5. Let $L = \{w \in \{0,1\}^* \; : \; w$ has an odd no. of 1's $\}$, and let A be the DFA with tabular representation:

A	0	1
$\rightarrow p$	p	q
$\star\, q$	q	p

Prove that $L = L(A)$. *Hint:* Do the $L(A) \subseteq L$ part of the proof by induction on the the length of the string processed by A. You need a mutual induction with a claim for state p and a claim for state q.

Solution:

To prove $w \in L(A) \Rightarrow w \in L$ we use an induction on length of w. We claim that

(1) If $\hat{\delta}(p, w) = p$, then w has even number of 1's.

(2) If $\hat{\delta}(p, w) = q$, then w has odd number of 1's.

Basis: $|w| = 0$ then $w = \epsilon$, so obviously (1) is true. Also, it is clear that $\hat{\delta}(p, \epsilon) = p \neq q$, so (2) is vacuously true.

IH: (1) and (2) are true for any string w of length n.

IS: Now prove (1) and (2) for $w = xa$, where $|x| = n$ and $a \in \Sigma$.

(1): If $\hat{\delta}(p, xa) = p$ then $\hat{\delta}(p, x)$ is p and $a = 0$ or q and $a = 1$, as can be seen from the transition diagram of A.

If $\hat{\delta}(p, x) = p$ then by the IH (1), the string x has even number of 1's. Therefore $w = x0$ which has even number of 1's.

If $\hat{\delta}(p, x) = q$ then by the IH (2), the string x has odd number of 1's, so $w = x1$ has even number of 1's.

(2): If $\hat{\delta}(p, xa) = q$ then $\hat{\delta}(p, x)$ is p and $a = 1$ or q and $a = 0$, as can be seen from the transition diagram of A.

If $\hat{\delta}(p, x) = p$ then by the IH (1), the string x has even number of 1's. Therefore $w = x1$ which has odd number of 1's.

If $\hat{\delta}(p, x) = q$ then by the IH (2), the string x has odd number of 1's, so $w = x0$ has odd number of 1's. This completes induction.

For the other direction, we have:

$$\begin{aligned} w \notin L(A) &\Rightarrow \hat{\delta}(p, w) = p \\ &\Rightarrow w \text{ has even number of 1's} \\ &\Rightarrow w \notin L \end{aligned}$$

So it never ends in a final state with an even number of 1's. Therefore, this direction is also proved, that is $L \subseteq L(A)$.