Axial Deformations

1.1 Introduction

An important aspect of the analysis and design of structures relates to the deformations caused by the loads applied to a structure. Clearly it is important to avoid deformations so large that they may prevent the structure from fulfilling the purpose for which it is intended. But the analysis of deformations may also help us in the determination of stresses. It is not always possible to determine the forces in the members of a structure by applying only the principle of statics. This is because statics is based on the assumption of undeformable, rigid structures. By considering engineering structures as deformable and analyzing the deformations in their various members, it will be possible to compute forces which are statically indeterminate. Also the distribution of stresses in a given member is indeterminate, even when the force in that member is known. To determine the actual distribution of stresses within a member, it is necessary to analyze the deformations which take place in that member. This chapter deals with the deformations of a structural member such as a rod, bar or a plate under *axial loading.*

Top

1.2 Free body diagram - Revisited

The first step towards solving an engineering problem is drawing the free body diagram of the element/structure considered.

Removing an existing force or including a wrong force on the free body will badly affect the equilibrium conditions, and hence, the analysis.

In view of this, some important points in drawing the free body diagram are discussed below.

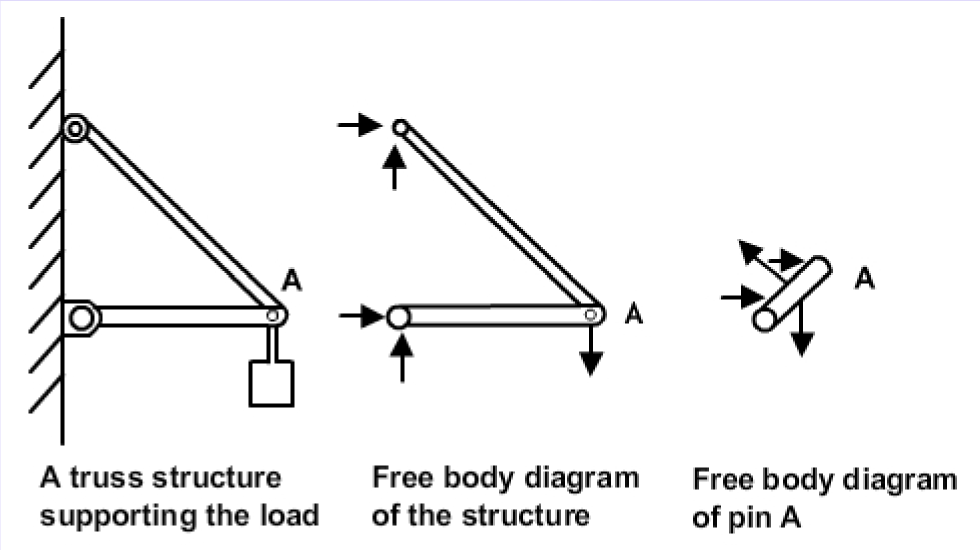


Figure 1.1

At the beginning, a clear decision is to be made by the analyst on the choice of the body to be considered for free body diagram.

Then that body is detached from all of its surrounding members including ground and only their forces on the free body are represented.

The weight of the body and other external body forces like centrifugal, inertia, etc., should also be included in the diagram and they are assumed to act at the centre of gravity of the body.

When a structure involving many elements is considered for free body diagram, the forces acting in between the elements should not be brought into the diagram.

The known forces acting on the body should be represented with proper magnitude and direction.

If the direction of unknown forces like reactions can be decided, they should be indicated clearly in the diagram.

After completing free body diagram, equilibrium equations from statics in terms of forces and moments are applied and solved for the unknowns.

Top

1.3 Normal, shear and bearing stress

1.3.1 Normal Stress:

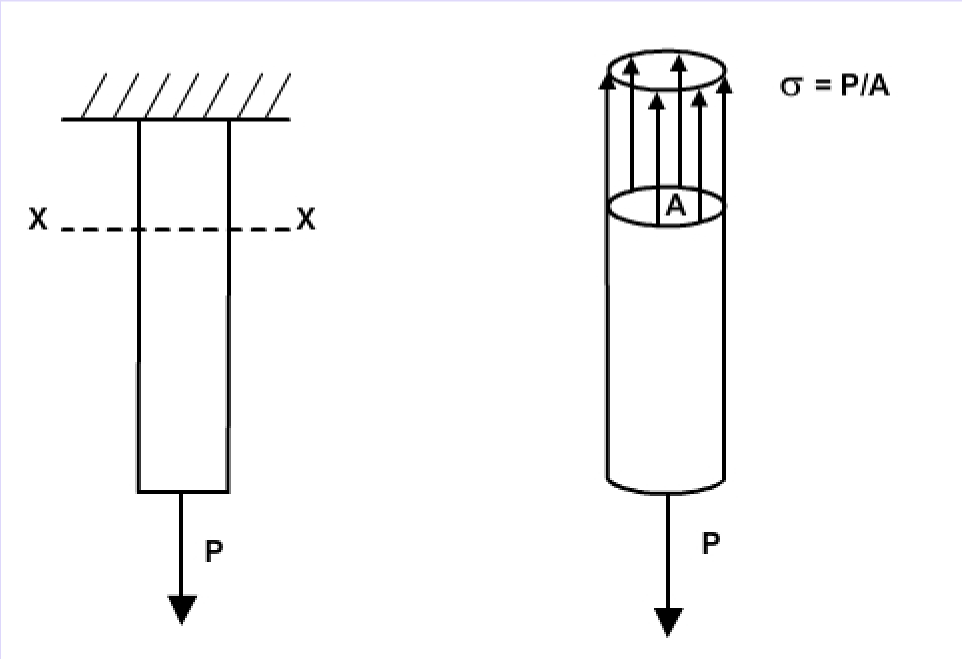


Figure 1.2

When a structural member is under load, predicting its ability to withstand that load is not possible merely from the reaction force in the member.

It depends upon the internal force, cross sectional area of the element and its material properties.

Thus, a quantity that gives the ratio of the internal force to the cross sectional area will define the ability of the material in with standing the loads in a better way.

That quantity, i.e., the intensity of force distributed over the given area or simply the force per unit area is called the *stress*.

$$\sigma = \frac{P}{A} \qquad 1.1$$

In SI units, force is expressed in newtons (N) and area in square meters. Consequently, the stress has units of newtons per square meter (N/m^2) or Pascals (Pa).

In figure 1.2, the stresses are acting normal to the section XX that is perpendicular to the axis of the bar. These stresses are called *normal stresses*.

The stress defined in equation 1.1 is obtained by dividing the force by the cross sectional area and hence it represents the average value of the stress over the entire cross section.

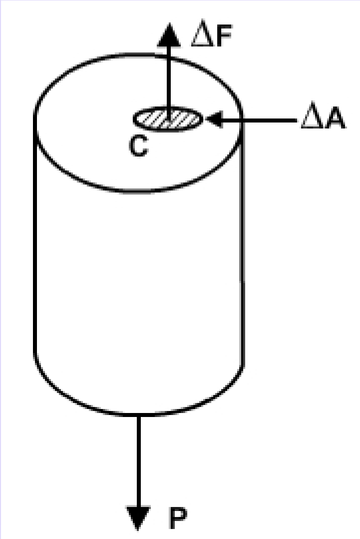


Figure 1.3

Consider a small area ΔA on the cross section with the force acting on it ΔF as shown in figure 1.3. Let the area contain a point C.

Now, the stress at the point C can be defined as,

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \qquad 1.2$$

The average stress values obtained using equation 1.1 and the stress value at a point from equation 1.2 may not be the same for all cross sections and for all loading conditions.

1.3.2 Saint - Venant's Principle:

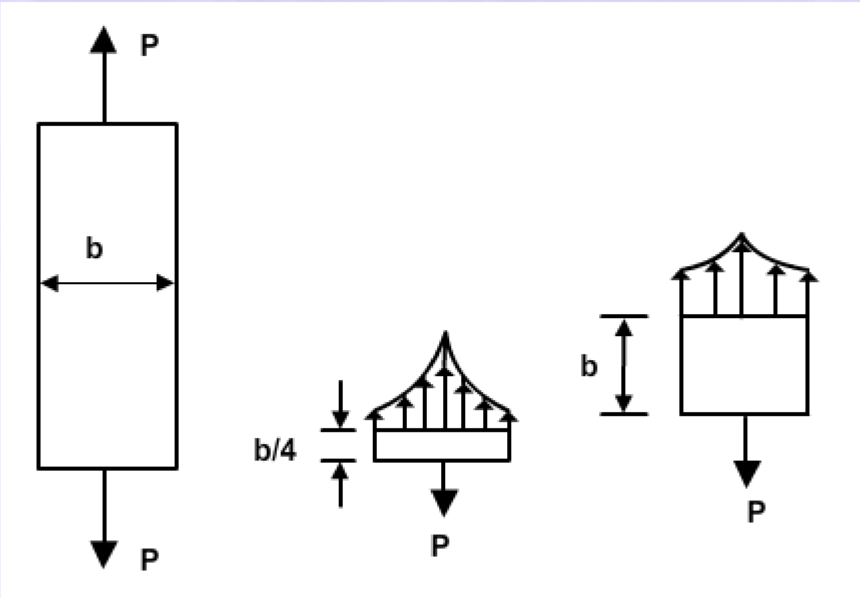


Figure 1.4

Consider a slender bar with point loads at its ends as shown in figure 1.4.

The normal stress distribution across sections located at distances *b/4* and *b* from one and of the bar is represented in the figure.

It is found from figure 1.4 that the stress varies appreciably across the cross section in the immediate vicinity of the application of loads.

The points very near the application of the loads experience a larger stress value whereas, the points far away from it on the same section has lower stress value.

The variation of stress across the cross section is negligible when the section considered is far away, about equal to the width of the bar, from the application of point loads.

Thus, except in the immediate vicinity of the points where the load is applied, the stress distribution may be assumed to be uniform and is independent of the mode of application of loads. This principle is called *Saint-Venant's principle.*

1.3.3 Shear Stress:

The stresses acting perpendicular to the surfaces considered are normal stresses and were discussed in the preceding section.

Now consider a bolted connection in which two plates are connected by a bolt with cross section A as shown in figure 1.5.

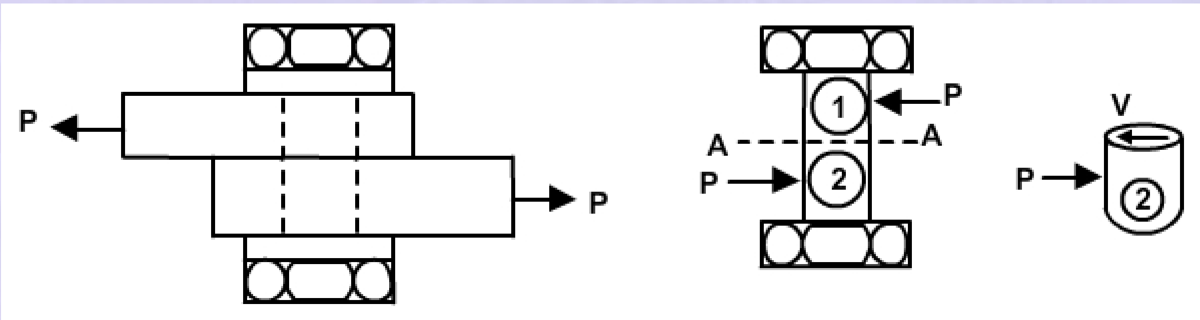


Figure 1.5

The tensile loads applied on the plates will tend to shear the bolt at the section AA.

Hence, it can be easily concluded from the free body diagram of the bolt that the internal resistance force V must act in the plane of the section AA and it should be equal to the external load P.

These internal forces are called shear forces and when they are divided by the corresponding section area, we obtain the *shear stress* on that section.

$$\tau = \frac{V}{A} \qquad 1.3$$

Equation 1.3 defines the average value of the shear stress on the cross section and the distribution of them over the area is not uniform.

In general, the shear stress is found to be maximum at the centre and zero at certain locations on the edge. This will be dealt in detail in shear stresses in beams (module 6).

In figure 1.5, the bolt experiences shear stresses on a single plane in its body and hence it is said to be under single shear.

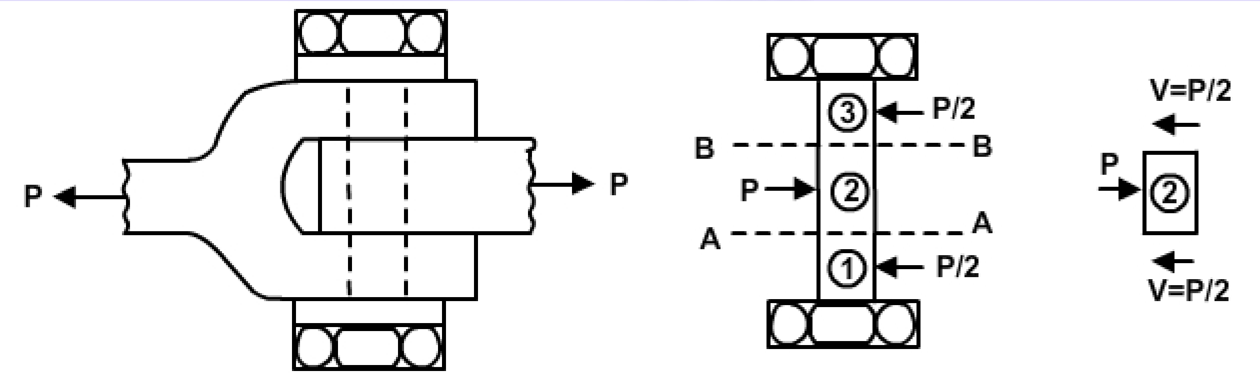


Figure 1.6

In figure 1.6, the bolt experiences shear on two sections AA and BB. Hence, the bolt is said to be under double shear and the shear stress on each section is

$$\tau = \frac{V}{A} = \frac{P}{2A} \qquad 1.4$$

Assuming that the same bolt is used in the assembly as shown in figure 1.5 and 1.6 and the same load P is applied on the plates, we can conclude that the shear stress is reduced by half in double shear when compared to a single shear.

Shear stresses are generally found in bolts, pins and rivets that are used to connect various structural members and machine components.

1.3.4 Bearing Stress:

In the bolted connection in figure 1.5, a highly irregular pressure gets developed on the contact surface between the bolt and the plates.

The average intensity of this pressure can be found out by dividing the load P by the projected area of the contact surface. This is referred to as the *bearing stress*.

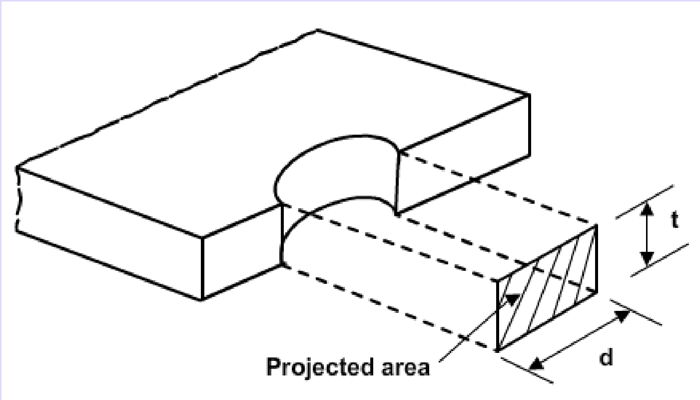


Figure 1.7

The projected area of the contact surface is calculated as the product of the diameter of the bolt and the thickness of the plate.

Bearing stress,

$$\sigma_b = \frac{P}{A} = \frac{P}{t \times d} \qquad 1.5$$

Example 1:

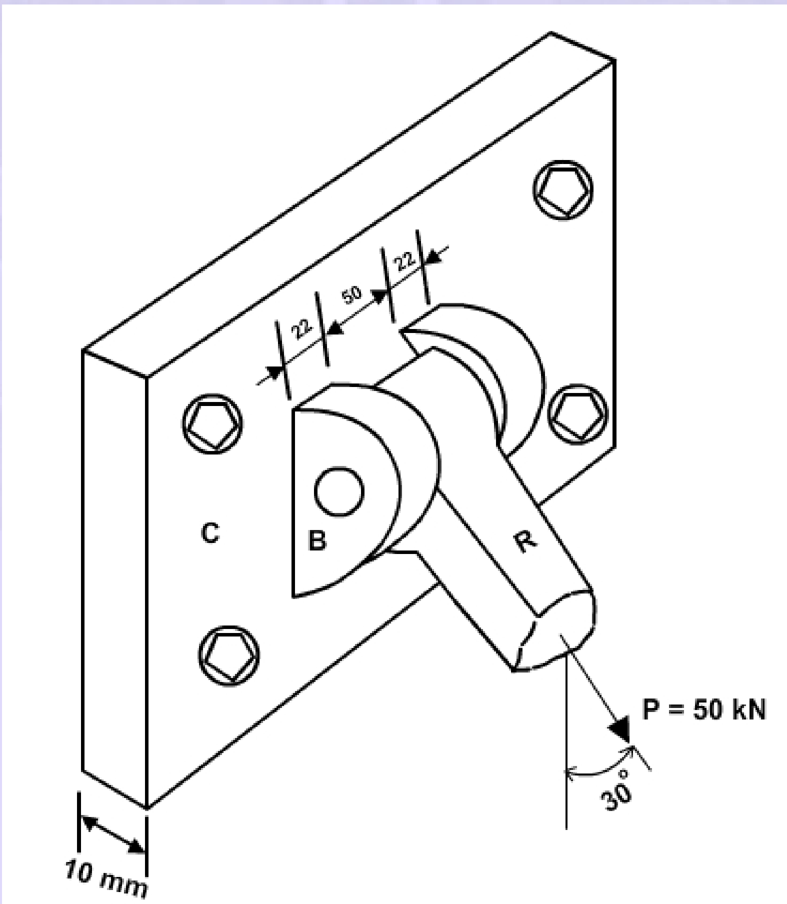


Figure 1.8

A rod R is used to hold a sign board with an axial load 50 kN as shown in figure 1.8. The end of the rod is 50 mm wide and has a circular hole for fixing the pin which is 20 mm diameter. The load from the rod R is transferred to the base plate C through a bracket B that is 22mm wide and has a circular hole for the pin. The base plate is 10 mm thick and it is connected to the bracket by welding. The base plate C is fixed on to a structure by four bolts of each 12 mm diameter. Find the shear stress and bearing stress in the pin and in the bolts.

Solution:

Shear stress in the pin = $\tau_{pin} = \frac{V}{A} = \frac{P}{2A} = \frac{\left(50\times10^3\right)/2}{\pi\left(0.02\right)^2/4} = 79.6 \text{ MPa}$

Force acting on the base plate = Pcosθ= 50 cos30^0 = 43.3 kN

Shear stress in the bolt, $\tau_{bolt} = \frac{P}{4A} = \frac{\left(43.3\times10^3\right)/4}{\pi\left(0.012\right)^2/4}$

$= 95.7 \text{ MPa}$

Bearing stress between pin and rod, $\sigma_b = \frac{P}{b\times d} = \frac{\left(50\times10^3\right)}{\left(0.05\right)\times\left(0.02\right)}$

$= 50 \text{ MPa}$

Bearing stress between pin and bracket = $\sigma_b = \frac{P/2}{b\times d} = \frac{\left(50\times10^3\right)/2}{\left(0.022\right)\times\left(0.02\right)}$

= 56.8 MPa

Bearing stress between plate and bolts = $\sigma_b = \frac{P/4}{t\times d} = \frac{\left(43.3\times10^3\right)/4}{\left(0.01\right)\times\left(0.012\right)}$

$= 90.2 \text{ MPa}$

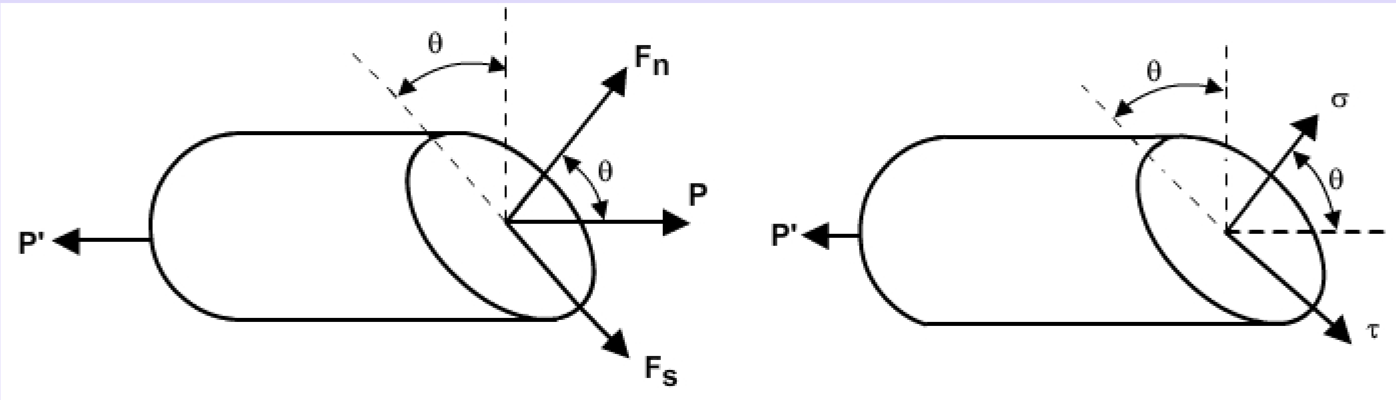
Top

1.4 Stress on inclined planes under axial loading:

When a body is under an axial load, the plane normal to the axis contains only the normal stress as discussed in section 1.3.1.

However, if we consider an oblique plane that forms an angle□ with normal plane, it consists shear stress in addition to normal stress.

Consider such an oblique plane in a bar. The resultant force P acting on that plane will



keep the bar in equilibrium against the external load P' as shown in figure 1.9.

Figure 1.9

The resultant force P on the oblique plane can be resolved into two components F_n and F_s that are acting normal and tangent to that plane, respectively.

If A is the area of cross section of the bar, A/cos□ is the area of the oblique plane. Normal and shear stresses acting on that plane can be obtained as follows.

$F_n = P\cos\theta$

$F_s = -P\sin\theta$ (Assuming shear causing clockwise rotation negative).

$$\sigma = \frac{P\cos\theta}{A/\cos\theta} = \frac{P}{A}\cos^2\theta \qquad 1.6$$

$$\tau = -\frac{P\sin\theta}{A/\cos\theta} = -\frac{P}{A}\sin\theta\cos\theta \qquad 1.7$$

Equations 1.6 and 1.7 define the normal and shear stress values on an inclined plane that makes an angle θ with the vertical plane on which the axial load acts.

From above equations, it is understandable that the normal stress reaches its maximum when $\theta = 0^o$ and becomes zero when $\theta = 90^o$.

But, the shear stress assumes zero value at $\theta = 0^o$ and $\theta = 90^o$ and reaches its maximum when $\theta = 45^o$.

The magnitude of maximum shear stress occurring at $\theta = 45^o$ plane is half of the maximum normal stress that occurs at $\theta = 0^o$ for a material under a uniaxial loading.

$$\tau_{max} = \frac{P}{2A} = \frac{\sigma_{max}}{2} \qquad 1.8$$

Now consider a cubic element A in the rod which is represented in two dimension as shown in figure 1.10 such that one of its sides makes an angle with the vertical plane.

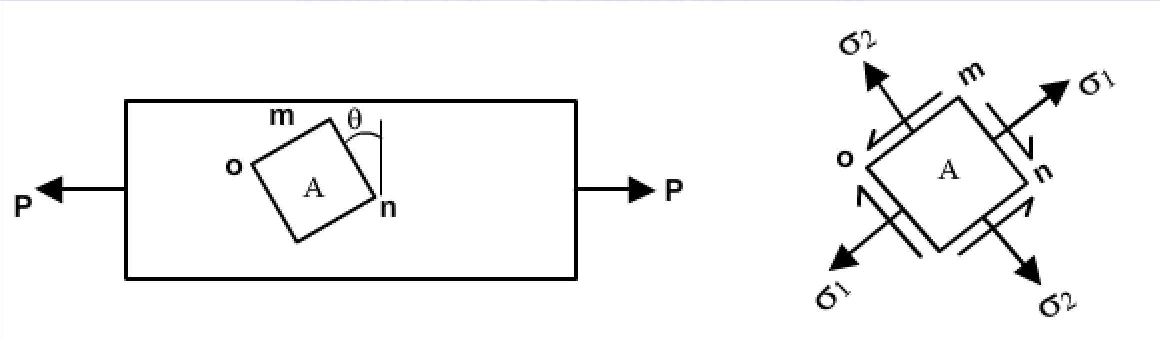


Figure 1.10

To determine the stresses acting on the plane *mn*, equations 1.6 and 1.7 are used as such and to knows the stresses on plane *om*, θ is replaced by $\theta + 90^o$.

Maximum shear stress occurs on both *om* and *mn* planes with equal magnitude and opposite signs, when *mn* forms 45^o angle with vertical plane.

Example 2:

A prismatic bar of sides 40 mm x 30 mm is axially loaded with a compressive force of 80 kN. Determine the stresses acting on an element which makes 30^0 inclination with the vertical plane. Also find the maximum shear stress value in the bar.

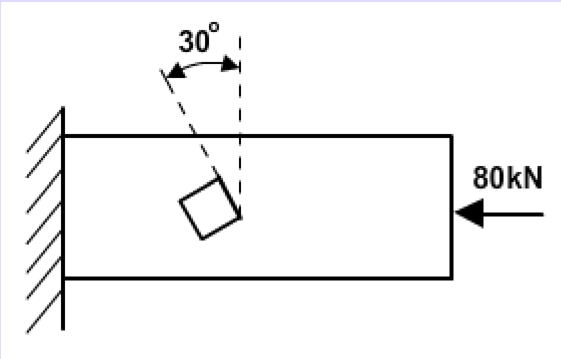


Figure 1.11

Solution:

Area of the cross section, A = 40 x 30 x 10^{-6} = 1.2 x 10^{-3} m^2

Normal stress on 30^0 inclined plane, $\sigma = \frac{P}{A}\cos^2\theta$

$$= \frac{-80\times10^3}{1.2\times10^{-3}}\times\cos^2 30^o = -50\text{ MPa}$$

Shear stress on 300 plane, $\tau = \frac{-P}{A}\sin\theta\cos\theta = \frac{80\times10^3}{1.2\times10^{-3}}\times\sin 30^0\times\cos 30^0$

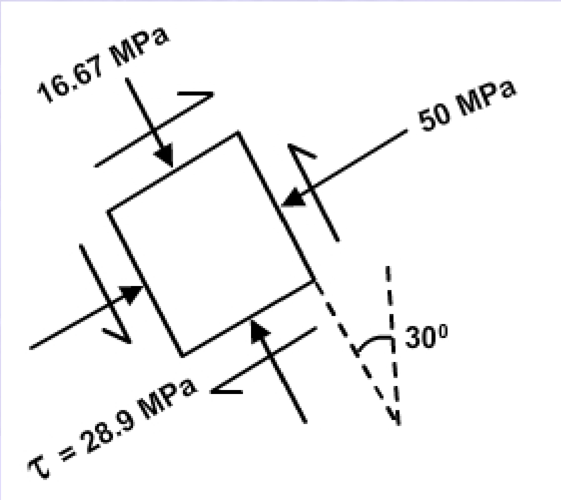
= 28.9 MPa [Counter clockwise]

Normal stress on 120^0 plane, $\sigma = \frac{-80\times10^3}{1.2\times10^{-3}}\cos^2 120^0 = -16.67\text{ MPa}$

Shear stress on 120^0 plane, $\tau = \frac{80\times10^3}{1.2\times10^{-3}}\times\sin 120^0\times\cos 120^0 = -28.9\text{ MPa}$ [Clock wise]

Maximum shear stress in the bar, $\tau_{max} = \frac{P}{2A} = \frac{80\times10^3}{2\times1.2\times10^{-3}}$

= $\pm$33.3 MPa



Top

1.5 Strain

The structural member and machine components undergo deformation as they are brought under loads.

To ensure that the deformation is within the permissible limits and do not affect the performance of the members, a detailed study on the deformation assumes significance.

A quantity called *strain* defines the deformation of the members and structures in a better way than the deformation itself and is an indication on the state of the material.

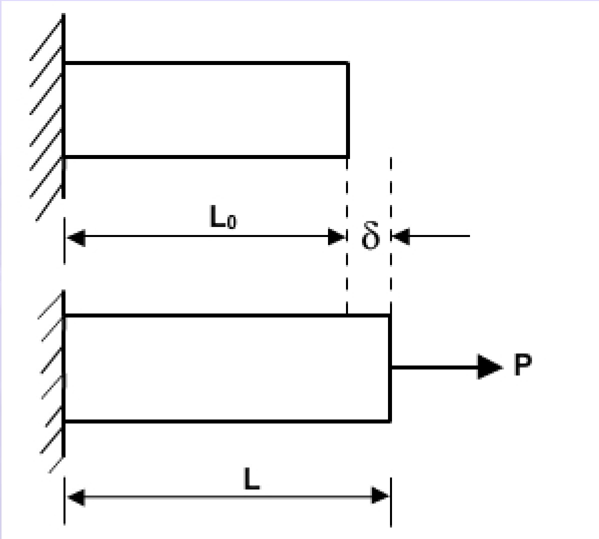


Figure 1.12

Consider a rod of uniform cross section with initial length L_0 as shown in figure 1.12. Application of a tensile load P at one end of the rod results in elongation of the rod by δ. After elongation, the length of the rod is L. As the cross section of the rod is uniform, it is appropriate to assume that the elongation is uniform throughout the volume of the rod. If the tensile load is replaced by a compressive load, then the deformation of the rod will be a contraction. The deformation per unit length of the rod along its axis is defined as the *normal strain*. It is denoted by ε

$$\varepsilon = \frac{\delta}{L} = \frac{L - L_0}{L} \qquad 1.9$$

Though the strain is a dimensionless quantity, units are often given in mm/mm, μm/m.

Example 3:

A circular hollow tube made of steel is used to support a compressive load of 500kN. The inner and outer diameters of the tube are 90mm and 130mm respectively and its length is 1000mm. Due to compressive load, the contraction of the rod is 0.5mm. Determine the compressive stress and strain in the post.

Solution

Force, $P = -500 \times 10^3\ \text{N}\,(\text{compressive})$

Area of the tube, $A = \frac{\pi}{4}\left[(0.13)^2 - (0.09)^2\right] = 6.912 \times 10^{-3}\text{m}^2$

Stress, $\sigma = \frac{P}{A} = \frac{-500 \times 10^3}{6.912 \times 10^{-3}} = -72.3\text{MPa}\,(\text{compressive})$

Strain, $\varepsilon = \frac{\delta}{L_0} = \frac{-0.5}{1000} = -5 \times 10^{-4}\,(\text{compressive})$

Top

1.6 Mechanical properties of materials

A tensile test is generally conducted on a standard specimen to obtain the relationship between the stress and the strain which is an important characteristic of the material.

In the test, the uniaxial load is applied to the specimen and increased gradually. The corresponding deformations are recorded throughout the loading.

Stress-strain diagrams of materials vary widely depending upon whether the material is ductile or brittle in nature.

If the material undergoes a large deformation before failure, it is referred to as ductile material or else brittle material.

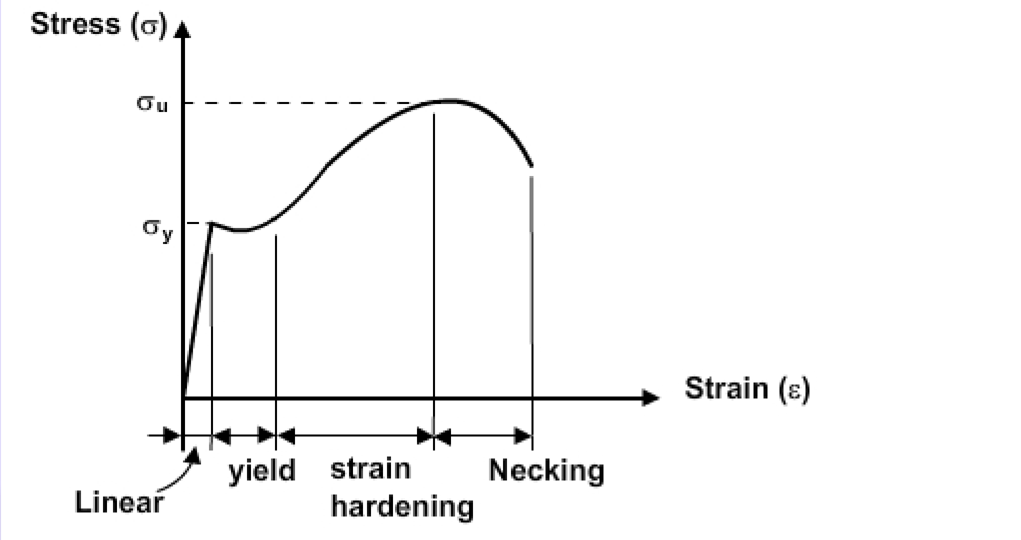


Figure 1.13

In figure 1.13, the stress-strain diagram of a structural steel, which is a ductile material, is given.

Initial part of the loading indicates a linear relationship between stress and strain, and the deformation is completely recoverable in this region for both ductile and brittle materials.

This linear relationship, i.e., stress is directly proportional to strain, is popularly known as *Hooke's law.*

$$\sigma = E\varepsilon \qquad 1.10$$

The co-efficient E is called the *modulus of elasticity* or *Young's modulus.*

Most of the engineering structures are designed to function within their linear elastic region only.

After the stress reaches a critical value, the deformation becomes irrecoverable. The corresponding stress is called the *yield stress* or yield strength of the material beyond which the material is said to start yielding.

In some of the ductile materials like low carbon steels, as the material reaches the yield strength it starts yielding continuously even though there is no increment in external load/stress. This flat curve in stress strain diagram is referred as perfectly plastic region.

The load required to yield the material beyond its yield strength increases appreciably and this is referred to strain hardening of the material.

In other ductile materials like aluminum alloys, the strain hardening occurs immediately after the linear elastic region without perfectly elastic region.

After the stress in the specimen reaches a maximum value, called *ultimate strength,* upon further stretching, the diameter of the specimen starts decreasing fast due to local instability and this phenomenon is called *necking*.

The load required for further elongation of the material in the necking region decreases with decrease in diameter and the stress value at which the material fails is called the *breaking strength*.

In case of brittle materials like cast iron and concrete, the material experiences smaller deformation before rupture and there is no necking.

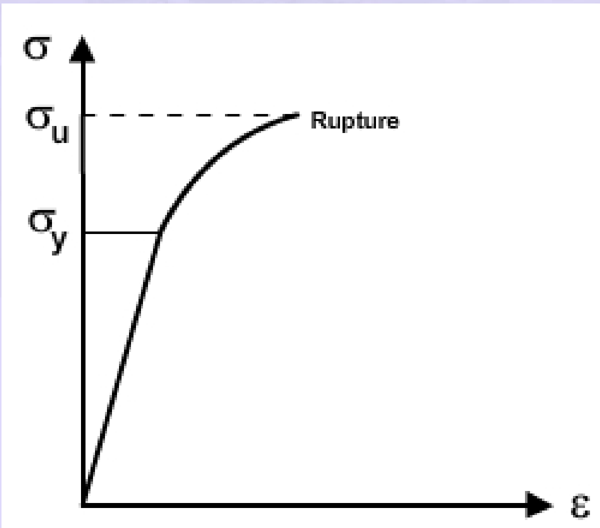


Figure 1.14

Top

1.7 True stress and true strain

In drawing the stress-strain diagram as shown in figure 1.13, the stress was calculated by dividing the load P by the initial cross section of the specimen.

But it is clear that as the specimen elongates its diameter decreases and the decrease in cross section is apparent during necking phase.

Hence, the actual stress which is obtained by dividing the load by the actual cross sectional area in the deformed specimen is different from that of the engineering stress that is obtained using undeformed cross sectional area as in equation 1.1

True stress or actual stress,

$$\sigma_{act} = \frac{P}{A_{act}} \qquad 1.11$$

Though the difference between the true stress and the engineering stress is negligible for smaller loads, the former is always higher than the latter for larger loads.

Similarly, if the initial length of the specimen is used to calculate the strain, it is called engineering strain as obtained in equation 1.9

But some engineering applications like metal forming process involve large deformations and they require actual or *true strains* that are obtained using the successive recorded lengths to calculate the strain.

$$\text{True strain} = \int_{L_0}^{L} \frac{dL}{L} = \ln\left(\frac{L}{L_0}\right) \qquad 1.12$$

True strain is also called as actual strain or natural strain and it plays an important role in theories of viscosity.

The difference in using engineering stress-strain and the true stress-strain is noticeable after the proportional limit is crossed as shown in figure 1.15.

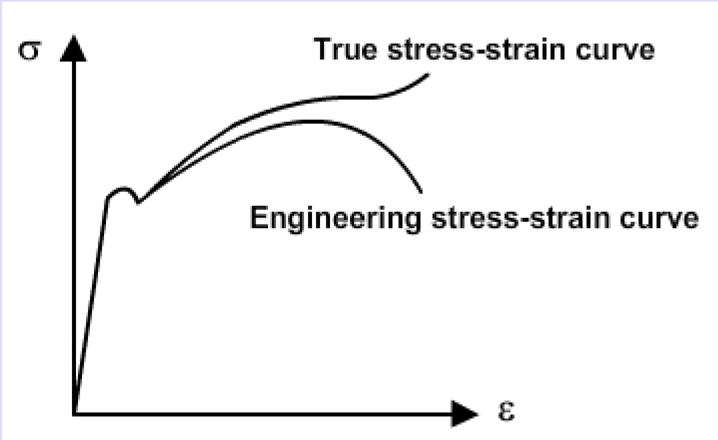


Figure 1.15

Top

1.8 Poissons ratio

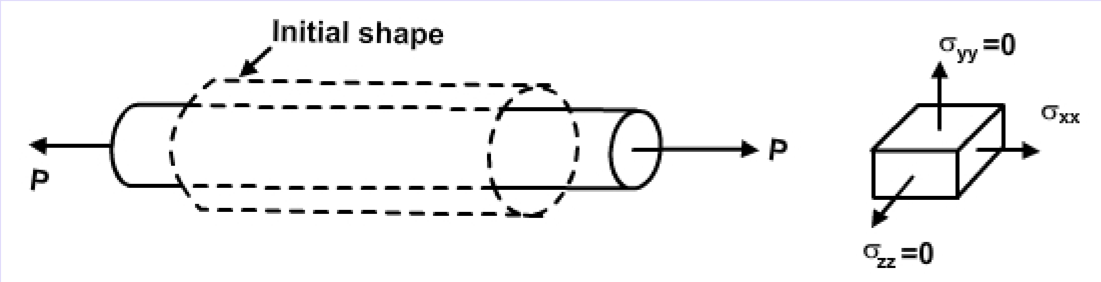


Figure 1.16

Consider a rod under an axial tensile load P as shown in figure 1.6 such that the material is within the elastic limit. The normal stress on x plane is $\sigma_{xx} = \frac{P}{A}$ and the associated longitudinal strain in the x direction can be found out from $\varepsilon_x = \frac{\sigma_{xx}}{E}$. As the material elongates in the x direction due to the load P, it also contracts in the other two mutually perpendicular directions, i.e., y and z directions.

Hence, despite the absence of normal stresses in y and z directions, strains do exist in those directions and they are called lateral strains.

The ratio between the lateral strain and the axial/longitudinal strain for a given material is always a constant within the elastic limit and this constant is referred to as *Poisson's ratio*. It is denoted by ν.

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} \qquad 1.13$$

Since the axial and lateral strains are opposite in sign, a negative sign is introduced in equation 1.13 to make ν positive.

Using equation 1.13, the lateral strain in the material can be obtained by

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_{xx}}{E} \qquad 1.14$$

Poisson's ratio can be as low as 0.1 for concrete and as high as 0.5 for rubber.

In general, it varies from 0.25 to 0.35 and for steel it is about 0.3.

Top

1.9 Elasticity and Plasticity

If the strain disappears completely after removal of the load, then the material is said to be in elastic region.

The stress-strain relationship in elastic region need not be linear and can be non-linear as in rubber like materials.

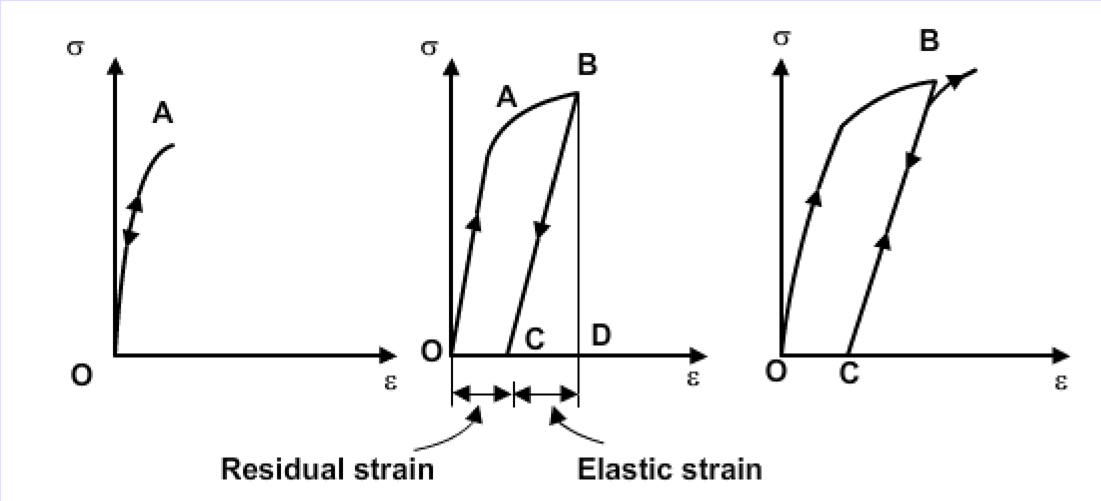


Figure 1.17

The maximum stress value below which the strain is fully recoverable is called the *elastic limit*. It is represented by point A in figure 1.17.

When the stress in the material exceeds the elastic limit, the material enters into plastic phase where the strain can no longer be completely removed.

To ascertain that the material has reached the plastic region, after each load increment, it is unloaded and checked for residual strain.

Presence of residual strain is the indication that the material has entered into plastic phase.

If the material has crossed elastic limit, during unloading it follows a path that is parallel to the initial elastic loading path with the same proportionality constant E.

The strain present in the material after unloading is called the residual strain or plastic strain and the strain disappears during unloading is termed as recoverable or elastic strain. They are represented by OC and CD, respectively in figure.1.17.

If the material is reloaded from point C, it will follow the previous unloading path and line CB becomes its new elastic region with elastic limit defined by point B.

Though the new elastic region CB resembles that of the initial elastic region OA, the internal structure of the material in the new state has changed.

The change in the microstructure of the material is clear from the fact that the ductility of the material has come down due to strain hardening.

When the material is reloaded, it follows the same path as that of a virgin material and fails on reaching the ultimate strength which remains unaltered due to the intermediate loading and unloading process.

Top

1.10 Creep and fatigue

In the preceding section, it was discussed that the plastic deformation of a material increases with increasing load once the stress in the material exceeds the elastic limit.

However, the materials undergo additional plastic deformation with time even though the load on the material is unaltered.

Consider a bar under a constant axial tensile load as shown in figure 1.18.

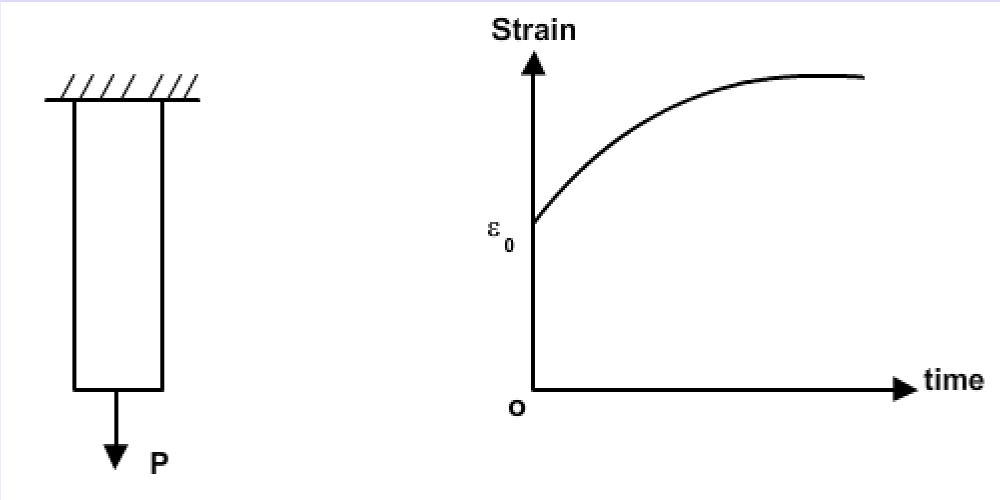


Figure 1.18

As soon as the material is loaded beyond its elastic limit, it undergoes an instant plastic deformation ε_0 at time t = 0.

Though the material is not brought under additional loads, it experiences further plastic deformation with time as shown in the graph in figure 1.18.

This phenomenon is called *creep*.

Creep at high temperature is of more concern and it plays an important role in the design of engines, turbines, furnaces, etc.

However materials like concrete, steel and wood experience creep slightly even at normal room temperature that is negligible.

Analogous to creep, the load required to keep the material under constant strain decreases with time and this phenomenon is referred to as *stress relaxation*.

It was concluded in section 1.9 that the specimen will not fail when the stress in the material is with in the elastic limit.

This holds true only for static loading conditions and if the applied load fluctuates or reverses then the material will fail far below its yield strength.

This phenomenon is known as *fatigue*.

Designs involving fluctuating loads like traffic in bridges, and reversing loads like automobile axles require fatigue analysis.

Fatigue failure is initiated by a minute crack that develops at a high stress point which may be an imperfection in the material or a surface scratch.

The crack enlarges and propagates through the material due to successive loadings until the material fails as the undamaged portion of the material is insufficient to withstand the load.

Hence, a polished surface shaft can take more number of cycles than a shaft with rough or corroded surface.

The number of cycles that can be taken up by a material before it fractures can be found out by conducting experiments on material specimens.

The obtained results are plotted as $\sigma - n$ curves as given in figure 1.19, which indicates the number of cycles that can be safely completed by the material under a given maximum stress.

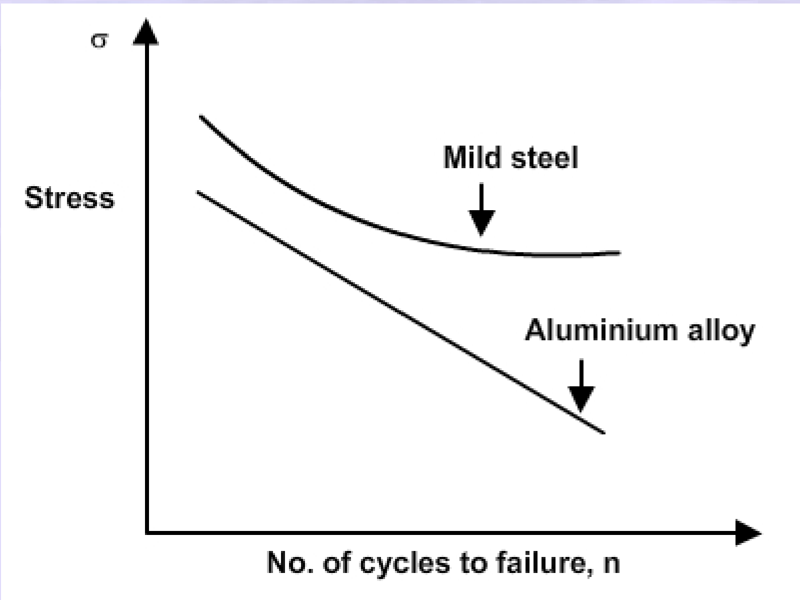


Figure 1.19

It is learnt from the graph that the number of cycles to failure increases with decrease in magnitude of stress.

For steels, if the magnitude of stress is reduced to a particular value, it can undergo an infinitely large number of cycles without fatigue failure and the corresponding stress is known as *endurance limit* or *fatigue limit*.

On the other hand, for non-ferrous metals like aluminum alloys there is no endurance limit, and hence, the maximum stress decreases continuously with increase in number of cycles. In such cases, the fatigue limit of the material is taken as the stress value that will allow an arbitrarily taken number of cycles, say 10^8 cycles.

Top

1.11 Deformation in axially loaded members

Consider the rod of uniform cross section under tensile load P along its axis as shown in figure 1.12.

Let that the initial length of the rod be L and the deflection due to load be δ. Using equations 1.9 and 1.10,

$$\frac{\delta}{L} = \varepsilon = \frac{\sigma}{E} = \frac{P}{AE}$$

$$\delta = \frac{PL}{AE} \qquad 1.15$$

Equation 1.15 is obtained under the assumption that the material is homogeneous and has a uniform cross section.

Now, consider another rod of varying cross section with the same axial load P as shown in figure 1.20.

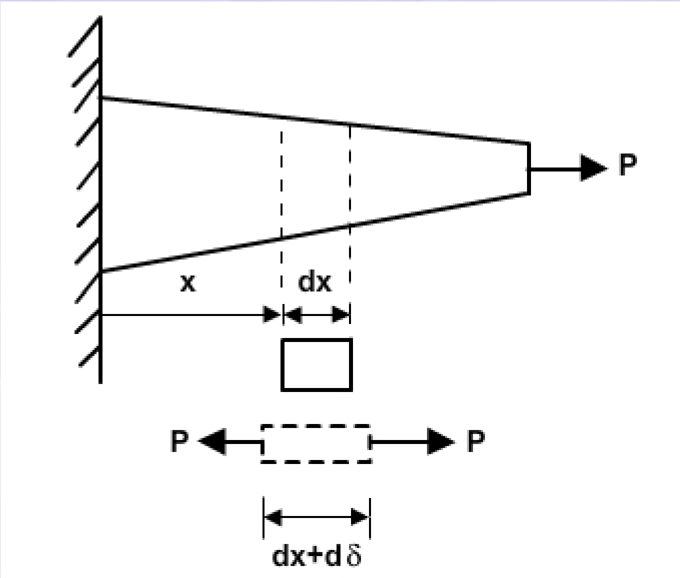


Figure 1.20

Let us take an infinitesimal element of length dx in the rod that undergoes a deflection $d\delta$ due to load P. The strain in the element is $\varepsilon = \frac{d\delta}{dx}$ and $d\delta = \varepsilon dx$

The deflection of total length of the rod can be obtained by integrating above equation, $\delta = \int \varepsilon dx$

$$\delta = \int_0^L \frac{Pdx}{EA(x)} \qquad 1.16$$

As the cross sectional area of the rod keeps varying, it is expressed as a function of its length.

If the load is also varying along the length like the weight of the material, it should also be expressed as a function of distance, i.e., P(x) in equation 1.16.

Also, if the structure consists of several components of different materials, then the deflection of each component is determined and summed up to get the total deflection of the structure.

When the cross section of the components and the axial loads on them are not varying along length, the total deflection of the structure can be determined easily by,

$$\delta = \sum_{i=1}^{n} \frac{P_i L_i}{A_i E_i} \qquad 1.17$$

Example 4:

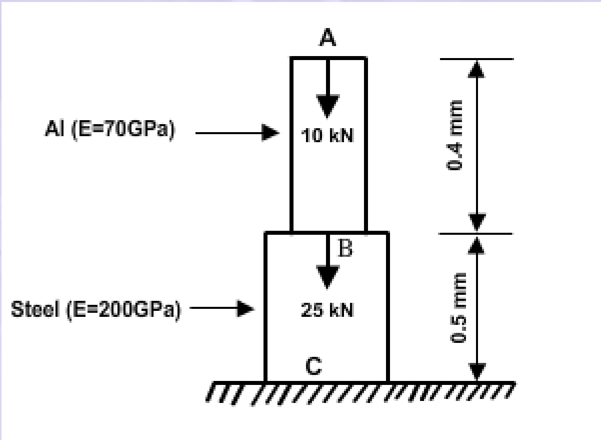
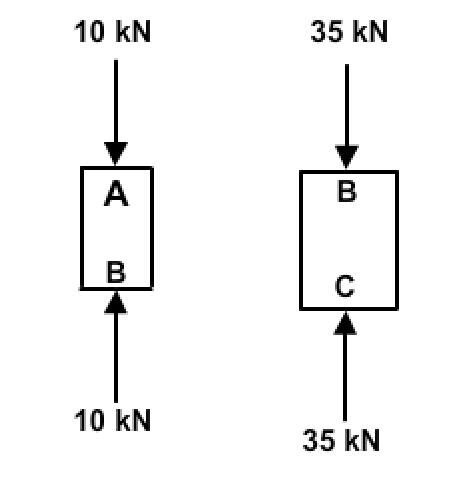


Figure 1.21

Consider a rod ABC with aluminum part AB and steel part BC having diameters 25mm and 50 mm respectively as shown figure 1.21. Determine the deflections of points A and B.

Solution:



Deflection of part AB = $\frac{P_{AB}L_{AB}}{A_{AB}E_{AB}} = -\frac{10\times10^3\times400}{\pi\times(0.025)^2/4\times70\times10^9}$

$= -0.1164$ mm

Deflection of part BC = $\frac{P_{BC}L_{BC}}{A_{BC}E_{BC}} = -\frac{35\times10^3\times500}{\pi\times(0.05)^2/4\times200\times10^9}$

$= -0.0446$ mm

Deflection point of B $= -0.0446$ mm

Deflection point of A $= (-0.1164) + (-0.0446)$

$= -0.161$mm

Top

1.12 Statically indeterminate problems.

Members for which reaction forces and internal forces can be found out from static equilibrium equations alone are called statically determinate members or structures.

Problems requiring deformation equations in addition to static equilibrium equations to solve for unknown forces are called *statically indeterminate* problems.

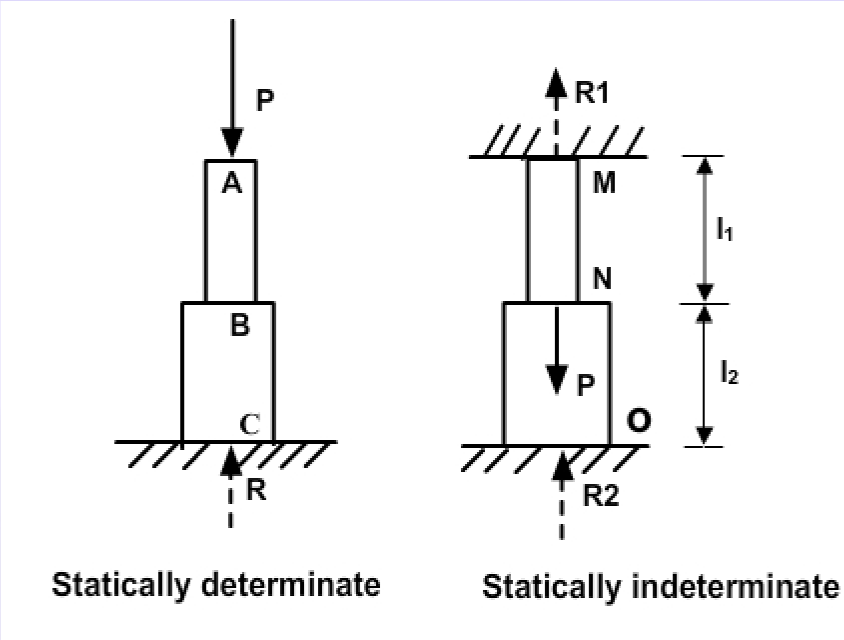


Figure 1.22

The reaction force at the support for the bar ABC in figure 1.22 can be determined considering equilibrium equation in the vertical direction.

$$\sum F_y = 0; \quad R - P = 0$$

Now, consider the right side bar MNO in figure 1.22 which is rigidly fixed at both the ends. From static equilibrium, we get only one equation with two unknown reaction forces R_1 and R_2.

$$-P + R_1 + R_2 = 0 \qquad 1.18$$

Hence, this equilibrium equation should be supplemented with a deflection equation which was discussed in the preceding section to solve for unknowns.

If the bar MNO is separated from its supports and applied the forces R_1, R_2 and P, then these forces cause the bar to undergo a deflection δ_{MO} that must be equal to zero.

$$\delta_{MO} = 0 \Rightarrow \delta_{MN} + \delta_{NO} = 0 \qquad 1.19$$

δ_{MN} and δ_{NO} are the deflections of parts MN and NO respectively in the bar MNO. Individually these deflections are not zero, but their sum must make it to be zero.

Equation 1.19 is called *compatibility equation*, which insists that the change in length of the bar must be compatible with the boundary conditions.

Deflection of parts MN and NO due to load P can be obtained by assuming that the material is within the elastic limit, $\delta_{MN} = \frac{R_1 l_1}{A_1 E}$ and $\delta_{NO} = \frac{R_2 l_2}{A_2 E}$.

Substituting these deflections in equation 1.19,

$$\frac{R_1 l_1}{A_1 E} - \frac{R_2 l_2}{A_2 E} = 0 \qquad 1.20$$

Combining equations 1.18 and 1.20, one can get,

$$R_1 = \frac{PA_1 l_2}{l_1 A_2 + l_2 A_1}$$
$$R_2 = \frac{PA_2 l_1}{l_1 A_2 + l_2 A_1} \qquad 1.21$$

From these reaction forces, the stresses acting on any section in the bar can be easily determined.

Example 5:

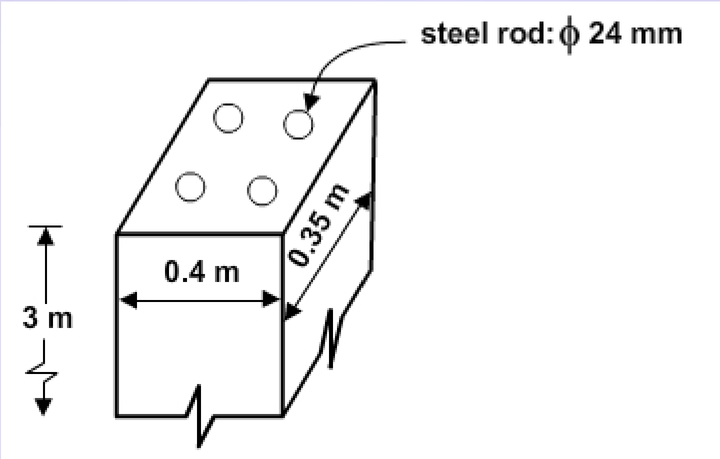


Figure 1.23

A rectangular column of sides $0.4\text{m} \times 0.35\text{m}$, made of concrete, is used to support a compressive load of 1.5MN. Four steel rods of each 24mm diameter are passing through the concrete as shown in figure 1.23. If the length of the column is 3m, determine the normal stress in the steel and the concrete. Take $E_{steel} = 200$ GPa and $E_{concrete} = 29$ GPa.

Solution:

P_s = Load on each steel rod

P_c = Load on concrete

From equilibrium equation,

$$P_c + 4P_s = P$$
$$P_c + 4P_s = 1.5 \times 10^3 \text{..........(a)}$$

Deflection in steel rod and concrete are the same.

$$\delta_{concrete} = \delta_{steel}$$

$$\frac{P_c \times 3}{(0.4) \times (0.35) \times 29 \times 10^9} = \frac{P_s \times 3}{\frac{\pi}{4} \times (0.024)^2 \times 200 \times 10^9}$$

$$P_c = 44.87 P_s \text{................(b)}$$

Combining equations (a) and (b),

$$P_s = 30.7\text{kN}$$
$$P_c = 1378\text{kN}$$

Normal stress on concrete= $\frac{P_c}{A_c} = \frac{1.378 \times 10^6}{(0.4)(0.35)} = 9.84\text{MPa}$

Normal stress on steel= $\frac{P_s}{A_s} = \frac{30.7 \times 10^3}{\frac{\pi}{4}(0.024)^2} = 67.86\text{MPa}$

Top

1.13 Thermal effect

When a material undergoes a change in temperature, it either elongates or contracts depending upon whether heat is added to or removed from the material.

If the elongation or contraction is not restricted, then the material does not experience any stress despite the fact that it undergoes a strain.

The strain due to temperature change is called *thermal strain* and is expressed as

$$\varepsilon_T = \alpha(\Delta T) \qquad 1.22$$

where α is a material property known as coefficient of thermal expansion and ΔT indicates the change in temperature.

Since strain is a dimensionless quantity and ΔT is expressed in K or ^{0}C, α has a unit that is reciprocal of K or ^{0}C.

The free expansion or contraction of materials, when restrained induces stress in the material and it is referred to as *thermal stress*.

Thermal stress produces the same effect in the material similar to that of mechanical stress and it can be determined as follows.

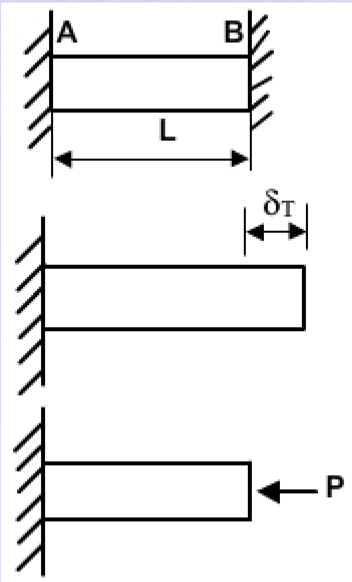


Figure 1.24

Consider a rod AB of length L which is fixed at both ends as shown in figure 1.24.

Let the temperature of the rod be raised by ΔT and as the expansion is restricted, the material develops a compressive stress.

In this problem, static equilibrium equations alone are not sufficient to solve for unknowns and hence is called statically indeterminate problem.

To determine the stress due to ΔT, assume that the support at the end B is removed and the material is allowed to expand freely.

Increase in the length of the rod δ_T due to free expansion can be found out using equation 1.22

$$\delta_T = \varepsilon_T L = \alpha(\Delta T)L \qquad 1.23$$

Now, apply a compressive load P at the end B to bring it back to its initial position and the deflection due to mechanical load from equation 1.15,

$$\delta_T = \frac{PL}{AE} \qquad 1.24$$

As the magnitude of δ_T and δ are equal and their signs differ,

$$\delta_T = -\delta$$

$$\alpha(\Delta T)L = -\frac{PL}{AE}$$

$$\text{Thermal stress, } \sigma_T = \frac{P}{A} = -\alpha(\Delta T)E \qquad 1.25$$

Minus sign in the equation indicates a compressive stress in the material and with decrease in temperature, the stress developed is tensile stress as ΔT becomes negative.

It is to be noted that the equation 1.25 was obtained on the assumption that the material is homogeneous and the area of the cross section is uniform.

Thermoplastic analysis assumes significance for structures and components that are experiencing high temperature variations.

Example 6:

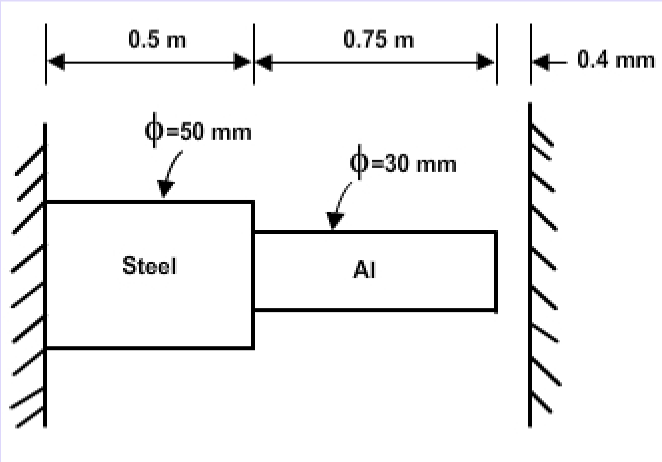


Figure 1.25

A rod consists of two parts that are made of steel and aluminum as shown in figure 1.25. The elastic modulus and coefficient of thermal expansion for steel are 200GPa and 11.7 x 10^{-6} per ^{0}C respectively and for aluminum 70GPa and 21.6 x 10^{-6} per ^{0}C respectively. If the temperature of the rod is raised by 50^0C, determine the forces and stresses acting on the rod.

Solution:

Deflection of the rod under free expansion,

$$\delta_T = \alpha(\Delta T)L$$
$$= \left(11.7\times10^{-6}\times50\times500\right)+\left(21.6\times10^{-6}\times50\times750\right)$$
$$= 1.1025 \text{ mm}$$

Restrained deflection of rod = 1.1025 - 0.4 = 0.7025 mm

Let the force required to make their elongation vanish be R which is the reaction force at the ends.

$$-\delta = \left(\frac{RL}{AE}\right)_{steel} + \left(\frac{RL}{AE}\right)_{Al}$$

$$\text{Area of steel rod} = \frac{\pi}{4}[0.05]^2 = 1.9635\times10^{-3}\,m^2$$

$$\text{Area of aluminium rod} = \frac{\pi}{4}[0.03]^2 = 0.7069\times10^{-3}\,m^2$$

$$-0.7025 = R\left[\frac{500}{1.9635\times10^{-3}\times200\times10^{9}} + \frac{750}{0.7069\times10^{-3}\times70\times10^{9}}\right]$$

Compressive force on the rod, R = − 42.076 kN

$$\text{Compressive stress on steel, } \sigma = \frac{P}{A} = \frac{-42.76\times10^{3}}{1.9635\times10^{-3}} = -21.8\,MPa$$

$$\text{Compressive stress on steel, } \sigma = \frac{P}{A} = \frac{-42.76\times10^{3}}{0.7069\times10^{-3}} = -60.5\,MPa$$

Top

1.14. Design considerations:

A good design of a structural element or machine component should ensure that the developed product will function safely and economically during its estimated life time.

The stress developed in the material should always be less than the maximum stress it can withstand which is known as ultimate strength as discussed in section 1.6.

During normal operating conditions, the stress experienced by the material is referred to as working stress or *allowable stress* or design stress.

The ratio of ultimate strength to allowable stress is defined as *factor of safety*.

$$\text{Factor of safety} = \frac{\text{Ultimate stress}}{\text{Allowable stress}} \qquad 1.26$$

Factor of safety can also be expressed in terms of load as,

$$\text{Factor of safety} = \frac{\text{Ultimate load}}{\text{Allowable load}} \qquad 1.27$$

Equations 1.26 and 1.27 are identical when a linear relationship exists between the load and the stress.

This is not true for many materials and equation 1.26 is widely used in design analysis.

Factor of safety take care of the uncertainties in predicting the exact loadings, variation in material properties, environmental effects and the accuracy of methods of analysis.

If the factor of safety is less, then the risk of failure is more and on the other hand, when the factor of safety is very high the structure becomes unacceptable or uncompetitive.

Hence, depending upon the applications the factor of safety varies. It is common to see that the factor of safety is taken between 2 and 3.

Stresses developed in the material when subjected to loads can be considered to be uniform at sections located far away from the point of application of loads.

This observation is called Saint Venant's principle and was discussed in section 1.3.

But, when the element has holes, grooves, notches, key ways, threads and other abrupt changes in geometry, the stress on those cross-sections will not be uniform.

These discontinuities in geometry cause high stresses concentrations in small regions of the material and are called stress raisers.

Experimentally it was found that the stress concentrations are independent of the material size and its properties, and they depend only on the geometric parameters.

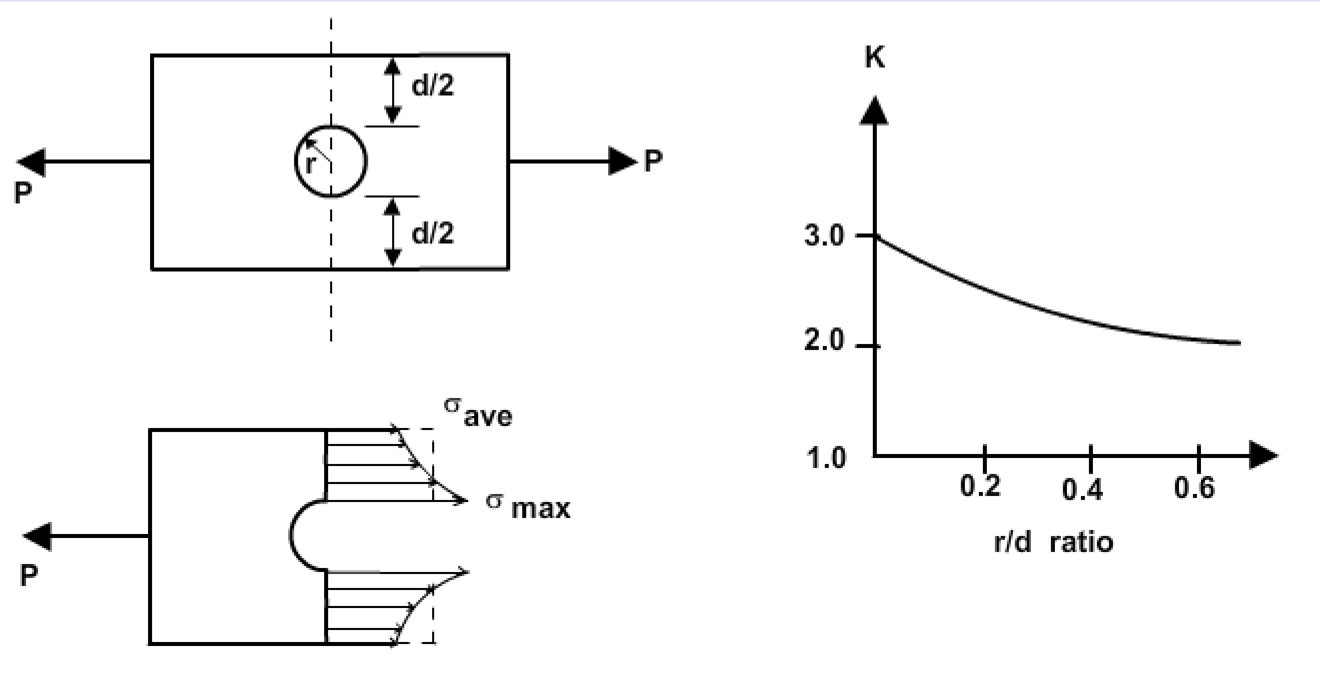


Figure 1.26

Consider a rectangular flat plate with a circular hole as shown in figure 1.26.

The stress distribution on the section passing through the centre of the hole indicates that the maximum stress occurs at the ends of the holes and it is much higher than the average stress.

Since the designer, in general, is more interested in knowing the maximum stress rather than the actual stress distribution, a simple relationship between the σ_{max} and σ_{ave} in terms of geometric parameters will be of practical importance.

Many experiments were conducted on samples with various discontinuities and the relationship between the *stress concentration factor* and the geometrical parameters are established, where

$$\text{Stress concentration factor, } K = \frac{\sigma_{max}}{\sigma_{ave}} \qquad 1.28$$

Hence, simply by calculating the average stress, $\sigma_{ave} = \frac{P}{A}$, in the critical section of a discontinuity, σ_{max} can be easily found and by multiplying σ_{ave} with K.

The variation of K in terms of *r*/d for the rectangular plate with a circular hole is given in figure 1.26.

It is to be noted that the expression in equation 1.28 can be used as long as σ_{max} is within the proportional limit of the material.

Example 7:

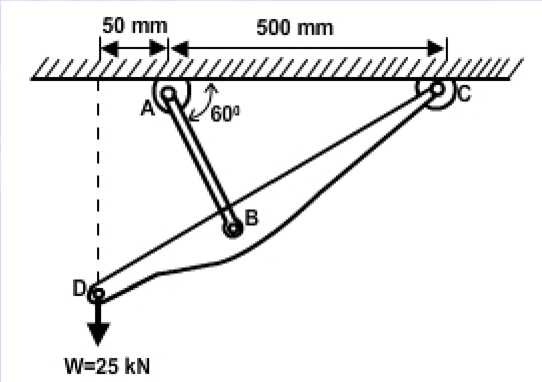
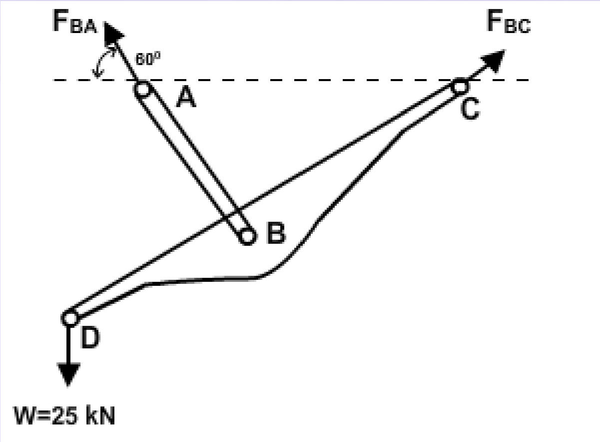


Figure 1.27

A rectangular link AB made of steel is used to support a load W through a rod CD as shown in figure 1.27. If the link AB is 30mm wide, determine its thickness for a factor of safety 2.5. The ultimate strength of steel may be assumed to be 450 MPa.

Solution:

Drawing free body diagram of the link and the rod,



Taking moment about C,

$$25 \times 550 - F_{BA} \sin 60^0 \times 500 = 0$$

$$F_{BA} = 31.75 \text{ kN}$$

Tension along link AB, $F_{BA} = 31.75$ kN.

$$\text{F.O.S} = \frac{\text{Ultimate stress}}{\text{allowable stress}}$$

Allowable stress in link AB, $\sigma_a = \dfrac{450 \times 10^6}{2.5} = 180 \text{ MPa}$

$$\text{Stress in link AB, } \sigma = \frac{\text{Tensile force}}{\text{Area}}$$

$$180 \times 10^6 = \frac{31.75 \times 10^3}{0.03 \times t}$$

Thickness of link AB, $t = 5.88$

$$t \approx 6 \text{ mm}$$

Top

1.15. Strain energy:

Strain energy is an important concept in mechanics and is used to study the response of materials and structures under static and dynamic loads.

Within the elastic limit, the work done by the external forces on a material is stored as deformation or strain that is recoverable.

On removal of load, the deformation or strain disappears and the stored energy is released. This recoverable energy stored in the material in the form of strain is called elastic strain energy.

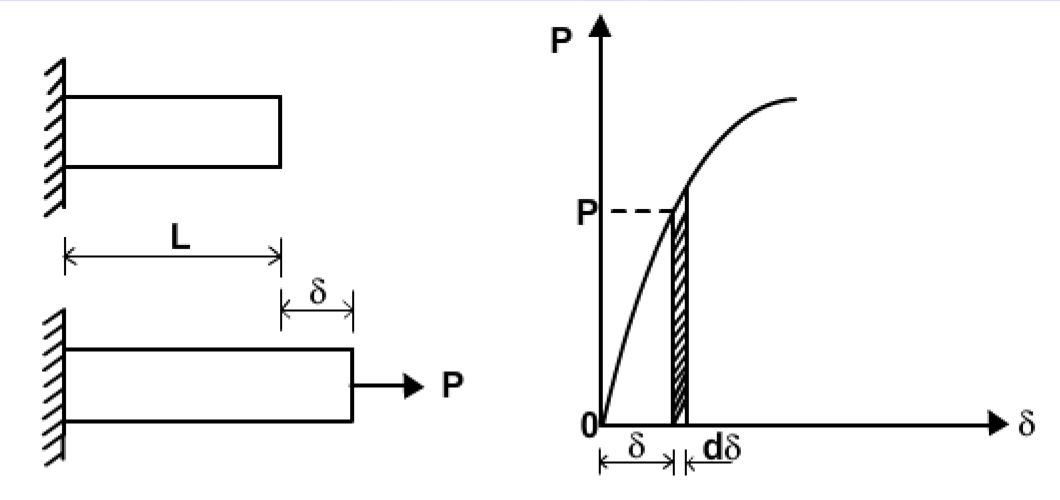


Figure 1.28

Consider a rod of uniform cross section with length L as shown in figure 1.28.

An axial tensile load P is applied on the material gradually from zero to maximum magnitude and the corresponding maximum deformation is δ.

Area under the load-displacement curve shown in figure 1.28 indicates the work done on the material by the external load that is stored as strain energy in the material.

Let dW be the work done by the load P due to increment in deflection dδ. The corresponding increase in strain energy is dU.

When the material is within the elastic limit, the work done due to dδ,

$$dW = dU = Pd\delta$$

The total work done or total elastic strain energy of the material,

$$W = U = \int_0^{\delta} Pd\delta \qquad 1.29$$

Equation 1.29 holds for both linear elastic and non-linear elastic materials.

If the material is linear elastic, then the load-displacement diagram will become as shown in figure 1.29.

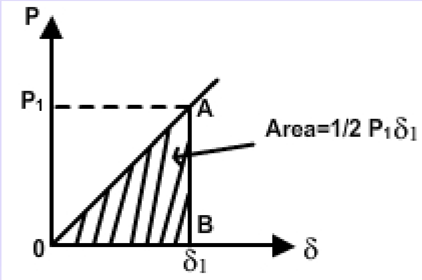


Figure 1.29

The elastic strain energy stored in the material is determined from the area of triangle OAB.

$$U = \frac{1}{2} P_1 \delta_1 \qquad 1.30$$

where $\delta_1 = \frac{P_1 L}{AE}$.

Since the load-displacement curve is a straight line here, the load P_1 can be expressed in terms of stiffness and deflection as $P_1 = k\delta_1$. Then equation 1.30 turns out to be,

$$U = \frac{1}{2} k {\delta_1}^2 \qquad 1.31$$

Work done and strain energy are expressed in N-m or joules (J).

Strain energy defined in equation 1.29 depends on the material dimensions.

In order to eliminate the material dimensions from the strain energy equation, strain energy density is often used.

Strain energy stored per unit volume of the material is referred to as *strain energy density*.

Dividing equation 1.29 by volume,

Strain energy density, $u = \int_0^{\varepsilon} \sigma d\varepsilon$ 1.32

Equation 1.32 indicates the expression of strain energy in terms of stress and strain, which are more convenient quantities to use rather than load and displacement.

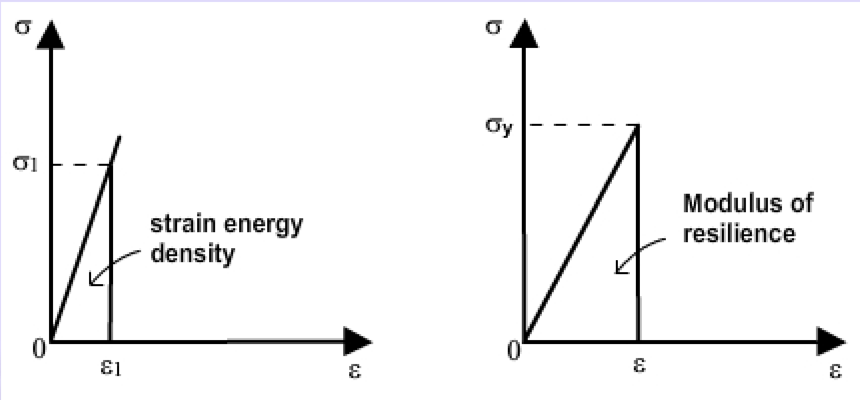


Figure 1.30

Area under the stress strain curve indicates the strain energy density of the material.

For linear elastic materials within proportional limit, equation 1.32 gets simplified as,

Strain energy density, $u = \frac{1}{2}\sigma_1\varepsilon_1$ 1.33

Using Hook's law, $\varepsilon_1 = \frac{\sigma_1}{E}$, strain energy density is expressed in terms of stress,

$$u = \frac{\sigma_1^{\ 2}}{2E} \quad 1.34$$

When the stress in the material reaches the yield stress σ_y, the strain energy density attains its maximum value and is called the *modulus of resilience*.

Modulus of resilience, $u_R = \frac{\sigma_Y^{\ 2}}{2E}$ 1.35

Modulus of resilience is a measure of energy that can be absorbed by the material due to impact loading without undergoing any plastic deformation.

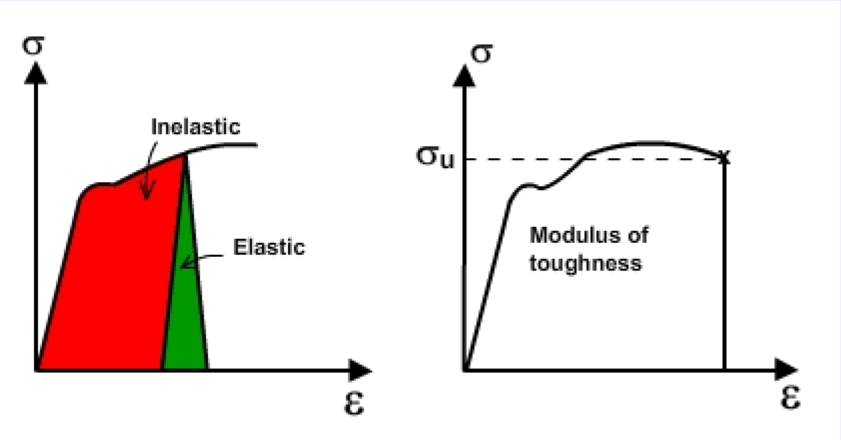


Figure 1.31

If the material exceeds the elastic limit during loading, all the work done is not stored in the material as strain energy.

This is due to the fact that part of the energy is spent on deforming the material permanently and that energy is dissipated out as heat.

The area under the entire stress strain diagram is called *modulus of toughness*, which is a measure of energy that can be absorbed by the material due to impact loading before it fractures.

Hence, materials with higher modulus of toughness are used to make components and structures that will be exposed to sudden and impact loads.

Example 8:

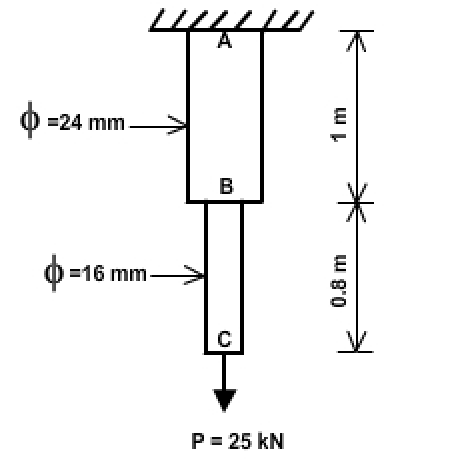


Figure 1.32

A 25 kN load is applied gradually on a steel rod ABC as shown in figure 1.32. Taking E=200 GPa, determine the strain energy stored in the entire rod and the strain energy density in parts AB and BC. If the yield strength of the material is σ_y=320MPa, find the maximum energy that can be absorbed by the rod without undergoing permanent deformation.

Solution:

$$\text{Strain energy density in part AB, } u_{AB} = \frac{\sigma_{AB}^2}{2E}$$

$$u_{AB} = \frac{1}{2\times 200\times 10^9}\left[\frac{25\times 10^3}{\frac{\pi}{4}(0.024)^2}\right]^2$$

$$= 7.63 \text{ kJ/m}^3$$

$$\text{Strain energy density in part BC, } u_{BC} = \frac{\sigma_{BC}^2}{2E}$$

$$u_{BC} = \frac{1}{2\times 200\times 10^9}\left[\frac{25\times 10^3}{\frac{\pi}{4}(0.016)^2}\right]^2$$

$$= 38.65 \text{ kJ/m}^3$$

Strain energy in the entire rod,

$$U = u_{AB}V_{AB} + u_{BC}V_{BC}$$

$$= 7.63\times 10^3\times\left[\frac{\Pi}{4}(0.024)^2\times 1\right] + 38.65\times 10^3\times\left[\frac{\Pi}{4}(0.016)^2\times 0.8\right]$$

$$U = 9.67\text{J}$$

The load that will produce yield stress in the material,

$$P = \sigma_y A_{BC} = 320\times 10^6\times\frac{\Pi}{4}(0.016)^2$$

$$P = 64.3\text{kN}$$

Maximum energy that can be stored in the rod,

$$U = \frac{1}{2E}\left[\left(\frac{P}{A_{AB}}\right)^2 \times V_{AB} + \left(\frac{P}{A_{BC}}\right)^2 \times V_{BC}\right]$$

$$= \frac{P^2}{2E}\left[\frac{L_{AB}}{A_{AB}} + \frac{L_{BC}}{A_{BC}}\right]$$

$$= \frac{\left(64.3\times10^3\right)^2}{2\times200\times10^9}\left[\frac{1}{\frac{\Pi}{4}(0.024)^2} + \frac{0.8}{\frac{\Pi}{4}(0.016)^2}\right]$$

$$= 63.97 J$$

Top

1.16 Impact loading

A static loading is applied very slowly so that the external load and the internal force are always in equilibrium. Hence, the vibrational and dynamic effects are negligible in static loading.

Dynamic loading may take many forms like fluctuating loads where the loads are varying with time and impact loads where the loads are applied suddenly and may be removed immediately or later.

Collision of two bodies and objects freely falling onto a structure are some of the examples of impact loading.

Consider a collar of mass M at a height h from the flange that is rigidly fixed at the end of a bar as shown in figure 1.33.

As the collar freely falls onto the flange, the bar begins to elongate causing axial stresses and strain within the bar.

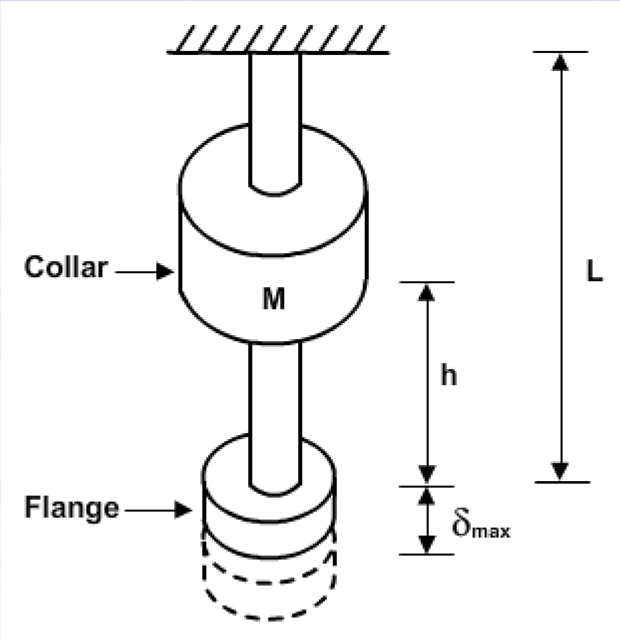


Figure 1.33

After the flange reaching its maximum position during downward motion, it moves up due to shortening of the bar.

The bar vibrates in the axial direction with the collar and the flange till the vibration dies out completely due to damping effects.

To simplify the complex impact loading analysis, the following assumptions are made.

The kinetic energy of the collar at the time of striking is completely transformed into strain energy and stored in the bar.

But in practice, not all the kinetic energy is stored in the material as some of the energy is dissipated out as heat and waves.

Hence, this assumption is conservative in the sense that the stress and deflection predicted by this way are higher than the actual values.

The second assumption is that after striking the flange, the collar and the flange move downward together without any bouncing.

This assumption is reasonable provided the weight of the collar is much larger than that of the bar.

The third assumption is that the stresses in the bar remain within linear elastic range and the stress distribution is uniform within the bar.

But, in reality, the stress distribution is not uniform since the stress waves generated due to impact loading travel through the bar.

Using the principle of conservation of energy, the kinetic energy of the collar is equated to the strain energy of the bar.

Assuming the height of fall h is much larger than the deformation of rod, and using equation 1.34,

$$\frac{1}{2}Mv^2 = \frac{\sigma_{max}^2 V}{2E} \qquad 1.36$$

where v is the velocity of the collar at strike ($v = \sqrt{2gh}$) and V is the volume of the material.

The maximum stress in the bar due to the impact load of mass M,

$$\sigma_{max} = \sqrt{\frac{Mv^2 E}{V}} \qquad 1.37$$

From above equation, it becomes clear that by increasing the volume of material, the effect of impact loading can be minimized.

Expressing strain energy in terms of deflection in equation 1.36,

$$\frac{1}{2}Mv^2 = \frac{\delta_{max}^2 EA}{2L}$$

$$\delta_{max} = \sqrt{\frac{Mv^2L}{EA}} \qquad 1.38$$

If the load of the collar is applied gradually on the bar i.e., under static loading, the static deflection δ_{st} will be,

$$\delta_{st} = \frac{MgL}{EA}$$

Substituting this in equation 1.38, relationship between the static deflection δ_{st} and the impact deflection δ_{max} is obtained.

$$\delta_{max} = \sqrt{2h\delta_{st}} \qquad 1.39$$

To represent the magnification of deflection due to impact load compared to that of static deflection for the same load, *impact factor* is used.

$$\text{Impact factor} = \frac{\delta_{max}}{\delta_{st}} \qquad 1.40$$

Alternately, the impact factor can be obtained from the ratio $\frac{\sigma_{max}}{\sigma_{st}}$.

The relationship between the stress σ_{st} developed in the bar due to static loading and the impact loading stress σ_{max} is determined as follows.

$$\sigma_{st} = E\varepsilon_{st} = \frac{E\delta_{st}}{L}$$

$$\sigma_{max} = E\varepsilon_{max} = \frac{E\delta_{max}}{L}$$

$$\sigma_{max} = \frac{E}{L}\sqrt{2h\delta_{st}}$$

$$\sigma_{max} = \sqrt{\frac{2hE\sigma_{st}}{L}} \qquad 1.41$$

Now, the effect of suddenly applied loads on materials or structures that forms a special case of impact loading is discussed.

In figure 1.33, if the collar is brought into contact on the top of the flange and released immediately, it is referred to as suddenly applied load.

The maximum stress produced in the bar due to suddenly applied load can be determined by replacing h by δ_{max} in equation 1.41

$$\sigma_{max} = 2\sigma_{st} \qquad 1.42$$

Hence, the stress developed in a material due to suddenly applied load is twice as large as that of gradually applied load.

Example 9:

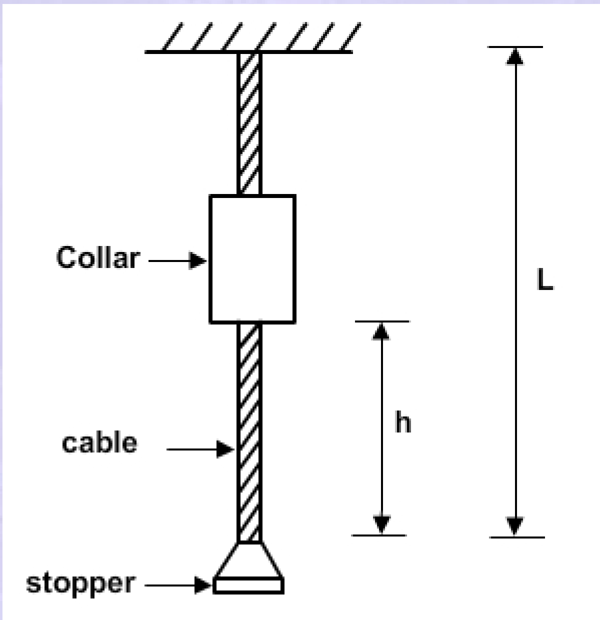


Figure 1.34

A 50 kg collar is sliding on a cable as shown in figure 1.34 from a height h = 1m. Its free fall is restrained by a stopper at the end of the cable. The effective cross-sectional area and the elastic modulus of the cable are taken to be 60 mm^2 and 150GPa respectively. If the maximum allowable stress in the cable due to impact load is 450MPa, calculate the minimum permissible length for the cable and the corresponding maximum deflection. Also find the impact factor.

Solution:

Maximum stress due to impact load, $\sigma_{max} = \sqrt{\dfrac{Mv^2E}{V}}$

Velocity, $v = \sqrt{2gh} = \sqrt{2 \times 9.81 \times 1} = 4.43 \text{ m/s}$

$$450 \times 10^6 = \sqrt{\frac{50 \times (4.43)^2 \times 150 \times 10^9}{\left(60 \times 10^{-6}\right) \times L}}$$

Minimum permissible length for the cable, L = 12.1 m

Static deflection, $\delta_{st} = \frac{MgL}{EA}$

$$= \frac{50 \times 9.81 \times 12.1}{150 \times 10^9 \times 60 \times 10^{-6}}$$

$= 0.656$ mm

Maximum deflection, $\delta_{max} = \sqrt{2h\delta_{st}}$

$$= \sqrt{2 \times 1000 \times 0.656}$$

$= 36$ mm

Impact factor, $= \frac{\delta_{max}}{\delta_{st}} = \frac{36}{0.656}$

$= 55$

Top

Stresses

Introduction

2.1 stress at a point

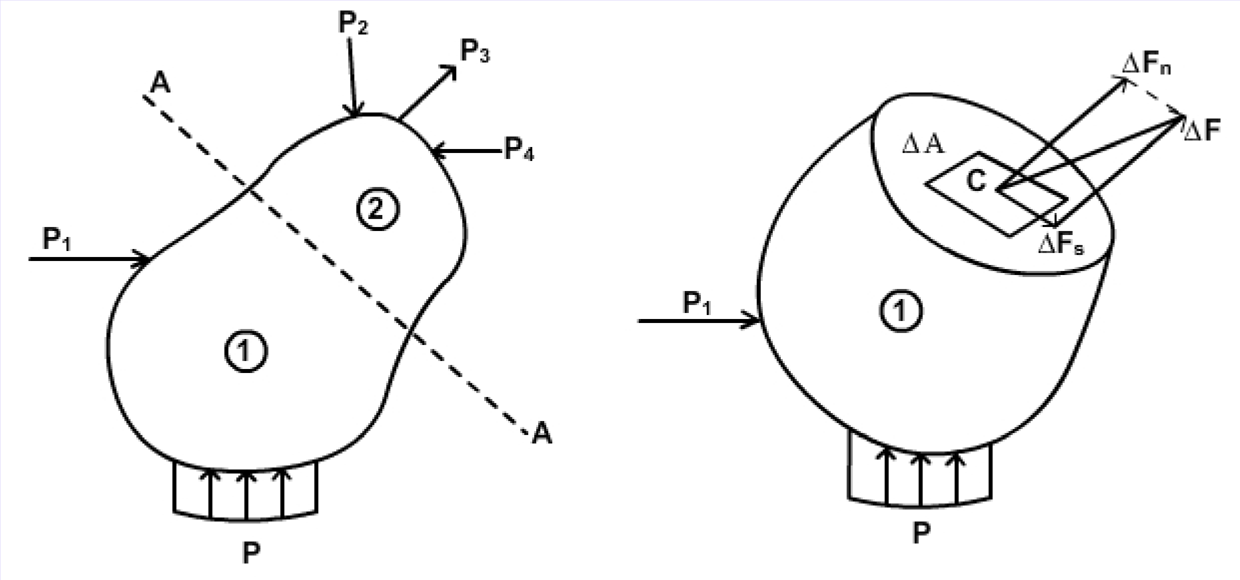


Figure 2.1

Consider a body in equilibrium under point and traction loads as shown in figure 2.1.

After cutting the body along section AA, take an infinitesimal area ΔA lying on the surface consisting a point C.

The interaction force between the cut sections 1 & 2, through ΔA is ΔF. Stress at the point C can be defined,

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \qquad 2.1$$

ΔF is resolved into ΔF_n and ΔF_s that are acting normal and tangent to ΔA.

$$\text{Normal stress, } \sigma_n = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A} \qquad 2.2$$

$$\text{Shear Stress, } \sigma_s = \lim_{\Delta A \to 0} \frac{\Delta F_s}{\Delta A} \qquad 2.3$$

Top

2.2 stress Tensor

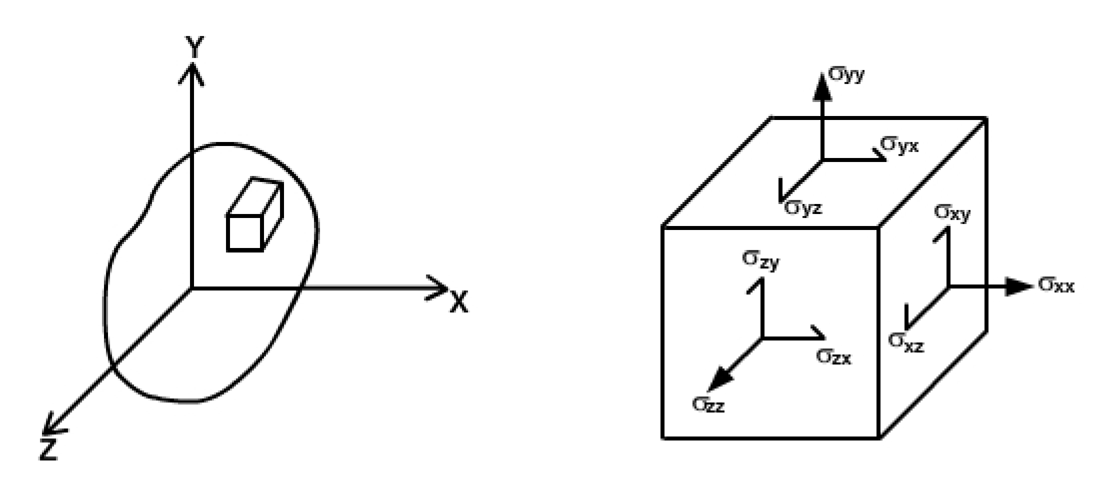


Figure 2.2

Consider the free body diagram of an infinitesimally small cube inside the continuum as shown in figure 2.2.

Stress on an arbitrary plane can be resolved into two shear stress components parallel to the plane and one normal stress component perpendicular to the plane.

Thus, stresses acting on the cube can be represented as a second order tensor with nine components.

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad 2.4$$

Is stress tensor symmetric?

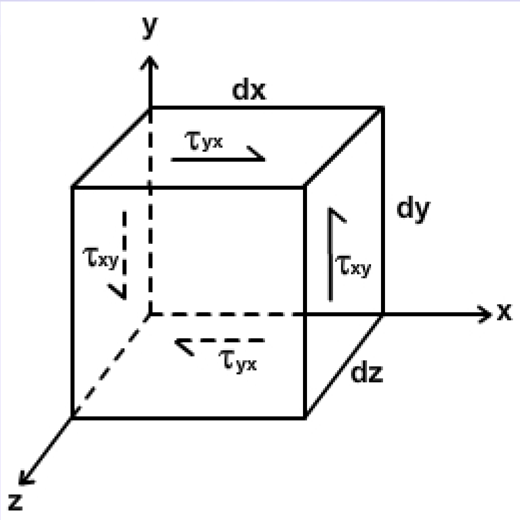


Figure 2.3

Consider a body under equilibrium with simple shear as shown in figure 2.3.

Taking moment about z axis,

$$M_z = \left(\tau_{yx} d_x d_z\right) d_y - \left(\tau_{xy} d_y d_z\right) d_x = 0$$

$$\boxed{\tau_{xy} = \tau_{yx}}$$

Similarly, $\tau_{xz} = \tau_{zx}$ and $\tau_{yz} = \tau_{zy}$.

Hence, the stress tensor is symmetric and it can be represented with six components, $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{xz}$ and τ_{yz}, instead of nine components.

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix} \qquad 2.5$$

Top

2.3 Equations of Equilibrium

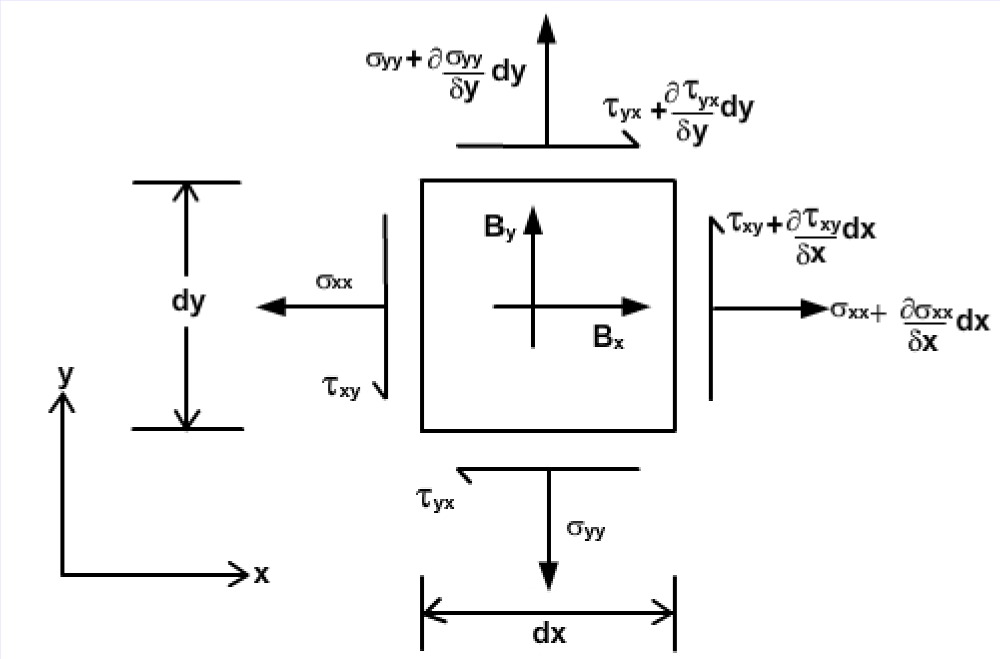


Figure 2.4

Consider an infinitesimal element of a body under equilibrium with sides $dx \times dy \times 1$ as shown in figure 2.4.

B_x, B_y are the body forces like gravitational, inertia, magnetic, etc., acting on the element through its centre of gravity.

$\sum F_x = 0$,

$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx\right)(d_y \times 1) - \sigma_{xx}(dy \times 1) + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{dy} dy\right)(dx \times 1) - \tau_{yx}(dx \times 1) + B_x(dx \times dy \times 1) = 0$$

Similarly taking $\sum F_y = 0$ and simplifying, equilibrium equations of the element in differential form are obtained as,

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + B_x = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + B_y = 0 \qquad 2.6$$

Extending this derivation to a three dimensional case, the differential equations of equilibrium become,

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+B_x=0$$
$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+B_y=0 \qquad 2.7$$
$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+B_z=0$$

When the right hand side of above equations is not equal to zero, then, they become equations of motion.

Equations of equilibrium are valid regardless of the materials nature whether elastic, plastic, viscoelastic etc.

In equation 2.7, since there are three equations and six unknowns (realizing $\tau_{xy}=\tau_{yx}$ and so on), all problems in stress analysis are statically indeterminate.

Hence, to solve for the unknown stresses, equilibrium equations are supplemented with kinematic requirements and constitutive equations.

Top

2.4 Different states of stress

Depending upon the state of stress at a point, we can classify it as uniaxial(1D), biaxial(2D) and triaxial(3D) stress.

2.4.1 One dimensional stress(Uniaxial)

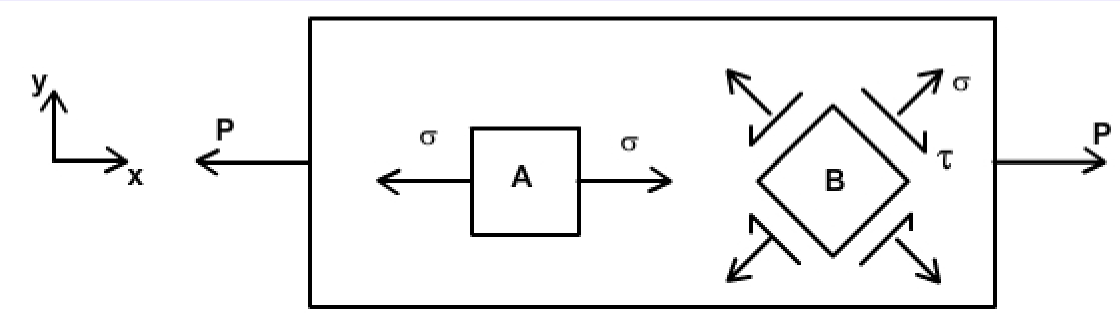


Figure 2.5

Consider a bar under a tensile load P acting along its axis as shown in figure 2.5.

Take an element A which has its sides parallel to the surfaces of the bar.

It is clear that the element has only normal stress along only one direction, i.e., x axis and all other stresses are zero. Hence it is said to be under uni-axial stress state.

Now consider another element B in the same bar, which has its slides inclined to the surfaces of the bar.

Though the element has normal and shear stresses on each face, it can be transformed into a uni-axial stress state like element A by transformation of stresses (will be discussed in section 2.5).

Hence, if the stress components at a point can be transformed into a single normal stress (principal stress as will be discussed later), then, the element is under uni-axial stress state.

Is an element under pure shear uni-axial?

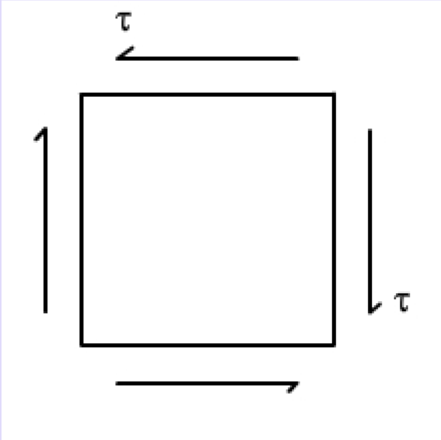


Figure 2.6

The given stress components cannot be transformed into a single normal stress along an axis but along two axes. Hence this element is under biaxial / two dimensional stress state.

2.4.2 Two dimensional stress (Plane stress)

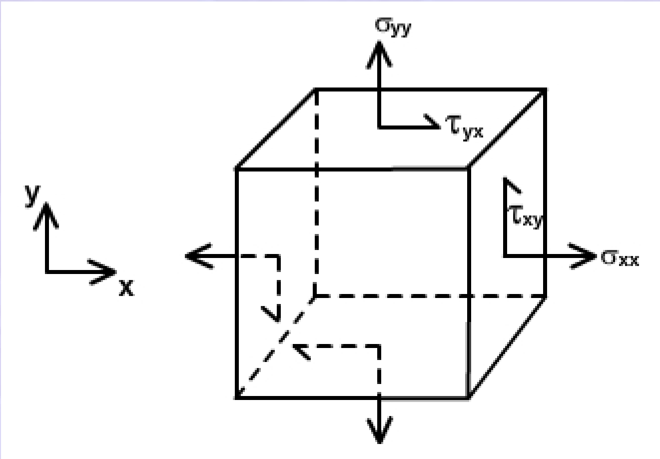


Figure 2.7

When the cubic element is free from any of the stresses on its two parallel surfaces and the stress components in the element can not be reduced to a uni-axial stress by transformation, then, the element is said to be in two dimensional stress/plane stress state. Thin plates under mid plane loads and the free surface of structural elements may experience plane stresses as shown below.

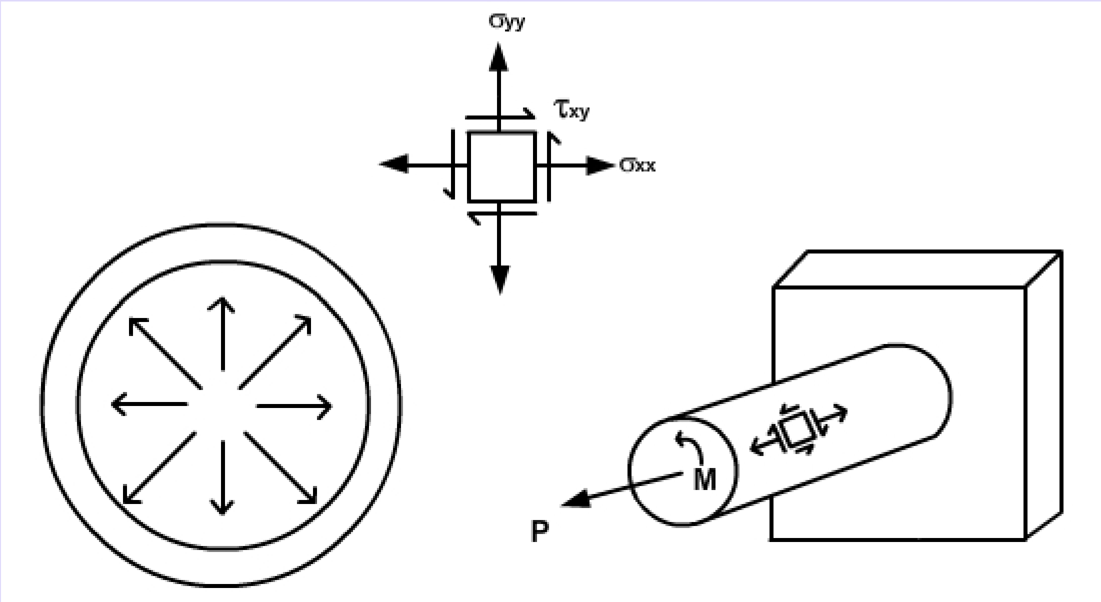


Figure 2.8

Top

2.5 Transformation of plane stress

Though the state of stress at a point in a stressed body remains the same, the normal and shear stress components vary as the orientation of plane through that point changes.

Under complex loading, a structural member may experience larger stresses on inclined planes then on the cross section.

The knowledge of maximum normal and shear stresses and their plane's orientation assumes significance from failure point of view.

Hence, it is important to know how to transform the stress components from one set of coordinate axes to another set of co-ordinates axes that will contain the stresses of interest.

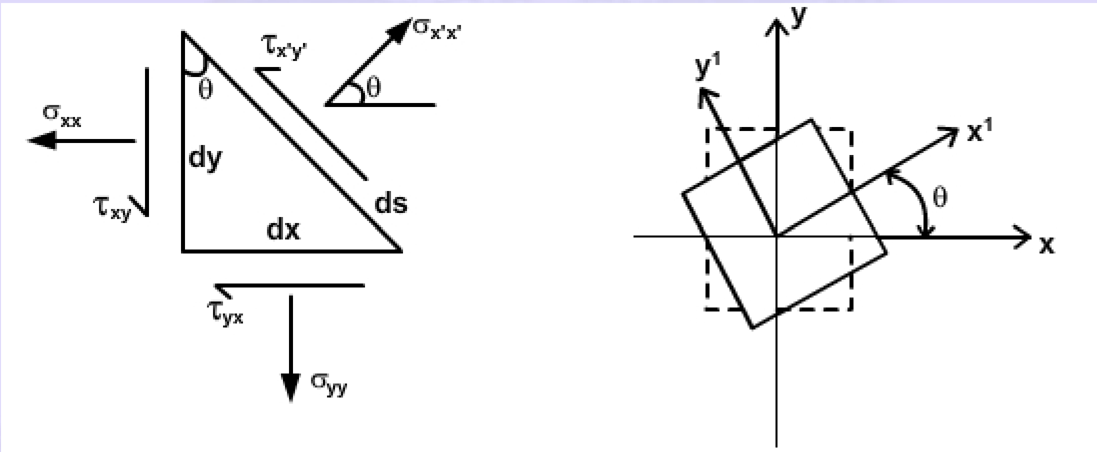


Figure 2.9

Consider a prismatic element with sides dx, dy and ds with their faces perpendicular to y, x and x' axes respectively. Thickness of the element is t.

$\sigma_{x'x'}$ and $\tau_{x'y'}$ are the normal and shear stresses acting on a plane inclined at an angle θ measured counter clockwise from x plane.

Under equilibrium, $\sum F_{x'} = 0$

$$\sigma_{x'x'}.t.ds - \sigma_{xx}.t.d_y.\cos\theta - \sigma_{yy}.t.d_x.\sin\theta - \tau_{xy}.t.d_y.\sin\theta - \tau_{yx}.t.d_x.\cos\theta = 0$$

Dividing above equation by t.ds and using $\frac{dy}{ds} = \cos\theta$ and $\frac{dx}{ds} = \sin\theta$,

$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

Similarly, from $\sum F_{y'} = 0$ and simplifying,

$$_{x'y'} = \left(\sigma_{yy} - \sigma_{xx}\right)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

Using trigonometric relations and simplifying,

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\tau_{x'y'} = \frac{-\left(\sigma_{xx} - \sigma_{yy}\right)}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \qquad 2.8$$

Replacing θ by $\theta + 90^0$, in $\sigma_{x'x'}$ expression of equation 2.8, we get the normal stress along y' direction.

$$\sigma_{y'y'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \qquad 2.9$$

Equations 2.8 and 2.9 are the transformation equations for plane stress using which the stress components on any plane passing through the point can be determined.

Notice here that,

$$\sigma_{xx} + \sigma_{yy} = \sigma_{x'x'} + \sigma_{y'y'} \qquad 2.10$$

Invariably, the sum of the normal stresses on any two mutually perpendicular planes at a point has the same value. This sum is a function of the stress at that point and not on the orientation of axes. Hence, this quantity is called *stress invariant* at that a point

Top

2.6 Principal stresses and maximum shear stress

From transformation equations, it is clear that the normal and shear stresses vary continuously with the orientation of planes through the point.

Among those varying stresses, finding the maximum and minimum values and the corresponding planes are important from the design considerations.

By taking the derivative of $\sigma_{x'x'}$ in equation 2.8 with respect to θ and equating it to zero,

$$\frac{d\sigma_{x'x'}}{d\theta} = -\left(\sigma_{xx} - \sigma_{yy}\right)\sin 2\theta + 2\tau_{xy}\cos 2\theta = 0$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \qquad 2.11$$

Here, θ_p has two values θ_{p1}, and θ_{p2} that differ by 90^0 with one value between 0^0 and 90^0 and the other between 90^0 and 180^0.

These two values define the *principal planes* that contain maximum and minimum stresses.

Substituting these two θ_p values in equation 2.8, the maximum and minimum stresses, also called as *principal stresses*, are obtained.

$$\sigma_{max,min} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau^2_{xy}} \qquad 2.12$$

The plus and minus signs in the second term of equation 2.12, indicate the algebraically larger and smaller principal stresses, i.e. maximum and minimum principal stresses.

In the second equation of 2.8, if $\tau_{x'y'}$ is taken as zero, then the resulting equation is same as equation 2.11.

Thus, the following important observation pertained to principal planes is made.

The shear stresses are zero on the principal planes

To get the maximum value of the shear stress, the derivative of $\tau_{x'y'}$ in equation 2.8 with respect to θ is equated to zero.

$$\frac{d\tau_{x'y'}}{d\theta} = -\left(\sigma_{xx} - \sigma_{yy}\right)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\tan 2\theta_s = \frac{-\left(\sigma_{xx} - \sigma_{yy}\right)}{2\tau_{xy}} \qquad 2.13$$

Hence, θ_s has two values, θ_{s1} and θ_{s2} that differ by 90^0 with one value between 0^0 and 90^0 and the other between 90^0 and 180^0.

Hence, the maximum shear stresses that occur on those two mutually perpendicular planes are equal in algebraic value and are different only in sign due to its complementary property.

Comparing equations 2.11 and 2.13,

$$\tan 2\theta_p = -\frac{1}{\tan 2\theta_s} \qquad 2.14$$

It is understood from equation 2.14 that the tangent of the angles $2\theta_p$ and $2\theta_s$ are negative reciprocals of each other and hence, they are separated by 90^0.

Hence, we can conclude that θ_p and θ_s differ by 45^0, i.e., the maximum shear stress planes can be obtained by rotating the principal plane by 45^0 in either direction.

A typical example of this concept is given in figure 2.10.

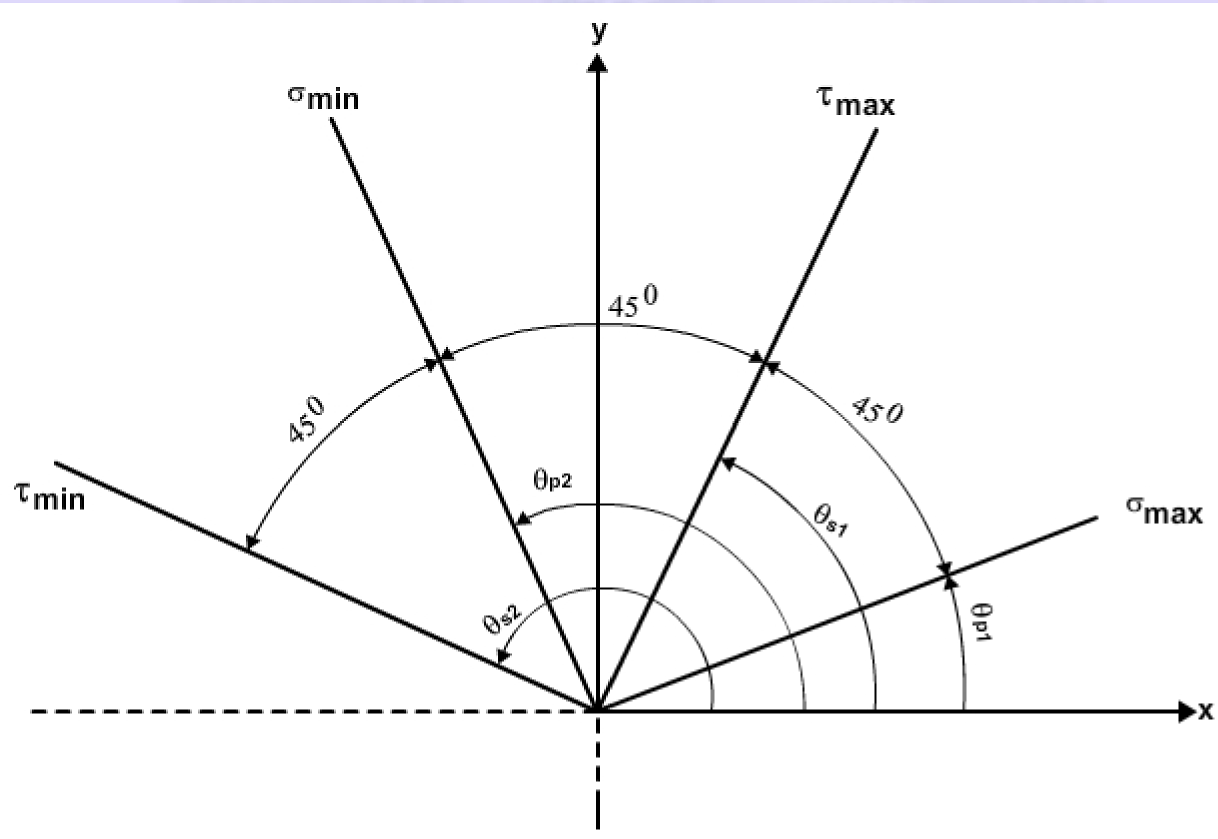


Figure 2.10

The principal planes do not contain any shear stress on them, but the maximum shear stress planes may or may not contain normal stresses as the case may be.

Maximum shear stress value is found out by substituting θ_s values in equation 2.8.

$$\tau_{max} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau^2_{xy}} \quad 2.15$$

Another expression for τ_{max} is obtained from the principal stresses,

$$\tau_{max} = \frac{\sigma_{max} - \sigma_{min}}{2} \quad 2.16$$

Example: (Knowledge in torsion and bending is necessary)

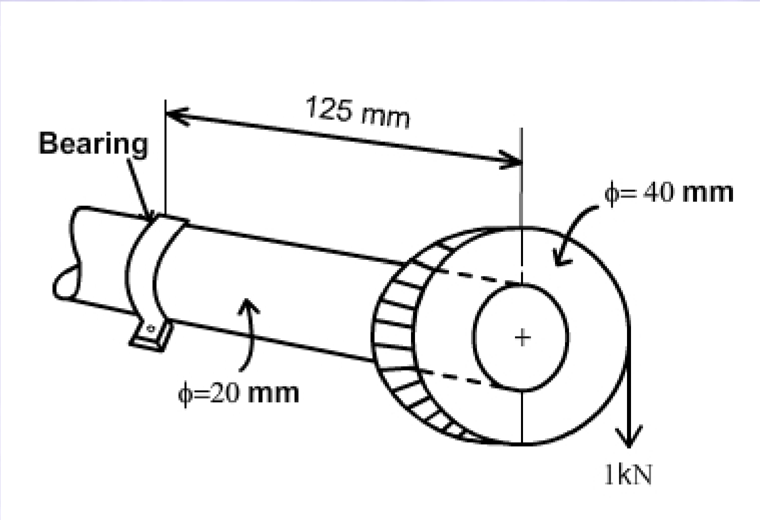
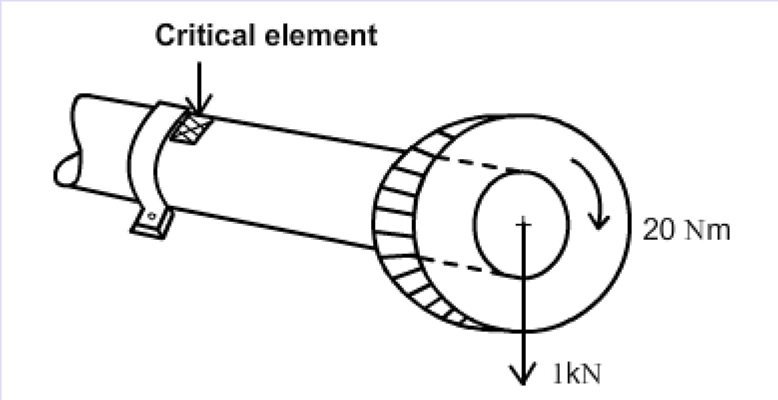


Figure 2.11

A gear with a shaft is used to transmit the power as shown in figure 2.11. The load at the gear tooth is 1kN. The diameter of the gear and the shaft are 40 mm and 20 mm respectively. Find the principal stresses and the maximum shear stress on an element, which you feel important on the shaft.

Solution:

The critical element of design interest lies on the top of the shaft, near to the bearing.

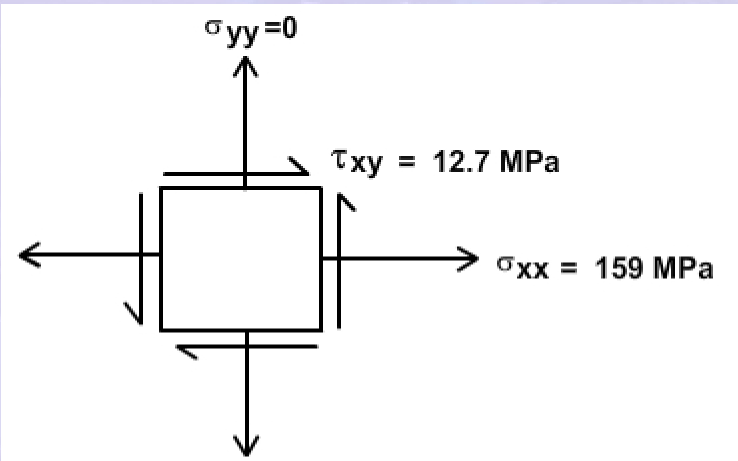


Transfer the load from the end of the gear to the centre of the shaft as a force-couple system with 1kN force and 20 N-m couple clockwise.

Bending stress, $\sigma_{max} = \frac{M}{I}y = \frac{(1\times10^3)\times(0.125)}{\Pi(0.02)^4/64}$

$= 159 \text{ MPa}$

Shearing stress, $\tau_{xy} = \frac{T}{J}r = \frac{20\times(0.01)}{\Pi(0.02)^4/32} = 12.7\text{Mpa}$



From equation 2.11, the principal plane,

$$\tan 2\theta_p = \frac{2\times12.7}{(159-0)}$$

$$\theta_{p1} = 4.5^0; \theta_{p2} = 90+4.5^0 = 94.5^0$$

Using equation 2.8, the principal stresses,

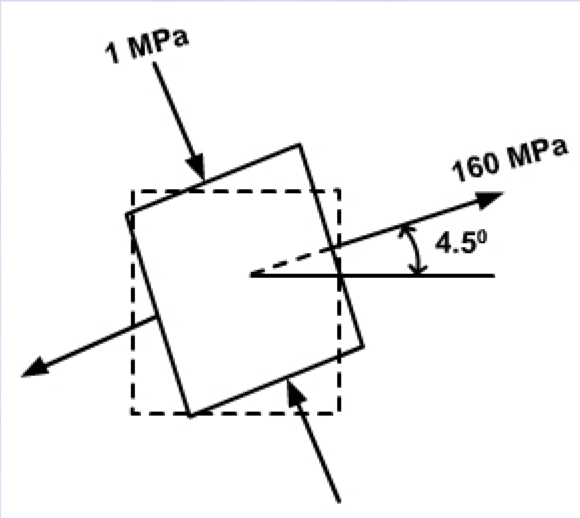
$$\sigma_{max} = \frac{(159+0)}{2} + \frac{(159-0)}{2}\cos\left(2\times 4.5^0\right) + 12.7\sin\left(2\times 4.5^0\right)$$
$$= 160 \text{ MPa}$$
$$\sigma_{min} = \frac{(159+0)}{2} + \frac{(159-0)}{2}\cos\left(2\times 94.5^0\right) + 12.7\sin\left(2\times 94.5^0\right)$$
$$= \text{-1 MPa}$$

Alternatively, using equation 2.12,

$$\sigma_{max} = \left(\frac{159+0}{2}\right) + \sqrt{\left(\frac{159+0}{2}\right) + (12.7)^2}$$
$$= 160 \text{ MPa}$$
$$\sigma_{min} = \text{-1 MPa}$$

Maximum Shear Stress, $\tau_{max} = \frac{160+1}{2} = 80.5 \text{ MPa}$

The principal planes and the stresses acting on them are shown below.



Top

2.7 Mohr's circle for plane stress

The transformation equations of plane stress 2.8 can be represented in a graphical form which is popularly known as *Mohr's circle*.

Though the transformation equations are sufficient to get the normal and shear stresses on any plane at a point, with Mohr's circle one can easily visualize their variation with respect to plane orientation θ.

Besides stress plots, Mohr's circles are used to plot strains, moment of inertia, etc., which follow the same transformation laws as do stresses.

2.7.1 Equations of Mohr's circle

Recalling transformation equations 2.8 and rearranging the terms

$$\begin{aligned}\sigma_{x'x'} - \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) &= \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \\ \tau_{x'y'} &= -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta\end{aligned} \qquad 2.17$$

A little consideration will show that the above two equations are the equations of a circle with $\sigma_{x'x'}$ and $\tau_{x'y'}$ as its coordinates and 2θ as its parameter.

If the parameter 2θ is eliminated from the equations, then the significance of them will become clear.

Squaring and adding equations 2.17 results in,

$$\left[\sigma_{x'x'} - \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right)\right]^2 + \tau^2_{x'y'} = \left[\frac{\sigma_{xx}-\sigma_{yy}}{2}\right]^2 + \tau^2_{xy} \qquad 2.18$$

For simple representation of above equation, the following notations are used.

$$\sigma_{ave} = \frac{\sigma_{xx}+\sigma_{yy}}{2};\ r = \left[\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau^2_{xy}\right]^{1/2} \qquad 2.19$$

Equation 2.18 can thus be simplified as,

$$\left(\sigma_{x'x'} - \sigma_{ave}\right)^2 + \tau^2_{x'y'} = r^2 \qquad 2.20$$

Equation 2.20 represents the equation of a circle in a standard form. This circle has $\sigma_{x'x'}$ as its abscissa and $\tau_{x'y'}$ as its ordinate with radius r.

The coordinate for the centre of the circle is $\sigma_{x'x'} = \sigma_{ave}$ and $\tau_{x'y'} = 0$.

2.7.2 Construction procedure

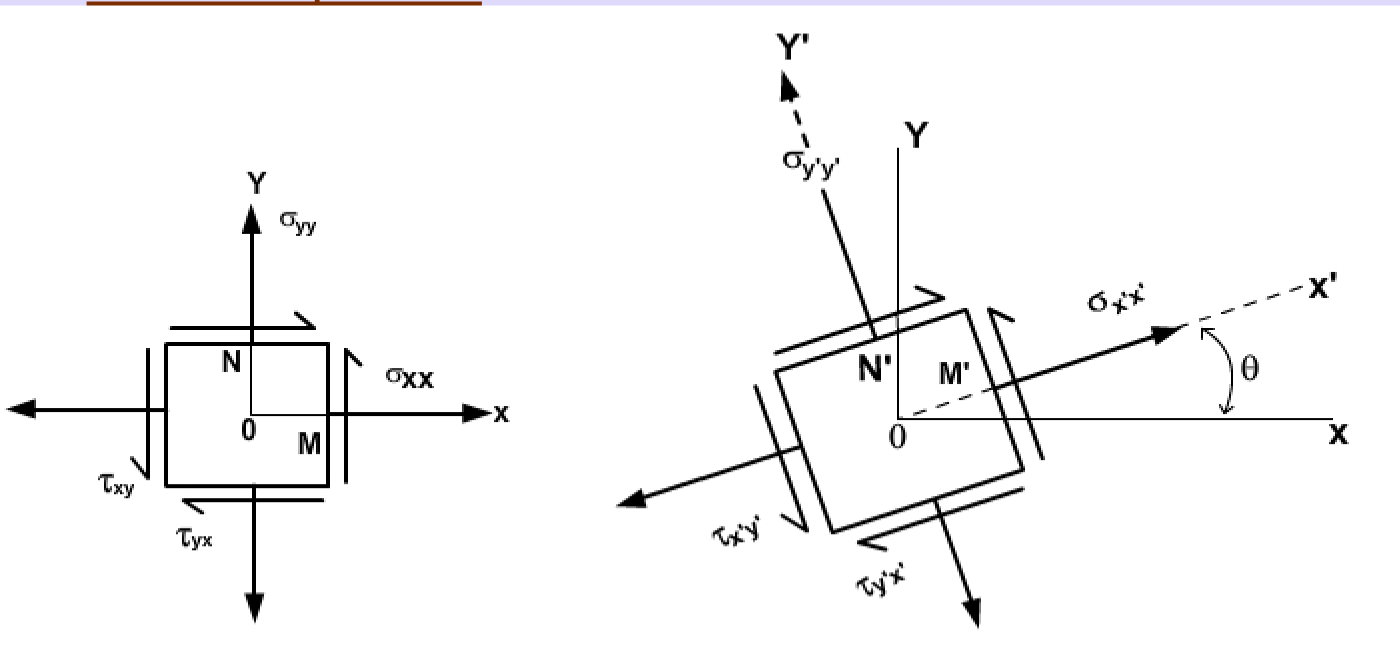


Figure 2.12

Sign convention: Tension is positive and compression is negative. Shear stresses causing clockwise moment about O are positive and counterclockwise negative.

Hence, τ_{xy} is negative and τ_{yx} is positive.

Mohr's circle is drawn with the stress coordinates σ_{xx} as its abscissa and τ_{xy} as its ordinate, and this plane is called the stress plane.

The plane on the element in the material with xy coordinates is called the physical plane.

Stresses on the physical plane M is represented by the point M on the stress plane with σ_{xx} and τ_{xy} coordinates.

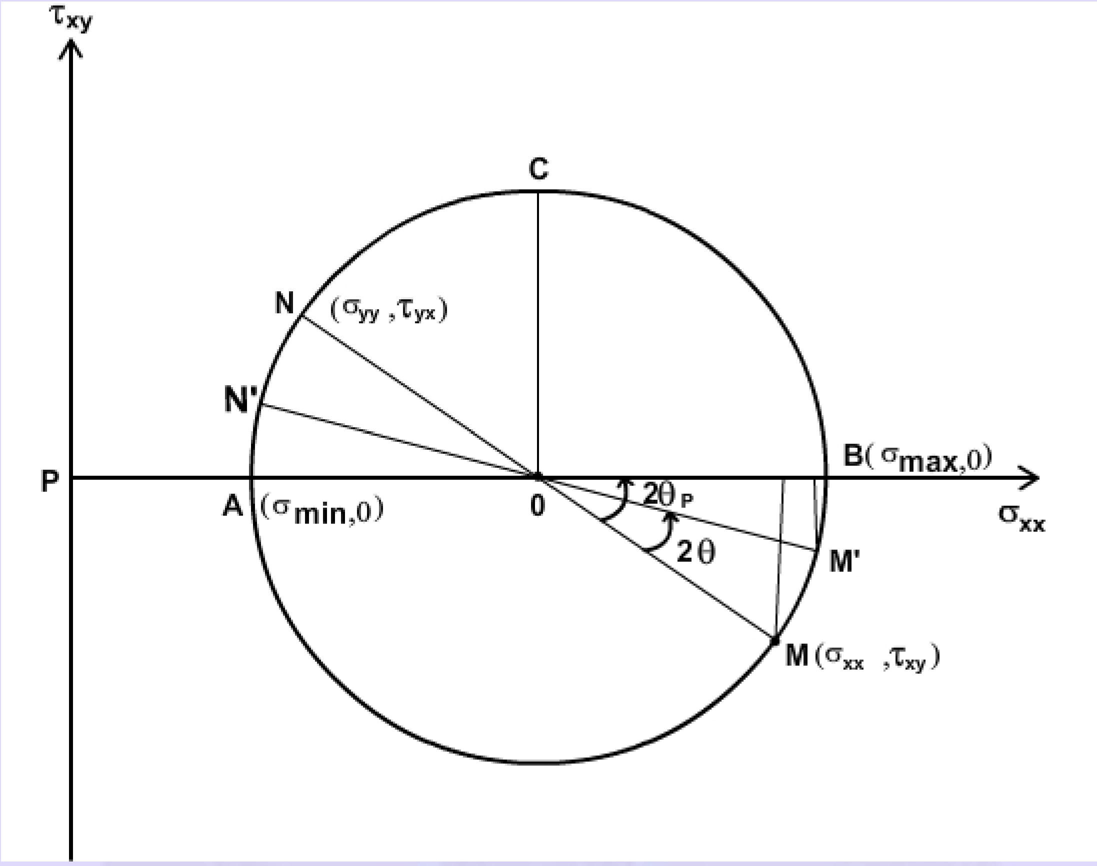


Figure 2.13

Stresses on the physical plane N, which is normal to M, is represented by the point N on the stress plane with σ_{yy} and τ_{yx}.

The intersecting point of line MN with abscissa is taken as O, which turns out to be the centre of circle with radius OM.

Now, the stresses on a plane which makes θ^0 inclination with x axis in physical plane can be determined as follows. Let that plane be M'.

An important point to be noted here is that a plane which has a θ^0 inclination in physical plane will make $2\theta^0$ inclination in stress plane.

Hence, rotate the line OM in stress plane by $2\theta^0$ counterclockwise to obtain the plane M'.

The coordinates of M' in stress plane define the stresses acting on plane M' in physical plane and it can be easily verified.

$$\sigma_{x'x'} = PO + r\cos\left(2\theta_p - 2\theta\right)$$
$$\sigma_{x'x'} = PO + r\left[\cos 2\theta_p \cos 2\theta + \sin 2\theta_p \sin 2\theta\right] \qquad 2.21$$

Where $PO = \frac{\sigma_{xx} + \sigma_{yy}}{2}$

$$r = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau^2_{xy}}$$

$$\cos 2\theta_p = \frac{\sigma_{xx} - \sigma_{yy}}{2r}$$

$$\sin 2\theta_p = \frac{\tau_{xy}}{2r}$$

Rewriting equation 2.21,

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \qquad 2.22$$

Equation 2.22 is same as equation 2.8.

This way it can be proved for shear stress $\tau_{x'y'}$ on plane M' (do it yourself).

Extension of line M'O will get the point N' on the circle which coordinate gives the stresses on physical plane N' that is normal to M'.

This way the normal and shear stresses acting on any plane in the material can be obtained using Mohr's circle.

Points A and B on Mohr's circle do not have any shear components and hence, they represent the principal stresses,

$$\sigma_{max,min} = PO \pm r = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau^2_{xy}}$$

The principal plane orientations can be obtained in Mohr's circle by rotating the line OM by $2\theta_p$ and $2\theta_p$+180^0 clockwise or counterclockwise as the case may be (here it is counter clock wise) in order to make that line be aligned to axis τ_{xy}=0.

These principal planes on the physical plane are obtained by rotating the plane m, which is normal to x axis, by θ_p and θ_p+90^0 in the same direction as was done in stress plane.

The maximum shear stress is defined by OC in Mohr's circle,

$$\tau_{max} = \pm r = \pm\sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau^2_{xy}}$$

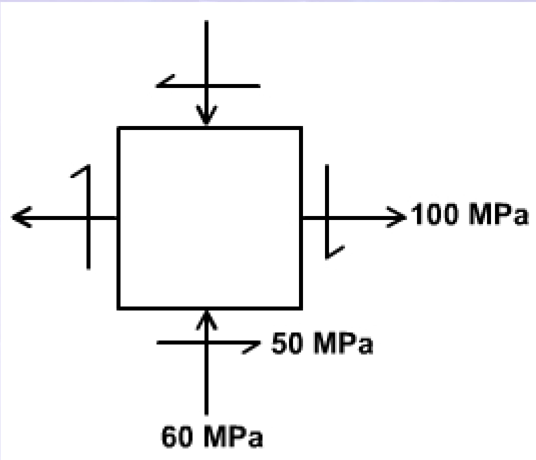
(or)

$$\tau_{max} = \pm\left(\frac{\sigma_{max} - \sigma_{min}}{2}\right)$$

It is important to note the difference in the sign convention of shear stress between analytical and graphical methods, i.e., physical plane and stress plane. The shear stresses that will rotate the element in counterclockwise direction are considered positive in the analytical method and negative in the graphical method. It is assumed this way in order to make the rotation of the elements consistent in both the methods.

Example:

For the state of plane stress given, determine the principal planes, the principal stresses and the maximum shear stress. Also find the stress components on the element after it is rotated by 20^0 counterclockwise.



Solution:

Analytical solution:

$$\sigma_{max,min} = \frac{100 + (-60)}{2} \pm \sqrt{\left[\frac{100 - (-60)}{2}\right]^2 + (-50)^2}$$
$$= 20 \pm 94.34 \text{ MPa}$$

Maximum principal stress = 114.34 MPa

Minimum principal stress = -74.4 MPa

Principal planes, $\tan 2\theta_p = \frac{2 \times (-50)}{100 - (-60)}$

θ_{P1} = -16^0

θ_{P2} = -16^0+90^0=74^0

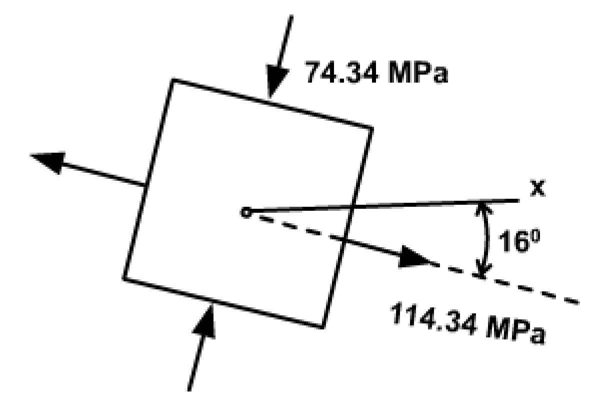
$$\text{Maximum shear stress} = \pm\left[\frac{114.34 - (-74.34)}{2}\right]$$
$$= \pm 94.34 \text{ MPa}$$

Normal stresses on the element after rotation by 20^0 counterclockwise

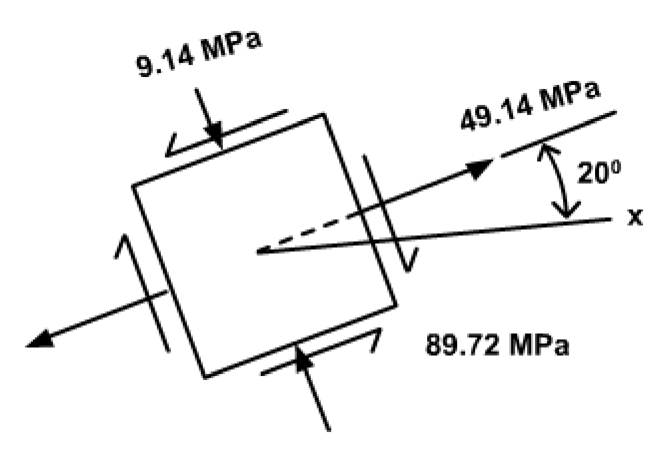
$$\sigma_{x'x'} = \frac{100+(-60)}{2} + \frac{100-(-60)}{2}\cos\left(2\times 20^0\right) + (-50)\sin\left(2\times 20^0\right)$$
$$= 49.14 \text{ MPa}$$

$$\sigma_{y'y'} = \frac{100+(-60)}{2} + \frac{100-(-60)}{2}\cos\left(2\times 110^0\right) + (-50)\sin\left(2\times 110^0\right)$$
$$= -9.14 \text{ MPa}$$

$$\tau_{x'y'} = -\left[\frac{100-(-60)}{2}\right]\sin\left(40^0\right) - 50\cos\left(40^0\right)$$
$$= -89.72 \text{ MPa}$$



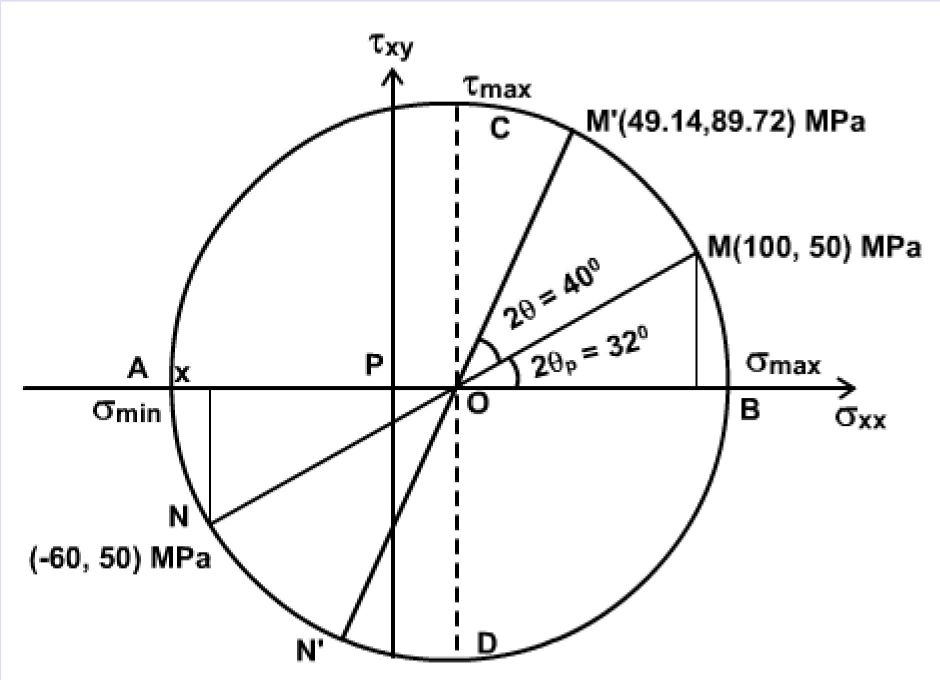
Principal stresses on the element



Stresses on the element after rotation by 20^0

II. Graphical Method:

Using the procedure discussed in the previous section, Mohr's circle is constructed as below.



$$\sigma_{max} = PB = 114.34 \text{ MPa}$$
$$\sigma_{min} = PA = -74.34 \text{ MPa}$$
$$\tau_{max} = OC \text{ or } OD = \pm 94.34 \text{ MPa}$$

Stresses on the element after rotating by 20^0 counterclockwise,

$$\sigma_{x'x'} = \text{abscissa of PM}' = 49.14 \text{ MPa}$$
$$\sigma_{y'y'} = \text{abscissa of PN}' = -9.14 \text{ MPa}$$
$$\tau_{x'y'} = \text{ordinate PM}' = 89.72 \text{ MPa}$$

Top

Torsion

Click here to check the Animation

Introduction:

Detailed methods of analysis for determining stresses and deformations in axially loaded bars were presented in the first two chapters. Analogous relations for members subjected to torque about their longitudinal axes are developed in this chapter. The constitutive relations for shear discussed in the preceding chapter will be employed for this purpose. The investigations are confined to the effect of a single type of action, i.e., of a torque causing a twist or torsion in a member.

The major part of this chapter is devoted to the consideration of members having circular cross sections, either solid or tubular. Solution of such elastic and inelastic problems can be obtained using the procedures of engineering mechanics of solids. For the solution of torsion problems having noncircular cross sections, methods of the mathematical theory of elasticity (or finite elements) must be employed. This topic is briefly discussed in order to make the reader aware of the differences in such solutions from that for circular members. Further, to lend emphasis to the difference in the solutions discussed, this chapter is subdivided into four distinct parts. It should be noted, however, that in practice, members for transmitting torque, such as shafts for motors, torque tubes for power equipment, etc., are predominantly circular or tubular in cross section. Therefore, numerous applications fall within the scope of the formulas derived in this chapter.
In this section, discussion is limited to torsion of circular bars only.

Top

Basic Assumptions

a. *Plane sections perpendicular to the axis of a circular member before application of torque remains plane even after application of torque.*

b. *Shear strains vary linearly from the central axis reaching a maximum value at the outer surface.*

c. *For linearly elastic material, Hooke's law is valid. Hence shear stress is proportional to shear strain.*

Top

Torsion Formula

Since shear strains varies linearly across the section,

$$\gamma = \frac{\gamma_{max}}{C} R$$

where γ is the shear strains at a point of raidus R, C is the radius of the member.

$$\therefore \text{ Torque, } T = \int_A \tau R \, dA$$

$$= \int_A G\gamma R \, dA$$

where $\tau = G\,\gamma$, the shear stress at any point at a distance R (Refer Figure 6.1)

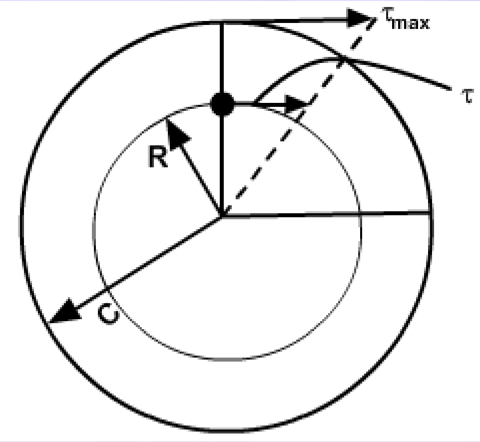


Figure 6.1

Hence writing in terms of shear stresses.

$$T = \int_A \frac{R}{C} \tau_{max} R dA$$

$$= \frac{\tau_{max}}{C} \int R^2 dA$$

$$\int R^2 dA = I_p$$

the Polar moment of Inertia of the circular section.

$$\therefore \; T = \frac{\tau_{max} I_p}{C}$$

$$\tau_{max} = \frac{TC}{I_p}$$

and

$$\tau = \frac{TR}{I_p}$$

Top

Stresses on Inclined Planes

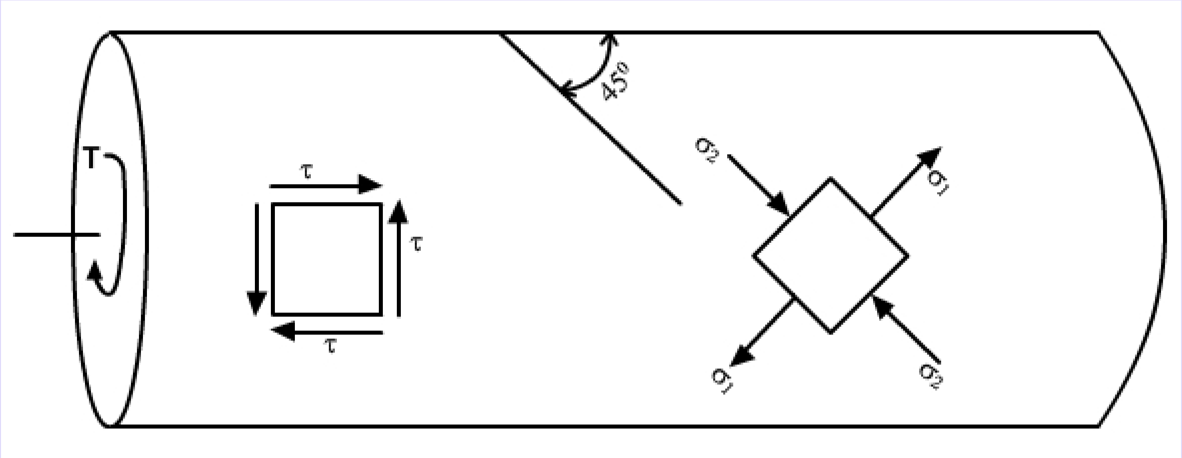


Figure 6.2

The shear stress at a point, on the surface acting on a plane perpendicular to the axis of the rod can be found out from the preceding analysis.

Using transformation law for, stresses the state of stress at a point on a plane at 45^0 to the axis can be found out. These stresses are found out to be

$$\sigma_1 = \tau_{max}$$
$$\sigma_2 = -\tau_{max}$$

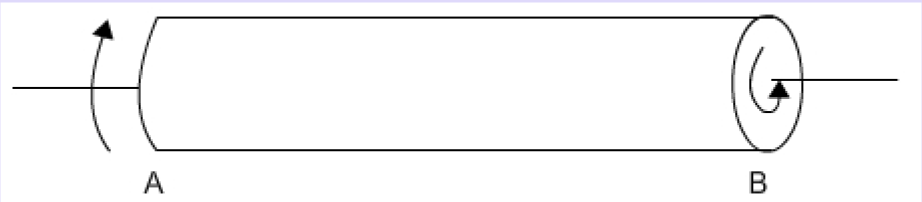
Ductile materials have lesser shear strength than tensile strength and hence fail through planes perpendicular to the axis.

Brittle materials have lesser tensile strength than shear strength. Hence they fail through planes inclined at 45^0 to the axis.

Top

Angle of Twist in Torsion

Consider a circular shaft subjected to torque. It is assumed that plane sections perpendicular to the axis remain plane even after loading. Isolating an element form such a member, (Refer Figure 6.3).



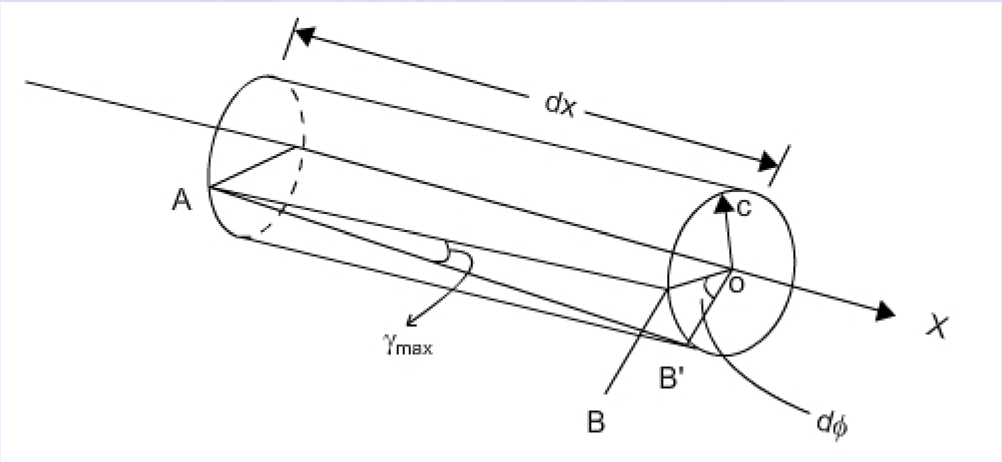


Figure 6.3

A line segment in the surface of the shaft, AB is initially parallel to the axis of the shaft. Upon application of torque, it occupies a new position AB'. The segment OB now occupy the position OB'.

From figure 16,

BB' = C dϕ

Also

$$BB' = dx\,\gamma_{max}$$

$$\frac{d\phi}{dx} = \frac{\gamma_{max}}{C}$$

Since $\tau = G\,\gamma$

$$\tau_{max} = G\,\gamma_{max}$$
$$\tau_{max} = \frac{TC}{GI_p}$$

$$\therefore \gamma_{max} = \frac{TC}{GI_p}$$

$$\therefore \frac{d\phi}{dx} = \frac{T}{GI_p}$$

This equation gives the relative angle of twist between any two sections of a shaft distance dx apart.

To find the total angle of twist ϕ between any two sections 1 and 2, all rotations of all elements between 1 and 2 should be summed up

$$\therefore \phi = \phi_2 - \phi_1 = \int \frac{Tdx}{I_p G}$$

Where

$$T = T(x)$$
$$I_p = I_p(x)$$
$$G = G(x)$$

When τ, I_p and G vary along the length of the shaft.

Top

Torsion of Circular Elastic Bars:

Formulae:

1. For solid circular member

Polar Moment of Inertia,

$$I_p = \frac{\pi C^4}{2} = \frac{\pi D^4}{32}$$

where, C is radius of the circular member. D is Diameter of the circular member.

2. For a circular tube:

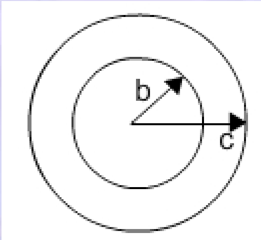


Figure 6.4

Polar moment of Inertia,

$$I_p = \frac{\pi c^4 - \pi b^4}{2}$$

where, c = outer radius of the tube.

b = Inner radius of the tube.

3. For very thin tubes:

where thickness t = c-b is very less. Then I_P can be approximated to,

$$I_p = 2\pi R_{avg} t$$

where

$$R_{avg} = \frac{b+c}{2},$$

4. (i) $$\boxed{\tau_{max} = \frac{TC}{I_P}}$$

where τ_{max} = Maximum shear stress in a circular cross section.

T = Torque in that section.

C = Radius of the section.

I_P = Polar Moment of Inertia.

Note: Shear stress linearly with radius.

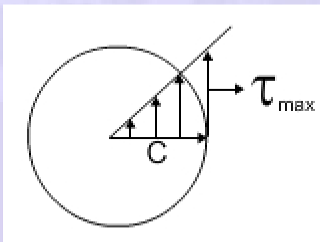


Figure 6.5

(ii) <u>Shear Stress (τ) at a distance R from the centre.</u>

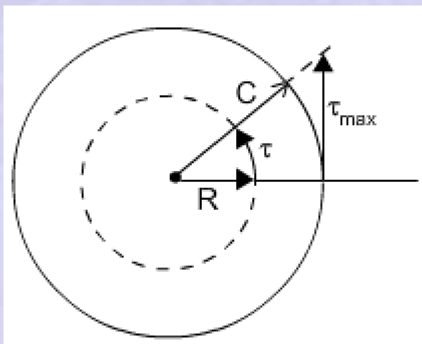


Figure 6.6

$$\tau = \tau_{max} \frac{R}{C}$$

$$\tau = \frac{TR}{I_P}$$

Note: J is also used to denote the polar moment of Inertia.

Click here to check the Animation

Assumptions

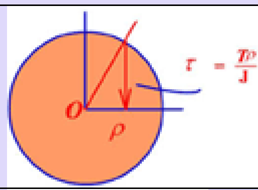
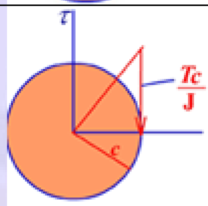
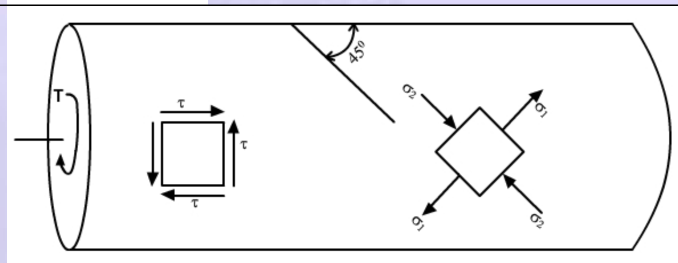
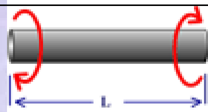
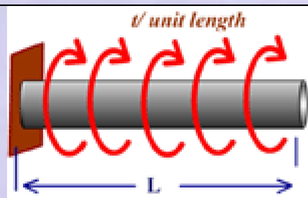
Stress Formula

Angle of twist

Maximum Stress

Top

Table of Formulae

S.No	Quantity	Formula	Diagram
1.	τ (Shear stress)	$\tau = \frac{T\rho}{J}$	
2.	τ_{max} (Max. shear stress)	$\tau_{max} = \frac{Tc}{J}$	
3.	σ_{max} (Max. normal stress)	$\sigma_{max} = \tau_{max}$	
4.	θ (angle of twist)	$\theta = \frac{TL}{GJ}$	
5.	θ (angle of twist)	$\theta = \int_L \frac{TL}{GJ}$	

Top

Misconceptions

Misconception 1

Torsional shear produces shear stresses on the cross section as shown, thus, the shear stresses should distort the cross section.

Fact:

No. The cross section is not distorted.

Click here to check the Animation

Misconception 2

If the bar is twisted, its length also changes.

Fact:

No. The length of the bar is unaffected.

Click here to check the Animation

Misconception 3

If there is a small slit made along the bar shown. Since all the properties remain almost the same, the bar will twist to the same extent.

Fact:

No.

Click here to check the Animation

Misconception 4

If a composite rod is subjected to torque T, each of the composite is inturn subjected to torsional load T.

Fact:

No.

Click here to check the Animation

Misconception 5

The total angle of the twist of the bar is same as the shear strain.

Fact:

No.

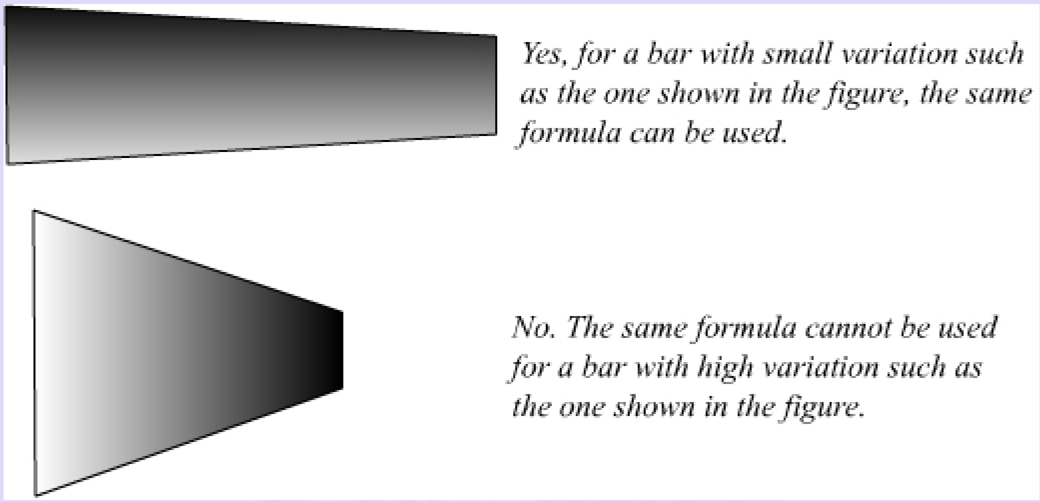
Click here to check the Animation

Misconception 6

For a bar of varying cross section, the same formula can be used!

Fact:

Yes, but for a bar with small variation and not for a bar with a steep variation.



Misconception 7

The bar becomes thinner as you twist it!

Fact:

No. From **assumption2**, we see that the cross section remains the same after twisting.

Misconception 8

The shear stress is maximum at the skin but it is a free surface. Therefore, should'nt the shear stress be zero?

Fact:

No.

Real Life Application

Shafts

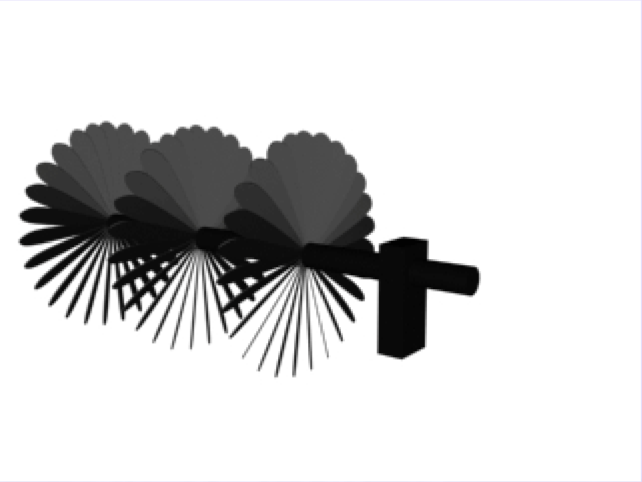
Torsion in a Nut

Torsion in a Helical spring

Torsion in a Two- way slab

Application1

Shafts

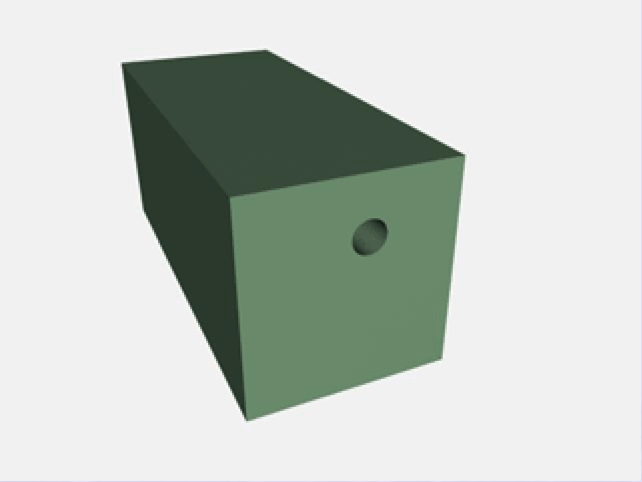


Click here to check the Animation

Top

Application 2

Torsion in a Nut

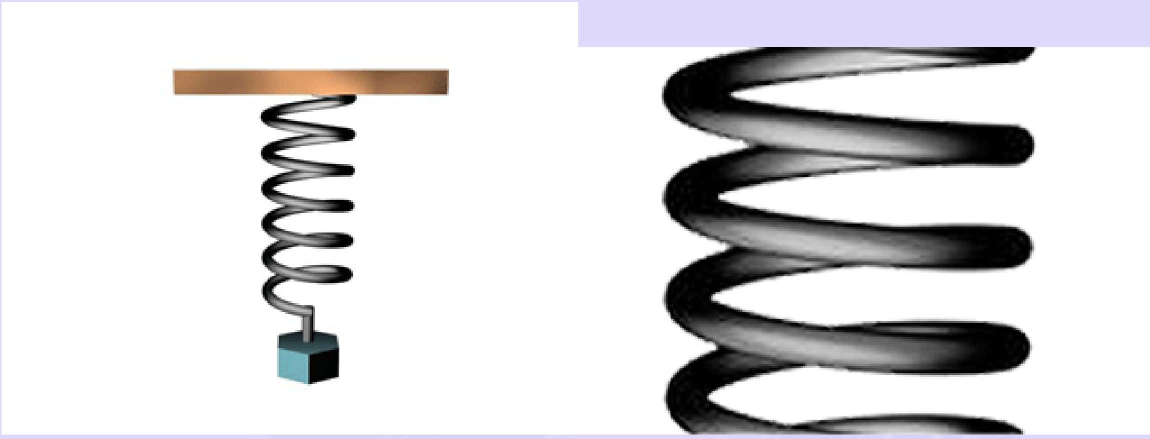


Click here to check the Animation

Top

Application 3

Torsion in a Helical spring

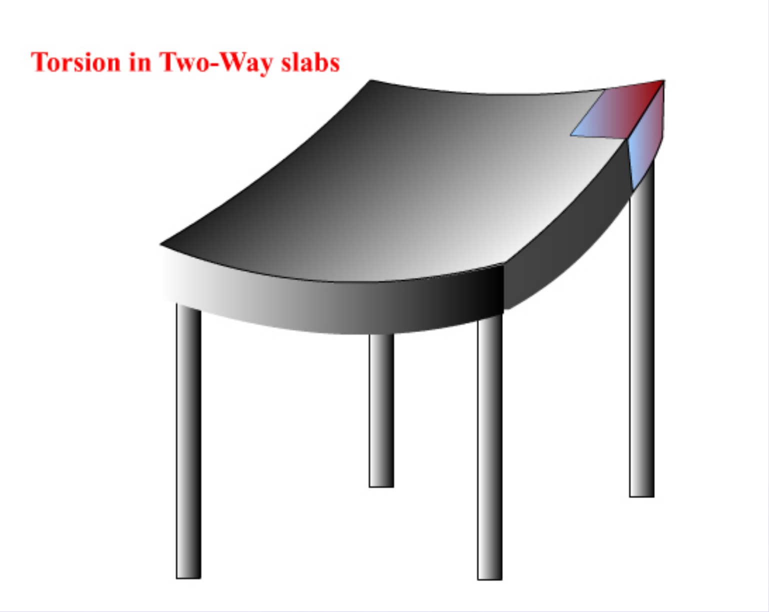


Click here to check the Animation

Top

Application 4

Torsion in a Two- way slab



Click here to check the Animation

Top

5. Beams

Introduction:

Apart from axial and torsional forces there are other types of forces to which members may be subjected. In many instances in structural and machine design, members must resist forces applied laterally or transversely to their axes. Such members are called beams. The main members supporting floors of buildings are beams, just as an axle of a car is a beam. Many shafts of machinery act simultaneously as torsion members and as beams. With modern materials, the beam is a dominant member of construction. The determination of the system of internal forces necessary for equilibrium of any beam segment will be the main objective of this chapter. For the axially or torsionally loaded members previously considered, only one internal force was required at an arbitrary section to satisfy the conditions of equilibrium. However, even for a beam with all forces in the same plane, i.e., a planar beam problem, a system of three internal force components can develop at a section. These are the axial force, the shear, and the bending moment. Determining these quantities is the focus of this chapter. The chapter largely deals with single beams. Some discussion of related problems of planar frames resisting axial forces, shears, and bending moments is also given. Only statically determinate systems will be fully analyzed for these quantities. Special procedures to be developed in subsequent chapters are required for determining reactions in statically indeterminate problems for complete solutions. Extensions to members in three-dimensional systems, where there are six possible internal force components, will be introduced in later chapters as ceded and will rely on the reader's knowledge of statics. In such problems at a section of a member there can be: an axial force, two shear components, two bending moment components, and a torque.

Top

5.1 Bending of Beams

5.1.1 Introduction:

Beams:

Bars Subjected to transverse loads.

Planar and slender members.

Supports:

Identified by the resistance offered to forces.

(a) Rollers/Links:

Resists forces in a direction along the line of action (Figure 5.1(a)).

(b) Pins:

Resists forces in any direction of the plane (Figure 5.1(b)).

(c) Fixed Support:

Resists forces in any direction (Figure 5.1(c)).

Resists moments.

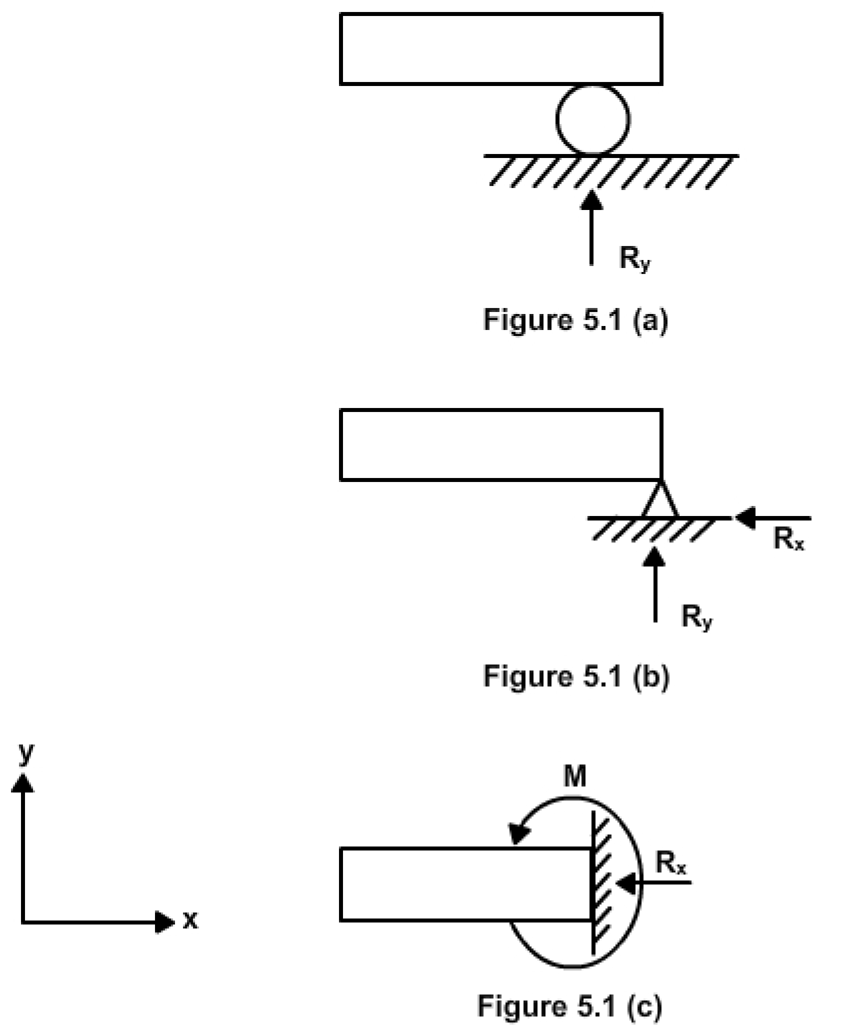


Figure 5.1 (a)

Figure 5.1 (b)

Figure 5.1 (c)

5.1.2 Classification of Beams

(a) Statically determinate or indeterminate.

Statically determinate - Equilibrium conditions sufficient to compute reactions.

Statically indeterminate - Deflections (Compatibility conditions) along with equilibrium equations should be used to find out reactions.

(b) Cross sectional Shapes - I,T,C or other cross sections.

(c) Depending on the supports used

1) Simply supported - pinned at one end and roller at the other (Figure 5.1.2(a))
2) Cantilever - fixed at one end and the other end free (Figure 5.1.2(b)).
3) Fixed beam - fixed at both ends (Figure 5.1.2(c)).

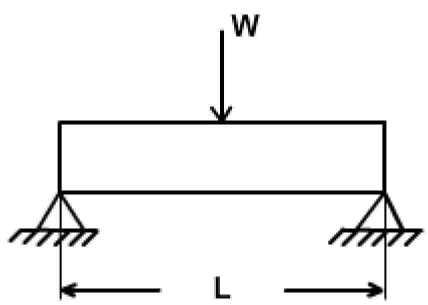


Figure 5.1.2(a)

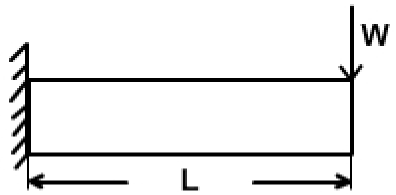


Figure 5.1.2(b)

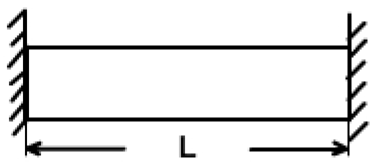


Figure 5.1.2(c)

Where W – loading acting, L – span.

5.1.3 Calculation of beam reactions

When all the forces are applied in a single plane, the three equations of static equilibrium are available for analysis.

$$\sum F_x = 0,\ \sum F_y = 0 \text{ and } \sum M_3 = 0$$

Employing these, the reactions at supports could be found out.

5.1.4 Procedures for Computing forces and moments

For a beam with all forces in one plane, three force components are internally developed.

- Axial force
- Shear
- Bending Moment

Procedures are to be established for finding these quantities.

Direct Method/Method of sections:

This method is illustrated by the following example

Find reactions at supports for the cantilever shown in figure 5.1.3(a) subjected to uniformly distributed load.

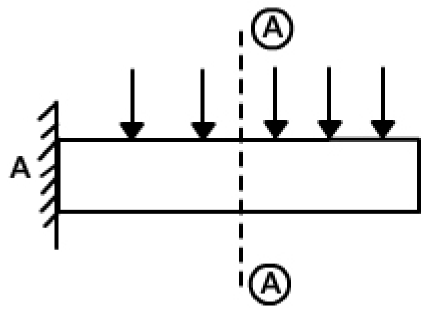


Figure 5.1.3(a)

Cut the cantilever along section A-A and obtain free body diagram as given in figure 5.1.3(b).

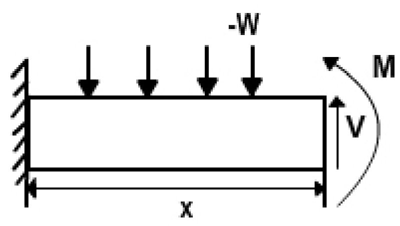


Figure 5.1.3(b)

The segment of the beam shown in figure 5.1.3(b) is in equilibrium under the action of external forces and internal forces and moments.

$$\sum F_y = 0$$

gives

$$V = -wx$$

$$\sum M = 0$$

$$\text{gives, } M - \frac{wx^2}{2} = 0$$

$$M = \frac{wx^2}{2}$$

Sign convention: Shear force

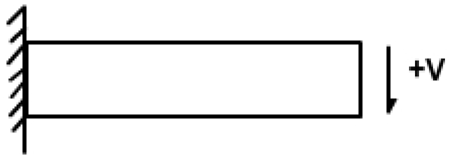


Figure 5.1.4(a)

Figure 5.1.4(b)

Figure 5.1.4(c)

Sign convention: Moments

Figure 5.1.5(a)

Figure 5.1.5(b)

As Shown in Figure 5.1.5(a), sagging (beam retains water) moment is positive, other wise bending moment is negative (Figure 5.1.5(b)).

5.1.5 Shear force and Bending Moment Diagrams (SFD & BMD)

Plot of shear and bending moment values on separate diagrams could be obtained.

Magnitude and location of different quantities can be easily visualized.

SFD & BMD are essential for designers to make decisions on the shape and size of a beam.

The worked out examples illustrate the procedure for plotting SFD and BMD by direct approach.

5.1.6 Shear force and BM Diagrams / (Alternate approach)

Beams Element: Differential equations of Equilibrium

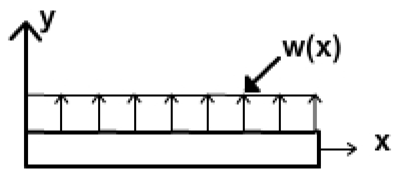


Figure 5.1.6

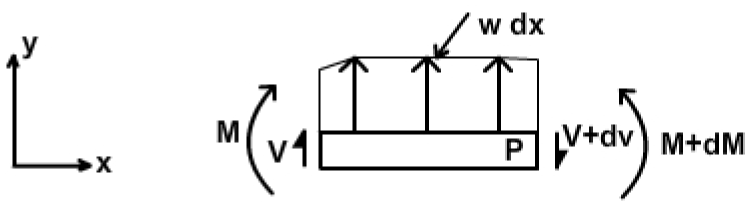


Figure 5.1.7

Free body diagram of element of length dx is shown Figure 5.17, which is cut from a loaded beam (Figure 5.1.6).

$\sum F_y = 0$ gives

$V + wdx - (V + dv) = 0$

$$\text{ie } \frac{dv}{dx} = w \qquad 5.1.1$$

$\sum M_p = 0$ gives

$M + dM - Vdx - M - (wdx).(dx/2) = 0$

$$\text{ie } \frac{dM}{dx} = V \qquad 5.1.2$$

Substituting equations 5.1.2 in 5.1.1

$$\frac{d^2M}{dx^2} = V \qquad 5.1.3$$

Integrating 5.1.2

$$V = \int_0^x wdx + c_1$$

Integrating 2

$$M = \int_0^x Vdx + C_2 + M_e$$

M_e - External moment acting.

Top

5.2 Theory of Bending

5.2.1 General theory

Plane sections normal to the axis before bending remain plane and normal after bending also, as shown in Figure 5.2.1.

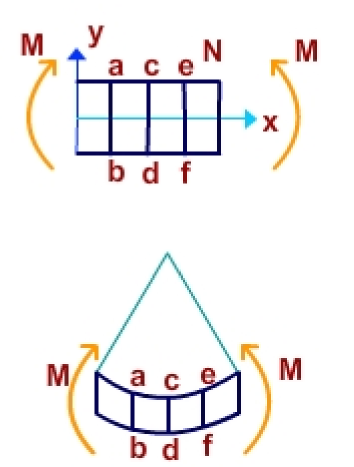


Figure 5.2.1

From Figure 5.2.1, ab, cd efs are sections which remain plane and normal. Beam is subjected to pure bending (no shear). Longitudinal top fibers are in compression and bottom fibres in tension.

Layer of fibres in between which is neither in tension or compression, is called the neutral surface. Neutral axis is the intersection of such a surface with the right section through the beam.

Assumptions of the theory of bending

Deflection of the beam axis is small compared to span of the beam.

Shear strains, along the plane xy are negligible.

Effect of shear stress in the plane $xy\left(\tau_{xy}\right)$ on normal stress $\left(\sigma_x\right)$ is neglected.

Note: Even through pure bending is assumed, distribution of normal stresses at any given cross section does not get significantly changed due to non uniform bending.

For pure bending of a beam, beam axis deforms into part of a circle of radius ρ; for an element defined by an infinitesimal angle d , the fiber length is given by (refer figure 5.2.2)).

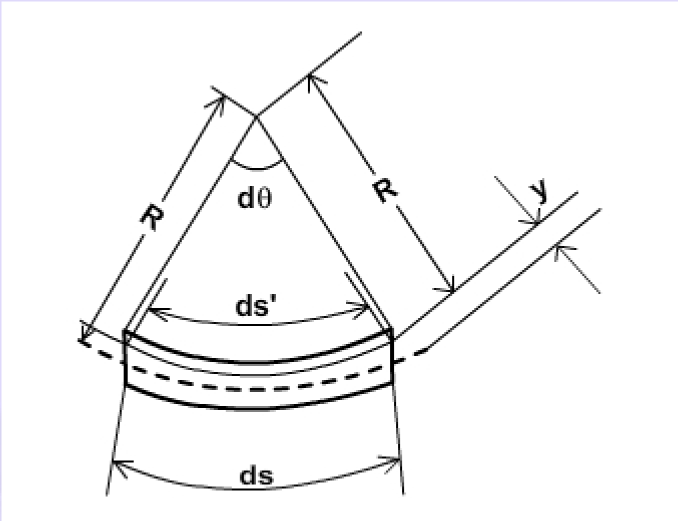


Figure 5.2.2

ds = R dθ

$$\frac{d\theta}{ds} = \frac{1}{R} = k \text{, where}$$

R - Radius of Curvature

k - Axis Curvature

For a fiber located at radius R' = R – y

$$ds' = (R - y)\,d\theta$$

Strain, $\varepsilon_x = \dfrac{ds - ds'}{ds}$

$$\varepsilon_x = -ky$$

5.2.2 Elastic Flexure Formula

By Hooke's Law,

$$\sigma_x = E\varepsilon_x = -Eky$$

$\sum F_x = 0$ gives

$$-Ek\int y dA = 0$$

i.e. neutral axis possess through the centroid of the cross section (Ref Figure 5.2.3.)

$\sum M = 0$ gives

$$M_z = E_k \int y^2.dA$$

$$= Ek\ I_z$$

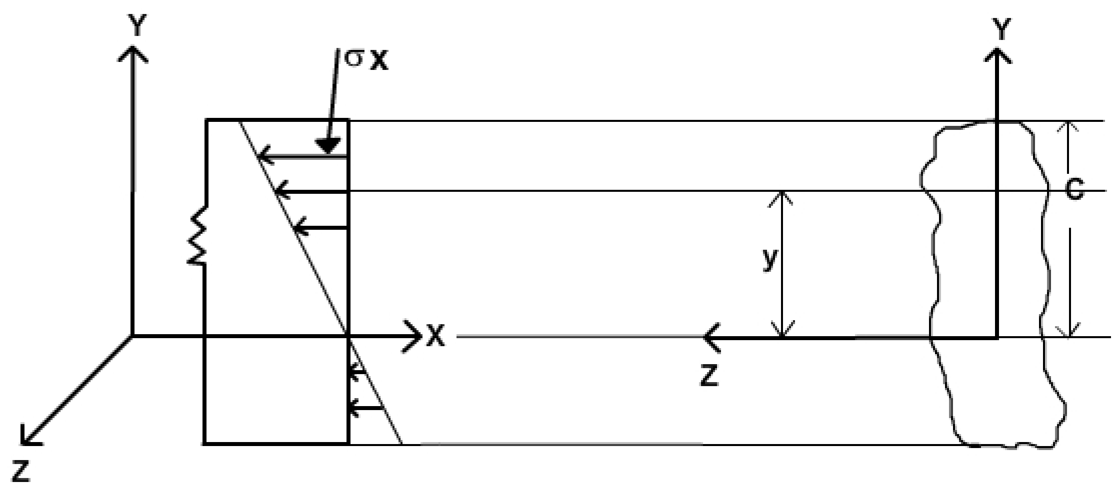


Figure 5.2.3

This gives

$$k = \frac{M}{E\ I_z}$$

$$\sigma_x = \frac{-M_z y}{I_z}$$

$$\sigma_{max} = \frac{M_z C}{I_z}$$

5.2.3 Beams of Composite Cross section

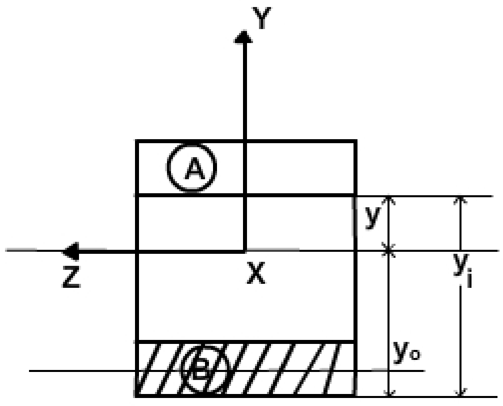


Figure 5.2.4

For beams of composite cross section $\sigma_x = -E_i k y$ for the i[th] material in the composite.

$y = y_i - y_0$

y_0 is the location of neutral axis from the bottom of the beam.

y_i is the location of neutral axis of the i[th] material. In the figure $y_i = y_A$, from this, we get

$$y_0 = \frac{\int_A E_i \, y_i \, dA}{\int E_i \, dA}$$

Where A the area of cross section of the corresponding material. The procedure for analyzing beams of composite cross section is illustrated in worked out examples.

Top

Deflection of beams

Introduction

The axis of a beam deflects from its initial position under action of applied forces. Accurate values for these beam deflections are sought in many practical cases: elements of machines must be sufficiently rigid to prevent misalignment and to maintain dimensional accuracy under load; in buildings, floor beams cannot deflect excessively to avoid the undesirable psychological effect of flexible floors on occupants and to minimize or prevent distress in brittle-finish materials; likewise, information on deformation characteristics of members is essential in the study of vibrations of machines as well as of stationary and flight structures.

Top

Deflection of Beams

(Solution Method by Direct Integration)

From Analytic geometry, Curvature of a line,

$$k = \frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{d^2y}{dx^2}\right)\right\}^{3/2}}$$

where x and y are co-ordinates of a point on the curve.

For small deflections,

$$k = \frac{d^2y}{dx^2}$$

Since, $\sigma_x = -My/EI$

and $k = -\varepsilon / y,$ and $\varepsilon = \sigma_x / E$

$$k = M/EI$$

$$\therefore M = EI\frac{d^2y}{dx^2}$$

where $M = M_y$

Hence,

$$EI\frac{d^2y}{dx^2} = M(x)$$

Since, $V = \frac{dM}{dx}$

$$EI\frac{d^3y}{dx^3} = V(x)$$

Since, $w = \frac{dv}{dx}$

$$EI\frac{d^4y}{dx^4} = w(x)$$

Boundary Conditions

Refer figure 5.2.7(a) – (d)

(a) Clamped Support:

$y(x_1) = 0;\ y'(x_1)=0;$

(b) Roller or Pinned Support:

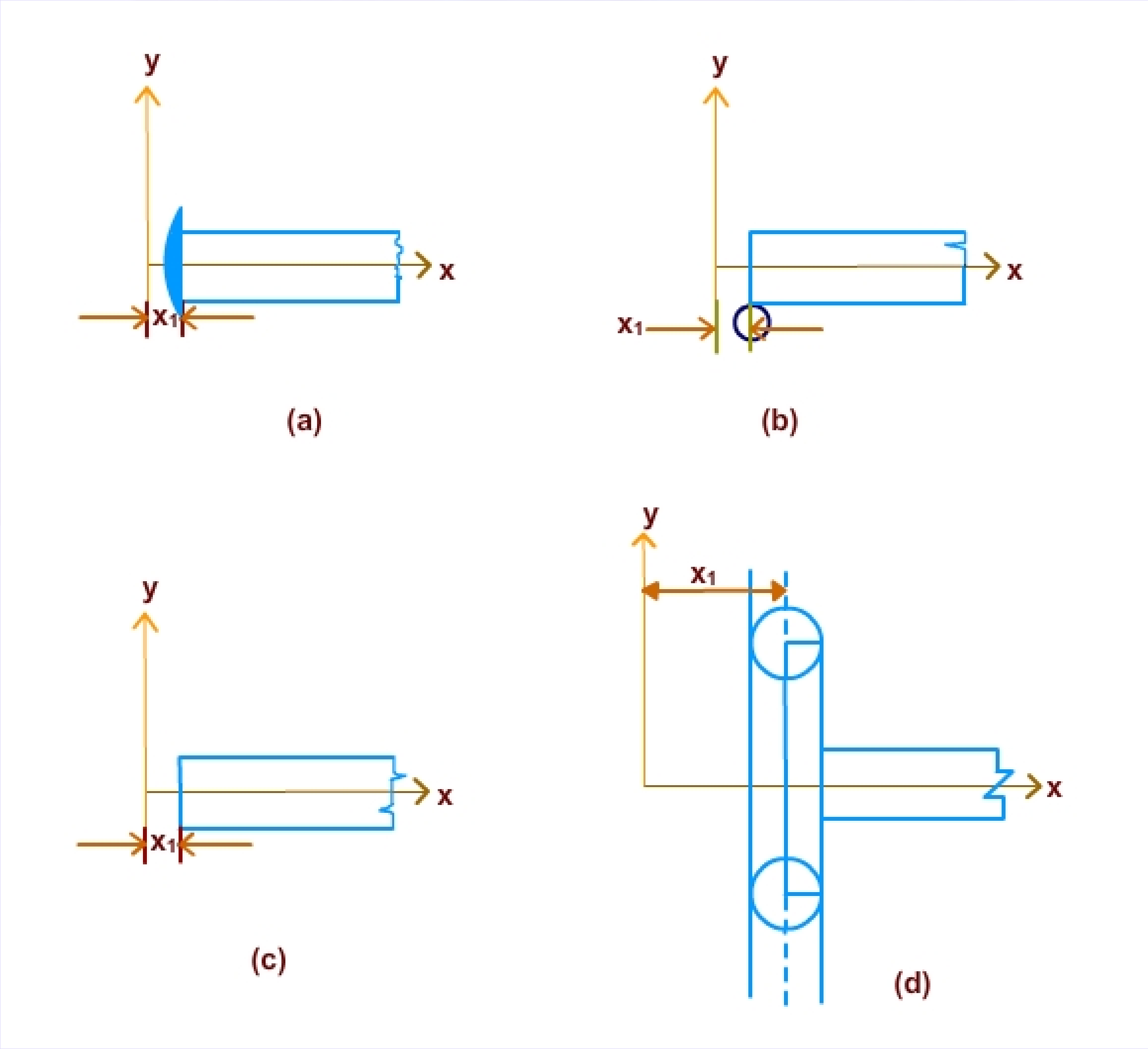
$y(x_1) = 0;\ M(x_1)=0;$

(c) Free end:

$M(x_1) = 0;\ V(x_1) = 0;$

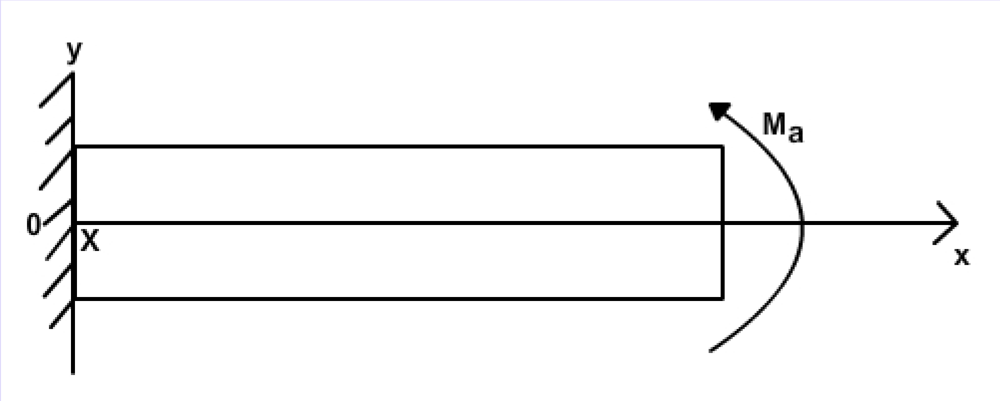
(d) Guided Support:

$y'(x_1) = 0;\ V(x_1)=0;$



Example:

Question: A Cantilever beam is subjected to a bending moment M at the force end. Take flexural rigidity to be constant and equal to EI. Find the equation of the elastic curve.



$$EI\frac{d^2y}{dx^2} = M_a$$

Integrating

EI dy/dx = $M_a x + C_1$

at x = 0; dy/dx = 0

which gives $C_1 = 0$

Integrating again,

$$EIy = \frac{M_a x^2}{2} + C_2$$

y = 0; at x = 0 gives

$C_2 = 0$

$$\therefore \ y = \frac{M_a x^2}{2EI}$$

which is the equation to the elastic curve.

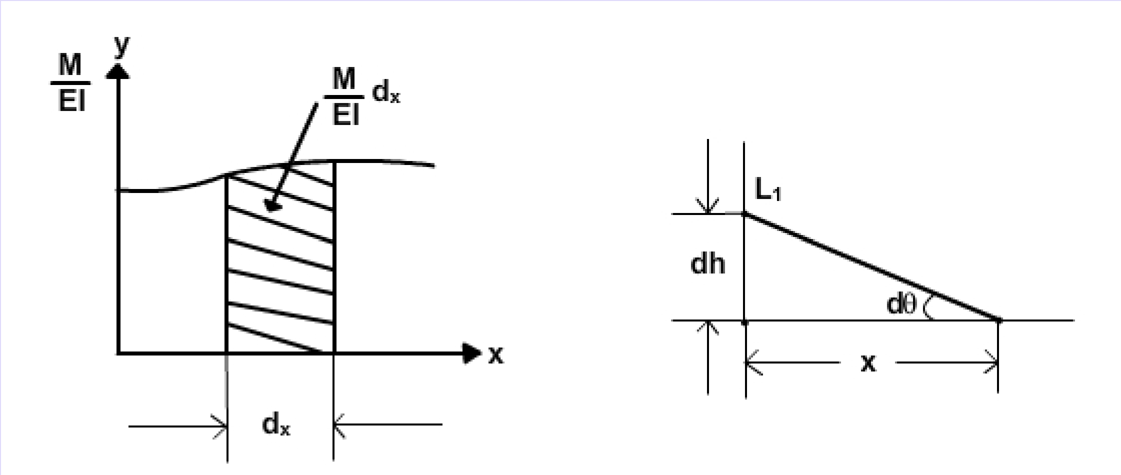
Top

Moment - Area Method for finding Beam Deflections

This method is used generally to obtain displacement and rotation at a single point on a beam.

This method makes use of the Moment - Area theorems given below.

Moment - Area Theorems



Refer above Figure

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{M}{EI} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Let dy/dx = θ

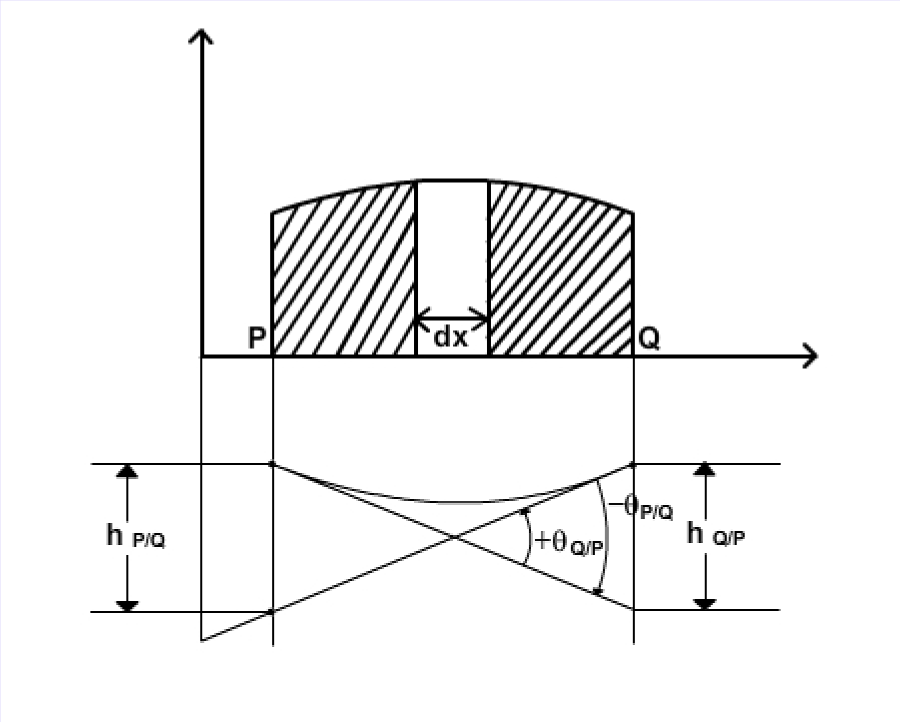
$$\frac{M}{EI} = \frac{d\theta}{dx}$$

$$\int d\theta = \int \frac{M}{EI}\, dx$$

Referring to figure down

$$\therefore\ \theta_{Q/P} = \theta_Q - \theta_P = \int_P^Q \frac{M}{EI}\, dx$$

This is the first moment area theorem, Where P and Q are any two sections on the beam. ie change in angle measured in radians between any two point P and Q on the elastic curve is equal to the M/EI area bounded by the ordinates through P and Q.



Referring to Figure, considering an element of the Elastic Curve,

dh = x . dθ

$$\therefore\ h_{Q/P} = \int_P^Q dh$$

$$= \int_P^Q x d\theta$$

$$= \int_P^Q \frac{M}{EI} x\, dx$$

This is the second moment Area theorem.

If $\overline{x_P}$ is the distance of centroid of the bending moment diagram between P and Q from P,

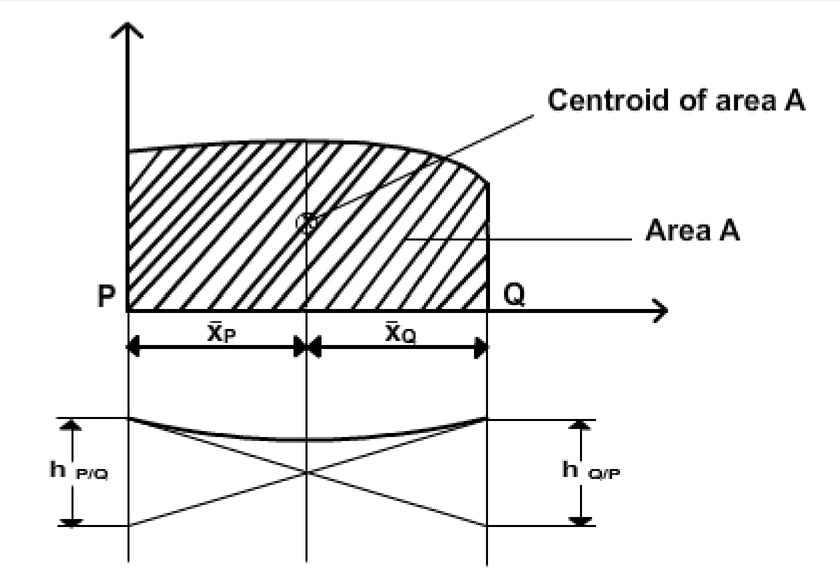
(Refer Figure)

then

$$h_{P/Q} = A\overline{x_P}$$

$$h_{Q/P} = A\overline{x_Q}$$

Here $h_{P/Q}$ is called the tangent deviation of the point P from a tangent at Q.

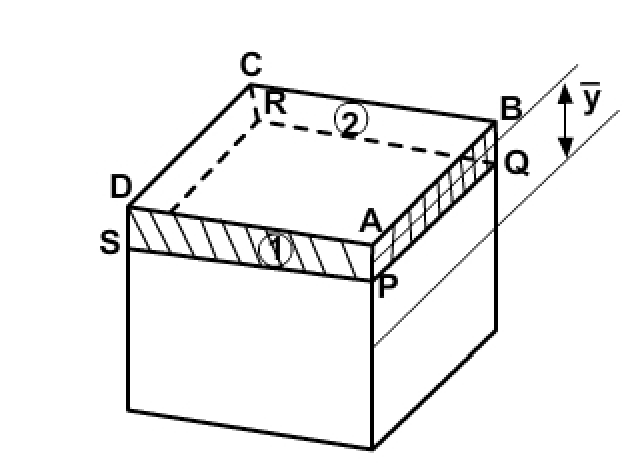


where A is the area of the BM diagram between P and Q. $\overline{x_p}$ and $\overline{x_Q}$ are as shown in figure.

Top

Shear Stresses in Beams

Shear force is related to change in bending moment between adjacent sections.



$$\frac{dM}{dx} = v$$

As shown in figure, consider a cut-out section from a beam.

At section 1 $\{$Area PADS$\}$

$$F_1 = \int_{A_1} \frac{-M_1\ y}{I}\, dA = \frac{-M_1}{I} \int_{A_1} y\ dA = \frac{M_1 Q}{I}$$

where y is the distance of any fiber from the neutral axis.

$Q = \int_{A_1} y.dA$ - First moment of area about the neutral axis $F_2 = \dfrac{M_2 Q}{I}$

if $M_2 = M_1 + dM$ and $F_2 = F_1 + dF$

then $dF = \dfrac{dM}{I} Q$

Force per unit length, $\dfrac{dF}{dx} = \dfrac{dM}{dx}\dfrac{Q}{I}$

This force per unit length is termed as the shear flow, q

Substituting for $\dfrac{dM}{dx} = V$;

$$q = \frac{VQ}{I}$$

Here I is the moment of Inertia of the entire cross-sectional area around the neutral axis.

Shear Stress Formula for Beams

The shear stress formula is obtained by modifying the shear flow formula.

$$\frac{dF}{dx} = \frac{dM}{dx}\frac{Q}{I}$$

Shear Stress,

$$\tau = \frac{1}{t}\frac{dF}{dx}$$

$$= \frac{VQ}{It} \qquad \text{(Refer figure)}$$

$$= \frac{q}{t}$$