FAF combat maths

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Abstract –

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1 Introduction

2 Theory

In the following some mathematical approaches to quantify combat strength will be discussed that enable the to predict the outcome of a battle by knowing the unit stats and numbers involved

2.1 Single unit combat score

For some first thoughts of combat strength the fight is modeled as two units that have the same range and beam like weapons. So there is no discrete damage, no bullet travel time and no dodging. So unit A dies at the moment the total damage dealt by unit B is the same as the hp h of unit A. Damage dmg dealt over time t can be calculated by:

$$dmg(t)_{unit} = d_{unit} \cdot t \tag{1}$$

d is the dps of the unit. We can calculate the time it takes to kill unit A by:

$$dmg(t_{death,A})_B \stackrel{!}{=} h_A \tag{2a}$$

$$t_{death,A} = \frac{h_A}{h_B} \tag{2b}$$

$$t_{death,B} = \frac{h_B}{d_A} \tag{2c}$$

So $t_{death,A}$ is the time of death of unit A. The same can be calculated the other way around and gives the time of death of unit B. So which ever unit has the greater time of death in the scenario wins the fight. The calculation:

$$\frac{t_{death,A}}{t_{death,B}} = \frac{h_A \cdot d_A}{h_B \cdot d_B} \tag{3}$$

results that unit A wins the fight if the quotient is bigger than one, otherwise it will lose. Therefore if the product of hp and DPS of a given unit is bigger than the one of the opponent it will win the fight. This way a combat score can be defined:

$$c_{unit} = d_{unit} \cdot h_{unit} \tag{4}$$

Which is a quantitative measure of combat strength

2.2 multi unit combat score

For multiple units facing each other beam weapons are assumed. The beam weapons ensure that there is no overkill and therefore no damage wasted. Also the units will start in an optimal position. So all units start firing at the same time. Also discretization of loosing fire capabilities due to single units of small groups dying are disregarded. So the loss of firepower is modeled in a continuous way. This implies also perfect focus fire. So all units are firing on the opposing same unit. This represents an approximation of large amount of units fighting on each side. First an army dps D is introduced that is proportional to the army hp H for our Armies A and B. Army A consists only units of type A and army B only consist of units of type B.

$$D_A(t) = \alpha H_A(t) \tag{5a}$$

$$D_B(t) = \beta H_B(t) \tag{5b}$$

The current $H_A(t)$ can be written as the original hp $H_A(0)$ minus the damage already dealt by the enemy:

$$H_A(t) = H_A(0) - \int_0^t D_B(t)dt \tag{6}$$

Same can be said for $H_B(t)$. The insertion of the definition of dps from (5a) and (5b) into the equation and the derivative of both equations by t results in a pair of coupled differential equation which are known as the Lanchester's laws:

$$\frac{dH_A(t)}{dt} = -\beta H_B(t) \quad (7a)$$

$$\frac{dH_B(t)}{dt} = -\alpha H_A(t) \quad (7b)$$

These Equations don't need to be solved. The multiplication of both equations with each other and the following integration of the equation results in the following expression:

$$\alpha(H_A(t))^2 - \beta(H_B(t))^2 = \Delta C \quad (8)$$

ΔC is a constant. This equation allows the calculation of the remaining hp of the winning army A $H_A(t \to \infty)$ under the condition that army B is annihilated. It is noteworthy, that the combat strength increases with the square of number of units. So if army A had double the amount of units it could kill army B four times in four fights. $\alpha(H_A(t))^2$ can be written as the product of hp $H_A(t)$ and dps $D_A(t)$ of the army:

$$\alpha(H_A(t))^2 = D_A(t) \cdot H_A(t) \quad (9)$$

This look very familiar to the original combat score for a single unit, despite introducing hp dependent dps. Also the hp and dps of the army can be derived from the unit stats with $N_A(t)$ being the number of units A at a given time:

$$D_A(t) \cdot H_A(t) = N_A(t) \cdot h_A \cdot N_A(t) \cdot d_A = \quad (10a)$$

$$N^2(t) \cdot c_A =: C_A(t) \quad (10b)$$

This is a combat score $C_A(t)$ for army A. An army with a higher initial score should defeat an army with a lower score. This can be tested in the game.

2.3 calculating efficiency of units

A relevant question for the game is which type of units are more more efficient. For instance: Will a Rhino get defeated by a group of Mantis of the same mass cost? Because of the non linear nature of the combat score and different mass cost for different units calculating combat score c_A per mass cost m_A for a given unit is not sufficient. First a norm for combat score c_A needs to be selected at which the units are compared at. For simplicity a value of 1 is chosen. Also the time is set to zero, because the time development is of no interest in this matter.

$$c_A \cdot N_A^2(0) \overset{!}{=} 1 \quad (11a)$$

$$N_A(0) = \frac{1}{\sqrt{c_A}} \quad (11b)$$

$N_A(0)$ multiplied by the mass cost of a single unit m_A gives us the mass cost of an army A at the combat strength of 1 returns the mass cost per combat score:

$$N_A(0) \cdot m_A = \frac{m_A}{\sqrt{c_A}} \quad (12a)$$

$$S_A := \frac{\sqrt{c_A}}{m_A} \quad (12b)$$

For better intuition we invert the fraction for the definition of the specific score S so a higher score equals a more efficient unit.

2.4 combat score for one unit against many

Up to this point only the scenario for large army facing each other is described. But the case of a single unit facing a large army can also be calculated. In this case the differential equations of lachnesters law 5 needs to be modified. Army A will consist of the single unit and army B of many units. The difference to the previous model is that in this case the single unit (army A) doesn't lose dps over time while army B does. This results in the following differential equations:

$$\frac{dH_A(t)}{dt} = -\beta H_B(t) \quad (13a)$$

$$\frac{dH_B(t)}{dt} = -\alpha H_A(0) \quad (13b)$$

so $\alpha H_A(t)$ becomes $\alpha H_A(0)$. This equation system can again be multiplied with each other which results in the following expression:

$$\alpha H_A(0) H_A(t) - \frac{\beta}{2} H_B^2(t) = \Delta C \quad (14a)$$

with the reintroduction of the single unit combat score similar to (10a):

$$N_A^2 \cdot c_A - \frac{N_B^2}{2} \cdot c_B = \Delta C \quad (14b)$$

N_A is 1 in this case. For the interpretation of this equation one needs to look at the scenario where army A and army B defeat annihilate each other and compare that to the equation (8) for large armies facing each other. In both cases this results in the constant being 0. The comparison of 14b with the equation for many units fighting each other reveals that having army A as a single units requires $\sqrt{2}$ times the units of army B to get a draw. So a single strong units get a bonus for fighting against many weak units because of the constant dps of the strong unit. If the numbers of both armies get increases, it is expected that the combat scores to converge to the description of the multi unit combat score.

2.5 One unit against few

For the case of for instance a single Rhino against a couple of Mantis the discretization of the firepower of the Mantis might not be negligible. So in this case it is necessary to model it as well, or at least approximate it. Doing this via the lanchesters laws would require the use of the delta distribution and the heavy side function in the differential equation. But the problem can be approached differently: A single unit A (mantis) always deals a certain amount of damage to unit B (rhino) up to the point it dies:

$$д_B = d_B t_B = d_B \frac{h_B}{d_A} \tag{15}$$

So the damage $\tilde{Д}_B$ army A (mantis) of number $N_B(0)$ deal to the Rhino up to the point of the first unit A (mantis) dying is proportional to $N_B(0)$:

$$\tilde{Д}_B(t_B) = N_B(0)\ д_B \tag{16}$$

So the amount of damage army A deals unit is it is annihilated can be calculated by summing up the damage done in the constant time frame of a death of a unit A (manits):

$$\tilde{Д}_B(t = N_0 t_B) = \sum_{k=0}^{N_B(0)} k \cdot д_B = \frac{д_B}{2}(N_B^2(0) + N_B(0)) \tag{17}$$

Using the time t until the Rhino kills the mantis army, approximated by $N_B(0) \cdot t_B$, the renaming health of the rhino can be calculated. This will be negative if the rhino loses the fight.

$$H_A(t = N_B(0)t_B) = H_A(0) - Д_B(t = \tilde{N}_B(0)t_B) \tag{18}$$

In the following it is assumed, that the rhino wins the fight or draws, so that the health of the rhino $H_A \geq 0$ The current problem with the model is that there is only a discrete time parameterization for $\tilde{Д}_B$ so the damage done at any point in time is unknown. This is equivalent to the question: Just how many mantis does it take to kill a Rhino? Or in another way; How many mantis are worth a Rhino? The answer to these questions will give a combat score that takes discretization of units into account.

For $H_A(t = N_B(0)t_B) = 0$ there would most likely not be a solution $N_B(0) \in \mathbb{N}_0$. The fractional part $\{(N_B(0)\}$ of the real solution could be interpreted as a an additional Unit A (mantis) with lower hp that is added to army A. But create a small error for the damage it would make until it dies. But that damage can be calculated. Under the assumption that the lower hp unit A (Mantis) gets focused first the damage that the army of Mantis can deal in that time $Д'_B$ can be calculated by:

$$Д'_B = \{N_B(0)\}\lceil N_B(0)\rceil\ д_B \tag{19}$$

$\{\}$ is the frac function and $\lceil\rceil$ is the ceiling function. Adding that damage to the one that will be dealt by the rest of army A (mantis) (17) results in:

$$Д_B(N_B(0)) = \underbrace{\sum_{k=0}^{\lfloor N_B(0)\rfloor} k \cdot д_B}_{=\tilde{Д}_B = \tilde{H}_A} + \underbrace{\{N_B(0)\}\lceil N_B(0)\rceil\ д_B}_{=Д'_B = H'_A} \tag{20}$$

For the case that army A (mantis) and Army B (single Rhino) annihilate each other $Д_B(N_B(0)) = H_A$ must be set. this equation needs to be resolved to $N_B(0)$ in order to get the amount of Mantis required. The left part of the sum returns $\lfloor N_B(0)\rfloor$: Herleitung aus formel 17

$$\tilde{H}_A = \sum_{k=0}^{\lfloor N_B(0)\rfloor} k \cdot д_B = \lfloor N_B(0)\rfloor(\lfloor N_B(0)\rfloor + 1)\frac{д_B}{2} \tag{21a}$$

$$\lfloor N_B(0)\rfloor = \frac{-1 + - \sqrt{1 + 8\frac{\tilde{H}_A}{д_B}}}{2} \tag{21b}$$

With equation (15) and $h_A d_A = c_A$

$$\lfloor N_B(0)\rfloor = \lfloor \frac{-1 + \sqrt{1 + 8\frac{c_A}{c_B}}}{2} \rfloor \tag{21c}$$

The right part of the sum can be resolved by $\{N_B(0)\}$:

$$H'_A = \{N_B(0)\}\lceil N_B(0)\rceil\ д_B \tag{22a}$$

$$\{N_B(0)\} = \frac{H'_A}{\lceil N_B(0)\rceil\ д_B} \tag{22b}$$

By using $H'_A = H_A - \tilde{H}_A$ and (17) we get:

$$\{N_B(0)\} = \frac{2H_A - (\lfloor N_B(0)\rfloor^2 + \lfloor N_B(0)\rfloor)\ д_B}{2\ д_B(\lfloor N_B(0)\rfloor + 1)} \tag{22c}$$

$$= \frac{H_A}{д_B \cdot \lfloor N_B(0)\rfloor + 1} + \frac{1}{2}(\lfloor N_B(0)\rfloor + 1) \tag{22d}$$

And finally:

$$N_B(0) = \lfloor N_B(0)\rfloor + \{N_B(0)\} \tag{23}$$

This way the number of unit A (mantis) and how much hp the additional one must have in order to kill single unit B (rhino) can be calculated.

The above can also be achieved by modifying the Lanchesters Laws. For this we will look a the 13 again:

$$\frac{dH_A(t)}{dt} = -\beta H_B(t) \tag{24a}$$

$$\frac{dH_B(t)}{dt} = -\alpha H_A(0) \tag{24b}$$

Here we can see that we get an error at 24a. This happens because the dps d_B Army A is taking is not dependent on the hp of Army B but on the number of units from army B still alive. So the dps Army A is taking rather is a step function over time than a linear one. If we look at the Integral of that step function we get a function that looks looks plotted like a polygonal chain that follows more or less a parabola. This function is the damage $Д_B$ done over time t by army B. So the solution to the differential equation is not a smooth function. But we can write it as a combination of a series and Integral.

$$Д_B(t) = \int_0^t D_B(t)dt = \int_0^t \beta H_B dt \tag{25a}$$

For a whole We want to look at it in time steps where a unit of army B dies t_B:

$$t = kt_B + \zeta \quad k \in \mathbb{N}_0,\ 0 \leq \frac{\zeta}{t_B} < 1 \tag{25b}$$

So the total damage $Д_B$ can be written as a series if we just want to look at times when a unit of Army B just died. If we want to have a result for a continuous time it is also required to add a last continuous damage integral to the series over the time span of ζ:

$$Д_B = d_B t_B \sum_{i=N_B(0)-(k-1)}^{N_B(0)} (i) + \int_{kt_B}^{t_B+\zeta} D_B dt \tag{25c}$$

$$Д_B = d_B[(k-1)N_B(0)t_B + (N_B(0)-k)\zeta] \tag{25d}$$

This last equation can be rewritten with the use of the floor operator $\lfloor x \rfloor$. t can be expressed as:

$$t = \underbrace{\left\lfloor \frac{t}{t_B} \right\rfloor}_{=k} t_B \underbrace{- \left\lfloor \frac{t}{t_B} \right\rfloor t_B + t}_{=\zeta} \tag{26}$$

This using this we get an equation that is more useful for direct calculation of values:

$$Д\ (t) = d_B \left(\left\lfloor \frac{t}{t_B} \right\rfloor - 1 \right) N_B(0) t_B$$
$$+ d_B \left(N_B(0) - \left\lfloor \frac{t}{t_B} \right\rfloor \right) \left(t - \left\lfloor \frac{t}{t_B} \right\rfloor t_B \right)$$

To model this we will write 24a in a different way:

$$h_A \cdot \frac{dN_A(t)}{dt} = -\beta h_B \cdot f(N_B(t))dt \tag{27}$$

in the previous model $f(N_B(t))$ would be $f(N) = N$. But we can also use a function that rounds the input to the next higher integer or a smooth approximation of that rounding function. The multiplication of both differential equations still works. Let F be the stammfunction of f.

$$c_A N_A(0) N_A(t) - \frac{c_B}{2}(F(N_B(t)) = c_A N_A^2(0) - \frac{c_B}{2} F(N_B(0)) \tag{28}$$

3 Testing the model

The calculation of the combat score above requires a lot of simplifications from the game. Likely the most significant are the lack of discretization of shots and unit hp. Also range differences are not taken into account. Therefore the score for units that rely on range advantage like Mongoose, Hoplite. They are expected to have a significant lower significant score S than close combat units like pillars and rhinos. Also missing of shots is not taken into account. This mostly affects artillery and missile launchers. So the model is expected to only apply for the main combat units of each faction.

参考文献