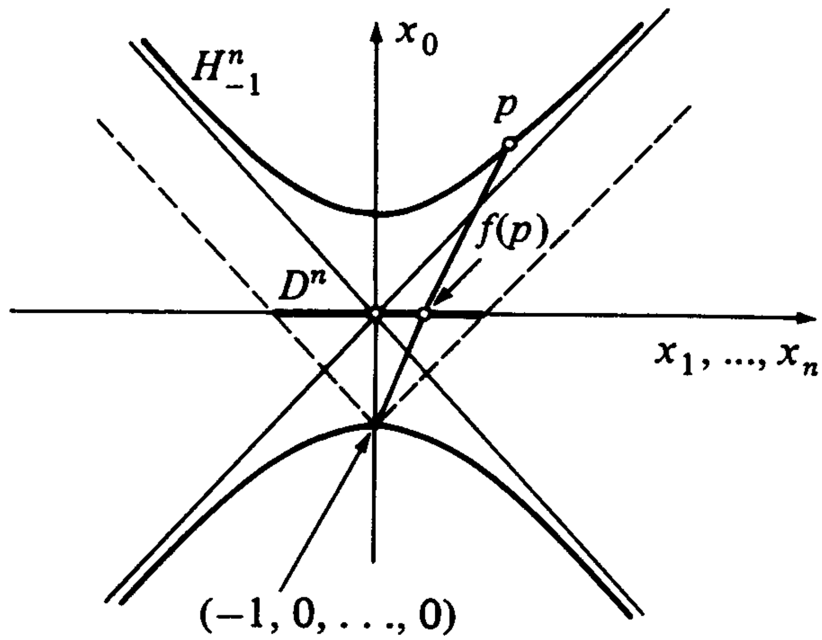

Figure 3

b) Show that:

$$- (dx_o)^2 + (dx_1)^2 + \cdots + (dx_n)^2$$
$$= \frac{4\{(du_1)^2 + \cdots + (du_n)^2\}}{(1 - \sum_\alpha u_\alpha^2)^2}.$$

Conclude that $f^{-1}: D^n \to H^n_{-1}$ induces on D the metric $g_{ij} = \frac{4\delta_{ij}}{(1-\Sigma_\alpha u_\alpha^2)^2}$. Therefore, D^n with the metric g_{ij} has constant curvature -1 (cf. Exercise 1(c)).

c) Show that the images by f of the non-empty intersections of affine hyperplanes P of L^{n+1} with H^n_{-1} are intersections with D^n of spheres (or planes, when P passes through p_o) contained in the hyperplane $x_o = 0$. Conclude that the umbilic hypersurfaces of H^n_{-1} (Cf. Exercise 6) are of the form $P \cap H^n_{-1}$.

8. (*Riemannian submersions*). A differentiable mapping $f: \overline{M}^{n+k} \to M^n$ is called *submersion* if f is surjective, and for all $\overline{p} \in \overline{M}$, $df_{\overline{p}}: T_{\overline{p}}\overline{M} \to T_{f(p)}M$ has rank n. In this case, for all $p \in M$, the *fiber* $f^{-1}(p) = F_p$ is a submanifold of $\overline{M}$ and a tangent vector of $\overline{M}$, tangent to some F_p, $p \in M$, is called a *vertical vector* of the submersion. If, in addition, $\overline{M}$

and M have Riemannian metrics, the submersion f is said to be *Riemannian* if, for all $p \in \overline{M}$, $df_p: T_p\overline{M} \to T_{f(p)}M$ preserves lengths of vectors orthogonal to F_p. Show that:

a) If $M_1 \times M_2$ is the Riemannian product, then the natural projections $\pi_i: M_1 \times M_2 \to M_i$, $i = 1, 2$, are Riemannian submersions.

b) If the tangent bundle TM is given the Riemannian metric as in Exercise 2 of Chap. 3, then the projection $\pi: TM \to M$ is a Riemannian submersion.

9. (*Connection of a Riemannian submersion*). Let $f: \overline{M} \to M$ be a Riemannian submersion. A vector $\overline{x} \in T_{\overline{p}}\overline{M}$ is *horizontal* if it is orthogonal to the fiber. The tangent space $T_{\overline{p}}\overline{M}$ then admits a decomposition $T_{\overline{p}}\overline{M} = (T_{\overline{p}}\overline{M})^h \oplus (T_{\overline{p}}\overline{M})^v$, where $(T_{\overline{p}}\overline{M})^h$ and $(T_{\overline{p}}\overline{M})^v$ denote the subspaces of horizontal and vertical vectors, respectively. If $X \in \mathcal{X}(M)$, the *horizontal lift* $\overline{X}$ of X is the horizontal field defined by $df_{\overline{p}}(\overline{X}(\overline{p})) = X(f(p))$.

a) Show that $\overline{X}$ is differentiable.

b) Let ∇ and $\overline{\nabla}$ be the Riemannian connections of M and $\overline{M}$ respectively. Show that

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \overline{(\nabla_X Y)} + \frac{1}{2}[\overline{X}, \overline{Y}]^v, \qquad X, Y \in \mathcal{X}(M),$$

where Z^v is the vertical component of Z.

c) $[\overline{X}, \overline{Y}]^v(\overline{p})$ depends only on $\overline{X}(\overline{p})$ and $\overline{Y}(\overline{p})$.

Hint for (b): Let $X, Y, Z \in \mathcal{X}(M)$. Let $T \in \mathcal{X}(\overline{M})$ be a vertical field. Observe that:

$$\left\langle \overline{X}, T \right\rangle = \left\langle \overline{Y}, T \right\rangle = \left\langle \overline{Z}, T \right\rangle = 0, \quad \overline{X}\left\langle \overline{Y}, \overline{Z} \right\rangle = X\langle Y, Z\rangle,$$

$$df[\overline{X}, T] = 0, \quad [X, Y] = [df\overline{X}, df\overline{Y}] = df[\overline{X}, \overline{Y}] \quad \text{and}$$

$$T\left\langle \overline{X}, \overline{Y} \right\rangle = 0.$$

Conclude that $\left\langle [\overline{X}, \overline{Y}], \overline{Z} \right\rangle = \langle [X, Y], Z\rangle$, $\left\langle [\overline{X}, T], \overline{Y} \right\rangle = 0$ and use the formula for the Riemannian connection as a function of the metric to obtain

$$\left\langle \overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z} \right\rangle = \langle \nabla_X Y, Z\rangle, \quad 2\left\langle \overline{\nabla}_{\overline{X}}\overline{Y}, T \right\rangle = \left\langle T, [\overline{X}, \overline{Y}] \right\rangle,$$

which implies (b).
Hint for (c): Use the fact that

$$\left\langle [\overline{X},\overline{Y}], T\right\rangle = \left\langle \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X}, T\right\rangle.$$

10. (*Curvature of a Riemannian submersion*). Let $f\colon \overline{M} \to M$ be a Riemannian submersion. Let $X, Y, Z, W \in \mathcal{X}(M)$, $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ be their horizontal lifts, and let R and $\overline{R}$ be the curvature tensors of M and $\overline{M}$ respectively. Prove that:

$$(a) \left\langle \overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{W}\right\rangle = \langle R(X,Y)Z,W\rangle - \frac{1}{4}\left\langle [\overline{X},\overline{Z}]^v, [\overline{Y},\overline{W}]^v\right\rangle$$

$$+\frac{1}{4}\left\langle [\overline{Y},\overline{Z}]^v, [\overline{X},\overline{W}]^v\right\rangle - \frac{1}{2}\left\langle [\overline{Z},\overline{W}]^v, [\overline{X},\overline{Y}]^v\right\rangle.$$

b) $K(\sigma) = \overline{K}(\overline{\sigma}) + \frac{3}{4}\left|[\overline{X},\overline{Y}]^v\right|^2 \geq \overline{K}(\overline{\sigma})$,
where σ is the plane generated by the orthonormal vectors $X, Y \in \mathcal{X}(M)$ and $\overline{\sigma}$ is the plane generated by $\overline{X}$, $\overline{Y}$.
Hint for (a): We shall use the notation of Exercise 9. Observe that $\overline{X}\left\langle \overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\right\rangle = X\,\langle \nabla_Y Z, W\rangle$. Therefore

$$\left\langle \overline{\nabla}_{\overline{X}}\overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\right\rangle = \overline{X}\left\langle \overline{\nabla}_{\overline{Y}}\overline{Z},\overline{W}\right\rangle - \left\langle \overline{\nabla}_{\overline{Y}}\overline{Z}, \overline{\nabla}_{\overline{X}}\overline{W}\right\rangle$$

$$= \langle \nabla_X\nabla_Y Z, W\rangle - \frac{1}{4}\left\langle [\overline{Y},\overline{Z}]^v, [\overline{X},\overline{W}]^v\right\rangle.$$

On the other hand, if $T \in \mathcal{X}(\overline{M})$ is vertical,

$$\left\langle \overline{\nabla}_T\overline{X},\overline{Y}\right\rangle = \left\langle \overline{\nabla}_{\overline{X}}T,\overline{Y}\right\rangle + \left\langle [T,\overline{X}],\overline{Y}\right\rangle = -\left\langle T, \overline{\nabla}_{\overline{X}}\overline{Y}\right\rangle.$$

Therefore,

$$\left\langle \overline{\nabla}_{[\overline{X},\overline{Y}]}\overline{Z},\overline{W}\right\rangle = \left\langle \overline{\nabla}_{[\overline{X},\overline{Y}]^h}\overline{Z},\overline{W}\right\rangle + \left\langle \overline{\nabla}_{[\overline{X},\overline{Y}]^v}\overline{Z},\overline{W}\right\rangle$$
$$= \left\langle \overline{\nabla}_{[X,Y]}Z, W\right\rangle - \frac{1}{2}\left\langle [\overline{X},\overline{Y}]^v, [\overline{Z},\overline{W}]^v\right\rangle.$$

Putting the above together, we obtain (a).