Aggregation of preferences: a review

Philippe VINCKE
Free University of Brussels, B-1050 Brussels, Belgium

This paper presents a review and a classification of the main *theoretical results* obtained up to now in the important field of the aggregation of preferences. (Let us mention that multicriteria analysis is not considered here, as it essentially consists of *methods* for the aggregation of preferences).

Section 1 describes a tableau, each square of which corresponds to a particular type of preference aggregation problem, and gives, for each of them, a list of references. The different types of problems are obtained by considering the various kinds of informations which can be obtained concerning the global preferences of a committee and the preferences of the individual members.

Sections 2, 3 and 4 present some comments respectively on Arrow's problem (three first columns of the tableau), on the theory of choice functions (fourth column) and voting procedures (fifth column).

In the bibliography, each reference is associated, in terms of its subject, to one particular square of the tableau.

1. Classification of the preference aggregation problems

Let A be a set of possible decisions: the elements of A can be investment projects, individuals, cars, menus in a restaurant,...: they will be called the *candidates*.

A committee of n individuals (the *voters*) has to rank them from the best to the worst, or to divide them into the good candidates and the others, or to elect one and only one candidate from A: these are in general the three main problems of decision-making.

In mathematical terms, 'to rank' means to build a complete preorder (reflexive, transitive and complete relation) on A. The problem of building a complete preorder by a committee of n individuals will correspond to the first column of the tableau 1.

It can happen that supplementary informations is required regarding the global ranking (for example, about the intensity of preferences), so that the committee has to provide 'more than a complete preorder'. We place these situations in the second column of the tableau.

On the other hand, due to the impossibility of obtaining a complete preorder in certain cases, the committee can sometimes give a global preference relation which has not the nice properties of a complete preorder (e.g. partial order, semiorder,...): the third column of the tableau corresponds to these cases where the answer of the committee is 'less than a complete preorder'.

The three first columns of the tableau represent what we could call 'Arrow's problem', that is the building of a global preference relation on the set of all candidates.

The fourth and fifth columns will respectively represent the problem of defining a partition of A into the 'good candidates' and the others ('problem of classification') and the problem of determining one candidate, considered as the best by the committee ('voting problem').

Similarly five kinds of information can, in general, be provided by each voter: a complete preorder, 'more than a complete preorder', 'less than a complete preorder', a partition into 'the good and the others', one candidate considered as the best by this voter.

In this way, we obtain a tableau of 25 squares which contains most of the works on preference aggregation. 12 of these squares correspond to interesting aggregation problems which have been considered in the literature. We give below, for each of them, the main references where these problems have been discussed.

A1: [1], [3], [12], [13], [14], [15], [17], [20], [25], [26], [27], [28], [33], [39], [40], [42], [44], [45], [49], [53], [55], [61], [63], [65], [67], [69], [71], [72], [73], [74], [75], [77], [79], [81], [94], [98];
A2: [10], [21], [22], [37], [43], [48], [50], [62], [82], [92], [96];
A3: [4], [59], [60];
A4: [32], [51], [52], [54], [55], [68], [90];

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Table 1

committee / n voters	Ranking			Classification	Voting
	Complete preorder	More than a complete preorder	Less than a complete preorder		
Complete preorders	A1	–	A4	C1	E1
More than complete preorders	A2	A3	–	–	E2
Less than complete preorders	A6	–	A5	C2	–
Classification	–	–	–	C3	–
Voting	–	–	–	–	E3

A5: [6], [12], [18], [29], [54], [56], [78];
A6: [5], [8], [9], [47], [64], [75], [88];
C1: [35], [41], [46], [58], [80], [83], [95];
C2: [11], [19], [31], [32], [34], [36], [57], [84], [87], [89];
C3: [41], [66];
E1: [7], [24], [38], [76], [85], [86], [93];
E2: [16], [23], [99];
E3: [7], [30].

Section 2 discusses squares A1 to A6, Section 3 discusses squares C1, C2 and C3 and finally Section 4 discusses squares E1, E2 and E3. The other squares await further research.

2. Arrow's problem

The square A1 corresponds to the problem of finding a global complete preorder (which will represent the preference of the committee) from n individual complete preorders (the preferences of the n voters): it is the classic problem of Arrow.

The main result concerning this problem is obviously Arrow's theorem which asserts the impossibility of finding a procedure which associates one complete preorder to each n-uple of complete preorders and which satisfies three conditions called unanimity, non-dictatorship and independence of irrelevant alternatives.

Several authors (Blau [12,15], Guha [40], Hansson [44,45],...) have proposed different interpretations of Arrow's theorem by modifying these conditions.

In fact, these interpretations take two different forms, each of them leading to a different line of research.

The first interpretation of Arrow's theorem is that the information contained in n complete preorders is not sufficient to determine one global complete preorder without ambiguity: information must be added. This interpretation leads to the works where the mathematical models of the individual preferences are 'more than complete preorders' (square A2). In particular, it is well known that the independence of irrelevant alternatives prohibites the interpersonal comparisons of preferences. Several authors have shown that by introducing some assumptions on these comparisons, it is possible to aggregate individual preferences and to satisfy 'rational' conditions of aggregation.

Keeney [59,60], for example, has established conditions which allow one to aggregate cardinal utilities into linear combinations of these utilities. Blackorby [10] has considered the situation where information is available about the relative importance of the individuals in the committee.

It is clear that when more information is available concerning the individual preferences, it is sometimes possible to obtain a global preference which is 'more than a complete preorder': this situation corresponds to the square A3 and is illustrated by the works on the aggregation of Von Neumann–Morgenstern's utilities into utilities of the same kind.

A second interpretation of Arrow's theorem is that to ask for a global complete preorder is too much when the individual preferences are complete preorders without any other information; so, we shall ask for a global preference relation which is 'less than a complete preorder'. This attitude corresponds to the square A4 where global preference can be a partial order, a quasi-transitive relation, an acyclic relation, a tournament,.... In some cases, the Arrow's impossibility theorem becomes a possibility theorem.

Several authors think (and they are often right) that it is not always possible, for an individual, to rank the candidates in a complete preorder: their works can be placed in the squares A5 and A6.

For A5, let us mention, as examples, the works of Batra and Pattanaik [6] which consider the aggregation of quasi-transitive relations, particularly with the majority rule and the works of Brown [18] in which quasi-transitive and acyclic relations are considered; Monjardet [78] has studied the aggregation of tournaments.

In general, the resolution of problems in A6 consists in finding the complete preorder which is closest (for a given distance measure) to individual preferences or closest to the global preference relation obtained in A5. Monjardet [75], for example, has studied the properties of complete preorders at minimum distance from a set of tournaments.

To be complete, we have also to mention that a lot of papers have been written about the cases where various restrictions of the domain of aggregation are introduced.

3. The classification problem

The papers on the problem of classifying the candidates into good and bad ones are essentially concerned with the theory of choice functions.

A choice function is a function C which associates to every subset B of A, a subset $C(B)$ of B. There is in fact, a relationship between the theory of choice functions and Arrow's problem. Indeed, to each preference relation R, a choice function can be associated such that, for instance

$$C(B)=\{a\in B \mid aRb,\ \forall b\in B\},$$

or

$$C(B)=\{a\in B \mid \nexists b\in B\colon bRa\}.$$

Results have been obtained concerning the conditions under which one can 'rationalize' a given choice function using a preference relation with specified properties.

Other results show the relationship between characteristics of R and those of C.

Different readings of Arrow's theorem can be formulated in terms of choice functions so that squares A4 and C1 on one hand, and squares A5 and C2 on the other hand, are treated simultaneously in most of the papers.

The square C3 corresponds to what Fishburn calls post-choice methods, in which good candidates are determined for each voter (eventually from their preference relation) before the good candidates for the committee are decided upon.

We have also to mention the interesting works of C. Plott [87] on the properties of 'path independence' of a selection procedure. In the papers of Pattanaik [83,84] conditions are found for the existence of choice functions for particular selection procedures such as 'majority rule' and 'non minority rule'.

4. The voting problem

For the problem of the election of one candidate by a committee, chiefly two fundamental problems have been considered in the literature. Let us illustrate each of them by an example.

Let $A=\{a,b,c\}$ and consider a committee of 9 voters, the preferences of which are as follows:

$a>b>c$ for 3 voters,
$b>c>a$ for 1 voter,
$c>a>b$ for 1 voter,
$a>c>b$ for 1 voter,
$c>b>a$ for 2 voters,
$b>a>c$ for 1 voter.

The election procedure is the following (it is known as the Lhuilier's voting procedure).

A candidate is elected if he is the best for more than half of the voters; if no candidate satisfies this condition, the committee elects the candidate who is ranked first or second by the greatest number of voters. In our example, b will be elected. However, let us remark that 5 voters out of 9 prefer a to b.

So, this procedure leads to a contradiction with a comparison of the candidates two by two: this situation is possible for every election procedure. On the other hand, it is well known that the method of paired comparisons can lead to cycles which do not allow the election of one best candidate.

The first fundamental problem thus consists in the impossibility of finding an election procedure which always leads to a solution and which is never in contradiction with any of the paired comparisons.

One solution to this problem is to find a procedure which minimizes the number of contradictions involving paired comparisons. Let us also mention the works of Kelly [61], Demeyer and Plott [20], where the probabilities of obtaining 'rational' results for given aggregation and selection procedures are studied.

Let now $A = \{a, b, c, d\}$ and consider a committee of three voters, the preferences of which are as follows:

$a > b > c > d$ for voter I,
$d > a > b > c$ for voter II,
$d > a > b > c$ for voter III.

One election procedure consists of giving 4 points to a candidate who is the best for a voter, 3 points when second, and so on, and to elect the candidate who obtains the greatest score. In our example, a will be elected.

Suppose now that voter III wants to favour candidate d and knows that a is a dangerous adversary. He can 'manipulate' his vote by declaring that his preference order is

$$d > b > c > a,$$

and in this case d is elected. This example illustrates the fundamental problem of manipulation of voting procedures.

The most important result in this field is that of Gibbard [38], who proves the impossibility of finding a non-dictatorial and non-manipulatable election procedure. Batteau and Blin [7] have generalized Gibbard's theorem to the case of collective manipulations. The examples given by Moon are good illustrations of the difficulties arising in the construction of election procedures.

Finally let us also mention the interesting work on the relationships between aggregation rules and properties of the decisive sets of voters (as in Brown [19]).

Bibliography *

[1] (A1) K.J. Arrow, Le principe de rationalité dans les décisions collectives, Economie Appl. (1952) 469–484.
[2] (G) K.J. Arrow, Utilities, attitudes, choices: a review note, Econometrica 26 (1958) 1–23.
[3] (A1) K.J. Arrow, Social Choice and Individual Values (Wiley, New York, 1963).
[4] (A3) J. Aumann, Measurable utility and the measurable choice theorem, in: La Décision, éditions du C.N.R.S. (1969).
[5] (A6) M. Barbut, Note sur les ordres totaux à distance minimum d'une relation binaire donnée, Math. Sci. Humaines 17 (1966) 47–48.
[6] (A5) R. Batra and P. Pattanaik, Transitive multi-stage majority decision with quasi-transitive individual preferences, Econometrica 40 (1972) 1121–1135.
[7] (E1) (E3) P. Batteau and J.-M. Blin, Sur le problème des procédures de scrutin garantissant une expression sincère des opinions, Math. Sci. Humaines 34 (1976) 45–60.
[8] (A6) J.-C. Bermond, Ordres à distance minimum d'un tournoi et graphes partiels sans circuits maximaux, Math. Sci. Humaines 37 (1972) 5–26.
[9] (A6) M. Besson, Rang moyen et agrégation de classements, RAIRO 9 (1975) 37–58.
[10] (A2) C. Blackorby, Degrees of cardinality and aggregate partial ordering, Econometrica 43 (1975) 845–852.
[11] (C2) D. Blair, Path independent social choice functions: a further result, Econometrica 43 (1975) 173–174.
[12] (A1) (A5) J.H. Blau, The existence of social welfare functions, Econometrica 25 (1957) 302–313.
[13] (A1) J.H. Blau, A direct proof of Arrow's theorem, Econometrica 40 (1972) 61–67.
[14] (A1) J.H. Blau, Neutrality, monotonicity and the right of veto: a comment, Econometrica 44 (1976) 603–604.
[15] (A1) J.H. Blau and R. Deb, Social decision functions and the veto, Econometrica 45 (1977) 871–880.
[16] (E2) J.M. Blin and M.A. Satterthwaite, On preferences, beliefs and manipulations within voting situations, Econometrica 45 (1977) 881–888.
[17] (A1) V. Bowman and C. Colantoni, Majority rule under transitivity constraints, Management Sci. 19 (1973) 1029–1041.
[18] (A5) D.J. Brown, An approximate solution to Arrow's problem, J. Economic Theory 9 (1974) 375–383.
[19] (C2) D.J. Brown, Aggregation of preferences, Quart. J. Economics 89 (1975) 456–469.

* See also the bibliography of [97]. N.B. (G) means "generalities".

[20] (A1) F. Demeyer and C.R. Plott, The probability of a cyclical majority, Econometrica 38 (1970) 345–354.
[21] (A2) R. Deschamps and L. Gevers, Separability, risk-bearing and social welfare judgments, European Econom. Rev. 10 (1977) 77–94.
[22] (A2) R. Deschamps and L. Gevers, Leximin and utilitarian rules: a joint characterization, J. Econom. Theory 17 (1978) 143–163.
[23] (E2) M. Dummett and R. Farquharson, Stability in voting, Econometrica 29 (1961) 33–43.
[24] (E1) R. Farquharson, Sophisticated voting and an example due to M. Kreweras, La Décision, éditions du C.N.R.S. (1969).
[25] (A1) J. Feldman, Pôles, intermédiaires et centres dans un groupe d'opinions, Math. Sci. Humaines 43 (1973) 39–54.
[26] (A1) K. Fine, Some necessary and sufficient conditions for representative decision on two alternatives, Econometrica 40 (1972) 1083–1090.
[27] (A1) B. Fine and K. Fine, Social choice and individual rankings, Rev. Econom. Studies 42 (1974) 303–322 and 459–475.
[28] (A1) P.C. Fishburn, Arrow's impossibility theorem: concise proof and infinite voters, J. Econom. Theory 2 (1970) 103–106.
[29] (A5) P.C. Fishburn, Intransitive individual indifference and transitive majority, Econometrica 38 (1970) 482–489.
[30] (E3) P.C. Fishburn, The theory of representative majority decision, Econometrica 39 (1971) 273–284.
[31] (C2) P.C. Fishburn, Transitive binary social choices and intraprofile conditions, Econometrica 41 (1973) 603–615.
[32] (A4) (C2) P.C. Fishburn, Impossibility theorems without the social completeness axiom, Econometrica 42 (1974) 695–704.
[33] (A1) P.C. Fishburn, On collective rationality and a generalized impossibility theorem, Rev. Econom. Studies 41 (1974) 445–457.
[34] (C2) P.C. Fishburn, Semiorders and choice functions, Econometrica 43 (1975) 975–977.
[35] (C1) P.C. Fishburn, Representable choice functions, Econometrica 44 (1976) 1033–1043.
[36] (C2) P.C. Fishburn, Multicriteria choice functions based on binary relations, J. Operations Res. 25 (1977) 989–1012.
[37] (A2) L. Gevers and Cl. D'Aspremont, Equity and the informational basis of collective choice, Rev. Econom. Studies XLIV (1977) 199–209.
[38] (E1) A. Gibbard, Manipulation of voting schemes: a general result, Econometrica 41 (1973) 587–601.
[39] (A1) L. Goodman and H. Markowitz, Social welfare functions based on individual rankings, Amer. J. Sociology 58 (1952) 257–262.
[40] (A1) A. Guha, Neutrality, monotonicity and the right of veto, Econometrica 40 (1972) 821–826.
[41] (C1) (C3) M. Guillaume, Analyse des choix collectifs: un théorème d'existence, Note interne.
[42] (A1) M. Hadjiat, Construction et étude axiomatique d'une procédure d'agrégation des préférences individuelles, Math. Sci. Humaines 52 (1975) 21–34.
[43] (A2) P. Hammond, Equity, Arrow's conditions, and Rawls' difference principle, Econometrica 44 (1976) 793–804.
[44] (A1) B. Hansson, Voting and group decision functions, Synthese 20 (1969) 526–537.
[45] (A1) B. Hansson, Group preferences, Econometrica 37 (1969) 50–54.
[46] (C1) H. Herzberger, Ordinal preference and rational choice, Econometrica 41 (1973) 187–237.
[47] (A6) C. Heuchenne, Un algorithme général pour trouver un sousensemble d'un certain type à distance minimum d'une partie donnée, Math. Sci. Humaines 30 (1970) 23–33.
[48] (A2) C. Hildreth, Alternative conditions for social orderings, Econometrica 21 (1953) 81–94.
[49] (A1) K. Inada, Alternative incompatible conditions for a social welfare function, Econometrica 23 (1955) 396–399.
[50] (A2) K. Inada, On the economic welfare function, Econometrica 32 (1964) 316–338.
[51] (A4) K. Inada, A note on the simple majority decision rule, Econometrica 32 (1964) 525–531.
[52] (A4) K. Inada, The simple majority decision rule, Econometrica 37 (1969) 490–506.
[53] (A1) E. Jacquet-Lagreze, L'agrégation des opinions individuelles, Inform. Sci. Humaines 4 (1969).
[54] (A4) (A5) E. Jacquet-Lagreze, Analyse d'opinions valuées et graphes de préférences, Math. Sci. Humaines, 33 (1971).
[55] (A1) (A4) E. Jacquet-Lagreze, Le problème de l'agrégation des préférences: une classe de procédures à seuil, Math. Sci. Humaines 43 (1973) 29–37.
[56] (A5) E. Jacquet-Lagreze, La modélisation des préférences, préordres, quasi-ordres et relations floues, thèse, SEMA, Rapport de recherche No. 80 (1975).
[57] (C2) D. Jamison and L. Lau, Semiorders and the theory of choice, Econometrica 41 (1973) 901–912.
[58] (C1) E. Kalai et al., Collective choice correspondence as admisible outcomes of social bargaining processes, Econometrica 44 (1976) 233–240.
[59] (A3) R.L. Keeney, Concepts of independance in multiattribute utility theory, in: Multiple Criteria Decision Making (University of South Carolina Press, Columbia, 1973).
[60] (A3) R.L. Keeney, Group preference axiomatization with cardinal utility, Management Sci. 23 (1976) 140–145.
[61] (A1) J.S. Kelly, Voting, anomalies, the number of voters and the number of alternatives, Econometrica 42 (1974) 239–252.
[62] (A2) G.H. Kramer, On a class of equilibrium conditions for majority rule, Econometrica 41 (1973) 285–298.
[63] (A1) G. Kreweras, Représentation polyédrique des préordres complets et application à l'agrégation des préférences, in: La Décision, éditions du C.N.R.S. (1969) 137–151.
[64] (A6) G. Kreweras, Les préordres totaux compatibles avec un ordre partiel, Math. Sci. Humaines 53 (1976) 5–30.
[65] (A1) K. Kuga and H. Nagatani, Voter antagonism and the paradox of voting, Econometrica 42 (1974) 1045–1068.
[66] (C3) S. Lhuilier, Examen du mode d'élection proposé à la Convention Nationale de France en février 1973 et adopté à Genève, Math. Sci. Humaines 54 (1976) 7–24.

[67] (A1) I. Little, Social choice and individual values, J. Political Econo. 50 (1952) 422–432.
[68] (A4) A. Mas-Colell and H. Sonnenschein, General impossibility theorems for group decisions, Rev. Econom. Studies 39 (1972) 185–192.
[69] (A1) K.O. May, A set of independent necessary and sufficient conditions for simple majority decisions, Econometrica 20 (1952) 680–684.
[70] (G) K.O. May, Transitivity, utility, and aggregation in preference patterns, Econometrica 22 (1954) 1–13.
[71] (A1) D. McGarvey, A theorem on the construction of voting paradoxes, Econometrica 21 (1953) 608–610.
[72] (A1) F. Mimiague and J.-M. Rousseau, Effet Condorcet: typologie et calculs de fréquences, Math. Sci. Humaines 43 (1973) 7–27.
[73] (A1) B. Monjardet, Remarques sur une classe de procédures de vote et les 'théorèmes de possibilité', in: La décision, éditions du C.N.R.S. (1969) 177–184.
[74] (A1) B. Monjardet, Quelques problèmes relatifs à la méthode des comparaisons par paires, Math. Sci. Humaines 37 (1972) 69–71.
[75] (A1) (A6) B. Monjardet, Tournois et ordres médians pour une opinion, Math. Sci. Humaines 43 (1973) 55–70.
[76] (E1) B. Monjardet, Lhuilier contre Condorcet au pays des paradoxes, Math. Sci. Humaines 54 (1976) 33–43.
[77] (A1) B. Monjardet, Une autre preuve du théorème d'Arrow, RAIRO 12 (3) (1978) 291–296.
[78] (A5) B. Monjardet, An axiomatic theory of tournament aggregation, Math. Operations Res. (1978) 334–351.
[79] (A1) J.W. Moon, A problem on rankings by committees, Econometrica 44 (1976) 241–246.
[80] (C1) A. Mukherji, The existence of choice functions, Econometrica 45 (1977) 889–894.
[81] (A1) Y. Murakami, A note on the general possibility theorem of the social welfare function, Econometrica 29 (1961) 244–246.
[82] (A2) D.K. Osborne, Irrelevant alternatives and social welfare, Econometrica 44 (1976) 1001–1015.
[83] (C1) P.K. Pattanaik, A note on democratic decision and the existence of choice sets, Rev. Econom. Studies 101 (1968) 1–9.
[84] (C2) P.K. Pattanaik, Sufficient conditions for the existence of a choice set under majority voting, Econometrica 38 (1970) 165–170.
[85] (E1) P.K. Pattanaik, Threats, counter-threats and strategic voting, Econometrica 44 (1976) 91–103.
[86] (E1) B. Peleg, Consistent voting systems, Econometrica 46 (1978) 153–161.
[87] (C2) C. Plott, Path independence, rationality and social voting, Econometrica 41 (1973) 1075–1091.
[88] (A6) G. Ribeill, Equilibre, équivalence, ordre et préordre à distance minimum d'un graphe complet, Math. Sci. Humaines 43 (1973) 71–106.
[89] (C2) R. Rosenthal, Voting majority sizes, Econometrica 43 (1975) 293–300.
[90] (A4) B. Roy, Critères multiples et modélisation des préférences; l'apport des relations de surclassement, Rev. Econom. Politique 1 (1974).
[91] (G) B. Roy, Aide à la décision. Critères Multiples pour Choisir, Trier, Ranger, to appear.
[92] (A2) R. Saponsnik, Social choice with continuous expression of individual preferences, Econometrica 43 (1975) 683–690.
[93] (E1) M. Satterthwaite, Strategy—proofness and Arrow's conditions", J. Econom. Theory 10 (1975) 187–217.
[94] (A1) A. Sen, A possibility theorem on majority decisions, Econometrica 34 (1966) 491–499.
[95] (C1) A. Sen, Quasi-transitivity, rational choice and collective decisions, Rev. Econom. Studies 36 (1969) 381–393.
[96] (A2) A. Sen, Interpersonal aggregation and partial comparability, Econometrica 38 (1970) 393–409; A correction, Econometrica 40 (1972) 959.
[97] (G) A. Sen, Social choice theory: a re-examination, Econometrica 45 (1977) 53–90.
[98] (A1) J.H. Smith, Aggregation of preferences and variable electorate, Econometrica 41 (1973) 1027–1043.
[99] (E1) A. Stoetzel, Un modèle de correspondance entre préférences individuelles et choix collectif, Math. Sci. Humaines 55 (1976) 5–23.