Chaos Theory: Is the Ergodic Theorem valid in Reality?

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Abstract. In this paper, I carry out my scientific research project, which I started in 2016 with the aim of experimentally testing the validity of the ergodic theorem in physical reality and presenting my new results from 2019 and 2020. Due to the limited scope, I cannot mention all the results and limited myself to the most relevant.

1. Introduction. The most important scientific outcome of the last 500 years is that we do not live in an arbitrary universe: Nothing does just happen; everything obeys the laws of nature that govern our universe mercilessly and without exception. In this ordered and determined world there seems to be no place for the term "chaos". In physics, chaos has long been viewed as an accumulation of measurement errors rather than a real physical principle. However, since the research of *Edward Lorenz* in the 1960s, this idea has been refuted [1]. Chaotic systems are characterized by the fact that their behavior is sensitively dependent on the initial conditions; an extremely small change in the initial conditions leads to such a large deviation in finite time periods, that neither the initial state, nor the scale of the disturbance is practically possible to reconstruct [2].

1.a What is the Ergodic Theorem? There are special models for describing the behavior of chaotic systems, one of which is the so-called *Ergodic Theorem*, which was formulated in a slightly modified version by *Ludwig Boltzmann* in 1887. It says that thermodynamic systems usually behave chaotically from a molecular level, what means that *the trajectory of the system in phase space comes as close as desired to any energetically possible point* (see Fig.1) [3]. It also makes statements about the time after which a point is passed, but this aspect is irrelevant for my research and the following explanations.[1]

1.b My Research Issue. My research issue concerns the ergodic theorem and its

[1] The time span in which the trajectory is inside a phase space region is proportional to the volume of the region.

scope. I wondered about following question:

What is the validity range of the Ergodic Theorem and what are the consequences of a limited validity?

However, this does not mean to which systems the ergodic theorem is applicable and which exceptions exist (this is already quiet clear since the discovery of *Spontaneous Symmetry Breaking* by Y.Nambu, M.Kobayashi and T.Maskawa) [4-6]. Instead, I would like to find out to what extent the theorem formulated as a mathematical model can actually be applied to the underlying systems in physical reality and whether the factors not taken into account in the model impair its applicability.

2. My Experimental Setup.

In order to answer my research question, I had to identify the factors that could potentially limit the validity of the ergodic theorem and construct an experimental setup through which the single factors can be viewed in isolation (see Fig.2).

2.a Identification of relevant factors. Before I designed the experimental setup, I had started with theoretical considerations based on extensive literature work. I identified six potential influences that differentiate physical reality from the mathematical model (see Tab.1).

2.b Selection of the research object. To answer the research question, I made a distinction between factors that can be eliminated by modifying the experimental setup and factors that are a fundamental part of our physical reality. I decided to investigate friction as the most fundamental aspect and chose a system that enables friction to be quantified and varied. Finally, I chose a *chaos pendulum* as my research object, as it is one of the simplest chaotic systems and can be measured with little effort, but can also be modeled mathematically [7]. To build a chaos pendulum, only two pendulum rods have to be coupled together, so that the second pendulum behaves chaotically. However, this choice also has disadvantages: Although the centrifugal force turned out to be negligible at the frequencies examined, there is a problem with the representation. For the dimension D of the phase space of a system with n degrees of freedom the following applies [8]:

$$D = 2n$$

A chaos pendulum has two degrees of

freedom; hence its phase space is four-dimensional [9]. After all, it takes four values to describe the state of a chaos pendulum: the angle between the rods, the angle between the upper rod and the suspension, and one angular velocity for each angle. However, creating four-dimensional phase spaces is a significant mathematic challenge that I was unable to overcome.

2.c Modification of the structure. To solve this problem, I modified the chaos pendulum. I installed a stepper motor that drives the top pendulum at a constant speed. In this way, the feedback between the pendulums is suppressed, whereby the upper angle and the associated angular velocity become irrelevant. Therefore, the pendulum has only one degree of freedom and consequently a two-dimensional phase space anymore. However, it was questionable whether the modified pendulum is still able to produce chaotic behavior, so I checked this before starting any friction-specific measurements. I varied the excitation frequency and studied common literature on phase spaces [10]. Then I attached a colored point to the second pendulum and measured its position using a video camera and an evaluation program [11]. Using simple trigonometry, I was able to calculate the angle from the positions:

$$\tan(\alpha) = \frac{y}{x}.$$

I just had to subtract the previous angle from the current angle and then divide the result by the time step to get an angular velocity, which I then plotted against the angle and got a phase space [8] (see Fig.3). In the phase space, the so-called *Feigenbaum scenario* could be clearly observed, small disturbances create a second rotation period that shifts a little further with each rotation. After all, the periods influence each other, which starts the chaos [12]. I also found out that this only happens at relatively high frequencies, at low frequencies the trajectory approaches the typical path of a simple driven pendulum. This enabled me to confirm the suitability of my setup and also to determine the frequency range that generates chaotic behavior.

3. Measuring Results.

Finally, I started with an one-year series of measurements to find out what influence a change in friction has on chaotic behavior and, ultimately, the validity of the ergodic theorem. However, before I started experimental verification, I developed a method to vary and quantify the influence of friction.

3.a Variation of friction. Instead of using oils of different viscosity to change the friction, I opted for the simpler method of varying the air resistance by mounting faces of different sizes in the direction of movement of the pendulum [13]. However, it was not certain whether there was a proportionality between the size of the face and the resulting damping. Therefore, I used a mathematical-experimental method to define a universal friction factor. First, I deflected the pendulum and, depending on the area, received a more or less strongly damped sinusoidal oscillation. The damping factor can be derived from the steepness of the line that results when the respective amplitudes are connected [14]. All necessary is inserting the high amplitude at the beginning for A_0, the time for t and the lower amplitude after t for A_t in

$$A_t = A_0 \cdot e^{-k*t}.$$

Term rewriting was first used to divide by A_0, whereby I obtained

$$\frac{A_t}{A_0} = e^{-k \cdot t},$$

then I calculated the natural logarithm by what e was omitted and I got the damping factor[2]:

$$\ln\left(\frac{A_t}{A_0}\right) = -k \cdot t.$$

I repeated this procedure for faces of different sizes and applied the various damping factors to the surfaces. It was clear: friction is equivalent to damping, there is clearly a proportionality (see Tab.2 and Fig.4). I was now able to vary friction in a reproducible manner.

3.b Investigation of friction. I started with a very simple measurement. Without artificially increasing the friction, I took advantage of the friction between the pendulum rods and observed its effect over long periods of time, on the order of a few days (see Fig.5 and 6). From these diagrams it can be concluded that friction slows down the chaotic behavior by causing the transition to a periodic state. An artificial increase in friction speeds up this process, an increase in frequency slows it down. This result confirmed my suspicion.

[2] Since it is a damping with the unit 1/s, -k results as a function of t.

However, there was also a surprise, because with higher friction, sometimes there was suddenly no more chaos at all. Only when I increased the frequency further did a Feigenbaum scenario occur again and ultimately chaos. Apparently, friction affects chaos in two different ways:

i) It leads to a faster transition to a periodic state.

ii) It shifts the chaos entry frequency backwards.

This realization was most unexpected, but it got even stranger: Although the damping is completely proportional to the friction (see Tab.2 and Fig.4), the damping is in no way proportional to the chaos entry frequency (see Tab.3 and Fig.7). It turns out that the friction initially acts quite proportionally, but there is an area in which the friction has a very sensitive effect on the chaos entry frequency. Then chaos entry frequency remains on a plateau.

3.c Interpretation of the results. What does this mean for the validity of the Ergodic Theorem? Because friction slows down chaotic behavior, the Ergodic Theorem is usually not fulfilled, after all, the half-life of chaos is so limited that never every energetically possible point is passed. However, since the influence of friction does not grow forever, but at some point, reaches a maximum value, whereas the excitation frequency can be increased further, the Ergodic Theorem is at least approximately fulfilled at high frequencies. Certainly, there is no unrestricted validity, the Ergodic Theorem is a mathematical model with limited applicability in reality.

4. Consequences of the Results.

Now I turn to the second part of the question posed at the beginning. Which consequences have the present results?

4.a Stability of Limit Cycles. For this purpose, I introduce the notion of *limit cycle.* A limit cycle is an isolated periodic solution of a chaotic system [15], it is characterized by the fact that neighboring trajectories diverge or converge. A limit cycle can also be described as an attractor in phase space that does not pull a system toward a point energy valley, but rather forces it into a particular cycle that it always strives to break, even when work is expended to break it - the counterpart, so to speak, of chaotic behavior [2]. But if it depends on the distance of the starting point of the trajectory from the limit cycle whether the system diverges or converges, and the distance lies on both the x-axis and the y-axis and thus represents a volume of energy in phase space, the further behavior of the system would then be sensitively dependent on the initial energy. Thus, I have provided a new limit-cycle oriented approach to classical chaos theory. However, if the course of the system depends on the distance to the limit cycle, i. e. on the energy difference, then there must be an individual limit above which trajectories diverge. One could think of this as a "catchment" as known from other attractors [16], but this conclusion leads to a dead end in the case of semistable limit cycles. If one draws the catchment area as a sphere surrounding the limit cycle, then this sphere contains completely different energy values that converge against the same state. This is senseful for stable limit cycles and point attractors, but not in this case. In fact, the result is that the catchment area does not have the character of a sphere, but of a point or a surface. However, if there are points near the limit cycle where the cycle is broken and the system returns to a chaotic state, what does this say about limit cycles? If the ergodic theorem is almost completely satisfied at high frequencies (see 3.b), the trajectory will hit one of the unstable points at $t \to \infty$ and decay. If it runs exactly back into itself, it is generally unstable, because any perturbation would grow exponentially. [2]. Limit cycles would thus also be inevitably unstable in reality, which could have far-reaching consequences for numerous chaotic systems such as our solar system, climate or stock exchange.

4.b Falsification Reasons. Of course, scientific work also includes the critical questioning of own hypotheses and the questioning of alternative explanations. These also exist for the facts of the limit cycles. For example, it would be conceivable that no unstable points or regions exist in the phase space, but the selection between diverging and converging is subject to chance instead of the energy difference. Superficially, this would be hardly distinguishable from a sensitive dependence, but then a trajectory that converges once could also follow the limit cycle forever. This sounds arbitrary, but it is a serious possibility, which could be described mathematically. In this case the phase space would be underlaid by a fractal pattern of starting points which produce diverging trajectories and starting points which produce converging trajectories. The distance between two such possible points is theoretically infinitesimal in a self-similar fractal, so what happens is left to random [17]. Possibly even the phase space itself would be fractal, D would then not necessarily have to be a natural, but also a decimal number, which could be calculated as *similarity dimension*:

$$D = -\frac{\log N}{\log \varepsilon},$$

where N is the number of versions of the set itself, reduced by the factor ε, of which it consists [18]. This would have interesting consequences, e. g. it would allow to interpret indeterminacy effects, since angle and angular velocity cannot be completely mapped in a less than four-dimensional phase space. However, this is currently a hypothesis - just like the instability of limit cycles.

4.c Verification of Stability. Whether the selection of the diverging and converging trajectories is subject to the energy difference, which would be accompanied by a general instability of limit cycles, or to chance, which would suggest a fractal phase space, is very difficult to verify experimentally. I calculated that the always occurring disturbances in chaotic systems make the verification by means of my experimental setup practically impossible. A computer simulation, on the other hand, could achieve sufficient precision. I therefore started to design a scheme for a program based on the programming language C++ [19]. It works according to the following principle: For the variables location $(x;\ y)$ and velocity $(vx;\ vy)$ initial values are entered as input. From these and the underlying physical laws, a value for the resulting force F and acceleration a is obtained, which is then divided into the components of the velocity change in x- and y-direction $(\Delta vx; \Delta vy)$. These can then be used to calculate the new velocities $(vx_{n+1};\ vy_{n+1})$ by simple addition, which is used to determine the new positions $(x_{n+1};\ y_{n+1})$ at the end. These then serve as initial values for the next iteration. After each iteration in the time span Δt, which was also specified at the beginning, the positions should also be plotted, so that I can track the position of the point in real time. This is of elementary importance for the methodology of data collection (see Fig.8).

5. Creation of a Simulation. In simulating my problem, I proceeded in several steps.

5.a Two-Body-Problem. The scheme of my simulation (see Fig.8) can be used to model numerous physical systems. I was faced with the problem that I had to verify my simulation, but this is not possible with my simulated system, because I need the already verified simulation to be able to solve it at all. Therefore, I started to apply it to an already known problem, the orbit of the moon around earth. For this I need only a handful of functions, I calculated

$$vx_t = vx_0 - G \cdot m_{\oplus} \cdot \frac{x}{r^3 * \Delta t}.$$

On the same principle I calculated also vy_t, then x_t could be calculated by

$$x_t = x_0 + vx \cdot \Delta t$$

Analogously also y_t. The orbital radius r of the moon is valid according to the Pythagorean theorem

$$r = \sqrt{x^2 + y^2}.$$

The entered start values correspond to the position of the moon in its perigee.[3] An optimal value for Δt can be obtained by variation, I chose $\Delta t = 100\ s$. This resulted in Fig.9. I then applied the simulation to another two-body problem, the orbit of the Earth around the Sun. For this I had just to insert new values:

$$vx_t = vx_0 - G \cdot m_{\odot} \cdot \frac{x}{r^3 \Delta t},$$

analogously for vy_t. Then I converted parameters to AU[4] and Fig.10. resulted.

5.b Three-Body-Problem. Far more difficult is the application to a three-body problem, but here nonlinearity occurs, which my program needs to be applicable to my pendulum [20]. I simulated the movement of the Sun, Earth and Mars under mutual attraction. The masses and orbital radii of every other body must be taken into account when calculating vx_t and vy_t from the superposition of all forces [21], e. g. for $vx_{\oplus}$:

$$vx_t = vx - G \cdot m_{\odot} \cdot \frac{x}{r_{\odot}^3 \Delta t} - G \cdot m_{♂} \cdot \frac{x}{r_{♂}^3 \Delta t}$$

By factoring out, I obtain:

$$vx_t = vx - \left[G \frac{x}{\Delta t} \left(\frac{m_{♂}}{r_{♂}^3} + \frac{m_{\odot}}{r_{\odot}^3} \right) \right]$$

Fig.1 resulted which showed some nonlinearities for the first time. The

[3] The data was converted to meters by multiplying by 10^3, since G is written in meters.

[4] 1 AU (Astronomical Unit) = 149.597.870.700 m

same procedure is to be followed with the y-component[5]. Afterwards I put the origin on the position of the earth by means of coordinate transformation, whereby the *opposition loops* received, which the Mars relative to the earth carries out (see. Fig.12) [22].

5.c Circular Motion. A circular motion is simpler than an astronomical three-body problem, a "one-body problem" so to speak, but it is also much more similar to my chaos pendulum and therefore relevant. One difference to the previous simulations is that I now worked with vectors and matrices instead of positions and velocities. From the angle of rotation of the pendulum I generated a rotation matrix, which I multiplied with the location vector [23], whereby it changes and generates the new position:

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

I checked the result of my simulation by calculating the velocities from the position vectors and plotting their x- and y-components separately. Thereby I got the typical harmonic oscillations, for v_x a sine curve, for v_y a cosine curve [24]. Thus, I could verify the function of my program also here several times.

5.d Coupled Circular Motion. Subsequently, I simulated also the coupled second pendulum by setting the origin of the second pendulum to x_t and y_t of the first pendulum. This allowed me to generate coupled circular motions - but since there are no forces acting yet, the feedbacks and thus the chaos remain absent (see Fig.12, 13, 14) [25]. However, regularities could be established, the number of loops within the path of the first pendulum equals the quotient of the excitation frequencies of the two pendulums minus 1.

5.d Damped Coupled Circular Motion. Another important step to simulate my system was introduce friction to be able to determine the influence of friction on chaos later. For this, I had to insert a linear friction factor μ when calculating the angular velocity of the second pendulum[6]:

$$\omega_{2_t} = \omega_{2_0} - \mu \cdot \omega_{2_0}$$

I first began to investigate the influence of friction on force-less coupled circular

[5] In this case, r is the distance to earth, not to the origin. Its calculated with $|d_{\oplus} - d_{♂}|$, if d is the distance to the origin.

[6] The friction of the first pendulum is negligible because it is driven by a stepper motor.

motions. Thereby it could be observed that the friction slows down the movement in a similar way as the chaos, and the two pendulums behave increasingly like one at higher friction. The higher the friction is, the faster the coupled circular motions become a simple circle. I wondered whether the time until the single circular motion occurs is proportional to the friction and tried to answer this question with my simulation. I simulated trajectories with varied μ and noted after how many iterations a simple circular motion applied (see Fig.16, 17, 18). The result was a curve with two asymptotes approaching the axes: Thus, for a friction of 0, it would take an infinite number of iterations; for a friction of 1, it is a single pendulum (see Tab.4 and Fig. 19).

6. Conclusion. At this point, I highlight my research results and distinguish between proven and conjectural results. The following results can be verified:

i) At my pendulum, chaos is frequency-dependent. (see 2.c)

ii) Friction shifts the Chaos entry frequency backward. (see 3.b)

iii) Friction leads to a transition to periodicity and breaks ergodicity. (see 3.b)

iv) At high frequencies, friction loses its influence and allows partial ergodicity. (see 3.a)

The following, however, is still an open question for which there are at least two different possible explanations, which I am currently pursuing (see 4.b):

v) Limit cycles could be unstable or the phase space has a fractal dimension. (see 4.a to 5)

The research question posed at the beginning can be answered comprehensively with these results: The Ergodic Theorem is not valid at most excitation frequencies, only at a few, particularly high frequencies it approaches validity due to decreasing influence of friction. However, the current state of research also leaves much room for further research. Therefore, I will further develop my computer simulation to answer also the question of the consequences and to prove the instability of limit cycles or the broken dimension of phase space. And above all, I will not be deterred by this complex topic, but will continue to research and bring order into chaos.