CATEGORY THEORY: A FIRST COURSE

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Abstract. These notes correspond to the Semester Course of Category Theory given in 2020 for Honours Students at the University of Cape Town.

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1. Introduction

As the Abstract says, these notes correspond to the Semester Course of Category Theory given in 2020 for Honours Students at the University of Cape Town. Describing this course one cannot avoid mentioning Mac Lane's "Categories for Working Mathematician", which is *the* book of category theory. Mac Lane's so elegant style of presenting ideas and results makes his book irreplaceable, not only for "working mathematicians who use category theory", but first of all for the experts in category theory; however, the beginners might need more formal definitions, more technicalities, and more exercises, covering much smaller basic material, which was the reason of writing these notes. Nevertheless reading them a good student should also read, at least partly, the first four chapters of Mac Lane's book.

Reading this text requires much less of general mathematical knowledge than Mac Lane's book does, but it requires at least familiarity with some introductory course of set theory; the exercises that require a bit of algebra are marked with *.

The 'collection of all sets' does not form a set; this problem is what one usually refers to as the *problem of size*. We will mostly ignore it here, assuming that in our entire considerations one can always replace "the collection of all" with "a collection of sufficiently many". A proper treatment of the problem of size would involve a considerable amount of material from mathematical logic, e.g. as much as needed to speak about models of set theory. We might, however, indicate that certain 'collections' we consider are not really sets by calling them (proper) classes, or just collections.

The notes consist of 20 sections (including this one) with exercises for almost all of them, the list of categories used with special abbreviated 'names', such as Sets for the category of sets, and selected questions and answers that came up in discussions with students.

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2. Categories

Definition 2.1. *A category $\mathbb{C}$ is a system $\mathbb{C} = (\mathbb{C}_0, \mathbb{C}_1, d_\mathbb{C}, c_\mathbb{C}, e_\mathbb{C}, m_\mathbb{C})$, in which:*

(a) *$\mathbb{C}_0$ and $\mathbb{C}_1$ are classes called the class of objects in $\mathbb{C}$ and the class of morphisms (or arrows) in $\mathbb{C}$, respectively;*
(b) *$d_\mathbb{C}$ and $c_\mathbb{C}$ are maps from $\mathbb{C}_1$ to $\mathbb{C}_0$ called domain and codomain respectively; for $f \in \mathbb{C}_1$, when $d_\mathbb{C}(f) = A$ and $c_\mathbb{C}(f) = B$, we write $f : A \to B$ and say that f is a morphism from A to B, or that the domain of f is A and the codomain of f is B;*
(c) *$e_\mathbb{C}$ is a map from $\mathbb{C}_0$ to $\mathbb{C}_1$ called identity; for $A \in \mathbb{C}_0$, we write $e_\mathbb{C}(A) = 1_A$ and say that 1_A is the identity morphism of A;*
(d) *$d_\mathbb{C}e_\mathbb{C} = 1_{\mathbb{C}_0} = c_\mathbb{C}e_\mathbb{C}$, that is, for every $A \in \mathbb{C}_0$, 1_A is a morphism from A to A;*
(e) *$m_\mathbb{C}$ is a map from $\mathbb{C}_1 \times_{(d_\mathbb{C}, c_\mathbb{C})} \mathbb{C}_1 = \{(g, f) \in \mathbb{C}_1 \times \mathbb{C}_1 \mid d_\mathbb{C}(g) = c_\mathbb{C}(f)\}$ to $\mathbb{C}_1$ called composition; accordingly, the class $\mathbb{C}_1 \times_{(d_\mathbb{C}, c_\mathbb{C})} \mathbb{C}_1$ is called the class of composable pairs of morphisms in $\mathbb{C}$, and the image $m_\mathbb{C}(g, f)$ of (g, f) under $m_\mathbb{C}$ is written as gf (or $g \circ f$) and called the composite of g and f;*
(f) *$1_B f = f = f 1_A$, for every morphism $f : A \to B$ in $\mathbb{C}$;*
(g) *$m_\mathbb{C}$ is associative, that is $h(gf) = (hg)f$ whenever (h, g) and (g, f) are composable pairs of morphisms in $\mathbb{C}$.*

When two morphisms in the same category have the same domain and the same codomain, we say that they are *parallel*.

A morphism $f : A \to A$ is also called an endomorphism of A.

Describing particular categories, one often uses one or more of the following conventions:

- In a category $\mathbb{C}$, the components $\mathbb{C}_0$, $\mathbb{C}_1$, $d_\mathbb{C}$, $c_\mathbb{C}$, $e_\mathbb{C}$, $m_\mathbb{C}$, are far from being independent from each other; for example one cannot change $e_\mathbb{C}$ with all the other components remain the same, as the standard argument $e(A) = e(A)e'(A) = e'(A)$ shows. Nevertheless one usually defines $e_\mathbb{C}$ independently, unless one uses an abbreviated description of a category, as in our convention (iii).

- Defining morphisms $f : A \to B$ in a particular category it might turn out that A and B are not really determined by f alone. This imperfection makes no harm, since one can always eliminate it replacing f with the triple (A, f, B). This is a counterpart of the convention in set theory, according to which a map $f : A \to B$ is often defined as a triple (A, G_f, B), where $G_f \subseteq A \times B$ is what is called the graph of f. Note, however, that the term "graph" has a different meaning in category theory (also different from its meaning in graph theory), which we will see later.
- Many categories are named after their objects, as it is done in Example 2.2 below. Moreover, having some experience constructing categories, one often describes them by describing only their objects, or, in less obvious cases, objects and morphisms; however, we will not do that in this section.

Consider some examples of categories:

Example 2.2. *The category* Sets *of sets is defined as follows:*

(a) $\mathbb{C}_0$ *is the class of all sets, and* $\mathbb{C}_1$ *the class of all maps of sets; domain, codomain, identity, and composition have, in this case, the same meaning as set theory, that is:*
(b) $f : A \to B$ *means that* f *is a map from* A *to* B*;*
(c) $1_A : A \to A$ *is the map defined by* $1_A(a) = a$*;*
(d) *given* $f : A \to B$ *and* $g : B \to C$*, the composite* $gf : A \to C$ *is defined by* $(gf)(a) = g(f(a))$*.*

Example 2.3. *A monoid is a triple* (M, e, m)*, in which* M *is a set,* e *an element in it, and* $m : M \times M \to M$ *a binary operation on* M *with*

$$m(e, x) = x = m(x, e) \text{ and } m(x, m(y, z)) = m(m(x, y), z),$$

for all x, y, $z \in M$*. The set* M *is called the underlying set of* (M, e, m)*, although one usually simply writes* $M = (M, e, m)$*, informally identifying this set with the monoid itself. It is often convenient to use either the additive notation, in which*

$$e = 0 \text{ and } m(x, y) = x + y$$

and the equalities above become

$$0 + x = x = x + 0 \text{ and } x + (y + z) = (x + y) + z,$$

or the multiplicative notation, in which

$$e = 1 \text{ and } m(x, y) = xy \text{ (or } m(x, y) = x \cdot y)$$

and the equalities above become

$$1x = x = x1 \text{ and } x(yz) = (xy)z.$$

While using this notation one says that the monoid M *is additive or multiplicative, respectively, and* e *and* m *are called zero and addition, or one (=identity) and multiplication, respectively. Any, say, multiplicative, monoid* M *can be viewed as a category as follows:*

(a) M_0 *is any one-element set, e.g., one can take* $M_0 = \{M\}$ *or* $M_0 = \{\emptyset\}$*, while* $M_1 = M$*, that is, the elements of* M *become morphisms of* M *viewed as a category;*
(b) *since* M_0 *has only one element, the maps* d_M *and* c_M *are uniquely determined and every pair of morphisms (=elements of M) is composable;*
(c) $e_M = 1$*, that is, the identity morphism of the unique object of* M *is* 1 *of* M*.*
(d) *the composition of* M *viewed as a category is the same as the multiplication of* M *considered as a monoid.*

Example 2.4. *A preorder is a pair (P, R), in which P is a set and R is a preorder relation on P, that is $R \subseteq P \times P$ is reflexive and transitive. One usually writes $x \leqslant y$ to mean $(x, y) \in R$, and then reflexivity and transitivity become*

$$x \leqslant x \text{ and } (x \leqslant y \;\&\; y \leqslant z) \Rightarrow x \leqslant z$$

(for all x, y, $z \in P$), respectively. Again, one usually simply writes $P = (P, R)$, informally identifying the underlying set P with the given preorder. Any preorder $P = (P, R)$ can be viewed as a category as follows:

(a) *$P_0 = P$ and $P_1 = R$, and each morphism in this category is of the form $(x, y) : x \to y$, for some $x \leqslant y$ in P;*
(b) *as follows from (a), there is no choice but to take $1_x = (x, x)$ and $(y, z)(x, y) = (x, z)$, for all x, y, $z \in P$ with $x \leqslant y \leqslant z$, which fully describes the identity and composition.*

In particular, this applies to ordered sets, which are preorders satisfying the anti-symmetry condition $(x \leqslant y \;\&\; y \leqslant x) \Rightarrow x = y$.

Example 2.5. *Let $\mathbb{C} = (\mathbb{C}_0, \mathbb{C}_1, d_{\mathbb{C}}, c_{\mathbb{C}}, e_{\mathbb{C}}, m_{\mathbb{C}})$ be a category and $\mathbb{S}_0$ and $\mathbb{S}_1$ subclasses of $\mathbb{C}_0$ and $\mathbb{C}_1$, respectively, satisfying the following conditions:*

(a) *If $f : A \to B$ is a morphism in $\mathbb{C}$ with f in $\mathbb{S}_1$, then A and B are in $\mathbb{S}_0$;*
(b) *If A is an object in $\mathbb{C}$ that is in $\mathbb{S}_0$, then 1_A is in $\mathbb{S}_1$;*
(c) *If (g, f) is a composable pair of morphisms in $\mathbb{C}$ with f and g in $\mathbb{S}_1$, then the composite gf also is in $\mathbb{S}_1$.*

Then the pair $(\mathbb{S}_0, \mathbb{S}_1)$ determines a category $\mathbb{S} = (\mathbb{S}_0, \mathbb{S}_1, d_{\mathbb{S}}, c_{\mathbb{S}}, e_{\mathbb{S}}, m_{\mathbb{S}})$, in which $\mathbb{S}_0$ and $\mathbb{S}_1$ are the same and $d_{\mathbb{S}}$, $c_{\mathbb{S}}$, $e_{\mathbb{S}}$, and $m_{\mathbb{S}}$ are induced by $d_{\mathbb{C}}$, $c_{\mathbb{C}}$, $e_{\mathbb{C}}$, and $m_{\mathbb{C}}$, respectively. We then say that $\mathbb{S}$ is a subcategory of $\mathbb{C}$, or, more precisely, that $\mathbb{S}$ is the subcategory of $\mathbb{C}$ determined by $\mathbb{S}_0$ and $\mathbb{S}_1$. Furthermore we say that:

- *$\mathbb{S}$ is a full subcategory of $\mathbb{C}$, if, for every A and B in $\mathbb{S}_0$ and every morphism $f : A \to B$ in $\mathbb{C}$, f is in $\mathbb{S}_1$; in this case one usually describes $\mathbb{S}$ as the full subcategory of $\mathbb{C}$ with objects in $\mathbb{S}_0$;*
- *$\mathbb{S}$ is a wide subcategory of $\mathbb{C}$, if $\mathbb{S}_0 = \mathbb{C}_0$; in this case one usually describes $\mathbb{S}$ as the wide subcategory of $\mathbb{C}$ with morphisms in $\mathbb{S}_1$;*

Example 2.6. *The opposite category $\mathbb{C}^{\mathrm{op}}$ of a category $\mathbb{C} = (\mathbb{C}_0, \mathbb{C}_1, d_{\mathbb{C}}, c_{\mathbb{C}}, e_{\mathbb{C}}, m_{\mathbb{C}})$ is constructed as $\mathbb{C}^{\mathrm{op}} = (\mathbb{C}_0, \mathbb{C}_1, c_{\mathbb{C}}, d_{\mathbb{C}}, e_{\mathbb{C}}, m^{\mathrm{op}}_{\mathbb{C}})$, where $m^{\mathrm{op}}_{\mathbb{C}}$ is defined by $m^{\mathrm{op}}_{\mathbb{C}}(f, g) = m_{\mathbb{C}}(g, f)$; note also that (f, g) is a composable pair of morphisms in $\mathbb{C}^{\mathrm{op}}$ if and only if (g, f) is a composable pair of morphisms in $\mathbb{C}$. One sometimes says "dual" instead of "opposite", but "dual" also has another meaning.*

Exercises

1. Given a set X, all maps from X to X form a monoid, which is a motivating example of a monoid. Explain that Example 2.2 similarly motivates Definition 2.1.
2. Let M and N be monoids, for which we will use the multiplicative notation. A monoid homomorphism $f : M \to N$ is a map f from M to N with $f(1) = 1$ and $f(xy) = f(x)f(y)$ for all x, $y \in M$. Define the category Mon of monoids similarly to Example 2.2, but replacing sets with monoids and replacing maps of sets with monoid homomorphisms. Do the same for a few, say, ten, examples of familiar types of mathematical structures on sets that admit a good notion of a structure preserving map. For instance, define the categories Preord and Ord of preordered sets (=preorders) and of ordered sets (=orders), where the preservation of (pre)order by a map f should be defined by $x \leqslant y \Rightarrow f(x) \leqslant f(y)$. Note: these categories will be used in exercises of other sections.

3. Show that the class $\mathcal{M}$ determines a wide subcategory of Sets with morphisms in $\mathcal{M}$ when $\mathcal{M}$ is the class of all maps that are:
(a) injective;
(b) surjective;
(c) bijective.

4. Define composable n-tuples ($n = 0, 1, 2, 3, 4, ...$) of morphisms in a category and show that each such n-tuple has a well-defined composite with no parentheses needed. Hint: define $f_n...f_1$ by induction, and then show that $(g_n...g_1)(f_m...f_1) = g_n...g_1f_m...f_1$.

5. Define the notion of opposite preorder and explain how it agrees with the notion of opposite category.

3. Algebraic categories

Definition 3.1. *Let Ω be a set equipped with a map $l_\Omega : \Omega \to \mathbf{N}$, where $\mathbf{N} = \{0, 1, 2, ...\}$ is the set of natural numbers; we will say that Ω is a signature. An Ω-algebra is a pair (A, v) in which A is a set and*

$$v : \Omega \to \bigcup_{n\in\mathbf{N}} A^{A^n}$$

is a map with $v(\omega) \in A^{A^{l(\omega)}}$, for all $\omega \in \Omega$. The set A is called the underlying set of (A, v).

Together with this definition, let us introduce some terminology, notation, and conventions:

- Given a set A, a map $A^n \to A$ is called an *n-ary operation* on A. Accordingly, when (A, v) is an Ω-algebra as in Definition 3.1, and $\omega \in \Omega$ has $l_\Omega(\omega) = n$, one says that ω is an *n-ary operation symbol*, or a *name for an n-ary operation*, or an *n-ary operator*, and that $v(\omega) : A^n \to A$ is an operation on A *determined by* (or *associated to*) ω, or the *valuation* of ω, or the *interpretation* of ω in (A, v). Operations on A of the form $v(\omega)$ (for some $\omega \in \Omega$) are called the *basic operations* on A. The signature Ω of (A, v) can also be called the *set of names of basic operations* for (A, v), or the *set of operators* for (A, v) (having in mind that l_Ω is also given). Logicians say "functional symbol" instead of "operation symbol".
- In the situation above, for $a_1, ..., a_n$ in A, one usually writes $\omega(a_1, ..., a_n)$ instead of $v(\omega)(a_1, ..., a_n)$. When $n = 0$, 1, and 2, this is usually further abbreviated as ω, ωa_1, and $a_1\omega a_2$, respectively. When n small:

One says	instead of
nullary	0-ary
unary	1-ary
binary	2-ary
ternary	3-ary

- Let ω be a nullary operator. The notation above suggests to write $\omega \in A$ to mean $v(\omega) \in A$. In fact $v(\omega)$ is a map from A^0, which is a one-element set, to A, and so, indeed, it can be identified with an element in A. This agrees with, say, linear algebra, where we write $0 \in V$ for any vector space V; cf. Example 2.3, where we had $0 \in M$ in the additive notation, and $1 \in M$ in the multiplicative notation. The authors who prefer not to use this convention, call nullary operations *constants*, and define constants of A as chosen elements of A.

- Once the notation and conventions above are freely used, we will also write informally $A = (A, v)$, ignoring therefore the distinction between the Ω-algebra (A, v) and its underlying set A. Such identifications will also be used for other mathematical structures, including (Ω, l_Ω) above, for which we will write $\Omega = (\Omega, l_\Omega)$.
- Furthermore, sometimes it is convenient to avoid using the symbol l_Ω completely and use the sets $\Omega_n = \{\omega \in \Omega \mid l_\Omega(\omega) = n\}$ $(n = 0, 1, 2, ...)$ instead. In other words, one can also think of Ω as a sequence $(\Omega_0, \Omega_1, \Omega_2, ...)$ of disjoint sets.

Definition 3.2. *Let A and B be Ω-algebras. An Ω-algebra homomorphism, or, simply, a homomorphism, $f : A \to B$ is a map $f : A \to B$ of the underlying sets with*

$$f(\omega(a_1, ...a_n)) = \omega(f(a_1), ..., f(a_n)),$$

for every $n = 0, 1, 2, ...,$ every $\omega \in \Omega_n$, and every $a_1, ..., a_n \in A$.

It is easy to see that:

- for every Ω-algebra A, the identity map 1_A is a homomorphism $1_A : A \to A$;
- for Ω-algebra homomorphisms $f : A \to B$ and $g : B \to C$, the (usual) composite $gf : A \to C$ is an Ω-algebra homomorphism;
- this defines a category, to be called the category of Ω-algebras; it will be denoted by $\mathsf{Alg}(\Omega)$.

Universal algebra studies various full subcategories of $\mathsf{Alg}(\Omega)$ for various Ω; some particular ones, including those considered in the rest of this section are studied in classical algebra.

Example 3.3. $\mathsf{Alg}(\emptyset) = \mathsf{Sets}$.

Example 3.4. *If Ω consists of one operator, and that operator is nullary, then* $\mathsf{Alg}(\Omega)$ *is called the category of pointed sets; we will simply write*

$$\mathsf{Alg}(\Omega) = \mathsf{Pointed\,Sets}.$$

This category can be identified with the category of pairs (X, x), where X is a set and $x \in X$.

Example 3.5. *If Ω consists of one operator, and that operator is binary, then* $\mathsf{Alg}(\Omega)$ *is called the category of magmas; we will write*

$$\mathsf{Alg}(\Omega) = \mathsf{Magmas}.$$

This category can be identified with the category of pairs (X, m), where X is a set and m a binary operation on X.

Example 3.6. *A magma (X, m) is called a semigroup if m is associative. This determines a full subcategory* $\mathsf{Semigroups}$ *of the category* Magmas.

Example 3.7. *Suppose Ω consists of two operators, one nullary and one binary. Then* $\mathsf{Alg}(\Omega)$ *can be identified with the category of triples (X, e, m), where X is a set, $e \in X$, and m is a binary operation on X. This makes the category* Mon *of monoids (cf. Exercise 2 of Section 2) a full subcategory of* $\mathsf{Alg}(\Omega)$.

Example 3.8. *A group is a four-tuple (G, e, m, i), in which (G, e, m) is a monoid, and i a unary operation on G with*

$$m(i(x), x) = e = m(x, i(x)),$$

for all $x \in G$, although it would be sufficient to require only one of these equalities (does not matter which one). This makes the category Groups *of groups a full subcategory of* $\mathsf{Alg}(\Omega)$, *where Ω consists of three operators, one nullary, one binary,* *and one unary. Note that, as in the case of monoids, one might use the multiplicative or the additive notation; for the first of them one writes $i(x) = x^{-1}$ while for the second one it becomes $i(x) = -x$.*

Example 3.9. *We obtain full subcategories* $\mathsf{CommSemigroups}$*, of commutative semigroups,* $\mathsf{CommMon}$*, of commutative monoids, and* Ab *of abelian (= commutative) groups, of the categories* $\mathsf{Semigroups}$*,* Mon*, and* Groups*, respectively, by requiring the commutativity condition $m(x, y) = m(y, x)$. A further possible condition is idempotency $m(x, x) = x$, creating the categories idempotent commutative semigroups and idempotent commutative monoids; one does not consider "idempotent commutative groups" since every such group would have only one element.*

Example 3.10. *Suppose $\Omega = \Omega_1$, that is, every operator in Ω is unary. Then the category $\mathsf{Alg}(\Omega)$ can be identified with the category of pairs (X, h), where X is a set and $h : \Omega \times X \to X$ is a map; this identification makes $h(\omega, x) = \omega x$, for all $x \in X$. If Ω is equipped with a monoid structure, say, multiplicative, then one might require*

$$1x = x \text{ and } \omega'(\omega x) = (\omega'\omega)x,$$

for all $x \in X$, and then call $X = (X, h)$ an Ω-set. This defines the category Sets^{Ω} of Ω-sets as a full subcategory of $\mathsf{Alg}(\Omega)$.

Exercises

1. Present a few, say, five, examples of familiar types mathematical structures, not considered above, as full subcategories of $\mathsf{Alg}(\Omega)$, for suitable Ω's. Have you already considered them for Exercise 2 of Section 2?
2. Have your examples for the previous exercise included vector spaces over a given field, or, more generally, modules over a given ring? If not, can you still make them Ω-algebras? Hint: use Example 3.10.
3. Let X be a set, $P(X)$ its power set (that is, the set of all subsets of X), and let $\cup$ = union, $\cap$ = intersection, and $+$ = symmetric difference be considered as binary operations on $P(X)$. Show that:
   (a) $(P(X), \emptyset, \cup)$ and $(P(X), X, \cap)$ are idempotent commutative monoids;
   (b) $(P(X), \emptyset, +, 1_{P(X)})$ is an abelian group.
   Remark: doing this exercise one is expected to know many more standard examples of groups and monoids of course.

4. Isomorphisms

In this and in fact also in the three following sections we 'are inside' an arbitrary but fixed category $\mathbb{C}$, which means that all objects and morphisms we will consider are in $\mathbb{C}$.

Definition 4.1. *A morphism $f : A \to B$ is said to be an isomorphism if it is invertible, that is, if there exists a morphism $g : B \to A$ with $gf = 1_A$ and $fg = 1_B$. If this is the case, then g above is called the inverse of f and we write $g = f^{-1}$.*

An endomorphism that is an isomorphism is also called an automorphism.

Of course the second sentence in Definition 4.1 assumes obvious that there is at most one g with $gf = 1_A$ and $fg = 1_B$. Indeed, if also $g'f = 1_A$ and $fg' = 1_B$, then $g' = g'1_B = g'(fg) = (g'f)g = 1_A g = g$. Other first obvious observations on isomorphisms are formulated in the following theorem, where actually a part of (b) repeats a part of (a), and both (a) and (b) only need straightforward calculations, essentially the same as what is needed to prove (c):

Theorem 4.2. *The class $\mathrm{Iso}(\mathbb{C})$ of all isomorphisms in $\mathbb{C}$ has the following properties:*

(a) *it contains all identity morphisms and is closed under composition and taking inverses of invertible morphisms;*
(b) *if $h = gf$ and two of the morphisms f, g, and h are isomorphisms, then so is the third one;*
(c) $1_A^{-1} = 1_A$, $(gf)^{-1} = f^{-1}g^{-1}$, *and* $(f^{-1})^{-1} = f$, *for every object A and every composable pair (g, f) of isomorphisms.* □

Definition 4.3. *We say that objects A and B are isomorphic and write $A \approx B$ if there exists an isomorphism from A to B (or, equivalently, from B to A).*

From Theorem 3.2(a), we immediately obtain:

Corollary 4.4. *The isomorphism relation $\approx$ is an equivalence relation on the class $\mathbb{C}_0$ of objects in $\mathbb{C}$.* □

The notion of isomorphism suggests introducing two 'extreme' types of categories:

Definition 4.5. *A given category is said to be:*

(a) *a groupoid, if every morphism in it is an isomorphism;*
(b) *a skeleton, if its objects satisfy the implication $A \approx B \Rightarrow A = B$.*

Exercises

1. It is easy to characterize isomorphisms in many categories. For example, a morphism in Sets is an isomorphism if and only if it is a bijective map. Prove this, and characterize isomorphisms in all other categories considered so far.
2. Describe all monoids of $\leqslant 3$ elements up to isomorphism.
3. Which monoids, viewed as categories, are groupoids?
4. Describe all preorders of $\leqslant 3$ elements up to isomorphism.
5. Which preorders viewed as categories are groupoids?
6. Which preorders viewed as categories are skeletons?
7. Describe all finite-dimensional vector spaces (over a given field) up to isomorphism.*
8. Describe all finite Boolean algebras up to isomorphism.*
9. Define the category of fields, describe all prime fields up to isomorphism, and prove that the full subcategory of fields with objects all prime fields is a groupoid.*

5. Initial and terminal objects

Definition 5.1. *An object Z in a category $\mathbb{C}$ is said to be:*

(a) *initial, if, for every object A in $\mathbb{C}$, there exists a unique morphism from Z to A;*
(b) *terminal, if, for every object A in $\mathbb{C}$, there exists a unique morphism from A to Z;*
(c) *zero, if it initial and terminal at the same time.*

Some authors use different terminology, or symbols 0 and 1, as shown in the table

Initial	Terminal
Universal	Couniversal
Left universal	Right universal
Left zero	Right zero
0	1

although the symbol “0” is sometimes used only for a zero object.

It is obvious that an object Z is:

- initial in $\mathbb{C}$ if and only if it is terminal in $\mathbb{C}^{\mathrm{op}}$;
- terminal in $\mathbb{C}$ if and only if it is initial in $\mathbb{C}^{\mathrm{op}}$.

One says that the notions of initial and terminal are *dual* to each other. Similarly, whatever is done, being it a definition, a construction, or an argument, in an abstract category, has its dual version. We will often simply say “dually”, either shortening or even omitting the details. Furthermore, some categorical notions, e.g. the notion of isomorphism, are self-dual.

Theorem 5.2. *Let Z and A be objects in a category $\mathbb{C}$. Then:*

(a) *if Z is initial, then A is initial if and only if A is isomorphic to Z in $\mathbb{C}$;*
(b) *if Z is terminal, then A is terminal if and only if A is isomorphic to Z in $\mathbb{C}$;*

Proof. (a): The “if” part is obvious. For the “only part” we observe:

- since both Z and A are initial, there are morphisms $f : Z \to A$ and $g : A \to Z$;
- since Z is initial, $gf = 1_Z : Z \to Z$;
- since A is initial, $fg = 1_A : A \to A$;
- that is, f and g are isomorphisms, inverse to each other.

(b) is dual to (a). □

When a mathematical construction is either defined as an initial object or defined as a terminal object in a certain category, one says that it is defined via a *universal property*, or defined as a *universal construction.* As follows from Theorem 5.2, the universal constructions are determined uniquely up to isomorphism. Defining by a universal property is a widely generalized idea of defining as “the smallest ... such that ...” or as the “the largest ... such that ...”, as the following example shows:

Example 5.3. *Recall that an ordered set (or a poset, where “po” is an abbreviation of “partial order”) $P = (P, \leqslant)$ is the same as a preorder $P = (P, \leqslant)$ in which $\leqslant$ is antisymmetric in the sense that $(x \leqslant y \;\&\; y \leqslant x) \Rightarrow x = y$ in it. An element z in P is:*

(a) *initial in P viewed as a category, if and only if it is the smallest element of the poset P, that is, if and only if $z \leqslant x$ for each $x \in P$;*
(b) *terminal in P viewed as a category, if and only if it is the largest element of the poset P, that is, if and only if $x \leqslant z$ for each $x \in P$.*

Example 5.4. *The category of sets has both initial and terminal objects. They are the empty set and any one-element set, respectively.*

Exercises

1. Give examples of categories, say, ten, where you can describe initial and terminal objects. Do such objects always exist?
2. Explain that the system of natural numbers can be defined as an initial object in the category of triples (X, e, f), where X is a set, e an element in X and f is a map from X to X.
3. Define other number systems by means of universal properties (in some cases there is more than one natural way to do that).*
4. When X is a basis of a vector space V, one also says that V is a free vector space on X. Define “free” by means of a universal property, and apply similar definitions to other algebraic structures; are there types of algebras for which you can construct free ones? Give examples showing that not every algebra is free.*

5. Given objects A and B in a category $\mathbb{C}$, let $\mathbb{D}$ be the category of triples (C, α, β), in which $\alpha : C \to A$ and $\beta : C \to B$ are morphisms in $\mathbb{C}$; a morphism $\gamma : (C, \alpha, \beta) \to (C', \alpha', \beta')$ in $\mathbb{D}$ is a morphism $\gamma : C \to C'$ with $\alpha'\gamma = \alpha$ and $\beta'\gamma = \beta$. A terminal object in $\mathbb{D}$, usually written as $(A \times B, \pi_1, \pi_2)$ is called the product of A and B. Dually, the product of A and B in $\mathbb{C}^{\mathrm{op}}$ is called the coproduct of A and B (in $\mathbb{C}$). Describe:

(a) products and coproducts in Sets; hint: use cartesian products and disjoint unions respectively;

(b) products and coproducts in Preord and in Ord; hint: use the previous exercise;

(c) products and coproducts in a preorder viewed as a category, and, in particular, in $(P(X), \subseteq)$ (where $P(X)$ is as in Exercise 3 of Section 3);

(d) products in $\mathsf{Alg}(\Omega)$; hint: to describe $A \times B$ in $\mathsf{Alg}(\Omega)$ use $A \times B$ in Sets with the Ω-algebra structure defined by

$$\omega((a_1, b_1), ..., (a_n, b_n)) = (\omega(a_1, ..., a_n), \omega(b_1, ..., b_n));$$

(e) products and coproducts in categories described in Examples 3.4 and 3.6-3.9.

6. Monomorphisms and epimorphisms

One can think of the notion of isomorphism as a generalization of the notion of invertible element in a monoid. Now we are going to generalize, similarly, the notions of left/right invertible and left/right cancellable element.

Again, we 'are inside' an arbitrary but fixed category $\mathbb{C}$.

Definition 6.1. *A morphism $f : A \to B$ is said to be:*

(a) *a monomorphism, if $fa = fa' \Rightarrow a = a'$ whenever a and a' are parallel morphisms with codomain A;*

(b) *an epimorphism, if $bf = b'f \Rightarrow b = b'$ whenever b and b' are parallel morphisms with domain B;*

(c) *a bimorphism if it is a monomorphism and epimorphism at the same time;*

(d) *a split monomorphism if there exist a morphism $g : B \to A$ with $gf = 1_A$;*

(e) *a split epimorphism if there exist a morphism $g : B \to A$ with $fg = 1_B$.*

Theorem 6.2. *For the five classes of morphisms in a given category introduced in Definition 6.1, we have:*

(a) *all these classes are closed under composition and contain all isomorphisms;*

(b) *if gf is a monomorphism, then so is f;*

(c) *if gf is an epimorphism, then so is g;*

(d) *if gf is a split monomorphism, then so is f;*

(e) *if gf is a split epimorphism, then so is g;*

(f) *if a morphism is monomorphism and a split epimorphism at the same time, then it is an isomorphism;*

(g) *if a morphism is epimorphism and a split monomorphism at the same time, then it is an isomorphism.* □

Proof. Let us prove only (f). If $f : A \to B$ is a split epimorphism, then there exists a morphism $g : B \to A$ with $fg = 1_B$. We also have $f(gf) = (fg)f = 1_B f = f = f1_A$, and, if f is a monomorphism, this gives $gf = 1_A$. □

Theorem 6.3. *A morphism $f : A \to B$ is an isomorphism in each of the following cases:*

(a) *B is an initial object and f is a monomorphism;*

(b) *A is a terminal object and f is an epimorphism.*

Proof. (a): Since B initial, there exist a morphism g from B to A. Applying initiality again, we also see that $fg = 1_B$, making f a split epimorphism. Therefore f is an isomorphism by Theorem 6.2(f).

(b) is dual to (a). □

Exercises

1. Prove all assertions of Theorem 6.2.
2. Explain that Theorem 6.3 improves the "only if" parts of Theorem 5.2.
3. Explain the first two sentences of this section.
4. Describe each of the five classes of morphisms from Definition 6.1 in Sets and in a preorder viewed as a category. In particular show that a morphism in Sets is:
   (a) a monomorphism if and only if it is an injective map;
   (b) an epimorphism if and only if it is a surjective map.
   Furthermore, almost every monomorphism in Sets is split (that is, is a split monomorphism). Why "almost"?
5. Explain that Axiom of Choice can be formulated as: every epimorphism in Sets is split (that is, is a split epimorphism).
6. Let $\mathbb{C}$ be a subcategory of either $\mathsf{Alg}(\Omega)$ (for some signature Ω), or of Preord (although preorder axioms will be irrelevant in this exercise). For a morphism $f : A \to B$ in $\mathbb{C}$, explain that (a) and (b) of Exercise 4 immediately imply:
   (a) if f is an injective map, then it is a monomorphism;
   (b) if f is a surjective map, then it is an epimorphism;
7. Describe each of the five classes of morphisms from Definition 6.1 in the following categories:
   (a) the category of pointed sets;
   (b) the category of vector spaces over a fixed field;*
   (c) the category of abelian groups.*
8. Describe monomorphisms, epimorphisms, and bimorphisms in Preord and in Ord. Would you say that these descriptions trivially agree?
9. Prove that the inclusion map from the additive monoid of natural numbers to the additive group of integers (considered as a monoid) is a bimorphism in Mon.
10. Modify the previous exercise to involve the ring of integers and the field of rational numbers.*
11. Explain that Theorem 6.3(a) applied in the situation of Exercise 2 of Section 5 gives the Induction Principle.

7. hom SETS

We write $\hom_{\mathbb{C}}(A, B)$, or simply $\hom(A, B)$, for the set of all morphisms from A to B in a category $\mathbb{C}$. Other authors might write "Hom" with capital "H", or any of the following:

$$\mathrm{mor}(A,B),\ \mathrm{mor}_{\mathbb{C}}(A,B),\ \mathrm{Mor}(A,B),\ \mathrm{Mor}_{\mathbb{C}}(A,B),\ \mathbb{C}(A,B),\ [A,B],\ \mathbb{C}[A,B],$$

to denote the same set.

Remark 7.1. *One can define a category in terms of these hom sets, as a system* $\mathbb{C} = (\mathbb{C}_0, (\hom_{\mathbb{C}}(A,B))_{(A,B)\in\mathbb{C}_0\times\mathbb{C}_0}, (e_A)_{A\in\mathbb{C}_0}, (m_{A,B,C})_{(A,B,C)\in\mathbb{C}_0\times\mathbb{C}_0\times\mathbb{C}_0})$*, in which* e_A *is an element of* $\hom_{\mathbb{C}}(A,A)$ $(A \in \mathbb{C}_0)$ *written as* 1_A*, and* $m_{A,B,C}$ *is a map* $\hom_{\mathbb{C}}(B,C) \times \hom_{\mathbb{C}}(A,B) \to \hom_{\mathbb{C}}(A,C)$ *written as* $m_{A,B,C}(g,f) = gf$*, and such that* $1_B f = f = f 1_A$ *and* $h(gf) = (hg)f$*, for all* $A, B, C, D \in \mathbb{C}_0$ *and* $f \in \hom_{\mathbb{C}}(A,B)$*,* $g \in \hom_{\mathbb{C}}(B,C)$*, and* $h \in \hom_{\mathbb{C}}(C,D)$*. It seems that in order* *to make this definition equivalent to Definition 2.1, one has to require the* hom-*disjointness condition*

$$\hom_{\mathbb{C}}(A,B) \cap \hom_{\mathbb{C}}(A',B') \neq \emptyset \Rightarrow (A = A' \ \&\ B = B'),$$

which was required in old text books (otherwise the domain and the codomain maps are not well-defined), but, as we will see later (Exercise 3 of Section 13), this condition can be avoided. Right now we can only say that once it is not satisfied we can make it satisfied by replacing each $f \in \hom_{\mathbb{C}}(A,B)$ with (A, f, B). This was already mentioned in convention (ii) in Section 2. Partly rephrasing it, let us point out that this is important, since, defining a particuler category $\mathbb{C}$, one often begins with $\mathbb{C}_0$ and then says "a morphism $f : A \to B$ in $\mathbb{C}$ is ...", not paying attention on the hom-*disjointness condition, which we will also do. In fact this was relevant already in Exercise 2 of Section 2.*

Definition 7.2. *Given a morphism $f : A \to B$ and an object X in the same category $\mathbb{C}$, we introduce induced maps*

(a) $\hom(X, f) : \hom(X, A) \to \hom(X, B)$ *defined by* $\hom(X, f)(a) = fa$;
(b) $\hom(f, X) : \hom(B, X) \to \hom(A, X)$ *defined by* $\hom(f, X)(b) = bf$.

Exercises

1. Explain that $\hom_{\mathbb{C}}(A,B) = \hom_{\mathbb{C}^{\mathrm{op}}}(B,A)$ makes the constructions 7.2(a) and 7.2(b) dual to each other, and observe dualities in other exercises of this section.

2. In the notation of Definition 7.2, show that:
(a) $\hom(X, 1_A)$ is the identity map of $\hom(X, A)$;
(b) given also a morphism $g : B \to C$, we have

$$\hom(X,g)\hom(X,f) = \hom(X,gf), \ \ \hom(f,X)\hom(g,X) = \hom(gf,X).$$

3. Given morphisms $f : A \to B$ and $g : C \to D$ define the induced map $\hom(g, f) : \hom(D, A) \to \hom(C, B)$ and explain how it is related to Definition 7.2.

4. Prove that the following conditions of a morphism $f : A \to B$ in a category $\mathbb{C}$ are equivalent:
(a) f is an isomorphism;
(b) the map $\hom(X, f)$ is bijective for every object X in $\mathbb{C}$;
(c) the map $\hom(f, X)$ is bijective for every object X in $\mathbb{C}$.

5. Reformulate Definition 6.1 in terms of hom sets. For, show that a morphism f in Sets is:
(a) a monomorphism if and only if the map $\hom(X, f)$ is injective for every object X in $\mathbb{C}$;
(b) an epimorphism if and only if the map $\hom(f, X)$ is injective for every object X in $\mathbb{C}$;
(c) a bimorphism if and only if the maps $\hom(X, f)$ and $\hom(f, X)$ are injective for every object X in $\mathbb{C}$;
(d) a split monomorphism if and only if the map $\hom(f, X)$ is surjective for every object X in $\mathbb{C}$;
(e) a split epimorphism if and only if the map $\hom(X, f)$ is surjective for every object X in $\mathbb{C}$.

6. Explain how the previous exercise can be used to prove Theorem 6.2 'set-theoretically' (see also Exercise 3 of Section 2).

8. Functors

Categories themselves form a category Cat (in fact 'smaller' categories form a 'larger' category), whose morphisms are called functors; they are defined as follows:

Definition 8.1. *Let $\mathbb{A}$ and $\mathbb{B}$ be categories. A functor $F : \mathbb{A} \to \mathbb{B}$ is a pair $F = (F_0, F_1)$, in which $F_0 : \mathbb{A}_0 \to \mathbb{B}_0$ and $F_1 : \mathbb{A}_1 \to \mathbb{B}_1$ are maps, satisfying the following conditions:*

(a) *if a is a morphism in $\mathbb{A}$ from A to A', then $F_1(a)$ is a morphism (in $\mathbb{B}$) from $F_0(A)$ to $F_0(A')$; it is written simply as $F(a) : F(A) \to F(A')$;*
(b) *$F(1_A) = 1_{F(A)}$, for every object A in $\mathbb{A}$;*
(c) *$F(a'a) = F(a')F(a)$, for every composable pair (a', a) of morphisms in $\mathbb{A}$.*

What remains to describe Cat is to say what is the identity functor of any category $\mathbb{A}$, and to say how to compose functors. Rather obviously, this is done as follows:

$$1_{\mathbb{A}} = (1_{\mathbb{A}_0}, 1_{\mathbb{A}_1}) \text{ and } GF = (G_0F_0, G_1F_1).$$

Convention (iii) in Section 2 on categories, obviously suggests a similar convention of functors, namely to describe only their 'object part'; at the beginning we will not use this convention but only omit checking conditions 8.1(b) and 8.1(c).

Consider some examples of functors:

Example 8.2. *If $\mathbb{S}$ is a subcategory of a category $\mathbb{C}$, there is the inclusion functor $\mathbb{S} \to \mathbb{C}$, which carries objects and morphisms of $\mathbb{S}$ to the same objects and morphisms in $\mathbb{C}$.*

Example 8.3. *For an arbitrary category $\mathbb{C}$ and an object X in $\mathbb{C}$, we have:*

(a) *the functor $\hom(X, -) = \hom_{\mathbb{C}}(X, -) : \mathbb{C} \to \mathsf{Sets}$, which carries any $f : A \to B$ from $\mathbb{C}$ to*
$$\hom(X, f) : \hom(X, A) \to \hom(X, B)$$
as defined in 7.2(a);
(b) *the functor $\hom(-, X) = \hom_{\mathbb{C}}(-, X) : \mathbb{C}^{\mathrm{op}} \to \mathsf{Sets}$, which carries any $f : B \to A$ from $\mathbb{C}^{\mathrm{op}}$ to*
$$\hom(f, X) : \hom(B, X) \to \hom(A, X)$$
as defined in 7.2(b).

Example 8.4. *Let M be a multiplicative monoid and X an M-set (see Example 3.10). Considering M as category, as in Example 2.3, X determines a functor $M \to \mathsf{Sets}$, constructed as follows:*

(a) *it carries the unique object of M to X;*
(b) *it carries any morphism u of M (that is, any element u of M) to the map $X \to X$, defined by $x \mapsto ux$.*

Moreover, every functor $F : M \to \mathsf{Sets}$ can be obtained this way by taking X to be the F-image of the unique object of M and putting $ux = F(u)(x)$.

Definition 8.5. *For a functor $F : \mathbb{A} \to \mathbb{B}$, we say that:*

(a) *F preserves monomorphisms, if, for every monomorphism m in $\mathbb{A}$, $F(m)$ is a monomorphism in $\mathbb{B}$;*
(b) *F reflects monomorphisms, if a morphism m in $\mathbb{A}$ is a monomorphism whenever so is $F(m)$.*

Preservation and reflection of epimorphisms, bimorphisms, split monomorphisms, split epimorphisms, and isomorphisms is defined similarly.

Definition 8.6. *Given a functor $F : \mathbb{A} \to \mathbb{B}$ and objects A and A' in $\mathbb{A}$, the map*

$$\hom_{\mathbb{A}}(A, A') \to \hom_{\mathbb{B}}(F(A), F(A'))$$

induced by F will be denoted by $F_{A,A'}$. A functor $F : \mathbb{A} \to \mathbb{B}$ is said to be

(a) *faithful, if each $F_{A,A'}$ is an injection;*

(b) *full, if each $F_{A,A'}$ is a surjection;*
(c) *f.f. (=fully faithful), if each $F_{A,A'}$ is a bijection;*
(d) *e.i.o. (=essentially injective on objects), if $F(A) \approx F(A') \Rightarrow A \approx A'$;*
(e) *e.s.o. (=essentially surjective on objects), if, for every object B in $\mathbb{B}$, there exists an object A in $\mathbb{A}$ with $F(A) \approx B$;*
(f) *e.b.o. (=essentially bijective on objects), if it is e.i.o. and e.s.o. at the same time.*

Exercises

1. Let $\mathbb{A}$ and $\mathbb{B}$ be categories, and B a fixed object in $\mathbb{B}$. Define the constant functor $\mathrm{Const}_B : \mathbb{A} \to \mathbb{B}$ associated to B.

2. If X is a fixed set and $F : \mathsf{Sets} \to \mathsf{Sets}$ a functor claimed to be defined by $F(A) = X \times A$, how would you define it on morphisms?

3. Let X be a fixed set. Show that, for the conditions (a)-(c) below, we have: (b) and (c) are equivalent to each other and imply (a), while (a) implies (b) and (c) whenever X is finite.
  (a) there exists a functor $F : \mathsf{Sets} \to \mathsf{Sets}$ with $F(A) = X \cup A$ for every set A;
  (b) there exists a functor $F : \mathsf{Sets} \to \mathsf{Sets}$ with $F(A) = X \cap A$ for every set A;
  (c) $X = \emptyset$.

4. Let Mon and Preord be as an Exercise 2 of Section 2. Explain that:
  (a) for monoids M and M', to give a functor $M \to M'$ is to give a monoid homomorphism $M \to M'$, and this determines an f.f. functor $\mathsf{Mon} \to \mathsf{Cat}$;
  (b) for preorders P and P', to give a functor $P \to P'$ is to give a preorder preserving map $P \to P'$, and this determines an f.f. functor $\mathsf{Preord} \to \mathsf{Cat}$.

5. Show that each of the six collections of functors, introduced in Definition 8.6, contains all isomorphisms (of categories) and is closed under the composition.

6. Prove the following preservation and reflection properties of functors:
  (a) Every functor preserves split monomorphisms, split epimorphisms, and isomorphisms;
  (b) Every faithful functor reflects monomorphisms, epimorphisms, and bimorphisms;
  (c) Every f.f. functor reflects split monomorphisms, split epimorphisms, and isomorphisms;
  (d) Every f.f. functor is e.i.o;
  (e) If a functor is f.f. and e.s.o., then it preserves monomorphisms, epimorphisms, and bimorphisms.

7. Show that:
  (a) there exist a full functor F and a morphism f in its domain category, such that $F(f)$ is an isomorphism while f is neither a monomorphism nor an epimorphism;
  (b) there exist an f.f. functor F that does not carry bimorphisms to monomorphisms.

8. Prove that the following conditions on a functor $F = (F_0, F_1)$ are equivalent:
  (a) F is an isomorphism (that is, an isomorphism in Cat);
  (b) F_0 and F_1 are bijections;
  (c) F_1 is a bijection;
  (d) F_0 is a bijection and F is fully faithful.

9. Define the underlying set functors $\mathsf{Mon} \to \mathsf{Sets}$ and $\mathsf{Preord} \to \mathsf{Sets}$ (also called *forgetful functors*), and show that:
   (a) both of them are faithful and preserve monomorphisms;
   (b) none of them is neither full nor e.i.o;
   (c) none of them reflects split monomorphisms and none of them reflects split epimorphisms;
   (d) the first of them reflects isomorphisms while the second one does not;
   (e) the first of them does not preserve bimorphisms and is not e.s.o.;
   (f) the second one preserves epimorphisms and is e.s.o.

   Note: in items (a) and (d), monoid axioms $1x = x = x1$ and $x(yz) = (xy)z$ play no role.

10. Any familiar category of mathematical structures on sets (as in Exercise 2 of Section 2) admits a suitable underlying set functor from it to Sets; try to analyze such underlying set functors by trying to establish properties similar to those of the previous exercise. More generally, do the same for various forgetful functors between categories of mathematical structures that associate weaker structures to given structures (e.g. the functor that associates to rings their additive groups).*

11. Try similar analysis for functors of Example 8.3 taking $\mathbb{C}$ to be one of familiar categories of mathematical structures.*

12. Make similar analysis for the functor F of Exercise 2. Some of it will require considering the following three cases separately: (a) $X = \emptyset$; (b) X is a one-element set; (c) X has more than one element.

9. Old terminology: covariant and contravariant functors

Functors, as we have defined them in Section 8, were originally called *covariant functors*, in contrast to *contravariant functors*, whose definition is obtained from Definition 8.1 by replacing:

- $F(a) : F(A) \to F(A')$ with $F(a) : F(A') \to F(A)$ in 8.1(a);
- $F(gf) = F(g)F(f)$ with $F(gf) = F(f)F(g)$ in 8.1(c).

The reason why it is better to avoid this terminology is that a contravariant functor $\mathbb{A} \to \mathbb{B}$ can either be seen as a functor $\mathbb{A}^{\mathrm{op}} \to \mathbb{B}$, or as a functor $\mathbb{A} \to \mathbb{B}^{\mathrm{op}}$, while one cannot identify these two functors with each other.

However, after abandoning this terminology, it is convenient to call the functors $\hom_{\mathbb{C}}(X, -) : \mathbb{C} \to \mathsf{Sets}$ and $\hom_{\mathbb{C}}(-, X) : \mathbb{C}^{\mathrm{op}} \to \mathsf{Sets}$, of Example 8.3, covariant hom functors and contravariant hom functors, respectively.

Exercises

1. Example 8.4 in fact says that, for a monoid M, to give a functor $M \to \mathsf{Sets}$ is to give an M-set. Rephrase it, explaining that covariant functors $M \to \mathsf{Sets}$ can be identified with left M-sets while contravariant functors $M \to \mathsf{Sets}$ can be identified with right M-sets. Hint: in Example 3.10 we wrote $h(\omega, x) = \omega x$, and "left" refers to ω written on the left in ωx – which is convenient in the covariant case, while writing $x\omega$ would be more convenient in the contravariant case (explain why).

2. For monoids M and M', explain that:
   (a) to give a covariant functor $M \to M'$ is, according to Exercise 4(a) of Section 8, to give a monoid homomorphism $M \to M'$;
   (b) to give a contravariant functor $M \to M'$ is to give a monoid antihomomorphism $M \to M'$; by a monoid antihomomorphism we mean a map f satisfying $f(1) = 1$ and $f(xy) = f(y)f(x)$ (for all $x, y \in M$) in the multiplicative notation.

3. For preorders P and P', explain that:
  (a) to give a covariant functor $P \to P'$ is, according to Exercise 4(b) of Section 8, to give a preorder preserving map $P \to P'$;
  (b) to give a contravariant functor $P \to P'$ is to give a preorder reversing map $P \to P'$; by a preorder reversing map we mean a map f satisfying $x \leqslant y \Rightarrow f(y) \leqslant f(x)$ (for all $x, y \in P$).

4. Explain that with an appropriate notion of composite of functors, covariant or contravariant, we have:
  (a) the composite of a covariant functor with a contravariant functor (in any order) is contravariant;
  (b) the composite of contravariant functors is covariant.

5. Explain that the identity functor of a category $\mathbb{C}$ can be considered either as a contravariant functor $\mathbb{C}^{\mathrm{op}} \to \mathbb{C}$ or as a contravariant functor $\mathbb{C} \to \mathbb{C}^{\mathrm{op}}$, and composinig with these functors (as in the previous exercise) defines certain obvious bijections between collections of covariant and contravariant functors.

10. Remarks on commutative squares

Postponing the formal definition of a *diagram*, let us agree that what we mean by a diagram is just a display of objects and morphisms in a given category, such as

$$\begin{tikzcd}
 & X_1 \arrow[r, "f_1"] & \cdots \arrow[r, "f_{m-1}"] & X_m \arrow[dr, "f_m"] & \\
A \arrow[ur, "f_0"] \arrow[dr, "g_0"'] & & & & B \\
 & Y_1 \arrow[r, "g_1"'] & \cdots \arrow[r, "g_{n-1}"'] & Y_n \arrow[ur, "g_n"'] &
\end{tikzcd}$$

calling it *commutative* (or saying that it *commutes*) if $f_m \ldots f_0 = g_n \ldots g_0$. In particular, a commutative square is a diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

with $f'a = bf$. The following theorem describes very basic simple properties of commutative squares; it is trivial, but worth mentioning.

Theorem 10.1. *In a given category:*

(a) *for every morphism $a : A \to A'$, the diagram*

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "a"'] & A \arrow[d, "a"] \\
A' \arrow[r, "1_{A'}"'] & A'
\end{tikzcd}$$

commutes;

(b) *if the first two of the following three squares commute, then the third one also commutes:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[d, "c"] \\ B' \arrow[r, "g'"'] & C' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "gf"] \arrow[d, "a"'] & C \arrow[d, "c"] \\ A' \arrow[r, "g'f'"] & C' \end{tikzcd}$$

(c) *if f and f' are isomorphisms and the first of the following two squares commutes, then the second one also commutes:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "f^{-1}"] \arrow[d, "b"'] & A \arrow[d, "a"] \\ B' \arrow[r, "(f')^{-1}"'] & A' \end{tikzcd}$$

□

Remark 10.2. *It would be convenient to formulate 10.1(b) rather informally, saying that if the two inner squares in the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[d, "c"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \end{tikzcd}$$

commute, then the outer square also commutes.

Remark 10.3. *Given a category $\mathbb{C}$, Theorem 10.1 in fact consructs the so-called arrow category $\mathsf{Ar}(\mathbb{C})$, whose objects are morphisms in $\mathbb{C}$ and whose morphisms are commutative squares; and 10.1(b) shows how to compose such morphisms. In the situation 10.1, informally identifying commutative squares with their pairs of horizontal arrows, we might write $1_a = (1_A, 1_A)$, $(g, g')(f, f') = (gf, g'f')$, and $(f, f')^{-1} = (f^{-1}, f'^{-1})$ – when f and f' are isomorphisms, which makes (f, f') to be an isomorphism.*

11. Natural transformations

Given categories $\mathbb{A}$ and $\mathbb{B}$, all functors from $\mathbb{A}$ to $\mathbb{B}$ form a category $\mathbb{B}^{\mathbb{A}}$, whose morphisms are called natural transformations; they are defined as follows:

Definition 11.1. *Let $F, G : \mathbb{A} \to \mathbb{B}$ be functors. A natural transformation $\tau : F \to G$ is a family $\tau = (\tau_A : F(A) \to G(A))_{A \in \mathbb{A}_0}$ of morphisms in $\mathbb{B}$ such that, for every morphism $a : A \to A'$ in $\mathbb{A}$, the diagram*

$$\begin{tikzcd} F(A) \arrow[r, "\tau_A"] \arrow[d, "F(a)"'] & G(A) \arrow[d, "G(a)"] \\ F(A') \arrow[r, "\tau_{A'}"'] & G(A'), \end{tikzcd}$$

called a naturality square, commutes.

As suggested by 10.1(a) and 10.1(b) (see also Remark 10.3), we also define $1_F = (1_{F(A)})_{A \in \mathbb{A}_0}$ and $\upsilon\tau = (\upsilon_A \tau_A)_{A \in \mathbb{A}_0}$, which completes our description of the functor category $\mathbb{B}^{\mathbb{A}}$. Also, from 10.1(c), we easily obtain:

Theorem 11.2. *Let $F, G : \mathbb{A} \to \mathbb{B}$ be functors. A natural transformation $\tau : F \to G$ is an isomorphism in $\mathbb{B}^{\mathbb{A}}$ if and only if $\tau_A : F(A) \to G(A)$ is an isomorphism in $\mathbb{B}$ for each object A in $\mathbb{A}$. If τ is an isomorphism, then $\tau^{-1} = (\tau_A^{-1})_{A \in \mathbb{A}_0}$.* □

Consider some examples of natural transformations:

Example 11.3. *Let $\mathbb{C}$ be category, $S : \mathbb{C} \to \mathsf{Sets}$ a functor, C an object in $\mathbb{C}$, and c an element of $S(C)$. Then $\sigma : \hom(C, -) \to S$ defined by $\sigma_A(f) = S(f)(c)$ is a natural transformation. This construction will be used in Theorem 14.1.*

Example 11.4. Sets^M *is a functor category. In detail: Let M be a multiplicative monoid and X and Y be M-sets viewed as functors $M \to \mathsf{Sets}$ (see Example 8.4). Then a natural transformation $X \to Y$ is nothing but an M-set homomorphism from X to Y, that is, a map $f : X \to Y$ with $f(ux) = uf(x)$ for all $u \in M$ and $x \in X$. So, Definitions 3.2 and 11.1 agree here.*

Example 11.5. *Let Ω be a signature, $\omega \in \Omega_n$, $\mathbb{C}$ any subcategory of $\mathsf{Alg}(\Omega)$, and $U : \mathbb{C} \to \mathsf{Sets}$ the (obviously defined) underlying set functor (cf. Exercises 9 and 10 of Section 8, although their results are not needed here). Let $U^n : \mathbb{C} \to \mathsf{Sets}$ be the functor defined by $U^n(A) = U(A)^n$, with $U^n(f) : U^n(A) \to U^n(B)$ defined, for any $f : A \to B$, by $U^n(f)(a_1, ..., a_n) = (f(a_1), ..., f(a_n))$. Then $\tau : U^n \to U$ defined by $\tau_A(a_1, ..., a_n) = \omega(a_1, ..., a_n)$ is a natural transformation.*

Thinking of categories as *sets equipped with morphisms between their elements*, we can make now a "functorial versions" of maps $\mathsf{hom}(X, f)$ and $\mathsf{hom}(f, X)$ in Example 8.3:

Example 11.6. (of functors) *For categories $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{X}$, a functor $F : \mathbb{A} \to \mathbb{B}$ determines:*

(a) *a functor $F^{\mathbb{X}} : \mathbb{A}^{\mathbb{X}} \to \mathbb{B}^{\mathbb{X}}$, which carries $\tau : \Phi \to \Phi'$ to $F\tau : F\Phi \to F\Phi'$, where $F\tau$ is defined by $(F\tau)_X = F(\tau_X)$.*

(b) *a functor $\mathbb{X}^F : \mathbb{X}^{\mathbb{B}} \to \mathbb{X}^{\mathbb{A}}$, which carries $\upsilon : \Psi \to \Psi'$ to $\upsilon F : \Psi F \to \Psi' F$, where υF is defined by $(\upsilon F)_A = \upsilon_{F(A)}$.*

Exercises

1. Let $\mathbb{A}$ and $\mathbb{B}$ be categories, and $b : B \to B'$ a morphism in $\mathbb{B}$. Explain that b can be considered as a natural transformation $\mathrm{Const}_B \to \mathrm{Const}_{B'}$ (see Exercise 1 of Section 8).
2. Let $x : X \to X'$ be a map of sets and $F, F' : \mathsf{Sets} \to \mathsf{Sets}$ functors defined by $F(A) = X \times A$ and $F'(A) = X' \times A$, respectively (see Exercise 2 of Section 8). Explain that x determines a natural transformation $F \to F'$.
3. Let $F, G : \mathbb{A} \to \mathbb{B}$ be functors. Assuming that $\mathbb{B}$ is a preorder viewed as a category, prove that there exist at most one natural transformation $F \to G$ and find a necessary and sufficient condition for its existence.
4. Let K be a field, $U(K)$ its multiplicative group (of non-zero elements), and $GL_n(K)$ the group of invertible $n \times n$ matrices with coefficients in K. Explain that U and GL_n can be viewed as functors from the category of fields to the category of groups, and that taking the determinants can be viewed as a natural transformation $GL_n \to U$.*
5. Explain that the natural transformation considered in the previous exercise can be obtained as a special case of the natural transformation constructed in Example 11.5.

12. Three remarks on isomorphic functors

The first remark is that the properties introduced in Definition 8.6 are invariant under the isomorphism in the following sense:

Theorem 12.1. *Let F and G be isomorphic functors, and $\mathfrak{K}$ be any of the six collections of functors introduced in Definition 8.6. Then $F \in \mathfrak{K} \Leftrightarrow G \in \mathfrak{K}$.*

Proof. Let $\tau : F \to G$ be an isomorphism of functors $\mathbb{A} \to \mathbb{B}$. Then the diagram

$$\begin{tikzcd} & \hom(F(A), F(A')) \arrow[dd, "f \mapsto \tau_{A'} f \tau_A^{-1}"] \\ \hom(A, A') \arrow[ur, "F_{A,A'}"] \arrow[dr, "G_{A,A'}"'] & \\ & \hom(G(A), G(A')) \end{tikzcd}$$

whose vertical arrow is a bijection (with the inverse map defined by $g \mapsto \tau_{A'}^{-1} g \tau_A$) commutes for all objects A and A' in $\mathbb{A}$. Therefore each $F_{A,A'}$ is injective if and only if each $G_{A,A'}$ is injective, and the same is true for surjectivity and bijectivity. This proves our theorem in the first three cases for $\mathfrak{K}$. In the other three cases just use the fact that $F(A) \approx G(A)$ for every object A in $\mathbb{A}$. □

The second remark, also formulated as a theorem, shows how to transport the functor structure from a functor $F : \mathbb{A} \to \mathbb{B}$ to a map $G : \mathbb{A} \to \mathbb{B}$ along isomorphisms $F(A) \approx G(A)$ given for all object A in $\mathbb{A}$:

Theorem 12.2. *Let $\mathbb{A}$ and $\mathbb{B}$ be categories, $F : \mathbb{A} \to \mathbb{B}$ a functor, $G : \mathbb{A}_0 \to \mathbb{B}_0$ a map, and $\tau = (\tau_A : F(A) \to G(A))_{A \in \mathbb{A}_0}$ a family of isomorphisms in $\mathbb{B}$. Then there exist a unique way to make G a functor from $\mathbb{A}$ to $\mathbb{B}$ that makes τ a natural transformation from F to G.*

Proof. To make G a functor that makes τ a natural transformation from F to G is to define $G(a) : G(A) \to G(A')$, for each $a : A \to A'$ in $\mathbb{A}$, in such a way that:

(a) $\tau_{A'} F(a) = G(a)\tau_A$,
(b) $G(1_A) = 1_{G(A)}$,
(c) $G(a'a) = G(a')G(a)$,

where a' is any morphisms whose domain is A'. However, (a) is equivalent to

$$G(a) = \tau_{A'} F(a) \tau_A^{-1},$$

which defines $G(a)$ and (used for each a), makes (b) and (c) hold. □

The third remark is in fact about the relationship between two notions of isomorphism, the abstract one (Definition 4.1) and the notion of isomorphism of categories. One can also say that it justifies the abstract notion of isomorphism:

Remark 12.3. *In Theorem 12.2, let us take $\mathbb{A} = \mathbb{B}$, $F = 1_{\mathbb{A}}$, and G to be bijective. Then:*

(a) *As follows from Theorem 12.1 and the assertion of Exercise 8 of Section 8, the functor G is an automorphism of $\mathbb{A}$.*
(b) *Roughly, the existence of the automorphism G should be understood as: whatever we do in a category $\mathbb{A}$, the same can be repeated after replacing all (or some) objects of $\mathbb{A}$ with isomorphic objects provided this replacement is a bijective map $\mathbb{A}_0 \to \mathbb{A}_0$.*

13. Equivalence and skeletons of categories

Definition 13.1. *Given categories $\mathbb{A}$ and $\mathbb{B}$, an equivalence (of categories) $\mathbb{A} \sim \mathbb{B}$ is a system (F, G, α, β) in which $F : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{A}$ are functors, and $\alpha : 1_{\mathbb{A}} \to GF$ and $\beta : 1_{\mathbb{B}} \to FG$ are isomorphisms. When such an equivalence* *exists, one says that the categories $\mathbb{A}$ and $\mathbb{B}$ are equivalent (to each other), and the functors F and G are said to be quasi-inverses of each other.*

Remark 13.2. *This definition has several standard variations, including the following ones, in which we use $\mathbb{A}$, $\mathbb{B}$, and (F, G, α, β) above. One might:*

- *write $(F, G, \alpha, \beta) : \mathbb{A} \sim \mathbb{B}$, or $(F, G) : \mathbb{A} \sim \mathbb{B}$;*
- *replace α, or β, or both α and β by their inverses in the system (F, G, α, β);*
- *say that (F, G) is a category equivalence when $1_{\mathbb{A}} \approx GF$ and $1_{\mathbb{B}} \approx FG$;*
- *say that F is a category equivalence when there exists G with $1_{\mathbb{A}} \approx GF$ and $1_{\mathbb{B}} \approx FG$.*

In this section we characterize category equivalences, which turns out to be far more sophisticated than the characterization of category isomorphisms given by Exercise 8 of Section 8.

Theorem 13.3. *The category equivalence is an equivalence relation on the collection of all categories. Moreover:*

(a) *Let $F : \mathbb{A} \to \mathbb{B}$ be an isomorphism of categories. Then $(F, F^{-1}, 1_{1_{\mathbb{A}}}, 1_{1_{\mathbb{B}}}) : \mathbb{A} \sim \mathbb{B}$ is an equivalence of categories.*
(b) *If $(F, G, \alpha, \beta) : \mathbb{A} \sim \mathbb{B}$ is an equivalence of categories, then so is $(G, F, \beta, \alpha) : \mathbb{B} \sim \mathbb{A}$.*
(c) *If $(F, G, \alpha, \beta_1) : \mathbb{A} \sim \mathbb{B}$ and $(K, L, \beta_2, \gamma) : \mathbb{B} \sim \mathbb{C}$ are equivalences of categories, then so is $(KF, GL, (G\beta_2F)\alpha, (K\beta_1L)\gamma) : \mathbb{A} \sim \mathbb{C}$, where $G\beta_2F$ is defined as $G(\beta_2F) = (G\beta_2)F$ (see Example 11.6), and $K\beta_1L$ is defined similarly.* $\square$

Lemma 13.4. *If a functor is an equivalence of categories, then it is fully faithful and essentially surjective on objects.*

Proof. The second assertion is obvious. To prove the first one, consider the commutative diagram

$$\begin{tikzcd} & \hom_{\mathbb{B}}(F(A), F(A')) \arrow[dr, "G_{F(A),F(A')}"] & \\ \hom_{\mathbb{A}}(A, A') \arrow[ur, "F_{A,A'}"] \arrow[rr, "(GF)_{A,A'}"'] & & \hom_{\mathbb{A}}(GF(A), GF(A')), \end{tikzcd}$$

for arbitrary two objects A and A' in $\mathbb{A}$, and observe:

- Since GF is isomorphic to $1_{\mathbb{A}}$, it is f.f. by Theorem 12.1.
- Since GF is f.f., $(GF)_{A,A'}$ is bijective.
- Since $(GF)_{A,A'}$ is bijective, $F_{A,A'}$ is injective.
- Since this is true for any two objects A and A' in $\mathbb{A}$, F is faithful.
- As follows from Theorem 13.3(b), since F is faithful (in each such situation), so is G.
- Since G is faithful, $G_{F(A),F(A')}$ is injective.
- Since $F_{A,A'}$ and $G_{F(A),F(A')}$ are injective while $(GF)_{A,A'}$ is bijective, $F_{A,A'}$ is bijective.

That is, F is f.f. $\square$

Definition 13.5. *Let $\mathbb{C}$ be a category. A skeleton of $\mathbb{C}$ is a 'maximal' full subcategory of $\mathbb{C}$ that is a skeleton, that is, a full subcategory of $\mathbb{C}$ such that, for every object C in $\mathbb{C}$, it has a unique object isomorphic to C.*

Theorem 13.6. *Let $\mathbb{C}$ be a category and $\mathbb{C}'$ a skeleton of $\mathbb{C}$. The inclusion functor $\mathbb{C}' \to \mathbb{C}$ is an equivalence of categories.*

Proof. We choose, for each object C in $\mathbb{C}$:

- an object $\Phi(C)$ in $\mathbb{C}'$ isomorphic to C (in $\mathbb{C}$), which, in particular, gives $C \in \mathbb{C}'_0 \Rightarrow \Phi(C) = C$, and
- an isomorphism $\tau_C : C \to \Phi(C)$, assuming, in particular, that $\tau_C = 1_C$ when $C \in \mathbb{C}'_0$.

Then we observe:

(a) Applying Theorem 12.2 to the identity functor $1_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$, the map $\Phi : \mathbb{C}_0 \to \mathbb{C}_0$, and the family $\tau = (\tau_C)_{C\in\mathbb{C}_0}$, we make Φ a functor from $\mathbb{C}$ to $\mathbb{C}$ and $\tau : 1_{\mathbb{C}} \to \Phi$ an isomorphism.
(b) Denoting the inclusion functor $\mathbb{C}' \to \mathbb{C}$ by F, we see that the functor Φ determines a functor $G : \mathbb{C} \to \mathbb{C}'$ with $FG = \Phi$ (since the Φ-images of all objects of $\mathbb{C}$ are in $\mathbb{C}'$ and $\mathbb{C}'$ is a full subcategory of $\mathbb{C}$).
(c) For any morphism $c : C_1 \to C_2$ in $\mathbb{C}'$, we have $GF(C_i) = C_i$ $(i = 1, 2)$ and $GF(c) = \tau_{C_2} c \tau_{C_1}^{-1} = 1_{C_2} c 1_{C_1}^{-1} = c$. Therefore $GF = 1_{\mathbb{C}'}$.

That is, $FG = \Phi \approx 1_{\mathbb{C}}$ and $GF = 1_{\mathbb{C}'}$. □

Given an equivalence $(F, G, \alpha, \beta) : \mathbb{A} \sim \mathbb{B}$ of categories, consider the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, shift left, "F"] \arrow[d, shift left, "P"] & \mathbb{B} \arrow[l, shift left, "G"] \arrow[d, shift left, "Q"] \\ \mathrm{Sk}(\mathbb{A}) \arrow[u, shift left, "S"] \arrow[r, shift left, "QFS"] & \mathrm{Sk}(\mathbb{B}), \arrow[u, shift left, "T"] \arrow[l, shift left, "PGT"] \end{tikzcd}$$

in which $\mathrm{Sk}(\mathbb{A})$ is any skeleton of $\mathbb{A}$ and $\mathrm{Sk}(\mathbb{B})$ is any skeleton of $\mathbb{B}$, while $(P, S) : \mathbb{A} \sim \mathrm{Sk}(\mathbb{A})$ and $(Q, T) : \mathbb{B} \sim \mathrm{Sk}(\mathbb{B})$ are equivalences of categories. We observe:

- As follows from Theorem 13.3, $(QFS, PGT) : \mathrm{Sk}(\mathbb{A}) \sim \mathrm{Sk}(\mathbb{B})$ is an equivalence of categories.
- After that, applying Lemma 13.4 (with Theorem 13.3(b)) and the result of Exercise 6(d) of Section 8, we conclude that the functors QFS and PGT are fully faithful and essentially bijective on objects.
- Being essentially bijective on objects functors between skeletons, QFS and PGT are bijective on objects.
- Being fully faithful and bijective on objects functors, QFS and PGT are isomorphisms of categories by the equivalence (a)⇔(b) in Exercise 8 of Section 8.
- On the other hand, just having two categories with equivalent skeletons, we could use Theorem 13.3 to conclude that they are equivalent to each other.

This proves:

Theorem 13.7. *For two categories $\mathbb{A}$ and $\mathbb{B}$, the following conditions are equivalent:*

(a) *$\mathbb{A}$ and $\mathbb{B}$ are equivalent categories;*
(b) *$\mathbb{A}$ has a skeleton $\mathbb{A}'$ and $\mathbb{B}$ has a skeleton $\mathbb{B}'$ such that $\mathbb{A}'$ and $\mathbb{B}'$ are equivalent categories;*
(c) *every skeleton of $\mathbb{A}$ is isomorphic to every skeleton of $\mathbb{B}$.*

In particular, a skeleton of a category is unique up to isomorphism. □

And the last assertion of Theorem 13.7 explains why it is reasonable to write $\mathrm{Sk}(\mathbb{A})$ for a skeleton of $\mathbb{A}$.

Now consider the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[d, shift left, "P"] & \mathbb{B} \arrow[d, shift left, "Q"] \\ \mathrm{Sk}(\mathbb{A}) \arrow[r, "QFS"'] \arrow[u, shift left, "S"] & \mathrm{Sk}(\mathbb{B}), \arrow[u, shift left, "T"] \end{tikzcd}$$

where F is only required to be fully faithful and essentially surjective on objects, while the vertical arrows are as before. Let us argue as follows:

- S and Q are also f.f. and e.s.o., by Lemma 13.4.
- Therefore so is QFS (see Exercise 5 of Section 8).
- Then, arguing as before, we conclude that QFS is an isomorphism.
- After that we put $S(QFS)^{-1}Q = G$ and calculate:

$$1_{\mathbb{A}} \approx SP \approx S(QFS)^{-1}QFSP \approx S(QFS)^{-1}QF \approx GF,$$

$$1_{\mathbb{B}} \approx TQ \approx TQFS(QFS)^{-1}Q \approx FS(QFS)^{-1}Q \approx FG,$$

which, together with Lemma 13.4 (see also Exercise 6(d) of Section 8 again), proves:

Theorem 13.8. *The following conditions on a functor are equivalent:*

(a) *it is an equivalence of categories;*
(b) *it is fully faithful and essentially bijective on objects;*
(c) *it is fully faithful and essentially surjective on objects.* □

Remark 13.9. *The crucial part of Theorem 13.8, which is the implication (c)⇒(a), can actually be proved directly, not using skeletons. For, given a fully faithful essentially surjective on object functor $F : \mathbb{A} \to \mathbb{B}$, we construct a functor $G : \mathbb{B} \to \mathbb{A}$ as follows:*

- *Define $G(B)$, for each object B of $\mathbb{B}$, as any object in $\mathbb{A}$ whose F-image is isomorphic to B.*
- *Fix an isomorphism $\beta_B : B \to FG(B)$.*
- *For each morphism $b : B \to B'$ in $\mathbb{B}$, define $G(b) : G(B) \to G(B')$ as the unique morphism in $\mathbb{A}$ with $FG(b) = \beta_{B'}b\beta_B^{-1}$. This makes G a functor from $\mathbb{B}$ to $\mathbb{A}$, and makes $\beta : 1_{\mathbb{B}} \to FG$ an isomorphism.*

After that, for each object A in $\mathbb{A}$, we define α_A as the unique morphism from A to $GF(A)$ in $\mathbb{A}$ with $F(\alpha_A) = \beta_{F(A)}$, which makes $\alpha : 1_{\mathbb{A}} \to GF$ an isomorphism. Indeed, given a morphism $a : A \to A'$ in $\mathbb{A}$, we have

$$\begin{aligned} F(GF(a)\alpha_A) = F(G(F(a)))F(\alpha_A) &= \beta_{F(A')}F(a)\beta_{F(A)}^{-1}\beta_{F(A)} \\ &= \beta_{F(A')}F(a) = F(\alpha_{A'})F(a) = F(\alpha_{A'}a), \end{aligned}$$

and so $GF(a)\alpha_A = \alpha_{A'}a$, which gives the naturality of α. Then the fact α is an isomorphism follows from the fact that F reflects isomorphisms (see Exercise 6(c) of Section 8) and Theorem 11.2.

Definition 13.10. *A full subcategory $\mathbb{B}'$ of a category $\mathbb{B}$ is said to be replete if it contains all objects B with $B \approx B'$ in $\mathbb{B}$ for some object B' in $\mathbb{B}'$. Accordingly, the replete image of a fully faithful functor $F : \mathbb{A} \to \mathbb{B}$, which we will briefly denote by $F(\mathbb{A})$, is the full subcategory of $\mathbb{B}$ whose objects are all objects B in $\mathbb{B}$ with $B \approx F(A)$ for some object A in $\mathbb{A}$.*

In the notation above, we can say that $F(\mathbb{A})$ is the smallest full replete subcategory of $\mathbb{B}$ containing F-images of all objects of $\mathbb{A}$. From Theorem 13.8(c)⇒(a) we obtain:

Corollary 13.11. *If $F : \mathbb{A} \to \mathbb{B}$ is a fully faithful functor, then the induced functor $F : \mathbb{A} \to F(\mathbb{A})$ is a category equivalence. More generally, the same will remain true if we will replace $F(\mathbb{A})$ with any full subcategory of it containing F-images of all objects of $\mathbb{A}$.*

Exercises.

1. Use Corollary 13.11 to establish an equivalence $\mathbb{A} \sim \mathbb{B}$ in the following cases:
   (a) $\mathbb{A} = \mathsf{Mon}$, and $\mathbb{B}$ is the full subcategory of Cat with objects all one-object categories.
   (b) $\mathbb{A} = \mathsf{Preord}$, and $\mathbb{B}$ is the full subcategory of Cat with objects all categories with no distinct parallel morphisms (such categories are sometimes called *coherent categories*).
   (c) For a fixed field K: $\mathbb{A}$ is the category whose objects are natural numbers 0, 1, 2, ..., and a morphism $n \to m$ is an $m \times n$ matrix with entries in K − and matrices are composed as usually; $\mathbb{B} = \mathsf{Vect}_K^{fin}$ is the category of finite-dimensional K-vector spaces. Hint: define a functor $F : \mathbb{A} \to \mathbb{B}$ by $F(n) = K^n$.*
   (d) $\mathbb{A}$ is the category of finite sets, and $\mathbb{B}$ is the opposite category of finite Boolean algebras. Hint: define $F : \mathbb{A} \to \mathbb{B}$ by $F(A) = P(A)$, where $P(A)$ is the Boolean algebra of all subsets of A.*

2. Given a set S, establish an equivalence $(F, G) : \mathbb{A} \sim \mathbb{B}$, in which:
   – the objects of $\mathbb{A}$ are S-indexed families of sets, and a morphism $u : (X_s)_{s\in S} \to (Y_s)_{s\in S}$ is a family $u = (u_s : X_s \to Y_s)_{s\in S}$ of maps;
   – the objects of $\mathbb{B}$ are pairs (X, p) in which X is a set and $p : X \to S$ is a map, and a morphism $v : (X, p) \to (Y, q)$ is a map $v : X \to Y$ with $qv = p$;
   – $F : \mathbb{A} \to \mathbb{B}$ is defined by
$$F((X_s)_{s\in S}) = (\bigcup_{s\in S} \{s\} \times X_s, p),$$
   where p is defined by $p(s, x) = s$;
   – $G : \mathbb{B} \to \mathbb{A}$ is defined by $G(X, p) = (p^{-1}(s))_{s\in S}$.

3. Use the previous exercise to establish an equivalence between Cat and its modified version in which categories are defined as in Remark 7.1 (not requiring the hom-disjointness condition).

4. Prove that the category Vect_K^{fin} is *self-dual*, that is, it is equivalent to its opposite category. Hint:
   (a) Considering K as an object in Vect_K^{fin}, use the functor $\hom(-, K)$ to make functors $F : (\mathsf{Vect}_K^{fin})^{\mathrm{op}} \to \mathsf{Vect}_K^{fin}$ and $G : \mathsf{Vect}_K^{fin} \to (\mathsf{Vect}_K^{fin})^{\mathrm{op}}$. In fact both F and G carry any finite-dimensional K-vector space V to its dual (in the sense of linear algebra) vector space V^*.
   (b) Given a K-vector space V, each $v \in V$ determines a K-linear map $\tilde{v} : V^* \to K$ defined by $\tilde{v}(f) = f(v)$. Show that this gives an injective linear map $V \to V^{**}$. Then conclude that, for a finite dimensional V, this map is an isomorphism.
   (c) Use (b) to show that (F, G) is a category equivalence.*

5. Prove that the category fAb of finite abelian groups is self-dual. Hint: instead of V^* used in the previous exercise, use, for any finite abelian group A, the abelian group of homomorphisms from A to $\mathbf{Q}/\mathbf{Z}$, where $\mathbf{Q}$ and $\mathbf{Z}$ are the additive groups of rational numbers and of integers, respectively.*

6. Use the way of establishing a category equivalence in Exercise 1(d) to establish the equivalences of Exercises 4 and 5. Does the converse make sense? Hint: for the converse, use a two-element set 2 as merely a set, which makes $P(A) \approx \hom(A, 2)$ (for any set A), and as a two-element Boolean algebra.*

7. Use the equivalence of Exercise 1(c) to establish the equivalence of Exercise 4.*

8. Prove that if $F : \mathbb{A} \to \mathbb{B}$ is a fully faithful functor, then an object A in $\mathbb{A}$ is:
(a) initial in $\mathbb{A}$ if $F(A)$ is initial in $\mathbb{B}$;
(b) terminal in $\mathbb{A}$ if $F(A)$ is terminal in $\mathbb{B}$.

9. Prove that if $F : \mathbb{A} \to \mathbb{B}$ is a category equivalence, then an object A in $\mathbb{A}$ is:
(a) initial in $\mathbb{A}$ if and only if $F(A)$ is initial in $\mathbb{B}$;
(b) terminal in $\mathbb{A}$ if and only if $F(A)$ is terminal in $\mathbb{B}$.

14. Yoneda Lemma and Yoneda Embedding

Theorem 14.1. ***("Yoneda Lemma: covariant form")*** *Let $\mathbb{C}$ be a category, C an object in $\mathbb{C}$, and $S : \mathbb{C} \to \mathsf{Sets}$ a functor. There are bijections*

$$\mathrm{Nat}(\hom(C, -), S) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} S(C)$$

inverse to each other, which are defined by $\alpha(\sigma) = \sigma_C(1_C)$ and $\beta(c)_A(f) = S(f)(c)$, between the set $\mathrm{Nat}(\hom(C, -), S)$ of all natural transformations $\hom(C, -) \to S$ and the set $S(C)$.

Proof. For $c \in S(C)$, we have $\alpha\beta(c) = \beta(c)_C(1_C) = S(1_C)(c) = c$.

For $\sigma : \hom(C, -) \to S$, in order to prove that $\beta\alpha(\sigma) = \sigma$, we need to prove that $(\beta\alpha(\sigma))_A(f) = \sigma_A(f)$, for every object A in $\mathbb{C}$ and every morphism $f : C \to A$. We have $(\beta\alpha(\sigma))_A(f) = S(f)(\alpha(\sigma)) = S(f)\sigma_C(1_C) = \sigma_A\hom(C, f)(1_C) = \sigma_A(f)$, where the equality $S(f)\sigma_C(1_C) = \sigma_A\hom(C, f)(1_C)$ follows from the commutativity of the naturality square

$$\begin{tikzcd} \hom(C, C) \arrow[r, "\sigma_C"] \arrow[d, "\hom(C,f)"'] & S(C) \arrow[d, "S(f)"] \\ \hom(C, A) \arrow[r, "\sigma_A"'] & S(A) \end{tikzcd}$$

used to calculate the image of 1_C under $S(f)\sigma_C = \sigma_A\hom(C, f)$. □

Remark 14.2. *Considering the commutative square above should actually be the first step towards the 'discovery' of Yoneda Lemma. Indeed, it immediately shows that a natural transformation $\sigma : \hom(C, -) \to S$ is completely determined by $\sigma_C(1_C)$, after which the whole proof of Yoneda Lemma becomes straightforward.*

Dually, that is, replacing $\mathbb{C}$ with $\mathbb{C}^{\mathrm{op}}$, we obtain:

Theorem 14.3. ***("Yoneda Lemma: contravariant form")*** *Let $\mathbb{C}$ be a category, C an object in $\mathbb{C}$, and $S : \mathbb{C}^{\mathrm{op}} \to \mathsf{Sets}$ a functor. There are bijections*

$$\mathrm{Nat}(\hom(-, C), S) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} S(C)$$

inverse to each other, which are defined by $\alpha(\sigma) = \sigma_C(1_C)$ and $\beta(c)_A(f) = S(f)(c)$, between the set $\mathrm{Nat}(\hom(-, C), S)$ of all natural transformations $\hom(-, C) \to S$ and the set $S(C)$. □

Example 14.4. *Given a functor $U : \mathbb{C} \to \mathbb{X}$ and an object X in $\mathbb{X}$, composing the hom functors* $\hom(X, -)$ *and* $\hom(-, X)$ *with U, we obtain functors*

$$\hom(X, U(-)) : \mathbb{C} \to \mathsf{Sets} \quad \text{and} \quad \hom(U(-), X) : \mathbb{C}^{\mathrm{op}} \to \mathsf{Sets},$$

and Theorems 14.1 and 14.3 give:

(a) *For $S = \hom(X, U(-))$, the bijections α and β of Theorem 14.1 become*

$$\mathrm{Nat}(\hom(C, -), \hom(X, U(-))) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \hom(X, U(C))$$

with $\alpha(\sigma) = \sigma_C(1_C)$ and $\beta(c)_A(f) = U(f)c$ for every morphism $c : X \to U(C)$ in $\mathbb{X}$, every object A in $\mathbb{C}$, and every morphism $f : C \to A$ in $\mathbb{C}$.

(b) *For $S = \hom(U(-), X)$, the bijections α and β of Theorem 14.3 become*

$$\mathrm{Nat}(\hom(-, C), \hom(U(-), X) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \hom(U(C), X)$$

with $\alpha(\sigma) = \sigma_C(1_C)$ and $\beta(c)_A(f) = cU(f)$ for every morphism $c : U(C) \to X$ in $\mathbb{X}$, every object A in $\mathbb{C}$, and every morphism $f : A \to C$ in $\mathbb{C}$.

Furthermore, taking $\mathbb{X} = \mathbb{C}$ and $U = 1_{\mathbb{C}}$, we obtain:

(c) *For $S = \hom(X, -)$, the bijections α and β of Theorem 14.1 become*

$$\mathrm{Nat}(\hom(C, -), \hom(X, -)) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \hom(X, C)$$

with $\alpha(\sigma) = \sigma_C(1_C)$ and $\beta(c)_A(f) = fc$ for all $c : X \to C$ and $f : C \to A$ in $\mathbb{C}$.

(d) *For $S = \hom(-, X)$, the bijections α and β of Theorem 14.3 become*

$$\mathrm{Nat}(\hom(-, C), \hom(-, X) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \hom(C, X)$$

with $\alpha(\sigma) = \sigma_C(1_C)$ and $\beta(c)_A(f) = cf$ for all $c : C \to X$ and $f : A \to C$ in $\mathbb{C}$. □

Remark 14.5. *We presented bijections described in Example 14.4(a) as a special case of bijections described in Theorem 14.1. However, one could also present bijections in Theorem 14.1 as a special case of bijections in Example 14.4(a). Indeed, in Exanple 14.4(a), take $\mathbb{X} = \mathsf{Sets}$, $S = U$, and X to be a one-element set; then* $\hom(X, U(-)) = \hom(X, S(-)) \approx S$ *and it is easy to check that the bijections described in Example 14.4(a) give the bijections described in Theorem 14.1.*

Theorem 14.6. ***("Yoneda embedding")*** *Given a category $\mathbb{C}$, the functor $Y = Y_{\mathbb{C}} : \mathbb{C} \to \mathsf{Sets}^{\mathbb{C}^{op}}$ defined by*

$$Y(C \xrightarrow{c} C') = (\hom(-, C) \xrightarrow{\hom(-,c)} \hom(-, C')),$$

where $\hom(-, c)$ *is the natural transformation defined by* $\hom(-, c)_A = \hom(A, c)$, *is fully faithful.*

Proof. For any two objects C and C' in $\mathbb{C}$, consider the map β of Example 14.4(d) for $X = C'$. We have

$$\beta(c)_A(f) = cf = \hom(A, c)(f) = \hom(-, c)_A(f) = Y(c)_A(f) = (Y_{C,C'}(c))_A(f),$$

for all $c : C \to C'$ and $f : A \to C$ in $\mathbb{C}$. Therefore $\beta = Y_{C,C'}$, which implies that $Y_{C,C'}$ is a bijection. □

Exercises

1. Let M a monoid. We could also consider M as an M-set, and, given any M-set S, form $\hom(M, S)$ in the category Sets^M of M-sets. Use Yoneda Lemma to establish a bijection $\hom(M, S) \approx S$.

2. What would you call the counterpart of the bijection above in linear algebra?*

3. Let M be a monoid and $\hat{M}$ the monoid of all maps (not just monoid homomorphisms) $M \to M$. Use Yoneda embedding to prove that the map $f : M \to \hat{M}$, defined by $f(x)(y) = xy$, is an injective homomorphism of monoids. How does the Yoneda embedding describe the image of f?

15. Representable functors and universal arrows

Definition 15.1. *Let $\mathbb{C}$ be category and $S : \mathbb{C} \to \mathsf{Sets}$ a functor. The category $\mathrm{El}_{\mathbb{C}}(S)$ of elements of S over $\mathbb{C}$ is the category of pairs (C, c), where C is an object in $\mathbb{C}$ and $c \in S(C)$; a morphism $f : (C, c) \to (C', c')$ in $\mathrm{El}_{\mathbb{C}}(S)$ is a morphism $f : C \to C'$ in $\mathbb{C}$ with $S(f)(c) = c'$.*

Theorem 15.2. *In the notation of Theorem 14.1, let $\sigma : \hom(C, -) \to S$ and $c \in S(C)$ correspond to each other via the bijection $\mathrm{Nat}(\hom(C, -), S) \approx S(C)$. Then the following conditions are equivalent:*

(a) *σ is an isomorphism;*
(b) *(C, c) is a initial object in $\mathrm{El}_{\mathbb{C}}(S)$.*

Proof. Just note that the following conditions are (subsequently) equivalent:

- condition (a);
- $\sigma_A : \hom(C, A) \to S(A)$ is bijective for every object A in $\mathbb{C}$;
- for every object A in $\mathbb{C}$ and every $a \in S(A)$, there exists a unique morphism $f : C \to A$ with $S(f)(c) = a$;
- for every object (A, a) in $\mathrm{El}_{\mathbb{C}}(S)$, there exists a unique morphism from (C, c) to (A, a),

and that the last of them is condition (b). $\square$

The dual definition and dual theorem are:

Definition 15.3. *Let $\mathbb{C}$ be category and $S : \mathbb{C}^{\mathrm{op}} \to \mathsf{Sets}$ a functor. The category $\mathrm{El}_{\mathbb{C}}(S)$ of elements of S over $\mathbb{C}$ is defined as $\mathrm{El}_{\mathbb{C}^{\mathrm{op}}}(S)^{\mathrm{op}}$. That is, a morphism $f : (C, c) \to (C', c')$ in $\mathrm{El}_{\mathbb{C}}(S)$ is a morphism $f : C \to C'$ in $\mathbb{C}$ with $S(f)(c') = c$.*

Theorem 15.4. *In the notation of Theorem 14.3, let $\sigma : \hom(-, C) \to S$ and $c \in S(C)$ correspond to each other via the bijection $\mathrm{Nat}(\hom(-, S), S) \approx S(C)$. Then the following conditions are equivalent:*

(a) *σ is an isomorphism;*
(b) *(C, c) is a terminal object in $\mathrm{El}_{\mathbb{C}}(S)$.*

Example 15.5. *Let us go back to situations (a) and (b) of Example 14.4. We have:*

(a) *For $S = \hom(X, U(-))$, the category $\mathrm{El}_{\mathbb{C}}(S)$, denoted by $(X \downarrow U)$, is the category of pairs (A, a), where A is an object in $\mathbb{A}$ and $a : X \to U(A)$ is a morphism in $\mathbb{X}$; a morphism $f : (A, a) \to (B, b)$ in this category is a morphism $f : A \to B$ in $\mathbb{A}$ with $U(f)a = b$, hence making the diagram*

$$\begin{tikzcd} U(A) \arrow[rr, "U(f)"] & & U(B) \\ & X \arrow[ul, "a"] \arrow[ur, "b"'] & \end{tikzcd}$$

commute. In this case the initial object of $\mathrm{El}_{\mathbb{C}}(S)$ *is called the universal arrow* $X \to U$*, and Theorem 15.2 says:*

Let $\sigma : \hom_{\mathbb{C}}(C, -) \to \hom_{\mathbb{X}}(X, U(-))$ *and* $c : X \to U(C)$ *correspond to each other via the bijection*

$$\mathrm{Nat}(\hom_{\mathbb{C}}(C, -), \hom_{\mathbb{X}}(X, U(-)) \approx \hom_{\mathbb{X}}(X, U(C)).$$

Then the following conditions are equivalent:

(a$_1$) *σ is an isomorphism;*

(a$_2$) *(C, c) is a universal arrow $X \to U$.*

(b) *For* $S = \hom(U(-), X)$*, the category* $\mathrm{El}_{\mathbb{C}}(S)$*, denoted by* $(U \downarrow X)$*, is the category of pairs* (A, a)*, where* A *is an object in* $\mathbb{A}$ *and* $a : U(A) \to X$ *is a morphism in* $\mathbb{X}$*; a morphism* $f : (A, a) \to (B, b)$ *in this category is a morphism* $f : A \to B$ *in* $\mathbb{A}$ *with* $bU(f) = a$*, hence making the diagram*

$$\begin{tikzcd} U(A) \arrow[rr, "U(f)"] \arrow[dr, "a"'] & & U(B) \arrow[dl, "b"] \\ & X & \end{tikzcd}$$

commute. In this case the terminal object of $\mathrm{El}_{\mathbb{C}}(S)$ *is called the universal arrow* $U \to X$*, and Theorem 15.4 says:*

Let $\sigma : \hom_{\mathbb{C}}(-, C) \to \hom_{\mathbb{X}}(U(-), X)$ *and* $c : U(C) \to X$ *correspond to each other via the bijection*

$$\mathrm{Nat}(\hom_{\mathbb{C}}(-, C), \hom_{\mathbb{X}}(U(-), X) \approx \hom_{\mathbb{X}}(U(C), X).$$

Then the following conditions are equivalent:

(a$_1$) *σ is an isomorphism;*

(a$_2$) *(C, c) is a universal arrow $U \to X$.*

Definition 15.6. *A functor* $S : \mathbb{C} \to \mathsf{Sets}$ *is said to be representable if there exists an object* C *in* $\mathbb{C}$ *with* $S \approx \hom(C, -)$*; the object* C *is then called a representing object for* S*.*

This definition is self-dual in the sense that it automatically implies that a functor $S : \mathbb{C}^{\mathrm{op}} \to \mathsf{Sets}$ should be said to be representable if there exists an object C in $\mathbb{C}$ with $S \approx \hom(-, C)$.

From the results above, we obtain:

Corollary 15.7. *We have:*

(a) *A functor* $S : \mathbb{C} \to \mathsf{Sets}$ *is representable if and only if the category* $\mathrm{El}_{\mathbb{C}}(S)$ *has an initial object;*

(b) *A functor* $S : \mathbb{C}^{\mathrm{op}} \to \mathsf{Sets}$ *is representable if and only if the category* $\mathrm{El}_{\mathbb{C}}(S)$ *has a terminal object.*

(c) *A functor of the form* $\hom_{\mathbb{X}}(X, U(-)) : \mathbb{C} \to \mathsf{Sets}$ *is representable if and only if there exists a universal arrow* $X \to U$*;*

(d) *A functor of the form* $\hom_{\mathbb{X}}(U(-), X) : \mathbb{C}^{\mathrm{op}} \to \mathsf{Sets}$ *is representable if and only if there exists a universal arrow* $U \to X$*.*

Remark 15.8. *Similarly to how it is described in Remark 14.5, it is easy to see that:*

(a) *Definition 15.1 can be presented as a special case of what is introduced in Example 15.5(a); briefly* $\mathrm{El}_{\mathbb{C}}(S) \approx (1 \downarrow U)$*;*

(b) *Theorem 15.2 can be deduced from the equivalence* $(a_1) \Leftrightarrow (a_2)$ *in Example 15.5(a);*

(c) *15.7(a) can be deduced from 15.7(c).*

Exercises.

1. Explain that one has to be careful in dualizing Remarks 14.5 and 15.8. Specifically, explain that, for a functor $S : \mathbb{C}^{\mathrm{op}} \to \mathsf{Sets}$, the existence of an isomorphism $\mathrm{El}_{\mathbb{C}}(S) \approx (U \downarrow 1)$ almost implies $S = 1$. Explain "almost" and how the symbol 1 is used here in two different ways.
2. For an object Z in a category $\mathbb{C}$, show that:
   (a) Z is initial in $\mathbb{C}$ if and only if it is a representing object for any functor $\mathbb{C} \to \mathsf{Sets}$ sending all objects of $\mathbb{C}$ to one-element sets;
   (b) Z is terminal in $\mathbb{C}$ if and only if it is a representing object for any functor $\mathbb{C}^{\mathrm{op}} \to \mathsf{Sets}$ sending all objects of $\mathbb{C}$ to one-element sets;
3. Show that the functor $1_{\mathsf{Sets}} : \mathsf{Sets} \to \mathsf{Sets}$ is representable, and use this to show that there are no non-identity natural transformations $1_{\mathsf{Sets}} \to 1_{\mathsf{Sets}}$.
4. Let X be a set, $\mathbb{C}$ a category of (familiar) mathematical structures, and $U : \mathbb{C} \to \mathsf{Sets}$ the underlying set functor (=the forgetful functor). Describe a universal arrow $X \to U$ for $\mathbb{C}$ being:
   (a) Sets, which makes $U = 1_{\mathsf{Sets}}$;
   (b) $\mathsf{Pointed\,Sets}$;
   (c) Sets^M, where M is a monoid;
   (d) $\mathsf{CommSemigroups}$;
   (e) $\mathsf{Semigroups}$;
   (f) Mon;
   (g) $\mathsf{CommMon}$;
   (h) Groups;
   (i) Ab;
   (j) the category of semimodules over a fixed semiring S (note that this includes the cases (g) and (i), and the case of modules and vector spaces);*
   (k) Preord, and the same construction works for Ord
5. Describe a universal arrow $X \to U$ for (X being an object in $\mathbb{X}$ and) $U : \mathbb{C} \to \mathbb{X}$ being the forgetful functor:
   (a) $\mathsf{Mon} \to \mathsf{Pointed\,Sets}$;
   (b) $\mathsf{Mon} \to \mathsf{Semigroups}$;
   (c) $\mathsf{Ab} \to \mathsf{CommMon}$;
6. Describe a universal arrow $U \to X$ for (X being an object in $\mathbb{X}$ and) $U : \mathbb{C} \to \mathbb{X}$ being:
   (a) as in (k) of Exercise 4;
   (b) as in (c) of Exercise 5;
   (c) the functor $\mathsf{Sets} \to \mathsf{Sets}$ defined, for a given fixed set S, by $C \mapsto C \times S$ (cf. Exercise 2 of Section 8).
7. Extend the results of Exercises 4(k) and 6(a) from Preord to Cat.
8. Extend the result of Exercise 6(c) from Sets to (Ord and) Preord, and then further to Cat.
9. Given functors $U : \mathbb{A} \to \mathbb{X}$ and $V : \mathbb{X} \to \mathbb{S}$, and universal arrows $(X, x) : S \to V$ and $(A, a) : X \to U$, prove that $(A, V(a)x) : S \to VU$ is a universal arrow.

16. Comma categories: general definition and special cases

Definition 16.1. *Given categories $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$, and functors*

$$\mathbb{A} \xrightarrow{F} \mathbb{C} \xleftarrow{G} \mathbb{B},$$

the comma category $(F \downarrow G)$ is the category of triples (A, B, c), in which A is an object in $\mathbb{A}$, B is an object in $\mathbb{B}$, and $c : F(A) \to G(B)$ is a morphism in $\mathbb{C}$; a *morphism* $(A, B, c) \to (A', B', c')$ *in* $(F \downarrow G)$ *is a pair* (a, b)*, in which* $a : A \to A'$ *is a morphism in* $\mathbb{A}$ *and* $b : B \to B'$ *is a morphism in* $\mathbb{B}$ *making the diagram*

$$\begin{tikzcd} F(A) \arrow[r, "c"] \arrow[d, "F(a)"'] & G(B) \arrow[d, "G(b)"] \\ F(A') \arrow[r, "c'"'] & G(B') \end{tikzcd}$$

commute. The morphisms compose obviously (cf. Remark 10.3).

The term "comma category" comes from the old symbol (F, G) used in earlier literature for $(F \downarrow G)$.

The following example in fact lists important special cases:

Example 16.2. *In the notation of Definition 16.1 we can take:*

(a) $\mathbb{A} = \mathbb{B} = \mathbb{C}$ *and* $F = G = 1_{\mathbb{C}}$*. Then* $(F \downarrow G) = \mathrm{Ar}(\mathbb{C})$*, the arrow category introduced in Remark 10.3.*

(b) $\mathbb{C}$ *to be indiscrete, that is, all hom sets in* $\mathbb{C}$ *are one-element sets. Then* $(F \downarrow G)$ *can be identified with* $\mathbb{A} \times \mathbb{B}$*, the product (defined in Exercise 5 of Section 5) of* $\mathbb{A}$ *and* $\mathbb{B}$ *in* Cat*. Of course the standard description of* $\mathbb{A} \times \mathbb{B}$ *should present it as the category of pairs* (A, B)*, in which* A *is an object in* $\mathbb{A}$ *and* B *is an object in* $\mathbb{B}$*; a morphism* $(A, B) \to (A', B')$ *in* $\mathbb{A} \times \mathbb{B}$ *is a pair* (a, b)*, in which* $a : A \to A'$ *is a morphism in* $\mathbb{A}$ *and* $b : B \to B'$ *is a morphism in* $\mathbb{B}$*.*

(c) $\mathbb{A} = \mathbf{1}$*, the terminal category (in which* $\mathbb{A}_0$ *and* $\mathbb{A}_1$ *are one-element sets), and let* C *be the* F*-image of the unique object of* $\mathbb{A}$*. Then* $(F \downarrow G)$ *can be identified with* $(C \downarrow G)$*, which is the same as the category of elements of* $\hom(C, G(-))$ *over* $\mathbb{B}$ *(see Example 15.5(a)).*

(d) $\mathbb{B} = \mathbf{1}$*, and let* C *be the* G*-image of the unique object of* $\mathbb{B}$*. Then* $(F \downarrow G)$ *can be identified with* $(F \downarrow C)$*, which is the same as the category of elements of* $\hom(F(-), C)$ *over* $\mathbb{A}$ *(see Example 15.5(b)).*

The following further special cases are also important:

(e) *if* $\mathbb{B} = \mathbb{C}$ *and* $G = 1_{\mathbb{C}}$ *in (c), then* $(C \downarrow G)$ *is also denoted by* $(C \downarrow \mathbb{C})$ *and called the coslice of* $\mathbb{C}$ *over* C*. Some authors also denote it by* $C \setminus \mathbb{C}$*.*

(f) *if* $\mathbb{A} = \mathbb{C}$ *and* $F = 1_{\mathbb{C}}$ *in (d), then* $(F \downarrow C)$ *is also denoted by* $(\mathbb{C} \downarrow C)$ *and called the slice of* $\mathbb{C}$ *over* C*. Some authors also denote it by* $\mathbb{C}/C$*.*

17. Products of categories and bifunctors

Given categories $\mathbb{A}$ and $\mathbb{B}$, we can form their product $\mathbb{A} \times \mathbb{B}$ constructed as, in Example 16.2, as the category of pairs (A, B), in which A is an object in $\mathbb{A}$ and B is an object in $\mathbb{B}$. As also briefly mentioned in Example 16.2, $\mathbb{A} \times \mathbb{B}$ is the product of $\mathbb{A}$ and $\mathbb{B}$ in Cat. More precisely, there are *projection functors*

$$\mathbb{A} \xleftarrow{P_1} \mathbb{A} \times \mathbb{B} \xrightarrow{P_2} \mathbb{B}$$

which carry $(a, b) : (A, B) \to (A', B')$ to $a : A \to A'$ and to $b : B \to B'$, respectively, and the triple $(\mathbb{A} \times \mathbb{B}, P_1, P_2)$ satisfies the universal property described in Exercise 5 of Section 5. This universal property in fact says, that given functors $F : \mathbb{X} \to \mathbb{A}$ and $G : \mathbb{X} \to \mathbb{B}$, there exists a unique functor $H : \mathbb{X} \to \mathbb{A} \times \mathbb{B}$ making (both triangles of) the diagram

$$\begin{tikzcd} & \mathbb{X} \arrow[dl, "F"'] \arrow[d, "H"] \arrow[dr, "G"] & \\ \mathbb{A} & \mathbb{A}\times\mathbb{B} \arrow[l, "P_1"] \arrow[r, "P_2"'] & \mathbb{B} \end{tikzcd}$$

commute. Clearly, H is defined by

$$H(x : X \to X') = ((F(x), G(x)) : (F(X), G(X)) \to (F(X'), G(X'))).$$

That is, to give a functor $\mathbb{X} \to \mathbb{A}\times\mathbb{B}$ is just to give a functor $\mathbb{X} \to \mathbb{A}$ and a functor $\mathbb{X} \to \mathbb{B}$. What about the functors *from* $\mathbb{A}\times\mathbb{B}$ to another category?

Definition 17.1. *A bifunctor (=functor of two variables) is a functor of the form $\mathbb{A}\times\mathbb{B} \to \mathbb{C}$, where $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ are arbitrary categories. Given such a bifunctor Φ we introduce the following two functors associated to it:*

(a) *the functor $\Phi_l : \mathbb{A} \to \mathbb{C}^{\mathbb{B}}$ defined by*

$$\Phi_l(a : A \to A') = \Phi(a, -) : \Phi(A, -) \to \Phi(A', -),$$

where $\Phi(A, -) : \mathbb{B} \to \mathbb{C}$ is the functor defined by

$$\Phi(A, -)(b : B \to B') = (\Phi(1_A, b) : \Phi(A, B) \to \Phi(A, B')),$$

$\Phi(A', -)$ is defined similarly, and the natural transformation $\Phi(a, -)$ is defined by $\Phi(a, -)_B = \Phi(a, 1_B)$.

(b) *the functor $\Phi_r : \mathbb{B} \to \mathbb{C}^{\mathbb{A}}$ defined by*

$$\Phi_r(b : B \to B') = \Phi(-, b) : \Phi(-, B) \to \Phi(-, B'),$$

where $\Phi(-, B) : \mathbb{A} \to \mathbb{C}$ is the functor defined by

$$\Phi(-, B)(a : A \to A') = (\Phi(a, 1_B) : \Phi(A, B) \to \Phi(A', B)),$$

$\Phi(-, B')$ is defined similarly, and the natural transformation $\Phi(-, b)$ is defined by $\Phi(-, b)_A = \Phi(1_A, b)$.

(The indices l and r stand for "left" and "right" respectively.)

Remark 17.2. *Note that, in the situation of Definition 17.1, we have:*

(a) $\Phi(1_A, b) = \Phi(A, b) = \Phi(A, -)(b) = \Phi_l(A)(b) = \Phi_r(b)_A$, *for every object A in $\mathbb{A}$ and every morphism b in $\mathbb{B}$;*

(b) $\Phi(a, 1_B) = \Phi(a, B) = \Phi(-, B)(a) = \Phi_r(B)(a) = \Phi_l(a)_B$, *for every morphism a in $\mathbb{A}$ and every object B in $\mathbb{B}$.*

Lemma 17.3. *A functor $\Phi : \mathbb{A}\times\mathbb{B} \to \mathbb{C}$ is determined by any of the following data:*

(a) *the functor $\Phi_l : \mathbb{A} \to \mathbb{C}^{\mathbb{B}}$ defined as in 17.1(a);*

(b) *the functor $\Phi_r : \mathbb{B} \to \mathbb{C}^{\mathbb{A}}$ defined as in 17.1(b);*

(c) *the pair $((\Phi_l)_0 : \mathbb{A}_0 \to (\mathbb{C}^{\mathbb{B}})_0, (\Phi_r)_0 : \mathbb{B}_0 \to (\mathbb{C}^{\mathbb{A}})_0)$ (in the notation of Definition 8.1), or, equivalently, the pair $((\Phi_l(A))_{A\in\mathbb{A}_0}, (\Phi_r(B))_{B\in\mathbb{B}_0})$ (which is the same as $((\Phi(A, -))_{A\in\mathbb{A}_0}, (\Phi(-, B))_{B\in\mathbb{B}_0})$).*

Specifically, for a morphism $(a, b) : (A, B) \to (A', B')$ in $\mathbb{A}\times\mathbb{B}$, we can reconstruct $\Phi(a, b)$ as follows:

(a′) $\Phi(a, b) = (\Phi_l(A')(b))\Phi_l(a)_B$ *or* $\Phi(a, b) = \Phi_l(a)_{B'}(\Phi_l(A)(b))$, *using Φ_l;*

(b′) $\Phi(a, b) = \Phi_r(b)_{A'}(\Phi_r(B)(a))$ *or* $\Phi(a, b) = (\Phi_r(B')(a))\Phi_r(b)_A$, *using Φ_r;*

(c′) $\Phi(a, b) = (\Phi_l(A')(b))(\Phi_r(B)(a))$ *or* $\Phi(a, b) = (\Phi_r(B')(a))(\Phi_l(A)(b))$, *using the data (c).*

Proof. Just use the equalities

$$(1_{A'}, b)(a, 1_B) = (a, b) = (a, 1_{B'})(1_A, b),$$

and relevant equalities from Remark 17.2. □

Theorem 17.4. *Given categories $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$. we have:*

(a) *For every functor $F : \mathbb{A} \to \mathbb{C}^{\mathbb{B}}$, there exist a unique functor $\Phi : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ with $\Phi_l = F$.*
(b) *For every functor $G : \mathbb{B} \to \mathbb{C}^{\mathbb{A}}$, there exist a unique functor $\Phi : \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ with $\Phi_r = F$.*
(c) *Let $((F_A : \mathbb{B} \to \mathbb{C})_{A\in\mathbb{A}_0}, (G_B : \mathbb{A} \to \mathbb{C})_{B\in\mathbb{B}_0})$ be a pair of families of functors, in which $F_A(B) = G_B(A)$ for all $A \in \mathbb{A}_0$ and $B \in \mathbb{B}_0$, and all diagrams of the form*

$$\begin{tikzcd}
 & G_B(A) \arrow[r, "G_B(a)"] \arrow[dl, equal] & G_B(A') \arrow[dr, equal] & \\
F_A(B) \arrow[d, "F_A(b)"'] & & & F_{A'}(B) \arrow[d, "F_{A'}(b)"] \\
F_A(B') \arrow[dr, equal] & & & F_{A'}(B') \arrow[dl, equal] \\
 & G_{B'}(A) \arrow[r, "G_{B'}(a)"'] & G_{B'}(A) &
\end{tikzcd}$$

commute. Then there exists a unique functor $\Phi : \mathbb{A}\times\mathbb{B} \to \mathbb{C}$ with $\Phi(A, -) = F_A$ and $\Phi(-, B) = G_B$ for all $A \in \mathbb{A}_0$ and $B \in \mathbb{B}_0$.

Proof. In each of these assertions, the uniqueness follows from Lemma 17.3, and so we need to prpve only the existence.

(a): For objects A in $\mathbb{A}$ and B in $\mathbb{B}$, we put $\Phi(A, B) = F(A)(B)$, and, for morphisms $a : A \to A'$ in $\mathbb{A}$ and $b : B \to B'$ in $\mathbb{B}$, consider the natuality square

$$\begin{tikzcd}
F(A)(B) \arrow[r, "F(a)_B"] \arrow[d, "F(A)(b)"'] & F(A')(B) \arrow[d, "F(A')(b)"] \\
F(A)(B') \arrow[r, "F(a)_{B'}"'] & F(A')(B');
\end{tikzcd}$$

we define $\Phi(a, b)$ as any of the two composites in it. That is, $\Phi(a, b)$ is defined as

$$\Phi(A, B) = F(A)(B) \xrightarrow{F(a)_{B'}(F(A)(b))=(F(A')(b))F(a)_B} F(A')(B') = \Phi(A', B').$$

The rest of the proof of (a) is a routine calculation, and we will only show that, given morphisms

$$(A, B) \xrightarrow{(a,b)} (A', B') \xrightarrow{(a',b')} (A'', B'')$$

in $\mathbb{A} \times \mathbb{B}$, we have $\Phi((a',b')(a,b)) = \Phi(a',b')\Phi(a,b)$. For, consider the diagram

$$\begin{tikzcd}
F(A)(B) \arrow[r, "F(a)_B"] \arrow[d, "F(A)(b)"'] \arrow[dr, "\Phi(a,b)"] & F(A')(B) \arrow[r, "F(a')_B"] \arrow[d, "F(A')(b)"] & F(A'')(B) \arrow[d, "F(A'')(b)"] \\
F(A)(B') \arrow[r, "F(a)_{B'}"'] \arrow[d, "F(A)(b')"'] & F(A')(B') \arrow[r, "F(a')_{B'}"] \arrow[d, "F(A')(b')"'] \arrow[dr, "\Phi(a',b')"] & F(A'')(B') \arrow[d, "F(A'')(b')"] \\
F(A)(B'') \arrow[r, "F(a)_{B''}"'] & F(A')(B'') \arrow[r, "F(a')_{B''}"'] & F(A'')(B''),
\end{tikzcd}$$

in which:

- the four small squares being naturality squares commutes;
- the four triangles commute by our definition of $\Phi(a,b)$ (and $\Phi(a',b')$);
- therefore, for example, the diagram

$$\begin{tikzcd}
F(A)(B) \arrow[d, "F(A)(b)"'] \arrow[dr, "\Phi(a,b)"] & & \\
F(A)(B') \arrow[d, "F(A)(b')"'] & F(A')(B') \arrow[dr, "\Phi(a',b')"] & \\
F(A)(B'') \arrow[r, "F(a)_{B''}"'] & F(A')(B'') \arrow[r, "F(a')_{B''}"'] & F(A'')(B'')
\end{tikzcd}$$

  commutes;
- since F and $F(A)$ are functors, we have $F(a')_{B''}F(a)_{B''} = F(a'a)_{B''}$ and $(F(A)(b'))(F(A)(b)) = F(A)(b'b)$;

This gives $\Phi(a',b')\Phi(a,b) = F(a'a)_{B''}(F(A)(b'b)) = \Phi(a'a,b'b) = \Phi((a',b')(a,b))$, as desired.

(b) and (c) can be proved similarly. Just note that, for (c), the naturality square used at the beginning of our proof of (a) should be replaced with the diagram displayed in in (c), and $\Phi(a,b)$ should be defined as

$$\Phi(A,B) \xrightarrow{G_{B'}(a)F_A(b)=F_{A'}(b)G_B(a)} \Phi(A',B'),$$

where $\Phi(A,B) = F_A(B) = G_B(A)$ and $\Phi(A',B') = F_{A'}(B') = G_{B'}(A')$. $\square$

Example 17.5. *Let us return again to Exercise 5 of Section 5, where the product $(A \times B, \pi_1, \pi_2)$ of objects A and B in a category $\mathbb{C}$ was defined via a universal property in fact saying that, for every object C and morphisms $\alpha : C \to A$ and $\beta : C \to B$, there exists a unique morphism $\gamma : C \to A \times B$ making the diagram*

$$\begin{tikzcd}
& C \arrow[dl, "\alpha"'] \arrow[d, "\gamma"] \arrow[dr, "\beta"] & \\
A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}$$

commute. We will write $\gamma = \langle \alpha, \beta \rangle$. Now, given morphisms $a : A \to A'$ and $b : B \to B'$, we can form the morphism $\langle a\pi_1, b\pi_2 \rangle : A \times B \to A' \times B'$. The *commutative diagram it is determined by can be displayed as*

$$\begin{tikzcd} A \arrow[d, "a"'] & A \times B \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\langle a\pi_1, b\pi_2\rangle"] & B \arrow[d, "b"] \\ A' & A' \times B' \arrow[l, "\pi_1'"] \arrow[r, "\pi_2'"'] & B', \end{tikzcd}$$

where both rows are product diagrams (that is, the represent the product of A and B and of A' and B', respectively). This suggests to write $\langle a\pi_1, b\pi_2\rangle = a \times b$. Next, we easily conclude that if all such products exist in $\mathbb{C}$ and if choose such a product for each pair (A, B) of objects in $\mathbb{C}$, then, associating $a \times b : A \times B \to A' \times B'$ to $(a, b) : (A, B) \to (A', B')$, we obtain a functor $\times : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. In this case, according to notation introduced in this section we can write:

(a) $A \times (-) : \mathbb{C} \to \mathbb{C}$, *for the functor defined by*
$$(A \times (-))(b : B \to B') = (A \times b : A \times B \to A \times B'),$$
where $A \times b = 1_A \times b$*;*

(b) $(-) \times B : \mathbb{C} \to \mathbb{C}$, *for the functor defined by*
$$((-) \times B)(a : A \to A') = (a \times B : A \times B \to A' \times B),$$
where $a \times B = a \times 1_B$*;*

(c) $\times_l : \mathbb{C} \to \mathbb{C}^{\mathbb{C}}$, *for the functor defined by*
$$\times_l(a : A \to A') = (a \times (-) : A \times (-) \to A' \times (-)),$$
where $a \times (-)$ is a natural transformation defined by $(a \times (-))_B = a \times 1_B$;

(d) $\times_r : \mathbb{C} \to \mathbb{C}^{\mathbb{C}}$, *for the functor defined by*
$$\times_r(b : B \to B') = ((-) \times b : (-) \times B \to (-) \times B'),$$
where $(-) \times b$ is a natural transformation defined by $((-) \times b)_A = 1_A \times b$.

Example 17.6. *Given again a category $\mathbb{C}$, consider the functor*

$$\hom : \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathsf{Sets}$$

defined by

$$\hom((a, b) : (A, B) \to (A', B')) = (\hom(a, b) : \hom(A, B) \to \hom(A', B')),$$

where $\hom(A, B)$ and $\hom(A', B')$ are as defined in Section 7, and $\hom(a, b)$ is defined by $(\hom(a, b))(f) = bfa$. In this case:

(a) $\hom(A, -)$ *and* $\hom(-, B)$ *are the same as in Section 7;*

(b) *the functor* $\hom_r : \mathbb{C} \to \mathsf{Sets}^{\mathbb{C}^{\mathrm{op}}}$ *is the same as the Yoneda embedding* $Y_{\mathbb{C}} : \mathbb{C} \to \mathsf{Sets}^{\mathbb{C}^{op}}$ *introduced in Theorem 14.6.*

Exercises.

1. Given categories $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ define the composition functor
$$\circ : \mathbb{C}^{\mathbb{B}} \times \mathbb{B}^{\mathbb{A}} \to \mathbb{C}^{\mathbb{A}}$$
with $\circ(G, F) = G \circ F = GF$, and explain that this functor was in fact inexplicitly used in Theorem 13.3(c) and at the end of the proof of Theorem 13.7.

2. Given categories $\mathbb{A}$ and $\mathbb{B}$, define the evaluation functor
$$\mathrm{Ev} = \mathrm{Ev}_{\mathbb{A},\mathbb{B}} : \mathbb{B}^{\mathbb{A}} \times \mathbb{A} \to \mathbb{B}$$
with $\mathrm{Ev}(F, A) = F(A)$.

3. Given a category $\mathbb{C}$, explain that the collection bijections

$$(\mathrm{Nat}(\hom(C,-),S) \rightleftarrows S(C))_{C\in\mathbb{C}_0, S\in(\mathsf{Sets}^{\mathbb{C}})_0}$$

from Theorem 14.1 make inverse to each other isomorphisms of functors $\mathsf{Sets}^{\mathbb{C}} \times \mathbb{C} \to \mathsf{Sets}$, one of which is the evaluation functor $\mathrm{Ev}_{\mathbb{C},\mathsf{Sets}}$.

4. For every two objects A and B in $\mathbb{C}$, chose their product diagram, and, using these chosen diagrams, examine the constructions introduced in Example 17.5 in the following cases:
(a) $\mathbb{C} = \mathsf{Sets}$;
(b) $\mathbb{C}$ is one of the other categories considered in Exercise 5 of Section 5;
(c) $\mathbb{C} = \mathsf{Cat}$.

5. Explain how Theorem 17.4 is related to Exercise 8 of Section 15.

18. Adjoint functors

We will define an *adjunction* in nine seemingly different ways, and then prove several simple theorems showing that these definitions are equivalent via certain straightforward bijections between the sets of structures involved.

Definition 18.1. *An adjunction $\mathbb{X} \to \mathbb{A}$ is any of the following kinds of structure on categories $\mathbb{X}$ and $\mathbb{A}$:*

(a) *A functor $U : \mathbb{A} \to \mathbb{X}$ and a family $((F(X), \eta_X))_{X\in\mathbb{X}_0}$, in which each $(F(X), \eta_X)$ is a universal arrow $X \to U$.*

(b) *A triple (F, U, η), in which $F : \mathbb{X} \to \mathbb{A}$ and $U : \mathbb{A} \to \mathbb{X}$ are functors, and $\eta : 1_{\mathbb{X}} \to UF$ a natural transformation such that each $(F(X), \eta_X)$ is a universal arrow $X \to U$.*

(c) *A functor $U : \mathbb{A} \to \mathbb{X}$ and a family $((F(X), \varphi_X))_{X\in\mathbb{X}_0}$, in which each $F(X)$ is an object in $\mathbb{A}$ and each φ_X is an isomorphism* $\hom_{\mathbb{A}}(F(X), -) \to \hom_{\mathbb{X}}(X, U(-))$.

(d) *A triple (F, U, φ), in which $F : \mathbb{X} \to \mathbb{A}$ and $U : \mathbb{A} \to \mathbb{X}$ are functors, and φ :* $\hom_{\mathbb{A}}(F(*), -) \to \hom_{\mathbb{X}}(*, U(-))$ *an isomorphism. Here* $\hom_{\mathbb{A}}(F(*), -)$ *and* $\hom_{\mathbb{X}}(*, U(-))$ *are the composites*

$$\mathbb{X}^{\mathrm{op}} \times \mathbb{A} \xrightarrow{F^{\mathrm{op}}\times\mathbb{X}} \mathbb{A}^{\mathrm{op}} \times \mathbb{A} \xrightarrow{\hom} \mathsf{Sets}$$

(where F^{op} is the functor dual to F, induced by F in the obvious way) and

$$\mathbb{X}^{\mathrm{op}} \times \mathbb{A} \xrightarrow{\mathbb{X}\times U} \mathbb{X}^{\mathrm{op}} \times \mathbb{X} \xrightarrow{\hom} \mathsf{Sets},$$

respectively.

(e) *A fourtuple $(F, U, \eta, \varepsilon)$, in which $F : \mathbb{X} \to \mathbb{A}$ and $U : \mathbb{A} \to \mathbb{X}$ are functors, and $\eta : 1_{\mathbb{X}} \to UF$ and $\varepsilon : FU \to 1_{\mathbb{A}}$ are natural transformation making the diagrams*

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FUF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} UFU \arrow[d, "U\varepsilon"'] & U \arrow[l, "\eta U"'] \arrow[dl, "1_U"] \\ U & \end{tikzcd}$$

commute.

(f) *A triple (F, U, ψ), in which $F : \mathbb{X} \to \mathbb{A}$ and $U : \mathbb{A} \to \mathbb{X}$ are functors, and* $\psi : \hom_{\mathbb{X}}(*, U(-)) \to \hom_{\mathbb{A}}(F(*), -)$ *an isomorphism.*

(g) *A functor $F : \mathbb{X} \to \mathbb{A}$ and a family $((U(A), \psi_A))_{A\in\mathbb{A}_0}$, in which each $U(A)$ is an object in $\mathbb{X}$ and each ψ_A is an isomorphism* $\hom_{\mathbb{X}}(-, U(A)) \to \hom_{\mathbb{A}}(F(-), A)$.

(h) *A triple* (F, U, ε), *in which* $F : \mathbb{X} \to \mathbb{A}$ *and* $U : \mathbb{A} \to \mathbb{X}$ *are functors, and* $\varepsilon : FU \to 1_{\mathbb{A}}$ *a natural transformation such that each* $(U(A), \varepsilon_A)$ *is a universal arrow* $F \to A$.

(i) *A functor* $F : \mathbb{X} \to \mathbb{A}$ *and a family* $((U(A), \varepsilon_A))_{A \in \mathbb{A}_0}$, *in which each* $(U(A), \varepsilon_A)$ *is a universal arrow* $F \to A$.

Theorem 18.2. ***('18.1(a)⇔18.1(b)')*** *Given a functor* $U : \mathbb{A} \to \mathbb{X}$ *and a family* $((F(X), \eta_X))_{X \in \mathbb{X}_0}$, *in which each* $(F(X), \eta_X)$ *is a universal arrow* $X \to U$, *there exists a unique way to define* F *also on morphisms such that it becomes a functor, and the family* $(\eta_X)_{X \in \mathbb{X}_0}$ *becomes a natural transformation* $\eta : 1_{\mathbb{X}} \to UF$.

Proof. If we could define F on morphisms as required, then, for every morphism $x : X \to Y$, the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "x"'] & UF(X) \arrow[d, "UF(x)"] \\ Y \arrow[r, "\eta_Y"'] & UF(Y) \end{tikzcd}$$

would commute. On the other hand, the universal property of $(F(X), \eta_X)$ shows that requiring this diagram to commute defines $F(x)$ as the unique morphism $F(X) \to F(Y)$, whose U-image composed with η_X gives $\eta_Y x$. After defining $F(x)$ this way for each morphism x in $\mathbb{X}$, we only need to show that $F(1_X) = 1_{F(X)}$ for each object X in $\mathbb{X}$, and that $F(yx) = F(y)F(x)$ for each composable pair (y, x) of morphisms in $\mathbb{X}$. Thanks to the universal property of $(F(X), \eta_X)$, is suffices to prove that $U(F(1_X))\eta_X = U(1_{F(X)})\eta_X$ and $U(F(yx))\eta_X = U(F(y)F(x))\eta_X$. We have:

$$U(F(1_X))\eta_X = \eta_X 1_X = U(1_{F(X)})\eta_X$$

and, for $x : X \to Y$ and $y : Y \to Z$:

$$U(F(yx))\eta_X = \eta_Z yx = UF(y)\eta_Y x = UF(y)UF(x)\eta_X = U(F(y)F(x))\eta_X,$$

as desired. □

From the equivalence (a_1)⇔(a_2) in Example 15.1(a) (see also Example 14.1(a)), we obtain:

Theorem 18.3. ***('18.1(a)⇔18.1(c)')*** *Given a functor* $U : \mathbb{A} \to \mathbb{X}$, *let* H *be the set of families* $((F(X), \eta_X))_{X \in \mathbb{X}_0}$ *satisfying 18.1(a) and* Φ *the set of families* $((F(X), \varphi_X))_{X \in \mathbb{X}_0}$ *satisfying 18.1(c). There is a bijection* $\mathsf{H} \approx \Phi$, *under which* $((F(X), \eta_X))_{X \in \mathbb{X}_0}$ *corresponds to* $((F(X), \varphi_X))_{X \in \mathbb{X}_0}$ *if and only if, for each* X, *they have the same* $F(X)$, *and* η_X *and* φ_X *determine each other by* $(\varphi_X)_A(f) = U(f)\eta_X$ *and* $\eta_X = (\varphi_X)_{F(X)}(1_{F(X)})$. □

Theorem 18.4. ***('18.1(c)⇔18.1(d)')*** *Given a functor* $U : \mathbb{A} \to \mathbb{X}$ *and a family* $((F(X), \varphi_X))_{X \in \mathbb{X}_0}$, *in which each* φ_X *is an isomorphism* $\hom_{\mathbb{A}}(F(X), -) \to \hom_{\mathbb{X}}(X, U(-))$, *there exists a unique way to define* F *also on morphisms such that it becomes a functor, and the family* $(\varphi_X)_{X \in \mathbb{X}_0}$ *becomes an isomorphism*

$$\hom_{\mathbb{A}}(F(*), -) \to \hom_{\mathbb{X}}(*, U(-)).$$

Proof. If we could define F on morphisms as required, then, for every morphism $x : X \to Y$, the diagram

$$\begin{tikzcd} \hom_{\mathbb{A}}(F(Y), -) \arrow[r, "\varphi_Y"] \arrow[d, "\hom_{\mathbb{A}}(F(x), -)"'] & \hom_{\mathbb{X}}(Y, U(-)) \arrow[d, "\hom_{\mathbb{X}}(x, U(-))"] \\ \hom_{\mathbb{A}}(F(X), -) \arrow[r, "\varphi_X"'] & \hom_{\mathbb{X}}(X, U(-)) \end{tikzcd}$$

would commute. On the other hand, since the Yoneda Embedding $\mathbb{A}^{op} \to \mathsf{Sets}^{\mathbb{A}}$ is fully faithful, requiring this diagram to commute defines $F(x)$. After defining $F(x)$ this way for each morphism x in $\mathbb{X}$, we only need to show that $F(1_X) = 1_{F(X)}$ for each object X in $\mathbb{X}$, and that $F(yx) = F(y)F(x)$ for each composable pair (y, x) of morphisms in $\mathbb{X}$. However, that is easy to check, and in we could simply apply Theorem 12.2 and the fact that the Yoneda Embedding $\mathbb{A}^{op} \to \mathsf{Sets}^{\mathbb{A}}$ is faithful. □

Let $((F(X), \eta_X))_{X\in\mathbb{X}_0}$ and $((F(X), \varphi_X))_{X\in\mathbb{X}_0}$ correspond to each other as in Theorem 18.3. According to Theorems 18.2 and 18.4, respectively, both of these families make F a functor $\mathbb{X} \to \mathbb{A}$, but do they make the same $F(x)$ for every $x : X \to Y$? To show that it is the case, consider the commutative diagram

$$\begin{tikzcd}
\hom_{\mathbb{A}}(F(Y), F(Y)) \arrow[r, "(\varphi_Y)_{F(Y)}"] \arrow[d, "\hom_{\mathbb{A}}(F(x),F(Y))"'] & \hom_{\mathbb{X}}(Y, UF(Y)) \arrow[d, "\hom_{\mathbb{X}}(x,UF(Y))"] \\
\hom_{\mathbb{A}}(F(X), F(Y)) \arrow[r, "(\varphi_X)_{F(Y)}"'] & \hom_{\mathbb{X}}(X, F(Y)),
\end{tikzcd}$$

which is the '$F(Y)$-component' of the commutative diagram used in the proof of Theorem 18.4. For $F(x)$, obtained as in that proof, we have

$$\begin{aligned} UF(x)\eta_X = (\varphi_X)_{F(Y)}(F(x)) = (\varphi_X)_{F(Y)}(\hom_{\mathbb{A}}(F(x), F(Y))(1_{F(Y)})) = \\ = \hom_{\mathbb{X}}(x, UF(Y))(\varphi_Y)_{F(Y)}(1_F(Y)) = \eta_Y x, \end{aligned}$$

and so it is indeed the same as $F(x)$ obtained as in the proof of Theorem 18.2.

This shows that 18.1(a)-18.1(d) all give the same notion of an adjunction. Dually, the same is true for 18.1(f)-18.1(i). Moreover, there is an obvious equivalence between 18.1(d) and 18.1(f): just take φ and ψ inverse of each other. We shall then write $(\varphi_X)_A = \varphi_{X,A}$ and $(\psi_A)_X = \psi_{X,A}$, and, from Theorem 18.3 and the dual result for ε and ψ, we obtain:

Corollary 18.5. *Let η be as in 18.1(b), φ as in 18.1(d) and related to η as in the theorems above, ψ the inverse of φ, and ε related to ψ dually to how η is related to φ. Then:*

$$\eta_X = \varphi_{X,F(X)}(1_{F(X)}),\ \ \varphi_{X,A}(f) = U(f)\eta_X,$$
$$\varepsilon_A = \psi_{U(A),A}(1_{U(A)}),\ \ \psi_{X,A}(u) = \varepsilon_A F(u),$$

for all $f : F(X) \to A$ in $\mathbb{A}$ and $u : X \to U(A)$ in $\mathbb{X}$. □

Theorem 18.6. ***(18.1(b) is 'equivalent' to other items of 18.1)*** *Under the assumptions of Corollary 18.5, the pair (η, ε) satisfies 18.1(e). Conversely, any pair $(\eta : 1_{\mathbb{X}} \to UF, \varepsilon : FU \to 1_{\mathbb{A}})$ of natural transformations satisfying 18.1(e) is obtained this way.*

Proof. Under the assumptions of Corollary 18.5, we calculate:

$$((\varepsilon F)(F\eta))_X = \varepsilon_{F(X)}F(\eta_X) = \psi_{X,F(X)}(\eta_X) = \psi_{X,F(X)}\varphi_{X,F(X)}(1_{F(X)}) = 1_{F(X)},$$

and, dually, $((U\varepsilon)(\eta U))_A = 1_{U(A)}$. That is, the pair (η, ε) satisfies 18.1(e).

Conversely, given natural transformations $\eta : 1_{\mathbb{X}} \to UF$ and $\varepsilon : FU \to 1_{\mathbb{A}}$, we can create φ out of η and ψ our of ε with all formulas of Corollary 18.5 satisfied. After that we will only need to prove that, if (η, ε) satisfies 18.1(e), then φ and ψ are inverse to each other. Moreover, as follows from Theorems 14.1 and 14.3 (or, just see Example 14.4), it suffices to prove that $(\psi_{X,F(X)}\varphi_{X,F(X)})(1_{F(X)}) = 1_{F(X)}$ and $(\varphi_{U(A),A}\psi_{U(A),A})(1_{U(A)}) = 1_{U(A)}$. But this follows from 18.1(e), since, as we have already seen, $(\psi_{X,F(X)}\varphi_{X,F(X)})(1_{F(X)}) = ((\varepsilon F)(F\eta))_X$, and, dually, $(\varphi_{U(A),A}\psi_{U(A),A})(1_{U(A)}) = ((U\varepsilon)(\eta U))_A$. □

Terminology and notation: When η, ε, φ, and ψ are as Corollary 18.5, one might say that $(F, U, \eta, \varepsilon, \varphi, \psi) : \mathbb{X} \to \mathbb{A}$, or just $(F, U, \eta, \varepsilon, \varphi, \psi)$, is an adjunction. More often, however, one replaces the sixtuple above either with $(F, U, \eta, \varepsilon)$ or with (F, U, φ, ψ), or even uses a triple involving F, G, and one of the letters η, ε, φ, or ψ. The shortest expression for the adjunction above is $F \dashv U$. The functors F and U are called the left and right adjoint (of U and of F), respectively. The natural transformations η and ε are called the unit and counit, respectively (of the given adjunction). One also says that (F, U) is an adjoint pair of functors. For some authors "right adjoint"="adjoint" and "left adjoint"="coadjoint"; for some others "left adjoint"="adjoint" and "right adjoint"="coadjoint".

Exercises.

1. Make precise and explain the following:
 (a) Initial and terminal objects in a category $\mathbb{C}$ can be described using left and right adjoints of the unique functor from $\mathbb{C}$ to a terminal category.
 (b) $F \dashv U$ is an adjunction if and only if so is $U^{op} \dashv F^{op}$.
 (c) If $F \dashv U$ is an adjunction, then $F \dashv U'$ is an adjunction if and only if $U \approx U'$.
 (d) If $F \dashv U$ and $G \dashv V$ are adjunctions and (G, F) is a composable pair of functors, then $GF \dashv UV$ is an adjunction. How is this related to Exercise 9 of Section 15?
 (e) If (F, G, α, β) is an equivalence of categories, then (F, G, α), (F, G, β^{-1}), (G, F, β), and (G, F, α^{-1}) are adjunctions, but (F, G, α, β) does not have to be (if it is the case, then it is called an *adjoint equivalence*).
 (f) Suppose $(F, U, \eta, \varepsilon) : \mathbb{X} \to \mathbb{A}$ is an adjunction, $\mathbb{X}'$ the full subcategory of $\mathbb{X}$ with objects all X in $\mathbb{X}$, for which η_X is an isomorphism, and $\mathbb{A}'$ the full subcategory of $\mathbb{A}$ with objects all A in $\mathbb{A}$, for which ε_A is an isomorphism. Then $(F, U, \eta, \varepsilon)$ induces an adjoint equivalence $\mathbb{X}' \sim \mathbb{A}'$, and it is the largest induced equivalence between full subcategories of $\mathbb{X}$ and of $\mathbb{A}$.
 (g) An adjunction Suppose $(F, U, \eta, \varepsilon) : \mathbb{X} \to \mathbb{A}$ can be described as an isomorphism $(F \downarrow 1_{\mathbb{A}}) \to (1_{\mathbb{X}} \downarrow U)$, making the diagram

$$\begin{tikzcd}
(F \downarrow 1_{\mathbb{A}}) \arrow[rr] \arrow[dr, "{(X,A,f) \mapsto (X,A)}"'] & & (1_{\mathbb{X}} \downarrow U) \arrow[dl, "{(X,A,u) \mapsto (X,A)}"] \\
& \mathbb{X} \times \mathbb{A} &
\end{tikzcd}$$

 commute.

2. Given an adjunction $(F, U, \eta, \varepsilon) : \mathbb{X} \to \mathbb{A}$, prove that:
 (a) F is faithful if and only if η_X is a monomorphism for each object X in $\mathbb{X}$;
 (b) F is full if and only if η_X is a split epimorphism for each object X in $\mathbb{X}$;
 (c) F is fully faithful if and only if η_X is an isomorphism for each object X in $\mathbb{X}$;
 (d) U is faithful if and only if ε_A is an epimorphism for each object A in $\mathbb{A}$;
 (e) U is full if and only if ε_A is a split monomorphism for each object A in $\mathbb{A}$;
 (f) U is fully faithful if and only if ε_A is an isomorphism for each object A in $\mathbb{A}$.

Hint: to prove (a) and (b) first prove that the diagram

$$\begin{tikzcd} & \hom(F(X'),F(X)) \arrow[dd, "\varphi_{X',F(X)}"] \\ \hom(X',X) \arrow[ur, "F_{X',X}"] \arrow[dr, "hom(X',\eta_X)"'] & \\ & \hom(X',UF(X)) \end{tikzcd}$$

commutes for all $X, X' \in \mathbb{X}_0$, and then use Exercises 5(a) and 5(e) of Section 7; use Theorem 6.2(f) to deduce (c) from (a) and (b); to deduce (d)-(f) from (a)-(c) use Exercise 1(b) of this section.

3. Replace general categories with ordered sets considered as categories and describe the resulting counterpart of Definition 18.1.

4. Describe adjoint functors obtained, via Theorem 18.2 (or its dual), from the universal arrows found in Exercises 4-8 of Section 15 (possibly excluding 4(j)).

5. For an adjunction $F \dashv U : \mathbb{X} \to \mathbb{A}$, prove that:
 (a) if X is an initial object in $\mathbb{X}$, then $F(X)$ is an initial object in $\mathbb{A}$;
 (b) if A is a terminal object in $\mathbb{A}$, then $U(A)$ is a terminal object in $\mathbb{X}$.
 (c) F preserves epimorphisms; then use Exercise 6(b) of Section 8 to conclude that, if F is faithful, then a morphism x in $\mathbb{X}$ is an epimorphism if and only if so is $F(x)$.
 (d) U preserves monomorphisms; then use Exercise 6(b) of Section 8 to conclude that, if U is faithful, then a morphism a in $\mathbb{A}$ is a monomorphism if and only if so is $U(a)$.

6. Explain that, as follows from Exercise 1(b), in Exercise 5 we have (a)$\Leftrightarrow$(b) and (c)$\Leftrightarrow$(d).

7. In the situation of Exercise 5, does U preserve initial objects and/or epimorphisms? Find examples and counter-examples.

8. Let $f : A \to B$ be a map of sets. In set theory one considers two induced maps between the power sets, which are
 – the map $P(A) \to P(B)$ carrying subsets of A to their images under f; in category this map is often denoted by $\exists_f$, since it can be presented as $X \mapsto \{b \in B \mid \exists_x (f(x) = b \wedge x \in X)\}$;
 – the map $P(B) \to P(A)$ carrying subsets of B to their inverse images under f; in category this map is often denoted by f^*.

Considering $P(A)$ and $P(B)$ as categories (using their inclusion orders), show that:
 (a) $\exists_f \dashv f^*$;
 (b) there is also an adjunction $f^* \dashv \forall_f$. Hint: define $\forall_f$ by

$$\forall_f (X) = \{b \in B \mid f(x) = b \Rightarrow x \in X\}.$$

Explain that (a) and (b) are dual to each other.

19. Graphs and diagrams

Informally, graphs, in the sense of category theory, are simply 'categories without identity morphisms and composition'. The formal definition is:

Definition 19.1. *A graph (also called a diagram scheme)* $\mathbb{G}$ *is a system* $\mathbb{G} = (\mathbb{G}_0, \mathbb{G}_1, d_{\mathbb{G}}, c_{\mathbb{G}})$*, in which:*

(a) *$\mathbb{G}_0$ and $\mathbb{G}_1$ are classes (usually actually sets) called the class of objects in $\mathbb{G}$ and the class of morphisms (or arrows) in $\mathbb{G}$, respectively;*
(b) *$d_{\mathbb{G}}$ and $c_{\mathbb{G}}$ are maps from $\mathbb{G}_1$ to $\mathbb{G}_0$ called domain and codomain respectively; for $f \in \mathbb{G}_1$, when $d_{\mathbb{G}}(f) = A$ and $c_{\mathbb{G}}(f) = B$, we write $f : A \to B$ and say that f is a morphism from A to B, or that the domain of f is A and the codomain of f is B.*

Graphs form a category Graphs*, in which a morphism $F : \mathbb{G} \to \mathbb{H}$ is a pair $F = (F_0, F_1)$, in which $F_0 : \mathbb{G}_0 \to \mathbb{H}_0$ and $F_1 : \mathbb{G}_1 \to \mathbb{H}_1$ are maps making the diagrams*

$$\begin{tikzcd} \mathbb{G}_1 \arrow[r, "F_1"] \arrow[d, "d_{\mathbb{G}}"'] & \mathbb{H}_1 \arrow[d, "d_{\mathbb{H}}"] \\ \mathbb{G}_0 \arrow[r, "F_0"'] & \mathbb{H}_0 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{G}_1 \arrow[r, "F_1"] \arrow[d, "c_{\mathbb{G}}"'] & \mathbb{H}_1 \arrow[d, "c_{\mathbb{H}}"] \\ \mathbb{G}_0 \arrow[r, "F_0"'] & \mathbb{H}_0 \end{tikzcd}$$

commute (equivalently, for every morphism $g : G \to G'$ in $\mathbb{G}$, $F_1(g)$ is a morphism (in $\mathbb{H}$) from $F_0(G)$ to $F_0(G')$; it is written simply as $F(g) : F(G) \to F(G')$).

Once graphs are defined, we can also introduce the formal definition of a diagram:

Definition 19.2. *Let $\mathbb{C} = (\mathbb{C}_0, \mathbb{C}_1, d_{\mathbb{C}}, c_{\mathbb{C}}, e_{\mathbb{C}}, m_{\mathbb{C}})$ be a category; the underlying graph of $\mathbb{C}$ is the graph $(\mathbb{C}_0, \mathbb{C}_1, d_{\mathbb{C}}, c_{\mathbb{C}})$. Given a graph $\mathbb{G}$ and a category $\mathbb{C}$, a diagram $D : \mathbb{G} \to \mathbb{C}$, also called a diagram in $\mathbb{C}$ over $\mathbb{G}$, is a morphism in* Graphs *from $\mathbb{G}$ to the underlying graph of $\mathbb{C}$.*

When a graph $\mathbb{G}$ has only a few objects and morphisms, it is convenient to display it as dots and arrows between them, dots for the objects and arrows for the morphisms. Given such a display, we can display a diagram $D : \mathbb{G} \to \mathbb{C}$ by replacing dots with (the names of) the images of the corresponding objects under D, and labeling arrows accordingly.

Remark 19.3. *Involving a graph in Definition 19.1 shows that we should not identify diagrams in $\mathbb{C}$ with sets of objects and morphisms in $\mathbb{C}$. For example*

$$C \xrightarrow{c} C \text{ and } C \xrightarrow{c} C \xrightarrow{c} C$$

display two different diagrams, and

$$C \xrightarrow{c} C \qquad C \xrightarrow{c} C \xrightarrow{c} C$$

displays yet another diagram. The graphs used in these three diagrams have two, three, and five objects, and one, two, and three morphisms, respectively.

Theorem 19.4. *There is an adjunction (F, U, η) :* Graphs $\to$ Cat*, in which:*

(a) *U :* Cat $\to$ Graphs *is the (obviously defined) underlying graph functor.*
(b) *For a graph $\mathbb{G}$, $F(\mathbb{G})$ is the category of paths of $\mathbb{G}$, in which the objects are the same as in $\mathbb{G}$, and a morphism $G \to G'$ is a sequence of the form*

$$G = G_0 \xrightarrow{g_1} \dots \xrightarrow{g_n} G_n = G',$$

written as $(g_n, ..., g_1) : G \to G'$, for each $n = 0, 1, 2, ...$; when $n = 0$, the sequence becomes empty and its existence forces $G = G'$, in which case it is written simply as 1_G, being the identity morphism of $F(\mathbb{G})$. The morphisms compose as $(g_n, ..., g_1)(f_m, ..., f_1) = (g_n, ..., g_1, f_m, ..., f_1)$. The category $F(\mathbb{G})$ is also called the free category on $\mathbb{G}$.
(c) *Each $\eta_{\mathbb{G}} : \mathbb{G} \to UF(\mathbb{G})$ is the (obviously defined) inclusion map.*

Proof. Let us only check the universal property of $\eta_{\mathbb{G}}$ (for an arbitrary graph $\mathbb{G}$), which is to check that, for every category $\mathbb{C}$ and every diagram $D : \mathbb{G} \to \mathbb{C}$, there exists a unique functor $\bar{D} : F(\mathbb{G}) \to \mathbb{C}$ with $\bar{D}\eta_{\mathbb{G}} = D$. If such a functor does exist, we must have

$$\bar{D}(g_n, ..., g_1) = \bar{D}(g_n)...\bar{D}(g_1) = \bar{D}\eta_{\mathbb{G}}(g_n)...\bar{D}\eta_{\mathbb{G}}(g_1) = D(g_n)...D(g_1),$$

which proves the uniqueness of $\bar{D}$ (under its existence). On the other hand, it is easy to check that we can define $\bar{D}$ by $\bar{D}(g_n, ..., g_1) = D(g_n)...D(g_1)$, which will give $\bar{D}\eta_{\mathbb{G}} = D$. $\square$

This theorem allows us to define the commutativity of a diagram formally:

Definition 19.5. *In the notation of Theorem 19.4 and its proof, a diagram D is said to be commutative if every two parallel morphisms in $F(\mathbb{G})$ have the same images under $\bar{D}$.*

Exercises.

1. Explain that the commutativity of a diagram in the sense of Section 10 is a special case of the commutativity of a diagram in the sense of Definition 19.5.
2. Prove that the diagram

$$\begin{tikzcd} A \arrow[r, "a"] & B \arrow[d, "b"] \\ D \arrow[u, "d"] & C \arrow[l, "c"] \end{tikzcd}$$

commutes if and only if all its arrows are isomorphisms and $dcba = 1_A$.
3. Let $D : \mathbb{G} \to \mathbb{A}$ be a diagram, $E : \mathbb{A} \to \mathbb{B}$ a functor, and U as in Theorem 19.4. Show that:
   (a) if D is commutative, then so is $U(E)D$;
   (b) if $U(E)D$ is commutative and E is faithful, then D is commutative.
4. Explain that Theorem 19.4 can be used in Exercise 4(f) of Section 15.
5. For a graph $\mathbb{G}$ and a category $\mathbb{C}$, define the category $\mathrm{Diag}(\mathbb{G}, \mathbb{C})$ of diagrams $\mathbb{G} \to \mathbb{C}$ and show that the bijection $\hom(F(\mathbb{G}), \mathbb{C}) \approx \hom(\mathbb{G}, U(\mathbb{C}))$, determined by the adjunction established in Theorem 19.4, extends to a category isomorphism $\mathbb{C}^{F}(\mathbb{G}) \approx \mathrm{Diag}(\mathbb{G}, \mathbb{C})$.
6. Find a graph $\mathbb{G}$ such that the category of graphs can be identified with the category $\mathrm{Diag}(\mathbb{G}, \mathsf{Sets})$.

20. Limits and colimits

When we think of diagrams over a 'very small' a graph, it is better denote our graph by G (rather than $\mathbb{G}$) and use small letters to denote its objects.

Definition 20.1. *Let G be a graph and $\mathbb{C}$ a category. Then:*

(a) *Given an object C in $\mathbb{C}$, the constant diagram $\Delta(C) = \mathrm{Const}_C : G \to \mathbb{C}$ (cf. Execrise 1 of Section 8) is defined by*

$$\Delta(C)(f : x \to y) = (1_C : C \to C).$$

(b) *Given a morphism $c : C \to C'$ in $\mathbb{C}$, the corresponding morphism $\Delta(c) : \Delta(C) \to \Delta(C')$ in $\mathrm{Diag}(G, \mathbb{C})$ (see Exercise 1 of Section 11 and Exercise 5 of Section 19), which is in fact the same a natural transformation of the corresponding functors $F(G) \to \mathbb{C}$, is defined by $\Delta(c)_x = c$.*

(c) *The diagonal functor $\Delta : \mathbb{C} \to \mathrm{Diag}(G, \mathbb{C})$ is determined by the data (a) and (b) above.*

Next, for a diagram $D : G \to \mathbb{C}$:

(d) *A cone (=limiting cone) over D is a natural transformation $\Delta(C) \to D$, for some object C in $\mathbb{C}$; explicitly, it is a system $(C, (\gamma_x : C \to D(x))_{x \in G_0})$ in which $D(f)\gamma_x = \gamma_y$ for each morphism $f : x \to y$ in G.*

(e) *A limit of D is a universal arrow $\Delta \to D$; it will be denoted by*

$$\lim D = (\lim D, (\pi_x : \lim D \to D(x))_{x \in G_0}).$$

According to the universal property of $\lim D$, *for every cone $(C, (\gamma_x : C \to D(x))_{x \in G_0})$ over D, there exists a unique morphism $\gamma : C \to \lim D$ with $\pi_x \gamma = \gamma_x$ for each object x in G. Here the morphisms π_x $(x \in G_0)$ are called the limit projections (sometimes just "projections")*

(f) *A cocone (=colimiting cone) over D is a natural transformation $D \to \Delta(C)$, for some object C in $\mathbb{C}$; explicitly, it is a system $(C, (\gamma_x : D(x) \to C)_{x \in G_0})$ in which $\gamma_y D(f) = \gamma_x$ for each morphism $f : x \to y$ in G.*

(g) *A colimit of D is a universal arrow $D \to \Delta$; it will be denoted by*

$$\operatorname{colim} D = (\operatorname{colim} D, (\iota_x : D(x) \to \operatorname{colim} D)_{x \in G_0}).$$

According to the universal property of $\operatorname{colim} D$, *for every cocone $(C, (\gamma_x : D(x) \to C)_{x \in G_0})$ over D, there exists a unique morphism $\gamma : \operatorname{colim} D \to C$ with $\gamma \iota_x = \gamma_x$ for each object x in G. Here the morphisms ι_x $(x \in G_0)$ are called the colimit injections.*

The same notions are used for functors (considered as diagrams).

Note that the colimit injections do not have to be injective in any sense, although in many 'classical' examples where $\mathbb{C}$ is a category of mathematical structures on sets they are.

Let us also mention some old terminology and notation, still kept in some 'non-category-theoretic' literature:

Now established	Old
limit	inverse limit
lim	$\varprojlim$
colimit	direct limit
colim	$\varinjlim$

and consider several special situations:

- Suppose $G_1 = \emptyset$, that is, G has no morphisms; let us also write $G_0 = S$. In this case, to give a diagram $D : G \to \mathbb{C}$ is to give an S-indexed family $(D_s)_{s \in S}$ of objects in $\mathbb{C}$ (where $D_s = D(s)$), and we write

$$\lim D = \prod_{s \in S} D_s;$$

this limit is called the *product* (also *cartesian product* or *direct product*) of the family $(D_s)_{s \in S}$, and its limit projections are called the product projections. When S is finite, say $S = \{1, ..., n\}$, we also write

$$\prod_{s \in S} D_s = D_1 \times ... \times D_n = D_1 \sqcap ... \sqcap D_n.$$

In particular, the product $A \times B$ considered in Example 17.5 (see also Exercise 5 of Section 5) is nothing but $D_1 \times D_2$ for $D_1 = A$ and $D_2 = B$.

Dually, for the colimit, called *coproduct*, or, sometimes, *sum*, we write

$$\operatorname{colim} D = \coprod_{s \in S} D_s = \sum_{s \in S} D_s$$

and

$$\sum_{s\in S} D_s = D_1 + ... + D_n = D_1 \sqcup ... \sqcup D_n \text{ (when } S = \{1, ..., n\});$$

coproducts of two objects are used in Exercise 5 of Section 5.

If $G_0 = S$ is also empty, then $\lim D = 1$ (a terminal object) and $\operatorname{colim} D = 0$ (an initial object) in $\mathbb{C}$.

- Suppose G is a graph of the form

$$\bullet \rightrightarrows \bullet\,,$$

that is, it has two objects and two parallel morphisms between them; let us write them as

$$\begin{tikzcd} 1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & 0. \end{tikzcd}$$

A diagram $D : G \to \mathbb{C}$ can be displayed as

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B, \end{tikzcd}$$

where $A = D(1)$, $B = D(0)$, $f = D(s)$, and $g = D(t)$. A cone over D can be presented as a triple (C, γ_0, γ_1), where $\gamma_0 : C \to B$ and $\gamma_1 : C \to A$ are morphisms in $\mathbb{C}$ with $f\gamma_1 = \gamma_0 = g\gamma_1$. However, since this makes γ_0 completely determined by D and γ_1, such a cone can be equivalently presented as a pair (C, h), where $h : C \to A$ is a morphism in $\mathbb{C}$ with $fh = gh$ (we can then take $\gamma_0 = hf$ and $\gamma_1 = h$). Accordingly, the limit of D, also called the *equalizer* of f and g, can be described as a pair (C, h) above whose universal property is: for every morphism $h' : C' \to A$ with $fh' = gh'$, there exists a unique morphism $k : C' \to C$ with $hk = h'$. The whole display is

$$\begin{tikzcd} & C' \arrow[dl, dotted, "k"'] \arrow[d, "h'"] & \\ C \arrow[r, "h"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B; \end{tikzcd}$$

one also says that the bottom row here is an equalizer diagram.

Dually, for the colimit of the same D, called the *coequalizer* of (f, g), the similar display is

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "h"] \arrow[d, "h'"'] & C \arrow[dl, dotted, "k"] \\ & C' & \end{tikzcd}$$

(indicating that (C, h) is the coequalizer of (f, g)), and the top row here is called a coequalizer diagram.

- Now suppose that G is a graph of the form

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

(having therefore three objects and two morphisms as displayed). Writing it as

$$\begin{tikzcd} & 2 \arrow[d, "v"] \\ 1 \arrow[r, "u"'] & 0 \end{tikzcd}$$

we can display a diagram $D : G \to \mathbb{C}$ as

$$\begin{tikzcd} & B \arrow[d, "b"] \\ A \arrow[r, "a"'] & X, \end{tikzcd}$$

where $X = D(0)$, $A = D(1)$, $B = D(2)$, $a = D(u)$, and $b = D(v)$. A cone over D can be presented as a quadruple $(C, \gamma_0, \gamma_1, \gamma_2)$, where $\gamma_0 : C \to X$, $\gamma_1 : C \to A$, and $\gamma_2 : C \to B$ are morphisms in $\mathbb{C}$ with $a\gamma_1 = \gamma_0 = b\gamma_2$. However, since this makes γ_0 completely determined by D, γ_1, and γ_2, such a cone can be equivalently presented as a triple (C, p, q), where $p : C \to A$ and $q : C \to B$ are morphisms in $\mathbb{C}$ with $ap = bq$ (we can then take $\gamma_0 = ap$, $\gamma_1 = a$, and $\gamma_2 = b$). Accordingly, the limit of D, also called the *pullback* of a and b, can be described as a triple (C, p, q) above whose universal property is: for every morphism $p' : C' \to A$ and every morphism $q' : C' \to B$ with $ap' = bq'$, there exists a unique morphism $k : C' \to C$ with $pk = p'$ and $qk = q'$. The whole display is

$$\begin{tikzcd} C' \arrow[dr, dotted, "k"] \arrow[drr, "q'"] \arrow[ddr, "p'"'] & & \\ & C \arrow[r, "q"'] \arrow[d, "p"] & B \arrow[d, "b"] \\ & A \arrow[r, "a"'] & X. \end{tikzcd}$$

Here, unlike the previous case, where G was isomorphic to its (obviously defined) dual graph, the dual display

$$\begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] & B \arrow[d, "j"'] \arrow[ddr, "j'"] & \\ A \arrow[r, "i"] \arrow[drr, "i'"'] & C \arrow[dr, dotted, "k"] & \\ & & C' \end{tikzcd}$$

presents 'new' (X, A, a, B, b) as a diagram, call it E, over a graph of the form

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

with a colimit of E, also called the *pushout* of a and b, and preserted as the triple (C, i, j).

Standard displays for pullbacks and pushouts are

$$\begin{tikzcd} A \times_X B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "b"] \\ A \arrow[r, "a"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "b"] \arrow[d, "a"'] & B \arrow[d, "\iota_2"] \\ A \arrow[r, "\iota_1"'] & A +_X B, \end{tikzcd}$$

where (a and b in the first square are different from a and b in the second one and) the first square presents $(A \times_X B, \pi_1, \pi_2)$ as a pullback of a and b, while the second one presents $(A +_X B, \iota_1, \iota_2)$ as a pushout of a and b.

More terminology involving the squares above:

– They are also called a pullback square or a cartesian square, and a pushout square or a cocartesian square, respectively. The terms "pullback diagram" and "pushout diagram" are also used.
– In the first of them: one says that π_1 is a pullback of b along a, and that π_2 is a pullback of a along b; one also says that $A \times_X B$ is a fibred product of A and B over X.
– In the second one: one says that ι_1 is a pushout of b along a, and that ι_2 is a pushout of a along b; one also says that $A +_X B$ is an amalgamated product (or amalgamated sum) of A and B over X.
– One also uses the terms "cofibred" and "coamalgamated" accordingly, and there are some other variations of this terminology in the literature, which we omit here.

- Old literature, especially the non-category-theoretic one, considers cases where G is an ordered set, usually infinite, regarded as a category and $D : G \to \mathbb{C}$ is a functor, covariant or contravariant, sometimes called a spectrum or just a system. Certain special terminology and notation are uses there, often assuming that G is directed, which means that for every finite subset S of G, there exists $g \in G$ with $s \leqslant g$ for all $s \in S$. There is a categorical counterpart of "directed", called "filtered", but we will not discuss this here.

Remark 20.2. *Introducing a new category $\mathbb{C}$, it is always important to know which limits and which colimits it admits, that is, for which diagrams $D : G \to \mathbb{C}$ the limit* $\lim D$ *and the colimit* $\operatorname{colim} D$ *exist. In particular, one says that:*

(a) $\mathbb{C}$ *is small complete if* $\lim D$ *exists whenever G is small. This terminology usually indicates that we distiguish between sets (possibly calling them small sets) and proper classes (possibly calling them large sets). The first exercise below in fact describes small limits in* Sets *(provided we assume that the objects of* Sets *are small sets). We could also assume the objects of* Sets *to be 'larger', and accordingly get* $\lim D$ *for larger G. But what we cannot do is to make G 'as large as' the category* Sets *itself. More precisely, one can even show if a category $\mathbb{C}$ admits a product of all its objects, then it is coherent (as defined in Exercise 1(b) of Section 13).*

(b) $\mathbb{C}$ *is finitely complete if* $\lim D$ *exists whenever G is finite (that is, G_0 and G_1 are finite sets).*

Exercise 11 below will show that the existence of small products and equalizers implies the existence of all small limits, and that the same is true with "finite" instead of "small". Exercises 12 and 13 will then show that the existence of finite products and equalizers can be replaced with the existence of pullbacks and a terminal object.

Exercises.

1. Describe limits in Sets. More specifically:
(a) First show that for a family $(A_s)_{s\in S}$ of sets, the product
$$\prod_{s\in S} A_s$$
can be described as the cartesian product in set theory, with the product projections π_s ($s \in S$) defined (also as in set theory) by $\pi_s((x_{s'})_{s'\in S}) = x_s$.
(b) Then (using or not the description above), show that for a graph G and a diagram $D : G \to \mathsf{Sets}$, the limit of D can be described as
$$\lim D = \{(d_x)_{x\in G_0} \in \prod_{x\in G_0} D(x) \,|\, \forall_{f:x\to y\in G_1}\, D(f)(d_x) = d_y\},$$
with the limit projections π_x ($x \in G_0$) defined by $\pi_x((d_{x'})_{x'\in G_0}) = d_x$.
(c) Use (b) to describe an equalizer of maps f and g from a set A to a set B. Then show that it can also be described as
$$\{a \in A \,|\, f(a) = g(a)\},$$
equipped with the inclusion map from it to A.
(d) Use (b) to describe a pullback of maps $f : A \to X$ and $g : B \to X$. Then show that it can also be described as
$$\{(a, b) \in A \times B \,|\, f(a) = g(b)\},$$
equipped with the map from it to A defined by $(a, b) \mapsto a$ and the map from it to B defined by $(a, b) \mapsto b$. In particular, when $f = g$, conclude that this pullback is nothing but the equivalence relation on A determined by f. Remark: in Sets and in general (when $f = g$) such a pullback is called a kernel pair of f.
(e) Show that the diagrams
$$\begin{tikzcd} f^{-1}(B) \arrow[r, "a\mapsto f(a)"] \arrow[d] & B \arrow[d] \\ A \arrow[r, "f"'] & X \end{tikzcd} \qquad \begin{tikzcd} A\cap B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & X, \end{tikzcd}$$
in Sets where B is a subset of X, A is a subset of X in the second diagram, and the unlabeled arrows are inclusion maps, are pullback diagrams.

2. Make descriptions similar to those in (a)-(d) of the previous exercise in all categories you are familiar with and their opposite categories.

3. Given categories $\mathbb{A}$ and $\mathbb{B}$, describe limits and colimits in $\mathbb{B}^{\mathbb{A}}$ using limits and colimits in $\mathbb{B}$.

4. Explain that, in the notation of Theorem 19.4 and its proof, we can write $\lim \bar{D} = \lim D$ and $\operatorname{colim} \bar{D} = \operatorname{colim} D$.

5. Explain that initial and terminal objects can be defined as 'empty colimit' and 'empty limit', respectively.

6. Explain that, for a functor $F : \mathbb{A} \to \mathbb{B}$, we can write:
(a) $\lim F = F(Z)$, if Z is an initial object in $\mathbb{A}$;
(b) $\operatorname{colim} F = F(Z)$, if Z is a terminal object in $\mathbb{A}$.

7. Define what it means for a functor to preserve (existing) limits and to preserve colimits, and show that, for an adjunction $F \dashv U$, F preserves colimits while U preserves limits.

8. Describe limits and colimits of diagrams in a preorder considered as a category. Hint: use Exercises 5(a) and 5(b) of Section 18.

9. Use the two previous exercises to explain that the functors $\exists_f$, f^*, and $\forall_f$, considered in Exercise 8 of Section 18, have the following preservation properties:

(a) $\exists_f$ preserves unions;
(b) f^* preserves unions, intersections, and complements;
(c) $\forall_f$ preserves intersections.

10. Let $\mathbb{C}$ be category and C an object in $\mathbb{C}$. Prove that the functor

$$\hom(C,-) : \mathbb{C} \to \mathsf{Sets}$$

preserves limits. Then use this and Exercise 3 to prove that the Yoneda embedding $\mathbb{C} \to \mathsf{Sets}^{\mathbb{C}^{op}}$ preserves limits.

11. Given a diagram $D : G \to \mathbb{C}$, consider the diagram displayed as

$$\begin{tikzcd}
 & D(c_G(f)) \arrow[r, "1_{D(c_G(f))}"] & D(c_G(f)) \\
L \arrow[r, "l"] & \prod_{x\in G_0} D(x) \arrow[u, "\pi_{c_G(f)}"] \arrow[d, "\pi_{d_G(f)}"'] \arrow[r, shift left, "p"] \arrow[r, shift right, "q"'] & \prod_{f\in G_1} D(c_G(f)) \arrow[u, "\pi_f"'] \arrow[d, "\pi_f"] \\
 & D(d_G(f)) \arrow[r, "D(f)"'] & D(c_G(f)),
\end{tikzcd}$$

where the vertical arrows are the suitable product projections, p is defined by requiring the top square to commute for every $f \in G_1$, q is defined by requiring the bottom square to commute for every $f \in G_1$, and the middle row is an equalizer diagram. Prove that $(L, (\pi_x l)_{x\in G_0})$ is a limit of D.

12. Explain that a pullback $A \times_X B$ becomes a product $A \times B$ when X is a terminal object, and that, in general, $A \times_X B$ in $\mathbb{C}$ can be described as a product in $(\mathbb{C} \downarrow X)$.

13. For morphisms $f, g : A \to B$ in a given category, prove that if

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "p"'] & B \arrow[d, "\langle 1_B, 1_B\rangle"] \\
A \arrow[r, "\langle f,g\rangle"'] & B \times B
\end{tikzcd}$$

is a pullback diagram, then (P, p) is an equalizer of f and g.

14. For a morphism $f : A \to B$ in a given category, prove that:

(a) f is a monomorphism if and only if

$$\begin{tikzcd}
A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

is a pullback diagram;

(b) f is an epimorphism if and only if

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "1_B"] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}$$

is a pushout diagram.

15. Consider again the commutative squares of Theorem 10.1, and show that:
(a) the square considered in Theorem 10.1(a) is a pullback square, and so are the squares considered in Theorem 10.1(c) (assuming that they commute and f and f' are isomorphisms);
(b) in Theorem 10.1(b): if the second commutative square in is a pullback square, then the first commutative square is a pullback square if and only if so is the third one.

16. Let $\mathbb{A}$ and $\mathbb{X}$ be categories, $U : \mathbb{A} \to \mathbb{X}$ be a functor, and X an object in $\mathbb{X}$. Explain that we can write $(X \downarrow U)^{\mathrm{op}} \approx (U^{\mathrm{op}} \downarrow X)$ 'canonically', and so a pair (A, α) is a universal arrow $X \to U$ if and only if it is a universal arrow $U^{\mathrm{op}} \to X$. Then explain that we can also write $\mathrm{colim}(D) = \lim(D^{\mathrm{op}})$ (and $\lim(D) = \mathrm{colim}(D^{\mathrm{op}})$).

17. Explain that kernels and cokernels of group homomorphisms are special cases of equalizers and coequalizers, respectively.*

18. Explain that the additive group of rational numbers can be described as a colimit of the diagram

$$\mathbf{Z} \xrightarrow{2} \mathbf{Z} \xrightarrow{3} \mathbf{Z} \xrightarrow{4} \mathbf{Z} \xrightarrow{5} \ldots,$$

where $\mathbf{Z}$ denotes the additive group of integers and each arrow-labeling number n denotes the homomorphism defined by $x \mapsto nx$. Generalize this to describe arbitrary fraction monoids and fraction rings as colimits (equipped with a suitable multiplication in the case of rings).*

List of categories

- Sets = the category of sets (Example 2.2).
- A monoid viewed as a category (Example 2.3).
- A preorder vied as a category (Example 2.4).
- $\mathbb{C}^{\mathrm{op}}$ = the opposite category of a category $\mathbb{C}$ (Example 2.6).
- Mon = the category of monoids (Exercise 2 of Section 2 and Example 3.7).
- Preord = the category of preordered sets (of preorders) (Exercise 2 of Section 2).
- Ord = the category of ordered sets (of orders) (Exercise 2 of Section 2).
- $\mathsf{Alg}(\Omega)$ = the category of Ω-algebras (below Definition 3.2).
- $\mathsf{Pointed\,Sets}$ = the category of pointed sets (Example 3.4).
- Magmas = the category of magmas (Example 3.5).
- $\mathsf{Semigroups}$ = the category of semigroups (Example 3.6).
- Groups = the category of groups (Example 3.8).
- $\mathsf{CommSemigroups}$ = the category of commutative semigroups (Example 3.9).
- $\mathsf{CommMon}$ = the category of commutative monoids (Example 3.9).
- Ab = the category of abelian groups (Example 3.9).
- Sets^M = the category of M-sets, where M is a monoid (Example 3.10, although Ω plays the role of M there).
- Cat = the category of categories (the beginning of Section 7).
- $\mathsf{Ar}(\mathbb{C})$ = the arrow category of $\mathbb{C}$ (Remark 10.3).
- $\mathbb{B}^{\mathbb{A}}$ = the category of all functors from $\mathbb{A}$ to $\mathbb{B}$ (the beginning of Section 11).
- Vect_K^{fin} = the category of finite-dimensional K-vector spaces, where K is a field (Exercise 1(c) of Section 13).
- fAb = the category of finite abelian groups (Exercise 5 of Section 13).
- $\mathrm{El}_{\mathbb{C}}(S)$ = the category of elements of S over $\mathbb{C}$ (Definitions 15.1 and 15.3).

- Comma categories $(F \downarrow G)$, $(C \downarrow G)$, $(F \downarrow C)$, $(C \downarrow \mathbb{C})$, and $(\mathbb{C} \downarrow C)$ (Section 16, see also Example 15.5).
- Graphs = the category of graphs (Definition 19.1).
- $\mathrm{Diag}(\mathbb{G}, \mathbb{C})$ = the category of diagrams $\mathbb{G} \to \mathbb{C}$ (Exercise 5 of Section 19).

Questions and answers

This section is devoted to questions of students (reformulated) and answers.

Question 1. What are morphisms in the category of triples considered in Exercise 2 of Section 5? Is it true that a morphism $u : (X, e, f) \to (X', e', f')$ is a map $u : X \to X'$ with $u(e) = e'$ and $uf = f'u$?

Answer. Yes. Having in mind convention (iii) above Example 2.2, choosing the right notion of morphism was a part of the exercise. And in this case the choice should have been motivated by Definition 3.2, since the triples considered in Exercise 2 clearly are Ω-algebras for a suitable Ω.

Question 2. Exercise 3 of Section 5 asks for universal properties of other number systems (apart from natural numbers); what can we do with $\mathbf{Q}$ (the system of rational numbers) and $\mathbf{R}$ (the system of real numbers, which seems to have much more structure)? Are there categories in which they are initial objects?

Answer. This is not a question of existence of such categories: for each mathematical object X one can obviously form a category $\mathbb{X}$ in which X is an initial object, e.g. by taking $\mathbb{X}_i = \{X\}$ $(i = 0, 1)$. It is the question of finding a *nice* such category. Here "nice" means that:

- it has a simple independent definition suggested by *most basic* properties of the object X of our interest;
- an object in it is initial if and only if it has *all* the properties we need X to have.

For example, constructing $\mathbf{Q}$, we obviously need to show that it is a field of characteristic 0 (since it should contain an isomorphic copy of the ring of integers) – and it is nice to observe that we can *define* it simply as an initial object in the category of fields of characteristic 0.

The story $\mathbf{R}$ is more complicated: its structure is not purely algebraic, in fact it has several natural structures and each of then can be used to find a nice category in which $\mathbf{R}$ is an initial object. For example, one can use the fact that $\mathbf{R}$ is a completion of $\mathbf{Q}$ as a metric space, but then meric spaces should be defined using $\mathbf{Q}$ instead of $\mathbf{R}$ (doing so, one defines $d(x, y) \leqslant q$ for each $q \in \mathbf{Q}$ instead of defining distances $d(x, y)$).

Question 3. Perhaps for the sake of clarity; when talking about $\hom_{\mathbb{C}}(X, -)$, is the "$-$" symbol just a "place holder" for an arbitrary object in $\mathbb{C}$? Is this done to avoid quantification over the objects of the category?

Answer. *Yes* to the first question, and *No* to the second one. And note that this kind of "place holder" notation has nothing to do category theory. For example, in high school mathematics we write, say, $y = f(x)$ and speak of a function f – but we could write $f(-)$ instead of f. This is convenient when we have to compose f with another function; for example if g is defined by $g(x) = f(x + 1)$, then it is convenient to write $g = f((-) + 1)$ (and of course "$-$" should not be understood here as "minus").

Question 4. Suppose we are in the category of sets and we have an object A in this category. Is the set-theoretic complement of A a well-defined object in the same category?

Answer. No, simply because the notion of set-theoretic complement of a set does not make sense. In set theory, given a set A, if we allow ourselves to form the set $\{x \,|\, x \notin A\}$, then we will also get $A \cup \{x \,|\, x \notin A\}$, which will be nothing but the forbidden 'set of all sets'. All we can do, is to define the set-theoretic complement of a given subset inside a given set; and yes, it can be defined categorically, but only up to isomorphism.

Question 5. In Theorem 14.1, the maps α and β are defined by $\alpha(\sigma) = \sigma_C(1_C)$ and $\beta(c)_A(f) = S(f)(c)$. What does this mean?

Answer. Well, α should be map from the set $\mathrm{Nat}(\hom(C, -), S)$ of all natural transformations $\hom(C, -) \to S$ to the set $S(C)$; therefore to define it is to define, for each natural transformation $\sigma : \hom(C, -) \to S$, the corresponding element $\alpha(\sigma)$ of $S(C)$. Here σ, being a natural transformation $\hom(C, -) \to S$, is a family $(\sigma_A)_{A \in \mathbb{C}_0}$ of maps $\sigma_A : \hom(C, A) \to S(A)$. In particular, one of the members of this family is the map $\sigma_C : \hom(C, C) \to S(C)$. We then define $\alpha(\sigma)$ as the image of 1_C under this map, which, of course, is written as $\alpha(\sigma) = \sigma_C(1_C)$.

This makes Theorem 14.1 amazing in a sense. Indeed, we take a 'large' family $\sigma = (\sigma_A)_{A \in \mathbb{C}_0}$, pick up only one element of it, namely σ_C, then apply it to just one element of $\hom(C, C)$, namely 1_C, and claim that sending σ to $\sigma_C(1_C)$ is a bijection (which means that the whole σ can be recovered from just one image of an element under just one of its members!).

Now about β. To define β is to define $\beta(c)$, for each $c \in S(c)$. But again, $\beta(c)$ being a natural transformation $\hom(C, -) \to S$ must be a family $(\beta(c)_A)_{A \in \mathbb{C}_0}$, in which each $\beta(c)_A$ is a map from the corresponding $\hom(C, A)$ to $S(A)$. And to define $\beta(c)_A$ is to define $\beta(c)_A(f)$, for each $f \in \hom(C, A)$. That is, to define β is define each $\beta(c)_A(f)$, and we define it by $\beta(c)_A(f) = S(f)(c)$.

Question 6. In the notation used in Question 5, how do we know that $\alpha\beta(c) = \beta(c)_C(1_C)$?

Answer. Well, $\alpha\beta(c)$ is nothing but $\alpha(\beta(c))$, and since α is defined by $\alpha(\sigma) = \sigma_C(1_C)$, taking σ to be $\beta(c)$ we obtain $\alpha(\beta(c)) = \beta(c)_C(1_C)$.

Question 7. Corollary 15.7 (see (a) and (b) there) characterizes representable functors in terms of categories of elements. Why are we interested in representable functors and why is this characterization useful?

Answer. Look at the Yoneda embedding $Y = Y_{\mathbb{C}} : \mathbb{C} \to \mathsf{Sets}^{\mathbb{C}^{op}}$. It makes an arbitrary category $\mathbb{C}$ equivalent to a full subcategory of the category $\mathsf{Sets}^{\mathbb{C}^{op}}$ that we can deal with almost as we deal with the category of sets. Saying "almost as we deal with the category of sets" refers to the fact we can generalize many notions and constructions from Sets to $\mathsf{Sets}^{\mathbb{C}^{op}}$ simply be defining them argumentwise. We can then use the argumentwise constructions in $\mathsf{Sets}^{\mathbb{C}^{op}}$ to make constructions in $\mathbb{C}$ using representability, which in fact allows us to copy some constructions from Sets to a general category!

For example, suppose we are interested to use this approach to define the product $A \times B$ of two objects A and B in a category $\mathbb{C}$. First we go from $\mathbb{C}$ to $\mathsf{Sets}^{\mathbb{C}^{op}}$, hence replacing A and B with $Y_{\mathbb{C}}(A) = \hom(-, A)$ and $Y_{\mathbb{C}}(B) = \hom(-, B)$, respectively. Then we define $Y_{\mathbb{C}}(A) \times Y_{\mathbb{C}}(B)$ by $(Y_{\mathbb{C}}(A) \times Y_{\mathbb{C}}(B))(X) = Y_{\mathbb{C}}(A)(X) \times Y_{\mathbb{C}}(B)(X)$ (which is what we mean by an argumentwise construction; we mean "for every object X", assuming that the reader will understand what do with morphisms);

here $Y_{\mathbb{C}}(A)(X)\times Y_{\mathbb{C}}(B)(X)$ is the ordinary cartesian product of the sets $Y_{\mathbb{C}}(A)(X) = \hom(X, A)$ and $Y_{\mathbb{C}}(B)(X) = \hom(X, B)$. If the functor $Y_{\mathbb{C}}\times Y_{\mathbb{C}}$ is not representable, then we simply say the product $A\times B$ does not exist in $\mathbb{C}$. But if it is representable, which means that $Y_{\mathbb{C}}(A) \times Y_{\mathbb{C}}(B) \approx Y_{\mathbb{C}}(C)$ for some object C in $\mathbb{C}$, then we say that $A \times B$ is that object C. It turns out that this determines $A \times B$ uniquely up to isomorphism.

However, before considering the Yoneda embedding we already had another excellent approach, namely using universal properties, to create categorical definitions. How to compare these two approaches? Well, this is what Corollary 15.7 does; more precisely, Corollary 15.7 shows that whatever is defined via representability can also be defined via a universal property. To illustrate this, let us return to our example above:

According to Corollary 15.7, the functor $Y_{\mathbb{C}}(A) \times Y_{\mathbb{C}}(B)$ is representable if and only if the category $\mathrm{El}_{\mathbb{C}}(Y_{\mathbb{C}}(A) \times Y_{\mathbb{C}}(B))$ has a terminal object. But the category $\mathrm{El}_{\mathbb{C}}(Y_{\mathbb{C}}(A)\times Y_{\mathbb{C}}(B))$ is the same as the category constructed in Exercise 5 of Section 5 to define $A\times B$ (see also Example 17.5). It is then easy to see that the two resulting definitions of $A \times B$ fully agree.

Remark: It seems that $A \times B$ defined via representability is just an object in $\mathbb{C}$, while the universal property defines it together with the projections $\pi_1 : A\times B \to A$ and $\pi_2 : A\times B \to B$. But fixing the projections π_1 and π_2 (which is important!) corresponds, by Yoneda Lemma, to fixing an isomorphism $Y_{\mathbb{C}}(A\times B) \to Y_{\mathbb{C}}(A)\times Y_{\mathbb{C}}(B)$ (which is necessary of course in order to use $A\times B$ defined via such an isomorphism).

Question 8. What are the nine types of structures introduced in Definition 18.1 good for?

Answer. The purpose of Section 18 is to show that these nine types of structures are equivalent to each other in a very strong sense: any structure of the type (a) can be 'canonically' transformed into a structure of the type (b), and the same is true for all other pairs of types, yielding bijections $\mathcal{A} \approx \mathcal{B} \approx \mathcal{C} \approx \mathcal{D} \approx \mathcal{E} \approx \mathcal{F} \approx \mathcal{G} \approx \mathcal{H} \approx \mathcal{I}$, where $\mathcal{A}$ is the collection of all structures of the type (a), $\mathcal{B}$ is the collection of all structures of the type (b), and so on. Therefore what we in fact deal with is a nine ways to define the same structure, which is called an adjunction. These adjunctions come up in many places in mathematics and having many equivalent ways to construct them helps to understand each special case. So, the question is, what are adjunctions good for and what are the important examples?

To answer this question, let us begin with an arbitrary functor $U : \mathbb{A} \to \mathbb{X}$ and ask ourselves if at any circumstances there is a way to construct a functor $F : \mathbb{X} \to \mathbb{A}$, associated to it. We know that:

- If U is an isomorphism, then there exists a unique functor $F : \mathbb{X} \to \mathbb{A}$ with $UF = 1_{\mathbb{X}}$ and $FU = 1_{\mathbb{A}}$.
- More generally, if U is an equivalence, then there exists a unique up an isomorphism functor $F : \mathbb{X} \to \mathbb{A}$ with $UF \approx 1_{\mathbb{X}}$ and $FU \approx 1_{\mathbb{A}}$.
- Exercise 1(e) of Section 18 tells that the case where U has a left adjoint is still more general, and Exercise 1(c) of Section 18 tells us that even in that case there is a unique up an isomorphism 'good' functor $F : \mathbb{X} \to \mathbb{A}$ associated to U. Here 'good' means *left adjoint*. Of course, dually we could speak of "right" instead of "left".

That is, an equivalence is a generalized isomorphism, and an adjunction is a generalized equivalence. After that we say that the notion of adjunction is good for defining 'generalizing inverses' of functors, and it remains to show some important examples. The following ones are determined by universal arrows considered in some Exercises of Section 15:

- the universal arrows to be described in Exercise 4 of Section 15 produce a left adjoint to the functor $U : \mathbb{A} \to \mathsf{Sets}$ in cases (a)-(k) of that exercise.
- the universal arrows to be described in Exercise 5 of Section 15 produce left adjoints to the forgetful functors $\mathsf{Mon} \to \mathsf{Pointed\,Sets}$, $\mathsf{Mon} \to \mathsf{Semigroups}$, and $\mathsf{Ab} \to \mathsf{CommMon}$.
- the universal arrows to be described in Exercise 6 of Section 15 produce right adjoints to the forgetful functors $\mathsf{Preord} \to \mathsf{Sets}$ and $\mathsf{Ab} \to \mathsf{CommMon}$, and to the functor $\mathsf{Sets} \to \mathsf{Sets}$ defined as in item (c) there.

Two other interesting examples (dual to each other in a sense), which in fact led to the development of categorical logic more than fifty years ago, are given in Exercise 8 of Section 18, and another one is described in Theorem 19.4. Furthermore, consider the diagonal functor $\Delta : \mathbb{C} \to \mathrm{Diag}(G, \mathbb{C})$ from Definition 20.1. As immediately follows from that definition, it has:

- a right adjoint $\lim : \mathrm{Diag}(\mathbb{G}, \mathbb{C}) \to \mathbb{C}$, whenever every diagram $G \to \mathbb{C}$ has a limit;
- a left adjoint $\mathrm{colim} : \mathrm{Diag}(\mathbb{G}, \mathbb{C}) \to \mathbb{C}$, whenever every diagram $G \to \mathbb{C}$ has a colimit.

But this list of important examples is very far from being complete...

Question 9. What do we mean by saying $\mathbf{Z}[x]$ is a free ring on $\{x\}$?

Answer. Given any category $\mathbb{A}$ of algebraic structures (by which we mean a full subcategory of the category $\mathsf{Alg}(\Omega)$, for some Ω, as defined in Section 3), we have the underlying set functor (=the forgetful functor) $U : \mathbb{A} \to \mathsf{Sets}$ (cf. Exercise 9 of Section 8). And, for a set X, a ("the") free algebra in $\mathbb{A}$ on X is a universal arrow $X \to U$ (see Example 15.5). Explicitly, a free algebra in $\mathbb{A}$ on X is a pair $(F(X), \eta_X)$ in which $F(X)$ is an object in $\mathbb{A}$ and η_X a map from X to (the underlying set of) $F(X)$ satisfying the following universal property:

For every object A in $\mathbb{A}$ and every map f from X to (the underlying set of) A, there exist a unique homomorphism $\bar{f} : F(X) \to A$ with $\bar{f}\eta_X = f$.

It is often convenient to assume that X is a subset of $F(X)$ and that η_X is the corresponding inclusion map. If so, then the universal property above becomes:

For every object A in $\mathbb{A}$, every map f from X to (the underlying set of) A uniquely extends to a unique homomorphism $\bar{f} : F(X) \to A$.

In particular, we could take $\mathbb{A}$ to be the category of rings (with 1), where Ω consists of:

- nullary 0 and 1,
- unary $-$,
- binary $+$ and $\cdot$,

and take X to be a one-element set $\{x\}$. Then, what is $F(X)$? Well, it turns out to be $\mathbf{Z}[x]$ = the ring of polynomials of one variable with coefficients in the ring $\mathbf{Z}$ of integers. Indeed, to a give a map from the set $\{x\}$ to a ring A is to pick up an element a in A, and what we have to show is:

For every element a in A, there exists a unique ring homomorphism $\mathbf{Z}[x] \to A$ sending x to a.

But we know this: the desired homomorphisms is defined by

$$p_0x^n + \ldots + p_n \mapsto p_0a^n + \ldots + p_n.$$

In fact we very often use this, specifically, when we define a ring homomorphism $\mathbf{Z}[x] \to A$ by $x \mapsto a$.

Question 10. In Section 14, why do we have two forms of Yoneda Lemma ("covariant" and "contravariant") and only one form of Yoneda embedding?

Answer. True, we could have either only one or both forms for both Yoneda Lemma and Yoneda embedding. However:

- Although the two forms of Yoneda Lemma very easily follow from each other, mentioning both explicitly is useful because each of them is used in many places in mathematics independently from the other one.
- On the other hand, the purpose of Yoneda embedding $\mathbb{C} \to \mathsf{Sets}^{\mathbb{C}^{\mathrm{op}}}$ (also used for many things!) is more specific in a sense: when we are interested in some category $\mathbb{C}$, in order to work in it we embed it in a larger but 'easier' category. Of course we could be interested in the opposite category $\mathbb{C}^{\mathrm{op}}$ of a given category $\mathbb{C}$ and need to use the Yoneda embedding $\mathbb{C}^{\mathrm{op}} \to \mathsf{Sets}^{\mathbb{C}}$. This does happen, for example in algebraic geometry and algebraic number theory, where it is done for $\mathbb{C}$ being the category of commutative algebras (with 1) over a field − but 'less frequently' in a sense. And again, we get this second form from the first one very easily: just substitute $\mathbb{C}^{\mathrm{op}}$ for $\mathbb{C}$ and use the fact that $(\mathbb{C}^{\mathrm{op}})^{\mathrm{op}} = \mathbb{C}$.

That is, the two forms of Yoneda Lemma are rather equally appealing, while for the Yoneda embedding the form $\mathbb{C} \to \mathsf{Sets}^{\mathbb{C}^{\mathrm{op}}}$ is clearly more appealing than $\mathbb{C}^{\mathrm{op}} \to \mathsf{Sets}^{\mathbb{C}}$.

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