7.25 A.C. GENERATOR

35. *With the help of a labelled diagram, explain the principle, construction and working of an a.c. generator. Derive the expression for the induced emf and current.*

A.C. Generator. *A generator or dynamo is a device which converts mechanical energy into electrical energy.* Actually, the name generator is a misnomer because it does not generate any energy. It just converts mechanical energy into electrical energy.

An ***a.c. generator*** is the one *which produces a current that alternates or changes its direction regularly after a fixed interval of time, i.e., it is a device which converts mechanical energy into alternating form of electrical energy.* The present form of a.c. generator is due to *Nikola Tesla,* the great Yugoslav scientist who built it in the year 1888.

Principle. The working of an a.c. generator is based on the principle of ***electromagnetic induction.*** *When a*

closed coil is rotated in a uniform magnetic field with its axis perpendicular to the magnetic field, the magnetic flux linked with the coil changes and an induced emf and hence a current is set up in it.

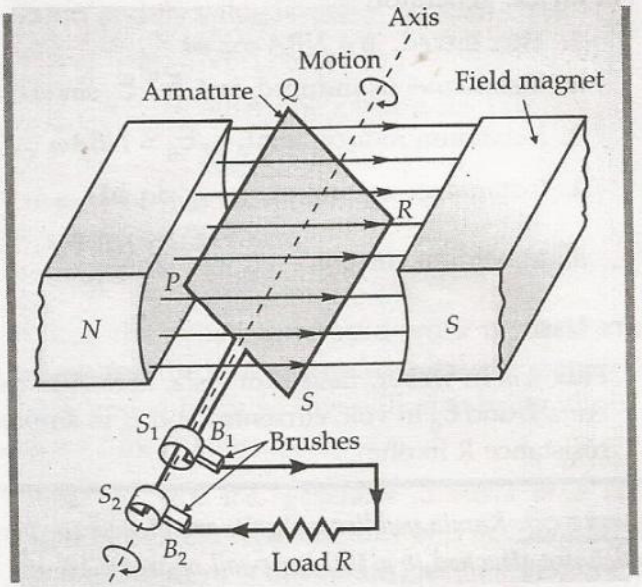


Fig. 7.50 A.C. Generator.

Construction. It essentially consists of the following main parts :

1. ***Field magnet.*** It is a permanent magnet of the horse shoe shape in a small dynamo (magneto) or it is a powerful electromagnet in a large dynamo. It produces a strong magnetic field in the region between its pole-pieces.

2. ***Armature.*** It consists of a rectangular coil *PQRS* having a large number of turns of insulated copper wire wound on a soft iron cylindrical core. The core is laminated one to avoid losses due to eddy currents. The soft iron core concentrates the lines of force to increase the flux density *B*. The armature can be rotated in the magnetic field of the field magnet about an axis perpendicular to field *B*.

3. ***Slip rings.*** The two ends of the armature coil are connected to two coaxial brass rings S_1 and S_2 called slip rings. The rings are rigidly fixed to same shaft which is used to rotate the coil. The slips are insulated from each other as well as from the shaft. As the armature coil rotates, the slip rings also rotate about the same axis of rotation.

4. ***Brushes.*** Two graphite or flexible metallic rods called brushes are lightly pressed against the two slip rings. The brushes B_1 and B_2 remain fixed in their positions and maintain sliding contacts with the rotatable slip rings S_1 and S_2 respectively. It is through these brushes that the current induced in the armature coil is fed to the external circuit by means of line wires.

5. ***Source of energy.*** The armature coil is rotated about its axis with the help of turbine or any other device connected to it. It is the rotational kinetic energy of the turbine which is ultimately converted into electrical energy by the a.c. generator.

Working. As the armature coil rotates, the magnetic flux linked with it changes and so an induced current flows through it. Suppose initially the coil *PQRS* be in the vertical position and it is rotated in the clockwise direction. The side *PQ* moves downward and *SR* moves upward. According to Fleming's right hand rule, the induced current flows from *Q* to *P* and from *S* to *R*. So during the first half rotation of the coil, the induced current flows in the direction *SRQP*, with brush B_1 acting as positive terminal and brush B_2 as negative terminal. During the second half-rotation, the side *PQ* moves upward and *SR* moves downward. The direction of induced current is reversed, *i.e.*, it flows along *PQRS*, so that the brush B_2 now functions as the positive terminal and brush B_1 as the negative terminal. Thus the direction of current in the external circuit is reversed after every half cycle. Hence alternating current is produced by the generator. Such a generator which generates alternating current is called an **a.c. generator** or an **alternator**.

Expression for induced emf. Let

N = number of turns in the coil

A = face area of each turn

B = magnitude of the magnetic field

θ = angle which normal to the coil makes with field $\vec{B}$ at any instant t

ω = the angular velocity with which coil rotates

Then the magnetic flux linked with the coil at any instant t will be

$$\phi = NBA \cos\theta = NBA \cos \omega t$$

By Faraday's flux rule, the induced emf is given by

$$\mathcal{E} = -\frac{d\phi}{dt} = -\frac{d}{dt}(NBA \cos \omega t) = NBA\, \omega \sin \omega t$$

or $$\mathcal{E} = \mathcal{E}_0 \sin \omega t$$

where $\mathcal{E}_0 = NBA\,\omega$. When a load of resistance R is connected across the terminals, a current I flows in the external circuit.

$$I = \frac{\mathcal{E}}{R} = \frac{\mathcal{E}_0 \sin \omega t}{R} = I_0 \sin \omega t$$

where $I_0 = \frac{\mathcal{E}_0}{R}$. Both current and voltage vary sinusoidally with time. The power dissipated in the load is supplied by the agent in rotating the coil in the magnetic field.

7.22 TRANSFORMER

32. *What is a transformer ? Explain the principle, construction, working and theory of a transformer. How is current affected in a transformer ? What are the various energy losses in a transformer ? How can they be reduced ?*

Transformer. *A transformer is an electrical device for converting an alternating current at low voltage into that at high voltage or vice versa.* If it increases the input voltage, it is called ***step up transformer*** and if it decreases the input voltage, it is called ***step down transformer***.

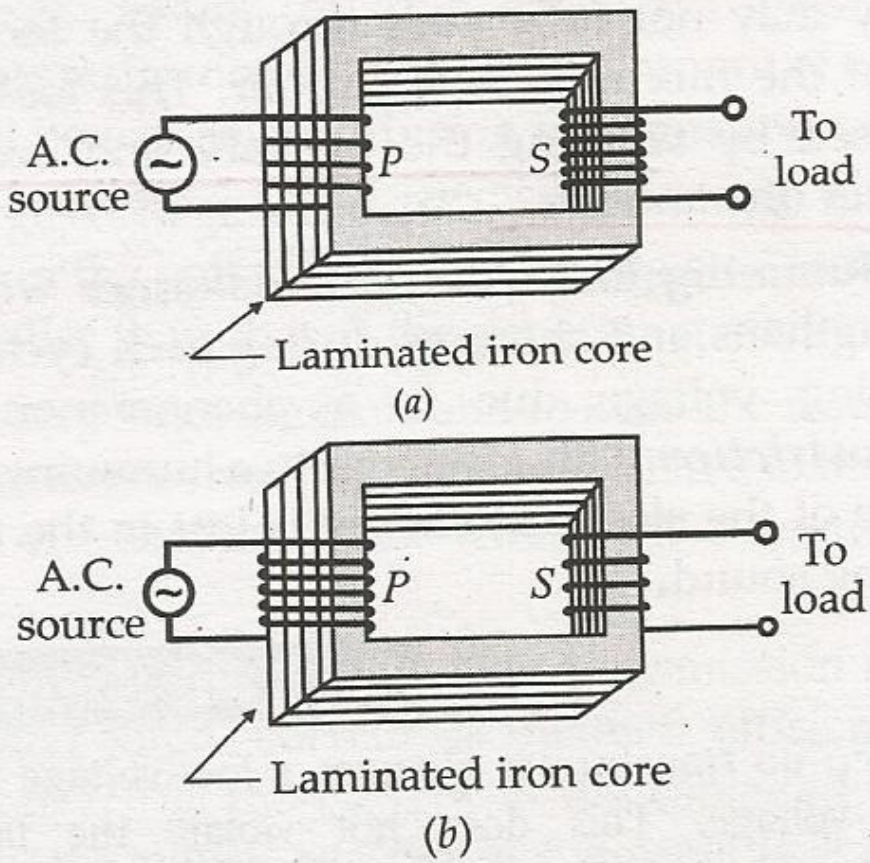


Fig. 7.47 (*a*) Step-up and (*b*) step-down, transformer.

Principle. *It works on the principle of mutual induction, i.e., when a changing current is passed through one of the two inductively coupled coils, an induced emf is set up in the other coil.*

Construction. A transformer essentially consists of two coils of insulated copper wire having different number of turns and wound on the same soft iron core.

The coil P to which electric energy is supplied is called the ***primary*** and the coil from which energy is drawn or output is obtained is called the ***secondary***. To prevent energy losses due to eddy currents, a laminated core is used. Because of high permeability of soft iron, the entire magnetic flux due to the current in the primary coil practically remains in the iron core and hence passes fully through the secondary. This also prevents the stray currents being generated in the conductors lying around and the consequent power loss.

Two types of arrangements are generally used for winding of primary and secondary coils in a transformer :

1. Core type. In the core type transformers, the primary and secondary coils are wound on separate limbs of the core so that the core is largely surrounded by the coils. Many of the modern transformers are of closed core type as shown in Fig. 7.48(*a*)

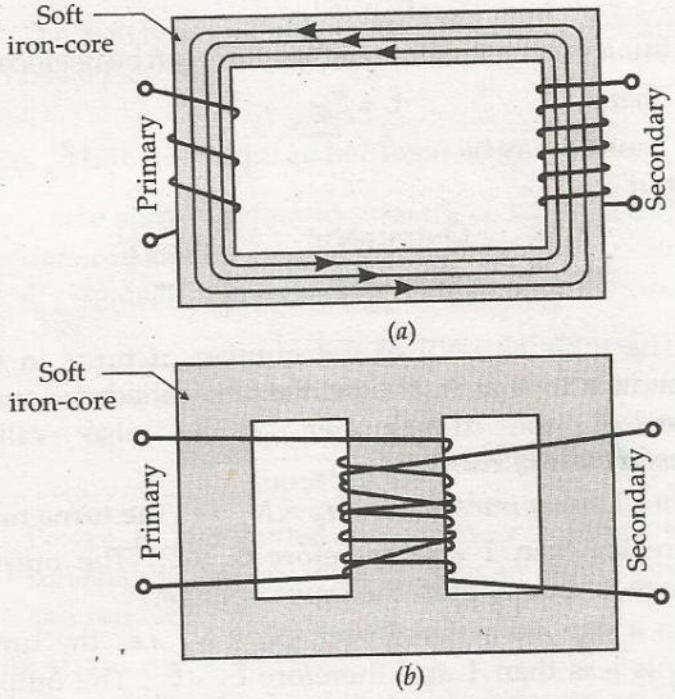


Fig. 7.48 (*a*) Core type and (*b*) Shell type transformer.

2. Shell type. In the shell type transformers, the primary and secondary coils are wound one over another on the same limb of the iron core. The coils are very largely surrounded by the iron core. Transformers used in radio and TV transmitters and receivers are of shell type, as shown in Fig. 7.48(*b*).

Working. As the alternating current flows through the primary, it generates an alternating magnetic flux in the core which also passes through the secondary. This changing flux sets up an induced emf in the secondary, also a self-induced emf in the primary. If there is no leakage of magnetic flux, then flux linked with each turn of the primary will be equal to that linked with each turn of the secondary.

• **Theory.** Consider the situation when no load is connected to the secondary, *i.e.*, its terminals are open. Let N_1 and N_2 be the number of turns in the primary and secondary respectively. Then

Induced emf in the primary coil, $\mathcal{E}_1 = -N_1 \frac{d\phi}{dt}$

Induced emf in the secondary coil, $\mathcal{E}_2 = -N_2 \frac{d\phi}{dt}$

where ϕ is the magnetic flux linked with each turn of the primary or secondary at any instant. Thus

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}$$

Let $\mathcal{E}$ be the emf applied to the primary. By Lenz's law, self-induced emf $\mathcal{E}_1$ opposes $\mathcal{E}$ in the primary coil.

$\therefore$ Resultant emf in the primary $= \mathcal{E} - \mathcal{E}_1$

This emf sends current I_1, through the primary coil of resistance R.

$$\therefore \quad \mathcal{E} - \mathcal{E}_1 = RI_1$$

But R is very small, so the term RI_1 can be neglected.

Then $$\mathcal{E} = \mathcal{E}_1$$

Thus $\mathcal{E}_1$ may be regarded as input emf and $\mathcal{E}_2$ as the output emf.

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{\text{Output emf}}{\text{Input emf}} = \frac{N_2}{N_1} \qquad \ldots(1)$$

The ratio N_2 / N_1, of the number of turns in the secondary to that in the primary, is called the ***turns ratio*** of the transformer. It is also called ***transformation ratio.***

In a step up transformer, $N_2 > N_1$, *i.e.*, the turns ratio is greater than 1 and therefore $\mathcal{E}_2 > \mathcal{E}_1$. The output voltage is greater than the input voltage.

In a step down transformer, $N_2 < N_1$,*i.e.*, the turns ratio is less than 1 and therefore $\mathcal{E}_2 < \mathcal{E}_1$. The output voltage is less than the input voltage.

It may be noted that equation (1) has been derived by using the following three assumptions :

1. The primary resistance and current are small.
2. The same flux links both with the primary and secondary windings as the flux leakage from the core is negligibly small.
3. The terminals of the secondary are open or the current taken from it is small.

Currents in primary and secondary. Assuming the transformer to be ideal one so that there are no energy losses, then

Input power = Output power or $\mathcal{E}_1 I_1 = \mathcal{E}_2 I_2$

where I_1 and I_2 are the currents in the primary and secondary, respectively.

Hence $$\frac{I_1}{I_2} = \frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1} \quad ...(2)$$

Thus a step up transformer steps up the voltage, but steps down the current exactly in the same ratio. Similarly, a step down transformer steps down the voltage but steps up the current exactly in the same ratio.

The efficiency of a transformer is defined as

$$\eta = \frac{\text{Power output}}{\text{Power input}} \times 100\% \quad ...(3)$$

The efficiency of real transformers is fairly high (90 – 99%) though not 100%.

Energy losses in transformers. The main causes for energy loss in transformers are as follows :

1. Copper loss. Some energy is lost due to heating of copper wires used in the primary and secondary windings. This power loss ($= I^2R$) can be minimised by using thick copper wires of low resistance.

2. Eddy current loss. The alternating magnetic flux induces eddy currents in the iron core which leads to some energy loss in the form of heat. This loss can be reduced by using laminated iron core.

3. Hysteresis loss. The alternating current carries the iron core through cycles of magnetisation and demagnetisation. Work is done in each of these cycles and is lost as heat. This is called hysteresis loss and can be minimised by using core material having narrow hysteresis loop.

4. Flux leakage. The magnetic flux produced by the primary may not fully pass through the secondary. Some of the flux may leak into air. This loss can be minimised by winding the primary and secondary coils over one another.

5. Humming loss. As the transformer works, its core lengthens and shortens during each cycle of the alternating voltage due to a phenomenon called ***magnetostriction***. This gives rise to a humming sound. So some of the electrical energy is lost in the form of humming sound.

Mathematical treatment of *LC*-oscillations. As shown in Fig. 7.46(*b*), suppose a capacitor of capacitance *C*, initially charged to q_0, be connected to an inductor of inductance *L*. As the circuit is closed, the charge on the capacitor begins to decrease, giving rise to a current in the circuit. As *q* decreases, *I* increases, so

$$I = -\frac{dq}{dt}$$

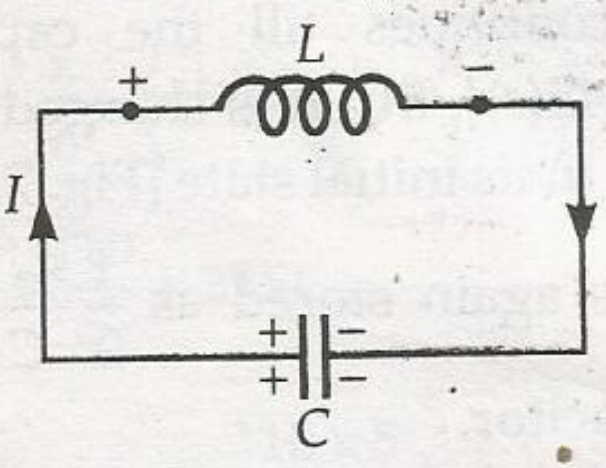


Fig. 7.46 (*b*) Polarity of induced emf across *L* as *I* increases in the circuit.

Induced emf across the inductor at any instant

$$= -L\frac{dI}{dt}$$

P.D. across the capacitor at that instant $= \frac{q}{C}$

According to Kirchhoff's loop rule,

$$-L\frac{dI}{dt} + \frac{q}{C} = 0$$

or

$$+L\frac{d^2q}{dt^2} + \frac{q}{C} = 0 \qquad \left[\because I = -\frac{dq}{dt},\ \frac{dI}{dt} = -\frac{d^2q}{dt^2}\right]$$

or $$\frac{d^2q}{dt^2} + \frac{1}{LC} q = 0$$

or $$\frac{d^2q}{dt^2} + \omega_0^2 q = 0$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$. This is a linear differential equation of second order. It has a general solution of the form :

$$q = A \cos \omega_0 t + B \sin \omega_0 t$$

At $t = 0$, $q = q_0$,

so $$q_0 = A \cos 0 + B \sin 0$$

or $$A = q_0$$

Also, $$\frac{dq}{dt} = -\omega_0 A \sin \omega_0 t + \omega_0 B \cos \omega_0 t$$

At $t = 0$, $q = q_0$ (maximum) and so $\frac{dq}{dt} = 0$

$\therefore$ $$0 = -\omega_0 A \sin 0 + \omega_0 B \cos 0$$

or $$B = 0$$

Hence $$q = q_0 \cos \omega_0 t$$

and $$I = -\frac{dq}{dt} = \omega_0 q_0 \sin \omega_0 t$$

Thus the charge on the capacitor plates oscillates simple harmonically with time with angular frequency,

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Frequency of oscillation, $f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$.

Conservation of energy in *LC*-oscillations. Just as the sum of potential and kinetic energies remains constant in SHM, the sum of energies stored in the capacitor and inductor remains constant during *LC*-oscillations. At any instant, the electrostatic energy stored in the capacitor is

$$U_E = \frac{1}{2} \cdot \frac{q^2}{C}$$

The magnetic energy stored in the inductor at any instant is

$$U_B = \frac{1}{2} LI^2$$

If there is no (resistive) loss of energy, then the total energy of the *LC*-circuit at any instant will be

$$U = U_E + U_B = \frac{1}{2} \cdot \frac{q^2}{C} + \frac{1}{2} LI^2$$

But $q = q_0 \cos \omega_0 t$ and $I = -\frac{dq}{dt} = \omega q_0 \sin \omega_0 t$

$$U = \frac{1}{2C} \cdot q_0^2 \cos^2 \omega_0 t + \frac{1}{2} L\omega_0^2 q_0^2 \sin^2 \omega_0 t$$

$$= \frac{1}{2C} \cdot q_0^2 \cos^2 \omega_0 t + \frac{1}{2} L \frac{1}{LC} q_0^2 \sin^2 \omega_0 t$$

$$= \frac{1}{2} \cdot \frac{q_0^2}{C} [\cos^2 \omega_0 t + \sin^2 \omega_0 t] = \frac{1}{2} \cdot \frac{q_0^2}{C}$$

$$= \frac{1}{2} C V_0^2 = \text{Initial energy, as expected}$$

Mechanical analogy for *LC*-oscillations. The *LC*-oscillations are similar to the oscillations of a mass-spring system. In the *LC*-system, the energy alternates between electrostatic and magnetic forms while in the mass-spring system, it alternates between potential and kinetic forms. The capacitor acts like a spring while the inductor acts like an inertial mass. The charge corresponds to displacement and the current corresponds to velocity.

The displacement x of oscillating mass satisfies the differential equation :

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0, \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

The charge q of the capacitor satisfies a similar equation :

$$\frac{d^2q}{dt^2} + \omega_0^2 q = 0, \quad \text{where} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

Clearly, x corresponds to q.

For a mechanical system,

$$F = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$

For an electrical system,

$$\mathcal{E} = -L\frac{dI}{dt} = -L\frac{d^2q}{dt^2}$$

On comparing the above two equations, we see that L is analogous to mass m *i.e.*, L is a measure of resistance to change in current.

Again, in case of LC system, $\omega_0 = \frac{1}{\sqrt{LC}}$

In case of mass-spring system, $\omega_0 = \sqrt{\frac{k}{m}}$

Obviously, $\frac{1}{C}$ is analogous to k. The force constant k is the force required to produce unit displacement while $\frac{1}{C}\left(=\frac{V}{q}\right)$ is the potential difference required to store a unit charge.

Table 7.1 Analogies between Mechanical and Electrical Quantities

Mechanical System	Electrical System
Mass m	Inductance L
Displacement x	Charge q
Velocity $v = \frac{dx}{dt}$	Current $I = \frac{dq}{dt}$
Force constant k	Reciprocal capacitance $\frac{1}{C}$
Mechanical energy $E = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$	Electromagnetic energy $U = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}LI^2$

7.15 POWER IN AN A.C. CIRCUIT

20. *Define power for an a.c. circuit. Derive an expression for the average of a series LCR-circuit connected to an a.c. source. Discuss the various special cases.*

Power in an a.c. circuit. *The rate at which electric energy is consumed in an electric circuit is called its power.* In a d.c. circuit, power is given by the product of voltage and current. But in an a.c. circuit, both voltage $\mathcal{E}$ and current I vary sinusoidally with time and are generally not in phase. So for an a.c. circuit, we define ***instantaneous power*** *as the product of the instantaneous voltage and instantaneous current.*

(I) Suppose in an a.c. circuit, the voltage and current at any instant are given by

$$\mathcal{E} = \mathcal{E}_0 \sin \omega t$$

and

$$I = I_0 \sin (\omega t - \phi)$$

where ϕ is the phase angle by which the voltage $\mathcal{E}$ leads the current I.

(II) The instantaneous power is given by

$$P = \mathcal{E} I = \mathcal{E}_0 I_0 \sin \omega t \,.\, \sin (\omega t - \phi)$$

$$= \mathcal{E}_0 I_0 \sin \omega t \,[\, \sin \omega t \cos \phi - \cos \omega t \sin \phi \,]$$

$$= \mathcal{E}_0 I_0 \,[\sin^2 \omega t \cos \phi - \sin \omega t \cos \omega t \sin \phi]$$

$$= \frac{\mathcal{E}_0 I_0}{2} [(1 - \cos 2\omega t) \cos \phi - \sin 2\omega t \sin \phi]$$

$$= \frac{\mathcal{E}_0 I_0}{2} [\cos \phi - (\cos 2\omega t \cos \phi + \sin 2\omega t \sin \phi)]$$

$$= \frac{\mathcal{E}_0 I_0}{2} [\, \cos \phi - \cos (2\omega t - \phi) \,]$$

If we assume the instantaneous power to remain constant for a small time dt, the work done during this time is

$$dW = P\,dt = \mathcal{E} I\,dt$$

Total work done over a complete cycle (*i.e.*, from $t = 0$ to $t = T$) is

$$W = \int_0^T \mathcal{E} I\,dt$$

Hence average power dissipated in the circuit over a complete cycle is

$$P_{av} = \frac{W}{T} = \frac{1}{T}\int_0^T \mathcal{E}\, I dt$$

$$= \frac{\mathcal{E}_0 I_0}{2T}\left[\int_0^T \cos\phi\,dt - \int_0^T \cos(2\omega t - \phi)dt\right]$$

$$= \frac{\mathcal{E}_0 I_0}{2T}[\cos\phi \mid t\,|_0^T - 0] = \frac{\mathcal{E}_0 I_0}{2T}[\cos\phi(T-0)]$$

or $$P_{av} = \frac{\mathcal{E}_0}{\sqrt{2}} \cdot \frac{I_0}{\sqrt{2}} \cdot \cos\phi$$

or $$P_{av} = \mathcal{E}_{rms}\, I_{rms} \cos\phi = \mathcal{E}_{rms}\, I_{rms} \cdot \frac{R}{Z}$$]

Special Cases

1. Pure resistive circuit. Here the voltage and current are in same phase, *i.e.*, $\phi = 0$ and $\cos\phi = 1$.

$$\therefore \quad P_{av} = \mathcal{E}_{rms} \cdot I_{rms} \times 1 = \mathcal{E}_{rms} \cdot I_{rms} = \frac{\mathcal{E}_{rms}^2}{R}$$

2. Pure inductive circuit. Here the voltage leads the current in phase by $\frac{\pi}{2}$, *i.e.*, $\phi = \frac{\pi}{2}$.

$$\therefore \quad P_{av} = \mathcal{E}_{rms} \cdot I_{rms} \cos\frac{\pi}{2} = 0$$

Thus the average power consumed in an inductive circuit over a complete cycle is zero.

3. Pure capacitive circuit. Here the voltage lags behind the current in phase by $\frac{\pi}{2}$, *i.e.*, $\phi = -\frac{\pi}{2}$.

$$\therefore \quad P_{av} = \mathcal{E}_{rms} \cdot I_{rms} \cos\left(-\frac{\pi}{2}\right) = 0$$

Thus the average power consumed in a capacitive circuit over a complete cycle is also zero.

4. Series *LCR*-circuit. For a series *LCR*-circuit, $P_{av} = \mathcal{E}_{rms}\, I_{rms} \cos\phi$, where $\phi = \tan^{-1}\frac{X_L - X_C}{R}$. Sometimes, ϕ may have a non-zero value for series *LR*-, *LC*- and *LCR*-circuits. So power is consumed in such circuits, but only in the resistor *R*.

5. Power dissipated at resonance in *LCR*-circuit. At resonance, $X_L = X_C$, and $\phi = 0$. So $\cos\phi = 1$, and $P_{av} = \mathcal{E}_{rms}\, I_{rms}$. That is, *maximum power is dissipated in the circuit (through R) at resonance.*

Average power associated with resistor. In case of a pure resistor, the voltage and current are always in same phase. So we can write the instantaneous values of voltage and current as follows :

$$V = V_0 \sin \omega t \quad \text{and} \quad I = I_0 \sin \omega t$$

Work done in small time dt will be

$$dW = P\,dt = VI\,dt = V_0\,I_0 \sin^2 \omega t\,dt$$

$$= \frac{V_0\,I_0}{2}(1 - \cos 2\,\omega t)\,dt$$

The average power dissipated per cycle in the resistor will be

$$P_{av} = \frac{W}{T} = \frac{1}{T}\int_0^T dW$$

$$= \frac{V_0 I_0}{2T}\int_0^T (1 - \cos 2\,\omega t)\,dt = \frac{V_0 I_0}{2T}\left[t - \frac{\sin 2\,\omega t}{2\omega}\right]_0^T$$

$$= \frac{V_0 I_0}{2T}[(T - 0) - 0] = \frac{V_0 I_0}{2} = \frac{V_0^{\,2}}{2R}$$

$$\text{or } P_{av} = \frac{V_0 I_0}{\sqrt{2}\,\sqrt{2}} = V_{rms}\,I_{rms} = \frac{V_{rms}^{\,2}}{R} \qquad \left[\because \frac{V_0}{\sqrt{2}} = V_{rms}\right]$$

Average power associated with an inductor. When a.c. is applied to an ideal inductor, current lags behind the voltage in phase by $\pi/2$ radian. So we can write the instantaneous values of voltage and current as follows :

$$V = V_0 \sin \omega t$$

and

$$I = I_0 \sin\left(\omega t - \frac{\pi}{2}\right)$$

$$= -I_0 \sin\left(\frac{\pi}{2} - \omega t\right) = -I_0 \cos \omega t$$

Work done in small time dt is

$$dW = P\,dt = -V_0\, I_0 \sin \omega t \cos \omega t\, dt$$

$$= -\frac{V_0\, I_0}{2} \sin 2\omega t\, dt$$

(iv) [The average power dissipated per cycle in the inductor is

$$P_{av} = \frac{W}{T} = \frac{1}{T}\int_0^T dW$$

$$= -\frac{V_0 I_0}{2T}\int_0^T \sin 2\omega t \, dt$$

$$= +\frac{V_0 I_0}{2T}\left[\frac{\cos 2\omega t}{2\omega}\right]_0^T = \frac{V_0 I_0}{4T\omega}\left[\cos\frac{4\pi}{T}t\right]_0^T$$

$$= \frac{V_0 I_0}{4T\omega}[\cos 4\pi - \cos 0] = \frac{V_0 I_0}{4T\omega}[1-1]$$

$$= \mathbf{0}.$$

Thus *the average power dissipated per cycle in an inductor is zero.*]

a.c. source does not dissipate any power.

Average power associated with a capacitor. When an a.c. is applied to a capacitor, the current leads the voltage in phase by $\pi/2$ radian. So we write the expressions for instantaneous voltage and current as follows :

$$V = V_0 \sin \omega t$$

and $$I = I_0 \sin\left(\omega t + \frac{\pi}{2}\right) = I_0 \cos \omega t$$

Work done in the circuit in small time dt will be

$$dW = P dt = VI\, dt = V_0 I_0 \sin \omega t \cos \omega t\, dt$$

$$= \frac{V_0 I_0}{2} \sin 2\omega t\, dt$$

The average power dissipated per cycle in the capacitor is

$$P_{av} = \frac{W}{T} = \frac{1}{T}\int_0^T dW = \frac{V_0 I_0}{2T}\int_0^T \sin 2\omega t\, dt$$

$$= \frac{V_0 I_0}{2T}\left[-\frac{\cos 2\omega t}{2\omega}\right]_0^T$$

$$= -\frac{V_0 I_0}{4T\omega}\left[\cos \frac{4\pi}{T} t\right]_0^T$$

$$= -\frac{V_0 I_0}{4T\omega}[\cos 4\pi - \cos 0]$$

$$= -\frac{V_0 I_0}{4T\omega}[1 - 1] = \mathbf{0}.$$

Thus *the average power dissipated per cycle in a capacitor is zero.*

7.11 SERIES *LCR*-CIRCUIT

13. *Using phasor diagram, derive an expression for the impedance of a series LCR-circuit.*

Series *LCR*-circuit. As shown in Fig. 7.27, suppose a resistance R, an inductance L and capacitance C are connected in series to a source of alternating emf $\mathcal{E}$ given by

$$\mathcal{E} = \mathcal{E}_0 \sin \omega t$$

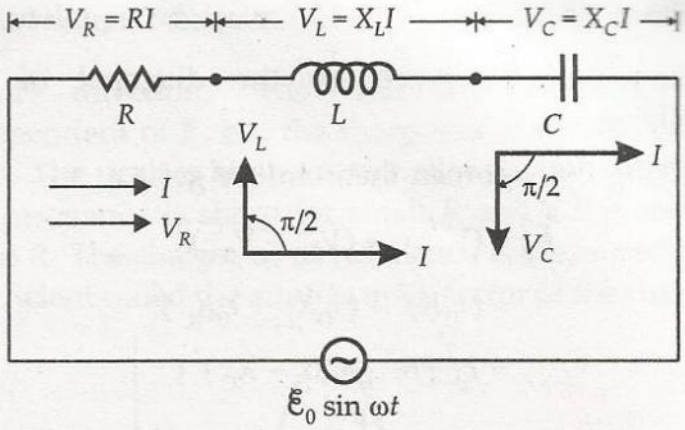


Fig. 7.27 A series *LCR*-circuit.

Let I be the current in the series circuit at any instant. Then

1. Voltage $\vec{V}_R = R\vec{I}$ across the resistance R will be *in phase* with current $\vec{I}$. So phasors $\vec{V}_R$ and $\vec{I}$ are in same direction, as shown in Fig. 7.28. The amplitude of $\vec{V}_R$ is
$$V_0^R = I_0 R$$
2. Voltage $\vec{V}_L = X_L \vec{I}$ across the inductance L is *ahead* of current I in phase by $\pi/2$ rad. So phasor $\vec{V}_L$ lies $\pi/2$ rad anticlockwise w.r.t. the phasor $\vec{I}$. Its amplitude is
$$V_0^L = I_0 X_L$$
3. Voltage $\vec{V}_C = X_C \vec{I}$ across the capacitance C *lags behind* the current $\vec{I}$ in phase by $\pi/2$ rad. So phasor $\vec{V}_C$ lies $\pi/2$ clockwise w.r.t. the phasor $\vec{I}$. Its amplitude is
$$V_0^C = I_0 X_C$$

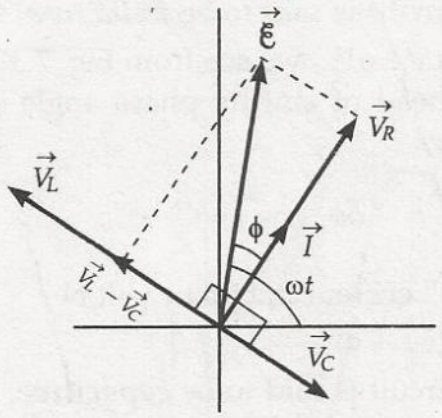


Fig. 7.28 Phasor diagram for a series LCR-circuit when $V_L > V_C$.

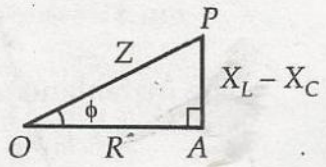


Fig. 7.29 Impedance triangle when $X_L > X_C$.

As $\vec{V}_L$ and $\vec{V}_C$ are in opposite directions, their resultant is $(\vec{V}_L - \vec{V}_C)$. By parallelogram law, the

resultant of $\vec{V}_R$ and $(\vec{V}_L - \vec{V}_C)$ must be equal to the applied emf $\vec{\mathcal{E}}$, given by the diagonal of the parallelogram.

Using Pythagorean theorem, we get

$$\mathcal{E}_0^2 = (V_0^R)^2 + (V_0^L - V_0^C)^2$$
$$= (I_0 R)^2 + (I_0 X_L - I_0 X_C)^2$$
$$= I_0^2 [R^2 + (X_L - X_C)^2]$$

or
$$I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + (X_L - X_C)^2}}$$

Clearly, $\sqrt{R^2 + (X_L - X_C)^2}$ is the *effective resistance of the series LCR-circuit which opposes or impedes the flow of current through it and is called its* ***impedance***. It is denoted by Z and its SI unit is ohm (Ω). Thus

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The relationship between the resistance R, inductive reactance X_L, capacitive reactance X_C and the impedance Z is shown in Fig. 7.29. The right angled Δ *OAP* is called the ***impedance triangle***.

Special Cases

1. When $X_L > X_C$ or $V_L > V_C$, we see from Fig. 7.28 that emf is ahead of current by phase angle ϕ which is given by

$$\tan\phi = \frac{X_L - X_C}{R} \quad \text{or} \quad \cos\phi = \frac{R}{Z}$$

The instantaneous current in the circuit will be

$$I = I_0 \sin(\omega t - \phi)$$

The series *LCR*-circuit is said to be ***inductive***.

2. When $X_L < X_C$ or $V_L < V_C$, we see from Fig. 7.30 that current is ahead of emf by phase angle ϕ which is given by

$$\tan\phi = \frac{X_C - X_L}{R} \quad \text{or} \quad \cos\phi = \frac{R}{Z}$$

The instantaneous current in circuit will be

$$I = I_0 \sin(\omega t + \phi)$$

The series *LCR*-circuit is said to be ***capacitive***.

3. When $X_L = X_C$ or $V_L = V_C$, $\phi = 0$, the emf and current will be in the same phase. The series *LCR*-circuit said to be ***purely resistive***.

It may also be noted that

$$I_0 = \frac{\mathcal{E}_0}{Z} \quad \text{or} \quad \frac{I_0}{\sqrt{2}} = \frac{\mathcal{E}_0}{\sqrt{2}Z} \quad \text{or} \quad I_{rms} = \frac{\mathcal{E}_{rms}}{Z}$$

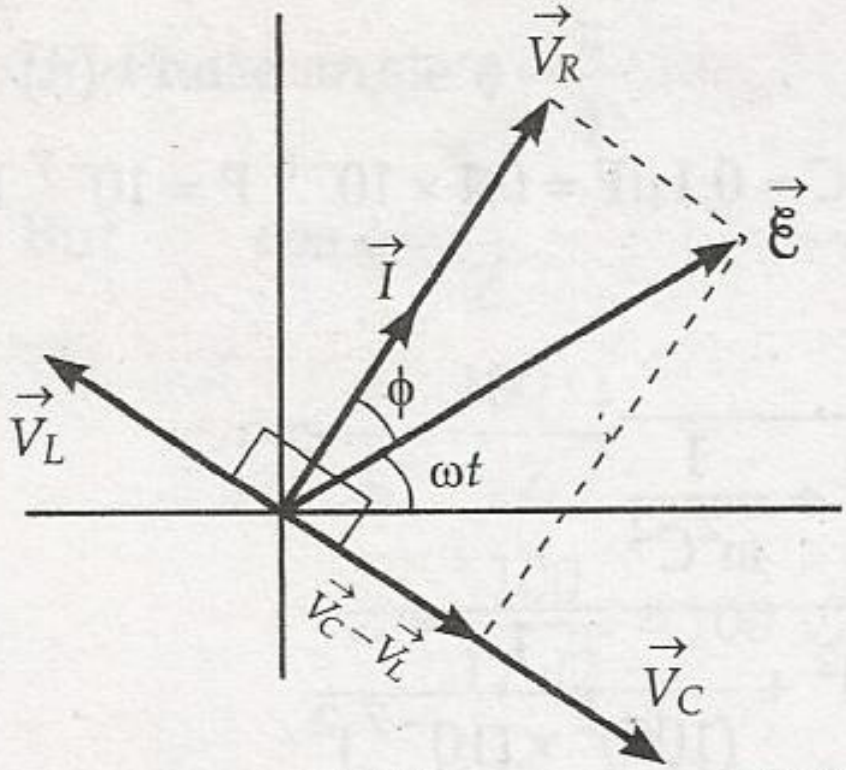


Fig. 7.30 Phasor diagram for a series *LCR*-circuit when $V_L < V_C$.

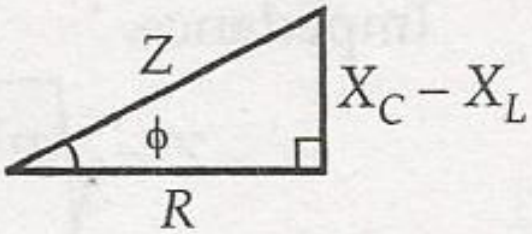


Fig. 7.31 Impedance triangle when $X_L < X_C$.

7.12 RESONANCE CONDITION OF A SERIES *LCR*-CIRCUIT

15. *What do you mean by the resonance condition of a series LCR-circuit ? Calculate its resonant frequency.*

Resonance condition of a series *LCR*-circuit. *A series LCR-circuit is said to be in the resonance condition when the current through it has its maximum value.*

The current amplitude I_0 for a series *LCR*-circuit is given by

$$I_0 = \frac{\mathcal{E}_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

Clearly, I_0 becomes zero both for $\omega \to 0$ and $\omega \to \infty$. The value of I_0 is maximum when

$$\omega L - \frac{1}{\omega C} = 0 \quad \text{or} \quad \omega = \frac{1}{\sqrt{LC}}$$

Then impedance, $Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = R$

Clearly the impedance is minimum. The circuit is purely resistive. The current and voltage are in the same phase and the current in the circuit is maximum. This condition of the *LCR*-circuit is called *resonance condition*. The frequency at which the current amplitude I_0 attains a peak value is called ***natural*** or ***resonant frequency*** of the *LCR*-circuit and is denoted by f_r.

Thus $\omega_r = 2\pi f_r = \frac{1}{\sqrt{LC}}$ or $f_r = \frac{1}{2\pi\sqrt{LC}}$.

The current amplitude at resonant frequency will be

$$I_0 = \frac{\mathcal{E}_0}{R}$$

7.13 SHARPNESS OF RESONANCE : Q-FACTOR

17. *What do you mean by sharpness of resonance in a series resonant circuit ? Find expression for Q-factor of the circuit.*

Sharpness of resonance : Q-Factor. Fig 7.32 shows the variation of current amplitude I_0 in a series LCR-circuit with angular frequency ω, for three different values of R. The current amplitude has a peak

at the resonant frequency $\omega_r = \frac{1}{\sqrt{LC}}$ and falls to zero in either direction. The resonant frequency is independent of R, but the sharpness of peak depends on R. The peak is higher for smaller values of R. Thus the resonance is sharp for small R and a flat one for large R. The sharpness of resonance is measured by a coefficient called the ***quality*** or ***Q-factor*** of the circuit.

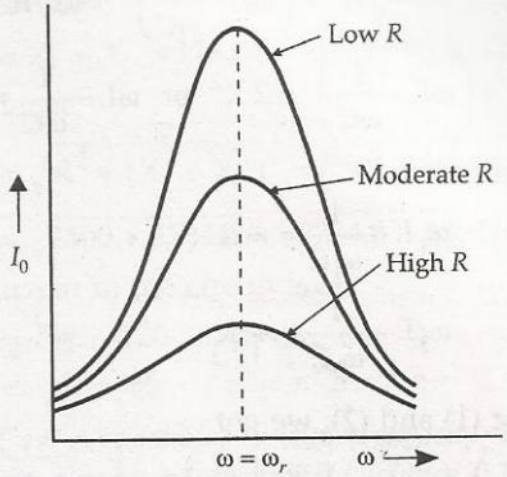


Fig. 7.32 Variation of current amplitude with frequency in an *LCR*-circuit.

The Q-factor of a ***series resonant*** *circuit is defined as the ratio of the resonant frequency to the difference in two frequencies taken on both sides of the resonant frequency such that at each frequency, the current amplitude becomes* $\frac{1}{\sqrt{2}}$ *times the value at resonant frequency.*

Mathematically, the Q-factor can be expressed as

$$Q = \frac{\omega_r}{\omega_2 - \omega_1} = \frac{\omega_r}{2\,\Delta\omega} = \frac{\text{Resonant frequency}}{\text{Bandwidth}}$$

where ω_1 and ω_2 are the frequencies at which the current falls to $\frac{1}{\sqrt{2}}$ times its resonant value, as shown in Fig. 7.33 and we have used

$$\omega_1 = \omega_r - \Delta\omega \; ; \quad \omega_2 = \omega_r + \Delta\omega$$

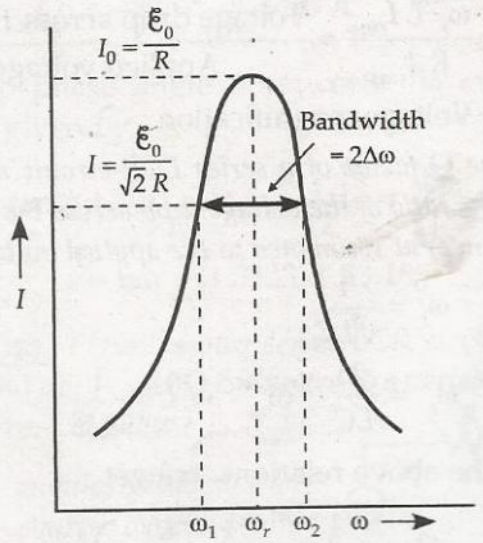


Fig. 7.33 Bandwidth of a series resonant circuit.

The frequency range $\omega_2 - \omega_1 = 2\Delta\omega$ is called ***band-width***. The larger the value of Q-factor, the smaller is the value of $2\Delta\omega$ or the bandwidth and sharper is the peak in the current. Q-factor is a *dimensionless quantity*.

Expression for Q-factor. Clearly at ω_r, the impedance is equal to R, while at ω_1 and ω_2 its value is $\sqrt{2}\ R$.

$$\therefore \quad Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} = \sqrt{2}\ R$$

or $$R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2 = 2R^2 \quad \text{or} \quad \omega L - \frac{1}{\omega C} = \pm R$$

We can write

$$\omega_1 L - \frac{1}{\omega_1 C} = -R \qquad ...(1)$$

and $$\omega_2 L - \frac{1}{\omega_2 C} = +R \qquad ...(2)$$

Adding (1) and (2), we get

$$(\omega_1 + \omega_2)L - \frac{1}{C}\left(\frac{\omega_1 + \omega_2}{\omega_1\ \omega_2}\right) = 0$$

or $$\omega_1\ \omega_2 = \frac{1}{LC}$$

Subtracting (1) from (2), we get

$$(\omega_2 - \omega_1)\ L + \frac{1}{C}\left(\frac{\omega_2 - \omega_1}{\omega_1 \omega_2}\right) = 2\ R$$

or $$(\omega_2 - \omega_1)\left(L + \frac{1}{C\omega_1\omega_2}\right) = 2\ R$$

or $$(\omega_2 - \omega_1)(L + L) = 2\ R \quad \left[\because \omega_1\omega_2 = \frac{1}{LC}\right]$$

or $$\omega_2 - \omega_1 = \frac{R}{L}$$

$$\therefore \quad Q = \frac{\omega_r}{2\Delta\omega} = \frac{\omega_r}{\omega_1 - \omega_2} = \frac{\omega_r L}{R} \qquad ...(3)$$

The above equation can also be written as

$$Q = \frac{\omega_r\ L\ I_{rms}}{R\ I_{rms}} = \frac{\text{Voltage drop across } L(\text{ or } C)}{\text{Applied voltage}}$$

= Voltage magnification

Thus the Q-factor of a series LCR-circuit may also be defined as the ratio of the voltage drop across the inductance (or capacitance) at resonance to the applied voltage.

As $$\omega_r = \frac{1}{\sqrt{LC}}$$

$$\therefore \quad \omega_r^2 = \frac{1}{LC} \quad \text{or} \quad \omega_r L = \frac{1}{\omega_r C}$$

Using the above relations, we get

$$Q = \frac{1}{\omega_r\ CR} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

From equation (3), we see that if Q-factor is large *i.e.*, if R is low or L is large, the bandwidth $2\Delta\omega$ is small. This means that the resonance is sharp or the series resonant circuit is more selective.

A.C. circuit containing resistance only. As shown in Fig. 7.4, suppose a resistor of resistance R is connected to a source of alternating emf $\mathcal{E}$ given by

$$\mathcal{E} = \mathcal{E}_0 \sin \omega t \qquad ...(1)$$

Such a circuit is known as a *purely resistive circuit.*

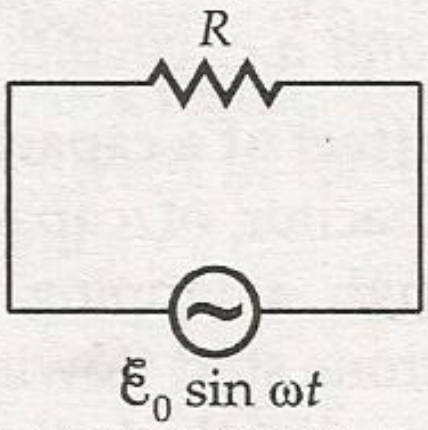


Fig. 7.4 A.C. through a resistor.

If I be the current in the circuit at instant t, then the potential drop across R will be IR. According to Kirchhoff's loop rule,

Instantaneous emf of the source

= Instantaneous p.d. across R

or $\mathcal{E}_0 \sin \omega t = IR$

or $$I = \frac{\mathcal{E}_0}{R} \sin \omega t$$

or $$I = I_0 \sin \omega t \qquad ...(2)$$

where $I_0 = \dfrac{\mathcal{E}_0}{R}$ = the maximum or peak value of a.c.

From equations (1) and (2), we note that both $\mathcal{E}$ and I are functions of $\sin \omega t$. Hence the emf $\mathcal{E}$ and current I are in *same phase* in a purely resistive circuit. This means that both $\mathcal{E}$ and I attain their zero, minimum and maximum values at the same respective times. This phase relationship is shown graphically in Fig. 7.5(*a*).

Figure 7.5(b) shows the phasor diagram for a resistive a.c. circuit. Both the phasors $\vec{\mathcal{E}}$ and $\vec{I}$ are in the same direction, making same angle ωt with x-axis. The phase angle between them is zero.

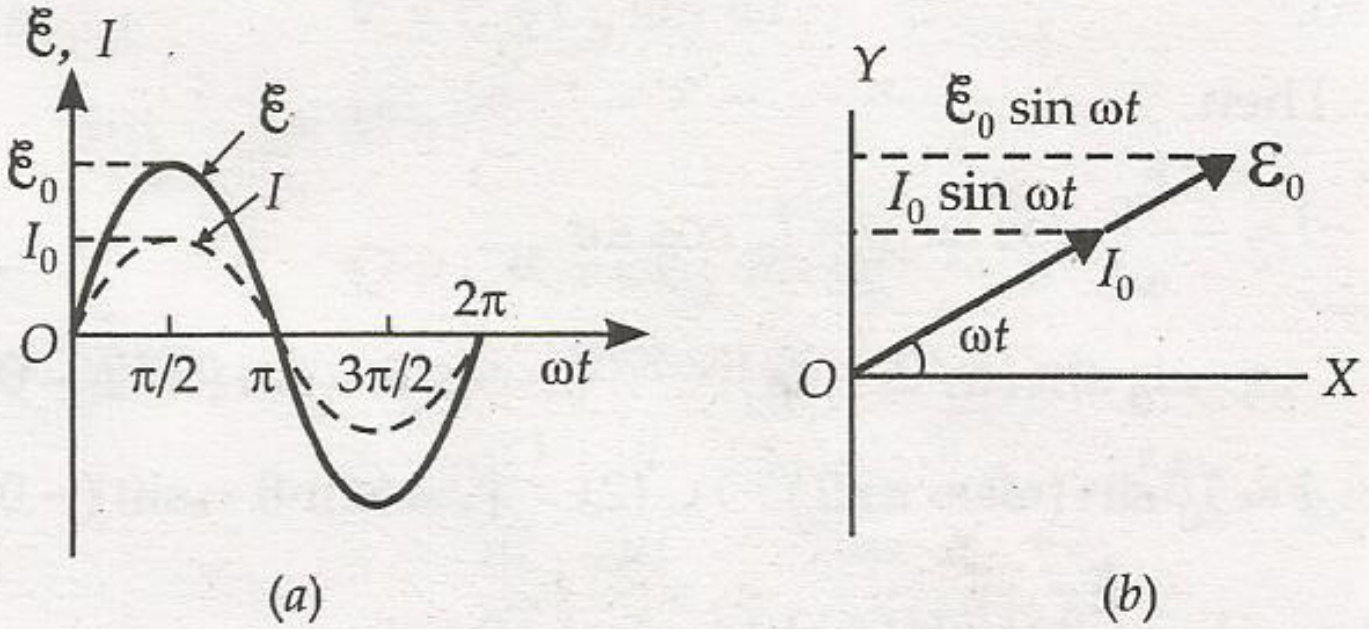


Fig. 7.5 (a) Graph of $\mathcal{E}$ and I versus ωt and (b) Phasor diagram, for a resistive a.c. circuit.

source. Give its units.

A.C. circuit containing only an inductor. Fig. 7.6 shows an inductor of inductance L connected to a source of alternating emf $\mathcal{E}$ given by

$$\mathcal{E} = \mathcal{E}_0 \sin \omega t \quad ...(1)$$

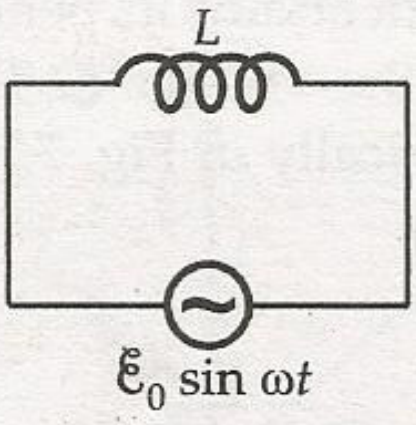


Fig. 7.6 A.C. through an inductor

We assume that the inductor has negligible resistance. Thus the circuit is *purely inductive a.c. circuit.*

As the alternating current flows through the inductor, a back emf $-L\frac{dI}{dt}$ is set up which opposes the applied emf.

$\therefore$ Net instantaneous emf $= \mathcal{E} - L\frac{dI}{dt}$

But this emf must be zero because there is no resistance in the circuit.

$$\therefore \quad \mathcal{E} - L\frac{dI}{dt} = 0 \quad \text{or} \quad \mathcal{E} = L.\frac{dI}{dt}$$

$$\text{or} \quad \mathcal{E}_0 \sin \omega t = L.\frac{dI}{dt}$$

$$\text{or} \quad dI = \frac{\mathcal{E}_0}{L} \sin \omega t \, . \, dt$$

$$\text{Integrating,} \quad \int dI = \int \frac{\mathcal{E}_0}{L} . \sin \omega t \, . \, dt$$

$$\text{or} \quad I = -\frac{\mathcal{E}_0}{\omega L} \cos \omega t + \text{constant}$$

As the applied emf is sinusoidal, we expect the current also to be sinusoidal. Thus the average of current I must be zero over a time period. Now the average of cos ωt is zero over a time period, hence the integration constant in the above equation must be zero.

Then

$$I = -\frac{\mathcal{E}_0}{\omega L}\cos \omega t = -I_0 \cos \omega t$$

$$= -I_0 \sin(\pi/2 - \omega t) \quad [\because \cos\theta = \sin(\pi/2 - \theta)]$$

or $I = I_0 \sin(\omega t - \pi/2)$ ...(2) $[\because -\sin\theta = \sin(-\theta)]$

where $I_0 = \dfrac{\mathcal{E}_0}{\omega L}$ = the peak value of a.c.

Phase relationship between $\mathcal{E}$ and I. On comparing equations (1) and (2), we find that the phase angle of current I is $\pi/2$ rad less than that of emf $\mathcal{E}$.

Thus in an inductive a.c. circuit, the voltage is ahead of the current in phase by 90° or the current lags behind the voltage in phase by 90°. This means that the voltage $\mathcal{E}$ attains its maximum value ($\mathcal{E}_0$) a quarter of cycle (time $T/4$) earlier than the current I, or the current attains its peak value (I_0) a quarter of cycle later than the voltage $\mathcal{E}$. This phase relationship is shown graphically in Fig. 7.7(*a*).

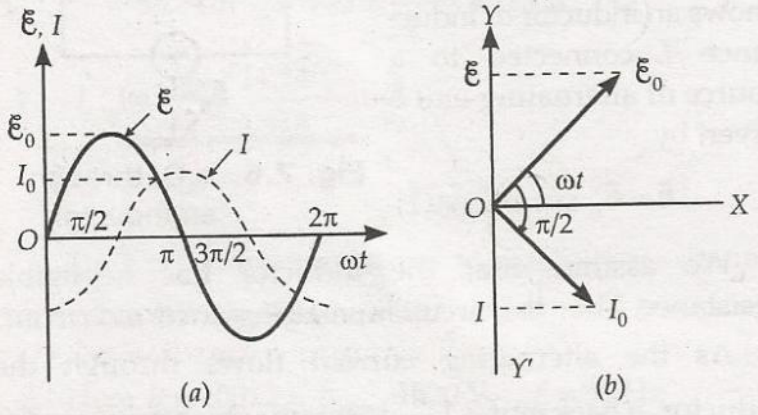


Fig. 7.7 (*a*) Graph of $\mathcal{E}$ and I versus ωt and (*b*) phasor diagram, for an inductive a.c. circuit.

Figure 7.7(*b*) shows the phasor diagram for an inductive a.c. circuit. The phasor $\vec{\mathcal{E}}$ makes an angle ωt with x-axis in anticlockwise direction. As the current lags behind the emf in phase by $\pi/2$ rad, so the current phasor $\vec{I}$ makes an angle $\pi/2$ rad with the phasor of $\vec{\mathcal{E}}$ in clockwise direction.

Inductive reactance. Comparing equation $I_0 = \mathcal{E}_0 / \omega L$ with the ohmic relation $I_0 = \mathcal{E}_0 / R$, we find that ωL plays the same role here as the resistance R in resistive case. *It is a measure of the effective resistance or opposition offered by the inductor to the flow of a.c. through it.* Such a non-resistive opposition to the flow of

current is called ***reactance***. In this case, it is called ***inductive reactance*** and is denoted by X_L.

$$\therefore \qquad X_L = \omega L = 2\pi fL$$

where f is the frequency of a.c. supply. The SI unit of inductive reactance is ohm (Ω).

For a.c., $X_L \propto f$

For d.c., $f = 0$, so $X_L = 0$

Thus an inductor allows d.c. flow through it easily but opposes the flow of a.c. through it. Obviously,

$$I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{\mathcal{E}_0}{\omega L\sqrt{2}} = \frac{\mathcal{E}_{rms}}{\omega L} = \frac{\mathcal{E}_{rms}}{X_L}$$

Variation of X_L with frequency. As $X_L \propto f$, so the graph of X_L versus f is a straight line with a positive slope. As f increases, X_L also increases.

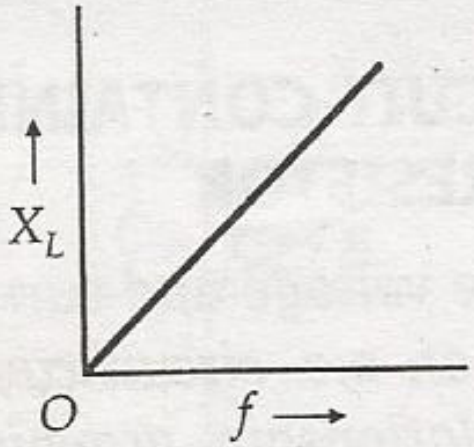


Fig. 7.8 Graph of X_L versus f.

source. Give its units.

A.C. circuit containing only a capacitor. As shown in Fig. 7.12, consider a pure capacitor C connected across a source of alternating emf $\mathcal{E}$ given by

$$\mathcal{E} = \mathcal{E}_0 \sin \omega t \qquad ...(1)$$

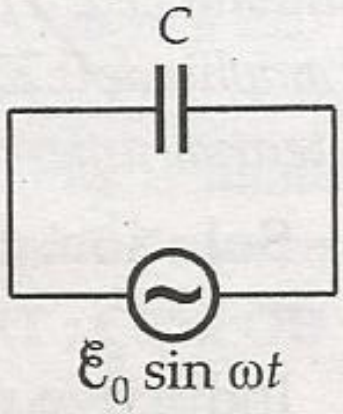


Fig. 7.12

Due to the continuous charging and discharging of the capacitor plates, a continuous but alternating current exists in the circuit.

At any instant,

P.D. across the capacitor plates = Applied emf

i.e., $$V = \mathcal{E} = \mathcal{E}_0 \sin \omega t$$

But $$V = \frac{Q}{C}$$

or $$Q = CV = C\mathcal{E}_0 \sin \omega t$$

∴ Current at any instant is

$$I = \frac{dQ}{dt} = \frac{d}{dt}(C\mathcal{E}_0 \sin \omega t) = \omega C \mathcal{E}_0 \cos \omega t$$

or $$I = I_0 \sin(\omega t + \pi/2) \qquad ...(2)$$

where $I_0 = \omega C \mathcal{E}_0 = \dfrac{\mathcal{E}_0}{1/\omega C}$ = the current amplitude.

Phase relationship between $\mathcal{E}$ and I. *On comparing equations (1) and (2), we find that in a capacitive a.c. circuit, the current leads the voltage or the voltage lags behind the current in phase by $\pi/2$ radian.* The phase relationship between $\mathcal{E}$ and I is shown graphically in Fig. 7.13(*a*). We see that the current reaches its maximum value earlier than the voltage by one-fourth of a period.

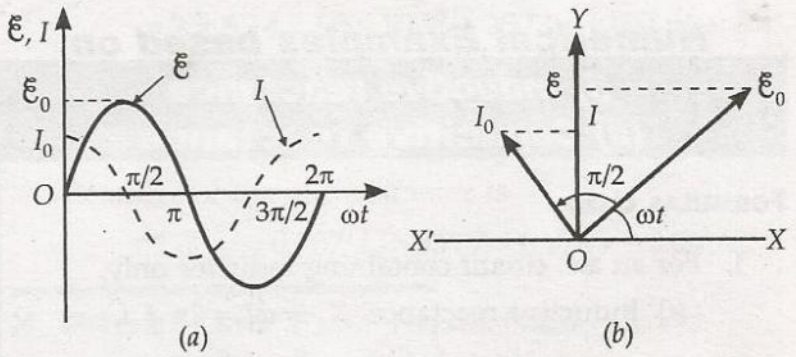


Fig. 7.13 (*a*) Graph of $\mathcal{E}$ and I versus ωt and (*b*) Phasor diagram, for a capacitive a.c. circuit.

Fig. 7.13(*b*) shows the phasor diagram for a capacitive a.c. circuit. The phasor $\vec{\mathcal{E}}$ makes an angle ωt with X-axis in anticlockwise direction. As the current leads the emf in phase by $\pi/2$ rad, so the current phasor $\vec{I}$ makes an angle $\pi/2$ rad with phasor $\vec{\mathcal{E}}$ in anticlockwise direction.

- **Capacitive reactance**. Comparing the relation,

$$I_0 = \frac{\mathcal{E}_0}{1/\omega C}$$

with the ohmic relation $I_0 = \dfrac{\mathcal{E}_0}{R}$, we find that the factor $\dfrac{1}{\omega C}$ is the effective resistance or *opposition offered by the capacitor to the flow of a.c. through it*. It is called *capacitive reactance* and is denoted by X_C.

Thus $$X_C = \frac{1}{\omega C} = \frac{1}{2\pi fC}$$

The SI unit of capacitive reactance is ohm (Ω).

For a.c., $X_C \propto \frac{1}{f}$

For d.c., $f = 0 \quad \therefore \quad X_C = \infty$

Thus a capacitor allows a.c. to flow through it easily but offers infinite resistance to the flow of d.c., *i.e, a capacitor blocks d.c.* Obviously,

$$I_{rms} = \frac{I_0}{\sqrt{2}} = \frac{\mathcal{E}_0}{1/\omega C \,.\, \sqrt{2}} = \frac{\mathcal{E}_{rms}}{1/\omega C} = \frac{\mathcal{E}_{rms}}{X_C}$$

Variation of capacitive reactance with frequency. Capacitive reactance,

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi fC}$$

i.e., $$X_C \propto \frac{1}{f}$$

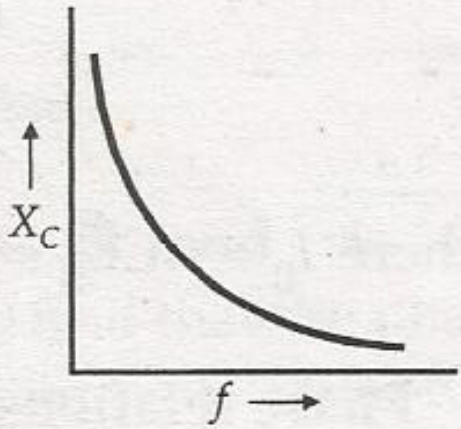


Fig. 7.14 Graph of X_C vs. f

Thus the capacitive reactance varies inversely with the frequency. As f increases, X_C decreases. Fig. 7.14 shows the variation of X_C with f.

Average value of a.c. over one complete cycle. The alternating current at any instant t is given by

$$I = I_0 \sin \omega t$$

Assuming the current remains constant for a small time dt, then the amount of charge that flows through the circuit in small time dt will be

$$dq = Idt = I_0 \sin \omega t \, . \, dt$$

The total charge that flows through the circuit in one complete cycle of a.c.,

$$q = \int dq = \int_0^T I_0 \sin \omega t \, dt$$

$$= I_0 \left[\frac{-\cos \omega t}{\omega} \right]_0^T = -\frac{I_0}{2\pi / T} \left[\cos \frac{2\pi}{T} t \right]_0^T$$

$$= -\frac{I_0 T}{2\pi} [\cos 2\pi - \cos 0] = -\frac{I_0 T}{2\pi} [1 - 1] = 0.$$

The average value of a.c. over one complete cycle of a.c.,

$$I_{av} = \frac{q}{T} = \mathbf{0}.$$

Thus *the average value of a.c. over a complete cycle of a.c. is zero.*

3. *Ordinary moving coil galvanometer used for d.c. cannot be used to measure an alternating current even if its frequency is low. Explain, why.*

Average value of a.c. It is defined as that *value of direct current which sends the same charge in a circuit in the same time as is sent by the given alternating current in its half time period.* It is denoted by

$$I_{av} \text{ or } I_m$$

- **Relation between average value and peak value of a.c.** The value of alternating current at any instant t is given by

$$I = I_0 \sin \omega t$$

This current can be assumed to remain constant for a small time dt. Then the amount of charge that flows through the circuit in small time dt is given by

$$dq = I \,.\, dt = I_0 \sin \omega t \,.\, dt$$

The total charge that flows through the circuit, say in the first half cycle, *i.e.*, from $t = 0$ to $t = T/2$ is given by

$$q = \int_0^{T/2} dq = \int_0^{T/2} I_0 \sin \omega t \, dt = I_0 \left[-\frac{\cos \omega t}{\omega} \right]_0^{T/2}$$

$$= -\frac{I_0}{2\pi/T} \left[\cos \frac{2\pi}{T} t \right]_0^{T/2}$$

$$= -\frac{I_0 T}{2\pi} [\cos \pi - \cos 0] \qquad \left[\because \omega = \frac{2\pi}{T} \right]$$

$$= -\frac{I_0 T}{2\pi} [-1 - 1] = \frac{I_0 T}{\pi}$$

$\therefore$ The average value of a.c. over the first half cycle is

$$I_{av} = \frac{\text{Charge}}{\text{Time}} = \frac{q}{T/2} = \frac{2q}{T} = \frac{2}{T} \cdot \frac{I_0 T}{\pi}$$

or

$$I_{av} = \frac{2}{\pi} I_0 = 0.637 \, I_0$$

(v) [*Thus the mean or average value of an alternating current is 2 / π or 0.637 times its peak value.* The similar relation can be proved for the alternating emf, which is

$$\mathcal{E}_{av} = \frac{2}{\pi}\mathcal{E}_0 = 0.637\,\mathcal{E}_0 .$$]

its peak value.

Root mean square or virtual or effective value of a.c. *It is defined as that value of a direct current which produces the same heating effect in a given resistor as is produced by the given alternating current when passed for the same time.* It is denoted by I_{rms}, I_v or by I_{eff}.

Relation between the effective and peak value of a.c. Suppose an alternating current $I = I_0 \sin \omega t$ be passed through a circuit of resistance R. Then the amount of heat produced in small time dt will be

$$dH = I^2 R\, dt$$

If T is the time period of a.c., then heat produced in one complete cycle will be

$$H = \int_0^T I^2 R dt$$

Let I_{eff} be the effective value of a.c. Then heat produced in time T must be

$$H = I_{eff}^2\, RT$$

$$\therefore \quad I_{eff}^2\, RT = \int_0^T I^2\, R\, dt \quad \text{or} \quad I_{eff}^2 = \frac{1}{T}\int_0^T I^2\, dt$$

But $\frac{1}{T}\int_0^T I^2 dt$ is the mean of the squares of the instantaneous values of a.c. over one complete cycle, hence the effective or virtual value of a.c. equals its root mean square value, *i.e.*,

$$I_{eff} = I_{rms} = \sqrt{\frac{1}{T}\int_0^T I^2\, dt}$$

Now $\int_0^T I^2\, dt = \int_0^T I_0^2 \sin^2 \omega t\, dt$

$$= I_0^2 \int_0^T \frac{1 - \cos 2\omega t}{2}\, dt$$

$$= \frac{I_0^2}{2}\left[t - \frac{\sin 2\,\omega t}{2\omega}\right]_0^T$$

$$= \frac{I_0^2}{2}\left[(T - 0) - \frac{1}{2\omega}\left|\sin \frac{4\pi}{T} t\right|_0^T\right]$$

$$= \frac{I_0^2}{2}\left[T - \frac{1}{2\omega}(\sin 4\pi - \sin 0)\right]$$

$$= \frac{I_0^2}{2}[T - 0] = \frac{I_0^2 T}{2}$$

$$\therefore \quad I_{eff} \text{ or } I_{rms} = \sqrt{\frac{1}{T} \cdot \frac{I_0^2 T}{2}}$$

or $$I_{eff} \text{ or } I_{rms} = \frac{1}{\sqrt{2}} I_0 = 0.707\ I_0$$

Thus the effective or rms value of an a.c. is $\frac{1}{\sqrt{2}}$ *times its peak value.*

Root mean square value of an alternating emf. *It is defined as that value of a steady voltage that produces the same amount of heat in a given resistance as is produced by the given alternating emf when applied to the same resistance for the same time.* It is also called ***virtual*** or ***effective*** value of the alternating emf. It is denoted by $\mathcal{E}_{rms}$ or $\mathcal{E}_{eff}$ or $\mathcal{E}_{v}$.

- **Relation between the rms value and the peak value of an alternating emf.** Suppose an alternating emf $\mathcal{E}$ applied to a resistance R is given by

$$\mathcal{E} = \mathcal{E}_0 \sin \omega t$$

(i) Heat produced in a small time dt will be

$$dH = \frac{\mathcal{E}^2}{R}\, dt = \frac{\mathcal{E}_0^2}{R} \sin^2 \omega t\, dt$$

(ii) Let T be the time period of the alternating emf. Then heat produced in time T will be

$$H = \int dH = \int_0^T \frac{\mathcal{E}_0^2}{R} \sin^2 \omega t\, dt$$

$$= \frac{\mathcal{E}_0^2}{R} \int_0^T \frac{(1 - \cos 2\omega t)}{2}\, dt = \frac{\mathcal{E}_0^2}{2R} \left[t - \frac{\sin 2\omega t}{2\omega} \right]_0^T$$

$$= \frac{\mathcal{E}_0^2}{2R} \left[(T - 0) - \frac{1}{2\omega} \left| \sin \frac{4\pi}{T} t \right|_0^T \right]$$

$$= \frac{\mathcal{E}_0^2}{2R} \left[T - \frac{1}{2\omega} \sin (4\pi - \sin 0) \right]$$

or $$H = \frac{\mathcal{E}_0^2}{2R} [T - 0] = \frac{\mathcal{E}_0^2 T}{2R}$$

(iv) If $\mathcal{E}_{rms}$ is the root mean square value of the alternating emf, then the amount of heat produced by it in the same resistance R in the time T will be

$$H = \frac{\mathcal{E}_{rms}^2 T}{R}$$

From the above two equations, we get

$$\frac{\mathcal{E}_{rms}^2 T}{R} = \frac{\mathcal{E}_0^2 T}{2R}$$

or

$$\mathcal{E}_{rms} = \frac{\mathcal{E}_0}{\sqrt{2}} = 0.707\,\mathcal{E}_0$$