General Relativity for the Experimentalist*

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Summary—**Einstein's general theory of relativity is broken down and simplified under limitations usually satisfied in experimentally realizable situations. Following the work of Møller,[1] an analogy between electromagnetism and gravitation is presented which allows calculation of various gravitational forces by considering the equivalent electromagnetic problem. A number of examples are included. Tensor formulation is not used except in the Appendix, where justification for the analogies is given.**

Introduction

WE SHALL assume that all gravitational effects are correctly described by Einstein's general theory of relativity.[2] Suppose we want to find the behavior of a system under the influence of gravitational and other forces in the proper general relativistic manner. We must first determine all the mass and energy in both the system being investigated and in the sources of the fields. Using these in a prescribed manner, we calculate the ten components of the energy-momentum tensor. Next, we put the energy-momentum tensor into Einstein's equations and solve these ten nonlinear differential equations for the ten components of the metric tensor. Then the metric tensor is substituted into the curvilinear equation of motion and it is solved to determine how the system moves.

Because of the difficulty of this process, there exist only a few solutions of Einstein's equation which are of experimental interest in that they describe some physically observable effect of general relativity. The process is so specialized and so difficult that it is practically impossible to attempt solution of a problem unless months of study on the specialized terminology, procedures, and conventions of the general relativity theorist have been completed. The most disappointing aspect is that in most cases, after struggling through the calculations, it will only be found that the effect calculated is too small to be observed.

Fortunately, it is possible to bypass this complicated procedure. By applying reasonable limitations to the systems involved, Einstein's equations can be linearized. These linearized equations can be examined and handled separately. One equation gives us the gravitational scalar potential; three others give a quantity which is similar to the magnetic vector potential; and the remainder, which describe the remaining properties of the gravitational field, can be handled by assuming that they represent a curvature to the space.

With these analogies, it is possible for anyone who has had electromagnetic theory to study a situation of experimental interest and to calculate the effects to be expected with sufficient accuracy to determine whether they warrant further study.[3]

There have been numerous books, papers,[4–6] articles,[7] and even advertisements[8] that have developed or mentioned a gravitational analog to the electric and magnetic fields. Some are based on classical ideas, some on Mach's principle, and others on Einstein's general theory of relativity. None based on Einstein's work, however, have been at the same time complete, rigorous, and free of tensors.

Analogies to Electricity and Magnetism

In Einstein's general theory of relativity there exist gravitational analogies to the electric and magnetic fields. We are already familiar with the analogy between the electric field of a charge and the gravitational field of mass. It is well known that the analogy breaks down almost immediately, because there are two kinds of charge and only one kind of mass and because two particles with the same type of charge repel, whereas two particles with the same type of mass attract. Nevertheless, if we are cognizant of these distinctions, we can still apply the analogy and obtain useful results.

In the Appendix it is shown that Einstein's equations not only can be made to show this well-known analogy between the electric field and the gravitational field, but they can also give a gravitational analog to the magnetic field. It has no name, but since it will be mentioned, we shall coin one and call it the "protational field," as it usually arises from the rotation of a mass. This "protational field" has the dimensions of angular velocity (radians/sec) and is closely related to coriolis-type forces which arise from the principle of general relativity.

When the analogy is carried out and all the constants

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[1] C. Møller, "The Theory of Relativity," Oxford University Press, London; 1952.

[2] A. Einstein, "The foundation of the general theory of relativity," *Ann. Phys.*, vol. 49, pp. 769–822; May 11, 1916. See also, A. Einstein, "The Principle of Relativity," Dover Publications, Inc., New York, N. Y.; 1923.

[3] If anyone using this approximate method comes upon a previously uncalculated effect that shows promise of being large enough to be observed, the author will be glad to repeat the calculations using the proper tensor formulation to ensure that the result was not produced by the approximations involved in simplifying the theory.

[4] D. W. Sciama, "On the origin of inertia," *Mon. Not. Roy. Astr. Soc.*, vol. 113, pp. 34–42; January, 1953.

[5] W. D. White, "Electromagnetic analogs for the gravitational fields in the vicinity of a satellite," PROC. IRE, vol. 46, pp. 920–922; May, 1958.

[6] W. Davidson, "General relativity and Mach's principle," *Mon. Not. Roy. Astr. Soc.*, vol. 117, pp. 212–224; February, 1957.

[7] G. P. Field, "Two source field theory," essay submitted to Gravitational Research Foundation, New Boston, N. H. Some of the terminology and symbols used were adopted from this essay.

[8] W. D. White, A.I.L. Advertising Monographs, PROC. IRE, vol. 46, p. 4A; November and December, 1958 and January, 1959.

are evaluated, we obtain an isomorphism between the gravitational and the electromagnetic quantities (see Table I).

Now, if we have a certain mass distribution and flow, all that is necessary is to find a similar charge and current distribution in electromagnetic texts, such as that of Smythe.[9] We then use the formulas derived for the electric and magnetic fields and make the substitutions in the electromagnetic formulas to obtain the gravitational formulas. Since we are using the linearized theory, superposition is valid, and fields for more complex bodies can be built up from the superposition of the fields of the parts.

Once we have calculated the fields generated by the mass density and currents, we can calculate the forces on a particle of mass m by a force equation which is analogous to the electromagnetic force equation,

$$\boldsymbol{F} = -m\nabla\chi - m\frac{\partial \boldsymbol{K}}{\partial t} + m\boldsymbol{v}\times(\nabla\times\boldsymbol{K}) = m\boldsymbol{G} + m(\boldsymbol{v}\times\boldsymbol{P}). \quad (1)$$

If the test body is spinning and has an angular momentum of $\boldsymbol{L}$, then the torque on it due to a "protational field" $\boldsymbol{P}$ will be by analogy

$$\boldsymbol{N} = \tfrac{1}{2}\boldsymbol{L}\times\boldsymbol{P}. \quad (2)$$

It should be emphasized that the previous discussion is approximate and is presented merely to provide a simple tool with which to make estimates. In deriving this analogy between some of the gravitational forces and the static and induction fields of electromagnetism, the following assumptions, among others, have been made:

1) The mass densities are normal (no dwarf stars).
2) All motions are much slower than the speed of light. (Often special relativistic effects will hide general relativistic effects.)
3) The kinetic or potential energy of all the bodies being considered is much smaller than their mass energy.
4) The fields are always weak enough so that superposition is valid.
5) The distances between objects is not so large that we have to take retardation into account. (This condition can be ignored when we have a stationary problem where the fields have already been prescribed and are not changing with time.)

To show how this analogy can be used, let us calculate a few simple examples.

Force between Two Masses

Suppose we have two particles with total masses M_1 and M_2. Then if we want to calculate the gravitational field due to mass M_1, we write down the electric field for a particle with charge Q_1:

$$\boldsymbol{E} = \frac{Q_1}{4\pi\epsilon r^2}\hat{r}. \quad (3)$$

Next we transform all quantities to the equivalent gravitational quantities and get

$$\boldsymbol{G} = \frac{-M_1}{4\pi\gamma r^2}\hat{r}. \quad (4)$$

Then the force on a particle of mass M_2 in the gravitational field of M_1 is

$$\boldsymbol{F} = M_2\boldsymbol{G} = \frac{-M_1M_2}{4\pi\gamma r^2}\hat{r}, \quad (5)$$

and if we transform γ into more familiar gravitational units, we get

$$\boldsymbol{F} = \frac{-GM_1M_2}{r^2}\hat{r}, \quad (6)$$

which is merely Newton's law.

TABLE I

	EM Symbol	Gravitational Symbol	Value or Definition
Force Vector	$-\boldsymbol{E}\rightarrow$	$\boldsymbol{G}$	$=-\nabla\chi-\frac{\partial \boldsymbol{K}}{\partial t}$
Solenoidal Force Vector	$-\boldsymbol{B}\rightarrow$	$\boldsymbol{P}$	$=\nabla\times\boldsymbol{K}$
Scalar Potential	$-\phi\rightarrow$	χ	$\approx-\frac{1}{4\pi\gamma}\int_V\frac{\mu}{r}dV$
Vector Potential	$-\boldsymbol{A}\rightarrow$	$\boldsymbol{K}$	$\approx-\frac{\eta}{4\pi}\int_V\frac{\mu\boldsymbol{v}}{r}dV$
Source Density	$\rho\rightarrow$	μ	$=\frac{dM}{dV}$
Source Quantity	$Q\rightarrow$	M	$=\int_V\mu dV$
Current Density	$\boldsymbol{j}\rightarrow$	$\boldsymbol{p}$	$=\mu\boldsymbol{v}$
Current Quantity	$\boldsymbol{I}\rightarrow$	T	$=\frac{dM}{dt}=\int_s\boldsymbol{p}\cdot\boldsymbol{n}dS$
Moment	$\boldsymbol{M}\rightarrow$	$\frac{1}{2}\boldsymbol{L}$	$=\frac{1}{2}I\omega$
Capacitivity of Space	$\epsilon\rightarrow$	γ	$=\frac{1}{4\pi G}=1.19\times10^{9}\frac{\text{kg-sec}^2}{\text{m}^3}$
Permeability of Space	$\mu\rightarrow$	η	$=\frac{16\pi G}{c^2}=3.73\times10^{-26}\frac{\text{m}}{\text{kg}}$

[9] W. R. Smythe, "Static and Dynamic Electricity," McGraw-Hill Book Co., Inc., New York, N. Y.; 1950.

Pinch Effect between Two Pipes

This example is included primarily to show why the gravitational equivalent of the magnetic field has never been observed.

Suppose that molten metal is flowing through two parallel pipes with spacing $d=0.1$ meter. To find their interaction due to the mass currents T_1 and T_2, we look at the equivalent magnetic case of two wires with a current I. There will exist a pinch effect caused by a force-per-unit length of

$$\frac{F}{L}=\frac{\mu I_1 I_2}{2\pi d}. \tag{7}$$

We then transform all the quantities to get the equivalent gravitational relation:

$$\frac{F}{L}=\frac{\eta T_1 T_2}{2\pi d}. \tag{8}$$

Let us be overly generous and assume that molten iridium is used so that each meter of pipe will contain about 200 kg and we can obtain a flow velocity of a kilometer per second. Then the mass current in each pipe will be

$$T_1 = T_2 = Mv = 2\times 10^5 \text{ kg/sec}.$$

The force between the two pipes due to the "protational" pinch effect will be about 2×10^{-15} newtons per meter of pipe.

If, for comparison, we also look at the gravitational attraction for the same two pipes and use Newton's law, we get a force of about 3×10^{-4} newtons per meter of pipe, so that the forces due to the pinch effect are hidden by the gravitational effect, which itself is not usually observable. This is quite discouraging, but it is the usual result of gravitational calculations. Furthermore, this estimate of the effect has saved us the labor of calculating it by using tensor quantities.

Satellites of Spinning Bodies

We shall now estimate the effect of the earth's rotation on an artificial satellite. Any effects will probably be hidden by the perturbations induced by inhomogeneous mass distributions, atmospheric friction, and even the light pressure from the sun; however, we shall calculate them anyway.

First we need to know the "protational field" of the earth. From Smythe[9] we find an expression for the external magnetic field produced by a ring current I at a latitude $\theta=\alpha$ on a spherical shell of radius R. By transforming the magnetic quantities into the equivalent gravitational quantities, we obtain an expression for the "protational field" of a rotating massive ring with a mass current T:

$$P_\theta = \frac{-\eta T \sin\alpha}{2R}\sum_{n=1}^{\infty}\frac{1}{(n+1)}\left(\frac{R}{r}\right)^{n+2} \cdot P_n^1(\cos\alpha)P_n^1(\cos\theta). \tag{9}$$

Since it is assumed that superposition is valid, we can construct the "protational field" of a solid spinning body by integrating over the volume:

$$P_\theta = \frac{-\eta\omega\sin\theta}{8\pi r^3}\int_V [\mu(\alpha, R)R^2\sin^2\alpha]R^2\sin\alpha d\alpha d\phi dR + \text{higher multipoles}. \tag{10}$$

Since $R\sin\alpha$ is the distance from the axis of rotation to the mass element, we see that the integral is merely the moment of inertia I of the body. Thus, in general, the "protational field" of any rotating body is approximately

$$P_\theta \approx \frac{-\eta I\omega\sin\theta}{8\pi r^3}. \tag{11}$$

Similarly, it can be shown that

$$P_r \approx -\frac{\eta I\omega\cos\theta}{4\pi r^3}. \tag{12}$$

Now that we know the "protational field" of a spinning body, such as the earth, we can calculate the effect of this field on a satellite.

A satellite in a polar orbit around a spinning body would experience a perturbing force due to the radial component of the "protational field." (See Fig. 1.) Neglecting space curvature, we find that this force would cause a precession of the orbit of an amount

$$\Omega = \frac{N}{L} = \frac{F_\phi r}{mrv_\theta} = P_r. \tag{13}$$

Averaging the effect over the whole orbit gives

$$\Omega = \frac{-2}{\pi}\frac{\eta I\omega}{4\pi r^3}. \tag{14}$$

Now, if we substitute numbers for the case of a satellite in a polar orbit around the earth,

$$I = 8.11\times10^{37} \text{ kg-m}^2$$
$$r = 7.4\times10^6 \text{ meters}$$
$$\eta = 3.73\times10^{-26} \text{ m/kg}$$
$$\omega = 7.29\times10^{-5} \text{ rad/sec},$$

we find a precession of the orbit of 5.5×10^{-14} rad/sec. This is equivalent to a period of rotation of the orbital

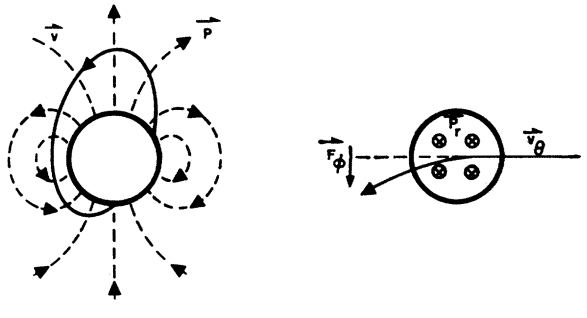

Fig. 1—Effect of earth's "protational field" on a satellite in a polar orbit.

plane of 27 billion years, which is indeed too slow to be seen.

A satellite in an equatorial orbit would experience a radial force due to the tangential component of the "protational field." This force would be completely hidden by the radial Newtonian gravitational force and would be observable only if the earth were completely symmetric and we knew its mass to the nth decimal place. If the satellite is spinning, however, then the spin axis of the satellite itself will experience a torque,

$$\boldsymbol{N} = \tfrac{1}{2}\boldsymbol{L} \times \boldsymbol{P} \tag{15}$$

due to spin-spin interaction. If we ignore space curvature effects, this torque will cause a precession of the spin axis by an amount,

$$\Omega = \frac{N}{L} = \frac{P}{2} \sin \beta. \tag{16}$$

For a satellite in an equatorial orbit of radius r around a spinning body with angular momentum $I\omega$, we get

$$\Omega = \frac{P_\theta}{2} = \frac{-\eta I\omega}{16\pi r^3} = -\frac{GI\omega}{c^2 r^3}. \tag{17}$$

The problem of the precession of a spinning satellite has been rigorously calculated in the proper relativistic manner by L. I. Schiff.[10] His equation for the precession rate was

$$\Omega = \frac{3GM}{2c^2 r}\omega_{\text{orbit}} - \frac{GI}{c^2 r^3}\omega_{\text{spin}}. \tag{18}$$

The first term, the largest, is due to the effects of space curvature, which we neglected. This term is independent of the spin of either the satellite or the planet. The second term is the one we calculated. If we substitute numbers for the case of a spinning satellite in equatorial orbit around the earth,

$$\begin{aligned} r &\approx 7.4 \times 10^6 \text{ meters} \\ \omega_{\text{spin}} &= 7.29 \times 10^{-5} \text{ rad/sec} \\ \omega_{\text{orbit}} &\approx 10^{-3} \text{ rad/sec} \\ I &= 8.11 \times 10^{37} \text{ km-m}^2 \\ M &= 5.98 \times 10^{24} \text{ kg}, \end{aligned}$$

we get

$$\Omega = (9.0 - 0.11) \times 10^{-13} \text{ rad/sec}. \tag{19}$$

Thus we see that the spin interaction term is a small percentage of the space curvature term.

Analogy to Electromagnetic Radiation

Since we have derived an analogy between electromagnetism and gravitation, we might naively suppose that this analogy also would hold for electromagnetic radiation. We might consider writing the Maxwell relations, transforming them to the equivalent gravitational relations, and then solving them to get the wave equation. If we do this using the analogy for the electric and magnetic fields, we shall find that the equations describe a wave with a propagation constant of half the speed of light, since in electromagnetic theory $1/\sqrt{\mu\epsilon} = c$ and in "graviprotational" theory $1/\sqrt{\gamma\eta} = c/2$. We are reasonably sure, however, that the velocity of propagation of gravitational energy will be the same as the speed of light, since the value obtained by Einstein for the rotation of the perihelion of Mercury depends upon this value. Thus we have another indication that our analogy is not perfect, but will give order-of-magnitude estimates only.

Despite the failure of this analogy, it is possible by taking more terms into account to show that Einstein's equation contains the proper wave equation. In the Appendix we obtain

$$\Delta^2 \phi_{\alpha\beta} - \frac{1}{c^2}\frac{\partial^2 \phi_{\alpha\beta}}{\partial t^2} = 0, \tag{20}$$

where $\phi_{\alpha\beta}$ is a quantity representing the gravitational potential of an accelerated mass. The interpretation of this equation is that an accelerated mass will emit gravitational waves which travel with the velocity of light.

No one has ever observed pure gravitational radiation, and from the examples at the end of this section, we shall see why. The observance of interaction in the induction or near-field zone of an accelerated mass is quite another matter. With a sensitive torsion balance, Cavendish[11] observed the attraction of one mass by another and measured the value of the gravitational constant (Fig. 2). If we swing the large masses back and

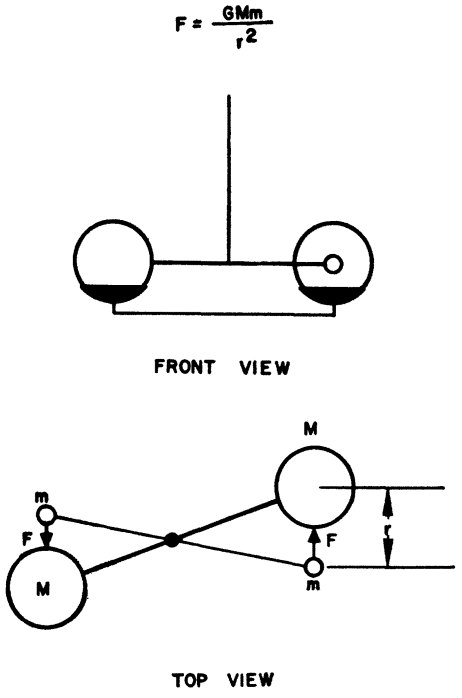

Fig. 2—Cavendish's experiment.

[10] L. I. Schiff, "Possible new experimental test of general relativity, theory," *Phys. Rev. Lett.*, vol. 4, pp. 215–217; March 1, 1960.

[11] H. Cavendish, *Phil. Trans. Roy. Soc.*, vol. 17, p. 469; 1798.

forth with the same period as the natural period of the torsion pendulum, it is easy to see that oscillations will build up in the pendulum and resonant absorption will occur in the near zone of the large masses.

It is well known that the induction fields are conservative and that if there were no resonant absorption, there would be no losses due to the near field. In electromagnetic theory, if we examine the fields at a large distance from the field generator, then the near field becomes negligible and all that remains is the radiation field, which is not conservative. A radiation field carries away energy, and the oscillations in the generator damp out as a result of the radiation losses. It is this gravitational equivalent of the radiation field that has never been observed, either directly or by radiation damping of a mechanically accelerating system.

In Landau and Lifshitz,[12] the wave equation for the gravitational potential is solved and transformed from a four-dimensional relationship into a temporal-spatial relationship. The general solution is

$$\phi_{ab} = \frac{2G}{c^4 r}\int_V \mu x_a x_b dV. \qquad a, b = 1, 2, 3. \tag{21}$$

By calculating the energy in a plane wave at large distances from the source and averaging over all directions, the total energy emitted per unit time in all directions by the accelerated mass is given by

$$-\frac{dE}{dt} = \frac{G}{45c^5}\sum_{a=1}^{3}\sum_{b=1}^{3}(\dddot{Q}_{ab})^2, \tag{22}$$

where

$$Q_{ab} = \int_V \mu(3x_a x_b - \delta_{ab} r^2)dV$$

is the mass quadrupole moment of the mass source.

Note that energy will not be radiated unless the source has an accelerated mass quadrupole moment. Thus, gravitational waves must be quadrupole radiation or higher multipole radiation. There is no dipole gravitational radiation; this is easily seen by physical arguments. Suppose that we grasp a charged particle and shake it, *i.e.*, accelerate it. Since it is the only moving charge in the area, it emits electromagnetic dipole radiation. Now suppose that we hold a particle with mass and shake it. As we rapidly accelerate the small particle in one direction, our large body, in order to conserve momentum, will slowly accelerate in the opposite direction. Because the "charge-to-mass" ratio in gravitation is unity, the two accelerated bodies will always radiate the same amount of dipole radiation, but they will be out of phase and therefore the dipole radiation will cancel.

Quantum Relations for Gravitational Radiation

Gravitational radiation never has been observed and general relativity has not been quantized; therefore, the following statements are only educated theoretical guesses.

1) Gravitational radiation is quantized. The elementary quanta have been named gravitons.
2) The spin of a graviton is 2. This is basically because gravitational radiation can only be of the quadrupole type. Photons, being dipole radiation, have a spin of 1.
3) The velocity of a graviton is the same as the velocity of a photon and is related to the frequency and wavelength in the same way:

$$c = f\lambda = 3 \times 10^8 \text{ m/sec.}$$

4) The energy and momentum of a graviton depends upon Planck's constant in the same way as does a photon:

$$E = hf$$
$$p = h/\lambda.$$

Gravitational Radiation from a Spinning Dumbbell

The simplest quadrupole mass source for the calculation of gravitational-radiation energy emission is two equal masses rotating about their center of mass.

We first calculate the mass quadrupole moment with respect to a fixed co-ordinate system. Let us assume that

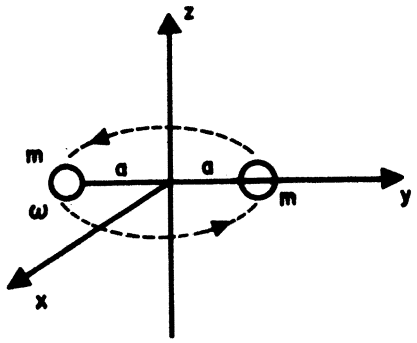

Fig. 3—Spinning dumbbell.

the spin axis is in the z-direction (Fig. 3), then

$$\begin{aligned} Q_{11} &= \int_V \mu(3x^2 - r^2)dV \\ &= 2m(2a^2\cos^2\omega t - a^2\sin^2\omega t) \\ Q_{21} = Q_{12} &= 3\int_V \mu xy dV = 6ma^2\cos\omega t\sin\omega t \\ Q_{22} &= \int_V \mu(3y^2 - r^2)dV \\ &= 2m(2a^2\sin^2\omega t - a^2\cos^2\omega t) \\ Q_{33} &= -2m(a^2\sin^2\omega t + a^2\cos^2\omega t) \\ &= -2ma^2, \end{aligned} \tag{23}$$

and all other quadrupole moments are zero.

Secondly, we calculate the third derivative, and we are left with only the x, y components,

[12] L. Landau and E. Lifshitz, "The Classical Theory of Fields," Addison-Wesley Publishing Company, Inc., Reading, Mass.; 1951.

$$\dddot{Q}_{11} = -\dddot{Q}_{22} = 24(2ma^2)\omega^3 \sin\omega t \cos\omega t$$
$$\dddot{Q}_{12} = \dddot{Q}_{21} = -12(2ma^2)\omega^3(\cos^2\omega t - \sin^2\omega t), \quad (24)$$

since the first derivative of the z component is zero.

The power radiated is then

$$-\frac{dE}{dt} = \frac{G}{45c^5}[\dddot{Q}_{11}^2 + \dddot{Q}_{12}^2 + \dddot{Q}_{22}^2 + \dddot{Q}_{21}^2] = \frac{8GI^2\omega^6}{5c^5}, \quad (25)$$

where $I = 2\ ma^2$ is the moment of inertia of the source.

Now we must substitute numbers into (25). For a 1-meter dumbbell weighing 1 metric ton and spinning at about 10,000 rpm, conditions which no known material can withstand, the power radiated is only

$$-\frac{dE}{dt} = 4.5 \times 10^{-33} \text{ watts}. \quad (26)$$

With numbers such as these, it is not surprising that this field has been of little interest to experimentalists.

From the exponents of I^2 and ω^6 in (25), it seems desirable, at first glance, to work with a higher rotational speed, even if it means that less mass could be used. However, we would find, when the strength of the material is considered, that it is more advantageous to lower the rotational speed and to use a greater mass. The ultimate in this procedure is represented by a rotating double star system. The rotational speed could be on the order of 1 month $\omega = 10^{-6}$ rad/sec, and the moment of inertia would then be roughly $I = ma^2 = 10^{30}\ \text{kg} \times 10^{12}\ \text{m}^2$; thus, the power radiated from a binary star is about 10^7 watts. This appears to be a large amount of power, but it would take 10^{10} billion years for the system to damp out as a result of radiation losses.

Space Curvature

The previous analogies have shown us how to calculate the *forces* exerted on a body as a result of the gravitational scalar and vector potential. However, if we are interested in the path of the particle under the influence of the forces, we encounter nonlinearities. It should be noted that we have not yet calculated the usual general relativistic effects, such as the precession of the perihelion of Mercury or the bending of light rays. This is because these effects are not a result of the gravitational-field components which have analogies in electromagnetic theory. Also, when the precession of a spinning satellite was calculated using the electromagnetic analogy, we obtained a result which was smaller than the space curvature effect which we neglected.

It will probably be true in most cases where the primary mass is large and the motions of the bodies have small velocities and accelerations that the only observable perturbations will be a result of the spatial tensor components of the gravitational field, which have no analogy in electromagnetic theory. The closest analogy which may be made is that these components of the gravitational field can be represented by assuming that the mass of the object somehow causes the space to be slightly curved. Then the concept of motion in a flat space under the influence of tensor forces can be replaced by the concept of forceless motion in a curved space.

In classical theory, once we have calculated the forces, we can solve the equation of motion:

$$\frac{d\boldsymbol{p}}{dt} = m\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}) \quad (27)$$

for $\boldsymbol{x} = \boldsymbol{x}(t)$. This equation is valid in flat space; however, if we want to include these other gravitational forces by assuming a curved space, then the ordinary rules of differentiation will not hold and we must use covarient differentiation. The usual equation of motion is really just the flat space approximation of the curvilinear equation of motion:

$$\frac{D_c p_a}{Dt} = \frac{dp_a}{dt} - \frac{1}{2}\sum_{b=1}^{3}\sum_{c=1}^{3}\left(\frac{\partial g_{bc}}{\partial x^a}\right)v^b p^c = F_a$$
$$a = 1, 2, 3, \quad (28)$$

where g_{bc} is the three-dimensional metric tensor, and the forces due to the gravitational vector and scalar potential are contained in $\boldsymbol{F}$, v^b is the velocity in the b direction and p^c is the momentum in the c direction.

This procedure should not be too unusual since we know that ordinary rules of differentiation hold only for flat cartesian coordinate systems. In curved coordinate systems, such as cylindrical or spherical systems, we have to use more general rules of differentiation. For instance, the general equation for the divergence of a vector is

$$\nabla\cdot\boldsymbol{A} = \frac{1}{(g_{11}g_{22}g_{33})^{1/2}}\left[\frac{\partial\sqrt{g_{22}g_{33}}\,A_1}{\partial x^1} + \frac{\partial\sqrt{g_{11}g_{33}}\,A_2}{\partial x^2} + \frac{\partial\sqrt{g_{11}g_{22}}\,A_3}{\partial x^3}\right] \quad (29)$$

if the metric tensor is

$$g_{ab} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}$$

And since the spatial metric tensor for cartesian coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$ is

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we get the familiar relation

$$\nabla\cdot\boldsymbol{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}. \quad (30)$$

For a spherical coordinate system, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, however, the spatial metric tensor is not constant, but

depends upon the position in the space,

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix},$$

and we find that calculating the divergence in this curved space is no longer a simple procedure:

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2}\frac{\partial(r^2 A_1)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_2)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}. \quad (31)$$

We also encounter similar problems in simple Newtonian mechanics when spherical or cylindrical coordinate systems are used. For instance, if we want to calculate the motion of a satellite, the equations of motion in cartesian coordinates (let $x^3 = z = 0$) are

$$m\ddot{x} = -\frac{GMm}{(x^2+y^2)^{3/2}}x$$

$$m\ddot{y} = -\frac{GMm}{(x^2+y^2)^{3/2}}y, \quad (32)$$

and in spherical coordinates (let $x^2 = \theta = \pi/2$) they are

$$m\ddot{r} - \frac{m}{2}\frac{\partial g_{33}}{\partial r}\dot{\phi}^2 = m\ddot{r} - mr\dot{\phi}^2 = -\frac{GMm}{r^2} \quad (33)$$

$$\frac{d}{dt}(r^2\dot{\phi}) = 0. \quad (34)$$

It can be seen that we have a term in (33) introduced by the metric tensor of the coordinate system; this is the familiar centrifugal "force." Thus, we can say either that the centrifugal "force" is a real force and that the coordinate space is flat and then use the ordinary equation of motion,

$$m\ddot{r} = \boldsymbol{F} = \boldsymbol{F}_{\text{centrifugal}} + \boldsymbol{F}_{\text{gravitational}},$$

or we can say that there is only one force, that due to the gravitational attraction. Since we are working in a curved coordinate space, however, we must use the curvilinear equation of motion:

$$m\ddot{r} - mr\dot{\phi}^2 = \boldsymbol{F} = \boldsymbol{F}_{\text{gravitational}}.$$

Metric Tensor Outside a Massive Body

We have shown that a curved coordinate system can be interpreted as a force. Now we shall attempt to explain how the remaining components of the gravitational force can be interpreted as a curved space. In practically every imaginable case, the spatial components of the metric tensor will be determined by the scalar potential and special relativity. For the sake of simplicity, we shall continue to ignore special relativity (although it should be taken into account for quantitatively correct results).

The metric tensor for a region of space with a scalar potential χ is given by the Schwarzschild metric tensor, which is a slight modification of the spherical coordinate metric tensor

$$g_{ab} = \begin{pmatrix} \left(1+\frac{2\chi}{c^2}\right)^{-1} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \quad (35)$$

In all cases of experimental interest $1 >> 2\chi/c^2$, and therefore,

$$\left(1+\frac{2\chi}{c^2}\right)^{\pm n} \approx \left(1 \mp n\frac{2\chi}{c^2}\right) \quad (36)$$

The metric tensor can be written for the common coordinate systems as follows: for spherical coordinate systems, $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$,

$$g_{ab} = \begin{pmatrix} 1-\frac{2\chi}{c^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}; \quad (36)$$

for cartesian coordinate systems, $x^1 = x$, $x^2 = y$, $x^3 = z$,

$$g_{ab} = \begin{pmatrix} 1-\frac{2\chi}{c^2} & 0 & 0 \\ 0 & 1-\frac{2\chi}{c^2} & 0 \\ 0 & 0 & 1-\frac{2\chi}{c^2} \end{pmatrix}; \quad (37)$$

and for cylindrical coordinate systems, $x^1 = \rho$, $x^2 = \phi$, $x^3 = z$,

$$g_{ab} = \begin{pmatrix} 1-\frac{2\chi}{c^2} & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1-\frac{2\chi}{c^2} \end{pmatrix}. \quad (38)$$

Relativistic Equation of Motion

From the Appendix, we can now write the proper equation of motion which uses this space curvature analogy:

$$\sum_{b=1}^{3}\frac{1}{\Gamma}\frac{d}{dt}(\Gamma g_{ab}\dot{x}^b) = \frac{1}{2}\sum_{b=1}^{3}\sum_{c=1}^{3}\frac{\partial g_{bc}}{\partial x^a}\dot{x}^b\dot{x}^c - \frac{\partial\chi}{\partial x^a} - \frac{\partial K_a}{\partial t} + \sum_{b=1}^{3}\left[\frac{\partial K_b}{\partial x^a} - \frac{\partial K_a}{\partial x^b}\right]v^b, \quad (39)$$

where g_{ab} is one of the metric tensors taken from the previous section, $a = 1, 2, 3$, and

$$\Gamma = \left\{\left[\left(1+\frac{2\chi}{c^2}\right)^{1/2} - \frac{K_a v^a}{c^2}\right]^2 - \frac{v^2}{c^2}\right\}^{-1/2}$$

$$\approx \left\{1+\frac{2\chi}{c^2} - \frac{v^2}{c^2}\right\}^{-1/2}$$

comes from the special and general relativistic corrections to the mass. Note that if $\chi = 0 = K_a$, then we get just the usual special relativistic correction to the mass:

$$m = m_0\Gamma = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{40}$$

Now that we have the equation of motion, all that is necessary is to calculate the gravitational scalar and vector potential by analogy to electromagnetism, calculate Γ and g_{ab} and put them in the equation, and then solve the equation by the method of successive approximations.

Measurements in a Curved Space

We should be familiar with the problems of operating in a curved space, since we live on a two-dimensional curved space—the earth. A straight line on the earth is the great circle route, because it is the shortest distance between two points in that two-dimensional space. If we make a triangle with these "straight lines," we find that the sum of the angles can range from π to 5π, depending upon the size of the triangle.

A more fundamental experiment is the parallel translation of a vector. Suppose we are on a flat surface and we place a test vector at one corner of a triangle. Then very carefully keeping the angle between the vector and the appropriate side of the triangle constant, we traverse the perimeter of the triangle and return to the starting point (see Fig. 4). The test vector obviously returns to the starting point with the initial orientation. Now try this same simple experiment with a vector moving about on a spherical triangle, as shown in Fig. 5. It will be obvious even to a flatlander inhabiting the surface of the sphere that the vector has rotated through an angle α as a result of its parallel translation around a closed path in the two-dimensional curved space. The size of the angle α will depend upon the amount of curvature of the space and the size of the triangle.

Effect of Space Curvature on a Satellite

It was pointed out in the section on satellites of spinning bodies that a proper solution of Einstein's equations for the precession of the spin axis of a satellite resulted in two terms. The smaller was a result of the gravitational equivalent of the magnetic field. The larger term was produced by the other components of the gravitational field. The precession of the spin axis of a satellite in a curved space cannot be calculated easily, but by using an analogy to two-dimensional curved space, we can see the qualitative reason for this precession.

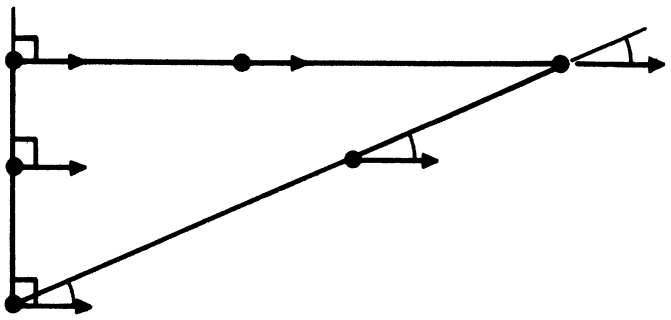

Fig. 4—Parallel translation in flat space.

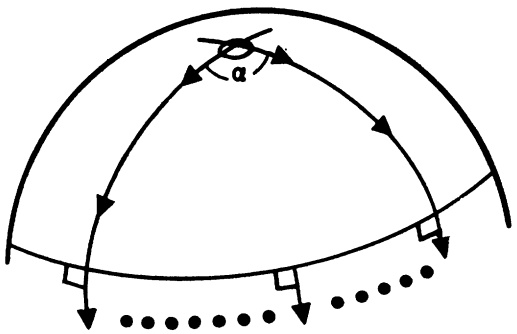

Fig. 5—Parallel translation on a sphere.

We said that these remaining components of the gravitational field of a mass in three-dimensional space could be examined by assuming a slightly curved space with a metric

$$g_{ab} = \begin{pmatrix} \dfrac{1}{1-\dfrac{2GM}{c^2 r}} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \tag{41}$$

where we have taken the scalar potential for a spherical mass $\chi = -GM/r$. Notice that if we are far away from the perturbing mass, then our space is flat again.

Now suppose that we look at a flatlander living on the edge of a two-dimensional massive circle in a two-dimensional flat universe (Fig. 6). If we assume that Einstein's law of gravitation holds in this two-dimensional space, then there will be a tensor gravitational field with which to contend. It will be easier for the flatlanders to ignore these tensor fields and assume that the massive circle warps the local area of his flat space and modifies his two-dimensional metric tensor slightly:

$$g_{ab} = \begin{pmatrix} \dfrac{1}{1-\dfrac{2GM}{c^2 r}} & 0 \\ 0 & r^2 \end{pmatrix}.$$

If a two-dimensional rocket is sent up and a two-dimensional satellite is put into orbit, then the flatlanders will notice that an axis of the satellite will rotate because of the local curvature of the space (Fig. 7). If the orbit is sufficiently distant, the extra contribution to the metric

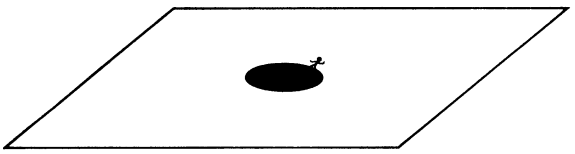

Fig. 6—Flat two-dimensional space.

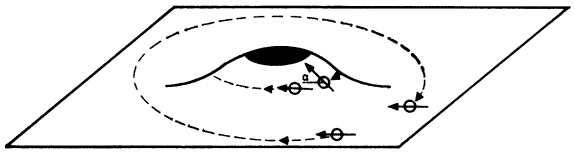

Fig. 7—Two-dimensional locally curved space.

tensor will be small and the satellite will travel through flat space with essentially no rotation.

If this picture is extended to three dimensions and the proper calculations are performed, we obtain the precession due to space curvature:

$$\Omega = \frac{3GM}{c^2 r}\,\omega_{\text{orbit}}$$

Calculation of the Orbit of Mercury

We start from the curvilinear equation of motion, where the only force is that due to the gradient of the scalar gravitational potential:

$$\sum_{b=1}^{3}\frac{1}{\Gamma}\frac{d}{dt}\left(\Gamma g_{ab}\frac{dx^b}{dt}\right) = \frac{1}{2}\sum_{b=1}^{3}\sum_{c=1}^{3}\frac{\partial g_{bc}}{\partial x^a}v^b v^c - \frac{\partial \chi}{\partial x^a} \tag{42}$$

where $a = 1, 2, 3$, and

$$\Gamma = \left(1 + \frac{2\chi}{c^2} - \frac{v^2}{c^2}\right)^{-1/2}.$$

Keeping only the nonvanishing terms, the three equations are:

$$\begin{aligned}
&\frac{1}{\Gamma}\frac{d}{dt}(\Gamma g_{11}\dot{r}) = \frac{1}{2}\left[\frac{\partial g_{11}}{\partial r}\dot{r}^2 + \frac{\partial g_{22}}{\partial r}\dot{\theta}^2 + \frac{\partial g_{33}}{\partial r}\dot{\phi}^2\right] - \frac{\partial \chi}{\partial r}\\
&\frac{1}{\Gamma}\frac{d}{dt}(\Gamma g_{22}\dot{\theta}) = \frac{1}{2}\left[\frac{\partial g_{33}}{\partial \theta}\dot{\phi}^2\right]\\
&\frac{1}{\Gamma}\frac{d}{dt}(\Gamma g_{33}\dot{\phi}) = 0.
\end{aligned} \tag{43}$$

The scalar potential, due to the sun, and its gradient are

$$\chi = -\frac{GM}{r}, \qquad \frac{\partial \chi}{\partial r} = \frac{GM}{r^2}.$$

Now if we define our coordinate axes so that the plane of the planetary orbit is in the equatorial plane, then

$$\theta = \frac{\pi}{2}, \qquad \dot{\theta} = 0, \qquad \sin\theta = 1, \qquad \cos\theta = 0.$$

Under these conditions, the components of the metric tensor for this coordinate system become

$$g_{ab} = \begin{pmatrix} \dfrac{1}{\left(1 - \dfrac{2GM}{c^2 r}\right)} & 0 & 0\\ 0 & r^2 & 0\\ 0 & 0 & r^2 \end{pmatrix} \tag{44}$$

When (44) is substituted into (43), two equations result:

$$\frac{1}{\Gamma}\frac{d}{dt}(\Gamma g_{11}\dot{r}) = \frac{1}{2}\frac{\partial g_{11}}{\partial r}\dot{r}^2 + r\dot{\phi}^2 - \frac{GM}{r^2} \tag{45}$$

$$\frac{d}{dt}(\Gamma r^2\dot{\phi}) = 0. \tag{46}$$

Eq. (46) expresses the conservation of angular momentum:

$$\Gamma r^2\dot{\phi} = \text{constant} = \frac{l\Gamma}{m_0 g_{11}},$$

where the measured angular momentum l is not a strict constant because the denotation of a "radius vector" used in the definition has an unambiguous meaning only in a flat space. If we substitute for $\dot{\phi}$ in the first equation, drop terms of higher order in v/c, and simplify, we get

$$g_{11}\ddot{r} - \frac{l^2}{m_0^2 r^3 g_{11}^2} + \frac{GM}{r^2} = 0. \tag{47}$$

Our problem now is to solve this equation, which is usually done by successive approximations. We shall find the task much easier if we use the mathematical shortcut of letting $r = 1/\sigma$ and calculating σ as a function of ϕ rather than t:

$$\begin{aligned}
&\frac{dr}{d\sigma} = -\frac{1}{\sigma^2} \qquad \frac{d\phi}{dt} = \frac{l}{m_0 r^2 g_{11}} = \frac{l\sigma^2}{m_0 g_{11}}\\
&\ddot{r} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d\phi}{dt}\frac{d}{d\phi}\left(\frac{d\phi}{dt}\frac{dr}{d\sigma}\frac{d\sigma}{d\phi}\right)\\
&\quad = -\frac{l^2\sigma^2}{m_0^2 g_{11}^2}\frac{d^2\sigma}{d\phi^2}.
\end{aligned} \tag{48}$$

Substituting and rearranging, we get

$$\left(1 - \frac{2GM}{c^2}\sigma\right)\frac{d^2\sigma}{d\phi^2} + \left(1 - \frac{2GM}{c^2}\sigma\right)^2\sigma - \frac{GMm_0^2}{l^2} = 0. \tag{49}$$

The zeroth approximation neglects all but the two largest terms and leaves us with

$$\sigma_0 = \frac{GMm_0^2}{l^2} = \frac{1}{r_0},$$

which is the equation for a circular orbit of radius r_0.

The first, or Newtonian approximation, neglects the

changes due to the metric tensor:

$$\frac{d^2\sigma}{d\phi^2} + \sigma - \sigma_0 = 0.$$

This equation has the solution

$$\sigma_1 = \sigma_0(1 + e\cos\phi),$$

where e is the eccentricity of the elliptic orbit.

We then put $\sigma = \sigma_1 + \sigma_2$ in (49), and after cancelling out equal terms and neglecting small terms, we obtain

$$\frac{d^2\sigma_2}{d\phi^2} + \sigma_2 - \frac{4GM}{c^2}\sigma_0^2 - \frac{6GM}{c^2}\sigma_0^2 e\cos\phi - \frac{2GM}{c^2}\sigma_0^2 e^2\cos^2\phi = 0. \quad (50)$$

Because of the $\cos\phi$ term, we have an equation for a driven oscillator which leads to a continually increasing change of σ with ϕ. Retaining only this term, we find the solution

$$\sigma = \sigma_1 + \sigma_2 = \frac{1}{r_0}\left(1 + e\cos\phi + \frac{3GM}{c^2}e\phi\sin\phi\right) \approx \frac{1}{r_0}\left\{1 + e\cos\left[\phi\left(1 - \frac{3GM}{c^2 r_0}\right)\right]\right\}. \quad (51)$$

It can be seen from (51) that after one revolution, the Newtonian orbit will shift by an amount

$$\Delta\phi = 2\pi\frac{3GM}{c^2 r_0} = \frac{6\pi GM}{c^2 a(1 - e^2)}, \quad (52)$$

where a is the length of the major axis.

This result of Einstein's theory cleared up a bothersome problem in celestial mechanics. The orbit of Mercury is well known and the major axis shifts 5599.74 ± 0.41 sec of arc per century. The perturbations introduced by the other planets in the solar system cause most of this shift, but careful calculations over many years gave the result that the maximum shift due to the planetary perturbations should be 5557.18 ± 0.85 sec of arc per century, leaving a discrepancy of 42.56 ± 0.94 sec of arc per century.

Eq. (52) gives us 42.9 sec of arc per century. This close agreement is a very strong argument in favor of Einstein's equations. Other theories of gravitation, when applied to Mercury's orbit, give an incorrect value or even the wrong sign.

Conclusion

At present, efforts are being made in a number of projects to measure gravitational effects. In the most active of these projects, investigators are attempting to measure the red shift in the frequency of light as it leaves the earth's gravitational field. Cranshaw[13] and others at Harwell, England, and Pound and Rebka[14] at Cambridge have already made measurements using the Mössbauer effect. The results agree with the predictions of general relativity. The Hughes Aircraft Company, National Bureau of Standards, and MIT are working on accurate clocks of various types to put in satellites to measure both special and general relativistic effects. Other experiments to test general relativity using space vehicles have been covered by Benedikt.[15] Kerns[16] at Berkeley and H. E. Fiala at the Hughes Aircraft Company have both made proposals to measure the speed of gravitational interaction. Weber, Zipoy, Forward and Sinsky[17,18] at the University of Maryland are working on the problem of the generation and detection of gravitational radiation.

It is hoped that someone with a practical turn of mind will think of an experiment to detect the gravitational equivalent of the magnetic field. This paper was designed to permit a preliminary evaluation of such ideas.

It is interesting to note that a good electromagnetic autotransformer has almost 100 per cent efficiency in transferring the ac motion of the charges in the primary wire to the charges in the second wire. But the high efficiency is a result of the high velocity of interaction between the charges, the low losses in the wires, and the high permeability of iron. A calculation of the efficiency of a "graviprotation" autotransformer would have to take all these practical considerations into account.

Appendix

Reduction of Einstein's Equations to Show the Analogy to Electromagnetism

The justification for the approximate analogies presented in the main body of the paper is presented here. It is assumed that the reader is familiar with tensor formulation, the summation convention, and the elementary procedures necessary for the handling of Einstein's equation. The procedure for linearizing Einstein's equation is included in all texts on general relativity.[19–21] The calculation of the energy-momentum tensor for slowly moving masses may be found in Møller.[21] The identification of the scalar and vector potentials and the three-dimensional metric tensor and

[13] T. E. Cranshaw, *et al.*, "Measurement of the gravitational red shift using the Mössbauer effect," *Phys. Rev. Lett.*, vol. 4, pp. 163–164; February 15, 1960.

[14] R. V. Pound and G. A. Rebka, Jr., "Apparent weight of photons," *Phys. Rev. Lett.*, vol. 4, pp. 337–341; April 1, 1960.

[15] E. T. Benedikt, "Advances in the Astronautical Sciences," Plenum Press, New York, N. Y., vol. 5, pp. 98–115; 1960.

[16] Q. A. Kerns, "Proposed laboratory measurement of the propagation velocity of gravitational interaction," Lawrence Rad. Lab., Univ. of California, Livermore, Tech. Rept. No. UCRL-8438; December, 1958.

[17] J. Weber, "Detection and generation of gravitational waves," *Phys. Rev.*, vol. 117, pp. 306–313; January 1, 1960.

[18] R. L. Forward, *et al.*, "Upper limit for interstellar millicycle gravitational radiation," *Nature*, vol. 189, p. 473; February 11, 1961.

[19] Landau and Lifshitz, *op. cit.*, p. 324 ff.

[20] A. S. Eddington, "The Mathematical Theory of Relativity," Cambridge University Press, New York, N. Y., p. 128 ff.; 1924.

[21] Møller, *op. cit.*, p. 313 ff.

their use in the equation of motion is also presented by Møller.[22]

We start with Einstein's equations:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta}.$$

Our first assumption will be that all velocities are small, so that special relativity can be neglected, and that all gravitational effects are weak. Then the metric tensor can be approximated by

$$g_{\alpha\beta} \approx \delta_{\alpha\beta} + h_{\alpha\beta},$$

where $\delta_{\alpha\beta}$ is the flat space metric and $h_{\alpha\beta}$ are the perturbations introduced by the masses. Using this form of the metric, the Ricci tensor and the curvature scalar can be calculated from the Christhoffel symbols:

$$R_{\alpha\beta} \approx -\tfrac{1}{2} h_{\alpha\beta}{}^{,\gamma}{}_{,\gamma} = -\tfrac{1}{2} \Box h_{\alpha\beta}$$

$$R = g^{\alpha\beta} R_{\alpha\beta} \approx -\tfrac{1}{2} \Box \delta^{\alpha\beta} h_{\alpha\beta} = -\tfrac{1}{2} \Box h, \qquad (53)$$

where in obtaining (53) we chose our coordinate system so that

$$[h_\alpha{}^\beta - \tfrac{1}{2}\delta_\alpha{}^\beta h]_{,\beta} = 0.$$

If we substitute the Ricci tensor and the curvature scalar into Einstein's equations, we obtain

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = -\tfrac{1}{2} \Box h_{\alpha\beta} + \tfrac{1}{4}\delta_{\alpha\beta} \Box h = \frac{8\pi G}{c^4} T_{\alpha\beta}. \qquad (54)$$

We now define the gravitational potential as

$$\phi_{\alpha\beta} = h_{\alpha\beta} - \tfrac{1}{2}\delta_{\alpha\beta} h;$$

substituting and rearranging, we get

$$\Box \phi_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}.$$

If we write out the d'Alembertian operator, we have

$$\Delta\phi_{\alpha\beta} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}. \qquad (55)$$

This is the basic equation upon which the analogies are based.

Scalar Potential

In the first approximation, we assume that all quantities are not varying with time and that the masses have low velocities or rotations. Then the time derivative of the gravitational potential is zero and all the components of the energy-momentum tensor are zero except

$$T_{oo} = \mu c^2.$$

Eqs. (55) reduce to

$$\Delta\phi_{oo} = -\frac{16\pi G}{c^2}\mu. \qquad (56)$$

This is essentially Poisson's equation, which has the solution

$$\phi_{oo} = +\frac{4G}{c^2}\int_V \frac{\mu}{r}\, dV.$$

If we define a gravitational capacitivity of the vacuum as

$$\gamma = (4\pi G)^{-1},$$

we get

$$-\frac{c^2\phi_{oo}}{4} = -\frac{1}{4\pi\gamma}\int_V \frac{\mu}{r}\, dV. \qquad (57)$$

By comparing (57) with the scalar potential of an electric charge density

$$\phi = +\frac{1}{4\pi\epsilon}\int_V \frac{\rho}{r}\, dV,$$

we see that we can construct the well-known gravitational analog to the scalar potential:

$$\chi = -\frac{c^2\phi_{oo}}{4} = -\frac{c^2(g_{oo}+1)}{2}.$$

Space Curvature

This first approximation (56) also determines the spatial metric. The existence of the component ϕ_{oo} results in an interval of the form

$$ds^2 = \left(1 - \frac{2\chi}{c^2}\right)(dx^2 + dy^2 + dz^2) - \left(1 + \frac{2}{c^2}\chi\right)c^2 dt^2. \qquad (58)$$

Thus the three-dimensional spatial metric will be of the form

$$g_{ab} = \begin{bmatrix} 1 - \frac{2\chi}{c^2} & 0 & 0 \\ 0 & 1 - \frac{2\chi}{c^2} & 0 \\ 0 & 0 & 1 - \frac{2\chi}{c^2} \end{bmatrix}.$$

In higher approximations that will be considered later, the additional terms in the spatial metric will be smaller than $2\chi/c^2$ by the order of $(v/c)^2$, and since we assume velocities much smaller than the speed of light, they will be of little experimental interest.

Vector Potential

In the next higher approximation, we still assume that the potential is not varying with time, but that the masses involved have appreciable velocity or rota-

[22] *Ibid.*, p. 246 ff. and p. 288 ff.

tion. Then the energy-momentum tensor will have the components

$$T_{oo} = \mu c^2 \left(\text{zero order in } \frac{v}{c}\right)$$

and

$$T_{ao} = -\mu c^2 \left(\frac{v_a}{c}\right) \left(\text{first order in } \frac{v}{c}\right).$$

We then have four equations remaining: one gives us the scalar potential obtained previously, and the other three are

$$\Delta\phi_{ao} = +\frac{16\pi G}{c^3}\mu v_a.$$

These equations have the solution

$$\phi_{ao} = -\frac{4G}{c^3}\int_V \frac{\mu v_a}{r}\,dV.$$

If we define a gravitational permeability of space by

$$\eta = \frac{16\pi G}{c^2},$$

then we can substitute and rearrange to get

$$c\phi_{ao} = -\frac{\eta}{4\pi}\int_V \frac{\mu v_a}{r}\,dV.$$

Thus we can identify a mass density flow $\boldsymbol{p}=\mu\boldsymbol{v}$ and a gravitational equivalent of the vector potential whose components are the three components of

$$K_a = c\phi_{ao} = cg_{ao},$$

and thereby arrive at the isomorphism of the equations

$$\boldsymbol{K} = -\frac{\eta}{4\pi}\int_V \frac{\boldsymbol{p}}{r}\,dV \qquad \boldsymbol{A} = \frac{\mu}{4\pi}\int_V \frac{\boldsymbol{j}}{r}\,dV. \tag{59}$$

Gravitational Radiation

Let us return to the basic equation (55):

$$\Delta\phi_{\alpha\beta} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi_{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\alpha\beta}. \tag{60}$$

As it stands, (60) is a wave equation for the gravitational potential $\phi_{\alpha\beta}$. The velocity of propagation is the same as the velocity of light. It is this equation, which results from the linearization of Einstein's equation, that gives credence to the statement that gravitational radiation exists. The solution of the wave equation and the calculation of the radiated power are straightforward and are carried out in Landau and Lifshitz.[19]

Equation of Motion

In the four-dimensional equation of motion (61), the gravitational effects are entirely in the metric tensor. The only forces explicitly stated are nongravitational forces. This four-dimensional equation of motion can be broken down and arranged so that it is a three-dimensional curvilinear spatial equation of motion. The gravitational effects resulting from the temporal components of the metric tensor are represented as forces due to a gravitational scalar and a gravitational vector potential. The spatial components of the metric tensor are used as the three-dimensional metric tensor.

The general equation of motion for a particle with only gravitational forces acting is given by Møller[23] as

$$\frac{dP_\alpha}{d\tau} - \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial x^\alpha}U^\beta P^\gamma = 0, \tag{61}$$

where

$$P_\alpha = m_0 U_\alpha = m_0 g_{\alpha\beta}U^\beta = m_0 g_{\alpha\beta}\Gamma\frac{dx^\beta}{dt}$$

and

$$\Gamma = \frac{dt}{d\tau} = \left\{\left[\left(1+\frac{2\chi}{c^2}\right)^{1/2} - \frac{K_a v^a}{c^2}\right]^2 - \frac{v^2}{c^2}\right\}^{-1/2}$$

The $\alpha=0$ equation gives us the conservation of mass-energy and the other three equations are

$$\begin{aligned}\Gamma\frac{d}{dt}\left(m_0\Gamma g_{ab}\frac{dx^b}{dt}\right) &+ \Gamma\frac{\partial}{\partial t}\left(m_0\Gamma g_{ao}\frac{dx^o}{dt}\right)\\ &+ \Gamma\frac{dx^b}{dt}\frac{\partial}{\partial x^b}\left(m_0\Gamma g_{ao}\frac{dx^o}{dt}\right)\\ &- \frac{1}{2}\frac{\partial g_{oo}}{\partial x^a}m_0\Gamma^2\left(\frac{dx^o}{dt}\right)^2 - \frac{\partial g_{bo}}{\partial x^a}m_0\Gamma^2\frac{dx^o}{dt}\frac{dx^b}{dt}\\ &- \frac{1}{2}\frac{\partial g_{bc}}{\partial x^a}m_0\Gamma^2\frac{dx^b}{dt}\frac{dx^c}{dt} = 0.\end{aligned}$$

Dividing through by $m_0\Gamma^2$, letting $dx^o/dt=c$, and neglecting higher-order terms, we get

$$\begin{aligned}\frac{1}{\Gamma}\frac{d}{dt}\left(\Gamma g_{ab}\frac{dx^b}{dt}\right) &= \frac{1}{2}\frac{\partial g_{bc}}{\partial x^a}\frac{dx^c}{dt}\frac{dx^b}{dt} + \frac{c^2}{2}\frac{\partial g_{oo}}{\partial x^a}\\ &\quad - c\frac{\partial g_{ao}}{\partial t} + c\left(\frac{\partial g_{bo}}{\partial x^a} - \frac{\partial g_{ao}}{\partial x^b}\right)\frac{dx^b}{dt}.\end{aligned}$$

We then use our definition of the gravitational scalar and vector potential

$$K_a = cg_{oa}$$

$$v^a = \frac{dx^a}{dt}$$

$$\chi = -\frac{c^2}{2}(g_{oo}+1)$$

$$g_{oo} = -1 - \frac{2\chi}{c^2}$$

[23] Møller, *op. cit.*, p. 290.

to get

$$\frac{1}{\Gamma}\frac{d}{dt}\left(\Gamma g_{ab}\frac{dx^b}{dt}\right)=\frac{1}{2}\frac{\partial g_{bc}}{\partial x^a}v^b v^c-\frac{\partial \chi}{\partial x^a}-\frac{\partial K_a}{dt}+v^b\left[\frac{\partial K_b}{\partial x^a}-\frac{\partial K_a}{\partial x^b}\right]. \quad (62)$$

The left-hand side of (62) is the acceleration of the particle in a curvilinear coordinate system. The first term on the right gives the fictional forces due to the choice of the coordinate system, the most familiar examples being the coriolis force and the centrifugal force. The second term is one component of $\nabla\chi$, the gravitational static attraction; the third term is one component of $\partial \boldsymbol{K}/\partial t$, the gravitational induction effect; and the fourth term is one component of $v\times(\nabla\times\boldsymbol{K})$, the gravitational equivalent of the Lorentz force.

A Matched Amplifier Using Two Cascaded Esaki Diodes*

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Summary—**The purpose of this paper is to introduce a new type of circuit for matched amplification using negative resistance devices. This circuit consists of a quarter-wave transmission line section whose input and output are paralleled by negative conductances. The characteristics of such an amplifier are discussed, and an expression for its noise figure is derived. The development of a 30-Mc amplifier using two Esaki diodes is described. Experimental results are presented, including curves of the characteristics as a function of frequency. A gain of 8.9 db was measured with a noise figure of 4.3 db.**

Introduction

AMPLIFICATION by a single negative resistance element has several disadvantages. First, such an amplifier necessarily presents a mismatch to the source. Second, amplification and reflection of noise power radiated by the load tends to contribute to its noise figure. The noise figure of such an amplifier is given by (1).

$$F=1+\frac{\overline{|i|^2}}{4kT_SG_S}+\frac{G_LT_L}{G_ST_S}. \quad (1)$$

In this expression, $\overline{|i|^2}$ is the equivalent noise current squared per cycle produced by the negative resistance device, G_S and G_L are the source and load conductances, and T_S and T_L are the source and load temperatures. It may be seen that in order to achieve a low noise figure, the factor G_LT_L must approach zero, regardless of the device used. If G_L becomes small, however, it is necessary to adjust the source and device conductances extremely close to the point of oscillation in order to achieve any power gain.[1]

These effects may be eliminated by the use of nonreciprocal circuit elements, or a hybrid with two negative resistance elements.[2,3] At lower frequencies, nonreciprocal passive elements are difficult to produce, and hybrids do not present the wide frequency range resistive load necessary to stabilize Esaki diodes. This paper describes another method of obtaining amplification using two negative resistance elements. The amplifier described is matched to both the source and load, and achieves a low noise figure with equal source and load conductances. Experimental work was done at 30 Mc, permitting the use of lumped circuits. Although Esaki diodes were used, the circuit techniques should be equally applicable to parametric and other negative resistance devices.

Theory of Operation

The basic circuit which was used consists of two conductances G and a $\frac{1}{4}$ wave transmission line section of impedance Z, as shown in Fig. 1. A 1-ohm source and load are assumed to simplify computation, so that the factors G and Z are normalized variables in all equations. The circuit may be easily analyzed using the fourpole matrix notation

$$\begin{Bmatrix}E_{in}\\I_{in}\end{Bmatrix}=\begin{pmatrix}A & B\\C & D\end{pmatrix}\begin{Bmatrix}E_{out}\\I_{out}\end{Bmatrix}. \quad (2)$$

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[2] M. E. Hines, "High-frequency negative-resistance circuit principles for Esaki diode applications," *Bell Sys. Tech. J.*, vol. 39, pp. 485–488; May, 1960.

[3] L. U. Kibler, "Directional bridge parametric amplifier," PROC. IRE, vol. 47, pp. 583–584; April, 1959.