Computing the mixed-strategy Nash equilibria

If (σ_1, σ_2) is a N.E. then

$$\begin{cases} \pi_1(\sigma_1, \sigma_2) = \overbrace{\pi_1(A, \sigma_2) = \pi_1(B, \sigma_2)} \\ \pi_2(\sigma_1, \sigma_2) = \pi_2(\sigma_1, C) = \pi_2(\sigma_1, D) \end{cases}$$

Theorem. At a Nash equilibrium in mixed strategies, a player must be indifferent between any two PURE strategies that she plays with positive probability.

$$= \left(\underbrace{\begin{pmatrix} A & B \\ p & 1-p \end{pmatrix}}_{\sigma_1}, \underbrace{\begin{pmatrix} C & D \\ q & 1-q \end{pmatrix}}_{\sigma_2} \right)$$

Player 2

		q		$1-q$	
		C		D	
	p A	**15**	**3**	**36**	**0**
Player 1	$1-p$ B	**34**	**0**	**30**	**2**

if Player 1 plays A his payoff would be $15q + 36(1-q) = 36 - 21q$

" " B " " $34q + 30(1-q) = 30 + 4q$

By the theorem it must be that $36 - 21q = 30 + 4q$

$6 = 25q$ $\quad \boxed{q = \frac{6}{25}}$

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$$\pi_2(\sigma_1, \sigma_2) = \pi_2(\sigma_1, C) = \pi_2(\sigma_1, D)$$

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		Player 2	
		q	1-q
		C	*D*
Player 1	p *A*	15 3	36 0
	1-p *B*	34 0	30 2

NASH EQUILIBRIUM

$$\left(\begin{pmatrix} A & B \\ \frac{2}{5} & \frac{3}{5} \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{6}{25} & \frac{19}{25} \end{pmatrix} \right)$$

IF player 2 plays C then $\pi_2 = 3p$

" D then $\pi_2 = 2(1-p) = 2 - 2p$

By theorem: need $3p = 2 - 2p$ $\quad 5p = 2$

$$p = \frac{2}{5}$$

Theorem. At a Nash equilibrium in mixed strategies, a player must be indifferent between any two strategies that she plays with positive probability.

Provides a necessary, *but not sufficient*, condition for a mixed-strategy profile to be a Nash equilibrium

Player 2

		D		E	
Player 1	A	6	0	0	4
	B	0	4	6	0
	C	4	0	4	2

$$\pi_1 \overbrace{\left(\begin{pmatrix} A & B & C \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right)}^{\text{NOT A Nash equilibrium}} =$$

$$\frac{1}{2}\frac{1}{2}6 + \frac{1}{2}\frac{1}{2}0 + \frac{1}{2}\frac{1}{2}0 + \frac{1}{2}\frac{1}{2}6 = \frac{6}{2} = 3$$

because

$$\pi_1\left(C, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) = 4\frac{1}{2} + \frac{4}{2} = 4$$

$$\left.\begin{aligned} \pi_1\left(A, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) &= 6\frac{1}{2} + 0\frac{1}{2} = 3 \\ \pi_1\left(B, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right) &= \frac{1}{2}0 + \frac{1}{2}6 = 3 \end{aligned}\right\}$$

The condition of the theorem is silent about $\pi_1\left(C, \begin{pmatrix} D & E \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right)$ because prob of C is zero

STRICTLY

Theorem: A strategy can be played with positive probability at a Nash equilibrium only if it survives the IDSDS procedure.

2

		D	q E	$1-q$ F
Step 1	A	~~**1 , 4**~~	~~**4 , 2**~~	~~**0 , 8**~~
1	p **B**	~~**4 , 0**~~	**2 , 1**	**2 , 0**
	$1-p$ **C**	~~**2 , 3**~~	**6 , 4**	**1 , 6**

strictly dominated by C

Step 2:
D now strictly
dominated by E

Player 1:

$B \rightarrow 2q + 2(1-q) = 2$

$C \rightarrow 6q + 1\cdot(1-q) = 1+5q$

need $2 = 1+5q$ $\boxed{q = \frac{1}{5}}$

Player 2:

$E \rightarrow 1\cdot p + 4(1-p) = 4-3p$

$F \rightarrow 0\cdot p + 6(1-p) = 6-6p$

need $4-3p = 6-6p$

$3p = 2$ $\boxed{p = \frac{2}{3}}$

$$\left(\begin{pmatrix} A & B & C \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \begin{pmatrix} D & E & F \\ 0 & \frac{1}{5} & \frac{4}{5} \end{pmatrix} \right)$$

Nash
equil

CARDINAL

IDSDS

Player 2

		D	E	F
Player 1	A	0, 1	3, 0	5, 0
	B	1, 2	1, 4	2, 3
	C	3, 0	0, 2	0, 1

Player 1 : B is not strictly dominated by A

B " " C

B $\underline{\text{is}}$ strictly dominated by $\overbrace{\begin{pmatrix} A & C \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}}^{\sigma_1}$:

- against D, B gives $\boxed{1}$, σ_1 gives: $\frac{1}{2}0 + \frac{1}{2}3 = \boxed{1.5}$
- E, " $\boxed{1}$, : $\frac{1}{2}3 + \frac{1}{2}0 = \boxed{1.5}$
- F, B gives $\boxed{2}$, σ_1 gives: $\frac{1}{2}5 + \frac{1}{2}0 = \boxed{2.5}$

q **Player 2** $1-q$

		D	E	~~F~~
	p A	0 1	3 0	~~5 0~~
Player 1	~~B~~	~~1 2~~	~~1 4~~	~~2 3~~
	$1-p$ C	3 0	0 2	~~0 1~~

Step 1

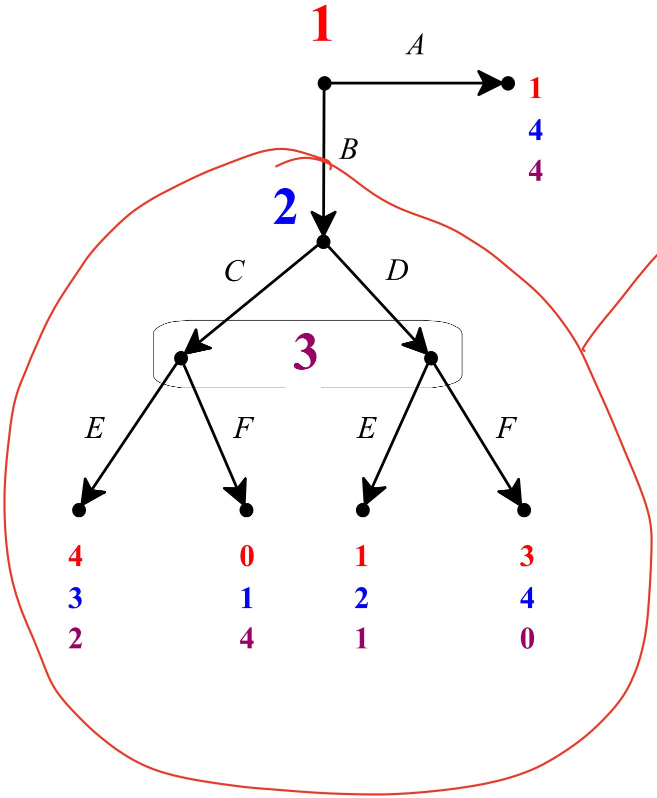
Step 2

In the reduced game after deletion of B

F is strictly dominated by $\underbrace{\begin{pmatrix} D & E \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}}_{\sigma_2}$

- F against A gives Player 2 $\boxed{0}$, σ_2 gives $\frac{1}{3}1+0\frac{2}{3}=\boxed{\frac{1}{3}}$
- " C $\boxed{1}$, σ_2 $\frac{1}{3}0+\frac{2}{3}2=\boxed{\frac{4}{3}}$

2, 2.5, 4/3



3

		q	1-q
		E	F
2	p C	3, 2	1, 4
	1-p D	2, 1	4, 0

No pure-strategy N.E.

Player 2:

$C \rightarrow 3q + 1(1-q) = 1+2q$

$D \rightarrow 2q + 4(1-q) = 4-2q$

$4-2q = 1+2q \quad 3 = 4q$

$\boxed{q = \frac{3}{4}}$

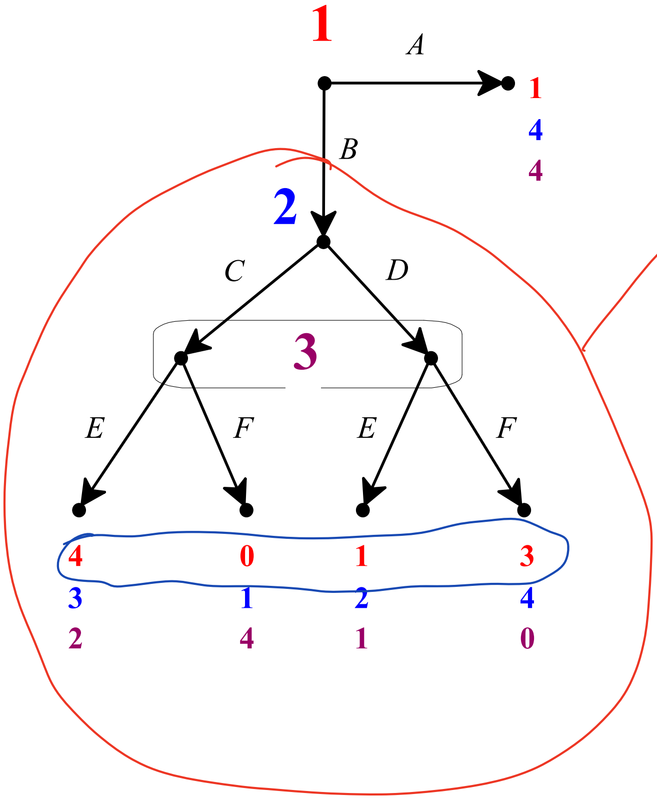
Player 3: $E \rightarrow 2p + 1(1-p) = 1+p$

$F \rightarrow 4p + 0(1-p) = 4p$

$4p = 1+p$

$3 = \quad p = \frac{1}{3}$

2, 2.5, 4/3



3

			q	1-q
			E	F
2	p	C	3, 2	1, 4
	1-p	D	2, 1	4, 0

No ure-strategy N.E.

N.E. : $\left(\left(\begin{array}{cc} C & D \\ \frac{1}{3} & \frac{2}{3} \end{array}\right), \left(\begin{array}{cc} E & F \\ \frac{3}{4} & \frac{1}{4} \end{array}\right)\right)$

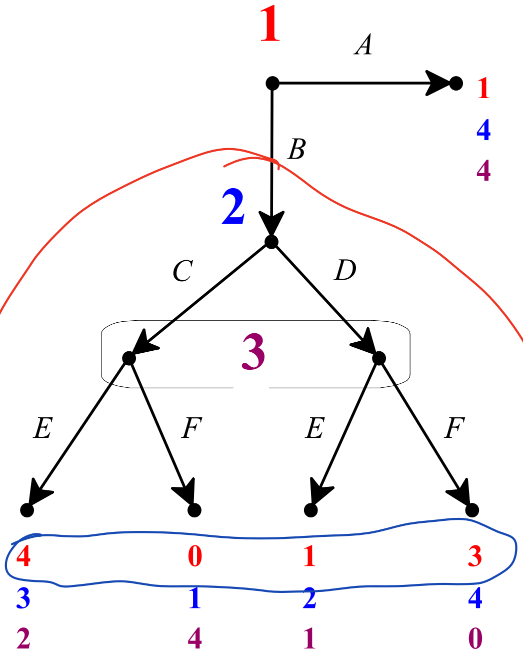
$$\pi_1 = \frac{1}{3}\frac{3}{4}4 + \frac{1}{3}\frac{1}{4}0 + \frac{2}{3}\frac{3}{4}1 + \frac{2}{3}\frac{1}{4}3$$

$$= 2$$

$$\pi_2 = 3\frac{3}{4} + 1\frac{1}{4} = 2.5$$

$$\pi_3 = 4\frac{1}{3} + 0\frac{2}{3} = \frac{4}{3}$$

2, 2.5, 4/3



			3	
		q		1-q
		E		F
2	p	C	3, 2	1, 4
	1-p	D	2, 1	4, 0

No ure-strategy N.E.

$$N.E. : \left(\overbrace{\begin{pmatrix} C & D \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}}^{\sigma_2}, \overbrace{\begin{pmatrix} E & F \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}}^{\sigma_3} \right)$$

$$\pi_1 = \frac{1}{3}\frac{3}{4}4 + \frac{1}{3}\frac{1}{4}0 + \frac{2}{3}\frac{3}{4}1 + \frac{2}{3}\frac{1}{4}3$$

$$= 2$$

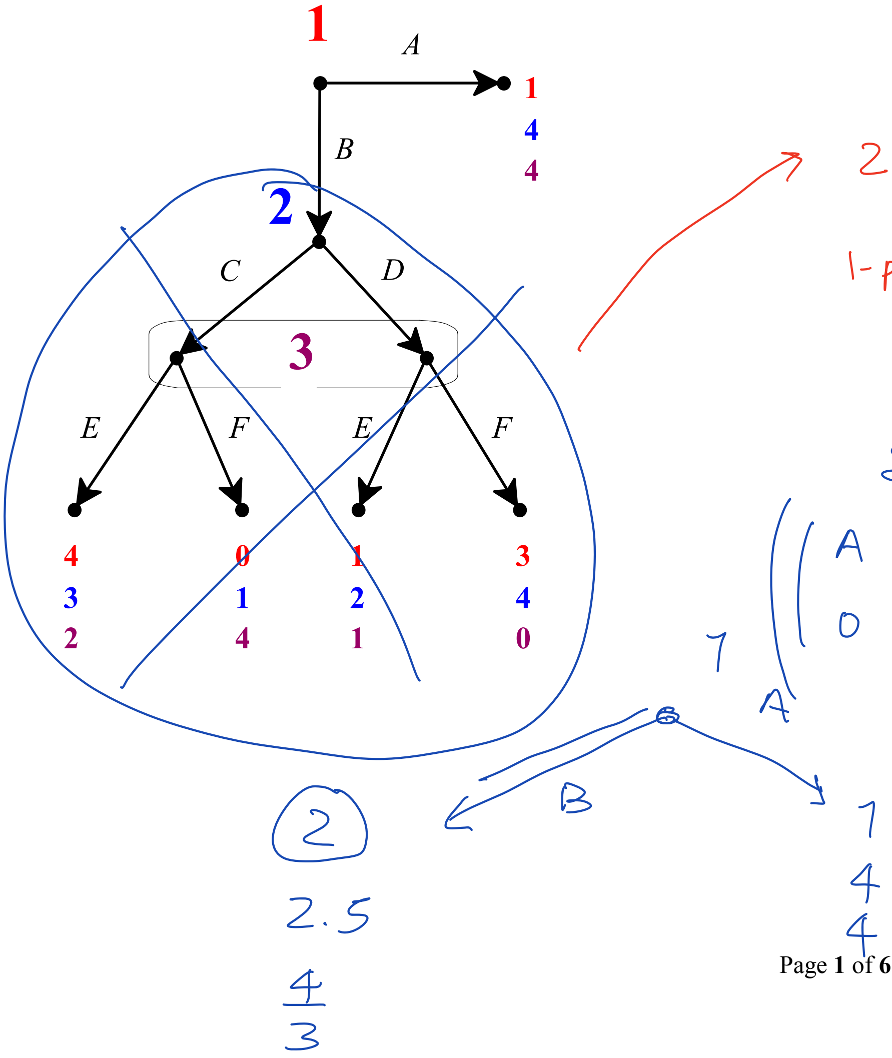
$$\pi_2 = 3\frac{3}{4} + 1\frac{1}{4} = 2.5$$ ← using the fact that $\pi_2(\sigma_2, \sigma_3) = \pi(C, \sigma_3)$

$$\pi_3 = 4\frac{1}{3} + 0\frac{2}{3} = \frac{4}{3}$$ ← using $\pi_3(\sigma_2, \sigma_3) = \pi_3(\sigma_2, F)$

2, 3)   3 (2,

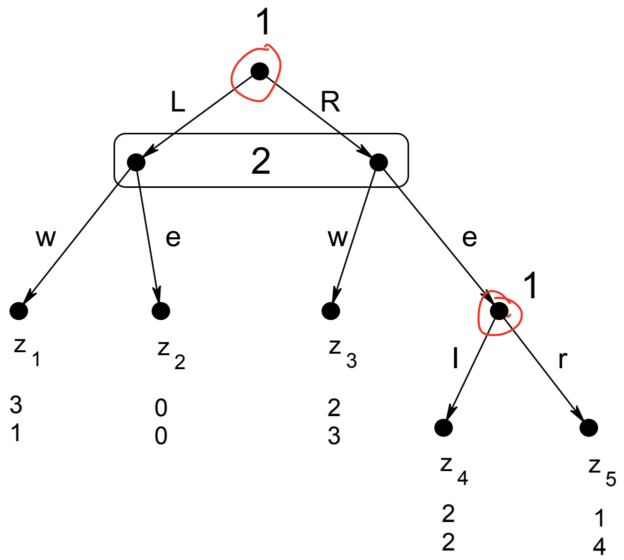
2, 2.5, 4/3

$$\begin{array}{cccc} & & & 3 \\ & & q & 1-q \\ & & E & F \\ 2 & p \quad C & 3,2 & 1,4 \\ & 1-p \quad D & 2,1 & 4,0 \end{array}$$



SPE :

$$7 \left(\begin{pmatrix} A & B \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} C & D \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \begin{pmatrix} E & F \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix} \right)$$



a mixed strategy for Player 1

$$\left(\begin{pmatrix} Ll & Lr & Rl & Rr \\ \frac{1}{12} & \frac{4}{12} & \frac{2}{12} & \frac{5}{12} \end{pmatrix}, \begin{pmatrix} w & e \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\right)$$

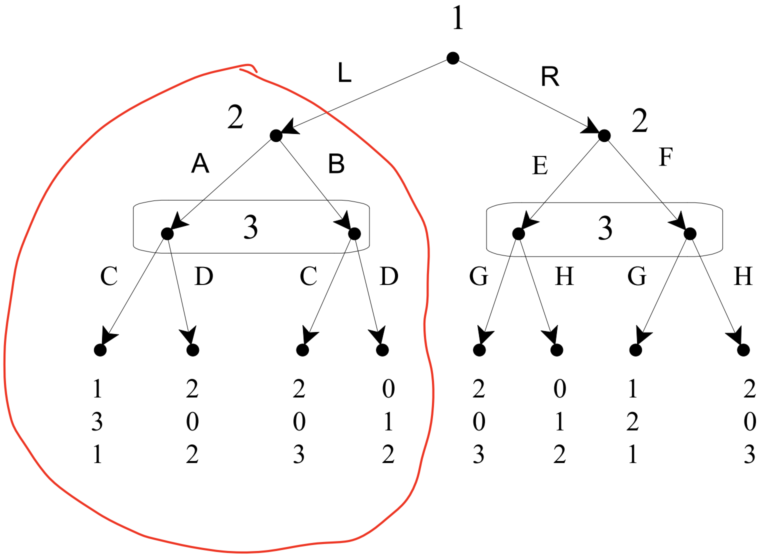
outcome	z_1	z_2	z_3	z_4	z_5
probability	$\frac{5}{36}$	$\frac{10}{36}$	$\frac{7}{36}$	$\frac{4}{36}$	$\frac{10}{36}$

$$\left(\underbrace{\begin{pmatrix} L & R & \vert & l & r \\ \frac{5}{12} & \frac{7}{12} & \vert & \frac{2}{7} & \frac{5}{7} \end{pmatrix}}_{\text{a behavioral strategy for Player 1}}, \begin{pmatrix} w & e \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}\right)$$

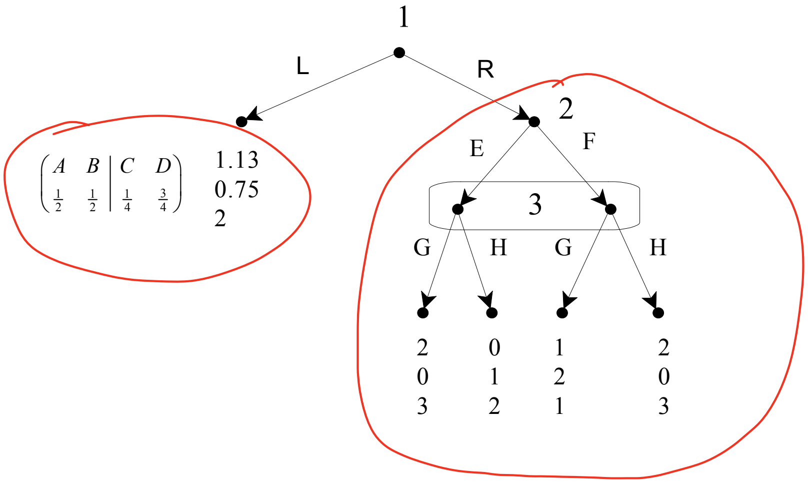
THEOREM (Kuhn, 1953). In an extensive game **with perfect recall**, mixed and behavior strategies are equivalent. [For a precise statement of "equivalence" see the textbook.]

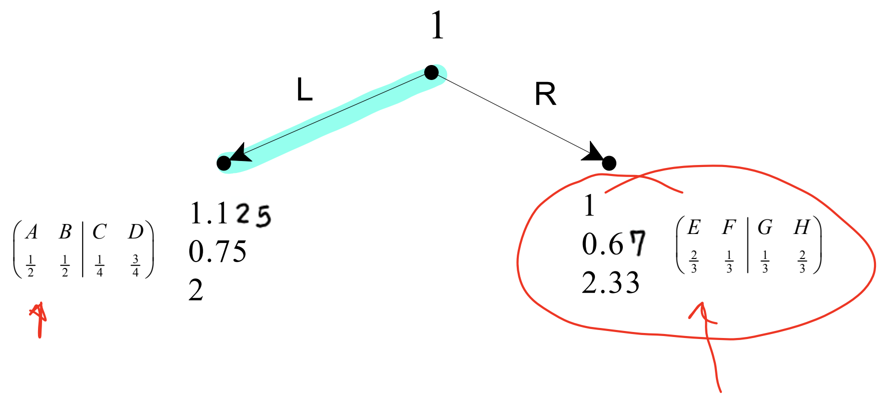
1.125

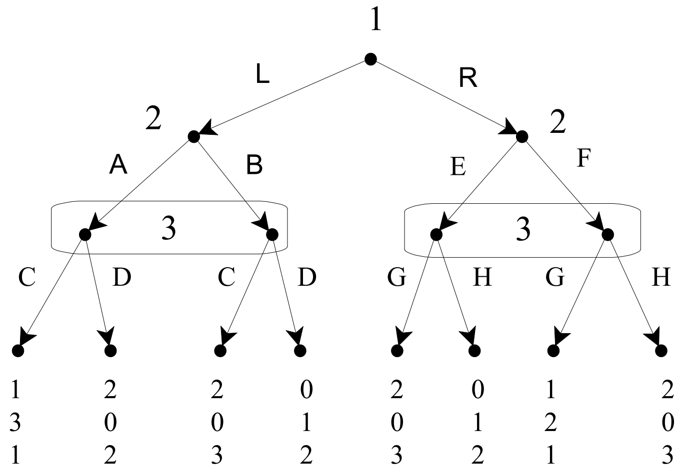
1.13, 0.75, 2



1, 0.67, 2.33







$$\text{SPE: } \begin{pmatrix} L & R \\ 1 & 0 \end{pmatrix}, \underbrace{\left(\begin{pmatrix} A & B \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} E & F \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} \right)}_{\text{behavioral strategy for 2}} , \left(\begin{pmatrix} C & D \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}, \begin{pmatrix} G & H \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \right)$$