Solutions to the exercises from T.M.Apostol, *Calculus*, vol. 1 assigned to doctoral students in years 2002-2003

andrea battinelli
dipartimento di scienze matematiche e informatiche "R.Magari"
dell'università di Siena
via del Capitano 15 - 53100 Siena
tel: +39-0577-233769/02 fax: /01/30
e-mail: battinelli @unisi.it
web: http//www.batman vai li

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VECTOR ALGEBRA

12.1 Historical introduction

12.2 The vector space of n-tuples of real numbers

12.3 Geometric interpretation for $n \leq 3$

12.4 Exercises

12.4.1 n. 1 (p. 450)

(a) $\mathbf{a} + \mathbf{b} = (5, 0, 9)$.

(b) $\mathbf{a} - \mathbf{b} = (-3, 6, 3)$.

(c) $\mathbf{a} + \mathbf{b} - \mathbf{c} = (3, -1, 4)$.

(d) $7\mathbf{a} - 2\mathbf{b} - 3\mathbf{c} = (-7, 24, 21)$.

(e) $2\mathbf{a} + \mathbf{b} - 3\mathbf{c} = (0, 0, 0)$.

12.4.2 n. 2 (p. 450)

The seven points to be drawn are the following:

$$\left(\frac{7}{3}, 2\right), \left(\frac{5}{2}, \frac{5}{2}\right), \left(\frac{11}{4}, \frac{13}{4}\right), (3, 4), (4, 7), (1, -2), (0, -5)$$

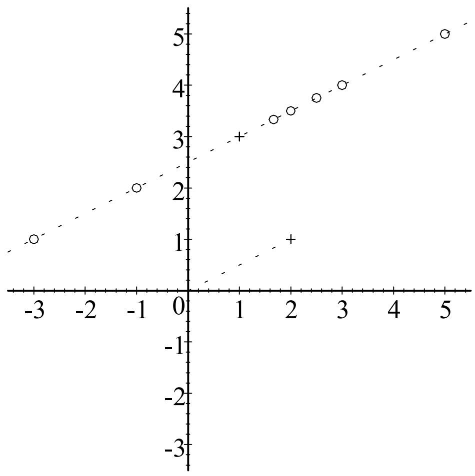
The purpose of the exercise is achieved by drawing, as required, a single picture, containing all the points (included the starting points A and B, I would say).

It can be intuitively seen that, by letting t vary in all $\mathbb{R}$, the straight line through point A with direction given by the vector $\mathbf{b} \equiv \overrightarrow{OB}$ is obtained.

12.4.3 n. 3 (p. 450)

The seven points this time are the following:

$$\left(\frac{5}{3}, \frac{10}{3}\right), \left(2, \frac{7}{2}\right), \left(\frac{5}{2}, \frac{15}{4}\right), (3,4), (5,5), (-1,2), (-3,1)$$



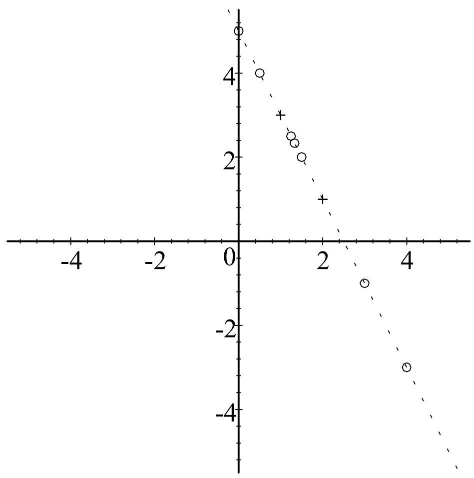
It can be intuitively seen that, by letting t vary in all $\mathbb{R}$, the straight line through B with direction given by the vector $\mathbf{a} \equiv \overrightarrow{OA}$ is obtained.

12.4.4 n. 4 (p. 450)

(**a**) The seven points to be drawn are the following:

$$\left(\frac{3}{2},2\right),\left(\frac{5}{4},\frac{5}{2}\right),\left(\frac{4}{3},\frac{7}{3}\right),(3,-1),(4,-3),\left(\frac{1}{2},4\right),(0,5)$$

The whole purpose of this part of the exercise is achieved by drawing a single picture, containing all the points (included the starting points A and B, I would say). This is made clear, it seems to me, by the question immediately following.

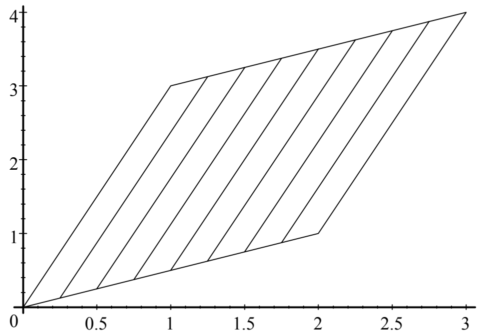


(**b**) It is hard not to notice that all points belong to the same straight line; indeed, as it is going to be clear after the second lecture, all the combinations are affine.

(**c**) If the value of x is fixed at 0 and y varies in $[0,1]$, the segment OB is obtained; the same construction with the value of x fixed at 1 yields the segment AD, where $\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{OB}$, and hence D is the vertex of the parallelogram with three vertices at O, A, and B. Similarly, when $x=\frac{1}{2}$ the segment obtained joins the midpoints of the two sides OA and BD; and it is enough to repeat the construction a few more times to convince oneself that the set

$$\left\{x\mathbf{a}+y\mathbf{b}:(x,y)\in[0,1]^2\right\}$$

is matched by the set of all points of the parallelogram $OADB$. The picture below is made with the value of x fixed at 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, $\frac{5}{4}$, $\frac{3}{2}$, and 2.



(**d**) All the segments in the above construction are substituted by straight lines, and the resulting set is the (infinite) stripe bounded by the lines containing the sides OB and AD, of equation $3x - y = 0$ and $3x - y = 5$ respectively.
(**e**) The whole plane.

12.4.5 n. 5 (p. 450)

$$x\begin{pmatrix}2\\1\end{pmatrix}+y\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

$$\begin{array}{llll} I & 2x+y=c_1 & 3I-II & 5x=3c_1-c_2 \\ II & x+3y=c_2 & 2II-I & 5y=2c_2-c_1 \end{array}$$

$$\begin{pmatrix}c_1\\c_2\end{pmatrix}=\frac{3c_1-c_2}{5}\begin{pmatrix}2\\1\end{pmatrix}+\frac{2c_2-c_1}{5}\begin{pmatrix}1\\3\end{pmatrix}$$

12.4.6 n. 6 (p. 451)

(**a**)

$$\mathbf{d}=x\begin{pmatrix}1\\1\\1\end{pmatrix}+y\begin{pmatrix}0\\1\\1\end{pmatrix}+z\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}x+z\\x+y+z\\x+y\end{pmatrix}$$

(**b**)

$$\begin{array}{lr} I & x+z=0 \\ II & x+y+z=0 \\ III & x+y=0 \end{array} \qquad \begin{array}{ll} I & x=-z \\ (\uparrow)\hookrightarrow II & y=0 \\ (\uparrow)\hookrightarrow III & x=0 \\ (\uparrow)\hookrightarrow I & z=0 \end{array}$$

(**c**)

$$\begin{array}{lr} I & x+z=1 \\ II & x+y+z=2 \\ III & x+y=3 \end{array} \qquad \begin{array}{ll} II-I & y=1 \\ II-III & z=-1 \\ (\uparrow)\hookrightarrow I & x=2 \end{array}$$

12.4.7 n. 7 (p. 451)

(**a**)

$$\mathbf{d} = x \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + z \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x+2z \\ x+y+z \\ x+y+z \end{pmatrix}$$

(**b**)

$$\begin{array}{rr} I & x+2z=0 \\ II & x+y+z=0 \\ III & x+y+z=0 \end{array} \qquad \begin{array}{cl} I & x=-2z \\ (\uparrow) \hookrightarrow II & y=z \\ z \leftarrow 1 & (-2,1,1) \end{array}$$

(**c**)

$$\begin{array}{rr} I & x+2z=1 \\ II & x+y+z=2 \\ III & x+y+z=3 \end{array} \qquad III-II \quad 0=1$$

12.4.8 n. 8 (p. 451)

(**a**)

$$\mathbf{d} = x \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} + z \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x+z \\ x+y+z \\ x+y \\ y \end{pmatrix}$$

(**b**)

$$\begin{array}{cl} I & x+z=0 \\ II & x+y+z=0 \\ III & x+y=0 \\ IV & y=0 \end{array} \qquad \begin{array}{ll} IV & y=0 \\ IV \hookrightarrow III & x=0 \\ (\uparrow) \hookrightarrow I & z=0 \\ II \text{ (check)} & 0=0 \end{array}$$

(**c**)

$$\begin{array}{cl} I & x+z=1 \\ II & x+y+z=5 \\ III & x+y=3 \\ IV & y=4 \end{array} \qquad \begin{array}{ll} IV & y=4 \\ IV \hookrightarrow III & x=-1 \\ (\uparrow) \hookrightarrow I & z=2 \\ II \text{ (check)} & -1+4+2=5 \end{array}$$

(**d**)

$$\begin{array}{cl} I & x+z=1 \\ II & x+y+z=2 \\ III & x+y=3 \\ IV & y=4 \end{array} \qquad II-I-IV \quad 0=-3$$

12.4.9 n. 9 (p. 451)

Let the two vectors u and v be both parallel to the vector w. According to the definition at page 450 (just before the beginning of § 12.4), this means that there are two real nonzero numbers α and β such that $u = \alpha w$ and $v = \beta w$. Then

$$u = \alpha w = \alpha \left(\frac{v}{\beta} \right) = \frac{\alpha}{\beta} v$$

that is, u is parallel to v.

12.4.10 n. 10 (p. 451)

Assumptions:

$$\begin{array}{ll} I & \mathbf{c} = \mathbf{a} + \mathbf{b} \\ II & \exists k \in \mathbb{R} \sim \{0\}\,,\ \mathbf{a} = k\mathbf{d} \end{array}$$

Claim:

$$(\exists h \in \mathbb{R} \sim \{0\}\,,\ \mathbf{c} = h\mathbf{d}) \Leftrightarrow (\exists l \in \mathbb{R} \sim \{0\}\,,\ \mathbf{b} = l\mathbf{d})$$

I present two possible lines of reasoning (among others, probably). If you look carefully, they differ only in the phrasing.

1.

$$\begin{array}{cc} & \exists h \in \mathbb{R} \sim \{0\}\,,\ \mathbf{c} = h\mathbf{d} \\ \underset{I}{\Leftrightarrow} & \exists h \in \mathbb{R} \sim \{0\}\,,\ \mathbf{a} + \mathbf{b} = h\mathbf{d} \\ \underset{II}{\Leftrightarrow} & \exists h \in \mathbb{R} \sim \{0\}\,, \exists k \in \mathbb{R} \sim \{0\}\,,\ k\mathbf{d} + \mathbf{b} = h\mathbf{d} \\ \Leftrightarrow & \exists h \in \mathbb{R} \sim \{0\}\,, \exists k \in \mathbb{R} \sim \{0\}\,,\ \mathbf{b} = (h-k)\,\mathbf{d} \\ (\mathbf{b} \neq \mathbf{0}) & h \neq k \\ \underset{l \equiv h-k}{\Leftrightarrow} & \exists l \in \mathbb{R} \sim \{0\}\,,\ \mathbf{b} = l\mathbf{d} \end{array}$$

2. ($\Rightarrow$) Since (by I) we have $\mathbf{b} = \mathbf{c} - \mathbf{a}$, if $\mathbf{c}$ is parallel to $\mathbf{d}$, then (by II) $\mathbf{b}$ is the difference of two vectors which are both parallel to $\mathbf{d}$; it follows that $\mathbf{b}$, which is nonnull, is parallel to $\mathbf{d}$ too.

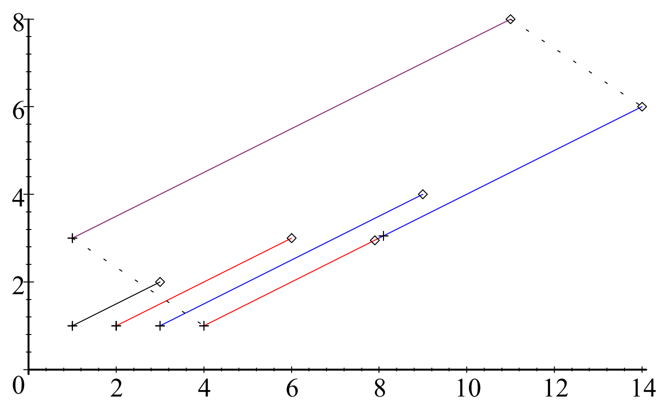
   ($\Leftarrow$) Since (by I) we have $\mathbf{c} = \mathbf{a} + \mathbf{b}$, if $\mathbf{b}$ is parallel to $\mathbf{d}$, then (by II) $\mathbf{c}$ is the sum of two vectors which are both parallel to $\mathbf{d}$; it follows that $\mathbf{c}$, which is nonnull, is parallel to $\mathbf{d}$ too.

12.4.11 n. 11 (p. 451)

(**b**) Here is an illustration of the first distributive law

$$(\alpha + \beta)\,\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$$

with $\mathbf{v} = (2,1)$, $\alpha = 2$, $\beta = 3$. The vectors $\mathbf{v}$, $\alpha\mathbf{v}$, $\beta\mathbf{v}$, $\alpha\mathbf{v} + \beta\mathbf{v}$ are displayed by means of representative oriented segments from left to right, in black, red, blue, and red-blue colour, respectively. The oriented segment representing vector $(\alpha + \beta)\,\mathbf{v}$ is above, in violet. The dotted lines are there just to make it clearer that the two oriented segments representing $\alpha\mathbf{v} + \beta\mathbf{v}$ and $(\alpha + \beta)\,\mathbf{v}$ are congruent, that is, the vectors $\alpha\mathbf{v} + \beta\mathbf{v}$ and $(\alpha + \beta)\,\mathbf{v}$ are the same.

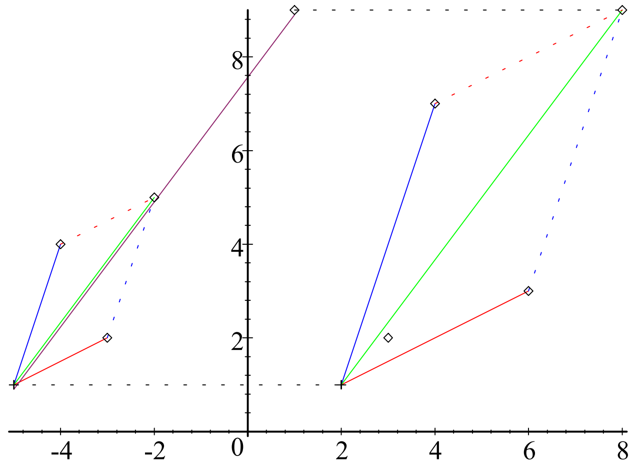


tails are marked with a cross, heads with a diamond

An illustration of the second distributive law

$$\alpha\,(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$$

is provided by means of the vectors $\mathbf{u} = (1,3)$, $\mathbf{v} = (2,1)$, and the scalar $\alpha = 2$. The vectors $\mathbf{u}$ and $\alpha\mathbf{u}$ are represented by means of blue oriented segments; the vectors $\mathbf{v}$ and $\alpha\mathbf{v}$ by means of red ones; $\mathbf{u} + \mathbf{v}$ and $\alpha\,(\mathbf{u} + \mathbf{v})$ by green ones; $\alpha\mathbf{u} + \alpha\mathbf{v}$ is in violet. The original vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u} + \mathbf{v}$ are on the left; the “rescaled” vectors $\alpha\mathbf{u}$, $\alpha\mathbf{v}$, $\alpha\,(\mathbf{u} + \mathbf{v})$ on the right. Again, the black dotted lines are there just to emphasize congruence.



tails are marked with a cross, heads with a diamond

12.4.12 n. 12 (p. 451)

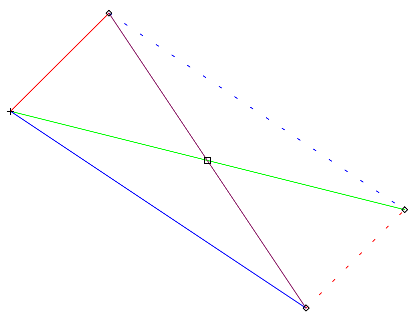
The statement to be proved is better understood if written in homogeneous form, with all vectors represented by oriented segments:

$$\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\overrightarrow{OB} \tag{12.1}$$

Since A and C are opposed vertices, the same holds for O and B; this means that the oriented segment OB covers precisely one diagonal of the parallelogram $OABC$, and AC precisely covers the other (in the rightward-downwards orientation), hence what needs to be proved is the following:

$$\overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OA}\right) = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OC}\right)$$

which is now seen to be an immediate consequence of the distributive properties. In the picture below, $\overrightarrow{OA}$ is in red, $\overrightarrow{OC}$ in blue, $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$ in green, and $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$ in violet.



The geometrical theorem expressed by equation (12.1) is the following:

Theorem 1 *The two diagonals of every parallelogram intersect at a point which divides them in two segments of equal length. In other words, the intersection point of the two diagonals is the midpoint of both.*

Indeed, the lefthand side of (12.1) is the vector represented by the oriented segment OM, where M is the midpoint of diagonal AC (violet), whereas the righthand side is the vector represented by the oriented segment ON, where N is the midpoint of diagonal AB (green). More explicitly, the movement from O to M is described as achieved by the composition of a first movement from O to A with a second movement from A towards C, which stops halfway (red plus half the violet); whereas the movement from A to N is described as a single movement from A toward B, stopping halfway (half the green). Since (12.1) asserts that $\overrightarrow{OM} = \overrightarrow{ON}$, equality between M and N follows, and hence a proof of (12.1) is a proof of the theorem

12.5 The dot product

12.6 Length or norm of a vector

12.7 Orthogonality of vectors

12.8 Exercises

12.8.1 n. 1 (p. 456)

(a) $\langle \mathbf{a}, \mathbf{b} \rangle = -6$

(b) $\langle \mathbf{b}, \mathbf{c} \rangle = 2$

(c) $\langle \mathbf{a}, \mathbf{c} \rangle = 6$

(d) $\langle \mathbf{a}, \mathbf{b} + \mathbf{c} \rangle = 0$

(e) $\langle \mathbf{a} - \mathbf{b}, \mathbf{c} \rangle = 4$

12.8.2 n. 2 (p. 456)

(a) $\langle \mathbf{a}, \mathbf{b} \rangle \mathbf{c} = (2 \cdot 2 + 4 \cdot 6 + (-7) \cdot 3)(3, 4, -5) = 7(3, 4, -5) = (21, 28, -35)$

(b) $\langle \mathbf{a}, \mathbf{b} + \mathbf{c} \rangle = 2 \cdot (2 + 3) + 4 \cdot (6 + 4) + (-7)(3 - 5) = 64$

(c) $\langle \mathbf{a}+\mathbf{b}, \mathbf{c}\rangle = (2+2)\cdot 3 + (4+6)\cdot 4 + (-7+3)\cdot(-5) = 72$

(d) $\mathbf{a}\langle \mathbf{b}, \mathbf{c}\rangle = (2,4,-7)\,(2\cdot 3 + 6\cdot 4 + 3\cdot(-5)) = (2,4,-7)\,15 = (30,60,-105)$

(e) $\frac{\mathbf{a}}{\langle \mathbf{b},\mathbf{c}\rangle} = \frac{(2,4,-7)}{15} = \left(\frac{2}{15}, \frac{4}{15}, -\frac{7}{15}\right)$

12.8.3 n. 3 (p. 456)

The statement is false. Indeed,

$$\langle \mathbf{a}, \mathbf{b}\rangle = \langle \mathbf{a}, \mathbf{c}\rangle \Leftrightarrow \langle \mathbf{a}, \mathbf{b}-\mathbf{c}\rangle = 0 \Leftrightarrow \mathbf{a} \perp \mathbf{b}-\mathbf{c}$$

and the difference $\mathbf{b}-\mathbf{c}$ may well be orthogonal to $\mathbf{a}$ without being the null vector. The simplest example that I can conceive is in $\mathbb{R}^2$:

$$\mathbf{a} = (1,0) \qquad \mathbf{b} = (1,1) \qquad \mathbf{c} = (1,2)$$

See also exercise 1, question (d).

12.8.4 n. 5 (p. 456)

The required vector, of coordinates (x,y,z) must satisfy the two conditions

$$\begin{aligned} \langle (2,1,-1), (x,y,z)\rangle &= 0 \\ \langle (1,-1,2), (x,y,z)\rangle &= 0 \end{aligned}$$

that is,

$$\begin{array}{lll} I & 2x+y-z=0 & \qquad I+II \quad 3x+z=0 \\ II & x-y+2z=0 & \qquad 2I+II \quad 5x+y=0 \end{array}$$

Thus the set of all such vectors can be represented in parametric form as follows:

$$\{(\alpha, -5\alpha, -3\alpha)\}_{\alpha\in\mathbb{R}}$$

12.8.5 n. 6 (p. 456)

$$\left.\begin{array}{c} \langle \mathbf{c}, \mathbf{b}\rangle = 0 \\ \mathbf{c} = x\mathbf{a}+y\mathbf{b} \end{array}\right\} \begin{array}{l} \Rightarrow \quad \langle x\mathbf{a}+y\mathbf{b}, \mathbf{b}\rangle = 0 \\ \Leftrightarrow \quad x\,\langle \mathbf{a}, \mathbf{b}\rangle + y\,\langle \mathbf{b}, \mathbf{b}\rangle = 0 \\ \Leftrightarrow \quad 7x+14y=0 \end{array}$$

Let $(x,y) \equiv (-2,1)$. Then $\mathbf{c} = (1,5,-4) \neq \mathbf{0}$ and $\langle (1,5,-4), (3,1,2)\rangle = 0$.

12.8.6 n. 7 (p. 456)

1 (**solution with explicit mention of coordinates**) From the last condition, for some α to be determined,

$$\mathbf{c} = (\alpha, 2\alpha, -2\alpha)$$

Substituting in the first condition,

$$\mathbf{d} = \mathbf{a} - \mathbf{c} = (2 - \alpha, -1 - 2\alpha, 2 + 2\alpha)$$

Thus the second condition becomes

$$1 \cdot (2 - \alpha) + 2 \cdot (-1 - 2\alpha) - 2\,(2 + 2\alpha) = 0$$

that is,

$$-4 - 9\alpha = 0 \qquad \alpha = -\frac{4}{9}$$

and hence

$$\begin{aligned} \mathbf{c} &= \frac{1}{9}\,(-4, -8, 8) \\ \mathbf{d} &= \frac{1}{9}\,(22, -1, 10) \end{aligned}$$

2 (**same solution, with no mention of coordinates**) From the last condition, for some α to be determined,

$$\mathbf{c} = \alpha\mathbf{b}$$

Substituting in the first condition,

$$\mathbf{d} = \mathbf{a} - \mathbf{c} = \mathbf{a} - \alpha\mathbf{b}$$

Thus the second condition becomes

$$\langle \mathbf{b}, \mathbf{a} - \alpha\mathbf{b} \rangle = 0$$

that is,

$$\langle \mathbf{b}, \mathbf{a} \rangle - \alpha \langle \mathbf{b}, \mathbf{b} \rangle = 0 \qquad \alpha = \frac{\langle \mathbf{b}, \mathbf{a} \rangle}{\langle \mathbf{b}, \mathbf{b} \rangle} = \frac{2 \cdot 1 + (-1) \cdot 2 + 2 \cdot (-2)}{1^2 + 2^2 + (-2)^2} = -\frac{4}{9}$$

and hence

$$\begin{aligned} \mathbf{c} &= \frac{1}{9}\,(-4, -8, 8) \\ \mathbf{d} &= \frac{1}{9}\,(22, -1, 10) \end{aligned}$$

12.8.7 n. 10 (p. 456)

(a) $\mathbf{b} = (1, -1)$ or $\mathbf{b} = (-1, 1)$.

(b) $\mathbf{b} = (-1, -1)$ or $\mathbf{b} = (1, 1)$.

(c) $\mathbf{b} = (-3, -2)$ or $\mathbf{b} = (3, 2)$.

(d) $\mathbf{b} = (b, -a)$ or $\mathbf{b} = (-b, a)$.

12.8.8 n. 13 (p. 456)

If $\mathbf{b}$ is the required vector, the following conditions must be satisfied

$$\langle \mathbf{a}, \mathbf{b} \rangle = 0 \qquad \|\mathbf{b}\| = \|\mathbf{a}\|$$

If $\mathbf{a} = \mathbf{0}$, then $\|\mathbf{a}\| = 0$ and hence $\mathbf{b} = \mathbf{0}$ as well. If $\mathbf{a} \neq \mathbf{0}$, let the coordinates of $\mathbf{a}$ be given by the couple (α, β), and the coordinates of $\mathbf{b}$ by the couple (x, y). The above conditions take the form

$$\alpha x + \beta y = 0 \qquad x^2 + y^2 = \alpha^2 + \beta^2$$

Either α or β must be different from 0. Suppose α is nonzero (the situation is completely analogous if $\beta \neq 0$ is assumed). Then the first equations gives

$$x = -\frac{\beta}{\alpha} y \tag{12.2}$$

and substitution in the second equation yields

$$\left(\frac{\beta^2}{\alpha^2} + 1 \right) y^2 = \alpha^2 + \beta^2$$

that is,

$$y^2 = \frac{\alpha^2 + \beta^2}{\frac{\alpha^2 + \beta^2}{\alpha^2}} = \alpha^2$$

and hence

$$y = \pm \alpha$$

Substituting back in (12.2) gives

$$x = \mp \beta$$

Thus there are only two solutions to the problem, namely, $\mathbf{b} = (-\beta, \alpha)$ and $\mathbf{b} = (\beta, -\alpha)$ In particular,

(a) $\mathbf{b} = (-2, 1)$ or $\mathbf{b} = (2, -1)$.

(b) $\mathbf{b} = (2, -1)$ or $\mathbf{b} = (-2, 1)$.

(c) $\mathbf{b} = (-2, -1)$ or $\mathbf{b} = (2, 1)$.

(d) $\mathbf{b} = (-1, 2)$ or $\mathbf{b} = (1, -2)$

12.8.9 n. 14 (p. 456)

1 (right angle in C; solution with explicit mention of coordinates) Let (x,y,z) be the coordinates of C. If the right angle is in C, the two vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$ must be orthogonal. Thus

$$\langle [(2,-1,1)-(x,y,z)], [(3,-4,-4)-(x,y,z)] \rangle = 0$$

that is,

$$\langle (2,-1,1), (3,-4,-4) \rangle + \langle (x,y,z), (x,y,z) \rangle - \langle (2,-1,1)+(3,-4,-4), (x,y,z) \rangle = 0$$

or

$$x^2+y^2+z^2-5x+5y+3z+6=0$$

The above equation, when rewritten in a more perspicuous way by "completion of the squares"

$$\left(x-\frac{5}{2}\right)^2+\left(y+\frac{5}{2}\right)^2+\left(x+\frac{3}{2}\right)^2=\frac{25}{4}$$

is seen to define the sphere of center $P=\left(\frac{5}{2},-\frac{5}{2},-\frac{3}{2}\right)$ and radius $\frac{5}{2}$.

2 (right angle in C; same solution, with no mention of coordinates) Let

$$\mathbf{a}\equiv\overrightarrow{OA}\quad \mathbf{b}\equiv\overrightarrow{OB}\quad \mathbf{x}\equiv\overrightarrow{OC}$$

(stressing that the point C, hence the vector $\overrightarrow{OC}$, is unknown). Then, orthogonality of the two vectors

$$\overrightarrow{CA}=\mathbf{a}-\mathbf{x}\qquad \overrightarrow{CB}=\mathbf{b}-\mathbf{x}$$

is required; that is,

$$\langle \mathbf{a}-\mathbf{x}, \mathbf{b}-\mathbf{x} \rangle = 0$$

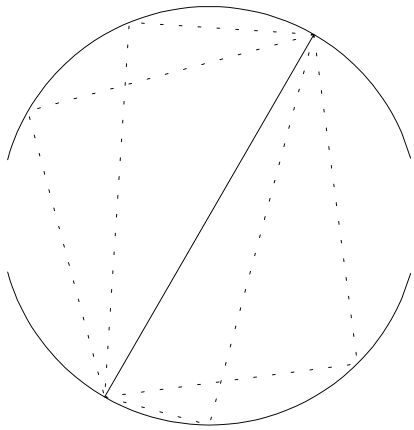
or

$$\langle \mathbf{a}, \mathbf{b} \rangle - \langle \mathbf{a}+\mathbf{b}, \mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{x} \rangle = 0$$

Equivalently,

$$\begin{aligned}
\|\mathbf{x}\|^2-2\left\langle \frac{\mathbf{a}+\mathbf{b}}{2}, \mathbf{x}\right\rangle+\left\|\frac{\mathbf{a}+\mathbf{b}}{2}\right\|^2 &= \left\|\frac{\mathbf{a}+\mathbf{b}}{2}\right\|^2-\langle \mathbf{a}, \mathbf{b} \rangle \\
\left\|\mathbf{x}-\frac{\mathbf{a}+\mathbf{b}}{2}\right\|^2 &= \left\|\frac{\mathbf{a}-\mathbf{b}}{2}\right\|^2
\end{aligned}$$

The last characterization of the solution to our problem shows that the solution set is the locus of all points having fixed distance ($\left\|\frac{\mathbf{a}-\mathbf{b}}{2}\right\|$) from the midpoint of the segment AB. Indeed, if π is any plane containing AB, and C is any point of the circle of π having AB as diameter, it is known by elementary geometry that the triangle ACB is rectangle in C.



3 (right angle in B; solution with explicit mention of coordinates) With this approach, the vectors required to be orthogonal are $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Since $\overrightarrow{BA} = (-1, 3, 5)$ and $\overrightarrow{BC} = (x-1, y+4, z+4)$, the following must hold

$$0 = \left\langle \overrightarrow{BA}, \overrightarrow{BC} \right\rangle = 1 - x + 3y + 12 + 5z + 20$$

that is,

$$x - 3y - 5z = 33$$

The solution set is the plane π through B and orthogonal to $(1, -3, -5)$.
4 (right angle in B; same solution, with no mention of coordinates) Proceeding as in the previous point, with the notation of point 2, the condition to be required is

$$0 = \langle \mathbf{a} - \mathbf{b}, \mathbf{x} - \mathbf{b} \rangle$$

that is,

$$\langle \mathbf{a} - \mathbf{b}, \mathbf{x} \rangle = \langle \mathbf{a} - \mathbf{b}, \mathbf{b} \rangle$$

Thus the solution plane π is seen to be through B and orthogonal to the segment connecting point B to point A.

5 (**right angle in** B) It should be clear at this stage that the solution set in this case is the plane π' through A and orthogonal to AB, of equation

$$\langle \mathbf{b}-\mathbf{a},\mathbf{x}\rangle = \langle \mathbf{b}-\mathbf{a},\mathbf{a}\rangle$$

It is also clear that π and π' are parallel.

12.8.10 n. 15 (p. 456)

$$\begin{array}{ll} I & c_1 - c_2 + 2c_3 = 0 \\ II & 2c_1 + c_2 - c_3 = 0 \end{array} \qquad \begin{array}{ll} I+II & 3c_1 + c_3 = 0 \\ I+2II & 5c_1 + c_2 = 0 \end{array}$$

$$\mathbf{c} = (-1,5,3)$$

12.8.11 n. 16 (p. 456)

$$\begin{array}{l} \mathbf{p} = (3\alpha, 4\alpha) \\ \mathbf{q} = (4\beta, -3\beta) \end{array} \qquad \begin{array}{ll} I & 3\alpha + 4\beta = 1 \\ II & 4\alpha - 3\beta = 2 \end{array}$$

$$\begin{array}{ll} 4II+3I & 25\alpha = 11 \\ 4I-3II & 25\beta = -2 \end{array} \qquad (\alpha,\beta) = \frac{1}{25}(11,-2)$$

$$\mathbf{p} = \frac{1}{25}(33,44) \qquad \mathbf{q} = \frac{1}{25}(-8,6)$$

12.8.12 n. 17 (p. 456)

The question is identical to the one already answered in exercise 7 of this section. Recalling solution 2 to that exercise, we have that $\mathbf{p} \equiv \overrightarrow{OP}$ must be equal to $\alpha\mathbf{b} = \alpha\overrightarrow{OB}$, with

$$\alpha = \frac{\langle \mathbf{a},\mathbf{b}\rangle}{\langle \mathbf{b},\mathbf{b}\rangle} = \frac{10}{4}$$

Thus

$$\begin{aligned} \mathbf{p} &= \left(\frac{5}{2},\frac{5}{2},\frac{5}{2},\frac{5}{2}\right) \\ \mathbf{q} &= \left(-\frac{3}{2},-\frac{1}{2},\frac{1}{2},\frac{3}{2}\right) \end{aligned}$$

12.8.13 n. 19 (p. 456)

It has been quickly seen in class that

$$\|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\langle \mathbf{a}, \mathbf{b}\rangle$$

substituting $-\mathbf{b}$ to $\mathbf{b}$, I obtain

$$\|\mathbf{a}-\mathbf{b}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 - 2\langle \mathbf{a}, \mathbf{b}\rangle$$

By subtraction,

$$\|\mathbf{a}+\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2 = 4\langle \mathbf{a}, \mathbf{b}\rangle$$

as required. You should notice that the above identity has been already used at the end of point 2 in the solution to exercise 14 of the present section.

Concerning the geometrical interpretation of the special case of the above identity

$$\|\mathbf{a}+\mathbf{b}\|^2 = \|\mathbf{a}-\mathbf{b}\|^2 \qquad \text{if and only if} \qquad \langle \mathbf{a}, \mathbf{b}\rangle = 0$$

it is enough to notice that orthogonality of $\mathbf{a}$ and $\mathbf{b}$ is equivalent to the property for the parallelogram $OACB$ (in the given order around the perimeter, that is, with vertex C opposed to the vertex in the origin O) to be a rectangle; and that in such a rectangle $\|\mathbf{a}+\mathbf{b}\|$ and $\|\mathbf{a}-\mathbf{b}\|$ measure the lengths of the two diagonals.

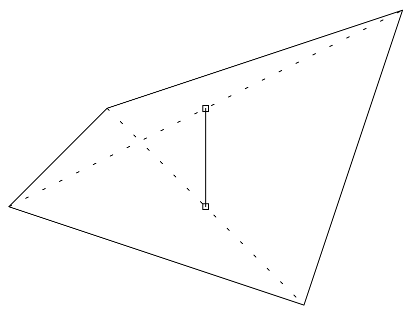
12.8.14 n. 20 (p. 456)

$$\begin{aligned}
\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 &= \langle \mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{b}\rangle + \langle \mathbf{a}-\mathbf{b}, \mathbf{a}-\mathbf{b}\rangle \\
&= \langle \mathbf{a}, \mathbf{a}\rangle + 2\langle \mathbf{a}, \mathbf{b}\rangle + \langle \mathbf{b}, \mathbf{b}\rangle + \langle \mathbf{a}, \mathbf{a}\rangle - 2\langle \mathbf{a}, \mathbf{b}\rangle + \langle \mathbf{b}, \mathbf{b}\rangle \\
&= 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2
\end{aligned}$$

The geometric theorem expressed by the above identity can be stated as follows:

Theorem 2 *In every parallelogram, the sum of the squares of the four sides equals the sum of the squares of the diagonals.*

12.8.15 n. 21 (p. 457)



Let A, B, C, and D be the four vertices of the quadrilateral, starting from left in clockwise order, and let M and N be the midpoint of the diagonals $\overrightarrow{AC}$ and $\overrightarrow{DB}$. In order to simplify the notation, let

$$\mathbf{u} \equiv \overrightarrow{AB}, \quad \mathbf{v} \equiv \overrightarrow{BC}, \quad \mathbf{w} \equiv \overrightarrow{CD}, \quad \mathbf{z} \equiv \overrightarrow{DA} = -(\mathbf{u}+\mathbf{v}+\mathbf{w})$$

Then

$$\begin{aligned}
\mathbf{c} &\equiv \overrightarrow{AC} = \mathbf{u}+\mathbf{v} \quad \mathbf{d} \equiv \overrightarrow{DB} = \mathbf{u}+\mathbf{z} = -(\mathbf{v}+\mathbf{w}) \\
\overrightarrow{MN} &= \overrightarrow{AN} - \overrightarrow{AM} = \mathbf{u} - \frac{1}{2}\mathbf{d} - \frac{1}{2}\mathbf{c} \\
2\overrightarrow{MN} &= 2\mathbf{u} + (\mathbf{v}+\mathbf{w}) - (\mathbf{u}+\mathbf{v}) = \mathbf{w}+\mathbf{u} \\
4\left\|\overrightarrow{MN}\right\|^2 &= \|\mathbf{w}\|^2 + \|\mathbf{u}\|^2 + 2\langle\mathbf{w},\mathbf{u}\rangle \\
\|\mathbf{z}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + 2\langle\mathbf{u},\mathbf{v}\rangle + 2\langle\mathbf{u},\mathbf{w}\rangle + 2\langle\mathbf{v},\mathbf{w}\rangle \\
\|\mathbf{c}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\langle\mathbf{u},\mathbf{v}\rangle \\
\|\mathbf{d}\|^2 &= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + 2\langle\mathbf{v},\mathbf{w}\rangle
\end{aligned}$$

We are now in the position to prove the theorem.

$$\begin{aligned}
& \left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + \|\mathbf{z}\|^2\right) - \left(\|\mathbf{c}\|^2 + \|\mathbf{d}\|^2\right) \\
=\ & \|\mathbf{z}\|^2 - \|\mathbf{v}\|^2 - 2\langle\mathbf{u},\mathbf{v}\rangle - 2\langle\mathbf{v},\mathbf{w}\rangle \\
=\ & \|\mathbf{u}\|^2 + \|\mathbf{w}\|^2 + 2\langle\mathbf{u},\mathbf{w}\rangle \\
=\ & 4\left\|\overrightarrow{MN}\right\|^2
\end{aligned}$$

12.8.16 n. 22 (p. 457)

Orthogonality of $x\mathbf{a}+y\mathbf{b}$ and $4y\mathbf{a}-9x\mathbf{b}$ amounts to

$$\begin{aligned}
0 &= \langle x\mathbf{a}+y\mathbf{b}, 4y\mathbf{a}-9x\mathbf{b}\rangle \\
&= -9\langle\mathbf{a},\mathbf{b}\rangle x^2 + 4\langle\mathbf{a},\mathbf{b}\rangle y^2 + \left(4\|\mathbf{a}\|^2 - 9\|\mathbf{b}\|^2\right)xy
\end{aligned}$$

Since the above must hold for every couple (x,y), choosing $x=0$ and $y=1$ gives $\langle\mathbf{a},\mathbf{b}\rangle = 0$; thus the condition becomes

$$\left(4\|\mathbf{a}\|^2 - 9\|\mathbf{b}\|^2\right)xy = 0$$

and choosing now $x=y=1$ gives

$$4\|\mathbf{a}\|^2 = 9\|\mathbf{b}\|^2$$

Since $\|\mathbf{a}\|$ is known to be equal to 6, it follows that $\|\mathbf{b}\|$ is equal to 4.

Finally, since **a** and **b** have been shown to be orthogonal,

$$\begin{aligned}\|2\mathbf{a}+3\mathbf{b}\|^2 &= \|2\mathbf{a}\|^2+\|3\mathbf{b}\|^2+2\langle 2\mathbf{a},3\mathbf{b}\rangle \\ &= 2^2\cdot 6^2+3^2\cdot 4^2+2\cdot 2\cdot 3\cdot 0 \\ &= 2^5\cdot 3^2\end{aligned}$$

and hence

$$\|2\mathbf{a}+3\mathbf{b}\| = 12\sqrt{2}$$

12.8.17 n. 24 (p. 457)
This is once again the question raised in exercises 7 and 17, in the general context of the linear space $\mathbb{R}^n$ (where $n\in\mathbb{N}$ is arbitrary). Since the coordinate-free version of the solution procedure is completely independent from the number of coordinates of the vectors involved, the full answer to the problem has already been seen to be

$$\begin{aligned}\mathbf{c} &= \frac{\langle \mathbf{b},\mathbf{a}\rangle}{\langle \mathbf{a},\mathbf{a}\rangle}\mathbf{a} \\ \mathbf{d} &= \mathbf{b}-\frac{\langle \mathbf{b},\mathbf{a}\rangle}{\langle \mathbf{a},\mathbf{a}\rangle}\mathbf{a}\end{aligned}$$

12.8.18 n. 25 (p. 457)
(**a**) For every $x\in\mathbb{R}$,

$$\begin{aligned}\|\mathbf{a}+x\mathbf{b}\|^2 &= \|\mathbf{a}\|^2+x^2\|\mathbf{b}\|^2+2x\langle \mathbf{a},\mathbf{b}\rangle \\ &= \|\mathbf{a}\|^2+x^2\|\mathbf{b}\|^2 \qquad \text{if } \mathbf{a}\perp\mathbf{b} \\ &\geq \|\mathbf{a}\|^2 \qquad \text{if } \mathbf{a}\perp\mathbf{b}\end{aligned}$$

(**b**) Since the norm of any vector is nonnegative, the following biconditional is true:

$$(\forall x\in\mathbb{R},\quad \|\mathbf{a}+x\mathbf{b}\|\geq\|\mathbf{a}\|)\Leftrightarrow\left(\forall x\in\mathbb{R},\quad \|\mathbf{a}+x\mathbf{b}\|^2\geq\|\mathbf{a}\|^2\right)$$

Moreover, by pure computation, the following biconditional is true:

$$\forall x\in\mathbb{R},\quad \|\mathbf{a}+x\mathbf{b}\|^2-\|\mathbf{a}\|^2\geq 0\Leftrightarrow\forall x\in\mathbb{R},\quad x^2\|\mathbf{b}\|^2+2x\langle \mathbf{a},\mathbf{b}\rangle\geq 0$$

If $\langle \mathbf{a},\mathbf{b}\rangle$ is positive, the trinomial $x^2\|\mathbf{b}\|^2+2x\langle \mathbf{a},\mathbf{b}\rangle$ is negative for all x in the open interval $\left(-\frac{2\langle \mathbf{a},\mathbf{b}\rangle}{\|\mathbf{b}\|^2},0\right)$; if it is negative, the trinomial is negative in the open interval $\left(0,-\frac{2\langle \mathbf{a},\mathbf{b}\rangle}{\|\mathbf{b}\|^2}\right)$. It follows that the trinomial can be nonnegative for every $x\in\mathbb{R}$ only if $\langle \mathbf{a},\mathbf{b}\rangle$ is zero, that is, only if it reduces to the second degree term $x^2\|\mathbf{b}\|^2$. In conclusion, I have proved that the conditional

$$(\forall x\in\mathbb{R},\quad \|\mathbf{a}+x\mathbf{b}\|\geq\|\mathbf{a}\|)\Rightarrow\langle \mathbf{a},\mathbf{b}\rangle=0$$

is true.

12.9 Projections. Angle between vectors in n-space

12.10 The unit coordinate vectors

12.11 Exercises

12.11.1 n. 1 (p. 460)

$$\langle \mathbf{a}, \mathbf{b} \rangle = 11 \qquad \|\mathbf{b}\|^2 = 9$$

The projection of **a** along **b** is

$$\frac{11}{9}\mathbf{b} = \left(\frac{11}{9}, \frac{22}{9}, \frac{22}{9} \right)$$

12.11.2 n. 2 (p. 460)

$$\langle \mathbf{a}, \mathbf{b} \rangle = 10 \qquad \|\mathbf{b}\|^2 = 4$$

The projection of **a** along **b** is

$$\frac{10}{4}\mathbf{b} = \left(\frac{5}{2}, \frac{5}{2}, \frac{5}{2}, \frac{5}{2} \right)$$

12.11.3 n. 3 (p. 460)

(**a**)

$$\begin{aligned} \cos \widehat{\mathbf{a}\,\mathbf{i}} &= \frac{\langle \mathbf{a}, \mathbf{i} \rangle}{\|\mathbf{a}\| \, \|\mathbf{i}\|} = \frac{6}{7} \\ \cos \widehat{\mathbf{a}\,\mathbf{j}} &= \frac{\langle \mathbf{a}, \mathbf{j} \rangle}{\|\mathbf{a}\| \, \|\mathbf{j}\|} = \frac{3}{7} \\ \cos \widehat{\mathbf{a}\,\mathbf{i}} &= \frac{\langle \mathbf{a}, \mathbf{k} \rangle}{\|\mathbf{a}\| \, \|\mathbf{k}\|} = -\frac{2}{7} \end{aligned}$$

(**b**) There are just two vectors as required, the unit direction vector **u** of **a**, and its opposite:

$$\begin{aligned} \mathbf{u} &= \frac{\mathbf{a}}{\|\mathbf{a}\|} = \left(\frac{6}{7}, \frac{3}{7}, -\frac{2}{7} \right) \\ -\frac{\mathbf{a}}{\|\mathbf{a}\|} &= \left(-\frac{6}{7}, -\frac{3}{7}, \frac{2}{7} \right) \end{aligned}$$

12.11.4 n. 5 (p. 460)
Let

$$\begin{aligned} A &\equiv (2,-1,1) & B &\equiv (1,-3,-5) & C &\equiv (3,-4,-4) \\ \mathbf{p} &\equiv \overrightarrow{BC} = (2,-1,1) & \mathbf{q} &\equiv \overrightarrow{CA} = (-1,3,5) & \mathbf{r} &\equiv \overrightarrow{AB} = (-1,-2,-6) \end{aligned}$$

Then

$$\begin{aligned} \cos\widehat{A} &= \frac{\langle -\mathbf{q},\mathbf{r}\rangle}{\|-\mathbf{q}\|\,\|\mathbf{r}\|} = \frac{35}{\sqrt{35}\sqrt{41}} = \frac{\sqrt{35}\sqrt{41}}{41} \\ \cos\widehat{B} &= \frac{\langle \mathbf{p},-\mathbf{r}\rangle}{\|\mathbf{p}\|\,\|-\mathbf{r}\|} = \frac{6}{\sqrt{6}\sqrt{41}} = \frac{\sqrt{6}\sqrt{41}}{41} \\ \cos\widehat{C} &= \frac{\langle -\mathbf{p},\mathbf{q}\rangle}{\|-\mathbf{p}\|\,\|\mathbf{q}\|} = \frac{0}{\sqrt{6}\sqrt{35}} = 0 \end{aligned}$$

There is some funny occurrence in this exercise, which makes a wrong solution apparently correct (or almost correct), if one looks only at numerical results. The angles in A, B, C, as implicitly argued above, are more precisely described as $B\widehat{A}C$, $C\widehat{B}A$, $A\widehat{C}B$; that is, as the three angles of triangle ABC. If some confusion is made between points and vectors, and/or angles, one may be led into operate directly with the coordinates of points A, B, C in place of the coordinates of vectors $\overrightarrow{BC}$, $\overrightarrow{AC}$, $\overrightarrow{AB}$, respectively. This amounts to work, as a matter of fact, with the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, therefore computing

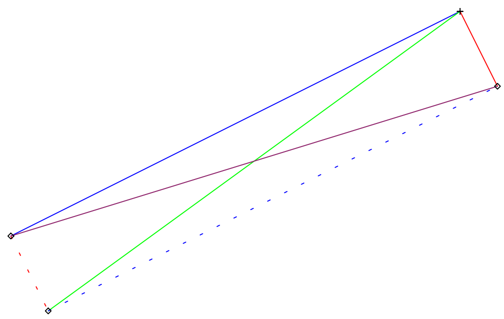
$$\begin{aligned} \frac{\left\langle \overrightarrow{OB},\overrightarrow{OC}\right\rangle}{\left\|\overrightarrow{OB}\right\|\,\left\|\overrightarrow{OC}\right\|} &= \cos B\widehat{O}C \qquad \text{an angle of triangle } OBC \\ \frac{\left\langle \overrightarrow{OC},\overrightarrow{OA}\right\rangle}{\left\|\overrightarrow{OC}\right\|\,\left\|\overrightarrow{OA}\right\|} &= \cos C\widehat{O}A \qquad \text{an angle of triangle } OCA \\ \frac{\left\langle \overrightarrow{OA},\overrightarrow{OB}\right\rangle}{\left\|\overrightarrow{OA}\right\|\,\left\|\overrightarrow{OB}\right\|} &= \cos A\widehat{O}B \qquad \text{an angle of triangle } OAB \end{aligned}$$

instead of $\cos B\widehat{A}C$, $\cos C\widehat{B}A$, $\cos A\widehat{C}B$, respectively. Up to this point, there is nothing funny in doing that; it's only a mistake, and a fairly bad one, being of conceptual type. The funny thing is in the numerical data of the exercise: it so happens that

1. points A, B, C are coplanar with the origin O; more than that,
2. $\overrightarrow{OA} = \overrightarrow{BC}$ and $\overrightarrow{OB} = \overrightarrow{AC}$; more than that,

3. $A\widehat{O}B$ is a right angle.

Point 1 already singles out a somewhat special situation, but point 2 makes $OACB$ a parallelogram, and point 3 makes it even a rectangle.



$\overrightarrow{OA}$ red, $\overrightarrow{OB}$ blue, $\overrightarrow{OC}$ green, AB violet

It turns out, therefore, that

$$\left\|\overrightarrow{OA}\right\| = \left\|\overrightarrow{BC}\right\| = \|\mathbf{p}\| \quad \left\|\overrightarrow{OB}\right\| = \left\|\overrightarrow{AC}\right\| = \|-\mathbf{q}\| = \|\mathbf{q}\| \quad \left\|\overrightarrow{OC}\right\| = \left\|\overrightarrow{AB}\right\| = \|\mathbf{r}\|$$

$$C\widehat{O}A = C\widehat{B}A \qquad B\widehat{O}C = B\widehat{A}C \qquad A\widehat{O}B = A\widehat{C}B$$

and such a special circumstance leads a wrong solution to yield the right numbers.
12.11.5 n. 6 (p. 460)
Since

$$\|\mathbf{a}+\mathbf{c}\pm\mathbf{b}\|^2 = \|\mathbf{a}+\mathbf{c}\|^2 + \|\mathbf{b}\|^2 \pm 2\langle\mathbf{a}+\mathbf{c},\mathbf{b}\rangle$$

from

$$\|\mathbf{a}+\mathbf{c}+\mathbf{b}\| = \|\mathbf{a}+\mathbf{c}-\mathbf{b}\|$$

it is possible to deduce

$$\langle\mathbf{a}+\mathbf{c},\mathbf{b}\rangle = 0$$

This is certainly true, as a particular case, if $\mathbf{c} = -\mathbf{a}$, which immediately implies

$$\begin{aligned}\widehat{\mathbf{ac}} &= \pi \\ \widehat{\mathbf{bc}} &= \pi - \widehat{\mathbf{ab}} = \frac{7}{8}\pi\end{aligned}$$

Moreover, even if $\mathbf{a}+\mathbf{c}=\mathbf{0}$ is not assumed, the same conclusion holds. Indeed, from

$$\langle \mathbf{c}, \mathbf{b} \rangle = - \langle \mathbf{a}, \mathbf{b} \rangle$$

and

$$\|\mathbf{c}\| = \|\mathbf{a}\|$$

it is easy to check that

$$\cos \widehat{\mathbf{bc}} = \frac{\langle \mathbf{b}, \mathbf{c} \rangle}{\|\mathbf{b}\| \|\mathbf{c}\|} = -\frac{\langle \mathbf{a}, \mathbf{c} \rangle}{\|\mathbf{b}\| \|\mathbf{a}\|} = -\cos \widehat{\mathbf{ab}}$$

and hence that $\widehat{\mathbf{bc}} = \pi \pm \widehat{\mathbf{ac}} = \frac{7}{8}\pi, \frac{9}{8}\pi$ (the second value being superfluous).

12.11.6 n. 8 (p. 460)

We have

$$\begin{aligned} \|\mathbf{a}\| &= \sqrt{n} \qquad \|\mathbf{b}_n\| = \sqrt{\frac{n(n+1)(2n+1)}{6}} \qquad \langle \mathbf{a}, \mathbf{b}_n \rangle = \frac{n(n+1)}{2} \\ \cos \widehat{\mathbf{a}\,\mathbf{b}_n} &= \frac{\frac{n(n+1)}{2}}{\sqrt{n}\sqrt{\frac{n(n+1)(2n+1)}{6}}} = \sqrt{\frac{\frac{n^2(n+1)^2}{4}}{\frac{n^2(n+1)(2n+1)}{6}}} = \sqrt{\frac{3}{2}\frac{n+1}{2n+1}} \\ \lim_{n\to+\infty} \cos \widehat{\mathbf{a}\,\mathbf{b}_n} &= \frac{\sqrt{3}}{2} \qquad \lim_{n\to+\infty} \widehat{\mathbf{a}\,\mathbf{b}_n} = \frac{\pi}{3} \end{aligned}$$

12.11.7 n. 10 (p. 461)

(**a**)

$$\begin{aligned} \langle \mathbf{a}, \mathbf{b} \rangle &= \cos\vartheta \sin\vartheta - \cos\vartheta \sin\vartheta = 0 \\ \|\mathbf{a}\|^2 &= \|\mathbf{b}\|^2 = \cos^2\vartheta + \sin^2\vartheta = 1 \end{aligned}$$

(**b**) The system

$$\begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

that is,

$$\begin{pmatrix} \cos\vartheta - 1 & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has the trivial solution as its unique solution if

$$\begin{vmatrix} \cos\vartheta - 1 & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta - 1 \end{vmatrix} \neq 0$$

The computation gives

$$\begin{aligned} 1+\cos^2\vartheta+\sin^2\vartheta-2\cos\vartheta &\neq 0 \\ 2\left(1-\cos\vartheta\right) &\neq 0 \\ \cos\vartheta &\neq 1 \\ \vartheta &\notin \{2k\pi\}_{k\in\mathbb{Z}} \end{aligned}$$

Thus if

$$\vartheta \in \{(-2k\pi,(2k+1)\,\pi)\}_{k\in\mathbb{Z}}$$

the only vector satisfying the required condition is $(0,0)$. On the other hand, if

$$\vartheta \in \{2k\pi\}_{k\in\mathbb{Z}}$$

the coefficient matrix of the above system is the identity matrix, and every vector in $\mathbb{R}^2$ satisfies the required condition.

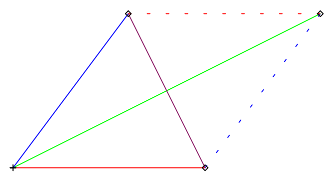
12.11.8 n. 11 (p. 461)

Let $OABC$ be a rhombus, which I have taken (without loss of generality) with one vertex in the origin O and with vertex B opposed to O. Let

$$\mathbf{a}\equiv\overrightarrow{OA}\quad\text{(red)}\quad \mathbf{b}\equiv\overrightarrow{OB}\quad\text{(green)}\quad \mathbf{c}\equiv\overrightarrow{OC}\quad\text{(blue)}$$

Since $OABC$ is a parallelogram, the oriented segments AB (blue, dashed) and OB are congruent, and the same is true for CB (red, dashed) and OA. Thus

$$\overrightarrow{AB}=\mathbf{b}\qquad\overrightarrow{CB}=\mathbf{a}$$



From elementary geometry (see exercise 12 of section 4) the intersection point M of the two diagonals OB (green) and AC (violet) is the midpoint of both.

The assumption that $OABC$ is a rhombus is expressed by the equality

$$\|\mathbf{a}\|=\|\mathbf{c}\| \qquad \text{((rhombus))}$$

The statement to be proved is orthogonality between the diagonals

$$\left\langle \overrightarrow{OB}, \overrightarrow{AC} \right\rangle = 0$$

that is,

$$\langle \mathbf{a} + \mathbf{c}, \mathbf{c} - \mathbf{a} \rangle = 0$$

or

$$\|\mathbf{c}\|^2 - \|\mathbf{a}\|^2 + \langle \mathbf{a}, \mathbf{c} \rangle - \langle \mathbf{c}, \mathbf{a} \rangle = 0$$

and hence (by commutativity)

$$\|\mathbf{c}\|^2 = \|\mathbf{a}\|^2$$

The last equality is an obvious consequence of ((rhombus)). As a matter of fact, since norms are nonnegative real numbers, the converse is true, too. Thus a parallelogram has orthogonal diagonals if **and only if** it is a rhombus.

12.11.9 n. 13 (p. 461)

The equality to be proved is straightforward. The "law of cosines" is often called

Theorem 3 (Carnot) *In every triangle, the square of each side is the sum of the squares of the other two sides, minus their double product multiplied by the cosine of the angle they form.*

The equality in exam can be readily interpreted according to the theorem's statement, since in every parallelogram $ABCD$ with $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{AC}$ the diagonal vector $\overrightarrow{CB}$ is equal to $\mathbf{a} - \mathbf{b}$, so that the triangle ABC has side vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{a} - \mathbf{b}$. The theorem specializes to Pythagoras' theorem when $\mathbf{a} \perp \mathbf{b}$.

12.11.10 n. 17 (p. 461)

(**a**) That the function

$$\mathbb{R}^n \to \mathbb{R}, \ \ \mathbf{a} \mapsto \sum_{i \in \mathbf{n}} |a_i|$$

is positive can be seen by the same arguments used for the ordinary norm; nonnegativity is obvious, and strict positivity still relies on the fact that a sum of concordant numbers can only be zero if all the addends are zero. Homogeneity is clear, too:

$$\begin{aligned} \forall \mathbf{a} &\in \mathbb{R}^n, \forall \alpha \in \mathbb{R}, \\ \|\alpha \mathbf{a}\| &= \sum_{i \in \mathbf{n}} |\alpha a_i| = \sum_{i \in \mathbf{n}} |\alpha| \, |a_i| = |\alpha| \sum_{i \in \mathbf{n}} |a_i| = |\alpha| \, \|\mathbf{a}\| \end{aligned}$$

Finally, the triangle inequality is much simpler to prove for the present norm (sometimes referred to as "the taxi-cab norm") than for the euclidean norm:

$$\begin{aligned}\forall \mathbf{a} &\in \mathbb{R}^n, \forall \mathbf{b} \in \mathbb{R}^n, \\ \|\mathbf{a}+\mathbf{b}\| &= \sum_{i\in\mathbf{n}} |a_i+b_i| \le \sum_{i\in\mathbf{n}} |a_i|+|b_i| = \sum_{i\in\mathbf{n}} |a_i| + \sum_{i\in\mathbf{n}} |b_i| \\ &= \|\mathbf{a}\| + \|\mathbf{b}\|\end{aligned}$$

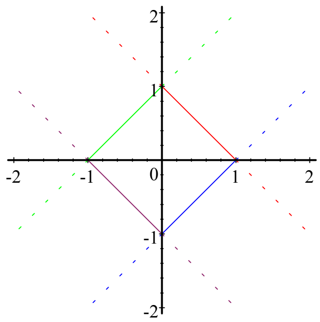
(**b**) The subset of $\mathbb{R}^2$ to be described is

$$\begin{aligned} S &\equiv \left\{(x,y)\in\mathbb{R}^2 : |x|+|y|=1\right\} \\ &= S_{++}\cup S_{-+}\cup S_{--}\cup S_{+-}\end{aligned}$$

where

$$\begin{array}{ll} S_{++} \equiv \left\{(x,y)\in\mathbb{R}_+^2 : x+y=1\right\} & \text{(red)} \\ S_{-+} \equiv \left\{(x,y)\in\mathbb{R}_-\times\mathbb{R}_+ : -x+y=1\right\} & \text{(green)} \\ S_{--} \equiv \left\{(x,y)\in\mathbb{R}_-^2 : -x-y=1\right\} & \text{(violet)} \\ S_{+-} \equiv \left\{(x,y)\in\mathbb{R}_+\times\mathbb{R}_- : x-y=1\right\} & \text{(blue)} \end{array}$$

Once the lines whose equations appear in the definitions of the four sets above are drawn, it is apparent that S is a square, with sides parallel to the quadrant bisectrices



(**c**) The function

$$f:\mathbb{R}^n\longrightarrow\mathbb{R},\ \ \mathbf{a}\mapsto\sum_{i\in\mathbf{n}}|a_i|$$

is nonnegative, but not positive (e.g., for $n=2$, $f(x,-x)=0\ \forall x\in\mathbb{R}$). It is homogeneous:

$$\begin{aligned}\forall \mathbf{a} &\in \mathbb{R}^n, \forall \alpha \in \mathbb{R}, \\ \|\alpha\mathbf{a}\| &= \left|\sum_{i\in\mathbf{n}} \alpha a_i\right| = \left|\alpha\sum_{i\in\mathbf{n}} a_i\right| = |\alpha|\left|\sum_{i\in\mathbf{n}} a_i\right| = |\alpha|\,\|\mathbf{a}\|\end{aligned}$$

Finally, f is subadditive, too (this is another way of saying that the triangle inequality holds). Indeed,

$$\begin{aligned}
\forall \mathbf{a} &\in \mathbb{R}^n, \forall \mathbf{b} \in \mathbb{R}^n, \\
\|\mathbf{a}+\mathbf{b}\| &= \left|\sum_{i\in\mathbf{n}} (a_i + b_i)\right| = \left|\sum_{i\in\mathbf{n}} a_i + \sum_{i\in\mathbf{n}} b_i\right| \leq \left|\sum_{i\in\mathbf{n}} a_i\right| + \left|\sum_{i\in\mathbf{n}} b_i\right| \\
&= \|\mathbf{a}\| + \|\mathbf{b}\|
\end{aligned}$$

12.12 The linear span of a finite set of vectors

12.13 Linear independence

12.14 Bases

12.15 Exercises

12.15.1 n. 1 (p. 467)

$$x\,(\mathbf{i}-\mathbf{j}) + y\,(\mathbf{i}+\mathbf{j}) = (x+y, y-x)$$

(a) $x+y=1$ and $y-x=0$ yield $(x,y) = \left(\frac{1}{2}, \frac{1}{2}\right)$.

(b) $x+y=0$ and $y-x=1$ yield $(x,y) = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

(c) $x+y=3$ and $y-x=-5$ yield $(x,y) = (4,-1)$.

(d) $x+y=7$ and $y-x=5$ yield $(x,y) = (1,6)$.

12.15.2 n. 3 (p. 467)

$$\begin{array}{lllll}
I & 2x+y=2 & \qquad & I+III & 3x=9 \\
II & -x+2y=-11 & & II+III & y=-4 \\
III & x-y=7 & & III \text{ (check)} & 3+4=7
\end{array}$$

The solution is $(x,y) = (3,-4)$.

12.15.3 n. 5 (p. 467)

(**a**) If there exists some $\alpha \in \mathbb{R} \sim \{0\}$ such that* $\mathbf{a} = \alpha\mathbf{b}$, then the linear combination $1\mathbf{a} - \alpha\mathbf{b}$ is nontrivial and it generates the null vector; **a** and **b** are linearly dependent.
(**b**) The argument is best formulated by counterposition, by proving that if **a** and **b** are linearly dependent, then they are parallel. Let **a** and **b** nontrivially generate the null vector: $\alpha\mathbf{a} + \beta\mathbf{b} = \mathbf{0}$ (according to the definition, at least one between α and β is nonzero; but in the present case they are both nonzero, since **a** and **b** have been assumed both different from the null vector; indeed, $\alpha\mathbf{a} + \beta\mathbf{b} = \mathbf{0}$ with $\alpha = 0$ and $\beta \neq 0$ implies $\mathbf{b} = \mathbf{0}$, and $\alpha\mathbf{a} + \beta\mathbf{b} = \mathbf{0}$ with $\beta = 0$ and $\alpha \neq 0$ implies $\mathbf{a} = \mathbf{0}$). Thus $\alpha\mathbf{a} = -\beta\mathbf{b}$, and hence $\mathbf{a} = -\frac{\beta}{\alpha}\mathbf{b}$ (or $\mathbf{b} = -\frac{\alpha}{\beta}\mathbf{a}$, if you prefer).

12.15.4 n. 6 (p. 467)

Linear independency of the vectors (a, b) and (c, d) has been seen in the lectures (proof of Steinitz' theorem, part 1) to be equivalent to non existence of nontrivial solutions to the system

$$\begin{aligned} ax + cy &= 0 \\ bx + dy &= 0 \end{aligned}$$

The system is linear and homogeneous, and has unknowns and equations in equal number (hence part 2 of the proof of Steinitz' theorem does not apply). I argue by the principle of counterposition. If a nontrivial solution $(\overline{x}, \overline{y})$ exists, both (a, c) and (b, d) must be proportional to $(\overline{y}, -\overline{x})$ (see exercise 10, section 8), and hence to each other. Then, from $(a, c) = h(b, d)$ it is immediate to derive $ad - bc = 0$. The converse is immediate, too

12.15.5 n. 7 (p. 467)

By the previous exercise, it is enough to require

$$\begin{aligned} (1+t)^2 - (1-t)^2 &\neq 0 \\ 4t &\neq 0 \\ t &\neq 0 \end{aligned}$$

12.15.6 n. 8 (p. 467)

(**a**) The linear combination

$$1\mathbf{i} + 1\mathbf{j} + 1\mathbf{k} + (-1)(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

is nontrivial and spans the null vector.
(**b**) Since, for every $(\alpha, \beta, \gamma) \in \mathbb{R}^3$,

$$\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k} = (\alpha, \beta, \gamma)$$

*Even if parallelism is defined more broadly – see the footnote in exercise 9 of section 4 – α cannot be zero in the present case, because both **a** and **b** have been assumed different from the null vector.

it is clear that

$$\forall (\alpha,\beta,\gamma) \in \mathbb{R}^3, \ \ (\alpha\mathbf{i}+\beta\mathbf{j}+\gamma\mathbf{k}=\mathbf{0}) \Rightarrow \alpha=\beta=\gamma=0$$

so that the triple $(\mathbf{i},\mathbf{j},\mathbf{k})$ is linearly independent.
(**c**) Similarly, for every $(\alpha,\beta,\gamma) \in \mathbb{R}^3$,

$$\alpha\mathbf{i}+\beta\mathbf{j}+\gamma(\mathbf{i}+\mathbf{j}+\mathbf{k}) = (\alpha+\gamma,\beta+\gamma,\gamma)$$

and hence from

$$\alpha\mathbf{i}+\beta\mathbf{j}+\gamma(\mathbf{i}+\mathbf{j}+\mathbf{k}) = \mathbf{0}$$

it follows

$$\alpha+\gamma=0 \quad \beta+\gamma=0 \quad \gamma=0$$

that is,

$$\alpha=\beta=\gamma=0$$

showing that the triple $(\mathbf{i},\mathbf{j},\mathbf{i}+\mathbf{j}+\mathbf{k})$ is linearly independent.
(**d**) The last argument can be repeated almost verbatim for triples $(\mathbf{i},\mathbf{i}+\mathbf{j}+\mathbf{k},\mathbf{k})$ and $(\mathbf{i}+\mathbf{j}+\mathbf{k},\mathbf{j},\mathbf{k})$, taking into account that

$$\begin{aligned} \alpha\mathbf{i}+\beta(\mathbf{i}+\mathbf{j}+\mathbf{k})+\gamma\mathbf{k} &= (\alpha+\beta,\beta,\beta+\gamma) \\ \alpha(\mathbf{i}+\mathbf{j}+\mathbf{k})+\beta\mathbf{j}+\gamma\mathbf{k} &= (\alpha,\alpha+\beta,\alpha+\gamma) \end{aligned}$$

12.15.7 n. 10 (p. 467)
(**a**) Again from the proof of Steinitz' theorem, part 1, consider the system

$$\begin{array}{ll} I & x+y+z=0 \\ II & y+z=0 \\ III & 3z=0 \end{array}$$

It is immediate to derive that its unique solution is the trivial one, and hence that the given triple is linearly independent.
(**b**) We need to consider the following two systems:

$$\begin{array}{ll} I & x+y+z=0 \\ II & y+z=1 \\ III & 3z=0 \end{array} \qquad \begin{array}{ll} I & x+y+z=0 \\ II & y+z=0 \\ III & 3z=1 \end{array}$$

It is again immediate that the unique solutions to the systems are $(-1,1,0)$ and $\left(0,-\frac{1}{3},\frac{1}{3}\right)$ respectively.

(**c**) The system to study is the following:

$$\begin{array}{lll} I & x+y+z=2 & III \qquad\qquad z=\frac{5}{3} \\ II & y+z=-3 & (\uparrow)\hookrightarrow II \quad y=-\frac{14}{3} \\ III & 3z=5 & (\uparrow)\hookrightarrow I \quad\ \ x=5 \end{array}$$

(**d**) For an arbitrary triple (a, b, c), the system

$$\begin{array}{ll} I & x+y+z=a \\ II & y+z=b \\ III & 3z=c \end{array}$$

has the (unique) solution $\left(a-b, b-\frac{c}{3}, \frac{c}{3}\right)$. Thus the given triple spans $\mathbb{R}^3$, and it is linearly independent (as seen at a).

12.15.8 n. 12 (p. 467)

(**a**) Let $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=\mathbf{0}$, that is,

$$\begin{aligned} x+z &= 0 \\ x+y+z &= 0 \\ x+y &= 0 \\ y &= 0 \end{aligned}$$

then y, x, and z are in turn seen to be equal to 0, from the fourth, third, and first equation in that order. Thus $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is a linearly independent triple.

(**b**) Any nontrivial linear combination $\mathbf{d}$ of the given vectors makes $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ a linearly dependent quadruple. For example, $\mathbf{d} \equiv \mathbf{a}+\mathbf{b}+\mathbf{c}=(2,3,2,1)$

(**c**) Let $\mathbf{e} \equiv (-1,-1,1,3)$, and suppose $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+t\mathbf{e}=\mathbf{0}$, that is,

$$\begin{aligned} x+z-t &= 0 \\ x+y+z-t &= 0 \\ x+y+t &= 0 \\ y+3t &= 0 \end{aligned}$$

Then, subtracting the first equation from the second, gives $y=0$, and hence t, x, and z are in turn seen to be equal to 0, from the fourth, third, and first equation in that order. Thus $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e})$ is linearly independent.

(**d**) The coordinates of $\mathbf{x}$ with respect to $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}\}$, which has just been seen to be a basis of $\mathbb{R}^4$, are given as the solution (s, t, u, v) to the following system:

$$\begin{aligned} s+u-v &= 1 \\ s+t+u-v &= 2 \\ s+t+v &= 3 \\ t+3v &= 4 \end{aligned}$$

It is more direct and orderly to work just with the table formed by the system (extended) matrix $\begin{pmatrix} \mathbf{A} & \mathbf{x} \end{pmatrix}$, and to perform elementary row operations and column exchanges.

$$\begin{pmatrix} s & t & u & v & \mathbf{x} \\ \\ 1 & 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 3 & 4 \end{pmatrix}$$

$\mathbf{a}_1 \leftrightarrow \mathbf{a}_3,\ {}_2\mathbf{a}' \leftarrow {}_2\mathbf{a}' - {}_1\mathbf{a}'$

$$\begin{pmatrix} u & t & s & v & \mathbf{x} \\ \\ 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 3 & 4 \end{pmatrix}$$

$\mathbf{a}_2 \leftrightarrow \mathbf{a}_3,\ {}_2\mathbf{a}' \leftrightarrow {}_3\mathbf{a}'$

$$\begin{pmatrix} u & s & t & v & \mathbf{x} \\ \\ 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 & 4 \end{pmatrix}$$

${}_4\mathbf{a}' \leftarrow {}_4\mathbf{a}' - {}_3\mathbf{a}'$

$$\begin{pmatrix} u & s & t & v & \mathbf{x} \\ \\ 1 & 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 3 \end{pmatrix}$$

Thus the system has been given the following form:

$$\begin{aligned} u + s - v &= 1 \\ s + t + v &= 3 \\ t &= 1 \\ 3v &= 3 \end{aligned}$$

which is easily solved from the bottom to the top: $(s, t, u, v) = (1, 1, 1, 1)$.

12.15.9 n. 13 (p. 467)

(**a**)

$$
\begin{aligned}
&\alpha\left(\sqrt{3},1,0\right)+\beta\left(1,\sqrt{3},1\right)+\gamma\left(0,1,\sqrt{3}\right)=(0,0,0)\\
\Leftrightarrow\ &\begin{array}{ll} I & \sqrt{3}\alpha+\beta=0\\ II & \alpha+\sqrt{3}\beta+\gamma=0\\ III & \beta+\sqrt{3}\gamma=0\end{array}\\
\Leftrightarrow\ &\begin{array}{ll} I & \sqrt{3}\alpha+\beta=0\\ \sqrt{3}II-III-I & 2\beta=0\\ III & \beta+\sqrt{3}\gamma=0\end{array}\\
\Leftrightarrow\ &(\alpha,\beta,\gamma)=(0,0,0)
\end{aligned}
$$

and the three given vectors are linearly independent.

(**b**)

$$
\begin{aligned}
&\alpha\left(\sqrt{2},1,0\right)+\beta\left(1,\sqrt{2},1\right)+\gamma\left(0,1,\sqrt{2}\right)=(0,0,0)\\
\Leftrightarrow\ &\begin{array}{ll} I & \sqrt{2}\alpha+\beta=0\\ II & \alpha+\sqrt{2}\beta+\gamma=0\\ III & \beta+\sqrt{2}\gamma=0\end{array}
\end{aligned}
$$

This time the sum of equations I and III (multiplied by $\sqrt{2}$) is the same as twice equation II, and a linear dependence in the equation system suggests that it may well have nontrivial solutions. Indeed, $\left(\sqrt{2},-2,\sqrt{2}\right)$ is such a solution. Thus the three given triple of vectors is linearly dependent.

(**c**)

$$
\begin{aligned}
&\alpha\,(t,1,0)+\beta\,(1,t,1)+\gamma\,(0,1,t)=(0,0,0)\\
\Leftrightarrow\ &\begin{array}{ll} I & t\alpha+\beta=0\\ II & \alpha+t\beta+\gamma=0\\ III & \beta+t\gamma=0\end{array}
\end{aligned}
$$

It is clear that $t=0$ makes the triple linearly dependent (the first and third vector coincide in this case). Let us suppose, then, $t\neq 0$. From I and III, as already noticed, I deduce that $\alpha=\gamma$. Equations II and III then become

$$
\begin{aligned}
t\beta+2\gamma &= 0\\
\beta+t\gamma &= 0
\end{aligned}
$$

a 2 by 2 homogeneous system with determinant of coefficient matrix equal to t^2-2. Such a system has nontrivial solutions for $t\in\left\{\sqrt{2},-\sqrt{2}\right\}$. In conclusion, the given triple is linearly dependent for $t\in\left\{0,\sqrt{2},-\sqrt{2}\right\}$.

12.15.10 n. 14 (p. 468)

Call as usual $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$ the four vectors given in each case, in the order.
(**a**) It is clear that $\mathbf{v} = \mathbf{u} + \mathbf{w}$, so that $\mathbf{v}$ can be dropped. Moreover, every linear combination of $\mathbf{u}$ and $\mathbf{w}$ has the form (x, y, x, y), and cannot be equal to $\mathbf{z}$. Thus $(\mathbf{u}, \mathbf{w}, \mathbf{z})$ is a maximal linearly independent triple.
(**b**) Notice that

$$\begin{aligned} \frac{1}{2}(\mathbf{u}+\mathbf{z}) &= \mathbf{e}^{(1)} \qquad \frac{1}{2}(\mathbf{u}-\mathbf{v}) = \mathbf{e}^{(2)} \\ \frac{1}{2}(\mathbf{v}-\mathbf{w}) &= \mathbf{e}^{(3)} \qquad \frac{1}{2}(\mathbf{w}-\mathbf{z}) = \mathbf{e}^{(4)} \end{aligned}$$

Since $(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z})$ spans the four canonical vectors, it is a basis of $\mathbb{R}^4$. Thus $(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z})$ is maximal linearly independent.
(**c**) Similarly,

$$\begin{aligned} \mathbf{u}-\mathbf{v} &= \mathbf{e}^{(1)} \qquad \mathbf{v}-\mathbf{w} = \mathbf{e}^{(2)} \\ \mathbf{w}-\mathbf{z} &= \mathbf{e}^{(3)} \qquad \mathbf{z} = \mathbf{e}^{(4)} \end{aligned}$$

and $(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z})$ is maximal linearly independent.

12.15.11 n. 15 (p. 468)

(**a**) Since the triple $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is linearly independent,

$$\begin{aligned} & \alpha(\mathbf{a}+\mathbf{b}) + \beta(\mathbf{b}+\mathbf{c}) + \gamma(\mathbf{a}+\mathbf{c}) = \mathbf{0} \\ \Leftrightarrow\ & (\alpha+\gamma)\mathbf{a} + (\alpha+\beta)\mathbf{b} + (\beta+\gamma)\mathbf{c} = \mathbf{0} \\ \Leftrightarrow\ & \begin{array}{ll} I & \alpha+\gamma=0 \\ II & \alpha+\beta=0 \\ III & \beta+\gamma=0 \end{array} \\ \Leftrightarrow\ & \begin{array}{ll} I+II-III & 2\alpha=0 \\ -I+II+III & 2\beta=0 \\ I-II+III & 2\gamma=0 \end{array} \end{aligned}$$

and the triple $(\mathbf{a}+\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{a}+\mathbf{c})$ is linearly independent, too
(**b**) On the contrary, choosing as nontrivial coefficient triple $(\alpha, \beta, \gamma) \equiv (1, -1, 1)$,

$$(\mathbf{a}-\mathbf{b}) - (\mathbf{b}+\mathbf{c}) + (\mathbf{a}+\mathbf{c}) = \mathbf{0}$$

it is seen that the triple $(\mathbf{a}-\mathbf{b}, \mathbf{b}+\mathbf{c}, \mathbf{a}+\mathbf{c})$ is linearly dependent.

12.15.12 n. 17 (p. 468)

Let $\mathbf{a} \equiv (0, 1, 1)$ and $\mathbf{b} \equiv (1, 1, 1)$; I look for two possible alternative choices of a vector $\mathbf{c} \equiv (x, y, z)$ such that the triple $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is linearly independent. Since

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = (\beta + \gamma x, \alpha + \beta + \gamma y, \alpha + \beta + \gamma z)$$

my choice of x, y, and z must be such to make the following conditional statement true for each $(\alpha, \beta, \gamma) \in \mathbb{R}^3$:

$$\left.\begin{array}{ll} I & \beta + \gamma x = 0 \\ II & \alpha + \beta + \gamma y = 0 \\ III & \alpha + \beta + \gamma z = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \alpha = 0 \\ \beta = 0 \\ \gamma = 0 \end{array}\right.$$

Subtracting equation III from equation II, I obtain

$$\gamma (y - z) = 0$$

Thus any choice of $\mathbf{c}$ with $y \neq z$ makes $\gamma = 0$ a consequence of II-III; in such a case, I yields $\beta = 0$ (independently of the value assigned to x), and then either II or III yields $\alpha = 0$. Conversely, if $y = z$, equations II and III are the same, and system I-III has infinitely many nontrivial solutions

$$\begin{array}{lcl} \alpha & = & -\gamma y - \beta \\ \beta & = & -\gamma x \\ \gamma & & \text{free} \end{array}$$

provided either x or y (hence z) is different from zero.

As an example, possible choices for $\mathbf{c}$ are $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 1)$, $(1, 1, 0)$.

12.15.13 n. 18 (p. 468)

The first example of basis containing the two given vectors is in point (c) of exercise 14 in this section, since the vectors $\mathbf{u}$ and $\mathbf{v}$ there coincide with the present ones. Keeping the same notation, a second example is $(\mathbf{u}, \mathbf{v}, \mathbf{w} + \mathbf{z}, \mathbf{w} - \mathbf{z})$.

12.15.14 n. 19 (p. 468)

(**a**) It is enough to prove that each element of T belongs to $\operatorname{lin} S$, since each element of $\operatorname{lin} T$ is a linear combination in T, and S, as any subspace of a vector space, is closed with respect to formation of linear combinations. Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be the three elements of S (in the given order), and let $\mathbf{a}$ and $\mathbf{b}$ be the two elements of T (still in the given order); it is then readily checked that

$$\mathbf{a} = \mathbf{u} - \mathbf{w} \qquad \mathbf{b} = 2\mathbf{w}$$

(**b**) The converse inclusion holds as well. Indeed,

$$\mathbf{v} = \mathbf{a} - \mathbf{b} \qquad \mathbf{w} = \frac{1}{2}\mathbf{b} \qquad \mathbf{u} = \mathbf{v} + \mathbf{w} = \mathbf{a} - \frac{1}{2}\mathbf{b}$$

Thus

$$\operatorname{lin} S = \operatorname{lin} T$$

Similarly, if $\mathbf{c}$ and $\mathbf{d}$ are the two elements of U,

$$\mathbf{c} = \mathbf{u} + \mathbf{v} \qquad \mathbf{d} = \mathbf{u} + 2\mathbf{v}$$

which proves that $\operatorname{lin} U \subseteq \operatorname{lin} S$. Inverting the above formulas,

$$\mathbf{u} = 2\mathbf{c} - \mathbf{d} \qquad \mathbf{v} = \mathbf{d} - \mathbf{c}$$

It remains to be established whether or not $\mathbf{w}$ is an element of $\operatorname{lin} U$. Notice that

$$\alpha\mathbf{c} + \beta\mathbf{d} = (\alpha + \beta, 2\alpha + 3\beta, 3\alpha + 5\beta)$$

It follows that $\mathbf{w}$ is an element of $\operatorname{lin} U$ if and only if there exists $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\begin{cases} I & \alpha + \beta = 1 \\ II & 2\alpha + 3\beta = 0 \\ III & 3\alpha + 5\beta = -1 \end{cases}$$

The above system has the unique solution $(3, -2)$, which proves that $\operatorname{lin} S \subseteq \operatorname{lin} U$. In conclusion,

$$\operatorname{lin} S = \operatorname{lin} T = \operatorname{lin} U$$

12.15.15 n. 20 (p. 468)

(**a**) The claim has already been proved in the last exercise, since from $A \subseteq B$ I can infer $A \subseteq \operatorname{lin} B$, and hence $\operatorname{lin} A \subseteq \operatorname{lin} B$.
(**b**) By the last result, from $A \cap B \subseteq A$ and $A \cap B \subseteq B$ I infer

$$\operatorname{lin} A \cap B \subseteq \operatorname{lin} A \qquad \operatorname{lin} A \cap B \subseteq \operatorname{lin} B$$

which yields

$$\operatorname{lin} A \cap B \subseteq \operatorname{lin} A \cap \operatorname{lin} B$$

(**c**) It is enough to define

$$A \equiv \{\mathbf{a}, \mathbf{b}\} \qquad B \equiv \{\mathbf{a} + \mathbf{b}\}$$

where the couple $(\mathbf{a}, \mathbf{b})$ is linearly independent. Indeed,

$$B \subseteq \operatorname{lin} A$$

and hence, by part (a) of last exercise,

$$\begin{aligned} \operatorname{lin} B &\subseteq \operatorname{lin} \operatorname{lin} A = \operatorname{lin} A \\ \operatorname{lin} A \cap \operatorname{lin} B &= \operatorname{lin} B \end{aligned}$$

On the other hand,

$$A \cap B = \emptyset \qquad \operatorname{lin} A \cap B = \{\mathbf{0}\}$$

12.16 The vector space $V_n(\mathbb{C})$ of n-tuples of complex numbers

12.17 Exercises

Chapter 13
APPLICATIONS OF VECTOR ALGEBRA TO ANALYTIC GEOMETRY

13.1 Introduction

13.2 Lines in n-space

13.3 Some simple properties of straight lines

13.4 Lines and vector-valued functions

13.5 Exercises

13.5.1 n. 1 (p. 477)

A direction vector for the line L is $\overrightarrow{PQ} = (4, 0)$, showing that L is horizontal. Thus a point belongs to L if and only if its second coordinate is equal to 1. Among the given points, (b), (d), and (e) belong to L.

13.5.2 n. 2 (p. 477)

A direction vector for the line L is $v \equiv \frac{1}{2}\overrightarrow{PQ} = (-2, 1)$. The parametric equations for L are

$$\begin{aligned} x &= 2 - 2t \\ y &= -1 + t \end{aligned}$$

If $t = 1$ I get point (a) (the origin). Points (b), (d) and (e) have the second coordinate equal to 1, which requires $t = 2$. This gives $x = -2$, showing that of the three points only (e) belongs to L. Finally, point (c) does not belong to L, because $y = 2$ requires $t = 3$, which yields $x = -4 \neq 1$.

13.5.3 n. 3 (p. 477)

The parametric equations for L are

$$\begin{aligned} x &= -3+h \\ y &= 1-2h \\ z &= 1+3h \end{aligned}$$

The following points belong to L:

$$(c) \quad (h=1) \qquad (d) \quad (h=-1) \qquad (e) \quad (h=5)$$

13.5.4 n. 4 (p. 477)

The parametric equations for L are

$$\begin{aligned} x &= -3+4k \\ y &= 1+k \\ z &= 1+6h \end{aligned}$$

The following points belong to L:

$$(b) \quad (h=-1) \qquad (e) \quad \left(h=\frac{1}{2}\right) \qquad (f) \quad \left(h=\frac{1}{3}\right)$$

A direction vector for the line L is $\overrightarrow{PQ}=(4,0)$, showing that L is horizontal. Thus a point belongs to L if and only if its second coordinate is equal to 1. Among the given points, (b), (d), and (e) belong to L.

13.5.5 n. 5 (p. 477)

I solve each case in a different a way.

(**a**)

$$\overrightarrow{PQ}=(2,0,-2) \qquad \overrightarrow{QR}=(-1,-2,2)$$

The two vectors are not parallel, hence the three points do not belong to the same line.

(**b**) Testing affine dependence,

$$\begin{array}{rlll} I & 2h+2k+3l=0 & I-2II+III & 2l=0 \\ II & -2h+3k+l=0 & I+II & 5k+4l=0 \\ III & -6h+4k+l=0 & 2I-III & 10h+5l=0 \end{array}$$

the only combination which is equal to the null vector is the trivial one. The three points do not belong to the same line.

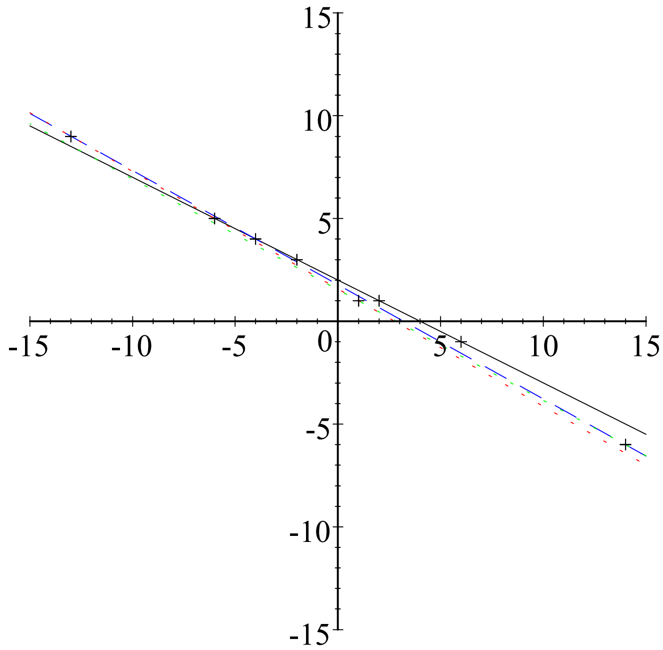
(**c**) The line through P and R has equations

$$\begin{aligned} x &= 2+3k \\ y &= 1-2k \\ z &= 1 \end{aligned}$$

Trying to solve for k with the coordinates of Q, I get $k = -\frac{4}{3}$ from the first equation and $k = -1$ from the second; Q does not belong to $L(P, R)$.

13.5.6 n. 6 (p. 477)

The question is easy, but it must be answered by following some orderly path, in order to achieve some economy of thought and of computations (there are $\binom{8}{2} = 28$ different oriented segments joining two of the eight given points. First, all the eight points have their third coordinate equal to 1, and hence they belong to the plane π of equation $z = 1$. We can concentrate only on the first two coordinates, and, as a matter of fact, we can have a very good hint on the situation by drawing a twodimensional picture, to be considered as a picture of the π



Since the first two components of $\overrightarrow{AB}$ are $(4, -2)$, I check that among the twodimensional projections of the oriented segments connecting A with points D to H

$$\begin{aligned} p_{xy}\overrightarrow{AD} &= (-4, 2) \qquad p_{xy}\overrightarrow{AE} = (-1, 1) \qquad p_{xy}\overrightarrow{AF} = (-6, 3) \\ p_{xy}\overrightarrow{AG} &= (-15, 8) \qquad p_{xy}\overrightarrow{AH} = (12, -7) \end{aligned}$$

only the first and third are parallel to $p_{xy}\overrightarrow{AB}$. Thus all elements of the set $P^1 \equiv \{A, B, C, D, F\}$ belong to the same (black) line L_{ABC}, and hence no two of them can both belong to a different line. Therefore, it only remains to be checked whether or not the lines through the couples of points (E, G) (red), (G, H) (blue), (H, E) (green) coincide, and whether or not any elements of P^1 belong to them. Direction vectors for these three lines are

$$\frac{1}{2}p_{xy}\overrightarrow{EG} = (-7, 4) \qquad \frac{1}{3}p_{xy}\overrightarrow{GH} = (9, -5) \qquad p_{xy}\overrightarrow{HE} = (-13, 7)$$

and as normal vectors for them I may take

$$n_{EG} \equiv (4, 7) \qquad n_{GH} \equiv (5, 9) \qquad n_{HE} \equiv (7, 13)$$

By requiring point E to belong to the first and last, and point H to the second, I end up with their equations as follows

$$L_{EG} : 4x + 7y = 11 \qquad L_{GH} : 5x + 9y = 16 \qquad L_{HE} : 7x + 13y = 20$$

All these three lines are definitely not parallel to L_{ABC}, hence they intersect it exactly at one point. It is seen by direct inspection that, within the set P^1, C belongs to L_{EG}, F belongs to L_{GH}, and neither A nor B, nor D belong to L_{HE}.

Thus there are three (maximal) sets of at least three collinear points, namely

$$P^1 \equiv \{A, B, C, D, F\} \qquad P^2 \equiv \{C, E, G\} \qquad P^3 \equiv \{F, G, H\}$$

13.5.7 n. 7 (p. 477)

The coordinates of the intersection point are determined by the following equation system

$$\begin{array}{ll} I & 1 + h = 2 + 3k \\ II & 1 + 2h = 1 + 8k \\ III & 1 + 3h = 13k \end{array}$$

Subtracting the third equation from the sum of the first two, I get $k = 1$; substituting this value in any of the three equations, I get $h = 4$. The three equations are consistent, and the two lines intersect at the point of coordinates $(5, 9, 13)$.

13.5.8 n. 8 (p. 477)

(**a**) The coordinates of the intersection point of $L(P; \mathbf{a})$ and $L(Q; \mathbf{b})$ are determined by the vector equation

$$P + h\mathbf{a} = Q + k\mathbf{b} \tag{13.1}$$

which gives

$$P - Q = k\mathbf{b} - h\mathbf{a}$$

that is,

$$\overrightarrow{PQ} \in \operatorname{span}\{\mathbf{a}, \mathbf{b}\}$$

13.5.9 n. 9 (p. 477)

$$X(t) = (1 + t, 2 - 2t, 3 + 2t)$$

(**a**)

$$\begin{aligned} d(t) &\equiv \|Q - X(t)\|^2 = (2 - t)^2 + (1 + 2t)^2 + (-2 - 2t)^2 \\ &= 9t^2 + 8t + 9 \end{aligned}$$

(**b**) The graph of the function $t \mapsto d(t) \equiv 9t^2 + 8t + 9$ is a parabola with the point $(t_0, d(t_0)) = \left(-\frac{4}{9}, \frac{65}{9}\right)$ as vertex. The minimum squared distance is $\frac{65}{9}$, and the minimum distance is $\frac{\sqrt{65}}{3}$.

(**c**)

$$X(t_0) = \left(\frac{5}{9}, \frac{26}{9}, \frac{19}{9}\right) \qquad Q - X(t_0) = \left(\frac{22}{9}, \frac{1}{9}, -\frac{10}{9}\right)$$

$$\langle Q - X(t_0), A\rangle = \frac{22 - 2 + 20}{9} = 0$$

that is, the point on L of minimum distance from Q is the orthogonal projection of Q on L.

13.5.10 n. 10 (p. 477)

(**a**) Let $A \equiv (\alpha, \beta, \gamma)$, $P \equiv (\lambda, \mu, \nu)$ and $Q \equiv (\varrho, \sigma, \tau)$. The points of the line L through P with direction vector $\overrightarrow{OA}$ are represented in parametric form (the generic point of L is denoted $X(t)$). Then

$$\begin{aligned} X(t) &\equiv P + At = (\lambda + \alpha t, \mu + \beta t, \nu + \gamma t) \\ f(t) &\equiv \|Q - X(t)\|^2 = \|Q - P - At\|^2 \\ &= \|Q\|^2 + \|P\|^2 + \|A\|^2 t - 2\langle Q, P\rangle + 2\langle P - Q, A\rangle t \\ &= at^2 + bt + c \end{aligned}$$

where

$$\begin{aligned} a &\equiv \|A\|^2 = \alpha^2 + \beta^2 + \gamma^2 \\ \frac{b}{2} &\equiv \langle P - Q, A\rangle = \alpha(\lambda - \varrho) + \beta(\mu - \sigma) + \gamma(\nu - \tau) \\ c &\equiv \|Q\|^2 + \|P\|^2 - 2\langle Q, P\rangle = \|P - Q\|^2 = (\lambda - \varrho)^2 + (\mu - \sigma)^2 + (\nu - \tau)^2 \end{aligned}$$

The above quadratic polynomial has a second degree term with a positive coefficient; its graph is a parabola with vertical axis and vertex in the point of coordinates

$$\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right) = \left(\frac{\langle Q - P, A\rangle}{\|A\|^2}, \frac{\|A\|^2 \|P - Q\|^2 - \langle P - Q, A\rangle^2}{\|A\|^2}\right)$$

Thus the minimum value of this polynomial is achieved at

$$t_0 \equiv \frac{\langle Q - P, A\rangle}{\|A\|^2} = \frac{\alpha(\varrho - \lambda) + \beta(\sigma - \beta) + \gamma(\tau - \nu)}{\alpha^2 + \beta^2 + \gamma^2}$$

and it is equal to

$$\frac{\|A\|^2 \|P - Q\|^2 - \|A\|^2 \|P - Q\|^2 \cos^2 \vartheta}{\|A\|^2} = \|P - Q\|^2 \sin^2 \vartheta$$

where

$$\vartheta \equiv \widehat{\overrightarrow{OA}, \overrightarrow{QP}}$$

is the angle formed by direction vector of the line and the vector carrying the point Q not on L to the point P on L.
(**b**)

$$\begin{aligned} Q - X(t_0) &= Q - (P + At_0) = (Q - P) - \frac{\langle Q - P, A\rangle}{\|A\|^2} A \\ \langle Q - X(t_0), A\rangle &= \langle Q - P, A\rangle - \frac{\langle Q - P, A\rangle}{\|A\|^2} \langle A, A\rangle = 0 \end{aligned}$$

13.5.11 n. 11 (p. 477)

The vector equation for the coordinates of an intersection point of $L(P; \mathbf{a})$ and $L(Q; \mathbf{a})$ is

$$P + h\mathbf{a} = Q + k\mathbf{a} \tag{13.2}$$

It gives

$$P - Q = (h - k)\mathbf{a}$$

and there are two cases: either

$$\overrightarrow{PQ} \notin \operatorname{span}\{\mathbf{a}\}$$

and equation (13.2) has no solution (the two lines are parallel), or

$$\overrightarrow{PQ} \in \operatorname{span}\{\mathbf{a}\}$$

that is,

$$\exists c \in \mathbb{R}, \ \ Q - P = c\mathbf{a}$$

and equation (13.2) is satisfied by all couples (h, k) such that $k - h = c$ (the two lines intersect in infinitely many points, i.e., they coincide). The two expressions $P + h\mathbf{a}$ and $Q + k\mathbf{a}$ are seen to provide alternative parametrizations of the same line. For each given point of L, the transformation $h \mapsto k(h) \equiv h + c$ identifies the parameter change which allows to shift from the first parametrization to the second.

13.5.12 n. 12 (p. 477)

13.6 Planes in euclidean n-spaces

13.7 Planes and vector-valued functions

13.8 Exercises

13.8.1 n. 2 (p. 482)

We have:

$$\overrightarrow{PQ} = (2, 2, 3) \qquad \overrightarrow{PR} = (2, -2, -1)$$

so that the parametric equations of the plane are

$$\begin{aligned} x &= 1 + 2s + 2t \\ y &= 1 + 2s - 2t \\ z &= -1 + 3s - t \end{aligned} \tag{13.3}$$

(**a**) Equating (x, y, z) to $\left(2, 2, \frac{1}{2}\right)$ in (13.3) we get

$$\begin{aligned} 2s + 2t &= 1 \\ 2s - 2t &= 1 \\ 3s - t &= \frac{3}{2} \end{aligned}$$

yielding $s = \frac{1}{2}$ and $t = 0$, so that the point with coordinates $\left(2, 2, \frac{1}{2}\right)$ belongs to the plane.

(**b**) Similarly, equating (x, y, z) to $\left(4, 0, \frac{3}{2}\right)$ in (13.3) we get

$$\begin{aligned} 2s + 2t &= 3 \\ 2s - 2t &= -1 \\ 3s - t &= \frac{1}{2} \end{aligned}$$

yielding $s = \frac{1}{2}$ and $t = 1$, so that the point with coordinates $\left(4, 0, \frac{3}{2}\right)$ belongs to the plane.

(**c**) Again, proceeding in the same way with the triple $(-3, 1, -1)$, we get

$$\begin{aligned} 2s + 2t &= -4 \\ 2s - 2t &= 0 \\ 3s - t &= -2 \end{aligned}$$

yielding $s = t = -1$, so that the point with coordinates $(-3, 1, -1)$ belongs to the plane.
(**d**) The point of coordinates $(3, 1, 5)$ does not belong to the plane, because the system

$$\begin{aligned} 2s + 2t &= 2 \\ 2s - 2t &= 0 \\ 3s - t &= 4 \end{aligned}$$

is inconsistent (from the first two equations we get $s = t = 1$, contradicting the third equation).
(**e**) Finally, the point of coordinates $(0, 0, 2)$ does not belong to the plane, because the system

$$\begin{aligned} 2s + 2t &= -1 \\ 2s - 2t &= -1 \\ 3s - t &= 1 \end{aligned}$$

is inconsistent (the first two equations yield $s = -\frac{1}{2}$ and $t = 0$, contradicting the third equation).

13.8.2 n. 3 (p. 482)

(**a**)

$$\begin{aligned} x &= 1 + t \\ y &= 2 + s + t \\ z &= 1 + 4t \end{aligned}$$

(**b**)

$$\begin{aligned} \mathbf{u} &= (1, 2, 1) - (0, 1, 0) = (1, 1, 1) \\ \mathbf{v} &= (1, 1, 4) - (0, 1, 0) = (1, 0, 4) \end{aligned}$$

$$\begin{aligned} x &= s + t \\ y &= 1 + s \\ z &= s + 4t \end{aligned}$$

13.8.3 n. 4 (p. 482)

(**a**) Solving for s and t with the coordinates of the first point, I get

$$\begin{array}{lll lll} I & s-2t+1=0 & & I+II-III & t+3=0 \\ II & s+4t+2=0 & & 2III-I-II & 2s-3=0 \\ III & 2s+t=0 & & \text{check on } I & \frac{3}{2}-6+1\neq 0 \end{array}$$

and $(0,0,0)$ does not belong to the plane M. The second point is directly seen to belong to M from the parametric equations

$$(1,2,0)+s\,(1,1,2)+t\,(-2,4,1) \tag{13.4}$$

Finally, with the third point I get

$$\begin{array}{lll lll} & & & I+II-III & t+3=2 \\ I & s-2t+1=2 & & 2III-I-II & 2s-3=-5 \\ II & s+4t+2=-3 & & \text{check on } I & -1+2+1=2 \\ III & 2s+t=-3 & & \text{check on } II & -1-4+2=-3 \\ & & & \text{check on } III & -2-1=-3 \end{array}$$

and $(2,-3,-3)$ belongs to M.

(**b**) The answer has already been given by writing equation (13.4):

$$P\equiv(1,2,0) \qquad \mathbf{a}=(1,1,2) \qquad \mathbf{b}=(-2,4,1)$$

13.8.4 n. 5 (p. 482)

This exercise is a replica of material already presented in class and inserted in the notes.

(**a**) If $p+q+r=1$, then

$$\begin{aligned} pP+qQ+rR &= P-(1-p)\,P+qQ+rR \\ &= P+(q+r)\,P+qQ+rR \\ &= P+q\,(Q-P)+r\,(R-P) \end{aligned}$$

and the coordinates of $pP+qQ+rR$ satisfy the parametric equations of the plane M through P, Q, and R..

(**b**) If S is a point of M, there exist real numbers q and r such that

$$S=P+q\,(Q-P)+r\,(R-P)$$

Then, defining $p\equiv 1-(q+r)$,

$$\begin{aligned} S &= (1-q-r)\,P+qQ+rR \\ &= pP+qQ+rR \end{aligned}$$

13.8.5 n. 6 (p. 482)

I use three different methods for the three cases.

(**a**) The parametric equations for the first plane π_1 are

$$\begin{array}{ll} I & x = 2 + 3h - k \\ II & y = 3 + 2h - 2k \\ III & z = 1 + h - 3k \end{array}$$

Eliminating the parameter h $(I - II - III)$ I get

$$4k = x - y - z + 2$$

Eliminating the parameter k $(I + II - III)$ I get

$$4h = x + y - z - 4$$

Substituting in III (or, better, in $4 \cdot III$),

$$4z = 4 + x + y - z - 4 - 3x + 3y + 3z - 6$$

I obtain a cartesian equation for π_1

$$x - 2y + z + 3 = 0$$

(**b**) I consider the four coefficients (a, b, c, d) of the cartesian equation of π_2 as unknown, and I require that the three given points belong to π_2; I obtain the system

$$\begin{aligned} 2a + 3b + c + d &= 0 \\ -2a - b - 3c + d &= 0 \\ 4a + 3b - c + d &= 0 \end{aligned}$$

which I solve by elimination

$$\begin{bmatrix} 2 & 3 & 1 & 1 \\ -2 & -1 & -3 & 1 \\ 4 & 3 & -1 & 1 \end{bmatrix}$$

$$\left(\begin{array}{c} {}_2\mathbf{a}' \leftarrow \frac{1}{2} \left({}_2\mathbf{a}' + {}_1\mathbf{a}' \right) \\ {}_3\mathbf{a}' \leftarrow {}_3\mathbf{a}' - 2\,{}_1\mathbf{a}' \end{array} \right) \qquad \begin{bmatrix} 2 & 3 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -3 & -3 & -1 \end{bmatrix}$$

$$\left({}_3\mathbf{a}' \leftarrow \frac{1}{2} \left({}_3\mathbf{a}' + 3\,{}_1\mathbf{a}' \right) \right) \qquad \begin{bmatrix} 2 & 3 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

From the last equation (which reads $-3c + d = 0$) I assign values 1 and 3 to c and d, respectively; from the second (which reads $b - c + d = 0$) I get $b = -2$, and from the first (which is unchanged) I get $a = 1$. The cartesian equation of π_2 is

$$x - 2y + z + 3 = 0$$

(notice that π_1 and π_2 coincide)
(**c**) This method requires the vector product (called cross product by Apostol), which is presented in the subsequent section; however, I have thought it better to show its application here already. The plane π_3 and the given plane being parallel, they have the same normal direction. Such a normal is

$$\mathbf{n} = (2, 0, -2) \times (1, 1, 1) = (2, -4, 2)$$

Thus π_3 coincides with π_2, since it has the same normal and has a common point with it. At any rate (just to show how the method applies in general), a cartesian equation for π_3 is

$$x - 2y + z + d = 0$$

and the coefficient d is determined by the requirement that π_3 contains $(2, 3, 1)$

$$2 - 6 + 1 + d = 0$$

yielding

$$x - 2y + z + 3 = 0$$

13.8.6 n. 7 (p. 482)
(**a**) Only the first two points belong to the given plane.
(**b**) I assign the values -1 and 1 to y and z in the cartesian equation of the plane M, in order to obtain the third coordinate of a point of M, and I obtain $P = (1, -1, 1)$ (I may have as well taken one of the first two points of part **a**). Any two vectors which are orthogonal to the normal $\mathbf{n} = (3, -5, 1)$ and form a linearly independent couple can be chosen as direction vectors for M. Thus I assign values arbitrarily to d_2 and d_3 in the orthogonality condition

$$3d_1 - 5d_2 + d_3 = 0$$

say, $(0, 3)$ and $(3, 0)$, and I get

$$\mathbf{d}^I = (-1, 0, 3) \qquad \mathbf{d}^{II} = (5, 3, 0)$$

The parametric equations for M are

$$\begin{aligned} x &= 1 - s + 5t \\ y &= -1 + 3t \\ z &= 1 + 3s \end{aligned}$$

13.8.7 n. 8 (p. 482)

The question is formulated in a slightly insidious way (perhaps Tom did it on purpose...), because the parametric equations of the two planes M and M' are written using the same names for the two parameters. This may easily lead to an incorrect attempt two solve the problem by setting up a system of three equations in the two unknown s and t, which is likely to have no solutions, thereby suggesting the wrong answer that M and M' are parallel. On the contrary, the equation system for the coordinates of points in $M \cap M'$ has *four* unknowns:

$$\begin{array}{ll} I & 1+2s-t=2+h+3k \\ II & 1-s=3+2h+2k \\ III & 1+3s+2t=1+3h+k \end{array} \tag{13.5}$$

I take all the variables on the lefthand side, and the constant terms on the righthand one, and I proceed by elimination:

$$\begin{array}{cccc|c} s & t & h & k & const. \\ 2 & -1 & -1 & -3 & 1 \\ -1 & 0 & -2 & -2 & 2 \\ 3 & 2 & -3 & -1 & 0 \end{array}$$

$$\begin{pmatrix} {}_1\mathbf{r}' \leftarrow {}_1\mathbf{r}' + 2\,{}_2\mathbf{r}' \\ {}_3\mathbf{r}' \leftarrow {}_3\mathbf{r}' + 3\,{}_2\mathbf{r}' \\ {}_1\mathbf{r}' \leftrightarrow {}_2\mathbf{r}' \end{pmatrix} \qquad \begin{array}{cccc|c} s & t & h & k & const. \\ -1 & 0 & -2 & -2 & 2 \\ 0 & -1 & -5 & -7 & 5 \\ 0 & 2 & -9 & -7 & 6 \end{array}$$

$$\begin{pmatrix} {}_3\mathbf{r}' \leftarrow {}_3\mathbf{r}' + 2\,{}_2\mathbf{r}' \\ {}_1\mathbf{r}' \leftrightarrow {}_2\mathbf{r}' \end{pmatrix} \qquad \begin{array}{cccc|c} s & t & h & k & const. \\ -1 & 0 & -2 & -2 & 2 \\ 0 & -1 & -5 & -7 & 5 \\ 0 & 0 & -19 & -21 & 16 \end{array}$$

A handy solution to the last equation (which reads $-19h - 21k = 16$) is obtained by assigning values 8 and -8 to h and k, respectively. This is already enough to get the first point from the parametric equation of M', which is apparent (though incomplete) from the righthand side of (13.5). Thus $Q = (-14, 3, 17)$. However, just to check on the computations, I proceed to finish up the elimination, which leads to $t = 11$ from the second row, and to $s = -2$ from the first. Substituting in the lefthand side of (13.5), which is a trace of the parametric equations of M, I do get indeed $(-14, 3, 17)$. Another handy solution to the equation corresponding to the last row of the final elimination table is $(h,k) = \left(-\frac{2}{5}, -\frac{2}{5}\right)$. This gives $R = \left(\frac{2}{5}, \frac{7}{5}, -\frac{3}{5}\right)$, and the check by means of $(s,t) = \left(-\frac{2}{5}, -\frac{1}{5}\right)$ is all right.

13.8.8 n. 9 (p. 482)

(**a**) A normal vector for M is

$$\mathbf{n} = (1,2,3) \times (3,2,1) = (-4,8,-4)$$

whereas the coefficient vector in the equation of M' is $(1, -2, 1)$. Since the latter is parallel to the former, and the coordinates of the point $P \in M$ do not satisfy the equation of M', the two planes are parallel.
(**b**) A cartesian equation for M is

$$x - 2y + z + d = 0$$

From substitution of the coordinates of P, the coefficient d is seen to be equal to 3. The coordinates of the points of the intersection line $L \equiv M \cap M''$ satisfy the system

$$\begin{cases} x - 2y + z + 3 = 0 \\ x + 2y + z = 0 \end{cases}$$

By sum and subtraction, the line L can be represented by the simpler system

$$\begin{cases} 2x + 2z = -3 \\ 4y = 3 \end{cases}$$

as the intersection of two different planes π and π', with π parallel to the y-axis, and π' parallel to the xz-plane. Coordinate of points on L are now easy to produce, e.g., $Q = \left(-\frac{3}{2}, \frac{3}{4}, 0\right)$ and $R = \left(0, \frac{3}{4}, -\frac{3}{2}\right)$

13.8.9 n. 10 (p. 483)

The parametric equations for the line L are

$$\begin{aligned} x &= 1 + 2r \\ y &= 1 - r \\ z &= 1 + 3r \end{aligned}$$

and the parametric equations for the plane M are

$$\begin{aligned} x &= 1 + 2s \\ y &= 1 + s + t \\ z &= -2 + 3s + t \end{aligned}$$

The coordinates of a point of intersection between L and M must satisfy the system of equations

$$\begin{aligned} 2r - 2s &= 0 \\ r + s + t &= 0 \\ 3r - 3s - t &= -3 \end{aligned}$$

The first equation yields $r = s$, from which the third and second equation give $t = 3$, $r = s = -\frac{3}{2}$. Thus $L \cap M$ consists of a single point, namely, the point of coordinates $\left(-2, \frac{5}{2}. -\frac{7}{2}\right)$.

13.8.10 n. 11 (p. 483)

The parametric equations for L are

$$\begin{aligned} x &= 1+2t \\ y &= 1-t \\ z &= 1+3t \end{aligned}$$

and a direction vector for L is $\mathbf{v} = (2,-1,3)$

(**a**) L is parallel to the given plane (which I am going to denote π_a) if $\mathbf{v}$ is a linear combination of $(2,1,3)$ and $\left(\frac{3}{4},1,1\right)$. Thus I consider the system

$$\begin{aligned} 2x + \frac{3}{4}y &= 2 \\ x+y &= -1 \\ 3x+y &= 3 \end{aligned}$$

where subtraction of the second equation from the third gives $2x = 4$, hence $x = 2$; this yields $y = -3$ in the last two equations, contradicting the first.

Alternatively, computing the determinant of the matrix having $\mathbf{v}$, $\mathbf{a}$, $\mathbf{b}$ as columns

$$\begin{vmatrix} 2 & 2 & \frac{3}{4} \\ -1 & 1 & 1 \\ 3 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 0 & -\frac{5}{4} \\ -1 & 1 & 1 \\ 6 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 4 & -\frac{5}{4} \\ 6 & -2 \end{vmatrix} = -\frac{1}{2}$$

shows that $\{\mathbf{v},\mathbf{a},\mathbf{b}\}$ is a linearly independent set.

With either reasoning, it is seen that L is not parallel to π_a.

(**b**) By computing two independent directions for the plane π_b, e.g., $\overrightarrow{PQ} = (3,5,2) - (1,1,-2)$ and $\overrightarrow{PR} = (2,4,-1) - (1,1,-2)$, I am reduced to the previous case. The system to be studied now is

$$\begin{aligned} 2x+y &= 2 \\ 4x+3y &= -1 \\ 4x+y &= 3 \end{aligned}$$

and the three equations are again inconsistent, because subtraction of the third equation from the second gives $y = -2$, whereas subtraction of the first from the third yields $x = \frac{1}{2}$, these two values contradicting all equations. L is not parallel to π_b.

(**c**) Since a normal vector to π_c has for components the coefficients of the unknowns in the equation of π_c, it suffices to check orthogonality between $\mathbf{v}$ and $(1,2,3)$:

$$\langle (2,-1,3),(1,2,3)\rangle = 9 \neq 0$$

L is not parallel to π_c.

13.8.11 n. 12 (p. 483)

Let R be any point of the given plane π, other than P or Q. A point S then belongs to M if and only if there exists real numbers q and r such that

$$S = P + q(Q - P) + r(R - P) \tag{13.6}$$

If S belongs to the line through P and Q, there exists a real number p such that

$$S = P + p(Q - P)$$

Then, by defining

$$q \equiv p \qquad r \equiv 0$$

it is immediately seen that condition (13.6) is satisfied.

13.8.12 n. 13 (p. 483)

Since every point of L belongs to M, M contains the points having coordinates equal to $(1,2,3)$, $(1,2,3) + t(1,1,1)$ for each $t \in \mathbb{R}$, and $(2,3,5)$. Choosing, e.g., $t = 1$, the parametric equations for M are

$$\begin{aligned} x &= 1 + r + s \\ y &= 2 + r + s \\ z &= 3 + r + 2s \end{aligned}$$

It is possible to eliminate the two parameters at once, by subtracting the first equaiton from the second, obtaining

$$x - y + 1 = 0$$

13.8.13 n. 14 (p. 483)

Let $\mathbf{d}$ be a direction vector for the line L, and let $Q \equiv (x_Q, y_Q, z_Q)$ be a point of L. A plane containing L and the point $P \equiv (x_P, y_P, z_P)$ is the plane π through Q with direction vectors $\mathbf{d}$ and $\overrightarrow{QP}$

$$\begin{aligned} x &= x_Q + hd_1 + k(x_P - x_Q) \\ y &= y_Q + hd_2 + k(y_P - y_Q) \\ z &= z_Q + hd_3 + k(z_P - z_Q) \end{aligned}$$

L belongs to π because the parametric equation of π reduces to that of L when k is assigned value 0, and P belongs to π because the coordinates of P are obtained from the parametric equation of π when h is assigned value 0 and k is assigned value 1. Since any two distinct points on L and P determine a unique plane, π is the only plane containing L and P.

13.9 The cross product

13.10 The cross product expressed as a determinant

13.11 Exercises

13.11.1 n. 1 (p. 487)

(a) $A \times B = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

(b) $B \times C = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.

(c) $C \times A = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

(d) $A \times (C \times A) = 8\mathbf{i} + 10\mathbf{j} + 4\mathbf{k}$.

(e) $(A \times B) \times C = 8\mathbf{i} + 3\mathbf{j} - 7\mathbf{k}$.

(f) $A \times (B \times C) = 10\mathbf{i} + 11\mathbf{j} + 5\mathbf{k}$.

(g) $(A \times C) \times B = -2\mathbf{i} - 8\mathbf{j} - 12\mathbf{k}$.

(h) $(A + B) \times (A - C) = 2\mathbf{i} - 2\mathbf{j}$.

(i) $(A \times B) \times (A \times C) = -2\mathbf{i} + 4\mathbf{k}$.

13.11.2 n. 2 (p. 487)

(a) $\frac{A\times B}{\|A\times B\|} = -\frac{4}{\sqrt{26}}\mathbf{i} + \frac{3}{\sqrt{26}}\mathbf{j} + \frac{1}{\sqrt{26}}\mathbf{k}$ or $-\frac{A\times B}{\|A\times B\|} = \frac{4}{\sqrt{26}}\mathbf{i} - \frac{3}{\sqrt{26}}\mathbf{j} - \frac{1}{\sqrt{26}}\mathbf{k}$.

(b) $\frac{A\times B}{\|A\times B\|} = -\frac{41}{\sqrt{2054}}\mathbf{i} - \frac{18}{\sqrt{2054}}\mathbf{j} + \frac{7}{\sqrt{2054}}\mathbf{k}$ or $-\frac{A\times B}{\|A\times B\|} = \frac{41}{\sqrt{2054}}\mathbf{i} + \frac{18}{\sqrt{2054}}\mathbf{j} - \frac{7}{\sqrt{2054}}\mathbf{k}$.

(c) $\frac{A\times B}{\|A\times B\|} = -\frac{1}{\sqrt{6}}\mathbf{i} - \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$ or $-\frac{A\times B}{\|A\times B\|} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} + \frac{1}{\sqrt{6}}\mathbf{k}$.

13.11.3 n. 3 (p. 487)

(a)

$$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= (2,-2,-3) \times (3,2,-2) = (10,-5,10) \\ \text{area}\, ABC &= \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{15}{2} \end{aligned}$$

(b)

$$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= (3,-6,3) \times (3,-1,0) = (3,9,15) \\ \text{area}\, ABC &= \frac{1}{2}\left\|\overrightarrow{AB} \times \overrightarrow{AC}\right\| = \frac{3\sqrt{35}}{2} \end{aligned}$$

(**c**)

$$\begin{aligned}\overrightarrow{AB}\times\overrightarrow{AC} &= (0,1,1)\times(1,0,1)=(1,1,-1)\\ \text{area}\,ABC &= \frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\frac{\sqrt{3}}{2}\end{aligned}$$

13.11.4 n. 4 (p. 487)

$$\overrightarrow{CA}\times\overrightarrow{AB}=(-\mathbf{i}+2\mathbf{j}-3\mathbf{k})\times(2\mathbf{j}+\mathbf{k})=8\mathbf{i}-\mathbf{j}-2\mathbf{k}$$

13.11.5 n. 5 (p. 487)

Let $\mathbf{a}\equiv(l,m,n)$ and $\mathbf{b}\equiv(p,q,r)$ for the sake of notational simplicity. Then

$$\begin{aligned}\|\mathbf{a}\times\mathbf{b}\|^2 &= (mr-nq)^2+(np-lr)^2+(lq-mp)^2\\ &= m^2r^2+n^2q^2-2mnrq+n^2p^2+l^2r^2\\ &\quad -2lnpr+l^2q^2+m^2p^2-2lmpq\\ \|\mathbf{a}\|^2\|\mathbf{b}\|^2 &= \left(l^2+m^2+n^2\right)\left(p^2+q^2+r^2\right)\\ &= l^2p^2+l^2q^2+l^2r^2+m^2p^2+m^2q^2\\ &\quad +m^2r^2+n^2p^2+n^2q^2+n^2r^2\\ (\|\mathbf{a}\times\mathbf{b}\|=\|\mathbf{a}\|\|\mathbf{b}\|) &\Leftrightarrow \left(\|\mathbf{a}\times\mathbf{b}\|^2=\|\mathbf{a}\|^2\|\mathbf{b}\|^2\right)\\ &\Leftrightarrow (lp+mq+nr)^2=0\\ &\Leftrightarrow \langle\mathbf{a},\mathbf{b}\rangle=0\end{aligned}$$

13.11.6 n. 6 (p. 487)

(**a**)

$$\langle\mathbf{a},\mathbf{b}+\mathbf{c}\rangle=\langle\mathbf{a},\mathbf{b}\times\mathbf{a}\rangle=0$$

because $\mathbf{b}\times\mathbf{a}$ is orthogonal to $\mathbf{a}$.

(**b**)

$$\langle\mathbf{b},\mathbf{c}\rangle=\langle\mathbf{b},(\mathbf{b}\times\mathbf{a})-\mathbf{b}\rangle=\langle\mathbf{b},(\mathbf{b}\times\mathbf{a})\rangle+\langle\mathbf{b},-\mathbf{b}\rangle=-\|\mathbf{b}\|^2$$

because $\mathbf{b}\times\mathbf{a}$ is orthogonal to $\mathbf{b}$. Thus

$$\langle\mathbf{b},\mathbf{c}\rangle<0 \qquad \cos\widehat{\mathbf{bc}}=-\frac{\|\mathbf{b}\|}{\|\mathbf{c}\|}<0 \qquad \widehat{\mathbf{bc}}\in\left]\frac{\pi}{2},\pi\right]$$

. Moreover, $\widehat{\mathbf{bc}}=-\pi$ is impossible, because

$$\begin{aligned}\|\mathbf{c}\|^2 &= \|\mathbf{b}\times\mathbf{a}\|^2+\|\mathbf{b}\|^2-2\|\mathbf{b}\times\mathbf{a}\|\|\mathbf{b}\|\cos\widehat{(\mathbf{b}\times\mathbf{a})\,\mathbf{b}}\\ &= \|\mathbf{b}\times\mathbf{a}\|^2+\|\mathbf{b}\|^2>\|\mathbf{b}\|^2\end{aligned}$$

since $(\mathbf{a},\mathbf{b})$ is linearly independent and $\|\mathbf{b}\times\mathbf{a}\|>0$.

(**c**) By the formula above, $\|\mathbf{c}\|^2=2^2+1^2$, and $\|\mathbf{c}\|=\sqrt{5}$.

13.11.7 n. 7 (p. 488)

(**a**) Since $\|\mathbf{a}\| = \|\mathbf{b}\| = 1$ and $\langle \mathbf{a}, \mathbf{b} \rangle = 0$, by Lagrange's identity (theorem 13.12.f)

$$\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 - \langle \mathbf{a}, \mathbf{b} \rangle^2 = 1$$

so that $\mathbf{a} \times \mathbf{b}$ is a unit vector as well. The three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ are mutually orthogonal either by assumption or by the properties of the vector product (theorem 13.12.d-e).

(**b**) By Lagrange's identity again,

$$\|\mathbf{c}\|^2 = \|\mathbf{a} \times \mathbf{b}\|^2 \|\mathbf{a}\|^2 - \langle \mathbf{a} \times \mathbf{b}, \mathbf{a} \rangle^2 = 1$$

(**c**) And again,

$$\begin{aligned} \|(\mathbf{a} \times \mathbf{b}) \times \mathbf{b}\|^2 &= \|\mathbf{a} \times \mathbf{b}\|^2 \|\mathbf{b}\|^2 - \langle \mathbf{a} \times \mathbf{b}, \mathbf{b} \rangle^2 = 1 \\ \|(\mathbf{a} \times \mathbf{b}) \times \mathbf{a}\|^2 &= \|\mathbf{c}\|^2 = 1 \end{aligned}$$

Since the direction which is orthogonal to $(\mathbf{a} \times \mathbf{b})$ and to $\mathbf{b}$ is spanned by $\mathbf{a}$, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{b}$ is either equal to $\mathbf{a}$ or to $-\mathbf{a}$. Similarly, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{a}$ is either equal to $\mathbf{b}$ or to $-\mathbf{b}$. Both the righthand rule and the lefthand rule yield now

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} = \mathbf{b} \qquad (\mathbf{a} \times \mathbf{b}) \times \mathbf{b} = -\mathbf{a}$$

(**d**)

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \times \mathbf{a} &= \begin{pmatrix} (a_3 b_1 - a_1 b_3)\, a_3 - (a_1 b_2 - a_2 b_1)\, a_2 \\ (a_1 b_2 - a_2 b_1)\, a_1 - (a_2 b_3 - a_3 b_2)\, a_3 \\ (a_2 b_3 - a_3 b_2)\, a_2 - (a_3 b_1 - a_1 b_3)\, a_1 \end{pmatrix} \\ &= \begin{pmatrix} (a_2^2 + a_3^2)\, b_1 - a_1 (a_3 b_3 + a_2 b_2) \\ (a_1^2 + a_3^2)\, b_2 - a_2 (a_1 b_1 + a_3 b_3) \\ (a_1^2 + a_2^2)\, b_3 - a_3 (a_1 b_1 + a_2 b_2) \end{pmatrix} \end{aligned}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are orthogonal,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} = \begin{pmatrix} (a_2^2 + a_3^2)\, b_1 + a_1 (a_1 b_1) \\ (a_1^2 + a_3^2)\, b_2 + a_2 (a_2 b_2) \\ (a_1^2 + a_2^2)\, b_3 + a_3 (a_3 b_3) \end{pmatrix}$$

Since $\mathbf{a}$ is a unit vector,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

The proof that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{b} = -\mathbf{a}$ is identical.

13.11.8 n. 8 (p. 488)

(**a**) From $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, either there exists some $h \in \mathbb{R}$ such that $\mathbf{a} = h\mathbf{b}$, or there exists some $k \in \mathbb{R}$ such that $\mathbf{b} = k\mathbf{a}$. Then either $\langle \mathbf{a}, \mathbf{b} \rangle = h \|\mathbf{b}\|^2$ or $\langle \mathbf{a}, \mathbf{b} \rangle = k \|\mathbf{a}\|^2$, that is, either $h \|\mathbf{b}\| = 0$ or $k \|\mathbf{a}\| = 0$. In the first case, either $h = 0$ (which means $\mathbf{a} = \mathbf{0}$), or $\|\mathbf{b}\| = 0$ (which is equivalent to $\mathbf{b} = \mathbf{0}$). In the second case, either $k = 0$ (which means $\mathbf{b} = \mathbf{0}$), or $\|\mathbf{a}\| = 0$ (which is equivalent to $\mathbf{a} = \mathbf{0}$). Geometrically, suppose that $\mathbf{a} \neq \mathbf{0}$. Then both the projection $\frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\|\mathbf{a}\|^2}\mathbf{a}$ of $\mathbf{b}$ along $\mathbf{a}$ and the projecting vector (of length $\frac{\|\mathbf{b} \times \mathbf{a}\|}{\|\mathbf{a}\|}$) are null, which can only happen if $\mathbf{b} = \mathbf{0}$.

(**b**) From the previous point, the hypotheses $\mathbf{a} \neq \mathbf{0}$, $\mathbf{a} \times (\mathbf{b} - \mathbf{c}) = \mathbf{0}$, and $\langle \mathbf{a}, \mathbf{b} - \mathbf{c} \rangle = 0$ imply that $\mathbf{b} - \mathbf{c} = \mathbf{0}$.

13.11.9 n. 9 (p. 488)

(**a**) Let $\vartheta \equiv \widehat{\mathbf{ab}}$, and observe that $\mathbf{a}$ and $\mathbf{c}$ are orthogonal. From $\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| |\sin \vartheta|$, in order to satisfy the condition

$$\mathbf{a} \times \mathbf{b} = \mathbf{c}$$

the vector $\mathbf{b}$ must be orthogonal to $\mathbf{c}$, and its norm must depend on ϑ according to the relation

$$\|\mathbf{b}\| = \frac{\|\mathbf{c}\|}{\|\mathbf{a}\| |\sin \vartheta|} \tag{13.7}$$

Thus $\frac{\|\mathbf{c}\|}{\|\mathbf{a}\|} \leq \|\mathbf{b}\| < +\infty$. In particular, $\mathbf{b}$ can be taken orthogonal to $\mathbf{a}$ as well, in which case it has to be equal to $\frac{\mathbf{a} \times \mathbf{c}}{\|\mathbf{a}\|^2}$ or to $\frac{\mathbf{c} \times \mathbf{a}}{\|\mathbf{a}\|^2}$. Thus two solutions to the problem are

$$\pm \left(\frac{7}{9}, -\frac{8}{9}, -\frac{11}{9} \right)$$

(**b**) Let (p, q, r) be the coordinates of $\mathbf{b}$; the conditions $\mathbf{a} \times \mathbf{b} = \mathbf{c}$ and $\langle \mathbf{a}, \mathbf{b} \rangle = 1$ are:

$$\begin{array}{ll} I & -2q - r = 3 \\ II & 2p - 2r = 4 \\ III & p + 2q = -1 \\ IV & 2p - q + 2r = 1 \end{array}$$

Standard manipulations yield

$$\begin{array}{ll} 2II + 2IV + III & 9p = 9 \\ (\uparrow) \hookrightarrow II & 2 - 2r = 4 \\ (\uparrow) \hookrightarrow I & -2q + 1 = 3 \\ \text{check on } IV & 2 + 1 - 2 = 1 \\ \text{check on } III & 1 - 2 = 1 \end{array}$$

that is, the unique solution is

$$\mathbf{b} = (1, -1, -1)$$

The solution to this exercise given by Apostol at page 645 is wrong.

13.11.10 n. 10 (p. 488)

Replacing **b** with **c** in the first result of exercise7,

$$(\mathbf{a} \times \mathbf{c}) \times \mathbf{a} = \mathbf{c}$$

Therefore, by skew-simmetry of the vector product, it is seen that the $\mathbf{c} \times \mathbf{a}$ is a solution to the equation

$$\mathbf{a} \times \mathbf{x} = \mathbf{c} \tag{13.8}$$

However, $\mathbf{c} \times \mathbf{a}$ is orthogonal to **a**, and hence it does not meet the additional requirement

$$\langle \mathbf{a}, \mathbf{x} \rangle = 1 \tag{13.9}$$

We are bound to look for other solutions to (13.8). If **x** and **z** both solve (13.8),

$$\mathbf{a} \times (\mathbf{x} - \mathbf{z}) = \mathbf{a} \times \mathbf{x} - \mathbf{a} \times \mathbf{z} = \mathbf{c} - \mathbf{c} = \mathbf{0}$$

which implies that $\mathbf{x} - \mathbf{z}$ is parallel to **a**. Thus the set of all solutions to equation (13.8) is

$$\left\{ \mathbf{x} \in \mathbb{R}^3 : \exists \alpha \in \mathbb{R},\ \mathbf{x} = \mathbf{a} \times \mathbf{c} + \alpha \mathbf{a} \right\}$$

From condition (13.9),

$$\langle \mathbf{a}, \mathbf{a} \times \mathbf{c} + \alpha \mathbf{a} \rangle = 1$$

that is,

$$\alpha = -\frac{1}{\|\mathbf{a}\|^2}$$

Thus the unique solution to the problem is

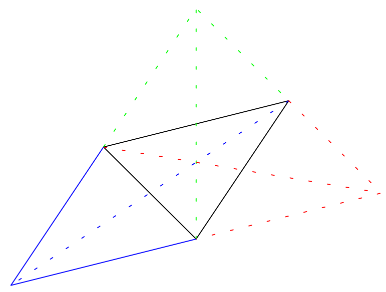
$$\mathbf{b} \equiv \mathbf{a} \times \mathbf{c} - \frac{\mathbf{a}}{\|\mathbf{a}\|^2}$$

13.11.11 n. 11 (p. 488)

(**a**) Let

$$\begin{aligned} \mathbf{u} &\equiv \overrightarrow{AB} = (-2, 1, 0) \\ \mathbf{v} &\equiv \overrightarrow{BC} = (3, -2, 1) \\ \mathbf{w} &\equiv \overrightarrow{CA} = (-1, 1, -1) \end{aligned}$$

Each side of the triangle ABC can be one of the two diagonals of the parallelogram to be determined.



If one of the diagonals is BC, the other is AD, where $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{AC} = \mathbf{u} - \mathbf{w}$. In this case

$$D = A + \mathbf{u} - \mathbf{w} = \text{“}B + C - A\text{”} = (0, 0, 2)$$

If one of the diagonals is CA, the other is BE, where $\overrightarrow{BE} = \overrightarrow{BC} + \overrightarrow{BA} = \mathbf{v} - \mathbf{u}$. In this case

$$E = B + \mathbf{v} - \mathbf{u} = \text{“}A + C - B\text{”} = (4, -2, 2)$$

If one of the diagonals is AB, the other is CF, where $\overrightarrow{CF} = \overrightarrow{CA} + \overrightarrow{CB} = -\mathbf{v} + \mathbf{w}$. In this case

$$F = A - \mathbf{v} + \mathbf{w} = \text{“}A + B - C\text{”} = (-2, 2, 0)$$

(**b**)

$$\begin{aligned}
\mathbf{u} \times \mathbf{w} &= \left(\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix}, - \begin{vmatrix} -2 & 0 \\ -1 & -1 \end{vmatrix}, \begin{vmatrix} -2 & 1 \\ -1 & 1 \end{vmatrix} \right) = (-1, -2, -1) \\
\|\mathbf{u} \times \mathbf{w}\| &= \sqrt{6} = \qquad \text{area}\,(ABC) = \frac{\sqrt{6}}{2}
\end{aligned}$$

13.11.12 *n. 12 (p. 488)*

$$\begin{aligned}
\mathbf{b}+\mathbf{c} &= 2\left(\mathbf{a}\times\mathbf{b}\right)-2\mathbf{b} \\
\left\langle \mathbf{a},\mathbf{b}+\mathbf{c}\right\rangle &= -2\left\langle \mathbf{a},\mathbf{b}\right\rangle = -4 \\
\cos\widehat{\mathbf{ab}} &= \frac{\left\langle \mathbf{a},\mathbf{b}\right\rangle}{\|\mathbf{a}\|\,\|\mathbf{b}\|} = \frac{1}{2} \\
\|\mathbf{a}\times\mathbf{b}\|^2 &= \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2\sin^2\widehat{\mathbf{ab}} = 12 \\
\|\mathbf{c}\|^2 &= 4\,\|\mathbf{a}\times\mathbf{b}\|^2 - 12\left\langle \mathbf{a}\times\mathbf{b},\mathbf{b}\right\rangle + 9\,\|\mathbf{b}\|^2 = 192 \\
\|\mathbf{c}\| &= 8\sqrt{3} \\
\left\langle \mathbf{b},\mathbf{c}\right\rangle &= 2\left\langle \mathbf{b},\mathbf{a}\times\mathbf{b}\right\rangle - 3\,\|\mathbf{b}\|^2 = 48 \\
\cos\widehat{\mathbf{bc}} &= \frac{\left\langle \mathbf{b},\mathbf{c}\right\rangle}{\|\mathbf{b}\|\,\|\mathbf{c}\|} = \frac{\sqrt{3}}{2}
\end{aligned}$$

13.11.13 *n. 13 (p. 488)*

(**a**) It is true that, if the couple $(\mathbf{a},\mathbf{b})$ is linearly independent, then the triple

$$\left(\mathbf{a}+\mathbf{b},\mathbf{a}-\mathbf{b},\mathbf{a}\times\mathbf{b}\right)$$

is linearly independent, too. Indeed, in the first place $\mathbf{a}$ and $\mathbf{b}$ are nonnull, since every n-tuple having the null vector among its components is linearly dependent. Secondly, $(\mathbf{a}\times\mathbf{b})$ is nonnull, too, because $(\mathbf{a},\mathbf{b})$ is linearly independent. Third, $(\mathbf{a}\times\mathbf{b})$ is orthogonal to both $\mathbf{a}+\mathbf{b}$ and $\mathbf{a}-\mathbf{b}$ (as well as to any vector in $\operatorname{lin}\{\mathbf{a},\mathbf{b}\}$), because it is orthogonal to $\mathbf{a}$ and $\mathbf{b}$; in particular, $(\mathbf{a}\times\mathbf{b})\notin\operatorname{lin}\{\mathbf{a}.\mathbf{b}\}$, since the only vector which is orthogonal to itself is the null vector. Fourth, suppose that

$$x\left(\mathbf{a}+\mathbf{b}\right)+y\left(\mathbf{a}-\mathbf{b}\right)+z\left(\mathbf{a}\times\mathbf{b}\right)=\mathbf{0} \tag{13.10}$$

Then z cannot be different from zero, since otherwise

$$\begin{aligned}
\left(\mathbf{a}\times\mathbf{b}\right) &= -\frac{x}{z}\left(\mathbf{a}+\mathbf{b}\right)-\frac{y}{z}\left(\mathbf{a}-\mathbf{b}\right) \\
&= -\frac{x+y}{z}\mathbf{a}+\frac{y-x}{z}\mathbf{b}
\end{aligned}$$

would belong to $\operatorname{lin}\{\mathbf{a},\mathbf{b}\}$. Thus $z=0$, and (13.10) becomes

$$x\left(\mathbf{a}+\mathbf{b}\right)+y\left(\mathbf{a}-\mathbf{b}\right)=\mathbf{0}$$

which is equivalent to

$$\left(x+y\right)\mathbf{a}+\left(x-y\right)\mathbf{b}=\mathbf{0}$$

By linear independence of $(\mathbf{a},\mathbf{b})$,

$$x+y=0 \qquad \text{and} \qquad x-y=0$$

and hence $x = y = 0$.
(**b**) It is true that, if the couple $(\mathbf{a}, \mathbf{b})$ is linearly independent, then the triple

$$(\mathbf{a}+\mathbf{b}, \mathbf{a}+\mathbf{a}\times\mathbf{b}, \mathbf{b}+\mathbf{a}\times\mathbf{b})$$

is linearly independent, too. Indeed, let (x, y, z) be such that

$$x(\mathbf{a}+\mathbf{b}) + y(\mathbf{a}+\mathbf{a}\times\mathbf{b}) + z(\mathbf{b}+\mathbf{a}\times\mathbf{b}) = \mathbf{0}$$

which is equivalent to

$$(x+y)\,\mathbf{a} + (x+z)\,\mathbf{b} + (y+z)\,\mathbf{a}\times\mathbf{b} = \mathbf{0}$$

Since, arguing as in the fourth part of the previous point, $y+z$ must be null, and this in turn implies that both $x+y$ and $x+z$ are also null. The system

$$\begin{aligned} x+y &= 0 \\ x+z &= 0 \\ y+z &= 0 \end{aligned}$$

has the trivial solution as the only solution.
(**c**) It is true that, if the couple $(\mathbf{a}, \mathbf{b})$ is linearly independent, then the triple

$$(\mathbf{a}, \mathbf{b}, (\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}))$$

is linearly independent, too. Indeed,

$$\begin{aligned} (\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) &= \mathbf{a}\times\mathbf{a} - \mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{a} - \mathbf{b}\times\mathbf{b} \\ &= -2\mathbf{a}\times\mathbf{b} \end{aligned}$$

and the triple

$$(\mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b})$$

is linearly independent when the couple $(\mathbf{a}, \mathbf{b})$ is so.
13.11.14 *n. 14 (p. 488)*
(**a**) The cross product $\overrightarrow{AB} \times \overrightarrow{AC}$ equals the null vector if and only if the couple $\left(\overrightarrow{AB}, \overrightarrow{AC}\right)$ is linearly dependent. In such a case, if $\overrightarrow{AC}$ is null, then A and C coincide and it is clear that the line through them and B contains all the three points. Otherwise, if $\overrightarrow{AC}$ is nonnull, then in the nontrivial null combination

$$x\overrightarrow{AB} + y\overrightarrow{AC} = \mathbf{0}$$

x must be nonnull, yielding

$$\overrightarrow{AB} = -\frac{y}{x}\overrightarrow{AC}$$

that is,

$$B = A - \frac{y}{x}\overrightarrow{AC}$$

which means that B belongs to the line through A and C.
(**b**) By the previous point, the set

$$\left\{P : \overrightarrow{AP} \times \overrightarrow{BP} = \mathbf{0}\right\}$$

is the set of all points P such that A, B, and P belong to the same line, that is, the set of all points P belonging to the line through A and B.

13.11.15 n. 15 (p. 488)

(**a**) From the assumption $(\mathbf{p} \times \mathbf{b}) + \mathbf{p} = \mathbf{a}$,

$$\begin{aligned} \langle \mathbf{b}, \mathbf{p} \times \mathbf{b} \rangle + \langle \mathbf{b}, \mathbf{p} \rangle &= \langle \mathbf{b}, \mathbf{a} \rangle \\ \langle \mathbf{b}, \mathbf{p} \rangle &= 0 \end{aligned}$$

that is, $\mathbf{p}$ is orthogonal to $\mathbf{b}$. This gives

$$\|\mathbf{p} \times \mathbf{b}\| = \|\mathbf{p}\| \|\mathbf{b}\| = \|\mathbf{p}\| \tag{13.11}$$

and hence

$$1 = \|\mathbf{a}\|^2 = \|\mathbf{p} \times \mathbf{b}\|^2 + \|\mathbf{p}\|^2 = 2\|\mathbf{p}\|^2$$

that is, $\|\mathbf{p}\| = \frac{\sqrt{2}}{2}$.
(**b**) Since $\mathbf{p}$, $\mathbf{b}$, and $\mathbf{p} \times \mathbf{b}$ are pairwise orthogonal, $(\mathbf{p}, \mathbf{b}, \mathbf{p} \times \mathbf{b})$ is linearly independent, and $\{\mathbf{p}, \mathbf{b}, \mathbf{p} \times \mathbf{b}\}$ is a basis of $\mathbb{R}^3$.
(**c**) Since $\mathbf{p}$ is orthogonal both to $\mathbf{p} \times \mathbf{b}$ (definition of vector product) and to $\mathbf{b}$ (point a), there exists some $h \in \mathbb{R}$ such that

$$(\mathbf{p} \times \mathbf{b}) \times \mathbf{b} = h\mathbf{p}$$

Thus, taking into account (13.11),

$$|h| \|\mathbf{p}\| = \|\mathbf{p} \times \mathbf{b}\| \|\mathbf{b}\| \left|\sin \frac{\pi}{2}\right| = \|\mathbf{p}\|$$

and $h \in \{1, -1\}$. Since the triples $(\mathbf{p}, \mathbf{b}, \mathbf{p} \times \mathbf{b})$ and $(\mathbf{p} \times \mathbf{b}, \mathbf{p}, \mathbf{b})$ define the same orientation, and $(\mathbf{p} \times \mathbf{b}, \mathbf{b}, \mathbf{p})$ defines the opposite one, $h = -1$.
(**d**) Still from the assumption $(\mathbf{p} \times \mathbf{b}) + \mathbf{p} = \mathbf{a}$,

$$\begin{aligned} \langle \mathbf{p}, \mathbf{p} \times \mathbf{b} \rangle + \langle \mathbf{p}, \mathbf{p} \rangle &= \langle \mathbf{p}, \mathbf{a} \rangle \\ \langle \mathbf{p}, \mathbf{a} \rangle &= \|\mathbf{p}\|^2 = \frac{1}{2} \\ \mathbf{p} \times (\mathbf{p} \times \mathbf{b}) + \mathbf{p} \times \mathbf{p} &= \mathbf{p} \times \mathbf{a} \\ \|\mathbf{p} \times \mathbf{a}\| &= \|\mathbf{p}\|^2 \|\mathbf{b}\| = \frac{1}{2} \end{aligned}$$

Thus

$$\mathbf{p} = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{q} \tag{13.12}$$

where $\mathbf{q} = 2\mathbf{p} - \mathbf{a}$ is a vector in the plane π generated by $\mathbf{p}$ and $\mathbf{a}$, which is orthogonal to $\mathbf{a}$. Since a normal vector to π is $\mathbf{b}$, there exists some $k \in \mathbb{R}$ such that

$$\mathbf{q} = k(\mathbf{b} \times \mathbf{a})$$

Now

$$\begin{aligned} \|\mathbf{q}\|^2 &= 4\|\mathbf{p}\|^2 + \|\mathbf{a}\|^2 - 4\|\mathbf{p}\|\|\mathbf{a}\|\cos\widehat{\mathbf{pa}} \\ &= 2 + 1 - 4\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} = 1 \\ \|\mathbf{q}\| &= 1 \qquad |k| = 1 \end{aligned}$$

If on the plane orthogonal to $\mathbf{b}$ the mapping $\mathbf{u} \mapsto \mathbf{b} \times \mathbf{u}$ rotates counterclockwise, the vectors $\mathbf{a} = \mathbf{p} + \mathbf{b} \times \mathbf{p}$, $\mathbf{b} \times \mathbf{p}$, and $\mathbf{b} \times \mathbf{a}$ form angles of $\frac{\pi}{4}$, $\frac{\pi}{2}$, and $\frac{3\pi}{4}$, respectively, with $\mathbf{p}$. On the other hand, the decomposition of $\mathbf{p}$ obtained in (13.12) requires that the angle formed with $\mathbf{p}$ by $\mathbf{q}$ is $-\frac{\pi}{4}$, so that $\mathbf{q}$ is discordant with $\mathbf{b} \times \mathbf{a}$. It follows that $k = -1$. I have finally obtained

$$\mathbf{p} = \frac{1}{2}\mathbf{a} - \frac{1}{2}(\mathbf{b} \times \mathbf{a})$$

13.12 The scalar triple product

13.13 Cramer's rule for solving systems of three linear equations

13.14 Exercises

13.15 Normal vectors to planes

13.16 Linear cartesian equations for planes

13.17 Exercises

13.17.1 n. 1 (p. 496)

(**a**) Out of the inifnitely many vectors satisfying the requirement, a distinguished one is

$$\mathbf{n} \equiv (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \times (\mathbf{j} + \mathbf{k}) = 7\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

(**b**)

$$\langle \mathbf{n}, (x, y, z) \rangle = 0 \qquad \text{or} \qquad 7x - 2y + 2z = 0$$

(**c**)

$$\langle \mathbf{n}, (x, y, z) \rangle = \langle \mathbf{n}, (1, 2, 3) \rangle \qquad \text{or} \qquad 7x - 2y + 2z = 9$$

13.17.2 n. 2 (p. 496)

(**a**)

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right)$$

(**b**) The three intersection points are:

$$X\text{-axis} : (-7, 0, 0) \qquad Y\text{-axis} : \left(0, -\frac{7}{2}, 0 \right) \qquad Z\text{-axis} : \left(0, 0, \frac{7}{2} \right).$$

(**c**) The distance from the origin is $\frac{7}{3}$.
(**d**) Intersecting with π the line through the origin which is directed by the normal to π

$$\begin{cases} x = h \\ y = 2h \\ z = -2h \\ x + 2y - 2z + 7 = 0 \end{cases}$$

yields

$$\begin{aligned} h + 4h + 4h + 7 &= 0 \\ h &= -\frac{7}{9} \\ (x, y, z) &= \left(-\frac{7}{9}, -\frac{14}{9}, \frac{14}{9} \right) \end{aligned}$$

13.17.3 n. 3 (p. 496)

A cartesian equation for the plane which passes through the point $P \equiv (1, 2, -3)$ and is parallel to the plane of equation

$$3x - y + 2z = 4$$

is

$$3(x-1) - (y-2) + 2(z+3) = 0$$

or

$$3x - y + 2z + 5 = 0$$

The distance between the two planes is

$$\frac{|4 - (-5)|}{\sqrt{9+1+4}} = \frac{9\sqrt{14}}{14}$$

13.17.4 n. 4 (p. 496)

$$\begin{aligned}
\pi_1 &: x + 2y - 2z = 5 \\
\pi_2 &: 3x - 6y + 3z = 2 \\
\pi_3 &: 2x + y + 2z = -1 \\
\pi_4 &: x - 2y + z = 7
\end{aligned}$$

(**a**) π_2 and π_4 are parallel because $\mathbf{n}_2 = 3\mathbf{n}_4$; π_1 and π_3 are orthogonal because $\langle \mathbf{n}_1, \mathbf{n}_3 \rangle = 0$.

(**b**) The straight line through the origin having direction vector $\mathbf{n}_4$ has equations

$$\begin{aligned}
x &= t \\
y &= -2t \\
z &= t
\end{aligned}$$

and intersects π_2 and π_4 at points C and D, respectively, which can be determined by the equations

$$\begin{aligned}
3t_C - 6(-2t_C) + 3t_C &= 2 \\
t_D - 2(-2t_D) + t_D &= 7
\end{aligned}$$

Thus $t_C = \frac{1}{9}$, $t_D = \frac{7}{6}$, $\overrightarrow{CD} = \left(\frac{7}{6} - \frac{1}{9}\right)\mathbf{n}_4$, and

$$\left\|\overrightarrow{CD}\right\| = \frac{19}{18}\|\mathbf{n}_4\| = \frac{19}{18}\sqrt{6}$$

Alternatively, rewriting the equation of π_4 with normal vector $\mathbf{n}_2$

$$3x - 6y + 3z = 21$$

$$dist(\pi, \pi') = \frac{|d_2 - d_4|}{\|\mathbf{n}_2\|} = \frac{19}{\sqrt{54}}$$

13.17.5 n. 5 (p. 496)

(**a**) A normal vector for the plane π through the points $P \equiv (1,1,-1)$, $Q \equiv (3,3,2)$, $R \equiv (3,-1,-2)$ is

$$\mathbf{n} \equiv \overrightarrow{PQ} \times \overrightarrow{QR} = (2,2,3) \times (0,-4,-4) = (4,8,-8)$$

(**b**) A cartesian equation for π is

$$x + 2y - 2z = 5$$

(**c**) The distance of π from the origin is $\frac{5}{3}$.

13.17.6 n. 6 (p. 496)

Proceeding as in points (a) and (b) of the previous exercise, a normal vector for the plane through the points $P \equiv (1,2,3)$, $Q \equiv (2,3,4)$, $R \equiv (-1,7,-2)$ is

$$\mathbf{n} \equiv \overrightarrow{PQ} \times \overrightarrow{RQ} = (1,1,1) \times (3,-4,6) = (10,-3,-7)$$

and a cartesian equation for it is

$$10x - 3y - 7z + 17 = 0$$

13.17.7 n. 8 (p. 496)

A normal vector to the plane is given by any direction vector for the given line

$$\mathbf{n} \equiv (2,4,12) - (1,2,3) = (1,2,9)$$

A cartesian equation for the plane is

$$(x-2) + 2(y-3) + 9(z+7) = 0$$

or

$$x + 2y + 9z = 55$$

13.17.8 n. 9 (p. 496)

A direction vector for the line is just the normal vector of the plane. Thus the parametric equations are

$$x = 2 + 4h \qquad y = 1 - 3h \qquad z = -3 + h$$

13.17.9 n. 10 (p. 496)

(**a**) The position of the point at time t can be written as follows:

$$x = 1 - t \qquad y = 2 - 3t \qquad z = -1 + 2t$$

which are just the parametric equations of a line.

(**b**) The direction vector of the line L is

$$\mathbf{d} = (-1, -3, 2)$$

(**c**) The hitting instant is determined by substituting the coordinates of the moving point into the equation of the given plane π

$$\begin{aligned} 2(1-t) + 3(2-3t) + 2(-1+2t) + 1 &= 0 \\ -7t + 7 &= 0 \end{aligned}$$

yielding $t = 1$. Hence the hitting point is $(0, -1, 1)$.

(**d**) At time $t = 3$, the moving point has coordinates $(-2, -7, 5)$. Substituting,

$$\begin{aligned} 2 \cdot (-2) + 3 \cdot (-7) + 2 \cdot 5 + d &= 0 \\ d &= 15 \end{aligned}$$

A cartesian equation for the plane π' which is parallel to π and contains $(-2, -7, 5)$ is

$$2x + 3y + 2z + 15 = 0$$

(**e**) At time $t = 2$, the moving point has coordinates $(-1, -4, 3)$. A normal vector for the plane π'' which is orthogonal to L and contains $(-1, -4, 3)$ is $-\mathbf{d}$. Thus

$$\begin{aligned} -1 \cdot -1 - 3 \cdot -4 + 2 \cdot 3 + d &= 0 \\ d &= -19 \end{aligned}$$

A cartesian equation for π'' is

$$(x+1) + 3(y+4) - 2(z-3) = 0$$

or

$$x + 3y - 2z + 19 = 0$$

13.17.10 n. 11 (p. 496)

From

$$\widehat{\mathbf{n}\,\mathbf{i}} = \frac{\pi}{3} \qquad \widehat{\mathbf{n}\,\mathbf{j}} = \frac{\pi}{4} \qquad \widehat{\mathbf{n}\,\mathbf{k}} = \frac{\pi}{3}$$

we get

$$\frac{\mathbf{n}}{\|\mathbf{n}\|} = \frac{1}{2}\left(1, \sqrt{2}, 1\right)$$

A cartesian equation for the plane in consideration is

$$(x-1) + \sqrt{2}\,(y-1) + (z-1) = 0$$

or

$$x + \sqrt{2}y + z = 2 + \sqrt{2}$$

13.17.11 n. 13 (p. 496)

First I find a vector of arbitrary norm which satisfies the given conditions:

$$\begin{aligned} l + 2m - 3n &= 0 \\ l - m + 5n &= 0 \end{aligned}$$

Assigning value 1 to n, the other values are easily obtained: $(l, m) = \left(-\frac{7}{3}, \frac{8}{3}\right)$. Then the required vector is $\left(-\frac{7}{\sqrt{122}}, \frac{8}{\sqrt{122}}, \frac{3}{\sqrt{122}}\right)$

13.17.12 n. 14 (p. 496)

A normal vector for the plane π which is parallel to both vectors $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$ is

$$\mathbf{n} \equiv (\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k}) = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

Since the intercept of π with the X-axis is $(2, 0, 0)$, a cartesian equation for π is

$$x - y + z = 2$$

13.17.13 n. 15 (p. 496)

I work directly on the coefficient matrix for the equation system:

$$\begin{array}{ccc|c} 3 & 1 & 1 & 5 \\ 3 & 1 & 5 & 7 \\ 1 & -1 & 3 & 3 \end{array}$$

With obvious manipulations

$$\begin{array}{ccc|c} 0 & 4 & -8 & -4 \\ 0 & 0 & 4 & 2 \\ 1 & -1 & 3 & 3 \end{array}$$

I obtain a unique solution $(x, y, z) = \left(\frac{3}{2}, 0, \frac{1}{2}\right)$.

13.17.14 n. 17 (p. 497)

If the line ℓ under consideration is parallel to the two given planes, which have as normal vectors $\mathbf{n}_1 \equiv (1,2,3)$ and $\mathbf{n}_2 \equiv (2,3,4)$, a direction vector for ℓ is

$$\mathbf{d} \equiv (2,3,4) \times (1,2,3) = (1,-2,1)$$

Since ℓ goes through the point $P \equiv (1,2,3)$, parametric eqautions for ℓ are the following:

$$x = 1 + t \qquad y = 2 - 2t \qquad z = 3 + t$$

13.17.15 n. 20 (p. 497)

A cartesian equation for the plane under consideration is

$$2x - y + 2z + d = 0$$

The condition of equal distance from the point $P \equiv (3,2,-1)$ yields

$$\frac{|6-2-2+d|}{3} = \frac{|6-2-2+4|}{3}$$

that is,

$$|2+d| = 6$$

The two solutions of the above equation are $d_1 = 4$ (corresponding to the plane already given) and $d_2 = -8$. Thus the required equation is

$$2x - y + 2z = 8$$

13.18 The conic sections

13.19 Eccentricity of conic sections

13.20 Polar equations for conic sections

13.21 Exercises

13.22 Conic sections symmetric about the origin

13.23 Cartesian equations for the conic sections

13.24 Exercises

13.25 Miscellaneous exercises on conic sections

Chapter 14
CALCULUS OF VECTOR-VALUED FUNCTIONS

Chapter 15
LINEAR SPACES

15.1 Introduction

15.2 The definition of a linear space

15.3 Examples of linear spaces

15.4 Elementary consequences of the axioms

15.5 Exercises

15.5.1 n. 1 (p. 555)

The set of all real rational functions is a real linear space. Indeed, let P, Q, R, and S be any four real polynomials, and let

$$f : x \mapsto \frac{P(x)}{Q(x)} \qquad g : x \mapsto \frac{R(x)}{S(x)}$$

Then for every two real numbers α and β

$$\alpha f + \beta g : x \mapsto \frac{\alpha P(x) S(x) + \beta Q(x) R(x)}{Q(x) S(x)}$$

is a well defined real rational function. This shows that the set in question is closed with respect to the two linear space operations of function sum and function multiplication by a scalar. From this the other two existence axioms (of the zero element and of negatives) also follow as particular cases, for $(\alpha, \beta) = (0, 0)$ and for $(\alpha, \beta) = (0, -1)$ respectively. Of course it can also be seen directly that the identically null function $x \mapsto 0$ is a rational function, as the quotient of the constant polynomials $\overline{P} : x \mapsto 0$

and $\overline{Q} : x \mapsto 1$. The remaining linear space axioms are immediately seen to hold, taking into account the general properties of the operations of function sum and function multiplication by a scalar

15.5.2 n. 2 (p. 555)

The set of all real rational functions having numerator of degree not exceeding the degree of the denominator is a real linear space. Indeed, taking into account exercise 1, and using the same notation, it only needs to be proved that if $\deg P \leq \deg Q$ and $\deg R \leq \deg S$, then

$$\deg\left[\alpha PS + \beta QR\right] \leq \deg QS$$

This is clear, since for every two real numbers α and β

$$\begin{aligned}
\deg\left[\alpha PS + \beta QR\right] &\leq \max\left\{\deg PS, \deg QR\right\} \\
\deg PS &= \deg P \deg S \leq \deg Q \deg S \\
\deg QR &= \deg Q \deg R \leq \deg Q \deg S
\end{aligned}$$

so that the closure axioms hold. It may be also noticed that the degree of both polynomials $\overline{P}$ and $\overline{Q}$ occurring in the representation of the identically null function as a rational function is zero.

15.5.3 n. 3 (p. 555)

The set of all real valued functions which are defined on a fixed domain containing 0 and 1, and which have the same value at 0 and 1, is a real linear space. Indeed, for every two real numbers α and β,

$$(\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = \alpha f(1) + \beta g(1) = (\alpha f + \beta g)(1)$$

so that the closure axioms hold. Again, the two existence axioms follow as particular cases; and it is anyway clear that the identically null function $x \mapsto 0$ achieves the same value at 0 and 1. Similarly, that the other linear space axioms hold is a straightforward consequence of the general properties of the operations of function sum and function multiplication by a scalar.

15.5.4 n. 4 (p. 555)

The set of all real valued functions which are defined on a fixed domain containing 0 and 1, and which achieve at 0 the double value they achieve at 1 is a real linear space. Indeed, for every two real numbers α and β,

$$(\alpha f + \beta g)(0) = \alpha f(0) + \beta g(0) = \alpha 2 f(1) + \beta 2 g(1) = 2(\alpha f + \beta g)(1)$$

so that the closure axioms hold. A final remark concerning the other axioms, of a type which has become usual at this point, allows to conclude.

15.5.5 n. 5 (p. 555)

The set of all real valued functions which are defined on a fixed domain containing 0 and 1, and which have value at 1 which exceeds the value at 0 by 1, is not a real linear space. Indeed, let α and β be any two real numbers such that $\alpha+\beta \neq 1$. Then

$$\begin{aligned}(\alpha f+\beta g)(1) &= \alpha f(1)+\beta g(1)=\alpha[1+f(0)]+\beta[1+g(0)] \\ &= \alpha+\beta+\alpha f(0)+\beta g(0)=\alpha+\beta+(\alpha f+\beta g)(0) \\ &\neq 1+(\alpha f+\beta g)(0)\end{aligned}$$

and the two closure axioms fail to hold (the above shows, however, that the set in question is an affine subspace of the real linear space). The other failing axioms are: existence of the zero element (the identically null function has the same value at 0 and at 1), and existence of negatives

$$f(1)=1+f(0) \Rightarrow (-f)(1)=-1+(-f)(0) \neq 1+(-f)(0)$$

15.5.6 n. 6 (p. 555)

The set of all real valued step functions which are defined on $[0,1]$ is a real linear space. Indeed, let f and g be any two such functions, so that for some nonnegative integers m and n, some increasing $(m+1)$-tuple $(\sigma_r)_{r\in\{0\}\cup\mathbf{m}}$ of elements of $[0,1]$ with $\sigma_0=0$ and $\sigma_m=1$, some increasing $(n+1)$-tuple $(\tau_s)_{s\in\{0\}\cup\mathbf{n}}$ of elements of $[0,1]$ with $\tau_0=0$ and $\tau_m=1$, some $(m+1)$-tuple $(\gamma_r)_{r\in\{0\}\cup\mathbf{m}}$ of real numbers, and some n-tuple $(\delta_s)_{s\in\{0\}\cup\mathbf{n}}$ of real numbers*,

$$f \equiv \gamma_0\chi_{\{0\}}+\sum_{r\in\mathbf{m}}\gamma_r\chi_{I_r} \qquad g \equiv \delta_0\chi_{\{0\}}+\sum_{s\in\mathbf{n}}\delta_s\chi_{J_s}$$

where for each subset C of $[0,1]$, χ_C is the characteristic function of C

$$\chi_C \equiv x \mapsto \left\langle \begin{array}{ll} 1 & \text{if } x\in C \\ 0 & \text{if } x\notin C \end{array}\right.$$

and $(I_r)_{r\in\mathbf{m}}$, $(J_s)_{s\in\mathbf{n}}$ are the partitions of $(0,1]$ associated to $(\sigma_r)_{r\in\{0\}\cup\mathbf{m}}$, $(\tau_s)_{s\in\{0\}\cup\mathbf{n}}$

$$\begin{array}{ll} I_r \equiv (\sigma_{r-1},\sigma_r] & (r\in\mathbf{m}) \\ J_s \equiv (\tau_{s-1},\tau_s] & (s\in\mathbf{n}) \end{array}$$

Then, for any two real numbers α and β,

$$\alpha f+\beta g=(\alpha\gamma_0+\beta\delta_0)\chi_{\{0\}}+\sum_{(r,s)\in\mathbf{m}\times\mathbf{n}}(\alpha\gamma_r+\beta\delta_s)\chi_{K_{rs}}$$

*I am using here the following slight abuse of notation for degenerate intervals: $(\sigma,\sigma]\equiv\{\sigma\}$; This is necessary in order to allow for "point steps".

where $(K_{rs})_{(r,s)\in\mathbf{m}\times\mathbf{n}}$ is the "meet" partition of $(I_r)_{r\in\mathbf{m}}$ and $(J_s)_{s\in\mathbf{n}}$

$$K_{rs} \equiv I_r \cap J_s \qquad\qquad ((r,s) \in \mathbf{m} \times \mathbf{n})$$

which shows that $\alpha f + \beta g$ is a step function too, with no more† than $m + n$ steps on $[0,1]$.

Thus the closure axioms hold, and a final, usual remark concenrning the other linear space axioms applies. It may also be noticed independently that the identically null function $[0,1] \mapsto \mathbb{R}$, $x \mapsto 0$ is indeed a step function, with just one step ($m = 1$, $\gamma_0 = \gamma_1 = 0$), and that the opposite of a step function is a step function too, with the same number of steps.

15.5.7 n. 7 (p. 555)

The set of all real valued functions which are defined on $\mathbb{R}$ and convergent to 0 at $+\infty$ is a real linear space. Indeed, by classical theorems in the theory of limits, for every two such functions f and g, and for any two real numbers α and β,

$$\begin{aligned} \lim_{x\to+\infty} (\alpha f + \beta g)(x) &= \alpha \lim_{x\to+\infty} f(x) + \beta \lim_{x\to+\infty} g(x) \\ &= 0 \end{aligned}$$

so that the closure axioms hold. Final remark concerning the other linear space axioms. It may be noticed independently that the identically null function $[0,1] \mapsto \mathbb{R}$, $x \mapsto 0$ indeed converges to 0 at $+\infty$ (and, for that matter, at any $x_0 \in \mathbb{R}\cup\{-\infty\}$).

15.5.8 n. 11 (p. 555)

The set of all real valued and increasing functions of a real variable is not a real linear space. The first closure axiom holds, because the sum of two increasing functions is an increasing function too. The second closure axiom does not hold, however, because the function αf is decreasing if f is increasing and α is a negative real number. The axiom of existence of the zero element holds or fails, depending on whether monotonicity is meant in the weak or in the strict sense, since the identically null function is weakly increasing (and, for that matter, weakly decreasing too, as every constant function) but not strictly increasing. Thus, in the former case, the set of all real valued and (weakly) increasing functions of a real variable is a convex cone. The axiom of existence of negatives fails, too. All the other linear space axioms hold, as in the previous examples.

15.5.9 n. 13 (p. 555)

The set of all real valued functions which are defined and integrable on $[0,1]$, with the integral over $[0,1]$ equal to zero, is a real linear space. Indeed, for any two such functions f and g, and any two real numbers α and β,

$$\int_0^1 (\alpha f + \beta g)(x)\,dx = \alpha \int_0^1 f(x)\,dx + \beta \int_0^1 g(x)\,dx = 0$$

†Some potential steps may "collapse" if, for some $(r,s) \in \mathbf{m} \times \mathbf{n}$, $\sigma_{r-1} = \tau_{s-1}$ and $\alpha\sigma_r + \beta\tau_s = \alpha\sigma_{r-1} + \beta\tau_{s-1}$.

15.5.10 n. 14 (p. 555)

The set of all real valued functions which are defined and integrable on $[0,1]$, with nonnegative integral over $[0,1]$, is a convex cone, but not a linear space. For any two such functions f and g, and any two nonnegative real numbers α and β,

$$\int_0^1 (\alpha f + \beta g)(x)\,dx = \alpha \int_0^1 f(x)\,dx + \beta \int_0^1 g(x)\,dx \geq 0$$

It is clear that if $\alpha < 0$ and $\int_0^1 (\alpha f)(x)\,dx > 0$, then $\int_0^1 (\alpha f)(x)\,dx < 0$, so that axioms 2 and 6 fail to hold.

15.5.11 n. 16 (p. 555)

First solution. The set of all real Taylor polynomials of degree less than or equal to n (including the zero polynomial) is a real linear space, since it coincides with the set of all real polynomials of degree less than or equal to n, which is already known to be a real linear space. The discussion of the above statement is a bit complicated by the fact that nothing is said concerning the point where our Taylor polynomials are to be centered. If the center is taken to be 0, the issue is really easy: every polynomial is the Taylor polynomial centered at 0 of itself (considered, as it is, as a real function of a real variable). This is immediate, if one thinks at the motivation for the definition of Taylor polynomials: best n-degree polynomial approximation of a given function. At any rate, if one takes the standard formula as a definition

$$\text{Taylor}_n\ (f)\ \text{ at } 0 \equiv x \mapsto \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} x^j$$

here are the computations. Let

$$P : x \mapsto \sum_{i=0}^{n} p_i x^{n-i}$$

Then

$$\begin{array}{lll}
P' & : \ x \mapsto \sum_{i=0}^{n-1} (n-i)\, p_i x^{n-1-i} & P'(0) = p_{n-1} \\
P'' & : \ x \mapsto \sum_{i=0}^{n-2} (n-i)(n-(i+1))\, p_i x^{n-2-i} & P''(0) = 2p_{n-2} \\
\vdots & \quad \vdots & \vdots \\
P^{(j)} & : \ x \mapsto \sum_{i=0}^{n-j} \frac{(n-i)!}{(n-i-j)!} p_i x^{n-j-i} & P^{(j)}(0) = j! p_{n-j} \\
\vdots & \quad \vdots & \vdots \\
P^{(n-1)} & : \ x \mapsto n! p_0 x + (n-1)! p_1 & P^{(n-1)}(0) = (n-1)! p_1 \\
P^{(n)} & : \ x \mapsto n! p_0 & P^{(n)}(0) = n! p_0
\end{array}$$

and hence

$$\sum_{j=0}^{n} \frac{P^{(j)}(0)}{j!} x^j = \sum_{j=0}^{n} \frac{j! p_{n-j}}{j!} x^j = \sum_{i=0}^{n} p_i x^{n-i} = P(x)$$

On the other hand, if the Taylor polynomials are meant to be centered at some $x_0 \neq 0$

$$\text{Taylor}_n\ (f)\ \text{ at } x_0 \equiv x \mapsto \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j$$

it must be shown by more lengthy arguments that for each polynomial P the following holds:

$$\sum_{j=0}^{n} \frac{P^{(j)}(x_0)}{j!} (x - x_0)^j = \sum_{i=0}^{n} p_i x^{n-i}$$

Second solution. The set of all real Taylor polynomials (centered at $x_0 \in \mathbb{R}$) of degree less than or equal to n (including the zero polynomial) is a real linear space. Indeed, let P and Q be any two such polynomials; that is, let f and g be two real functions of a real variable which are m and n times differentiable in x_0, and let $P \equiv \text{Taylor}_m\ (f)$ at x_0, $Q \equiv \text{Taylor}_n\ (g)$ at x_0, that is,

$$P : x \mapsto \sum_{i=0}^{m} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i \qquad Q : x \mapsto \sum_{j=0}^{n} \frac{g^{(j)}(x_0)}{j!} (x - x_0)^j$$

Suppose first that $m = n$. Then for any two real numbers α and β

$$\begin{aligned} \alpha P + \beta Q &= \sum_{k=0}^{m} \left\{ \alpha \left[\frac{f^{(k)}(x_0)}{k!} \right] + \beta \left[\frac{g^{(k)}(x_0)}{k!} \right] \right\} (x - x_0)^k \\ &= \sum_{k=0}^{m} \frac{(\alpha f + \beta g)^{(k)}(x_0)}{k!} (x - x_0)^k \\ &= \text{Taylor}_m\,(\alpha f + \beta g)\ \text{ at } x_0 \end{aligned}$$

Second, suppose (without loss of generality) that $m > n$. In this case, however,

$$\alpha P + \beta Q = \sum_{k=0}^{n} \frac{\alpha f^{(k)}(x_0) + \beta g^{(k)}(x_0)}{k!} (x - x_0)^k + \sum_{k=n+1}^{m} \frac{\alpha f^{(k)}(x_0)}{k!} (x - x_0)^k$$

a polynomial which can be legitimately considered the Taylor polynomial of degree n at x_0 of the function $\alpha f + \beta g$ only if all the derivatives of g of order from $n+1$ to m are null at x_0. This is certainly true if g itself a polynomial of degree n. In fact, this is true only in such a case, as it can be seen by repeated integration. It is hence necessary, in addition, to state and prove the result asserting that each Taylor polynomial of any degree and centered at any $x_0 \in \mathbb{R}$ can be seen as the Taylor polynomial of itself.

15.5.12 n. 17 (p. 555)

The set $\mathcal{S}$ of all solutions of a linear second-order homogeneous differential equation

$$\forall x \in (a,b), \qquad y''(x) + P(x)\, y'(x) + Q(x)\, y(x) = 0$$

where P and Q are given everywhere continuous real functions of a real variable, and (a,b) is some open interval to be determined together with the solution, is a real linear space. First of all, it must be noticed that $\mathcal{S}$ is nonempty, and that its elements are indeed real functions which are everywhere defined (that is, with $(a,b) = \mathbb{R}$), due to the main existence and solution continuation theorems in the theory of differential equations. Second, the operator

$$\mathcal{L} : \mathcal{D}_2 \to \mathbb{R}^{\mathbb{R}}, \ \ y \mapsto y'' + Py' + Qy$$

where $\mathbb{R}^{\mathbb{R}}$ is the set of all the real functions of a real variable, and $\mathcal{D}_2$ is the subset of $\mathbb{R}^{\mathbb{R}}$ of all the functions having a second derivative which is everywhere defined, is linear:

$$\forall y \in \mathcal{D}_2, \, \forall z \in \mathcal{D}_2, \, \forall \alpha \in \mathbb{R}, \, \forall \beta \in \mathbb{R},$$

$$\begin{aligned} \mathcal{L}(\alpha y + \beta z) &= (\alpha y + \beta z)'' + P(\alpha y + \beta z)' + Q(\alpha y + \beta z) \\ &= \alpha(y'' + Py' + Qy) + \beta(z'' + Pz' + Qz) \\ &= \alpha \mathcal{L}(y) + \beta \mathcal{L}(z) \end{aligned}$$

Third, $\mathcal{D}_2$ is a real linear space, since

$$\forall y \in \mathcal{D}_2, \, \forall z \in \mathcal{D}_2, \, \forall \alpha \in \mathbb{R}, \, \forall \beta \in \mathbb{R},$$

$$\alpha y + \beta z \in \mathcal{D}_2$$

by standard theorems on the derivative of a linear combination of differentiable functions, and the usual remark concerning the other linear space axioms. Finally, $\mathcal{S} = \ker \mathcal{L}$ is a linear subspace of $\mathcal{D}_2$, by the following standard argument:

$$\mathcal{L}(y) = 0 \wedge \mathcal{L}(z) = 0 \Rightarrow \mathcal{L}(\alpha y + \beta z) = \alpha \mathcal{L}(y) + \beta \mathcal{L}(z) = 0$$

and the usual remark.

15.5.13 n. 18 (p. 555)

The set of all bounded real sequences is a real linear space. Indeed, for every two real sequences $\mathbf{x} \equiv (x_n)_{n \in \mathbb{N}}$ and $\mathbf{y} \equiv (y_n)_{n \in \mathbb{N}}$ and every two real numbers α and β, if $\mathbf{x}$ and $\mathbf{y}$ are bounded, so that for some positive numbers ε and η and for every $n \in \mathbb{N}$ the following holds:

$$|x_n| < \varepsilon \qquad |y_n| < \eta$$

then the sequence $\alpha \mathbf{x} + \beta \mathbf{y}$ is also bounded, since for every $n \in \mathbb{N}$

$$|\alpha x_n + \beta y_n| \le |\alpha|\, |x_n| + |\beta|\, |y_n| < |\alpha|\, \varepsilon + |\beta|\, \eta$$

Thus the closure axioms hold, and the usual remark concerning the other linear space axioms applies.

15.5.14 n. 19 (p. 555)

The set of all convergent real sequences is a real linear space. Indeed, for every two real sequences $\mathbf{x} \equiv (x_n)_{n\in\mathbb{N}}$ and $\mathbf{y} \equiv (y_n)_{n\in\mathbb{N}}$ and every two real numbers α and β, if $\mathbf{x}$ and $\mathbf{y}$ are convergent to $\overline{x}$ and $\overline{y}$ respectively, then the sequence $\alpha\mathbf{x}+\beta\mathbf{y}$ converges to $\alpha\overline{x}+\beta\overline{y}$. Thus the closure axioms hold. Usual remark concerning the other linear space axioms.

15.5.15 n. 22 (p. 555)

The set U of all elements of $\mathbb{R}^3$ with their third component equal to 0 is a real linear space. Indeed, by linear combination the third component remains equal to 0; U is the kernel of the linear function $\mathbb{R}^3 \to \mathbb{R},\ (x,y,z) \mapsto z$.

15.5.16 n. 23 (p. 555)

The set W of all elements of $\mathbb{R}^3$ with their second or third component equal to 0 is not a linear subspace of $\mathbb{R}^3$. For example, $(0,1,0)$ and $(0,0,1)$ are in the set, but their sum $(0,1,1)$ is not. The second closure axiom and the existence of negatives axiom fail too. The other axioms hold. W is not even an affine subspace, nor it is convex; however, it is a cone.

15.5.17 n. 24 (p. 555)

The set π of all elements of $\mathbb{R}^3$ with their second component which is equal to the third multiplied by 5 is a real linear space, being the kernel of the linear function $\mathbb{R}^3 \to \mathbb{R},\ (x,y,z) \mapsto 5x-y$.

15.5.18 n. 25 (p. 555)

The set ℓ of all elements (x,y,z) of $\mathbb{R}^3$ such that $3x+4y=1$ and $z=0$ (the line through the point $P \equiv (-1,1,0)$ with direction vector $v \equiv (4,-3,0)$) is an affine subspace of $\mathbb{R}^3$, hence a convex set, but not a linear subspace of $\mathbb{R}^3$, hence not a linear space itself. Indeed, for any two triples (x,y,z) and (u,v,w) of $\mathbb{R}^3$, and any two real numbers α and β

$$\left.\begin{array}{r} 3x+4y=1 \\ z=0 \\ 3u+4v=1 \\ w=0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 3(\alpha x+\beta u)+4(\alpha y+\beta v)=\alpha+\beta \\ z+w=0 \end{array}\right.$$

Thus both closure axioms fail for ℓ. Both existence axioms also fail, since neither the null triple, nor the opposite of any triple in ℓ, belong to ℓ. The associative and distributive laws have a defective status too, since they make sense in ℓ only under quite restrictive assumptions on the elements of $\mathbb{R}^3$ or the real numbers appearing in them.

15.5.19 n. 26 (p. 555)

The set r of all elements (x,y,z) of $\mathbb{R}^3$ which are scalar multiples of $(1,2,3)$ (the line through the origin having $(1,2,3)$ as direction vector) is a real linear space. For any two elements $(h, 2h, 3h)$ and $(k, 2k, 3k)$ of r, and for any two real numbers α and β, the linear combination

$$\alpha (h, 2h, 3h) + \beta (k, 2k, 3k) = (\alpha h + \beta k) (1, 2, 3)$$

belongs to r. r is the kernel of the linear function $\mathbb{R}^3 \to \mathbb{R}^2,\ (x, y, z) \mapsto (2x - y, 3x - z)$.

15.5.20 n. 27 (p. 555)

The set of solutions of the linear homogenous system of equations

$$A (x, y, z)' = \mathbf{0}$$

is the kernel of the linear function $\mathbb{R}^3 \to \mathbb{R},\ (x, y, z) \mapsto A (x, y, z)'$ and hence a real linear space.

15.5.21 n. 28 (p. 555)

The subset of $\mathbb{R}^n$ of all the linear combinations of two given vectors $\mathbf{a}$ and $\mathbf{b}$ is a vector subspace of $\mathbb{R}^n$, namely span $\{\mathbf{a}, \mathbf{b}\}$. It is immediate to check that every linear combination of linear combinations of $\mathbf{a}$ and $\mathbf{b}$ is again a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

15.6 Subspaces of a linear space

15.7 Dependent and independent sets in a linear space

15.8 Bases and dimension

15.9 Exercises

15.9.1 n. 1 (p. 560)

The set S_1 of all elements of $\mathbb{R}^3$ with their first coordinate equal to 0 is a linear subspace of $\mathbb{R}^3$ (see exercise 5.22 above). S_1 is the coordinate YZ-plane, its dimension is 2. A standard basis for it is $\{(0, 1, 0), (0, 0, 1)\}$. It is clear that the two vectors are linearly independent, and that for every real numbers y and z

$$(0, y, z) = y (0, 1, 0) + z (0, 0, 1)$$

15.9.2 n. 2 (p. 560)

The set S_2 of all elements of $\mathbb{R}^3$ with the first and second coordinate summing up to 0 is a linear subspace of $\mathbb{R}^3$ (S_2 is the plane containing the Z-axis and the bisectrix of the even-numbered quadrants of the XY-plane). A basis for it is $\{(1,-1,0),(0,0,1)\}$; its dimension is 2. It is clear that the two vectors are linearly independent, and that for every three real numbers x, y and z such that $x+y=0$

$$(x,y,z) = x(1,-1,0) + z(0,0,1)$$

15.9.3 n. 3 (p. 560)

The set S_3 of all elements of $\mathbb{R}^3$ with the coordinates summing up to 0 is a linear subspace of $\mathbb{R}^3$ (S_3 is the plane through the origin and normal vector $n \equiv (1,1,1)$). A basis for it is $\{(1,-1,0),(0,-1,1)\}$; its dimension is 2. It is clear that the two vectors are linearly independent, and that for every three real numbers x, y and z such that $x+y+z=0$

$$(x,y,z) = x(1,-1,0) + z(0,-1,1)$$

15.9.4 n. 4 (p. 560)

The set S_4 of all elements of $\mathbb{R}^3$ with the first two coordinates equal is a linear subspace of $\mathbb{R}^3$ (S_4 is the plane containing the Z-axis and the bisectrix of the odd-numbered quadrants of the XY-plane). A basis for it is $\{(1,1,0),(0,0,1)\}$; its dimension is 2. It is clear that the two vectors are linearly independent, and that for every three real numbers x, y and z such that $x=y$

$$(x,y,z) = x(1,1,0) + z(0,0,1)$$

15.9.5 n. 5 (p. 560)

The set S_5 of all elements of $\mathbb{R}^3$ with all the coordinates equal is a linear subspace of $\mathbb{R}^3$ (S_5 is the line through the origin and direction vector $d \equiv (1,1,1)$). A basis for it is $\{d\}$; its dimension is 1.

15.9.6 n. 6 (p. 560)

The set S_6 of all elements of $\mathbb{R}^3$ with the first coordinate equal either to the second or to the third is not a linear subspace of $\mathbb{R}^3$ (S_6 is the union of the plane S_4 of exercise 4 and the plane containing the Y-axis and the bisectrix of the odd-numbered quadrants of the XZ-plane). For example, $(1,1,0)$ and $(1,0,1)$ both belong to S_6, but their sum $(2,1,1)$ does not.

15.9.7 n. 7 (p. 560)

The set S_7 of all elements of $\mathbb{R}^3$ with the first and second coordinates having identical square is not a linear subspace of $\mathbb{R}^3$ (S_7 is the union of the two planes containing the Z-axis and one of the bisectrices of the odd and even-numbered quadrants of the XY-plane). For example, $(1,-1,0)$ and $(1,1,0)$ both belong to S_7, but their sum

$(2,0,0)$ or their semisum $(1,0,0)$ do not. S_7 is not an affine subspace of $\mathbb{R}^3$, and it is not even convex. However, S_7 is a cone, and it is even closed with respect to the external product (multiplication by arbitrary real numbers).

15.9.8 n. 8 (p. 560)

The set S_8 of all elements of $\mathbb{R}^3$ with the first and second coordinates summing up to 1 is not a linear subspace of $\mathbb{R}^3$ (S_8 is the vertical plane containing the line through the points $P \equiv (1,0,0)$ and $Q \equiv (0,1,0)$) For example, for every $\alpha \neq 1$, $\alpha(1,0,0)$, and $\alpha(0,1,0)$ do not belong to S_8. S_8 is an affine subspace of $\mathbb{R}^3$, since

$$\forall (x,y,z) \in \mathbb{R}^3,\ \forall (u,v,w) \in \mathbb{R}^3,\ \forall \alpha \in \mathbb{R}$$

$$(x+y=1) \wedge (u+v=1) \Rightarrow [(1-\alpha)x + \alpha u] + [(1-\alpha)y + \alpha v] = 1$$

15.9.9 n. 9 (p. 560)

See exercise 26, p.555.

15.9.10 n. 10 (p. 560)

The set S_{10} of all elements (x,y,z) of $\mathbb{R}^3$ such that

$$\begin{aligned} x+y+z &= 0 \\ x-y-z &= 0 \end{aligned}$$

is a line containing the origin, a subspace of $\mathbb{R}^3$ of dimension 1. A base for it is $\{(0,1,-1)\}$. If (x,y,z) is any element of S_{10}, then

$$(x,y,z) = y\,(0,1,-1)$$

15.9.11 n. 11 (p. 560)

The set S_{11} of all polynomials of degree not exceeding n, and taking value 0 at 0 is a linear subspace of the set of all polynomials of degree not exceeding n, and hence a linear space. If P is any polynomial of degree not exceeding n,

$$P : t \mapsto p_0 + \sum_{h \in \mathbf{n}} p_h t^h \tag{15.1}$$

then P belongs to S_{11} if and only if $p_0 = 0$. It is clear that any linear combination of polynomials in S_{11} takes value 0 at 0. A basis for S_{11} is

$$\mathcal{B} \equiv \left\{t \mapsto t^h\right\}_{h \in \mathbf{n}}$$

Indeed, by the general principle of polynomial identity (which is more or less explicitly proved in example 6, p.558), a linear combination of $\mathcal{B}$ is the null polynomial, if and only if all its coefficients are null. Moreover, every polynomial belonging to S_{11} is a linear combination of $\mathcal{B}$. It follows that $\dim S_{11} = n$.

15.9.12 n. 12 (p. 560)

The set S_{12} of all polynomials of degree not exceeding n, with first derivative taking value 0 at 0, is a linear subspace of the set of all polynomials of degree not exceeding n. Indeed, for any two suach polynomials P and Q, and any two real numbers α and β,

$$(\alpha P + \beta Q)'(0) = \alpha P'(0) + \beta Q'(0) = \alpha \cdot 0 + \beta \cdot 0 = 0$$

A polynomial P as in (15.1) belongs to S_{12} if and only if $p_1 = 0$. A basis for S is

$$\mathcal{B} \equiv \left\{ \left(t \mapsto t^{h+1}\right)_{h \in \mathbf{n-1}}, (t \mapsto 1) \right\}$$

That $\mathcal{B}$ is a basis can be seen as in the previous exercise. Thus $\dim S_{12} = n$.

15.9.13 n. 13 (p. 560)

The set S_{13} of all polynomials of degree not exceeding n, with second derivative taking value 0 at 0, is a linear subspace of the set of all polynomials of degree not exceeding n. This is proved exactly as in the previous exercise (just replace $'$ with $''$). A polynomial P as in (15.1) belongs to S_{13} if and only if $p_2 = 0$. A basis for S is

$$\mathcal{B} \equiv \left\{ \left(t \mapsto t^{h+2}\right)_{h \in \mathbf{n-2}}, (t \mapsto 1), (t \mapsto t) \right\}$$

That $\mathcal{B}$ is a basis can be seen as in the exercise 11. Thus $\dim S_{13} = n$.

15.9.14 n. 14 (p. 560)

The set S_{14} of all polynomials of degree not exceeding n, with first derivative taking value at 0 which is the opposite of the polynomial value at 0, is a linear subspace of the set of all polynomials of degree not exceeding n. A polynomial P as in (15.1) belongs to S_{14} if and only if $p_0 + p_1 = 0$. If P and Q are any two polynomials in S, and α and β are any two real numbers,

$$\begin{aligned} (\alpha P + \beta Q)(0) + (\alpha P + \beta Q)'(1) &= \alpha P(0) + \beta Q(0) + \alpha P'(0) + \beta Q'(0) \\ &= \alpha [P(0) + P'(0)] + \beta [Q(0) + Q'(0)] \\ &= \alpha \cdot 0 + \beta \cdot 0 \end{aligned}$$

A basis for S_{14} is

$$\mathcal{B} \equiv \left\{ t \mapsto 1 - t, \left(t \mapsto t^{h+1}\right)_{h \in \mathbf{n-1}} \right\}$$

Indeed, if $\boldsymbol{\alpha}$ is the n-tuple of coefficients of a linear combination R of $\mathcal{B}$, then

$$R : t \mapsto \alpha_1 - \alpha_1 t + \sum_{h \in \mathbf{n-1}} \alpha_{h+1} t^{h+1}$$

By the general principle of polynomial identity, the n-tuple $\boldsymbol{\alpha}$ of coefficients of any linear combination of $\mathcal{B}$ spanning the null vector must satisfy the conditions

$$\begin{aligned}\alpha_1 &= 0 \\ \alpha_{h+1} &= 0 \qquad (h \in \mathbf{n}-\mathbf{1})\end{aligned}$$

that is, the combination must be the trivial one. If P is any polynomial such that $p_0 + p_1 = 0$, P belongs to $\operatorname{span}\mathcal{B}$, since by the position

$$\begin{aligned}\alpha_1 &= p_0 = -p_1 \\ \alpha_{h+1} &= p_{h+1} \qquad (h \in \mathbf{n}-\mathbf{1})\end{aligned}$$

the linear combination of $\mathcal{B}$ with $\boldsymbol{\alpha}$ as n-tuple of coefficients is equal to P. It follows that $\mathcal{B}$ is a basis of S_{14}, and hence that $\dim S = n$.

15.9.15 n. 15 (p. 560)

The set S_{15} of all polynomials of degree not exceeding n, and taking the same value at 0 and at 1 is a linear subspace of the set of all polynomials of degree not exceeding n, and hence a linear space. This can be seen exactly as in exercise 3, p.555. A polynomial P belongs to S_{15} if and only if

$$p_0 = p_0 + \sum_{h \in \mathbf{n}} p_h$$

that is, if and only if

$$\sum_{h \in \mathbf{n}} p_h = 0$$

A basis for S_{15} is

$$\mathcal{B} \equiv \left\{ (t \mapsto 1), \left(t \mapsto (1-t)\, t^h\right)_{h \in \mathbf{n}-\mathbf{1}} \right\}$$

and the dimension of S_{15} is n.

15.9.16 n. 16 (p. 560)

The set S_{16} of all polynomials of degree not exceeding n, and taking the same value at 0 and at 2 is a linear subspace of the set of all polynomials of degree not exceeding n, and hence a linear space. This can be seen exactly as in exercise 3, p.555. A polynomial P belongs to S_{16} if and only if

$$p_0 = p_0 + \sum_{h \in \mathbf{n}} 2^h p_h$$

that is, if and only if

$$\sum_{h \in \mathbf{n}} 2^h p_h = 0$$

A basis for S_{16} is

$$\mathcal{B} \equiv \left\{ (t \mapsto 1), \left(t \mapsto (2-t)\, t^h \right)_{h \in \mathbf{n-1}} \right\}$$

and the dimension of S_{16} is n.

15.9.17 n. 22 (p. 560)

(**b**) It has alredy been shown in the notes that

$$\begin{aligned} \operatorname{lin} S &\equiv \bigcap_{\substack{W \text{ subspace of } V \\ S \subseteq W}} W \\ &= \left\{ \mathbf{v} : \exists n \in \mathbb{N},\ \exists \mathbf{u} \equiv (u_i)_{i \in \mathbf{n}} \in V^n,\ \exists a \equiv (\alpha_i)_{i \in \mathbf{n}} \in F^n,\ \mathbf{v} = \sum_{i \in \mathbf{n}} \alpha_i u_i \right\} \end{aligned}$$

hence if T is a subspace, and it contains S, then T contains $\operatorname{lin} S$.
(**c**) If $S = \operatorname{lin} S$, then of course S is a subspace of V because $\operatorname{lin} S$ is so. Conversely, if S is a subspace of V, then S is one among the subspaces appearing in the definition of $\operatorname{lin} S$, and hence $\operatorname{lin} S \subseteq S$; since $S \subseteq \operatorname{lin} S$ always, it follows that $S = \operatorname{lin} S$.
(**e**) If S and T are subspaces of V, then by point c above $\operatorname{lin} S = S$ and $\operatorname{lin} T = T$, and hence

$$S \cap T = \operatorname{lin} S \cap \operatorname{lin} T$$

Since the intersection of any family of subspaces is a subspace, $S \cap T$ is a subspace of V.
(**g**) Let $V \equiv \mathbb{R}^2$, $S \equiv \{(1,0), (0,1)\}$, $T \equiv \{(1,0), (1,1)\}$. Then

$$\begin{aligned} S \cap T &= \{(1,0)\} \\ L(S) &= L(T) = \mathbb{R}^2 \\ L(S \cap T) &= \{(x,y) : y = 0\} \end{aligned}$$

15.9.18 n. 23 (p. 560)

(**a**) Let

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R},\ x \mapsto 1 \\ g &: \mathbb{R} \to \mathbb{R},\ x \mapsto e^{ax} \\ h &: \mathbb{R} \to \mathbb{R},\ x \mapsto e^{bx} \\ \mathbf{0} &: \mathbb{R} \to \mathbb{R},\ x \mapsto 0 \end{aligned}$$

and let $(u, v, w) \in \mathbb{R}^3$ be such that

$$uf + vg + wh = \mathbf{0}$$

In particular, for $x = 0$, $x = 1$, and $x = 2$, we have

$$\begin{aligned} u + v + w &= 0 \\ u + e^{a} v + e^{b} w &= 0 \\ u + e^{2a} v + e^{2b} w &= 0 \end{aligned} \tag{15.2}$$

The determinant of the coefficient matrix of the above system is

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ 1 & e^{a} & e^{b} \\ 1 & e^{2a} & e^{2b} \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1 \\ 0 & e^{a} - 1 & e^{b} - 1 \\ 0 & e^{2a} - 1 & e^{2b} - 1 \end{vmatrix} \\ &= \left(e^{a} - 1\right)\left(e^{b} - 1\right)\left[e^{b} + 1 - \left(e^{a} + 1\right)\right] \\ &= \left(e^{a} - 1\right)\left(e^{b} - 1\right)\left(e^{b} - e^{a}\right) \end{aligned}$$

Since by assumption $a \neq b$, at most one of the two numbers can be equal to zero. If none of them is null, system (15.2) has only the trivial solution, the triple (f, g, h) is linearly independent, and $\dim \operatorname{lin} \{f, g, h\} = 3$. If $a = 0$, then $g = f$; similarly, if $b = 0$, then $h = f$. Thus if either a or b is null then (f, g, h) is linearly dependent (it suffices to take $(u, v, w) \equiv (1, -1, 0)$ or $(u, v, w) \equiv (1, 0, -1)$, respectively).

It is convenient to state and prove the following (very) simple

Lemma 4 *Let X be a set containing at least two distinct elements, and let $f : X \to \mathbb{R}$ be a nonnull constant function. For any function $g : X \to \mathbb{R}$, the couple (f, g) is linearly dependent if and only if g is constant, too.*

Proof. If g is constant, let $\operatorname{Im} f \equiv \{y_f\}$ and $\operatorname{Im} g \equiv \{y_g\}$. Then the function $y_g f - y_f g$ is the null function. Conversely, if (f, g) is linearly dependent, let $(u, v) \in \mathbb{R}^2 \sim (0, 0)$ be such that $uf + vg$ is the null function. Thus

$$\forall x \in X,\ u y_f + v g(x) = 0$$

Since f is nonnull, $y_f \neq 0$. This yields

$$v = 0 \Rightarrow u = 0$$

and hence $v \neq 0$. Then

$$\forall x \in X,\ g(x) = -\frac{u}{v} y_f$$

and g is constant, too. ■

It immediately follows from the above lemma (and from the assumption $a \neq b$) that if either $a = 0$ or $b = 0$, then $\dim \operatorname{lin} \{f, g, h\} = 2$.

(**b**) The two functions

$$f : x \mapsto e^{ax} \qquad g : x \mapsto x e^{ax}$$

are linearly independent, since

$$\begin{aligned}[\forall x\in\mathbb{R},\ \alpha e^{x}+\beta x e^{ax}=0] &\Leftrightarrow [\forall x\in\mathbb{R},\ (\alpha+\beta x)\,e^{ax}=0]\\ &\Leftrightarrow [\forall x\in\mathbb{R},\ (\alpha+\beta x)=0]\\ &\Leftrightarrow \alpha=\beta=0\end{aligned}$$

Notice that the argument holds even in the case $a=0$, Thus $\dim\operatorname{lin}\{f,g\}=2$.
(**c**) Arguing as in point a, let $(u,v,w)\in\mathbb{R}^3$ be such that

$$uf+vg+wh=\mathbf{0}$$

where

$$f:x\mapsto 1\qquad g:x\mapsto e^{ax}\qquad h:x\mapsto xe^{ax}$$

In particular, for $x=0$, $x=1$, and $x=-1$, we have

$$\begin{aligned}u+v+w &= 0\\ u+e^{a}v+e^{a}w &= 0\\ u+e^{-a}v-e^{-a}w &= 0\end{aligned}\tag{15.3}$$

a homogeneous system of equations whose coefficient matrix has determinant

$$\begin{aligned}\begin{vmatrix}1 & 1 & 1\\ 1 & e^{a} & e^{a}\\ 1 & e^{-a} & -e^{-a}\end{vmatrix} &= \begin{vmatrix}1 & 1 & 1\\ 0 & e^{a}-1 & e^{a}-1\\ 0 & e^{-a}-1 & -e^{-a}-1\end{vmatrix}\\ &= -\left(e^{a}-1\right)\left(e^{-a}+1+e^{-a}-1\right)\\ &= 2e^{-a}\left(1-e^{a}\right)\end{aligned}$$

Thus if $a\neq 0$ the above determinant is different from zero, yielding $(u,v,w)=(0,0,0)$. The triple (f,g,h) is linearly independent, and $\dim\operatorname{lin}\{f,g,h\}=3$. On the other hand, if $a=0$ then $f=g$ and $h=\mathrm{Id}_{\mathbb{R}}$, so that $\dim\operatorname{lin}\{f,g,h\}=2$.
(**f**) The two functions

$$f:x\mapsto\sin x\qquad g:x\mapsto\cos x$$

are linearly independent. Indeed, for every two real numbers α and β which are not both null,

$$\begin{gathered}\forall x\in\mathbb{R},\ \alpha\sin x+\beta\cos x=0\\ \Updownarrow\\ \forall x\in\mathbb{R},\ \tfrac{\alpha}{\sqrt{\alpha^2+\beta^2}}\sin x+\tfrac{\beta}{\sqrt{\alpha^2+\beta^2}}\cos x=0\\ \Updownarrow\\ \forall x\in\mathbb{R},\ \sin(\gamma+x)=0\end{gathered}$$

where

$$\gamma \equiv \operatorname{sign}\beta \arccos \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$$

since the sin function is not identically null, the last condition cannot hold. Thus $\dim \operatorname{lin}\{x \mapsto \sin x, x \mapsto \cos x\} = 2$.

(**h**) From the trigonometric addition formulas

$$\forall x \in \mathbb{R} \quad \cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$

Let then

$$f : x \mapsto 1 \quad g : x \mapsto \cos x \qquad g : x \mapsto \sin^2 x$$

The triple (f, g, h) is linearly dependent, since

$$f - g + 2h = \mathbf{0}$$

By the lemma discussed at point **a**, $\dim \operatorname{lin}\{f, g, h\} = 2$.

15.10 Inner products. Euclidean spaces. Norms

15.11 Orthogonality in a euclidean space

15.12 Exercises

15.12.1 n. 9 (p. 567)

$$
\begin{aligned}
\langle \mathbf{u}_1, \mathbf{u}_2 \rangle &= \int_{-1}^{1} t\,dt = \left.\frac{t^2}{2}\right|_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0 \\
\langle \mathbf{u}_2, \mathbf{u}_3 \rangle &= \int_{-1}^{1} t + t^2\,dt = 0 + \left.\frac{t^3}{3}\right|_{-1}^{1} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3} \\
\langle \mathbf{u}_3, \mathbf{u}_1 \rangle &= \int_{-1}^{1} 1 + t\,dt = \left.t\right|_{-1}^{1} + 0 = 1 - (-1) = 2 \\
\|\mathbf{u}_1\|^2 &= \int_{-1}^{1} 1\,dt = 2 \qquad \|\mathbf{u}_1\| = \sqrt{2} \\
\|\mathbf{u}_2\|^2 &= \int_{-1}^{1} t^2\,dt = \frac{2}{3} \qquad \|\mathbf{u}_2\| = \sqrt{\frac{2}{3}} \\
\|\mathbf{u}_3\|^2 &= \int_{-1}^{1} 1 + 2t + t^2\,dt = 2 + \frac{2}{3} \qquad \|\mathbf{u}_3\| = \sqrt{\frac{8}{3}} \\
\cos \widehat{\mathbf{u}_2\,\mathbf{u}_3} &= \frac{\frac{2}{3}}{\sqrt{\frac{2}{3}}\sqrt{\frac{8}{3}}} = \frac{1}{2} \qquad \cos \widehat{\mathbf{u}_1\,\mathbf{u}_3} = \frac{2}{\sqrt{2}\sqrt{\frac{8}{3}}} = \frac{\sqrt{3}}{2} \\
\widehat{\mathbf{u}_2\,\mathbf{u}_3} &= \frac{\pi}{3} \qquad \widehat{\mathbf{u}_1\,\mathbf{u}_3} = \frac{\pi}{6}
\end{aligned}
$$

15.12.2 n. 11 (p. 567)

(a) Let for each $n \in \mathbb{N}$

$$
I_n \equiv \int_0^{+\infty} e^{-t} t^n dt
$$

Then

$$
\begin{aligned}
I_0 &= \int_0^{+\infty} e^{-t} dt = \lim_{x \to +\infty} \int_0^{x} e^{-t} dt \\
&= \lim_{x \to +\infty} -e^{-t}\Big|_0^x = \lim_{x \to +\infty} -e^{-x} + 1 \\
&= 1
\end{aligned}
$$

and, for each $n \in \mathbb{N}$,

$$\begin{aligned} I_{n+1} &= \int_0^{+\infty} e^{-t} t^{n+1} dt = \lim_{x \to +\infty} \int_0^x e^{-t} t^{n+1} dt \\ &= \lim_{x \to +\infty} \left\{ -e^{-t} t^{n+1} \Big|_0^x + (n+1) \int_0^x e^{-t} t^n dt \right\} \\ &= \lim_{x \to +\infty} \left\{ -e^{-x} x^{n+1} - 0 \right\} + (n+1) \lim_{x \to +\infty} \left\{ \int_0^x e^{-t} t^n dt \right\} \\ &= 0 + (n+1) \int_0^{+\infty} e^{-t} t^n dt \\ &= (n+1)\, I_n \end{aligned}$$

The integral involved in the definition of I_n is always convergent, since for each $n \in \mathbb{N}$

$$\lim_{x \to +\infty} e^{-x} x^n = 0$$

It follows that

$$\forall n \in \mathbb{N}, \quad I_n = n!$$

Let now

$$f : t \mapsto \alpha_0 + \sum_{i \in \mathbf{m}} \alpha_i t^i \qquad g : t \mapsto \beta_0 + \sum_{i \in \mathbf{n}} \beta_j t^j$$

be two real polynomials, of degree m and n respectively. Then the product fg is a real polynomial of degree $m+n$, containing for each $k \in \mathbf{m}+\mathbf{n}$ a monomial of degree k of the form $\alpha_i t^i \beta_j t^j = \alpha_i \beta_j t^{i+j}$ whenever $i + j = k$. Thus

$$fg = \alpha_0 \beta_0 + \sum_{k \in \mathbf{m}+\mathbf{n}} \left(\sum_{\substack{i \in \mathbf{m},\, j \in \mathbf{n} \\ i+j=k}} \alpha_i \beta_j \right) t^k$$

and the scalar product of f and g is results in the sum of $m+n+1$ converging integrals

$$\begin{aligned}
\langle f,g\rangle &= \int_0^{+\infty} e^{-t}\left[\alpha_0\beta_0 + \sum_{k\in\mathbf{m+n}}\left(\sum_{\substack{i\in\mathbf{m},\, j\in\mathbf{n}\\ i+j=k}}\alpha_i\beta_j\right)t^k\right]dt \\
&= \alpha_0\beta_0\int_0^{+\infty} e^{-t}dt + \sum_{k\in\mathbf{m+n}}\left(\sum_{\substack{i\in\mathbf{m},\, j\in\mathbf{n}\\ i+j=k}}\alpha_i\beta_j\right)\int_0^{+\infty} e^{-t}t^k dt \\
&= \alpha_0\beta_0 I_0 + \sum_{k\in\mathbf{m+n}}\left(\sum_{\substack{i\in\mathbf{m},\, j\in\mathbf{n}\\ i+j=k}}\alpha_i\beta_j\right)I_k \\
&= \alpha_0\beta_0 + \sum_{k\in\mathbf{m+n}}\left(\sum_{\substack{i\in\mathbf{m},\, j\in\mathbf{n}\\ i+j=k}}\alpha_i\beta_j\right)k!
\end{aligned}$$

(**b**) If for each $n\in\mathbb{N}$

$$x_n : t \mapsto t^n$$

then for each $m\in\mathbb{N}$ and each $n\in\mathbb{N}$

$$\begin{aligned}
\langle \mathbf{x}_m,\mathbf{x}_n\rangle &= \int_0^{+\infty} e^{-t}t^m t^n dt \\
&= I_{m+n} = (m+n)!
\end{aligned}$$

(**c**) If

$$f : t\mapsto (t+1)^2 \qquad g : t\mapsto t^2+1$$

then

$$\begin{aligned}
fg &: t\mapsto t^4+2t^3+2t^2+2t+1 \\
\langle f,g\rangle &= \int_0^{+\infty} e^{-t}\left(t^4+2t^3+2t^2+2t+1\right)dt \\
&= I_4+2I_3+2I_2+2I_1+I_0 \\
&= 4!+2\cdot 3!+2\cdot 2!+2\cdot 1!+1 \\
&= 43
\end{aligned}$$

(**d**) If

$$f : t \mapsto t + 1 \qquad g : t \mapsto \alpha t + \beta$$

then

$$\begin{aligned} fg &: t \mapsto \alpha t^2 + (\alpha + \beta) t + \beta \\ \langle f, g \rangle &= \alpha I_2 + (\alpha + \beta) I_1 + \beta I_0 \\ &= 3\alpha + 2\beta \end{aligned}$$

and

$$\langle f, g \rangle = 0 \Leftrightarrow (\alpha, \beta) = (2\gamma, -3\gamma) \qquad (\gamma \in \mathbb{R})$$

that is, the set of all polynomials of degree less than or equal to 1 which are orthogonal to f is

$$\mathcal{P}_1 \cap \operatorname{perp}\{f\} = \{g_\gamma : t \mapsto 2\gamma t - 3\gamma\}_{\gamma \in \mathbb{R}}$$

15.13 Construction of orthogonal sets. The Gram-Schmidt process

15.14 Orthogonal complements. projections

15.15 Best approximation of elements in a euclidean space by elements in a finite-dimensional subspace

15.16 Exercises

15.16.1 n. 1 (p. 576)

(**a**) By direct inspection of the coordinates of the three given vectors. it is seen that they all belong to the plane π of equation $x - z = 0$. Since no two of them are parallel, $\operatorname{lin}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} = \pi$ and $\dim \operatorname{lin}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} = 2$. A unit vector in π is given by

$$\mathbf{v} \equiv \frac{1}{3}(2, 1, 2)$$

Every vector belonging to the line $\pi \cap \operatorname{perp}\{\mathbf{v}\}$ must have the form

$$(x, -4x, x)$$

with norm $3\sqrt{2}\,|x|$. Thus a second unit vector which together with $\mathbf{v}$ spans π, and which is orthogonal to $\mathbf{v}$, is

$$\mathbf{w} \equiv \frac{\sqrt{2}}{6}\,(1,-4,1)$$

The required orthonormal basis for $\operatorname{lin}\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\}$ is $\{\mathbf{v},\mathbf{w}\}$.
(**b**) The answer is identical to the one just given for case a.

15.16.2 n. 2 (p. 576)

(**a**) **First solution**. It is easily checked that

$$\begin{aligned} \mathbf{x}_2 &\notin \operatorname{lin}\{\mathbf{x}_1\} \\ \mathbf{x}_3 &\notin \operatorname{lin}\{\mathbf{x}_1,\mathbf{x}_2\} \\ \mathbf{0} &= \mathbf{x}_1-\mathbf{x}_2+\mathbf{x}_3-\mathbf{x}_4 \end{aligned}$$

Thus $\dim W \equiv \dim\operatorname{lin}\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\} = 3$, and three vectors are required for any basis of W. The following vectors form an orthonormal one:

$$\begin{aligned}
\mathbf{y}_1 &\equiv \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \frac{\sqrt{2}}{2}\,(1,1,0,0) \\
\mathbf{y}_2 &\equiv \frac{\mathbf{x}_2-\langle\mathbf{x}_2,\mathbf{y}_1\rangle\,\mathbf{y}_1}{\|\mathbf{x}_2-\langle\mathbf{x}_2,\mathbf{y}_1\rangle\,\mathbf{y}_1\|} = \frac{(0,1,1,0)-\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\,(1,1,0,0)}{\left\|(0,1,1,0)-\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\,(1,1,0,0)\right\|} \\
&= \frac{\left(-\frac{1}{2},\frac{1}{2},1,0\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}} = \frac{\sqrt{6}}{6}\,(-1,1,2,0) \\
\mathbf{y}_3 &\equiv \frac{\mathbf{x}_3-\langle\mathbf{x}_3,\mathbf{y}_1\rangle\,\mathbf{y}_1--\langle\mathbf{x}_3,\mathbf{y}_2\rangle\,\mathbf{y}_2}{\|\mathbf{x}_3-\langle\mathbf{x}_3,\mathbf{y}_1\rangle\,\mathbf{y}_1--\langle\mathbf{x}_3,\mathbf{y}_2\rangle\,\mathbf{y}_2\|} = \frac{(0,0,1,1)-\mathbf{0}-2\frac{\sqrt{6}}{6}\frac{\sqrt{6}}{6}\,(-1,1,2,0)}{\left\|(0,0,1,1)-\mathbf{0}-2\frac{\sqrt{6}}{6}\frac{\sqrt{6}}{6}\,(-1,1,2,0)\right\|} \\
&= \frac{\left(\frac{1}{3},-\frac{1}{3},\frac{1}{3},1\right)}{\sqrt{\frac{1}{9}+\frac{1}{9}+\frac{1}{9}+1}} = \frac{\sqrt{3}}{6}\,(1,-1,1,3)
\end{aligned}$$

Second solution. More easily, it suffices to realize that

$$\operatorname{lin}\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\} = H_{(1,-1,1,-1),0} \equiv \left\{(x,y,z,t)\in\mathbb{R}^4 : x-y+z-t=0\right\}$$

is the hyperplane of $\mathbb{R}^4$ through the origin, with $(1,-1,1,-1)$ as normal vector, to directly exhibit two mutually orthogonal unit vectors in $\operatorname{lin}\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\}$

$$\mathbf{u} \equiv \frac{\sqrt{2}}{2}\,(1,1,0,0) \qquad \mathbf{v} \equiv \frac{\sqrt{2}}{2}\,(0,0,1,1)$$

It remains to find a unit vector in $H_{(1,-1,1,-1),0} \cap \operatorname{perp}\{\mathbf{u},\mathbf{v}\}$. The following equations characterize $H_{(1,-1,1,-1),0} \cap \operatorname{perp}\{\mathbf{u},\mathbf{v}\}$:

$$\begin{aligned} x-y+z-t &= 0 \\ x+y &= 0 \\ z+t &= 0 \end{aligned}$$

yielding (by addition) $2(x+z)=0$, and hence

$$\mathbf{w} \equiv \frac{1}{2}(1,-1,-1,1) \qquad \text{or} \qquad \mathbf{w} \equiv -\frac{1}{2}(1,-1,-1,1)$$

Thus an orthonormal basis for $\operatorname{lin}\{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4\}$ is $\{\mathbf{u},\mathbf{v},\mathbf{w}\}$.
(**b**) It is easily checked that

$$\begin{aligned} \mathbf{x}_2 &\notin \operatorname{lin}\{\mathbf{x}_1\} \\ \mathbf{0} &= 2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3 \end{aligned}$$

Thus $\dim W \equiv \dim \operatorname{lin}\{x_1,x_2,x_3\} = 2$, and two vectors are required for any basis of W. The following vectors form an orthonormal one:

$$\begin{aligned} \mathbf{y}_1 &\equiv \frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \frac{\sqrt{3}}{3}(1,1,0,1) \\ \mathbf{y}_2 &\equiv \frac{\mathbf{x}_2 - \langle \mathbf{x}_2,\mathbf{y}_1\rangle \mathbf{y}_1}{\|\mathbf{x}_2 - \langle \mathbf{x}_2,\mathbf{y}_1\rangle \mathbf{y}_1\|} = \frac{(1,0,2,1) - 2\frac{\sqrt{3}}{3}\frac{\sqrt{3}}{3}(1,1,0,1)}{\left\|(0,1,1,0) - 2\frac{\sqrt{3}}{3}\frac{\sqrt{3}}{3}(1,1,0,1)\right\|} \\ &= \frac{\left(\frac{1}{3},-\frac{2}{3},2,\frac{1}{3}\right)}{\sqrt{\frac{1}{9}+\frac{4}{9}+4+\frac{1}{9}}} = \frac{\sqrt{42}}{42}(1,-2,6,1) \end{aligned}$$

15.16.3 n. 3 (p. 576)

$$\begin{aligned} \langle \mathbf{y}_0,\mathbf{y}_0\rangle &= \int_0^\pi \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\pi}}\,dt = \pi\frac{1}{\pi} = 1 \\ \langle \mathbf{y}_n,\mathbf{y}_n\rangle &= \int_0^\pi \sqrt{\frac{2}{\pi}}\cos nt\sqrt{\frac{2}{\pi}}\cos nt\,dt = \frac{2}{\pi}\int_0^\pi \cos^2 nt\,dt \end{aligned}$$

Let

$$I_n \equiv \int_0^\pi \cos^2 nt dt = \int_0^\pi \cos nt \cos nt\,dt$$

integrating by parts

$$\begin{aligned} I_n &= \cos nt \frac{\sin nt}{n}\Big|_0^\pi - \int_0^\pi -n \sin nt \frac{\sin nt}{n}\,dt \\ &= \int_0^\pi \sin^2 nt\,dt = \int_0^\pi 1-\cos^2 nt\,dt \\ &= \pi - I_n \end{aligned}$$

Thus

$$\begin{aligned} I_n &= \frac{\pi}{2} \\ \langle \mathbf{y}_n, \mathbf{y}_n \rangle &= 1 \end{aligned}$$

and every function $\mathbf{y}_n$ has norm equal to one.

To check mutual orthogonality, let us compute

$$\begin{aligned} \langle \mathbf{y}_0, \mathbf{y}_n \rangle &= \int_0^\pi \frac{1}{\sqrt{\pi}} \cos nt \, dt = \frac{1}{\sqrt{\pi}} \left. \frac{\sin nt}{n} \right|_0^\pi = 0 \\ \langle \mathbf{y}_m, \mathbf{y}_n \rangle &= \int_0^\pi \cos mt \cos nt \, dt \\ &= \cos mt \left. \frac{\sin nt}{n} \right|_0^\pi - \int_0^\pi -m \sin mt \frac{\sin nt}{n} \, dt \\ &= \frac{m}{n} \int_0^\pi \sin mt \sin nt \, dt \\ &= \frac{m}{n} \sin mt \left. \left(-\frac{\cos nt}{n} \right) \right|_0^\pi - \frac{m}{n} \int_0^\pi m \cos mt \left(-\frac{\cos nt}{n} \right) dt \\ &= \left(\frac{m}{n} \right)^2 \langle y_m, y_n \rangle \end{aligned}$$

Since m and n are distinct positive integers, $\left(\frac{m}{n}\right)^2$ is different from 1, and the equation

$$\langle \mathbf{y}_m, \mathbf{y}_n \rangle = \left(\frac{m}{n} \right)^2 \langle \mathbf{y}_m, \mathbf{y}_n \rangle$$

can hold only if $\langle \mathbf{y}_m, \mathbf{y}_n \rangle = 0$.

That the set $\{\mathbf{y}_n\}_{n=0}^{+\infty}$ generates the same space generated by the set $\{\mathbf{x}_n\}_{n=0}^{+\infty}$ is trivial, since, for each n, y_n is a multiple of x_n.

15.16.4 n. 4 (p. 576)

We have

$$\begin{aligned} \langle \mathbf{y}_0, \mathbf{y}_0 \rangle &= \int_0^1 1 dt = 1 \\ \langle \mathbf{y}_1, \mathbf{y}_1 \rangle &= \int_0^1 3 \left(4t^2 - 4t + 1 \right) dt = 3 \left(\left. \frac{4}{3} t^3 - 2t^2 + t \right|_0^1 \right) \\ &= 1 \\ \langle \mathbf{y}_2, \mathbf{y}_2 \rangle &= \int_0^1 5 \left(36t^4 - 72t^3 + 48t^2 - 12t + 1 \right) dt \\ &= 5 \left(\left. \frac{36}{5} t^5 - 18t^4 + 16t^3 - 6t^2 + t \right|_0^1 \right) \\ &= 1 \end{aligned}$$

which proves that the three given functions are unit vectors with respect to the given inner product.

Moreover,

$$\begin{aligned}
\langle \mathbf{y}_0, \mathbf{y}_1 \rangle &= \int_0^1 \sqrt{3}\,(2t-1)\,dt = \sqrt{3}\left(t^2 - t\Big|_0^1\right) = 0 \\
\langle \mathbf{y}_0, \mathbf{y}_2 \rangle &= \int_0^1 \sqrt{5}\left(6t^2 - 6t + 1\right)dt = \sqrt{5}\left(2t^3 - 3t^2 + t\Big|_0^1\right) = 0 \\
\langle \mathbf{y}_1, \mathbf{y}_2 \rangle &= \int_0^1 \sqrt{15}\left(12t^3 - 18t^2 + 8t - 1\right)dt \\
&= \sqrt{15}\left(3t^4 - 6t^3 + 4t^2 - t\Big|_0^1\right) = 0
\end{aligned}$$

which proves that the three given functions are mutually orthogonal. Thus $\{y_1, y_2, y_3\}$ is an orthonormal set, and hence linearly independent.

Finally,

$$\mathbf{x}_0 = \mathbf{y}_0 \qquad \mathbf{x}_1 = \frac{1}{2}\mathbf{y}_0 + \frac{\sqrt{3}}{2}\mathbf{y}_1 \qquad \mathbf{x}_2 = \left(1 - \frac{\sqrt{5}}{30}\right)\mathbf{y}_1 + \frac{\sqrt{3}}{3}\mathbf{y}_1 + \frac{\sqrt{5}}{30}\mathbf{y}_2$$

which proves that

$$\text{lin}\,\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\} = \text{lin}\,\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$$

Chapter 16
LINEAR TRANSFORMATIONS AND MATRICES

16.1 Linear transformations

16.2 Null space and range

16.3 Nullity and rank

16.4 Exercises

16.4.1 n. 1 (p. 582)

T is linear, since for every $(x, y) \in \mathbb{R}^2$, for every $(u, v) \in \mathbb{R}^2$, for every $\alpha \in \mathbb{R}$, for every $\beta \in \mathbb{R}$,

$$\begin{aligned} T[\alpha(x,y)+\beta(u,v)] &= T[(\alpha x+\beta u, \alpha y+\beta v)] \\ &= (\alpha y+\beta v, \alpha x+\beta u) \\ &= \alpha(y,x)+\beta(v,u) \\ &= \alpha T[(x,y)]+\beta T[(u,v)] \end{aligned}$$

The null space of T is the trivial subspace $\{(0,0)\}$, the range of T is $\mathbb{R}^2$, and hence

$$\operatorname{rank} T = 2 \qquad \operatorname{nullity} T = 0$$

T is the orthogonal symmetry with respect to the bisectrix r of the first and third quadrant, since, for every $(x, y) \in \mathbb{R}^2$, $T(x, y) - (x, y)$ is orthogonal to r, and the midpoint of $[(x, y), T(x, y)]$, which is $(x + y, x + y)$, lies on r.

16.4.2 n. 2 (p. 582)

T is linear, since for every $(x,y)\in\mathbb{R}^2$, for every $(u,v)\in\mathbb{R}^2$, for every $\alpha\in\mathbb{R}$, and for every $\beta\in\mathbb{R}$,

$$\begin{aligned}
T\left[\alpha\left(x,y\right)+\beta\left(u,v\right)\right] &= T\left[\left(\alpha x+\beta u,\alpha y+\beta v\right)\right] \\
&= \left(\alpha y+\beta v,\alpha x+\beta u\right) \\
&= \alpha\left(y,x\right)+\beta\left(v,u\right) \\
&= \alpha T\left[\left(x,y\right)\right]+\beta T\left[\left(u,v\right)\right]
\end{aligned}$$

The null space of T is the trivial subspace $\{(0,0)\}$, the range of T is $\mathbb{R}^2$, and hence

$$\operatorname{rank} T = 2 \qquad \operatorname{nullity} T = 0$$

T is the orthogonal symmetry with respect to the X-axis.

16.4.3 n. 3 (p. 582)

T is linear , since for every $(x,y)\in\mathbb{R}^2$, for every $(u,v)\in\mathbb{R}^2$, for every $\alpha\in\mathbb{R}$, and for every $\beta\in\mathbb{R}$,

$$\begin{aligned}
T\left[\alpha\left(x,y\right)+\beta\left(u,v\right)\right] &= T\left[\left(\alpha x+\beta u,\alpha y+\beta v\right)\right] \\
&= \left(\alpha x+\beta u,0\right) \\
&= \alpha\left(x,0\right)+\beta\left(u,0\right) \\
&= \alpha T\left[\left(x,y\right)\right]+\beta T\left[\left(u,v\right)\right]
\end{aligned}$$

The null space of T is the Y-axis, the range of T is the X-axis, and hence

$$\operatorname{rank} T = 1 \qquad \operatorname{nullity} T = 1$$

T is the orthogonal projection on the X-axis.

16.4.4 n. 4 (p. 582)

T is linear, since for every $(x,y)\in\mathbb{R}^2$, for every $(u,v)\in\mathbb{R}^2$, for every $\alpha\in\mathbb{R}$, and for every $\beta\in\mathbb{R}$,

$$\begin{aligned}
T\left[\alpha\left(x,y\right)+\beta\left(u,v\right)\right] &= T\left[\left(\alpha x+\beta u,\alpha y+\beta v\right)\right] \\
&= \left(\alpha x+\beta u,\alpha x+\beta u\right) \\
&= \alpha\left(x,x\right)+\beta\left(u,u\right) \\
&= \alpha T\left[\left(x,y\right)\right]+\beta T\left[\left(u,v\right)\right]
\end{aligned}$$

The null space of T is the Y-axis, the range of T is the bisectrix of the I and III quadrant, and hence

$$\operatorname{rank} T = \operatorname{nullity} T = 1$$

16.4.5 n. 5 (p. 582)

T is not a linear function, since, e.g., for x and u both different from 0,

$$\begin{array}{rcl} T(x+u,0) & = & \left(x^2+2xy+u^2,0\right) \\ T(x,0)+T(u,0) & = & \left(x^2+u^2,0\right) \end{array}$$

16.4.6 n. 6 (p. 582)

T is not linear; indeed, for every $(x,y)\in\mathbb{R}^2$, for every $(u,v)\in\mathbb{R}^2$, for every $\alpha\in\mathbb{R}$, for every $\beta\in\mathbb{R}$,

$$\begin{array}{rcl} T[\alpha(x,y)+\beta(u,v)] & = & T[(\alpha x+\beta u,\alpha y+\beta v)] \\ & = & \left(e^{\alpha x+\beta u},e^{\alpha y+\beta v}\right) \\ & = & \left[(e^x)^\alpha\cdot(e^u)^\beta,(e^y)^\alpha\cdot(e^v)^\beta\right] \\ \alpha T[(x,y)]+\beta T[(u,v)] & = & \alpha(e^x,e^y)+\beta(e^u,e^v) \\ & = & (\alpha e^x+\beta e^u,\alpha e^y+\beta e^v) \end{array}$$

so that, e.g., when $x=y=0$ and $u=v=\alpha=\beta=1$,

$$\begin{array}{rcl} T[(0,0)+(1,1)] & = & (e,e) \\ T[(0,0)]+T[(1,1)] & = & (1+e,1+e) \end{array}$$

16.4.7 n. 7 (p. 582)

T is not an affine function, but it is not linear. Indeed, for every $(x,y)\in\mathbb{R}^2$, for every $(u,v)\in\mathbb{R}^2$, for every $\alpha\in\mathbb{R}$, and for every $\beta\in\mathbb{R}$,

$$\begin{array}{rcl} T[\alpha(x,y)+\beta(u,v)] & = & T[(\alpha x+\beta u,\alpha y+\beta v)] \\ & = & (\alpha x+\beta u,1) \\ \alpha T[(x,y)]+\beta T[(u,v)] & = & \alpha(x,1)+\beta(u,1) \\ & = & (\alpha x+\beta u,\alpha+\beta) \end{array}$$

16.4.8 n. 8 (p. 582)

T is affine, but not linear; indeed, for every $(x,y)\in\mathbb{R}^2$, for every $(u,v)\in\mathbb{R}^2$, for every $\alpha\in\mathbb{R}$, and for every $\beta\in\mathbb{R}$,

$$\begin{array}{rcl} T[\alpha(x,y)+\beta(u,v)] & = & T[(\alpha x+\beta u,\alpha y+\beta v)] \\ & = & (\alpha x+\beta u+1,\alpha y+\beta v+1) \\ \alpha T[(x,y)]+\beta T[(u,v)] & = & \alpha(x+1,y+1)+\beta(u+1,v+1) \\ & = & (\alpha x+\beta u+\alpha+\beta,\alpha y+\beta v+\alpha+\beta) \end{array}$$

16.4.9 n. 9 (p. 582)

T is linear, since for every $(x,y) \in \mathbb{R}^2$, for every $(u,v) \in \mathbb{R}^2$, for every $\alpha \in \mathbb{R}$, and for every $\beta \in \mathbb{R}$,

$$\begin{aligned} T[\alpha(x,y)+\beta(u,v)] &= T[(\alpha x+\beta u, \alpha y+\beta v)] \\ &= (\alpha x+\beta u-\alpha y-\beta v, \alpha x+\beta u+\alpha y+\beta v) \\ &= \alpha(x-y, x+y)+\beta(u-v, u+v) \\ &= \alpha T[(x,y)]+\beta T[(u,v)] \end{aligned}$$

The null space of T is the trivial subspace $\{(0,0)\}$, the range of T is $\mathbb{R}^2$, and hence

$$\operatorname{rank} T = 2 \qquad \operatorname{nullity} T = 0$$

The matrix representing T is

$$\begin{aligned} A &\equiv \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \sqrt{2}\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} \end{aligned}$$

Thus T is the composition of a counterclockwise rotation by an angle of $\frac{\pi}{4}$ with a homothety of modulus $\sqrt{2}$.

16.4.10 n. 10 (p. 582)

T is linear, since for every $(x,y) \in \mathbb{R}^2$, for every $(u,v) \in \mathbb{R}^2$, for every $\alpha \in \mathbb{R}$, and for every $\beta \in \mathbb{R}$,

$$\begin{aligned} T[\alpha(x,y)+\beta(u,v)] &= T[(\alpha x+\beta u, \alpha y+\beta v)] \\ &= (2(\alpha x+\beta u)-(\alpha y+\beta v), (\alpha x+\beta u)+(\alpha y+\beta v)) \\ &= \alpha(2x-y, x+y)+\beta(2u-v, u+v) \\ &= \alpha T[(x,y)]+\beta T[(u,v)] \end{aligned}$$

The null space of T is the trivial subspace $\{(0,0)\}$, the range of T is $\mathbb{R}^2$, and hence

$$\operatorname{rank} T = 2 \qquad \operatorname{nullity} T = 0$$

The matrix representing T is

$$A \equiv \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

the characteristic polynomial of A is

$$\lambda^2 - 3\lambda + 3$$

and the eigenvalues of A are

$$\lambda_1 \equiv \frac{3+\sqrt{3}i}{2} \qquad \lambda_2 \equiv \frac{3-\sqrt{3}i}{2}$$

An eigenvector associated to λ_1 is

$$z \equiv \begin{pmatrix} 2 \\ 1-\sqrt{3}i \end{pmatrix}$$

The matrix representing T with respect to the basis

$$\{\operatorname{Im} z, \operatorname{Re} z\} = \left\{ \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

is

$$\begin{aligned} B &\equiv \begin{bmatrix} \frac{3}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{3}{2} \end{bmatrix} = \sqrt{3} \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} \end{aligned}$$

Thus T is the composition of a counterclockwise rotation by an angle of $\frac{\pi}{6}$ with a homothety of modulus $\sqrt{3}$.

16.4.11 n. 16 (p. 582)

T is linear, since for every $(x, y, z) \in \mathbb{R}^3$, for every $(u, v, w) \in \mathbb{R}^3$, for every $\alpha \in \mathbb{R}$, for every $\beta \in \mathbb{R}$,

$$\begin{aligned} T[\alpha(x,y,z) + \beta(u,v,w)] &= T[(\alpha x + \beta u, \alpha y + \beta v, \alpha z + \beta w)] \\ &= (\alpha z + \beta w, \alpha y + \beta v, \alpha x + \beta u) \\ &= \alpha(z,y,x) + \beta(w,v,u) \\ &= \alpha T[(x,y,z)] + \beta T[(u,v,w)] \end{aligned}$$

The null space of T is the trivial subspace $\{(0,0,0)\}$, the range of T is $\mathbb{R}^3$, and hence

$$\operatorname{rank} T = 3 \qquad \operatorname{nark} T = 0$$

16.4.12 n. 17 (p. 582)

T is linear (as every projection on a coordinate hyperplane), since for every $(x, y, z) \in \mathbb{R}^3$, for every $(u, v, w) \in \mathbb{R}^3$, for every $\alpha \in \mathbb{R}$, for every $\beta \in \mathbb{R}$,

$$\begin{aligned} T[\alpha(x,y,z) + \beta(u,v,w)] &= T[(\alpha x + \beta u, \alpha y + \beta v, \alpha z + \beta w)] \\ &= (\alpha x + \beta u, \alpha y + \beta v, 0) \\ &= \alpha(x,y,0) + \beta(u,v,0) \\ &= \alpha T[(x,y,z)] + \beta T[(u,v,w)] \end{aligned}$$

The null space of T is the Z-axis, the range of T is the XY-plane, and hence

$$\operatorname{rank} T = 2 \qquad \operatorname{nark} T = 1$$

16.4.13 n. 23 (p. 582)

T is linear, since for every $(x,y,z) \in \mathbb{R}^3$, for every $(u,v,w) \in \mathbb{R}^3$, for every $\alpha \in \mathbb{R}$, for every $\beta \in \mathbb{R}$,

$$\begin{aligned} T[\alpha(x,y,z)+\beta(u,v,w)] &= T[(\alpha x+\beta u, \alpha y+\beta v, \alpha z+\beta w)] \\ &= (\alpha x+\beta u+\alpha z+\beta w, 0, \alpha x+\beta u+\alpha y+\beta v) \\ &= [\alpha(x+z), 0, \alpha(x+y)] + [\beta(u+w), 0, \beta(u+v)] \\ &= \alpha(x,y,z)+\beta(u,v,w) \\ &= \alpha T[(x,y,z)] + \beta T[(u,v,w)] \end{aligned}$$

The null space of T is the axis of central symmetry of the $(+,-,-)$-orthant and of the $(-,+,+)$-orthant, of parametric equations

$$x = t \qquad y = -t \qquad z = -t$$

the range of T is the XZ-plane, and hence

$$\operatorname{rank} T = 2 \qquad \operatorname{nark} T = 1$$

16.4.14 n. 25 (p. 582)

Let $\mathcal{D}_1(-1,1)$ or, more shortly, $\mathcal{D}_1$ be the linear space of all real functions of a real variable which are defined and everywhere differentiable on $(-1,1)$. If

$$T : \mathcal{D}_1 \to \mathbb{R}^{(-1,1)}, \; f \mapsto (x \mapsto xf'(x))$$

then T is a linear operator. Indeed,

$$\begin{aligned} T(f+g) &= \big(x \mapsto x[f+g]'(x)\big) \\ &= (x \mapsto x[f'(x)+g'(x)]) \\ &= (x \mapsto xf'(x)+xg'(x)) \\ &= (x \mapsto xf'(x)) + (x \mapsto +xg'(x)) \\ &= T(f)+T(g) \end{aligned}$$

Moreover,

$$\begin{aligned} \ker T &= \{f \in \mathcal{D}_1 : \forall x \in (-1,1),\; xf'(x) = 0\} \\ &= \{f \in \mathcal{D}_1 : \forall x \in (-1,0)\cup(0,1),\; f'(x) = 0\} \end{aligned}$$

By an important though sometimes overlooked theorem in calculus, every function f which is differentiable in $(-1,0)\cup(0,1)$ and continuous in 0, is differentiable in 0 as well, provided $\lim_{x\to 0} f'(x)$ exists and is finite, in which case

$$f'(0) = \lim_{x\to 0} f'(x)$$

Thus

$$\ker T = \{f \in \mathcal{D}_1 : f' = \mathbf{0}\}$$

(here $\mathbf{0}$ is the identically null function defined on $(-1,1)$). If f belongs to $\ker T$, by the classical theorem due to Lagrange,

$$\forall x \in (-1,1), \exists \vartheta_x \in (0,x)$$

$$f(x) = f(0) + xf'(\vartheta_x)$$

and hence f is constant on $(-1,1)$. It follows that a basis of $\ker T$ is given by the constant function $(x \mapsto 1)$, and $\text{nullity}\, T = 1$. Since $T(\mathcal{D}_1)$ contains, e.g., all the restrictions to $(-1,1)$ of the polynomials with vanishing zero-degree monomial, and it is already known that the linear space of all polynomials has an infinite basis, the dimension of $T(\mathcal{D}_1)$ is infinite.

16.4.15 n. 27 (p. 582)

Let $\mathcal{D}_2$ be the linear space of all real functions of a real variable which are defined and everywhere differentiable on $\mathbb{R}$. If

$$\mathcal{L} : \mathcal{D}_2 \to \mathbb{R}^{\mathbb{R}}, \ \ y \mapsto y'' + Py' + Q$$

where P and Q are real functions of a real variable which are continuous on $\mathbb{R}$, it has been shown in the solution to exercise 17, p.555, that $\mathcal{L}$ is a linear operator. Thus $\ker \mathcal{L}$ is the set of all solutions to the linear differential equation of the second order

$$\forall x \in \mathbb{R}, \qquad y''(x) + P(x)\, y'(x) + Q(x)\, y(x) = 0 \tag{16.1}$$

By the uniqueness theorem for Cauchy's problems in the theory of differential equations, for each $(y_0, y_0') \in \mathbb{R}^2$ there exists a unique solution to equation (16.1) such that $y(0) = y_0$ and $y'(0) = y_0'$. Hence the function

$$\varphi : \ker \mathcal{L} \to \mathbb{R}^2, \ \ y \mapsto \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix}$$

is injective and surjective. Moreover, let u be the solution to equation (16.1) such that $(u(0), u'(0)) = (1,0)$, and let v be the solution to equation (16.1) such that $(v(0), v'(0)) = (0,1)$. Then for each $(y_0, y_0') \in \mathbb{R}^2$, since $\ker \mathcal{L}$ is a linear subspace of $\mathcal{D}_2$, the function

$$y : x \mapsto y_0 u(x) + y_0' v(x)$$

is a solution to equation (16.1), and by direct inspection it is seen that $y(0) = y_0$ and $y'(0) = y_0'$. In other words, $\varphi^{-1}((y_0, y_0'))$, and

$$\ker \mathcal{L} = \text{span}\{u, v\}$$

Finally, u and v are linearly independent; indeed, since $u(0) = 1$ and $v(0) = 0$, $\alpha u(x) + \beta v(x) = 0$ for each $x \in \mathbb{R}$ can only hold (by evaluating at $x = 0$) if $\alpha = 0$; in such a case, from $\beta v(x) = 0$ for each $x \in \mathbb{R}$ and $v'(0) = 1$ it is easily deduced that $\beta = 0$ as well. Thus $\text{nullity}\, \mathcal{L} = 2$.

16.5 Algebraic operations on linear transformations

16.6 Inverses

16.7 One-to-one linear transformations

16.8 Exercises

16.8.1 n. 15 (p. 589)

T is injective (or one to one), since

$$T(x,y,z) = T(x',y',z') \Leftrightarrow \begin{cases} x = x' \\ 2y = 2y' \\ 3z = 3z' \end{cases} \Leftrightarrow (x,y,z) = (x',y',z')$$

$$T^{-1}(u,v,w) = \left(u, \frac{v}{2}, \frac{w}{3}\right)$$

16.8.2 n. 16 (p. 589)

T is injective, since

$$T(x,y,z) = T(x',y',z') \Leftrightarrow \begin{cases} x = x' \\ y = y' \\ x+y+z = x'+y'+z' \end{cases} \Leftrightarrow (x,y,z) = (x',y',z')$$

$$T^{-1}(u,v,w) = (u, v, w-u-v)$$

16.8.3 n. 17 (p. 589)

T is injective, since

$$T(x,y,z) = T(x',y',z') \Leftrightarrow \begin{cases} x+1 = x'+1 \\ y+1 = y'+1 \\ z-1 = z'-1 \end{cases} \Leftrightarrow (x,y,z) = (x',y',z')$$

$$T^{-1}(u,v,w) = (u-1, v-1, w+1)$$

16.8.4 n. 27 (p. 590)
Let

$$\mathbf{p} = x \mapsto p_0 + p_1 x + p_2 x^2 + \cdots + p_{n-1} x^{n-1} + p_n x^n$$

$$\begin{aligned} DT(\mathbf{p}) &= D[T(\mathbf{p})] = D\left[x \mapsto \int_0^x \mathbf{p}(t)\ dt\right] \\ &= x \mapsto \mathbf{p}(x) \end{aligned}$$

the last equality being a consequence of the fundamental theorem of integral calculus.

$$\begin{aligned} TD(\mathbf{p}) &= T[D(\mathbf{p})] = T\left[x \mapsto p_1 + 2p_2 x + \cdots (n-1) p_{n-1} x^{n-2} + np_n x^{n-1}\right] \\ &= x \mapsto \int_0^x p_1 + 2p_2 t + \cdots (n-1) p_{n-1} t^{n-2} + np_n t^{n-1}\, dt \\ &= x \mapsto p_1 t + p_2 t^2 + \cdots p_{n-1} t^{n-1} + p_n t^n \Big|_0^x \\ &= x \mapsto p_1 x + p_2 x^2 + \cdots p_{n-1} x^{n-1} + p_n x^n \\ &= \mathbf{p} - p_0 \end{aligned}$$

Thus TD acts as the identity map only on the subspace W of V containing all polynomials having the zero degree monomial (p_0) equal to zero. $\operatorname{Im} TD$ is equal to W, and $\ker TD$ is the set of all constant polynomials, i.e., zero-degree polynomials.

16.9 Linear transformations with prescribed values

16.10 Matrix representations of linear transformations

16.11 Construction of a matrix representation in diagonal form

16.12 Exercises

16.12.1 n. 3 (p. 596)
(**a**) Since $T(\mathbf{i}) = \mathbf{i} + \mathbf{j}$ and $T(\mathbf{j}) = 2\mathbf{i} - \mathbf{j}$,

$$\begin{aligned} T(3\mathbf{i} - 4\mathbf{j}) &= 3T(\mathbf{i}) - 4T(\mathbf{j}) = 3(\mathbf{i} + \mathbf{j}) - 4(2\mathbf{i} - \mathbf{j}) \\ &= -5\mathbf{i} + 7\mathbf{j} \\ T^2(3\mathbf{i} - 4\mathbf{j}) &= T(-5\mathbf{i} + 7\mathbf{j}) = -5T(\mathbf{i}) + 7T(\mathbf{j}) \\ &= -5(\mathbf{i} + \mathbf{j}) + 7(2\mathbf{i} - \mathbf{j}) = 9\mathbf{i} - 12\mathbf{j} \end{aligned}$$

(**b**) The matrix of T with respect to the basis $\{\mathbf{i}, \mathbf{j}\}$ is

$$A \equiv \begin{bmatrix} T(\mathbf{i}) & T(\mathbf{j}) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

and the matrix of T^2 with respect to the same basis is

$$A^2 \equiv \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

(**c**) **First solution** (matrix for T). If $\mathbf{e}_1 = \mathbf{i} - \mathbf{j}$ and $\mathbf{e}_2 = 3\mathbf{i} + \mathbf{j}$, the matrix

$$P \equiv \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$$

transforms the (canonical) coordinates $(1, 0)$ and $(0, 1)$ of $\mathbf{e}_1$ and $\mathbf{e}_2$ with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ into their coordinates $(1, -1)$ and $(3, 1)$ with respect to the basis $\{\mathbf{i}, \mathbf{j}\}$; hence P is the matrix of coordinate change from basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ to basis $\{\mathbf{i}, \mathbf{j}\}$, and P^{-1} is the matrix of coordinate change from basis $\{\mathbf{i}, \mathbf{j}\}$ to basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. The operation of the matrix B representing T with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ can be described as the combined effect of the following three actions: 1) coordinate change from coordinates w.r. to basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ into coordinates w.r. to the basis $\{\mathbf{i}, \mathbf{j}\}$ (that is, multiplication by matrix P); 2) transformation according to T as expressed by the matrix representing T w.r. to the basis $\{\mathbf{i}, \mathbf{j}\}$ (that is, multiplication by A); 3) coordinate change from coordinates w.r. to the basis $\{\mathbf{i}, \mathbf{j}\}$ into coordinates w.r. to basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ (that is, multiplication by P^{-1}). Thus

$$\begin{aligned} B = P^{-1}AP &= \frac{1}{4}\begin{bmatrix} 1 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} -2 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} -7 & -1 \\ 1 & 7 \end{bmatrix} \end{aligned}$$

Second solution (matrix for T). The matrix B representing T with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ is

$$B \equiv \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix}$$

provided $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ are meant as coordinate vectors (α, β) and (γ, δ) with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. Since, on the other hand, with respect to the basis $\{\mathbf{i}, \mathbf{j}\}$ we have

$$\begin{aligned} T(\mathbf{e}_1) &= T(\mathbf{i} - \mathbf{j}) = T(\mathbf{i}) - T(\mathbf{j}) = (\mathbf{i} + \mathbf{j}) - (2\mathbf{i} - \mathbf{j}) \\ &= -\mathbf{i} + 2\mathbf{j} \\ T(\mathbf{e}_2) &= T(3\mathbf{i} + \mathbf{j}) = 3T(\mathbf{i}) + T(\mathbf{j}) = 3(\mathbf{i} + \mathbf{j}) + (2\mathbf{i} - \mathbf{j}) \\ &= 5\mathbf{i} + 2\mathbf{j} \end{aligned}$$

it suffices to find the $\{\mathbf{e}_1, \mathbf{e}_2\}$-coordinates (α, β) and (γ, δ) which correspond to the $\{\mathbf{i}, \mathbf{j}\}$-coordinates $(-1, 2)$ and $(5, 2)$. Thus we want to solve the two equation systems (in the unknowns (α, β) and (γ, δ), respectively)

$$\begin{aligned} \alpha \mathbf{e}_1 + \beta \mathbf{e}_2 &= -\mathbf{i} + 2\mathbf{j} \\ \gamma \mathbf{e}_1 + \delta \mathbf{e}_2 &= 5\mathbf{i} + 2\mathbf{j} \end{aligned}$$

that is,

$$\begin{cases} \alpha + 3\beta = -1 \\ -\alpha + \beta = 2 \end{cases} \qquad \begin{cases} \gamma + 3\delta = 5 \\ -\gamma + \delta = 2 \end{cases}$$

The (unique) solutions are $(\alpha, \beta) = \frac{1}{4}(-7, 1)$ and $(\gamma, \delta) = \frac{1}{4}(-1, 7)$, so that

$$B = \frac{1}{7} \begin{bmatrix} -7 & -1 \\ 1 & 7 \end{bmatrix}$$

Continuation (matrix for T^2). Since T^2 is represented by a scalar diagonal matrix with respect to the initially given basis, it is represented by the same matrix with respect to every basis (indeed, since scalar diagonal matrices commute with every matrix of the same order, $P^{-1}DP = D$ for every scalar diagonal matrix D).

16.12.2 n. 4 (p. 596)

$$\begin{aligned} T : (x, y) \text{ (reflection w.r.t. to the } Y\text{-axis)} &\mapsto (-x, y) \\ \text{(length doubling)} &\mapsto (-2x, 2y) \end{aligned}$$

Thus T may be represented by the matrix

$$\mathbf{A}_T \equiv \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

and hence T^2 by the matrix

$$\mathbf{A}_T^2 \equiv \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

16.12.3 n. 5 (p. 596)

a)

$$\begin{aligned} T(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) &= T(\mathbf{k}) + T(\mathbf{j} + \mathbf{k}) + T(\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= (2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}) + \mathbf{i} + (\mathbf{j} - \mathbf{k}) \\ &= 3\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} \end{aligned}$$

The three image vectors $T(\mathbf{k})$, $T(\mathbf{j}+\mathbf{k})$, $T(\mathbf{i}+\mathbf{j}+\mathbf{k})$ form a linearly independent triple. Indeed, the linear combination

$$\begin{aligned} xT(\mathbf{k})+yT(\mathbf{j}+\mathbf{k})+zT(\mathbf{i}+\mathbf{j}+\mathbf{k}) &= x(2\mathbf{i}+3\mathbf{j}+5\mathbf{k})+y\mathbf{i}+z(\mathbf{j}-\mathbf{k}) \\ &= (2x+y)\mathbf{i}+(3x+z)\mathbf{j}+(5x-z)\mathbf{k} \end{aligned}$$

spans the null vector if and only if

$$\begin{aligned} 2x+y &= 0 \\ 3x+z &= 0 \\ 5x-z &= 0 \end{aligned}$$

which yields $(II+III)$ $x=0$, and hence (by substitution in I and II) $y=z=0$. Thus the range space of T is $\mathbb{R}^3$, and $\operatorname{rank} T$ is 3. It follows that the null space of T is the trivial subspace $\{(0,0,0)\}$, and $\operatorname{nark} T$ is 0.

b) The matrix of T with respect to the basis $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ is obtained by aligning as columns the coordinates w.r.t. to $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ of the image vectors $T(\mathbf{i})$, $T(\mathbf{j})$, $T(\mathbf{k})$. The last one is given in the problem statement, but the first two need to be computed.

$$\begin{aligned} T(\mathbf{j}) &= T(\mathbf{j}+\mathbf{k}-\mathbf{k}) = T(\mathbf{j}+\mathbf{k})-T(\mathbf{k}) \\ &= -\mathbf{i}-3\mathbf{j}-5\mathbf{k} \\ T(\mathbf{i}) &= T(\mathbf{i}+\mathbf{j}+\mathbf{k}-\mathbf{j}-\mathbf{k}) = T(\mathbf{i}+\mathbf{j}+\mathbf{k})-T(\mathbf{j}+\mathbf{k}) \\ &= -\mathbf{i}+\mathbf{j}-\mathbf{k} \end{aligned}$$

$$\mathbf{A}_T \equiv \begin{pmatrix} -1 & -1 & 2 \\ 1 & -3 & 3 \\ -1 & -5 & 3 \end{pmatrix}$$

16.12.4 n. 7 (p. 597)

(**a**)

$$T(4\mathbf{i}-\mathbf{j}+\mathbf{k}) = 4T(\mathbf{i})-T(\mathbf{j})+T(\mathbf{k}) = (0,-2)$$

Since $\{T(\mathbf{j}), T(\mathbf{k})\}$ is a linearly independent set,

$$\ker T = \{\mathbf{0}\} \qquad \operatorname{rank} T = 2$$

(**b**)

$$\mathbf{A}_T = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

(**c**)

Let $\mathcal{C} = (w_1, w_2)$

$$\begin{aligned} T(\mathbf{i}) &= 0w_1 + 0w_2 \\ T(\mathbf{j}) &= 1w_1 + 0w_2 \\ T(\mathbf{k}) &= -\frac{1}{2}w_1 + \frac{3}{2}w_2 \end{aligned}$$

$$(\mathbf{A}_T)_{\mathcal{C}} = \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 0 & 0 & \frac{3}{2} \end{pmatrix}$$

(**d**)

Let $\mathcal{B} \equiv \{\mathbf{j}, \mathbf{k}, \mathbf{i}\}$ and $\mathcal{C} = \{w_1, w_2\} = \{(1,1), (1,-1)\}$. Then

$$(\mathbf{A}_T)_{\mathcal{B},\mathcal{C}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

16.12.5 n. 8 (p. 597)

(**a**) I shall distinguish the canonical unit vectors of $\mathbb{R}^2$ and $\mathbb{R}^3$ by marking the former with an underbar. Since $T(\underline{\mathbf{i}}) = \mathbf{i} + \mathbf{k}$ and $T(\underline{\mathbf{j}}) = -\mathbf{i} + \mathbf{k}$,

$$\begin{aligned} T\left(2\underline{\mathbf{i}} - 3\underline{\mathbf{j}}\right) &= 2T(\underline{\mathbf{i}}) - 3T(\underline{\mathbf{j}}) = 2(\mathbf{i} + \mathbf{k}) - 3(\mathbf{i} - \mathbf{k}) \\ &= -\mathbf{i} + 5\mathbf{k} \end{aligned}$$

For any two real numbers x and y,

$$\begin{aligned} T\left(x\underline{\mathbf{i}} + y\underline{\mathbf{j}}\right) &= xT(\underline{\mathbf{i}}) + yT(\underline{\mathbf{j}}) = x(\mathbf{i} + \mathbf{k}) + y(\mathbf{i} - \mathbf{k}) \\ &= (x+y)\,\mathbf{i} + (x-y)\,\mathbf{k} \end{aligned}$$

It follows that

$$\begin{aligned} \operatorname{Im} T &= \operatorname{lin}\{\mathbf{i}, \mathbf{k}\} \\ \operatorname{rank} T &= 2 \\ \operatorname{nullity} T &= 2 - 2 = 0 \end{aligned}$$

(**b**) The matrix of T with respect to the bases $\{\underline{\mathbf{i}}, \underline{\mathbf{j}}\}$ and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is

$$A \equiv \left[\begin{array}{cc} T(\underline{\mathbf{i}}) & T(\underline{\mathbf{j}}) \end{array}\right] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}$$

(**c**) **First solution**. If $\mathbf{w}_1 = \underline{\mathbf{i}} + \underline{\mathbf{j}}$ and $\mathbf{w}_2 = \underline{\mathbf{i}} + 2\underline{\mathbf{j}}$, the matrix

$$P \equiv \left[\begin{array}{cc} \mathbf{w}_1 & \mathbf{w}_2 \end{array}\right] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

transforms the (canonical) coordinates $(1,0)$ and $(0,1)$ of $\mathbf{w}_1$ and $\mathbf{w}_2$ with respect to the basis $\{\mathbf{w}_1,\mathbf{w}_2\}$ into their coordinates $(1,1)$ and $(1,2)$ with respect to the basis $\{\underline{\mathbf{i}},\underline{\mathbf{j}}\}$; hence P is the matrix of coordinate change from basis $\{\mathbf{w}_1,\mathbf{w}_2\}$ to basis $\{\underline{\mathbf{i}},\underline{\mathbf{j}}\}$, and P^{-1} is the matrix of coordinate change from basis $\{\underline{\mathbf{i}},\underline{\mathbf{j}}\}$ to basis $\{\mathbf{w}_1,\mathbf{w}_2\}$. The operation of the matrix B of T with respect to the bases $\{\mathbf{w}_1,\mathbf{w}_2\}$ and $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ can be described as the combined effect of the following two actions: 1) coordinate change from coordinates w.r. to basis $\{\mathbf{w}_1,\mathbf{w}_2\}$ into coordinates w.r. to the basis $\{\underline{\mathbf{i}},\underline{\mathbf{j}}\}$ (that is, multiplication by matrix P); 2) transformation according to T as expressed by the matrix representing T w.r. to the bases $\{\underline{\mathbf{i}},\underline{\mathbf{j}}\}$ and $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ (that is, multiplication by A). Thus

$$\begin{aligned} B = AP &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

Second solution (matrix for T). The matrix B representing T with respect to the bases $\{\mathbf{w}_1,\mathbf{w}_2\}$ and $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ is

$$B \equiv \left[\ T(\mathbf{w}_1) \quad T(\mathbf{w}_2)\ \right]$$

where

$$\begin{aligned} T(\mathbf{w}_1) &= T(\underline{\mathbf{i}}+\underline{\mathbf{j}}) = T(\underline{\mathbf{i}}) + T(\underline{\mathbf{j}}) = (\mathbf{i}+\mathbf{k}) + (\mathbf{i}-\mathbf{k}) \\ &= \mathbf{i} \\ T(\mathbf{w}_2) &= T(\underline{\mathbf{i}}+2\underline{\mathbf{j}}) = T(\underline{\mathbf{i}}) + 2T(\underline{\mathbf{j}}) = (\mathbf{i}+\mathbf{k}) + 2(\mathbf{i}-\mathbf{k}) \\ &= 3\mathbf{i} - \mathbf{k} \end{aligned}$$

Thus

$$B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

(**c**) The matrix C representing T w.r. to bases $\{\mathbf{u}_1,\mathbf{u}_2\}$ and $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ is diagonal, that is,

$$C = \begin{bmatrix} \gamma_{11} & 0 \\ 0 & \gamma_{22} \\ 0 & 0 \end{bmatrix}$$

if and only if the following holds

$$T(\mathbf{u}_1) = \gamma_{11}\mathbf{v}_1 \qquad T(\mathbf{u}_2) = \gamma_{22}\mathbf{v}_2$$

There are indeed very many ways to achieve that. In the present situation the simplest way seems to me to be given by defining

$$\begin{array}{ll} \mathbf{u}_1 \equiv \underline{\mathbf{i}} & \mathbf{v}_1 \equiv T(\mathbf{u}_1) = \mathbf{i} + \mathbf{k} \\ \mathbf{u}_2 \equiv \underline{\mathbf{j}} & \mathbf{v}_2 \equiv T(\mathbf{u}_2) = \mathbf{i} - \mathbf{k} \\ & \mathbf{v}_3 \equiv \text{any vector such that } \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \text{ is lin. independent} \end{array}$$

thereby obtaining

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

16.12.6 n. 16 (p. 597)

We have

$$\begin{array}{ll} D(\sin) = \cos & D^2(\sin) = D(\cos) = -\sin \\ D(\cos) = -\sin & D^2(\cos) = D(-\sin) = -\cos \\ D(\mathrm{id}\cdot\sin) = \sin + \mathrm{id}\cdot\cos & D^2(\mathrm{id}\cdot\sin) = D(\sin + \mathrm{id}\cdot\cos) = 2\cos - \mathrm{id}\cdot\sin \\ D(\mathrm{id}\cdot\cos) = \cos - \mathrm{id}\cdot\sin & D^2(\mathrm{id}\cdot\cos) = D(\cos - \mathrm{id}\cdot\sin) = -2\sin - \mathrm{id}\cdot\cos \end{array}$$

and hence the matrix representing the differentiation operator D and its square D^2 acting on $\operatorname{lin}\{\sin, \cos, \mathrm{id}\cdot\sin, \mathrm{id}\cdot\cos\}$ with respect to the basis $\{\sin, \cos, \mathrm{id}\cdot\sin, \mathrm{id}\cdot\cos\}$ is

$$A = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad A^2 = \begin{bmatrix} -1 & 0 & 0 & -2 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$