**

(iv) Explain why this value of x gives a minimum for L. **1**

$\sin\alpha = \frac{x}{AE}$ $\quad \sin\beta = \frac{9-x}{EB}$

α & $\beta < 90$ $\therefore \sin\alpha = \sin\beta$ when $\alpha = \beta$

$\alpha = \beta$

(iii) $\frac{5}{x} = \frac{7}{9-x}$

$45 - 5x = 7x$

$12x = 45$

$x = 3.75$km

Question 16 continues on page 16

i) $AE = \sqrt{x^2+5^2}$

$EB = \sqrt{(9-x)^2+7^2}$

$L = AE + EB$

$= \sqrt{x^2+25} + \sqrt{49+(9-x)^2}$

ii) $\frac{dL}{dx} = \frac{2x}{2\sqrt{x^2+25}} - \frac{2(9-x)}{2\sqrt{49+(9-x)^2}} = 0$

$\frac{x}{\sqrt{x^2+25}} - \frac{(9-x)}{\sqrt{49+(9-x)^2}} = 0$

$\frac{x}{AE} - \frac{9-x}{EB} = 0$

$\frac{x}{AE} = \frac{9-x}{EB}$

$\sin\alpha = \sin\beta$

iv) because $\frac{dL}{dx} = 0$

also $L'(3) < 0$

$L'(4) > 0$

3   3.75   4

\   —   / $\therefore$ minimum

$0<a<4$

Question 16 (continued)

(b) A geometric series has first term a and limiting sum 2. **3**

Find all possible values for a.

$S=\frac{a}{1-r}$

$2=\frac{a}{1-r}$ $\quad -1<r<1$

$a=2(1-r)$

(c) In the triangle ABC, the point M is the mid-point of BC. The point D lies on AB and

$$BD = DA + AC.$$

The line that passes through the point C and is parallel to MD meets BA produced at E.

$\frac{BM}{MC}=\frac{BD}{DE}$ (ratio of intercepts parallel lines)

$BM=MC$ (given)

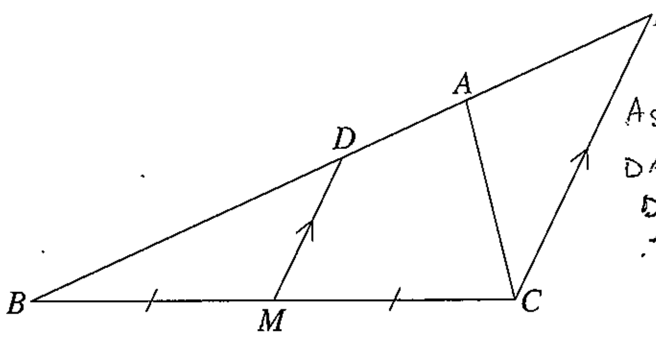
$\therefore BD=DE$

As $BD=DA+AC$

$DA+AC=DE$

$DA+AC=DA+AE$

$\therefore AC=AE$



Copy or trace this diagram into your writing booklet.

(i) Prove that $\triangle ACE$ is isosceles. **3**

(ii) The point F is chosen on BC so that AF is parallel to DM. **2**

let $\angle AEC = x$

$\angle ACE = x$ (base $\angle$s, isos $\triangle$)

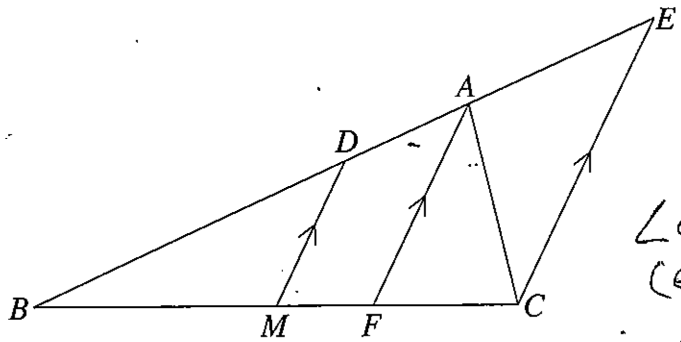
$\angle CAF = x$ (alt $\angle$s $AF \parallel EC$)

$\angle CAB = \angle AEC + \angle ACE$ (ext $\angle$ $\triangle$)

$\therefore \angle CAB = 2x$

$\therefore \angle DAF = 2x - x$
$= x$

Hence AF bisects $\angle BAC$



Show that AF bisects $\angle BAC$.

End of paper