first passage percolation project

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1 preliminaries and notation

We define $\mathcal{E}$ the set of all edges of our grid (lets say $\mathbb{Z}^2$), τ some positive random variable, and $\{\tau_e\}_{e\in\mathcal{E}}$ a collection of positive i.i.d copies of τ (so $\tau_e \underset{\mathrm{d}}{=} \tau$). Define the passage time

$$T_n(\tau) = \inf_{\gamma} \sum_{e\in\gamma} \tau_e$$

where the infimum is over all the paths γ which goes from $(0,0)$ to $(0,n)$ (so the fastest we can get from the origin to $(0,n)$ if τ_e is the time it takes to cross the edge e).

We denote $\Gamma_{n,\tau}$ the minimal path in the above (with some deterministic law for the case of multiple minimal paths, while existence is well known) so we have $T_n(\tau) = \sum\limits_{e\in\Gamma_{n,\tau}} \tau_e$. The basic result in the study of first passage percolation is the existence of the limit

$$\mu(\tau) = \lim_{n\to\infty} \frac{T_n(\tau)}{n} \tag{1.1}$$

And here we are interested in what happens when we add to each τ_e some small number (this is called the shifted weights model), i.e we study the real valued function $\mu_\tau(t) := \mu(\tau + t)$ for some fixed τ (it is well defined as long as $\tau + t$ is positive a.s), which is known to be a concave function. In particular we will be interested in its relation to the asyimptotical size of $\Gamma_{n,\tau}$. For that we define

$$\lambda(\tau) = \frac{|\Gamma_{n,\tau}|}{n}$$

where while it is not known if the limit always exist (so λ may not be defined everywhere) we do know that the liminf and limsup of the above are between the right and left dirivative of μ_τ at 0 (which exists since its concave), in particular if μ_τ is differentiable at 0 then λ is well defined and equal to $\mu'_\tau(0)$, or in general if we define $\lambda_\tau(t) = \lambda(\tau + t)$ then $\lambda_\tau = \mu'_\tau$ whenever μ' exist (which is a.e since μ is concave).

2 the result

our main theorem is

Theorem 1 : Assume that λ_τ is continuous in some interval $(-r, r)$ than we have that it is constant in that interval and μ_τ is linear in it.

Note that (since if a concave function is differentiable in an interval the derivative is continuous there) this is equivalent to the statement that μ_τ is differentiable in an interval iff it is linear in it. This is surprising since we may expect a concave function to be differentiable in general, but some known result also suggest that we expect μ_τ to be strictly concave and in particular not linear (see Theorem 2.2 in 1)

Now our main theorem will follow from the existence of the (random) functional ϕ_t (for any $t \in \mathbb{R}$ satisfying $\tau + t > 0$ a.s) such that

$$\phi_t(x+t) = \mu_\tau(t) \tag{a}$$
$$\phi_t t = t\lambda_\tau(t) \tag{b}$$
$$\forall t, s \quad \phi_t(x+t) \leq \phi_s(x+t) \tag{c}$$

Where here and in what follows $x+t$ is the function $x \to x+t$, t is the constant t function (and in general we will always denote the variable of our function with x, everything else will be a parameter), and the second and third condition obviously only need to hold when $\lambda_\tau(t), \phi_t, \phi_s$ are well defined.

This construction is the only non trivial part of the proof, and as such it is of course not mine, but rater the result of a (great) work by Erik Bates (see section 2.2 in 2 for more), expect we work with functional instead of measures (but this is obviously equivalent).
To summarize the idea is that for any n we can define the functional

$$\phi_{n,t}(f) = \sum_{e \in \Gamma_{n,\tau+t}} f(\tau_e)$$

and then using compactness argument to get that $\frac{\phi_{n,t}}{n}$ has some weak * convergent subsequunce, and it's limit is our desired functional ϕ_t.
Note that by definition of weak * convergence

$$\phi_t(x+t) = \lim_{k\to\infty} \frac{1}{n_k} \sum_{e \in \Gamma_{n_k,\tau+t}} \tau_e + t = \lim_{k\to\infty} \frac{T_{n_k}(\tau+t)}{n_k} = \mu_\tau(t)$$
$$\phi_t(t) = t\phi_f(1) = t \lim_{k\to\infty} \frac{1}{n_k} \sum_{e \in \Gamma_{n_k,\tau+t}} 1 = t \lim_{k\to\infty} \frac{|\Gamma_{n_k,\tau+t}|}{n_k} = t\lambda_\tau(t)$$

and we clearly have that $\phi_t(x+t) \le \phi_s(x+t)$ for all t, s by definition of $\Gamma_{n,\tau+t}$ as the minimal path for $\tau + t$. So our functional dose have the desired properties.

Now first we claim that if $\lambda_\tau(t)$ is continuous in $(-r, r)$ than we have for all $c \in (-r, r)$

$$\phi_c(x) = \phi_0(x) = \mu_\tau(0) \tag{2.1}$$

Proof : using linearity of ϕ and its properties we have (for small $t > 0$)

$$\phi_{c+t}(x) + t\lambda_\tau(c+t) \underset{\text{b}}{=} \phi_{c+t}(x+t) \underset{\text{c}}{\le} \phi_c(x+t) \underset{\text{b}}{=} \phi_c(x) + t\lambda_\tau(c)$$
$$\phi_{c-t}(x) - t\lambda_\tau(c-t) \underset{\text{b}}{=} \phi_{c-t}(x-t) \underset{\text{c}}{\le} \phi_c(x-t) \underset{\text{b}}{=} \phi_c(x) - t\lambda_\tau(c)$$

so from the above we get

$$\lambda_\tau(c-t) - \lambda_\tau(c) \le \frac{\phi_c(x) - \phi_{c-t}(x)}{t} \underset{\text{c}}{\le} 0 \underset{\text{c}}{\le} \frac{\phi_{c+t}(x) - \phi_0(x)}{t} \le \lambda_\tau(c+t) - \lambda_\tau(c)$$

And since λ_τ is continuous at c we can take the limit $t \to 0$ to get that the function $h(t) = \phi_t(x)$ is differentiable at c and its derivative there is $h'(c) = 0$. Implying that $h(c) = \phi_c(x)$ is indeed constant in the interval. an interesting corollary form the above is that for any $t \in (-r, r)$ we have

$$\mu_\tau(t) \underset{\text{a}}{=} \phi_t(x+t) = \phi_t(x) + \phi_t(t) \underset{2.1}{=} \phi_0(x) + \phi_t(t) \underset{\text{a,b}}{=} \mu_\tau(0) + t\lambda_\tau(t)$$

From which we get $\frac{\mu_\tau(t) - \mu_\tau(0)}{t} = \lambda_\tau(t)$, and since we only need continuity of λ_τ and by definition $\lambda_{\tau+c}(t) = \lambda_\tau(c+t)$ (and the same for μ_τ) we can conclude

$$\frac{\mu_\tau(c+t) - \mu_\tau(c)}{t} = \frac{\mu_{\tau+c}(t) - \mu_{\tau+c}(0)}{t} = \lambda_{\tau+c}(t) = \lambda_\tau(c+t) \tag{2.2}$$

Which by taking t to 0 gives $\mu'_\tau(c) = \lambda_\tau(c)$, and in particular μ_τ is differentiable in $(-r, r)$ and inserting this remark back to the above (2.2) with $c = 0$ gives $\mu_\tau(t) = \mu_\tau(0) + t\mu'_\tau(t)$. Solving this simple ODE will indeed give us that μ_τ is linear and $\mu'_\tau = \lambda_x i$ is constant, as we claimed.

3 References

[1] Krishnan, A., F. Rassoul-Agha and T. Seppalainen. "Geodesic length and shifted weights in first-passage percolation." (2021).

[2] Bates, E.. "Empirical Measures, Geodesic Lengths, and a Variational Formula in First-Passage Percolation." (2020).