Gravitational pull on an infinite plane

In this abstract we want to discuss, how gravity would act if the earth were an infinite plane. Therefore, we will consider two things to be true:

- The earth is an infinite plane
- Newtons formula of gravitational attraction holds

With these two postulates, we can calculate the gravitational pull on other masses and, from that, derive the thickness d of the plane. Using newtons formula

$$F = G\frac{mM}{r^2} \tag{1}$$

and $F = mg$, we recieve

$$g = G\frac{M}{r^2}$$

Now we want to consider the gravitational pull on an external mass of height h above the plane. Using figure 1, we derive $r^2 = x^2 + (h+l)^2$. Since we assume the plane to be symmetric via rotations, we can assume that all horizontal parts of the attractive force cancel out. Therefore, we just want to consider the downwards part of the gravitational pull. We receive:

$$g_{down} = G\frac{M}{x^2 + (h+l)^2} * \frac{h+l}{\sqrt{x^2 + (h+l)^2}}$$

where the second fraction represents the downwards part of the pull. Now we assume a tiny element of volume V and density ρ is responsible for this pull. Substituting $\rho = M/V$, we receive:

$$g_{down} = G\rho\frac{V(h+l)}{(x^2 + (h+l)^2)^{\frac{3}{2}}}$$

Substituting our tiny element of volume V by an infinitesimal element dV and integrating, the result is

$$g_{down} = G\rho\int_{\mathbb{R}^3}\frac{(h+l)}{(x^2 + (h+l)^2)^{\frac{3}{2}}}\,dV$$

Using polar coordinates:

$$\begin{aligned} g_{down} &= G\rho\int_0^{2\pi}\int_0^{\infty}\int_0^{d}\frac{x(h+l)}{(x^2 + (h+l)^2)^{\frac{3}{2}}}\,dl\,dx\,d\phi \\ &= 2\pi G\rho\int_0^{d}\int_0^{\infty}\frac{x(h+l)}{(x^2 + (h+l)^2)^{\frac{3}{2}}}\,dx\,dl \\ &\overset{(*)}{=} 2\pi G\rho\int_0^{d}\frac{|h+l|}{h+l} \end{aligned}$$

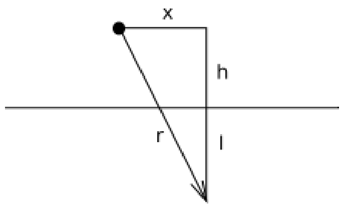

Figure 1: Vector diagram of the gravitational pull

Where in ($*$) we took the friendly help of wolframalpha. Since we assume the weight to be above the plane, we see that $h + l$ is always positive, resulting in

$$g_{down} = 2\pi G \rho d \tag{2}$$

Note that this term is independent of the height h, as long as one is above the surface. Another interesting aspect is, that even if you dig beneath the surface, the gravitational pull recedes linearly, until it completely vanishes at the depth of $d/2$. This means that an infinite plane would not tend to collapse into a sphere, but is gravitationally stable. We now want to calculate the depth d of the plane, therefore we set $g_{down} = 9,81ms^{-2}$ and $G = 6,674 * 10^{-11} m^3 kg^{-1} s^{-2}$. A quick google search says, that the density of the earth is approx. $\rho = 5510kgm^{-3}$. Solving

$$d = \frac{g_{down}}{2\pi G \rho}$$

we get $d = 4245,7km$.

We see now that an infinite plane is gravitationally stable and creates a homogenous, finite downwards pull (Einstein would love that!). Also, assuming this plane has the same density as the density calculated for the round earth, the plane must be approximately 4250km thick. Many other problems still arise with this model, but gravitational stability is not one of them.