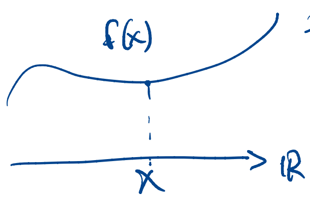
What is a manifold?

Well, in calc 1 & 2 we take $f: \mathbb{R} \to \mathbb{R}$ and say

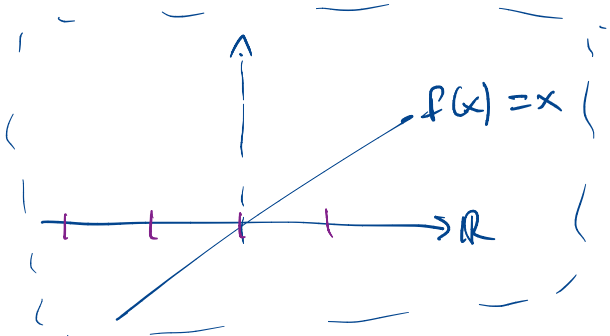


$f(x)$ is continuous

Then in Multivariable calc we consider $f(x,y)$ and we can vary x, y. And we are studying differentiation & integration of $f(x,y)$ in $\mathbb{R}^2$

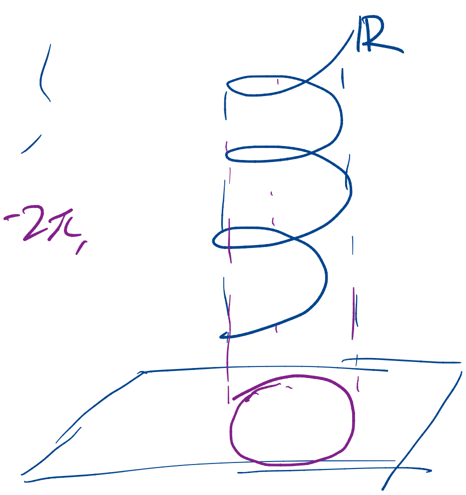
BUT, can we change the domain? And if so, how?

Well consider $f(x) = x$ as follows



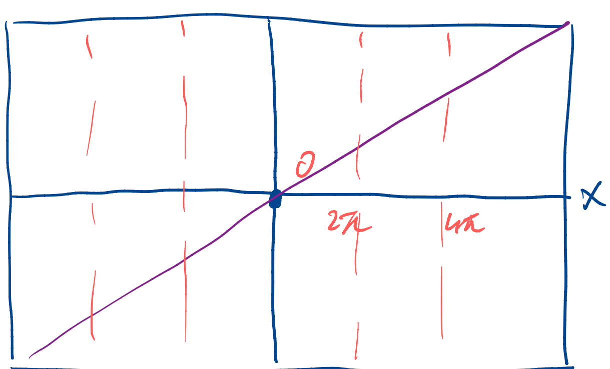
- Consider points -4π, -2π, 0, 2π, etc.

- Now imagine taking $\mathbb{R}$ & turning our number line into a spiral

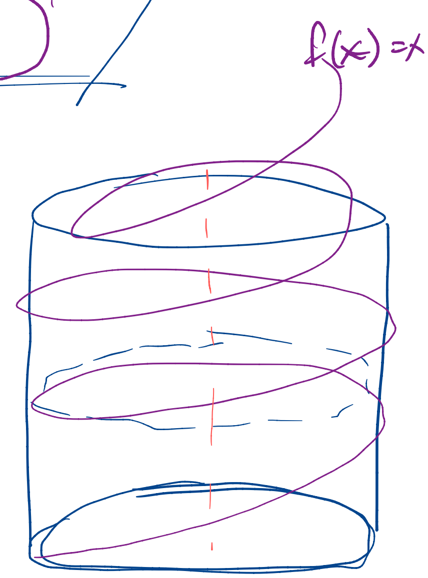


- we could map our spiral onto a plane.

- Imagine rolling up our plane to get the spiral. etc

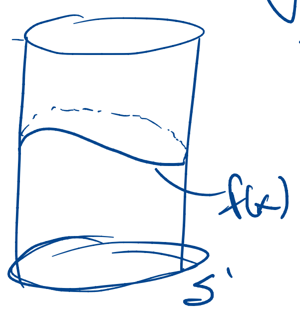


$\Longrightarrow$



But, there is one slight problem; After "wrapping" our sheet up, we see each x actually maps to more than one f(x). This violates our functions for functions in single & multivariable calculus.

Thus, we narrow our focus to segments of the $\mathbb{R}$ line on the cylinder where we can then apply our tools from single & multi. var. Calc onto a function with a different domain.



Thus! We need the concept of manifolds to translate integration & differentiation from $\mathbb{R}^n$ to ? some new domain, say spiral, S^1 :).

A brief primer on proof writing

I won't go into this too much, but the essence of my notes are to show definitions, and use manipulations, deductions, and other mathematical techniques to show a result satisfies the definition of some theorem.

Then we conclude what we want to prove

Let's quickly go through some prereqs

Sequences
- monotonicity, strictly increasing / decreasing
- boundedness & convergence

↳ informally: A sequence converges to a lim a iff, going far enough along the sequence, we can make a_n as close as we like to a.

Question to think about: Is the statement true?

↳ Every sequence has a monotonic subsequence

Back to sequence convergence.

Def. a_n converges iff $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n > N$ $|a_n - a| < \varepsilon$

Usually when a professor presents the def. of convergence they follow it up with an example

Example Claim, $(3 - 4/n)$ converges to 3.

Proof Let $\varepsilon > 0$ be arbitrary.

rough work: we need to set an N s.t. all $n > N$, $|a_n - a| < \varepsilon$

plugging in $a_n = 3 - 4/n$, and $a = 3$ we get

$|(3 - 4/n) - 3| < \varepsilon$, $4/n < \varepsilon$

↳ set $N = \lceil 4/\varepsilon \rceil$

What happens when $n > 4/\varepsilon$? and how did we get it?

we know from our rough work $4/n < \varepsilon$, $\frac{4}{\varepsilon} < n$ since $n, \varepsilon > 0$

so set $N = \lceil 4/\varepsilon \rceil$, ensures $|a_n - a| < \varepsilon$ always for $n > N$

We see that for example

$5/\varepsilon > 4/\varepsilon$, thus $\frac{4}{5/\varepsilon} < \varepsilon$ since

$$\frac{4}{5/\varepsilon} = \frac{4\varepsilon}{5} < \varepsilon$$

so for any $n > 4/\varepsilon$ our inequality $|a_n - a| < \varepsilon$ holds

Not every proof rough work will be this easy. Some times we have to use clever mathematical techniques

Continuation of Analysis Preliminaries

Sequences

Many fields rely on analysis of Cauchy sequences

A cauchy sequence satisfies def.

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } \forall n, m > N, \ |a_n - a| < \varepsilon$$

Series

An example

Theorem If $\sum a_n$ converges then $(a_n) \to 0$

Let's define a <u>diagram</u>.

A diagram consists of first

1. a set of $X = \{X_1, \dots X_n\}$
   where X_i is a set

2. a set $\{ X_i \xleftarrow{f^{\alpha}_{ij}} X_j \}_{\alpha \in A_{ij}}$

We could draw these diagrams as

$$\begin{tikzcd}
& X_1 \arrow[dl, shift left] \arrow[dr] & & X_5 \\
X_2 \arrow[ur, shift left] & & X_3 \arrow[dl] & X_6 \\
& X_4 \arrow[ul] & &
\end{tikzcd}$$

as we can see we have X_1 through $X_6 \in X_n$.

Define a <u>path</u> in a diagram (X, T) which consists of subset $P \subseteq T$, $P = \{f_1, \dots f_n\}$ such that the source of f_j is the target of f_{j-1} for all $j \in \{2, \dots m\}$

so an example is X_1 to X_3 & X_3 to X_4. since the target of f_1 is the source of f_2, both X_3.

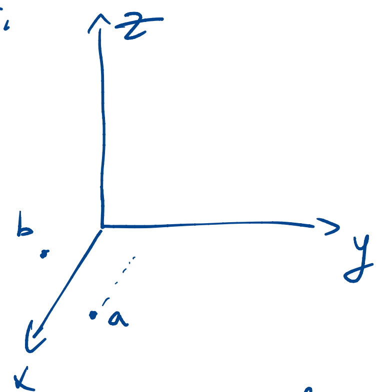
A diagram commutes whenever for any two paths $P = \{f_1, \dots, f_m\}$ & $Q = \{g_1, \dots g_k\}$ with source of f_1 is source of g_1 & target of g_k is f_m

so $X_1 \to X_3 \to X_4$ and $X_1 \to X_2 \to X_4$.

then $f_m \circ \dots \circ f_1 = g_k \circ \dots \circ g_1$

3. Distance example in $\mathbb{R}^3$, $d = \sqrt{x^2 + y^2 + z^2}$, distance between points

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

5. 

consider $(3, 1, 1)$ pt. a, facing the yz-plane two steps fwd, 2 steps left, ending at pt. b.

7. midpoint of $(-1, 3, 9)$ $(5, 6, -3)$

add coordinates divided by 2.

$$\left(\frac{5 + (-1)}{2}, \frac{6+3}{2}, \frac{-3+9}{2}\right) \Rightarrow \left(2, \tfrac{9}{2}, 3\right)$$

Displacement Vectors

Dot product & cross product examples

The lines $(2t+1)\vec{i} + (1-3t)\vec{j} + (2+2t)\vec{k}$ and
$(2t-3)\vec{i} + (7-4t)\vec{j} + (-2+at)\vec{k}$ are perpendicular

Find $\vec{a}$. Answer

Recall dot product $= 0$ if the lines are orthogonal.

for each line, recall we are only concerned with the direction vectors

$l_1: (2, -3, 2)$ and $l_2: (2, -4, a)$

Remember, evaluating the equations for some t would give us a point on the line, and taking the dot product of two points is meaningless. Okay, so

$$(2, -3, 2) \cdot (2, -4, a) =$$

$$4 + 12 + 2a = 0, \quad \text{thus } \boxed{a = -8}$$

Cross product

If $\vec{w} = 3\vec{i} - \vec{j} + 2\vec{k}$, $\vec{v} = \vec{i} + 4\vec{j}$, find a unit vector perpendicular to both $\vec{w}$ & $\vec{v}$

recall the cross product of $\vec{w}$ & $\vec{v}$ will give us a third vector perpendicular to both $\vec{w}$ & $\vec{v}$.

$$\begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 1 & 4 & 0 \end{vmatrix} = i(0-8) - j(0-2) + k(12-(-1))$$

$$a = -8 \quad b = 2 \quad c = 13$$

then divide by length, so $\sqrt{a^2 + b^2 + c^2} = \sqrt{237}$

Easy

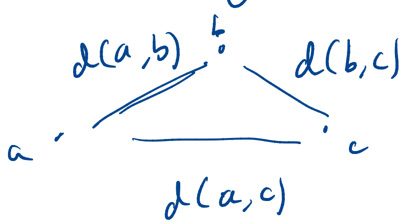
Vector Algebra Essentials

Magnitude of vector in $\mathbb{R}^3$, notation, $|av| = |a||v| = a\sqrt{\alpha^2 + \beta^2 + \gamma^2}$

unit vector $N = \frac{v}{|v|}$, and $N = av$ when $|v| < 1$, $a > 1$

- cross product
- Dot
- triangle inequality

The triangle inequality, $d(a,c) \leq d(a,b) + d(b,c)$, for points



triangle inequality is transparent with a diagram

Triangle Inequality for Vectors

$$||\vec{x} + \vec{y}|| \leq ||\vec{x}|| + ||\vec{y}||$$

triangle inequality with points & vectors holds

Dot product $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}| \cdot \cos\theta$, θ = angle between x & y.

remark $\vec{x} \cdot \vec{y} = 0$, means perpendicular

Some useful algebraic properties of dot product

1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
2. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$
3. $k(\vec{a} \cdot \vec{b}) = k(\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b})$

We can also find angles between two vectors with dot product equation yes $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}, \quad \theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}\right)$$

magnitude must be none zero AND

what if $\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|} > 1$, then $\cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}\right) \to$ DNE. ?

well, there is a theorem called the cauchy schwarz inequality which shows $\vec{a} \cdot \vec{b} \le \|\vec{a}\| \cdot \|\vec{b}\| \to$

Def Cross Product let $\vec{x}, \vec{y}$ be two vectors in $\mathbb{R}^3$

define cross product as.

$$\vec{x} \times \vec{y} = \det \begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = \begin{array}{ll} i\,(x_2 y_3 - x_3 y_2) & , x \text{ dim} \\ -j\,(x_1 y_3 - x_3 y_1) & , y \text{ dim} \\ +k\,(x_1 y_2 - x_2 y_1) & \end{array}$$

$$= [x_2 y_3 - x_3 y_2,\ -x_1 y_3 + x_3 y_1,\ x_1 y_2 - x_2 y_1] \quad z \text{ dim}$$

so the cross product produces another vector. perpendicular to both $\vec{x}$, & $\vec{y}$.

where $i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$

where i, j, k serves as a basis in $\mathbb{R}^3$

Let's talk some examples

so recall def of dot product $\vec{a} \cdot \vec{b}$ as $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$= a_1 b_1 + a_2 b_2 + a_3 b_3$ is a scalar. Just looking at this it may be hard to see the significance of the formula

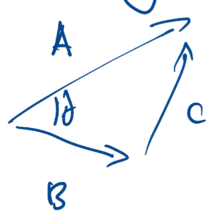
Well, geometrically the dot product states $\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$

1) what does this mean? well $\vec{A}\cdot\vec{A} = |A|^2\cos\theta = |A|^2$

dot product of a vector with itself is the magnitude squared

This is true by def. of dot product as $\vec{A}\cdot\vec{A} = a_1^2 + a_2^2 + a_3^3 = |A|^2$

2) we may also use the law of cosines to understand vectors



$\vec{C} = \vec{A} - \vec{B}$

then ② $|\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta$

great, then from above we know $\vec{C}\cdot\vec{C} = |\vec{C}|^2 = (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})$

① $= \vec{A}\cdot\vec{A} - 2\vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{B} = |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A}\cdot\vec{B}$

comparing ① and ② we get

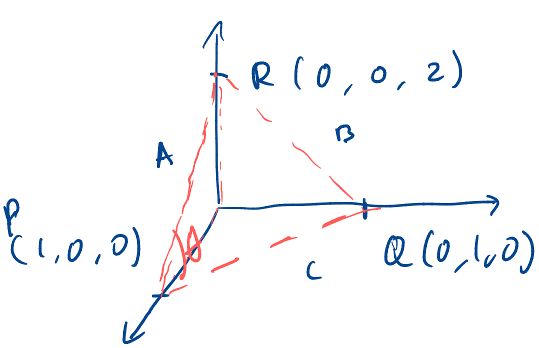
$\cancel{|\vec{A}|^2} + \cancel{|\vec{B}|^2} - 2|\vec{A}||\vec{B}|\cos\theta = \cancel{|\vec{A}|^2} + \cancel{|\vec{B}|^2} - 2\vec{A}\cdot\vec{B}$

thus if we believe in law of cosines

$2\vec{A}\cdot\vec{B} = 2|\vec{A}||\vec{B}|\cos\theta$

$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$ which is our geometric def. of dot

Application: computing lengths and angles



$\vec{A} = \begin{bmatrix}1\\0\\2\end{bmatrix}, \quad \vec{B} = \begin{bmatrix}0\\1\\2\end{bmatrix}, \quad \vec{C} = \begin{bmatrix}1\\1\\0\end{bmatrix}$

$\vec{A}\cdot\vec{C} = \begin{bmatrix}1\\0\\2\end{bmatrix}\cdot\begin{bmatrix}1\\1\\0\end{bmatrix} = 1$

$\cos\theta = \frac{1}{|\vec{A}||\vec{C}|}$ $\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}\sqrt{2}}\right)$, $\theta = 71.5°$

regarding the sign of $\vec{A} \cdot \vec{B}$

> 0 if $\theta < 90°$,
$= 0$ if $\theta = 90°$
< 0 if $\theta > 90°$

Given the zero case, the dot product is also useful for detecting orthogonality.

We can represent a line l in $\mathbb{R}^3$.

$$d(t) = \left\{ x(t), y(t), z(t) \;\middle|\; \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = t \left[\vec{v} \right] + p \right\}$$

if we let $\vec{v} = [v_1, v_2, v_3]$ $\quad p = (a_1 \; a_2 \; a_3)$

$$l(t) = \left\{ x(t), y(t), z(t) \;\middle|\; \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = t \left[\vec{v} \right] + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 + tv_1 \\ a_2 + tv_2 \\ a_3 + v_3 \end{bmatrix} \right.$$

This is how we parameterize a line in $\mathbb{R}^3$

A plane is the simplest 2D surface. We can define a plane (in $\mathbb{R}^3$) with a normal vector, $\vec{n}$, and a point (on the plane).

Recall the equation of a plane $Ax + By + Cz = D$ and the normal vector is $\vec{n} = [A \; B \; C]$ $\vec{n}$ is orthogonal to the plane. We can define a plane as $x + y + z = 3$

$\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 3\} : P$

④ Parameterization of Ellipse $(x = a\cos(t),\ y = b\sin(t))$

Now let's do some examples

Ex what curve is represented by $x = \sin(2t)$, $y = \cos(2t)$ $t \in [0, 2\pi]$?, $\sin^2(t) + \cos^2(t) = 1$, $x^2 + y^2 = 1$

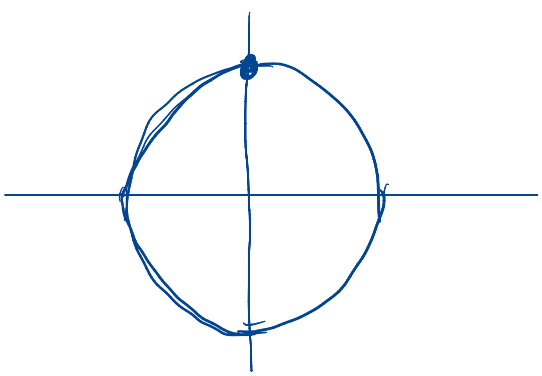
$\sin^2(2t) + \sin^2(2t) = 1$

normally which the parameterization of a circle, x corresponds to cosine & y corresponds to sin

What happens if we swap these points?

Ex: Let $r(t) = [f(t), g(t)]$ such that $f(t) = t^2 - 2t$, $g(t) = t + 1$

Sketch the curve based on above equations



recall:

$$\sqrt{(x(t) - 0)^2 + (y(t) - 0)^2}$$

$(d(p_1 = (x_1, y_1),\ p_2 = (x_2, y_2))$

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{1}$$

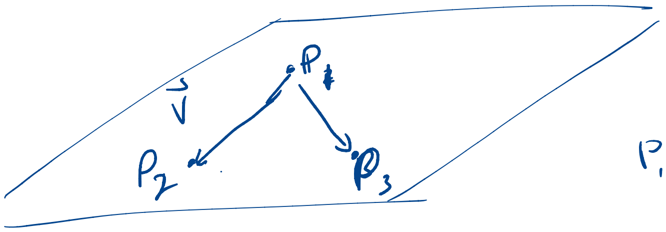
↳ the difference, is that the starting point for how this curve is "drawn" or "created", is at $t = 0$

$$\cancel{\sin^2(2(0))} + \cancel{\cos^2(2(0))} = 1$$

$0 \qquad 1$

thus the coordinates, $x = 0$, $y = 1$ form the starting point, & the circle is drawn clock wise as opposed to counter-clockwise

We can also define a plane using 3 non-collinear points.



find the plane through P_1, P_2, P_3, given $P_1(0,1,1)$, $P_2(1,0,1)$, $P_3(1,1,0)$.

we need two vectors, so lets connect $P_1 P_2$ and $P_2 P_3$ to get $\vec{v}$ & $\vec{w}$

$$\vec{n} = \begin{vmatrix} i & j & k \\ & \vec{v} & \\ & \vec{w} & \end{vmatrix} = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = i[-1 \times -1 - 0 \cdot 0] - j[1(-1) - 0(1)] + k[1(0) - (-1)]$$

$$= [1, 1, 1]$$

thus $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Example 2

find the equation for the plane P with normal vector parallel to $x + y - z = 5$, passing through point $(2, 4, 6)$

thus P has form $kx + ky - kz = D$, or

$x + y - z = D/k$, with solution $(2, 4, 6)$ so

$2 + 4 - 6 = 0$, thus ... $D = 0$?

$P: \boxed{x + y - z = 0}$

Now, this is not the only way to solve this question.

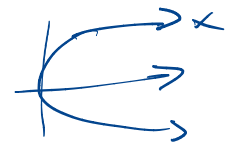
Okay, now we may formally define parameterized equations

Def The 1-D parameterization (parameterize a curve)

We have to know some types of well known curves.

① Parabola, defined by $y = ax^2$, or $y = ax^2 + bx + c$

or even $x = ay^2 + by + c$



② circle, $x^2 + y^2 = r^2$, or if offset from the centre. $(x-a)^2 + (y-b)^2 = r^2$

③ Ellipse: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = r$, or an offset from origin ellipse by $\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$

④ Hyperbola: centred at origin $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1$

then the hard version of $\left(\frac{x-h}{a}\right)^2 - \left(\frac{y-k}{b}\right)^2 = 1$

⑤ Straight line $y = mx + b$

Example graphing

What does $x^2 + 3y^2 + 2x - 12y + 10 = 0$ represent?

We need to use complete the square technique on both x^2 & y^2, so $x^2 + 2x + 1 + 3y^2 - 12y + 12 = 3$

$(x+1)^2 + 3(y-2)^2 = 3 \Rightarrow \boxed{\left(\frac{x+1}{\sqrt{3}}\right)^2 + \left(\frac{y-2}{1}\right)^2 = 1}$

Question: $\frac{(x-2)^2}{4} - \frac{(y+2)^2}{9} = 1$, which type of curve is this?

Parameterization

① General Case: For a curve in $\mathbb{R}^2$ given by a function. $y=f(x)$, $x\in(a,b)$. We have a natural parameterization.

$$\alpha(t)=(t,\ f(t)),\quad t\in(a,b)$$

by setting $y=f(x)$. $f(x=t)=f(t)$

and $a(t)$ gives you the parameterization of the curve defined by $y=f(x)$ from $x=a$ to $x=b$

② Line segment.

For a line from point a to point b, we utilize formula $\vec{v}=t\vec{u}+b$.

$$\vec{v}=b-a\begin{bmatrix}v_1\\v_2\end{bmatrix}\qquad b=\begin{bmatrix}B_1\\B_2\end{bmatrix}\Rightarrow a(t)=\begin{bmatrix}B_1+tv_1\\B_2+tv_2\end{bmatrix}$$

③ parametrize a circle

recall equation of a circle, centred at $(0,0)$.

$$x^2+y^2=R^2,\quad x=R\cos(t),\quad y=R\sin(t)$$

$$[R\cos(t)]^2+[R\sin(t)]^2=R^2$$

half circle parametrization, $[0,\pi]$, quarter circle $[0,\pi/2]$

full circle $[0,2\pi]$. think of the unit circle...

We advance t within our interval to get R & create our parameterized circle

ex $t=0$ $\quad \left(x=R\cos(t)\big|_{t=0}=R,\ y=R\sin(t)\big|_{t=0}=0\right)$

Another Problem : parameterization of a circle not centred at origin

Ex. this circle is centred at (h, k), then recall

$(x-h)^2 + (y-k)^2 = 1$

Step 1 change in variable

set $\alpha = x-h$, $\beta = y-k$, thus we form a new equation $\alpha^2 + \beta^2 = 1$, with a circle centred at $(0,0)$ relative to alpha & beta

We can parameterize $\alpha = x-h \Rightarrow x = \alpha + h \Rightarrow x(t) = \cos(t) + h$

$\beta = y-k \Rightarrow y = \beta + k \Rightarrow y(t) = \sin(t) + k$

Couple more examples

Ex. sketch the curve with parametric eq'n $x = \sin(t)$, $y = \sin^2(t)^2$.

Step 1 What kind of curve is this? Parabola. This comes from algebraic manipulation of the general case (change of variables).

sol'n $\sin(t) = \alpha \Rightarrow x \Rightarrow \alpha \Rightarrow y = x^2$

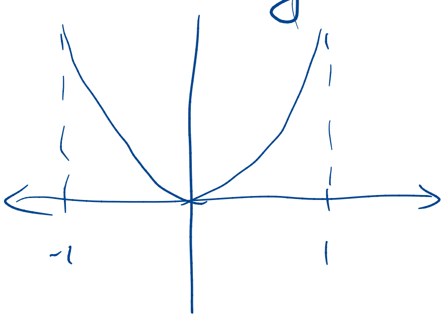
$y = \alpha^2$

parameterization comes from setting $x = \alpha$ and $y = f(x) = \alpha^2$.
this is the natural parameterization

Step 2 figure of the domain of the parabola, & range.

range of x, ... since $t \in \mathbb{R}$, then $\sin(t) \in [-1, 1]$

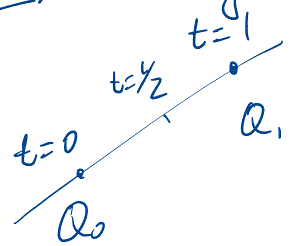
thus $0 \leq y = x^2 \leq 1$



thus as $t \to \infty$, our particle goes back & forth tracing the parabola.

More on Parametric equations of Lines & Curves

Ex Line through $Q_0(-1, 2, 2)$, $Q_1(1, 3, -1)$



$Q(t)$ = moving point @ constant speed.

The position at time t, given by $Q(t)$ is expressed by

$Q(t) = t\,Q_0Q_1 = t\,\langle 2, 1, -3\rangle$

We can break down $Q(t)$ into components $\langle x(t), y(t), z(t)\rangle$

$$\begin{cases} x(t) = -1 + 2t \\ y(t) = 2 + t \\ z(t) = 2 + (-3t) \end{cases}$$

, which are components of $Q(t) = t\overrightarrow{Q_0Q_1} + Q_0$

Q: Intersection with plane $x + 2y + 4z = 7$. where does Q_0, Q_1 intersect the plane? Does it even intersect?

Case 1 the points are in the plane?

$Q_0 : (-1, 2, 2)$ $\quad x + 2y + 4z = (-1) + 2\cdot 2 + 4\cdot 2 = 11 > 7$ Not in plane

$Q_1 : (1, 3, -1)$ $\quad 1 + 6 + (-4) = 3 < 7$, Q_1 not on plane

Well... given that Q_0 & Q_1 evaluate to 11 & 3, while the plane has value 7. To go from 3 to 11 we must cross 7 at some point

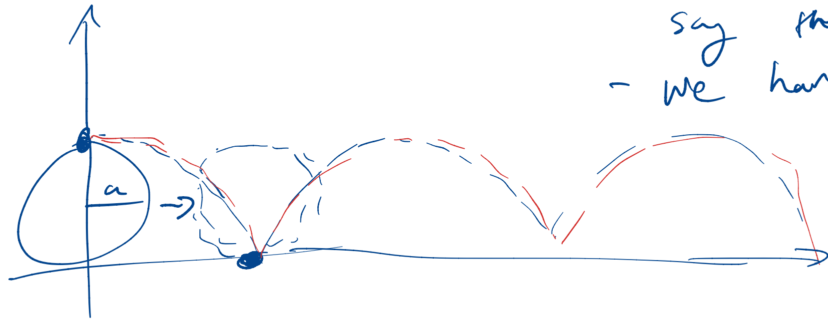
$\therefore$ they are on opposite sides of the plane

To solve intersection, let's plug in the parameterized eq'n of the line, $x(t) + 2y(t) + 4z(t) = (-1 + 2t) + 2(2 + t) + 4(2 - 3t) = 7$

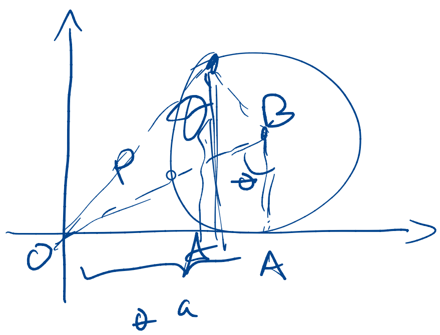
solving for t, we get $t = \frac{1}{2}$, makes sense since 7 is in between 3 & 11.

Cycloids

- Consider a wheel rolling on the ground say the x axis
- we have radius a,

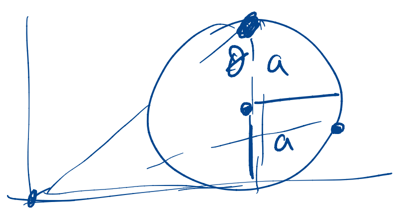


Question, what is the position $(x(\theta), y(\theta))$ where θ is the angle the wheel rotated.



- we want $\vec{OP}$, looks scary but we can break it down

$\vec{OP} = \vec{OA} + \vec{AB} + \vec{BP}$ individually they are easier to think about



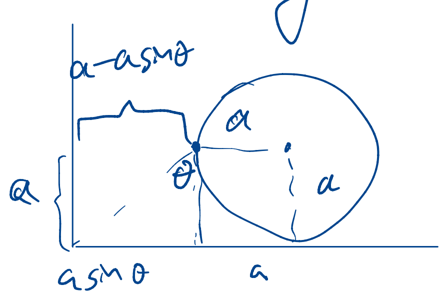
$$\vec{OP} = \langle \underbrace{a\theta - a\sin\theta}_{x(\theta)}, \underbrace{a - a\cos\theta}_{y(\theta)} \rangle$$

Approximating the behavior of the curve at the bottom requires taylor approximation

for t small $f(t) = f(0) + t f'(0) + \frac{t^2}{3} f''(0) + \frac{t^3}{6} f'''(0) \ldots$

$\sin(\theta) \approx \theta - \frac{\theta^3}{6}$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$

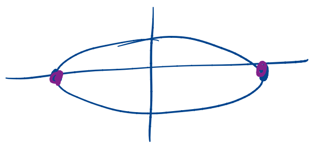
$x(\theta) \approx \frac{\theta^3}{6}$, $y(\theta) \approx \frac{\theta^2}{2}$, $|x| < |y|$ and the slope y/x tends to infinity near the origin



Calculus of Parameterized Curves , $\frac{dy}{dt}$, chain rule.

so $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, note this requires $\frac{dx}{dt} \neq 0$

otherwise we will have an undefined slope. (vertical tangent)

Ex.  purple points have ∞ slope

Example sketch graph using calculus.

Given $\alpha(t) = [x(t) = t^2,\ y(t) = t^3 - 3t]$

$x = t^2$, $t = \sqrt{x}$, $y = (\sqrt{x})^3 - 3\sqrt{x}$, $y = x^{3/2} - 3x^{1/2}$

Now using single-var calculus, first derivative & second der. tests, we can sketch using $y = x^{3/2} - 3x^{1/2}$

How can we use our parameterized method? The analytical way

step 1: tangents, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$

- note, case for vertical tangent at $t = 0$, denominator $= 0$
- next, horizontal tangent at $3t^2 - 3 = 0$, $t = 1, -1$

when $t = 1$ $x = 1$, $y = \pm 2$

so two horizontal tangents at $(1, 2)$ & $(1, -2)$

when $t = -1$ & 1 respectively

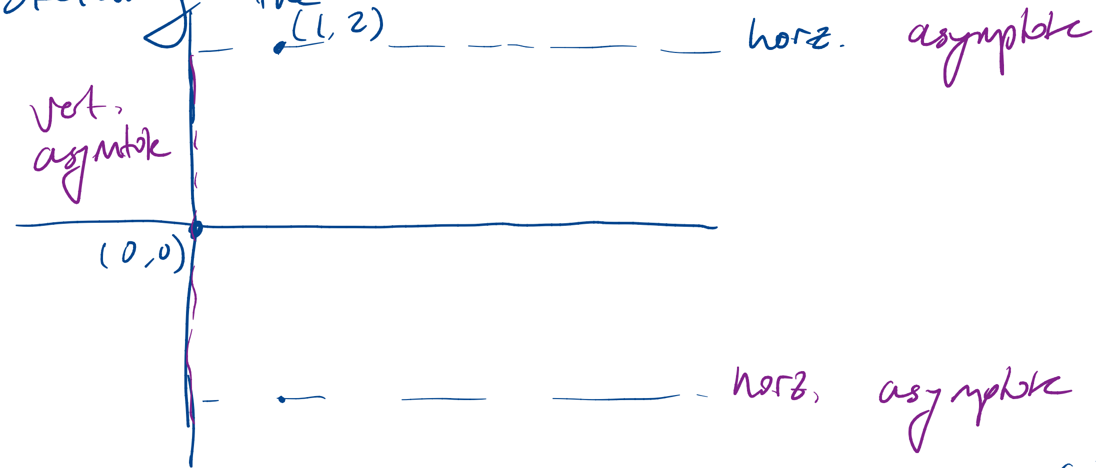
step 2: Increasing / decreasing , x, y intersection

$\frac{dy}{dx} = \frac{3}{2}\left(t - \frac{1}{t}\right) = 0$, $t - \frac{1}{t} = 0$, $t = 1$

$y = t^3 - 3t = 0$

$t = \pm\sqrt{3}$, at $t = \sqrt{3}$, $x = 3$, $(3, 0)$ x-int.

sketching the curve



slopes step 1 factor $y = t^3 - 3t = t(t^2 - 3) \to$ $t=0$, $t = \pm\sqrt{3}$

to give us the intervals for analysis

step 3 Concavity, second derivative test

Another simpler example

find the tangent for $\alpha(\theta) = [x(\theta), y(\theta)]$, $x(\theta) = r(\theta - \sin\theta)$, $y(\theta) = r(1 - \cos\theta)$

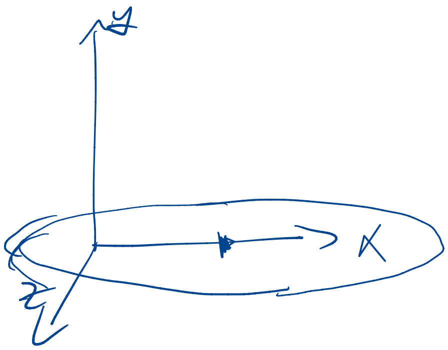
Q1: Find the tangent at $\alpha(\theta)$ where $\theta = \pi/3$

Q2: when is the tangent vertical?

More practice

given $\vec{w} = 2\vec{i} - 2\vec{j} + 3\vec{k}$, find a unit vector parallel to $\vec{w}$

Give Parameterizations for a circle of radius 5 in 3-space perpendicular to the y axis centred at (3, 0, 0)



would look something like... so since it is perpendicular to y axis, $y=0$

Answer: $x = 3 + 5\cos t,\ y = 0,\ z = 5\sin t,$
$0 \le t \le 2\pi$.

Note lines may not intersect in 3D space, ex

$l(t) = (c + 4t, 2 - t, 3 + t)$ & $m(t) = (4t, 1 - 8t, 4 + 4t)$

More Practice:

Find a unit vector $\vec{u}$ from $P = (6, 9)$ towards $Q = (30, 16)$

$30 - 6 = 24$ $\quad \frac{7}{24}$ $\quad d = \sqrt{7^2 + 24^2}$

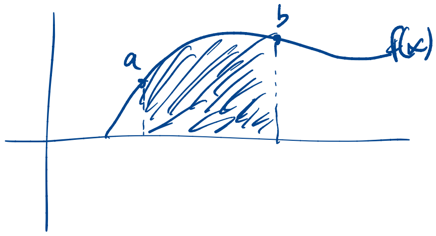
$16 - 9 = 7$

b) find vector of length 150 pointing in the same direction

c) perpendicular to

$\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Areas under parametric functions



(area)

if this curve $f(x)$ is bounded by parametric eq'n $\alpha(x)$, we can still find the area under $\alpha(\theta) = [x(\theta), y(\theta)]$

Suppose we represent $y=f(x)$ differently $\Rightarrow$ represent $y=f(x)$ as parametric equation ie, $\alpha(t) = [x(t), y(t)]$)

$\alpha(t)$, $t \in [\alpha, \beta]$, ie $t=\alpha \to \underline{[x(\alpha), y(\alpha)]}$

α

$t=\beta \to [x(\beta), y(\beta)]$

then find the area under $y=f(x)$, $x \in [x(\alpha), x(\beta)]$

our "normal" method of integration is $A = \int_a^b f(x)\,dx$

now we rewrite it as a parametric equation, representing x & y as fxns of t.

$$\int_a^b (y=y(t))\, d(x=x(t)), \quad \int_{t=\alpha}^{t=\beta} \quad \text{where } x(t)\Big|_{t=\beta}^{t=\alpha} = b$$

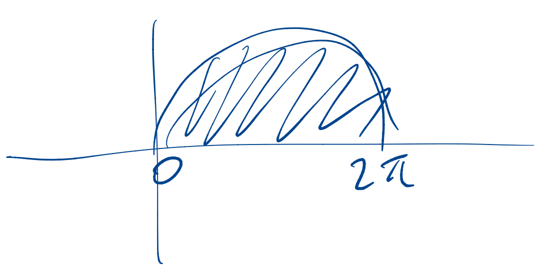
$$= \int_\alpha^\beta y(t)\, x'(t)\, dt$$

, using chain rule we can find the area under the curve represented by a parametric equation

still uses fundamental theorem of calculus & reimann sums

Ex find the area under the curve $\alpha(\theta) = \begin{bmatrix} x(\theta) = r(\theta - \sin\theta) \\ y(\theta) = r(1-\cos\theta) \end{bmatrix}$

$\theta \in [0, 2\pi]$



Solution

$$\int_0^{2\pi} r(1-\cos\theta)\, x'(\theta)\, dt \quad , \quad \theta = 2\pi \Rightarrow x(\theta)\Big|_{\theta 2\pi}$$

$$r(2\pi - \sin(2\pi)) = 2\pi r$$

$$x = 2\pi r$$

$$= \int_0^{2\pi} r(1-\cos\theta)\, r(1-\cos\theta)\, d\theta$$

$$= \boxed{3\pi r^2}$$

Example Arc Length: recall how to compute arc length

$y = f(x) \qquad a \le x \le b$

$x = a$ to $x = b$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Now use this concept for parametric curves.

Recall $\quad y = g(t) \quad \Rightarrow \quad$ End points

$x = f(t) \qquad a = x(t)\Big|_{t=\alpha}$

$b = x(t)\Big|_{t=\beta}$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \iff \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx(t)$$

recall $\quad \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$dx(t) = x'(t)\, dt$$

$$\hookrightarrow \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2} \cdot x'(t)\, dt$$

$$= \int_\alpha^\beta \sqrt{\left(1 + \left(\frac{dy}{dt}\right)^2 \Big/ \left(\frac{dx}{dt}\right)^2\right)\left(\frac{dx}{dt}\right)^2} \cdot dt$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex Compute the arc length of a circle

$x = \sin(t)$ $\quad t \in [0, 2\pi]$ $\quad \alpha(t)$ is a circle

$y = \cos(t)$

Solution using calculus to confirm $2\pi r = 2\pi(1) = 2\pi$

using $L = \int_{\alpha}^{\beta} \sqrt{f'(t)^2 + g'(t)^2}\, dt$

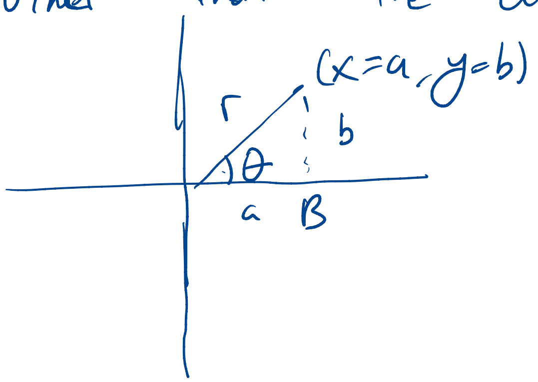
$$\int_0^{2\pi} \sqrt{(\cos(t))^2 + (-\sin(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{1}\, dt = 1(2\pi) \boxed{= 2\pi}$$

what if we change the range to $0, 4\pi$, then the formula simply becomes $\int_0^{4\pi} \sqrt{1}\, dt \boxed{= 4\pi}$

Polar Coordinates

allow us to represent coordinate points other than the classical way (x, y).



given θ & r I can uniquely represent a point in the $\mathbb{R}^2$ plane

$|AB| = b$, $|BO| = a$ $\Rightarrow r^2 = \sqrt{a^2 + b^2}$ $\quad (0,0)$

$\theta = \cos^{-1}\left(\frac{|BO|}{r}\right)$, θ = angle between r,

$\theta = \sin^{-1}\left(\frac{|AB|}{r}\right)$

$$\boxed{\begin{aligned} x &= r \cdot \cos\theta \\ y &= r \cdot \sin\theta \end{aligned}}$$

} essentially this is the conversion from (x, y) to polar coordinates

Let $p = (r = 2, \theta = \pi/3)$, convert p to cartesian coordinates

$x = 2 \cdot \cos(\pi/3)$, $y = 2 \cdot \sin(\pi/3)$, $(x, y) = (1, \sqrt{3})$

Now let's talk about the <u>curve in terms of polar form</u>

$(x-1)^2 + y^2 = 1$, $r = 2\cos\theta$ <u>blah</u>, how did we get $r = 2\cos\theta$?

<u>Answer</u>: recall polar: $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2 + y^2}$ or $r^2 = x^2 + y^2$

$(x-1)^2 + y^2 = \sin^2 + \cos^2$, $\cos\theta = \frac{x}{r}$ so

$2\cos\theta$ must be $\frac{2x}{r}$

→ if $r = 2\cos\theta$ then $2x = x^2 + y^2$, complete the square

$$x^2 - 2x + y^2 = 0$$

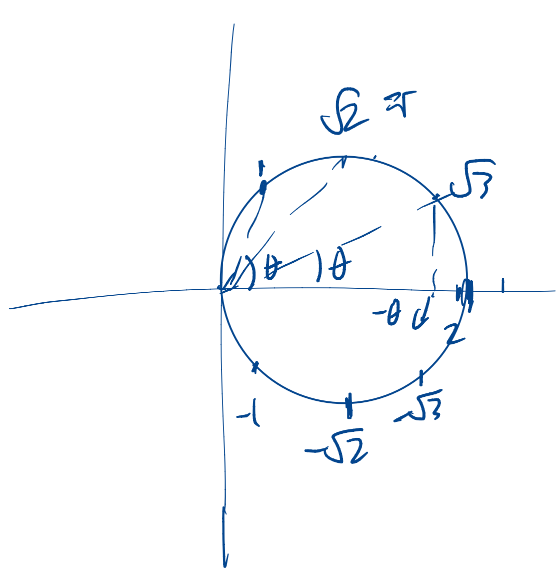
$$x^2 - 2x + 1 + y^2 = 1$$

$$\boxed{(x-1)^2 + y^2 = 1}$$

So what's going on here? Let's plot some points to get a better understanding

θ	$2\cos\theta$
0	1
$\pi/6$	$(\sqrt{3}/2) \cdot 2 = \sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
π	0
$+2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

r is the distance from origin to the point



Arc length tells us the distance travelled by a particle if we know its velocity (speed & direction).

$$\int_a^b \text{"speed"}\, dt = \int_a^b \|r'(t)\|\, dt = \text{arc length}$$

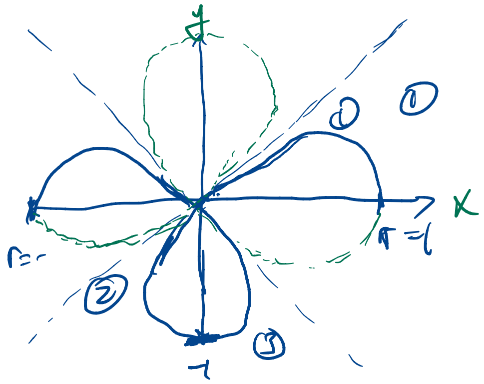
$$= \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_a^b \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2}\, dt$$

~~Example~~ Draw $r = \cos(2\theta)$, $0 \le \theta \le 2\theta \Rightarrow 0 \le \theta \le \pi$

The period of this function is π. $\cos(2\theta)$ draws the curve twice as fast as $\cos\theta$.

① first range to study $\theta \in [0, \pi/4] \Rightarrow r \downarrow$, $r \ge 0$

max at $\theta = 0$, min at $\theta = \pi/4$,



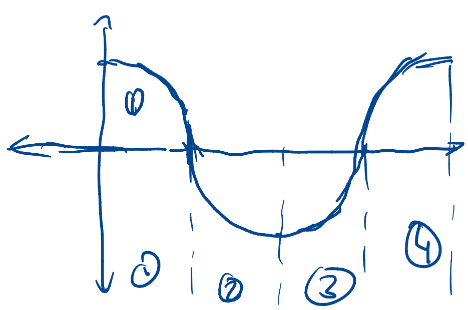
② second range $\theta \in [\pi/4, \frac{\pi}{2}]$

$|r| \uparrow$, $r \downarrow$, $r \le 0$

Starts at zero, ends ~~at~~ -1

③ $\theta \in [\frac{\pi}{2}, \frac{3\pi}{4}]$

④ $\theta \in [\frac{3\pi}{4}, \pi]$,



Final portion radius is non-negative

decreasing slope for

use $x = r\cos\theta$
$y = r\sin\theta$ to confirm your graph

Computing Tangents to Polar curves

$r = f(\theta) \to$ polar curve function

recall $\alpha(t)$ tangent $= \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, then for polar curves we have

$\alpha(\theta) = [x(\theta), y(\theta)]$, tangent $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$ by chain rule

$$= \frac{\frac{d(r \cdot \sin\theta)}{d\theta}}{\frac{d(r\cos\theta)}{d\theta}}$$

- also note here we have to use product rule since r is a function of θ thus the equation becomes ...

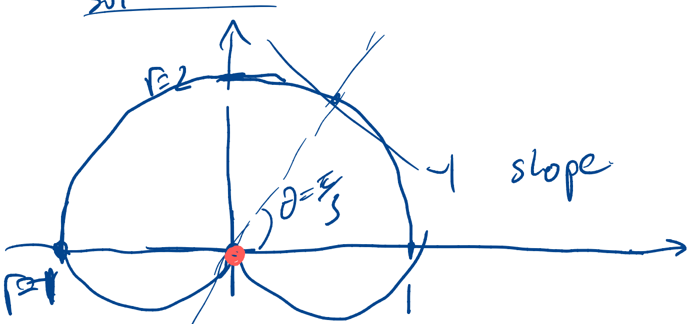
$$= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta + r(\sin\theta)}$$

- now we can check horizontal tangents by solving $r'\sin\theta + r\cos\theta$

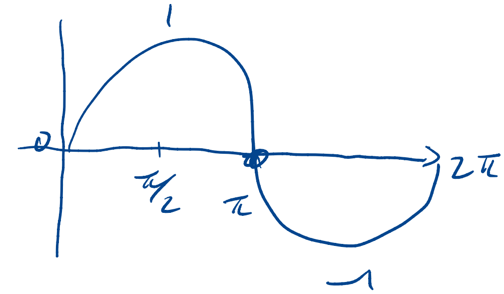
Example $r = 1 + \sin\theta$, $\theta \in [0, 2\pi]$, tangent at $\theta = \pi/3$?

remember, since trigonometric fns are periodic functions, we only need to understand its behaviour over a relevant interval.

Sol'n:



graph of sine fn



since $\sin\theta$ ranges from $[1, -1]$, we know $r = 1 + \sin\theta$ ranges from $[0, 2]$, so it starts at $r = 1$ to, then as $\pi/2$ it goes to $r = 2$, at π it goes back to 1

Drawing the graph gives us a visual representation of where the tangents might be, since from polar coordinates, $y = r\sin\theta$,

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}, \quad \frac{dr}{d\theta} = r' = \cos\theta$$

$$= \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos^2\theta - (1+\sin\theta)\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} \boxed{= -1}$$

horizontal tangent when horizontal tangent $= 0$, num $= 0$

$$\frac{dy}{d\theta} = \cos\theta\sin\theta + (1+\sin\theta)\cos\theta,$$

$$= \underbrace{\cos\theta}\underbrace{(1+2\sin\theta)} = 0$$

$$\theta = \pi/2, \ 3\pi/2, \ 7\pi/6, \ \frac{11\pi}{6}$$

or $\frac{dy}{d\theta}$ when denominator $= 0$

$$\underbrace{\cos^2\theta} - (1+\sin\theta)(\sin\theta)$$

$$= 1 - \sin^2\theta - (1+\sin\theta)(\sin\theta)$$

$$= (1+\sin\theta)(1-2\sin\theta) = 0$$

$$\theta = \frac{3\pi}{2}, \ \pi/6, \ \frac{5\pi}{6}$$, note that at $\frac{3\pi}{2}$ both the num. & denom. $= 0$, $\dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = 0/0$.

Further analysis is required through L'Hospital's rule

$$\lim_{\theta\to\frac{3\pi}{2}} \frac{(dy)'}{(dx)'} = \frac{1}{3}\lim_{\theta\to\frac{3\pi}{2}} \frac{(\cos\theta)'}{(1+\sin\theta)'} = \frac{1}{3}\lim_{\theta\to\frac{3\pi}{2}} \frac{-\sin\theta}{\cos\theta} = \infty,$$

vertical tangent

Left hand side limit $\neq$ right hand side limit

since $\cos\theta$ remains at 0 & $\sin\theta$ is non zero so the limit blows up to infinity

Another Example Computing arc length for a polar coordinate

Polar curve in terms of $xy \Rightarrow x(\theta) = f(\theta)\cos\theta$, $y(\theta) = f(\theta)\sin\theta$

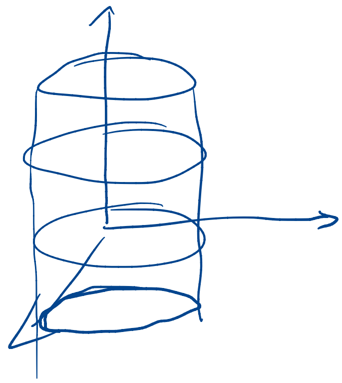
$[r = f(\theta)] \Longleftrightarrow x\theta = [x(\theta), y(\theta)]$

More on Arc Length & Arc length parameterization

Chapter

Describing $x^2+y^2=1$ in 3D? Notice the eq'n does not have a z component, thus the object does not depend on the value z.

Consider level sets $z=0, 1, z=2$ etc..... what is $x^2+y^2=1$?



$x^2+y^2=1$ defines an open end cylinder in $\mathbb{R}^3$

what about $y^2+z^2=1$ in 3D?

Thus the circle is on the y-z plane so the eqn $y^2+z^2=1$ defines an open ended cylinder along x

Ex sketch $x^2+y^2/9+\frac{z^2}{4}=1$

step 1 relate it to a 2D curve

step 2 draw the leveled curves

soln: the 2D curve, $x^2+\frac{y^2}{9}=1-\frac{z^2}{4}$, hm, notice this could be related to the ellipse, where the right hand side $1-\frac{z^2}{4}$ tells us the size of the ellipse

Analysis since all $x, y \in \mathbb{R}$ x^2+y^2 must be positive $1-\frac{z^2}{4}$ must also be >0, this imposes a restriction on the size of the ellipse, so $z^2 \leq 4$

thus $-2 \leq z \leq 2$ describes the range of z

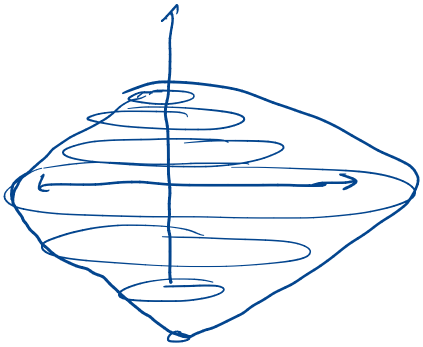
Now lets draw this curve for some level sets.

picking extreme points, -2 & $2 = z$

$z=-2$, $x^2, y^2=0$, $(x,y) \Rightarrow (0,0)$

then at $z=0$, $x^2+\frac{y^2}{9}=1$. this actually produces the biggest ellipse.

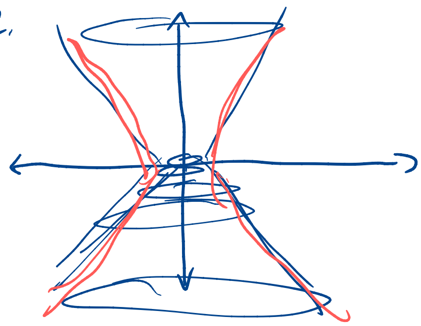
Ellipsoid ?



2. what about $\boxed{\frac{x^2}{4}+y^2=1+\frac{z^2}{4} \quad (x^2/4+y^2-z^2/4=1)}$

This is still an ellipse related 3D shape

2.



The collection of level sets "blows-up" from the origin outward.

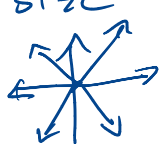
Openended cone shape

3. Example $x^2/4+y^2-z=0$

$\frac{x^2}{4}+y^2-\frac{z^2}{4}=0 \implies \frac{x^2}{4}+y^2=\frac{z^4}{4}$, still related to ellipse

$\frac{z^2}{4}\geq 0$ for all $z\in\mathbb{R} \implies$ there is no restriction for z.

what about $g(z)$ (radius) minimum value?, yes when $z=0$. so we have a min. size of the ellipse from the minimum size of the ellipse, or the single point in 3D. $(0,0,0)$.



Example

$4x^2-y^2+2z^2+4=0$, sketch

$-4x^2+y^2+2z^2=4$

$-4x^2+y^2=4+2z^2$, this is something we're familiar with.

Consider 2D $y^2-4x^2=g(z)$, $g(z)$ is a fn minimum at 4, to ∞.

This looks closer to the generic form of a hyperbola, $\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2=1$

where $g(z)$ tells us how far the curves of the parabola are we can solve for $\frac{x^2}{a}$ when $y=0$, say $g(z)=1$ then

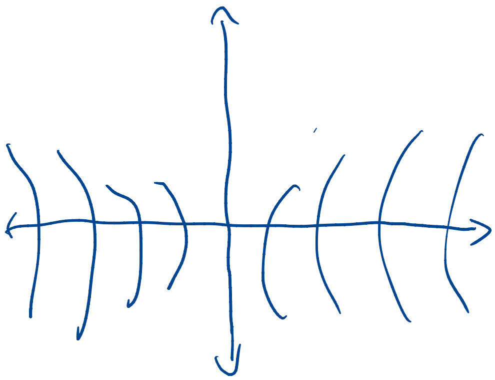
$\frac{x^2}{a^2}=1$, $x^2=a^2$, $x=\pm\sqrt{a}$ then $\left(\frac{x}{a}\right)^2=g(z)$, $\boxed{x^2=g(z)\cdot a^2}$

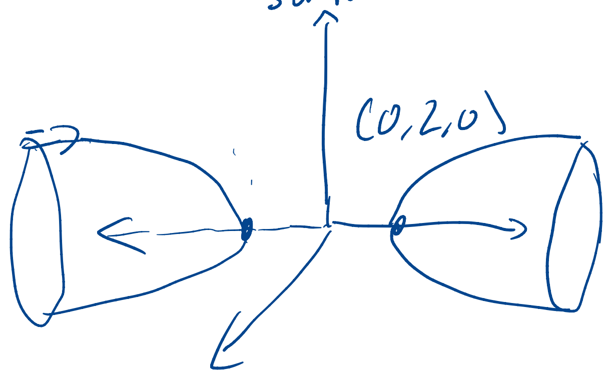
$x=\pm a\sqrt{g(z)}$

We solved for x^2 in standard form,

thus as x increases $g(z)$ increases

Now we draw the level set and surface

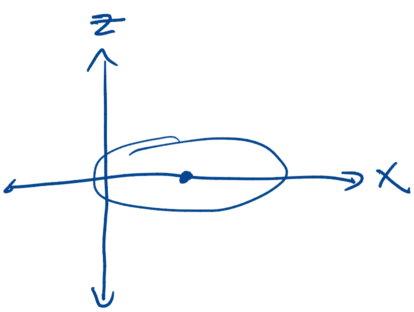




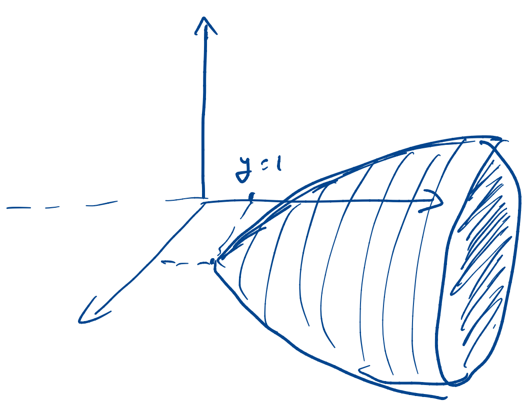
Example: sketch $x^2+2z^2-6x-y+10=0$, from algebraic perspective we should complete the square.

$x^2-6x+9-y+2z^2+1=0$

$\underbrace{(x-3)^2+2z^2}_{\text{ellipse}}=y-1$



so $(x-3)^2+2z^2$ produces an ellipse centred at $x=3$, $z=0$, with $y-1$ determining the size of the ellipse, and $y-1$ must always be ≥ 0. So the ellipse starts at $y=1$ & grows from there



Consider

1. $f(x,y) = z = \frac{1}{x^2+y^2}$, which points lie on the curve?

A $(1,0,1)$, B $(\sqrt{8}, 1, 3)$, C $(-3, 2, -3)$, D $(1, 1, \frac{1}{2})$

A, D lie on z since they are solutions to $f(x,y) = z$

2. Is function $f_1 = \frac{y}{3x^2+10}$ continuous at all points on the disk $x^2+y^2 \leq 9$?

$f_1(x,y)$ is continuous at all points where the denominator $\neq 0$

since $3x^2+10 \geq 10$, $f_1(x,y)$ is continuous at all points on the real plane.

Now consider $x^2+y^2 \leq 9$, which is a subset of the real plane

since $f_1(x,y)$ is continuous everywhere on the real plane, it is also continuous on the disk $x^2+y^2 \leq 9$

3. Is $f(x,y) = \begin{cases} x^2+y^2 & \text{when } x,y \neq 0 \\ 6 & \text{when } x,y = (0,0) \end{cases}$ continuous at $(0,0)$? NO.

4. Describe the shape / behaviour of each function?

i) $f(x,y,z) = \sin x$, y, z can vary. from $[-1, 1]$. $x \in \mathbb{R}$ $[-\infty, \infty]$

this function only depends on x, so its level surface are planes parallel to the y,z plane. The value of the function on each plane is constant & equal to the sine of the x-coor.

ii) $f(x,y,z) = y + \sin x$,

this fxn depends on x & y, so its level surfaces are not parallel to any of the coordinate planes

sinusodal curves in the xy plane expanded in the z direction.

iii) $f(x,y,z) = x^2+y^2+z^2-4$

- the fxn is a sphere centred at origin with radius 2

iv) $f(x,y,z) = x^2 + y^2 - z^2$

- Two cones with vertices at $(0,0,0)$ centred about the z axis

v) $f(x,y,z) = 5x^2 + 3y^2 - z$

- An elliptical paraboloid

Q5 Values of a, b that make $f(x,y)$ continuous

$$f(x,y) = \begin{cases} 2 + ax + by & y \geq 4 \\ 3 + 2x + 3y, & y < 4 \end{cases}$$

$2 + ax + by = 3 + 2x + 3y$, then plug in $y=4$

$2 + ax + 4b = 3 + 2x + 12$, $a = 3.25 - b$

$ax + 4b = 2x + 13$, $b = 3.25 - a$

Q6 Describe the surface described by $\sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2} = 5^2$

A sphere centred at $(1,2,3)$ with radius of 5?

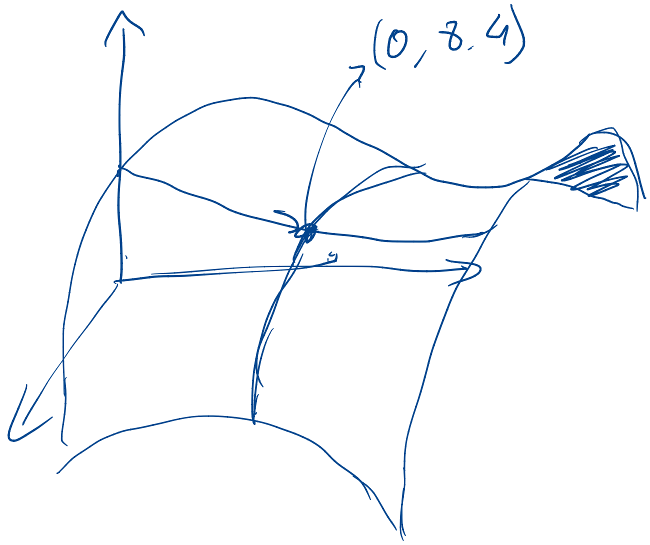
Q7 Let $f(x,y) = \begin{cases} \frac{15x}{x} y & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$, describe the continuity of the function.

The fxn is continuous everywhere

Q8 Applied Question! Let $P = f(A, r, N)$ describe mortgage payments on a house where A is the amt of \$ borrowed, r is interest rate, and N is # of yrs. Given $\left.\frac{\partial P}{\partial r}\right|_{(83{,}000,\ 14,\ 30)} = 65.70$

what is the significance of 65.70?

Q9 Given a saddle surface of $z = f(x,y)$



what is the sign of $f_x(0,8)$?

what is the sign of $f_y(0,8)$?

If the saddle point is...

Also learn how to read contour diagrams of 3 vars

Q1 Use difference quotients to estimate $f_x(6,1)$ & $f_y(6,1)$, $f(x,y) = \frac{x^2}{y+2}$

Q2: Find the PD z_x & z_y if $z = 5x^7 + 3^y + x^y$

Q1 parallel to plane $x+y-z=5$, through point $(2,4,6)$

The plane has equation in form $x+y-z=D$ with $(2,4,6)$ as a possible sol'n. Thus $2+4-6=D$ so $D=0$. Therefore, the eq'n of the plane is

$$\boxed{x+y-z=0}$$

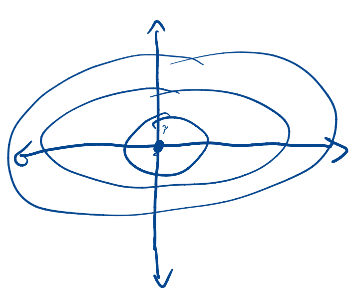
Q2 sketch the surface. Explain

$$\frac{-x^2}{4} - y^2 + z^2 = 1,$$

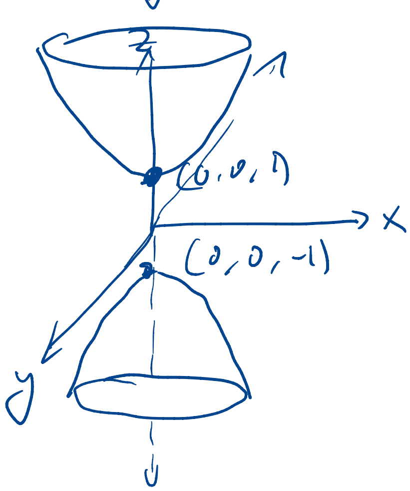
notice

$$-\frac{x^2}{4} - y^2 = 1 - z^2$$

$\frac{x^2}{4} + y^2 = z^2 - 1$, since $\frac{x^2}{4} + y^2$ is only ever positive $z^2 - 1 \geq 0$ so $z^2 \geq 1$. Also note the equation follows the form of a hyperboloid. At $(x,y) = (0,0)$, $z = \pm 1$ so the "starting" point for our sketch can be



given $\frac{x^2}{4} + y^2 = f(z)$, we can analyze the level sets as $f(z) = 0, 1, 2$ & so on. These would essentially be ellipses centred at the origin, growing in size as $f(z)$ increases.



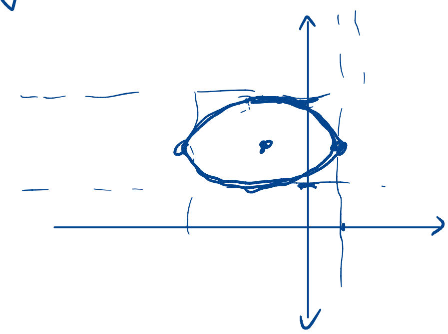
We can then apply this to 3D as a hyperboloid opening up & down on the z-axis

b) $x^2+3y^2+2x-12y+10=0$. First, let us complete the square

$$x^2+2x+1+3y^2-12y+9+3=3$$

$$=(x+1)^2+3(y-2)^2=3$$

Notice this follows the form of an ellipse in the xy plane centred at $(-1,2)$, y takes on min/max values when $(y-2)^2=1$, so at $y=1$, $y=3$



x takes on min/max values when $(x+1)^2=3$, so $x+1=\pm\sqrt{3}$

$x=\sqrt{3}-1$

$x=-1-\sqrt{3}$

c) A curve parameterized by $x(t)=t^2-3$, $y(t)=t^3-4t$ for $-2\le t\le 2$.

i) $$\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{3t^2-4}{2t}=\frac{dy}{dx}$$

horiz. tangent, $\frac{dy}{dx}=0$, so when $3t^2-4=0$, $t=\pm 2/\sqrt{3}$

thus $(x,y)=\left(4/3-3,\ \frac{8}{3\sqrt{3}}-\frac{4(2)}{\sqrt{3}}\right)=\left(-5/3,\ \frac{-16}{3\sqrt{3}}\right)$, $t=2/\sqrt{3}$

and $(x,y)=\left(-5/3,\ \frac{16}{3\sqrt{3}}\right)$ when $t=-2/\sqrt{3}$

when $t=0$ the denominator of $\frac{dy}{dx}\to 0$ so there is vertical tangent at $t=0$, $y=0$, $x=-3$, $(-3,0)$

d) Q: Let $\vec{r}(t)$ be a fxn representing the intersxn curve of the sphere $x^2 + y^2 + z^2 = 9$ and the plane $2x + y + 2z = 0$. Show $\vec{r}(t)$ is perpendicular to $\vec{r}'(t)$

Answer

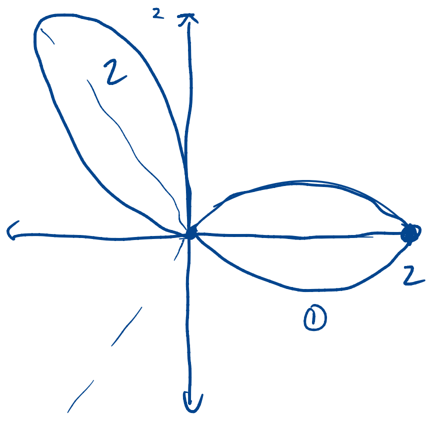
How show that the first derivative of the fxn representing the intersxn of sphere & plane is parallel to the original function?

Observations: $\vec{r}(t)$ represents a circle, which has general form $x^2 + y^2 = r^2$. For a level curve (such as the one describing the intersection of a sphere & plane) the gradient of the surface is always perpendicular to the curve.

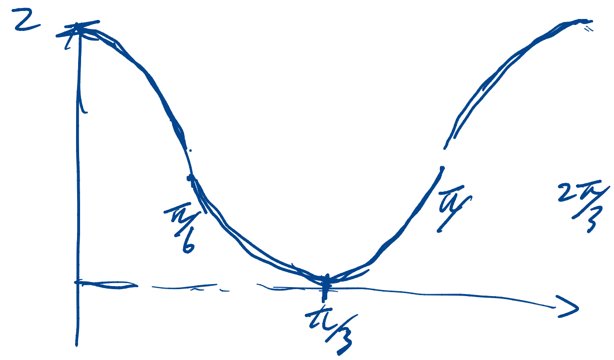
3. Sketch polar curve $r = 1 + \cos(3\theta)$

This is based on $\cos(3\theta)$

θ	$\cos 3\theta$



horizontal



The distance from origin is maximized at $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

Example 4 Let C be the curve of intersection of the following surfaces

$x^2+y^2+5z^2=9$ and $y=\sqrt{z^2+y^2}$

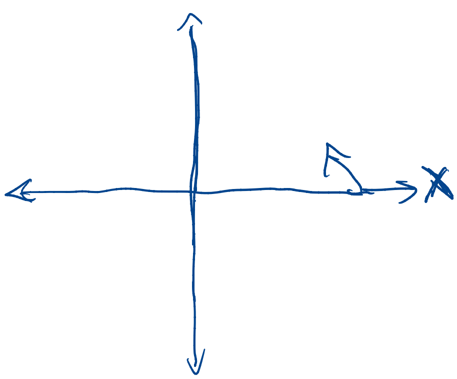
a) Find a vector function $\vec{r}(t)$ that represents C. (vector form)

b) Find the length of intersection curve C. (line integral)

a) A possible parameterization is with cylindrical coordinates

$x=r\cos\theta,\ y=r\sin(\theta),\ z=z$, where r is the radial distance from the z axis & θ is the angle measured counter clockwise from the positive x-axis

for a given r, z



2. Substituting into $r^2+5z^2=9$ which we can solve for $r=\sqrt{9-5z^2}$, subbing into the second surface to get

Q6 the lim of $f(x,y)$ as $(x,y) \to (0,0)$ is evaluated approaching from the left & right side with $\frac{\cot(x^2+y^2)}{2x^2+2y^2}$

As $f(x,y)$ approaches $(0,0)$ $2x^2+2y^2$ evaluates to 0. With $\cot(0^2+0^2)$ is infinity. Thus we have $\infty/0$ which is undefined.. This does not equal to $f(0,0)$ which is $= \frac{1}{2}$

thus the function is not continuous at $(0,0)$

Q8 a) $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 y^2} = 0$ since denominator is of higher order

b) $\lim_{(x,y)\to 2,2} \frac{y-x}{2-y+\ln(0.5x)}$, using L'Hopital's rule, derivative of $\frac{d(y-x)}{dx} = -1$ $\frac{d(2-y+\ln(0.5x))}{dx} = \frac{0.5}{x}$

thus limit evaluates to $\frac{-1}{\left(\frac{0.5}{2}\right)}$ $\boxed{= -2}$

c) $\lim_{(x,y)\to(0,0)} \underbrace{\pi^2(x^4+y^4)}_{\to 0} \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$

since $x^4+y^4=0$, the ultimate product $=0$.

thus the limit $\boxed{=0}$

Limits and Continuity

$\lim_{(x,y)\to a} f(x,y)$ if limit exists, then $\lim_{(x,y)\to a} f(x,y) = N < \infty$ $\forall \varepsilon > 0$,

$\exists \delta > 0$, such that $\underbrace{\|(x,y) - a\|}$ $< \delta$

$$\|(x,y) - (x_0, y_0)\| < \delta < \delta \Leftrightarrow \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

then $\|f(x,y) - N\| < \varepsilon \Leftrightarrow \sqrt{(f(x,y) - N)^2} < \varepsilon$

By non-rigorous way, we can say, if we can control our input, we can make our outputs arbitrarily close.

$f(x,y)$ is continuous at point $a = (x_0, y_0) \in \mathbb{R}^2$ if $\lim_{(x,y)\to a} f(x,y) = f(a)$

Okay, we'll go into examples, so I will assume you are familiar with basic delta epsilon proofs.

Ex. $x^2 + y^2 = 1$ in 3D space

Lec 6 Limits and Continuity Continued

① showing DNE for $f(x,y)$

$\lim_{(x,y)\to a} f(x,y) = \text{DNE}$, try different curves $y=g(x)$ or $x=h(y)$ such that passing through the given point $a=(x_0,y_0)$, thus

$$\lim_{(x,y)\to a} f(x,y)\Big|_{y=g(x)} \neq \lim_{(x,y)\to a} f(x,y)\Big|_{x=h(y)},$$ thus limit DNE

So we are essentially reusing ideas from single variable calculus. By showing there is a direction approaching a where the two limits from different directions do not agree, we show the limit DNE

We can not do the same thing for showing the limit exists.

② Remark: showing limit exists is harder!, ex. many curves resulting in $f(x,y)=a$ is not enough!
we still can not conclude limit exists

we should use delta epsilon or squeeze theorem

Squeeze theorem example

suppose: $\lim_{(x,y)\to a} f(x,y) = 0$? where $\frac{x^2+y^2}{\sin(xy)} = f(x,y)$

this is a non trivial case, where we can't use L'Hopital's rule, since this is multivariable calc

① Find upper & lower bound for f near point a, ex

$g(x,y) \le f(x,y) \le h(x,y)$

You want to make these functions simple functions

② Then take lim of upper & lower bound.

You hope they give you the same answer.

Proving Limits in $\mathbb{R}^2$

Squeeze theorem becomes much more useful

Limits are monotone? Does that follow from def. of limits

Evaluate $f(x,y) = \frac{x^2y^2}{x^2+y^2}$, $\lim_{(x,y)\to(0,0)} f(x,y)$

figuring out lower bound $g(x)$ & upper $h(x)$

note $0 < x^2 < x^2+y^2$ thus $0 < \frac{1}{x^2+y^2} < \frac{1}{x^2}$

multiplying all three terms by x^2y^2 ...

$$0 < \frac{x^2y^2}{x^2+y^2} < \frac{\cancel{x^2}y^2}{\cancel{x^2}} \Rightarrow 0 < \frac{x^2y^2}{x^2+y^2} < y^2$$

$$0 < y^2 < x^2+y^2$$

$$0 = \frac{0}{x^2+y^2} < \frac{y^2}{x^2+y^2} \leq 1$$

When is a fxn not continuous?

Q: $f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,y) \neq 0,0 \\ 0 & (x,y) = 0 \end{cases}$

Does the limit exist?

<u>recall:</u> the value of sin takes values between $-1, 1$

$$-1 < \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) < 1$$, multiplying by (x^2+y^2)

$$-(x^2+y^2) \le x^2+y^2 \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) \le x^2+y^2$$

since x^2+y^2 is continuous we may plug as both sides of the limit go to 0,

$$\lim_{(x,y)\to(0,0)} x^2+y^2 \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) = 0$$

What if we see limit doesn't exist at $(0,0)$?

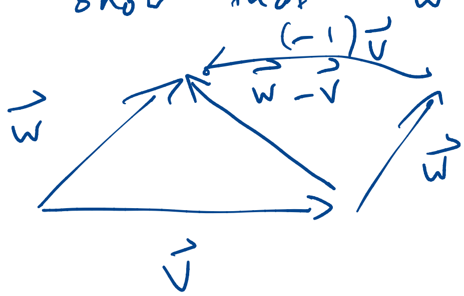
$$g(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Just show that $\lim\limits_{(x,y)\to(0,0)}$ differs depending on how we approach it. ex. $\lim\limits_{(x,0)\to(0,0)} \neq \lim\limits_{(x,x)\to(0,0)}$

$$0 \neq 1$$

13.1 Displacement vectors

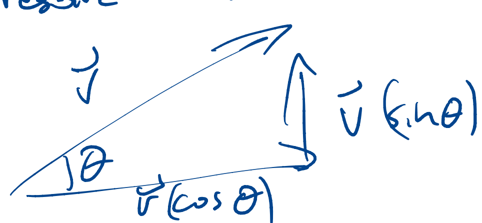
show that $\vec w - \vec v = \vec w + (-1)\vec v$, why?



Because the combined displacement of $\vec w - \vec v$ is the same as $\vec w + (-1)\vec v$

Vector Components

We may resolve vectors into components by



$$\vec v = (v\cos\theta)\vec i + (v\sin\theta)\vec j$$

Parameterization is hiding some secret, what are the big ideas of multivariable calculus?

Angle between vectors leads to defining the dot product

recall cosine law

$$\underbrace{\|\vec{u}-\vec{v}\|^2}\ = \underbrace{\|\vec{u}\|^2 + \|\vec{v}\|^2}\ - 2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

expanding this

$$\|\vec{u}\|^2 = u_1^2 + u_2^2 + u_3^2 \quad , \quad \|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

Now considering $\|\vec{u}-\vec{v}\|^2$, we get

$$= (u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2$$

and expanding and cancelling terms we get

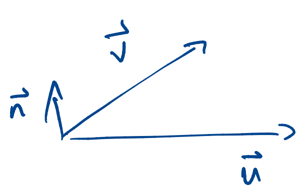
$$-2u_1 v_1 - 2u_2 v_2 - 2u_3 v_3 = -2\|\vec{u}\|\|\vec{v}\| \cos\theta$$

thus the angle between two non-zero

Algebraic Cross Product

Geometric Cross product

For
$$\vec{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}$$
$$\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$$

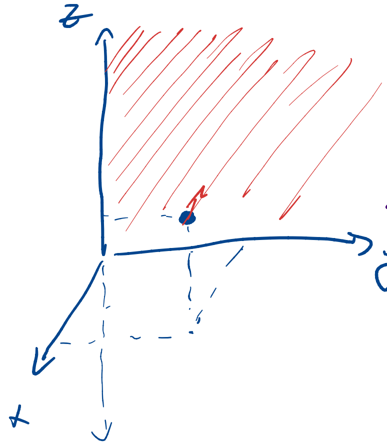


$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2)\,\hat{i} + (u_1 v_3 - u_3 v_1)\,\hat{j} + (u_1 v_2 - u_2 v_1)\,\hat{k}$$

Vector Equation of Line : $\vec{r} = \vec{r}_0 + t\vec{v}$

Review. Consider $(3, 4, -2)$ facing the yz plane.

walk 5, turn right, walk 2. What are new coordinates?



$\langle x,\ 3-5,\ z=-2$ $\qquad \boxed{\langle -2, 6, -2 \rangle}$ $\qquad$ E

$\langle\ , y,\ \rangle \qquad 4+2=6$

$\langle\ ,\ , z \rangle$

ii) Which vectors are perpendicular to both $\vec{i} + \hat{j}$ & $2\vec{i} + 2\vec{j} + \vec{k}$

use cross product $\qquad$ call $\hat{i} + \hat{j}$ a and $2\vec{i} + 2\vec{j} + \vec{k}$ b

$\langle 1, 1, 0 \rangle$

$\langle 2, 2, 1 \rangle$

$a \times b = (a_1 i + a_2 j + a_3 k) \times (b_1 i + b_2 j + b_3 k)$

Expanding using distributivity

↳ $a_1 b_1 (i \times i) + a_1 b_2 (i \times j) + a_1 b_3 (i \times k) + a_2 b_1 (j \times i) + a_2 b_2 (j \times j) + a_2 b_3 (j \times k) + a_3 b_1 (k \times i) + a_3 b_2 (k \times j) + a_3 b_3 (k \times k)$

the cross products of the unit vectors are easier to imagine

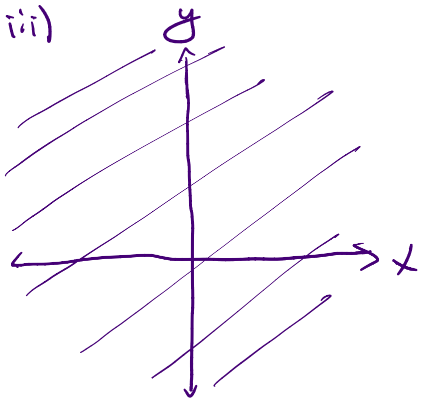
$= a_1 b_1 0 + a_1 b_2 k - a_1 b_3 j - a_2 b_1 k + a_2 b_2 0 + a_2 b_3 i + a_3 b_1 j + a_3 b_2 i + a_3 b_3 0$

Recall $\vec{A} \times \vec{B} = \|\vec{A}\| \|\vec{B}\| \sin\theta$ and the basis vectors satisfy

$i \times j = k, \quad j \times i = k, \quad j \times k = i, \quad k \times j = -i, \quad k \times i = j, \quad i \times k = j$

note the anti-commutative property $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ and the zero vector property $a \times b = 0$, and the Right-hand Rule.

iii)



for a given z, slope is positive

$z = 6 - 2x + 3y$

$2x + z - 6 = 3y$, positive slope

B

iv) Let $\vec{a} = \vec{i} + \vec{j} + 3\vec{k}$ and $\vec{b} = \vec{i} - 3\vec{j} + 5\vec{k}$. Find the angle between $\vec{a}$ & $\vec{b}$

using a 3×3 determinant the cross product is given by

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 3 \\ 1 & -3 & 5 \end{vmatrix}$$

$$= i \begin{vmatrix} 1 & 3 \\ -3 & 5 \end{vmatrix} - j \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} + k \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix}$$

$$= (5 - (-9))i - (5 - 3)j + (-3 - 1)k$$

$$= 14\vec{i} - 2\vec{j} - 4\vec{k}$$

now use the dot

$1 + (-3) + 15 = 13$

$$13 = \left(\sqrt{1^2 + 1^2 + 3^2}\right)\left(\sqrt{1^2 + (-3)^2 + 5^2}\right)\cos(\theta)$$

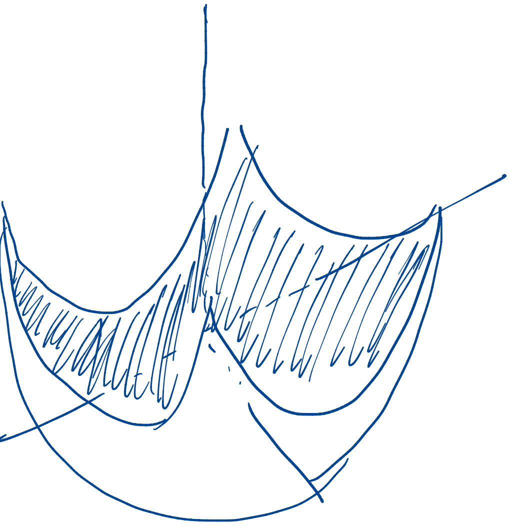
$$13 = (\sqrt{11})(\sqrt{35})\cos\theta$$

$$\cos^{-1}\left(\frac{13}{\sqrt{11}\sqrt{35}}\right) = \theta$$

A

Graphs and surfaces 12.2

the shape of $f(x,y) = x^2 + y^2$



describe how these functions are different

a) x^2+y^2+3, raised from the origin by 3

b) $h(x,y) = 5-x^2-y^2$, opening upside down starting from $(0,0,5)$

c) $k(x,y) = x^2+(y-1)^2$, shifted up along the y axis by 1 $(0,1,0)$.

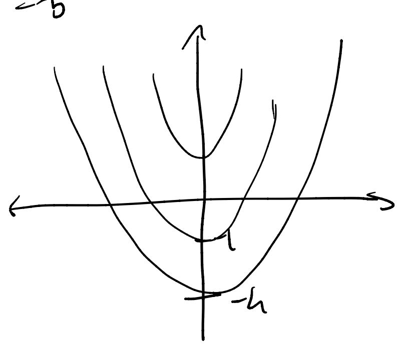
Describe $G(x,y) = e^{-(x^2+y^2)}$

Like a peak out from the origin to $(0,0,1)$ when $x,y = 0,0$, and since it can't go negative it is asymptotic to the xy plane

Describe the cross sections of $g(x,y) = x^2-y^2$

the cross sections with $g(x,y)$ with y fixed as $y=b$ are $z = x^2-b^2$, they all parabolas open upward starting at

$z=-b^2$



fixed x are $z=g(a,y)$

$a^2-y^2 = g(a,y)$

thus there are upward opening parabolas in the x directions and downward-opening parabolas in the y-directions

thus the surface is saddle shaped

when one variable is missing

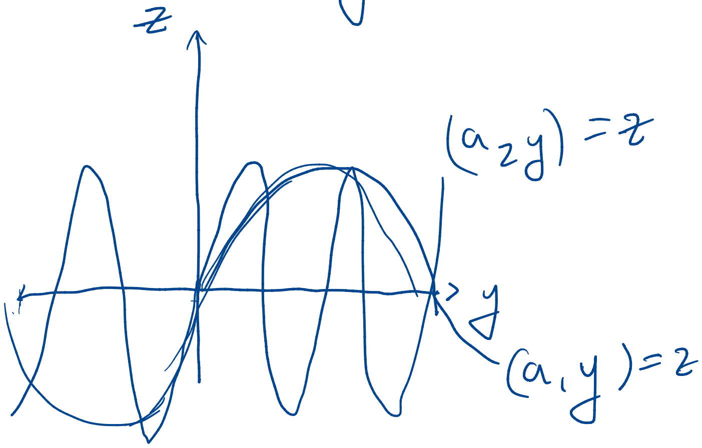
ex. for ~~f(x,y)~~ $z = x^2$, letting y vary up & down the y axis, we sweep through all the ys & $z = x^2$ is "duplicated" throughout

or for $x^2 + y^2 = 1$ in 3D is a cylinder over the z-axis

Consider all of the following with x fixed

2\. 1) $z = \sin(xy)$

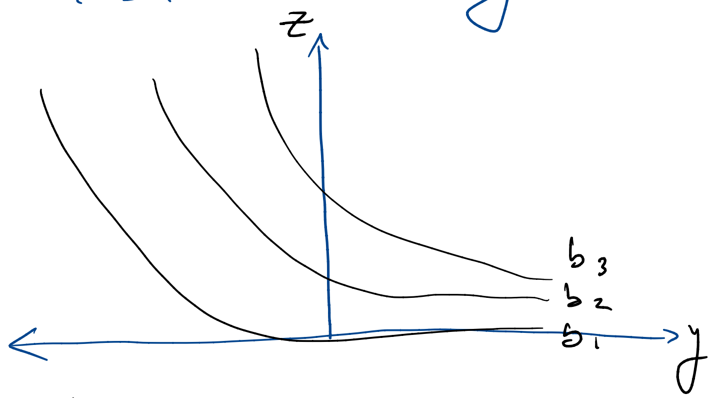
$z = \sin(ay)$



2) $z = e^{x-y}$

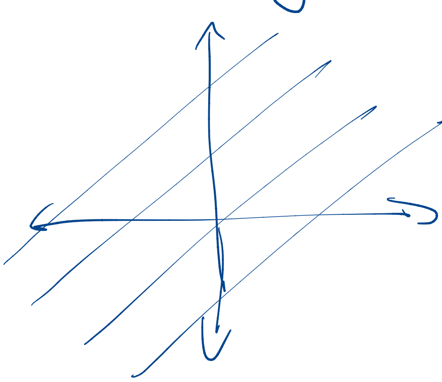
$z = e^{b-y}$

b_1, b_2, b_3 increasing

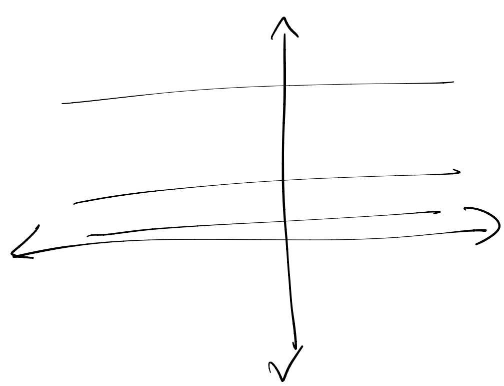


3) $z = 1 + x + y$

$z = 1 + a + y$



4) $z = x^2$, given no y, y can vary as much as it wants & $z = b^2$ is constant



nothing negative

3. Verify if the following exists

a) $\lim_{(x,y)\to(0,0)} 2x^2 |y|^3 \sin(\ln(x^2+y^2))$

$$\overset{=0}{2(0)|0|^3} \sin(\ln(0^2+0^2))$$

$$= (0)(0)\sin(\ln|0|) \quad \ln(0) \text{ is undefined}$$

We have to analyze the limit. notice that $2x^2|y|^3 \sin(\ln(x^2+y^2))$ has a sin component

the sin fn, where defined will oscillate between $(-1, 1)$

thus $-2x^2|y|^3 \leq 2x^2|y|^3 \sin(\ln(x^2+y^2)) \leq 2x^2|y|^3$

using squeeze theorem we will show both the upper & lower bounds converge to 0 as $(x,y)\to(0,0)$

$\lim_{(x,y)\to(0,0)} -2x^2|y|^3 = 0$ and $\lim_{(x,y)\to(0,0)} 2x^2|y|^3 = 0$

are both straightforward as x, y both go to 0, & the limit in a polynomial sense will be 0.

Therefore, by squeeze theorem $\lim_{(x,y)\to(0,0)} 2x^2|y|^3 \sin(\ln(x^2+y^2)) = 0$

b) if limit exists, evaluate

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} + e^{x^2+y}$$

Considering each term seperately: L_1: $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$ L_2: $\lim_{(x,y)\to(0,0)} e^{x^2+y}$

we can analyze L_1 with polar coordinate substitution. Recall

$x = r\cos\theta$, $y = r\sin\theta$

L_1:

$$\lim_{r\to 0} \frac{2r\cos\theta\,(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^4} = \lim_{r\to 0} \frac{2\cancel{r}^3\cos\theta\,\sin^2\theta}{\cancel{r^2}\cos^2\theta + \cancel{r}^4\sin^4\theta}$$

$$= \lim_{r\to 0} \frac{2r\cos\theta\,\sin^2\theta}{\cos^2\theta + r^2\sin^4\theta} = 0$$

thus as $r\to 0$ $L_1 \to 0$

For $L_2 = \lim_{(x,y)\to(0,0)} e^{x^2+y} = e^0 = 1$

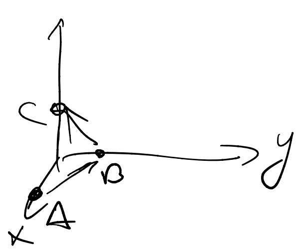
Combining the limits $L_1 + L_2 = 0 + 1 \boxed{=1}$

4.a) Equation of 3 points $A(1,0,0)$ $B(0,1,0)$ $C(0,0,1)$

express answer in $Ax + By + Cz = D$

b) consider a different set of points $A(5,1,0)$, $B(10,2,0)$

Vector $\overrightarrow{AB}$, $\overrightarrow{BC}$, find the norm



$$\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

vector $\overrightarrow{AB}$ is $\langle 1, 1, 0\rangle$, $\overrightarrow{BC}$ is $\langle 0, 1, 1\rangle$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= \langle 1, -1, 1\rangle$$

$x + (-1)y + z = D$, plugging in a point on the plane to get

$1 - 0 - 0 = 1$

$$\boxed{x - y + z = 1}$$

c) can't find a unique plane, as the solution set for A, B, is a line

8) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function given by $f(x,y) = \ln(x^2+y^2-4)$

find the domain & range of f)

the domain exists for all $x^2+y^2-4 \neq 0$

it D.N.E when $x^2+y^2 = 4$, the logarithm can't be negative. Thus the domain of the function f is well-defined over $x^2+y^2 > 4$

The output gives all values in $\mathbb{R}$, thus the range of $f(x,y)$ is $\mathbb{R}$

b) describe the level curve $f(x,y) = c$ when $c=1$

$\ln(x^2+y^2-4) = 1$, or

$e^{\ln(x^2+y^2-4)} = e^1$, or $x^2+y^2-4 = e$, or

$e+4 = x^2+y^2$, which determines the level curve when $c=1$. This is a circle centred at $(0,0)$ with radius $\sqrt{e+4}$

c) Describe the cross section of $f(x,y)$ when $y=2$.

fixing y to 2, f becomes $\ln(x^2+2^2-4) = \ln(x^2)$

$\boxed{= 2\ln(|x|), \quad x \neq 0}$

d) Is the graph of f a level surface of some function $g(x,y,z)$?

Yes, we may write $z = \ln(x^2+y^2-4)$ to group all the variables together with $z - \ln(x^2+y^2-4)$, then let $g(x,y,z) = z - \ln(x^2+y^2-4)$, thus the graph of f corresponds to the level surface $g(x,y,z) = 0$

6. The pressure of gas in a storage container, in atmospheres, is given by $p = f(n, T, v) = \frac{82nT}{V}$ where n is the amount of gas in kilomoles, T is the temp of the gas, in Kelvin & v is the volume of the storage container, in L

a) find the formula for the level surface containing the point (1, 200, 41)

given (1, 200, 41) we have $p = \frac{82(1)(200)}{41} = 400$

so the level surface containing this point is given by

$\frac{82nT}{V} = 400$

b) Describe the level surfaces of P algebraically for $P>0$. Using your formula & viewing V as a fn of n & T, what is the general shape of the cross sections of the form $n=c$, where c is constant?

suppose $P = P_0 > 0$ is a fixed constant, giving us $P_0 = \frac{82nT}{V}$, rearranging for V we get $V = \frac{82}{P_0}nT$

$V = \frac{82c}{P_0}T$, so the cross section for $n=c$ is line with slope $\frac{82c}{P_0}$

Review of Manifold Analysis Prereqs - cont.

2. Let $r^2 = \sin\theta$ be a polar eqn. Find a point (r,θ) other than $(0,0)$ which lies on the polar curve.

Such a point (r,θ) satisfies $r^2 = \sin\theta$
plugging in $\theta = \pi/2$, we get $r^2 = 1$, thus
$(1, \pi/2)$ and $(-1, 3\pi/2)$ are solutions

3. Find parametric equation of the line through $(-8, 0, 2)$ and $(3, 2, -4)$.

Parametric equations of x, y, z in t should allow us to move from one point to another. The following is a parameterization of a line through $(-8, 0, 2)$, $(3, 2, -4)$

$$x = -11t - 8, \quad y = -2t, \quad z = 6t + 2$$

b) parametrize a line through the origin & perpendicular to the vector $(1, 5, 2)$.

$$y = -\tfrac{1}{5}t, \quad z = -\tfrac{1}{2}t, \quad x = t$$

4. a) $x^2 + 4y^2 - 6x + 8y = 3$
b) $4x^2 + 8x + (y-3)^2 = 12$
c) $2y + x^2 = 2x + 5$
d) $2x + 7 = y^2 + 2y$

What graph does each equation generate?

a) complete the square

$$x^2 - 6x + 4y^2 + 8y - 3 = 0$$
$$(x-3)^2 + 4(y+1)^2 - 16 = 0$$

Ellipse centred at $(3, -1)$

b) $4(x+1)^2 + (y-3)^2 - 4 = 12$

ellipse centred at $(-1, 3)$

c) $x^2 - 2x + 2y - 5 = 0$

$y = -\frac{x^2}{2} + x + 5/2$, parabola opening up, vertex at $(0, 5/2)$

d) $x = \frac{y^2 + 2y - 7}{2}$, a rightward opening parabola, with vertex at $(-7/2, -1)$

5. a) Consider the parameterization $x(t) = t^2 - 3$, $y(t) = t^3 - 4t$ for $-2 \le t \le 2$.

Given a expression for dy/dx in terms of t

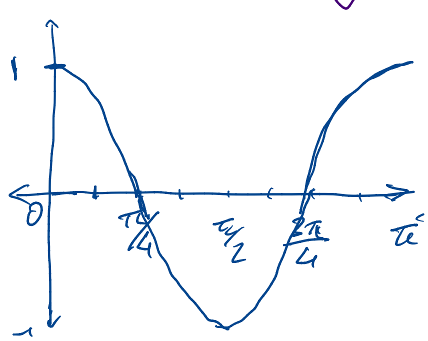
$$\boxed{\frac{dy/dt}{dx/dt} = \frac{3t^2 - 4}{2t}}$$

b) Find all (x,y) at horizontal / vert. tangents.

horizontal tangent when $\frac{dy}{dx} = 0$, which gives points $t = -2/3$, $t = 2/3$ resulting in (x,y) coordinates of $(-13/9, \frac{-20}{27})$ and $(1/9, \frac{-8}{27})$

The curve has a vertical tangent when $\frac{dy}{dx}$'s denominator $= 0$, so when $t = 0$, resulting in point $(-3, 0)$

6. Sketch a polar curve $r = \cos(2\theta)$ for $0 \le \theta \le \pi$ and indicate any horizontal & vert. tangents



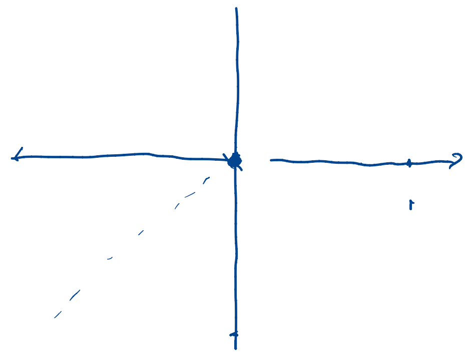
- The equation $r = \cos 2\theta$ describes a rose curve with two petals

- The # of petals in the curve is determined by the coefficient on θ inside the function over the interval $[0, 2\pi)$

- if the coeff. is even, the # of petals is 2x the coeff.
- If odd, # of petals = coeff.

This is a 2θ fxn over the interval $[0, \pi]$. thus the rose curve has 2 petals using a few key points, we find r for given theta

θ	r
0	1
$\pi/4$	0
$\pi/2$	-1
$3\pi/4$	0
π	1



- we can then plot the polar coordinates starting ~~at~~ at origin & measuring θ counterclockwise from the positive x-axis.

- to find the horizontal & vertical tangent lines, we need to find the values of θ for which $\frac{dr}{d\theta} = 0$ and vertical when $\frac{dr}{d\theta}$ is undefined.

Question 7 Let C be the curve of the intersection of the following surfaces

A $x^2 + y^2 + z^2 = 2$ & B $z = \sqrt{x^2 + y^2}$

Find a vector function $\vec{r}(t)$ that represent C.

The two given surfaces are a sphere A, & a cone B from equation

Example 10: Let $f(x,y) = x^2/y$. find total derivative of $f(x,y)$ and $f_{xy}(-1,1)$

Answer: the total derivative of a function is given by $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$.

pt1 Solving for the partial derivatives

$$\frac{\partial f}{\partial x} = \frac{2x}{y}, \quad \frac{\partial f}{\partial y} = \frac{-x^2}{y^2},$$ thus the total derivative is

$$df = \frac{2x}{y} dx - \frac{x^2}{y^2} dy$$

pt 2 f_{xy} is the second derivative w.r.t x & then y

$$\frac{\partial^2 f}{\partial x^2} = \frac{2}{y},$$ then wrt y $$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{-2}{y^2},$$ thus

$$f_{xy}(-1,1) = \frac{-2}{1} \boxed{= -2}$$

Q2: Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a fn given by

$$f(x,y) = \begin{cases} \frac{x^4}{x^2+y^2} + 2x + 3y & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Is $f(x,y)$ continuous around the neighborhood of pt $(0,0)$

recall the fn is cont. if the $\lim (x,y) \to (0,0)$ is equal to the value of the fn at that point. So we want to show

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

$f(0,0) = 0$ by pt. 2 of the piecewise and

$\lim_{(x,y)\to(0,0)} x^2 + y^2 + 2x + 2y$ also $= 0$. Therefore $f(x,y)$ is continuous at $(0,0)$.

Q3 Find the limit, or show it doesn't exist

a) $$\lim_{(x,y)\to(0,0)} (x^2+y^2)\cos\left(\frac{1}{\sqrt{x^3+y^3}}\right)$$

Consider as $(x,y)\to(0,0)$ when $x,y<0$ then the term $\sqrt{x^3+y^3}$ is not defined in the real plane.

Since the value of the lim depends on the path (x,y) takes in the complex plane as it approaches $(0,0)$.

Thus the limit is not well defined & we say it does not exist

Further explanation: As (x,y) approaches $(0,0)$, the term $\frac{1}{\sqrt{x^3+y^3}}$ in the cosine function becomes very large, which causes $\cos\left(\frac{1}{\sqrt{x^3+y^3}}\right)$ to oscillate rapidly between -1 & 1. The rate of oscillation depends on the path (x,y) takes as it approaches $(0,0)$. x,y could approach at the same rate, or different rates, x faster than so the value ~~of~~ the function as (x,y) approaches $(0,0)$ can vary

b) $$\lim_{(x,y)\to(1,1)} \frac{y-x}{1-y+\ln(x)}$$

As $(x,y)\to(1,1)$ the denominator approaches 0, which causes the function to approach ∞ or $-\infty$ depending on the values of x,y

Similar to the first part.

More Questions

Q1 Given mortgage payment function $P = f(A, r, N)$ and

$f(10, 7, 20) = 775.29$ $\quad f(10, 8, 20) = 836.44$ $\quad f(10, 7, 25) = 706.77$

$f(12, 7, 20) = 930.35$ $\quad f(12, 8, 20) = 1003.72$ $\quad f(12, 7, 25 = 848.13$

A ↳ Estimate value of $\left.\frac{\partial f}{\partial r}\right|_{(10, 7, 20)}$

differentiation in fxn given change in r. at a given $N = 20$

$$\frac{df}{dr} = \frac{f(10, 8, 20) - f(10, 7, 20)}{8-7} = \boxed{\frac{836.44 - 775.29}{1}}$$

Q2 Given table of $f(x,y)$ values, find $f_x(1,2)$ estimate

x \ y	1.5	2	2.5
0	36	35	34
1	38	37	35
2	44	42	38

find $\frac{df}{dx}$, at $(1,2)$. thus

x changes y stays same

To estimate the P.D. of $f\_x(1,2)$ we can use pt. around $(1,2)$

$$f\_x(1,2) = \frac{f(2,2) - f(0,2)}{2} = \frac{42-35}{2} = 3.5$$

Q3 Ideal gas law equation $PV = RT$

at $T = 305°K$, $P = 1$, $V = 0.01$, find $\partial P / \partial V$

How does pressure change with volume

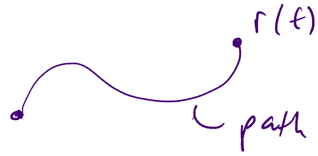
$$P = \frac{RT}{V} = RTV^{-1}$$

$$\frac{dp}{dv} = -RTV^{-2}$$, since $RT = PV$, plugging in we get

$$= \frac{-PV}{V^2} = \frac{dp}{dv} = \frac{-P}{V}$$

More Ch. 1 to Ch. 2 review questions

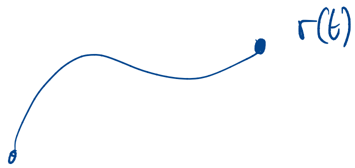
Q: let $r(t)$ be pos. of particle.



let $a(t) = [3, 0, 2t]$ be its acceleration, and $\vec{v}(t)$ at $t=0$, $[2, 1, -1]$.

Find the $v(t)$? and find time t at which the tangent is parallel to the xy plane.

A: velocity parallel to xy plane??



- the parameterized equation with t gives me the path of the particle.
- we know, the first derivative (of position) is velocity & the 2nd deriv. is the acceleration
  - ↳ OR we can integrate acceleration to get the antiderivative which is velocity

thus $\int a(t)\,dt = \int d\,v(t)$, since $a = [3, 0, 2t]$

we get ↳ $= [\int 3\,dt, \int 0\,dt, \int 3t\,dt] = [3t + c, c, t^2 + c] = v(t)$

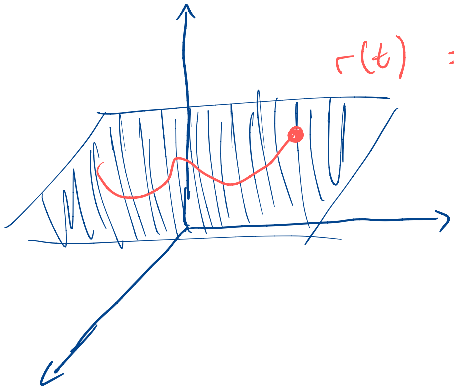
Then, we're also provided the initial condition at $t=0$

$v(0) = [2, 1, -1]$, which we can use to solve for c

the middle c

$c = 1$ this is similar to analysis I topics

thus $v(t) = [3t + 2, 1, t^2 - 1$



$r(t) = [x(t), y(t), c]$, which is the case for this question

So why is ~~z~~ constant?

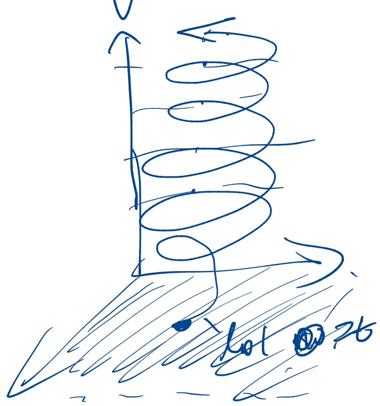
then the level surface is unchange & $r(t)$ is along a plane parallel to the xy plane

this is the trivial scenario.

For the purposes of answering this question we can pick any z constant, & vary xy. Let pick $z=0$, thus $t^2-1=0$. $\Rightarrow v(t)\big|_{t=1} = [3t+2, 1, 0]$

We don't need to find every z, only a simple case.

In theory an infinite # of paths (positions) exist parallel to the xy plane. Consider the following graph



this $r(t)$ path has repeated positions $r(t)$ parallel to xy

An argument for a general $r(t)$ that is parallel could involve the dot product

$$\vec{v} = [a_1, a_2, a_3] \cdot [0, 0, z] = a_1(0) + a_2(0) + a_3(z) = 0$$

$$= a_3(z) = 0$$

Follow up question: $\vec{r}(t) = [x(t), y(t), z(t)]$, with $\vec{r}(t)$ as a curve on surface $x^2 + y^2 - z^2 = 1$

show that if $\vec{r}(0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ then

$$\vec{r}\,'(0) \cdot \vec{r}(0) = 0.$$

Well, algebraically this means ...

$\vec{r}(0) \cdot \vec{r}'(0) = x(0)x'(0) + y(0)\,y'(0) + z(0)\,z'(0)$

Back to the question. $r(t) = [[x(t), y(t), z(t)]]$ is on the curve $x^2+y^2-z^2=1 \Rightarrow x^2(t) + y^2(t) - z^2(t) = 1$ for all t.. from eq'n of surface, we have

$\Rightarrow z(t)\,z'(t) = x(t)x'(t) + y(t)\,y'(t)$, then sub $t=0$ to get

$\Rightarrow t=0, \quad z(0)\,z'(0) = x(0)\cdot x'(0) + y(0)\,y'(0)$ ✓

recall : Given $\vec{r}(t)\big|_{t=0} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ from the problem statement and note especially that

$z(0) = 0.$

then given $r(0)\cdot\vec{r}'(0) = \underbrace{x(0)\,x'(0) + y(0)\cdot y'(0)}_{z(0)\,z'(0)} + \underbrace{z(0)\,z'(0)}_{0}$

$= 2\,z(0)\cdot z'(0)$? how

$\frac{d}{dt}\left(x^2(t) + y^2(t) - z^2(t)\right) = \frac{d}{dt}1,$

$2x(t)\,x'(t) + 2y(t)\,y'(t) - 2z(t)\,z(t) = \frac{d}{dt}1$

$\Rightarrow t=0, \quad z(0)\,z'(0) = x(0)\cdot x'(0) + y(0)\,y'(0)$ ✓

then recall : Given $\underbrace{\vec{r}(t=0) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}$

$z(0) = 0$

Limit Qs

$$\lim_{(x,y)\to(0,0)} \frac{\tan(x^2+y^2)}{3x^2+3y^2}$$

recall $\frac{\sin\alpha}{\cos\alpha} = \tan\alpha$

$$= \frac{\frac{\sin(x^2+y^2)}{\cos(x^2+y^2)}}{3x^2+3y^2}$$

let x^2, y^2 be something?

or $x^2+y^2 = \beta$?

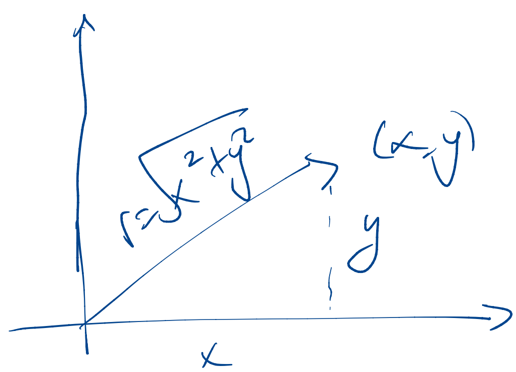
we change to polar coordinates

$$= \frac{\sin(x^2+y^2)}{(3x^2+3y^2)\cos(x^2+y^2)}$$

recall $x = r\cos\theta$ $\quad r = \sqrt{x^2+y^2}$

$y = r\sin\theta$ $\quad \theta = \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)$

ie



$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

$\tan\theta = \frac{x}{y}$

Then the limit becomes

$$\lim_{r\to 0} \frac{\sin(r^2)}{3r^2\cos(r^2)} \longrightarrow \text{single variable limit}$$

recall the result $\lim_{r\to 0} \frac{\sin(r^2)}{r^2} = 1$ & $\lim_{r\to 0} \cos(r^2) = 1$

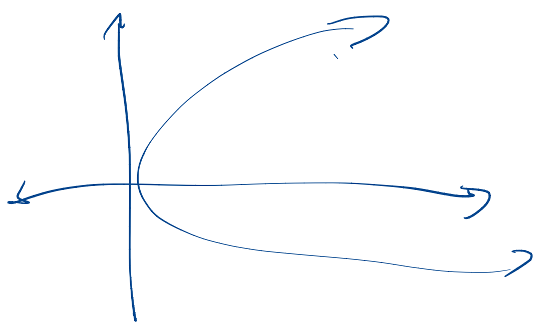
$$\boxed{\lim_{r\to 0} \frac{\sin(r^2)}{3r^2} \cdot \frac{1}{\cos r^2} = \frac{1}{3}}$$

Q4: $\lim_{(x,y)\to} \frac{y^2 \sin x}{x^2+y^4}$, sol'n ? , consider 1^0, $y=0$.

$y=0$ (along the x axis)

first path , next

$x=y^2$, sideways parabola



$$\lim_{(y^2,y)\to(0,0)} \frac{y^2 \sin(y^2)}{y^4+y^4} = $$

$$\lim_{\to(0,0)} \frac{y^2 \sin(y^2)}{2y^{4}}$$

$$\lim_{\to 0,0} \frac{\sin(y^2)}{2y^2} = 1$$ which we know from single var.

Conclude; we have found two limits $\to(0,0)$ that do not exist! since limits approaching from all directions must agree!

also remember $\lim_{t\to 0} \frac{\sin(t)}{t} = 1$ identity

Quick squeeze theorem question

$\lim_{x,y\to 0,0} \frac{x^2y^2}{x^2+y^2}$ $x^2+y^2 \geq x^2$, Answer: thus by transitivity ,

$\frac{x^2y^2}{x^2+y^2} \leq \frac{\cancel{x^2}y^2}{\cancel{x^2}} = y^2$, then by squeeze theorem

$-y^2 \leq \frac{x^2y^2}{x^2+y^2} \leq y^2$, $\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0}$

Q5 more limits

$$f(x,y) = \begin{cases} \dfrac{xy + 2x^3 + y^3}{x^2 + 3y^2} & f(x,y) \neq (0,0) \\ 0 & f(x,y) = (0,0) \end{cases}$$

a) check if $f(x,y)$ is continuous at $(0,0)$?
b) Can we compute $f_x(0,0)$ $f_y(0,0)$?

a) WTS $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})$ by definition

For this question
$\lim_{(x,y) \to (0,0)} f(x,y) = f(0,0) = 0$, so we are checking if the limit gives you ~~zero~~ when the limit goes to zero.

Let's start by checking if the limit exists from certain directions.
Remember (to prove something false, we only need to find one case where it isn't true) :)

Let's first test the $y=x$ direction, why? because this reduces the 2D limit to a 1-D limit

$$\lim_{(x,y) \to (0,0)} f(x,y) \Big|_{y=x} = \lim_{x \to 0} \frac{x^2 + 2x^3 + x^3}{x^2 + 3x^2}$$

$$= \lim_{x \to 0} \frac{x^2 + 3x^3}{4x^2} = \lim_{x \to 0} \frac{1+3x}{4} = \boxed{\frac{1}{4}}$$

$\Rightarrow$ This means!! if the limit exists, it must be $\frac{1}{4}$, because limits as $(x,y) \to 0$ must all be $1/4$.

Consequently, if something at (x,y) or $(x,y) \to 0$ does not evaluate to 0, the the limit DNE

NOTICE, we have $f(0) = 0$, & $\frac{1}{4} \neq 0$.

So the function is discontinuous at $(0,0)$ & is therefore not differentiable at $(0,0)$.

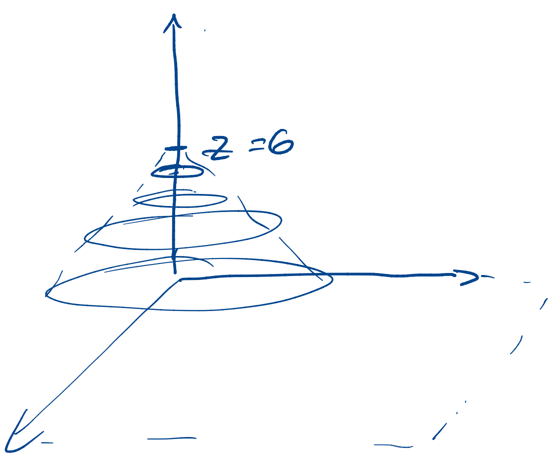
Q6 Graph: $z = f(x,y) = 2\sqrt{9 - x^2/2 - y^2}$

Answer $\left(\frac{z}{2}\right)^2 = 9 - \frac{x^2}{2} - y^2$ level set is an ellipse

$9 - \frac{z^2}{4} \geq 0 \Rightarrow \frac{z^2}{4} \leq 9 \Rightarrow \cancel{-3 \leq \frac{z}{2}} \leq 3$, $\{z \geq 0\} \cap \{-6 \leq z \leq 6\}$

$\Rightarrow 0 \leq z \leq 6$

when $z = 6$, $\frac{x^2}{2} + y^2 = 9 - 9 = 0$, $(x,y) = (0,0)$



$x^2 + y^2 = 9 - \frac{25}{4} = \frac{11}{4}$, level set $z=5$ check

Based on these level sets we may draw our final surface

We may not necessarily know the curve of the shape though unless we use differential geometry

A follow up; consider the surface, cut by some plane.

$\{z = f(x,y)\} \cap \{\text{plane} \mid x = 2\}$, parameterize the resulting curve

Answer: $\{\text{fix } x = 2\} \Leftrightarrow \{\text{intersection with plane } x = 2\}$

this gives us $\{(x=2, y, z = 2\sqrt{9 - \frac{2^2}{2} - y^2}\} \Rightarrow \{(2, y, 2\sqrt{7 - y^2}\}$

Now we may parameterize the intersection

$$\boxed{r(t) = (2, t, 2\sqrt{7 - t^2})}$$

Given the parameterization, we may find the tangent

$\vec{r}'(t) = \left(0, 1, \frac{-2t}{\sqrt{7-t^2}}\right)$

Thus the equation of the tangent at a certain point is

$l(t) = p + t\vec{v}$

---

recall we define the partial derivative of

$f(x,y) \Rightarrow \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$

and the total derivative as Df or $\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]$

let $f(x,y,z)$ then

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \Rightarrow \nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$

Example. $f(x,y) = x^3 + x^2 y$

$\left.\frac{\partial f}{\partial x}\right|_{(1,1)} =$

step1 $\frac{\partial f}{\partial x} = 3x^2 + 2xy$, treat y as constant

step2 evaluate at point $x=1$ $y=1$

$= 3 + 2 \boxed{= 5}$

thus $\frac{\partial f}{\partial y} = 0 + x^2$ at $x=1, y=1$, $\boxed{= 1}$

The partial derivative at the same point doesn't need to evaluate to the same thing

Examples continued

$x^3+y^3+z^3+6xyz=1$, this is a more difficult example since we can't isolate z in terms of x, y

① Apply $\frac{\partial}{\partial z}$ on both sides of the equation

$$\frac{\partial}{\partial z}(x^3+y^3+z^3+6xyz)=\frac{\partial}{\partial z}1$$

$$=0+0+3z^2+6xy=0$$, great :)

Here we interpretted z as a variable.

Can we interpret z as $z=f(x,y)$? yes we can, though our ∂f computation is different. Consider $\frac{\partial z}{\partial x}$

$$\frac{\partial}{\partial x}(x^3+y^3+z^3+6xyz)=\frac{\partial}{\partial x}1$$, don't try to solve $z=f(x,y)$

$$\hookrightarrow \frac{\partial}{\partial x}(x^3+y^3+z^3(x,y)+6xyz(x,y))=0$$ rewrite $z=z(x,y)$

note since z is a function of x, then I shouldn't have 0.

$$=3x^2+0+\underbrace{3z^2\frac{\partial z}{\partial x}}_{\text{chain rule}}+6xy\frac{\partial z}{\partial x}+6yz=0$$

$$=3x^2+\frac{\partial z}{\partial x}(3z^2+6xy)+6yz=0$$

$$\frac{\partial z}{\partial x}=\frac{-3x^2-6yz}{3z^2+6xy}$$

Moving onto higher order derivatives

Jun 6.

Example $f(x,y) = x^3 + x^2y^3 - 2y^2$, how many 3^{rd} order derivative results can we have? $2 \cdot 2 \cdot 2$? $= 8$

Also, mixed ∂. all agree with each other? Proof!

ex $f_{xy} = f_{yx}$

Then, chain rule with fxn composition. Consider

$u = f(x,y) \quad x = g(s,t) \quad y = h(s,t)$

$u = f(g,h)$, then $\frac{du}{dt} = \frac{du}{dx} \cdot \frac{dx}{dt} + \frac{du}{dy} \cdot \frac{dy}{dt}$

since $~~x~~, y$ both contain t.

then with the second derivative, we get ... & product rule arises

$$\frac{d}{dt}\left(\frac{du}{dt}\right) = \frac{d}{dt}\left(\underline{\frac{du}{dx}} \cdot \frac{dx}{dt} + \underline{\frac{du}{dy} \cdot \frac{dy}{dt}}\right)$$

Now let's discuss tangent planes in multi dimensions

When we restrict a tangent plane in 1 direction via

Tangent Plane$|_{x\text{-direction}}$ = tangent line defined by $\frac{df}{dx}$

Single var

Def. recall tangent line

$f(x)$ be a single var. fxn, the tangent line from $f(x)$ at point (x_0, y_0) is $y - y_0 = \underbrace{f'(x_0)}_{\text{slope}}(x - x_0)$, recall eq'n of a line,

or

$$y = f'(x_0) \cdot x - f'(x_0)x + y_0$$

multi-var

Def. Suppose f has cont. partial derivatives, $f(x,y) = z$, we define the eq'n of the tangent plane to the surface $z = f(x,y)$ at points $p(x_0, y_0, z_0)$ is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

where the span of $(d_1, d_2) = P$.

Ex: Let $z = f(x,y) = 2x^2 + y^2$, find T.P. at $(1,1,3)$

Recall the definition $z - z_0 = f_x'(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$

$\frac{df}{dx} = 4x \quad \frac{df}{dy} = 2y$

$\left.\frac{df}{dx}\right|_{\substack{x=1\\y=1}} = f_x(1,1) = 4 \quad \left.\frac{df}{dy}\right|_{(1,1)} = 2$

$z - 3 = 4(x-1) + 2(y-1)$

$$\underbrace{z = 4x + 2y - 3}_{\text{tangent plane}}$$

z can give you a local approximation around the point $(1,1,3)$, ex. $(1.1, 1.05, 3)$

Next Example Show $f(x,y) = xe^{xy}$ is differentiable at $(1,1,0)$ and find its linearization (ie: linear approx / local approx.)

use it to approx $f(1, 1, -0.1)$

Step 1 Differentiability

check if $f(x,y)$ is diff at $(1,0)$

$f_x = \frac{df}{dx} = xe^{xy}y + e^{xy}$

$f_y = x^2e^{xy}$, thus f_x, f_y exist at $(0,1)$

$$\left.\begin{aligned}&\lim_{(x,y)\to(1,0)} f_x(x,y) = f_x(1,0)\\ &\lim_{xy\to 1,0} e^{xy} + xe^{xy}y\end{aligned}\right\}$$

evaluation of limit is a straight-forward substitution. Now we just evaluate the linear approximation

$z - 1 = f_x(x - x_0) + f_y(y - y_0)$

$\underbrace{f(x,y) \approx x + y}_{\text{local approx}} \qquad f(1.1, -0.1) \approx x + y\Big|_{\substack{x=1.1\\y=-0.1}} = 1.1 + (-0.1) \approx 1$

$\boxed{\text{real value} = 0.98542}$

Interpretation of the q: the first derivative, velocity is perpendicular to the position. Algebraically this means

$r(t) \cdot \vec{r}(t) = [x(t), y(t), z(t)]$

More on Continuity

Def of Continuity from Analysis:

A fn, $f: A \to \mathbb{R}^m$ where $A \in \mathbb{R}^n$ is continuous at $c \in A$ iff for any $\varepsilon > 0$, $\exists$ $\delta > 0$ such that whenever I choose $x \in V_\delta(c) \cap A$ then $f(x) \in V_\varepsilon(f(c))$, otherwise stated x in a delta neighbourhood of C, & $\in A$, then the image under the function f is in the epsilon-neighbourhood of $f(c)$.

Examples of continuous functions

a) the projection of $\mathbb{R}^n \to \mathbb{R}^\varepsilon$

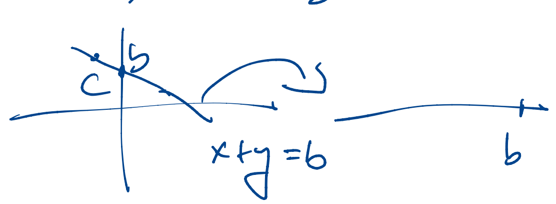
b) the identity map $\mathbb{R}^n \to \mathbb{R}^n$

c) Any constant function

d) $s: \mathbb{R}^{2n} \to \mathbb{R}^n$, by $s(x,y)$, $x, y \in \mathbb{R}^n$, $x+y$

x & y are component vectors

Proofs? proving d) case $n=1$, summing 2 numbers, fix $c \in \mathbb{R}^2$, for $\varepsilon > 0$, set $b := s(c)$ (b to the image of c)



$s^{-1}(b) = \{(x,y) : x+y=b\}$, describes

$y = -x + b$

Lectures on Differentiation

How can we think about the derivative? By def,

① $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(x)$ "the slope of the line between x & c as x approaches c"

or

② f is diff. @ c iff $\exists$ a real number $f'(c) \in \mathbb{R}$ s.t.

$$\lim_{h \to 0} \frac{f(c+h) - f(c) - f'(c) \cdot h}{h} = 0$$

otherwise stated: "we can approximate $f(c+h)$ using $f(c)$"

recall approximation in single var.

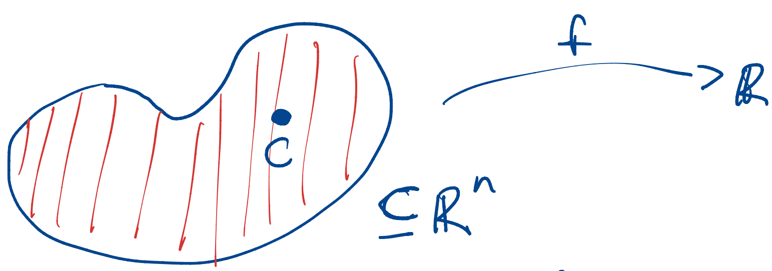
$$f(c+h) \approx f(c) + f'(c) \cdot h + h \cdot (\text{higher order}) \; (h.o.t.)$$

linear approx. formula. some exceptions such as $e^{-\frac{1}{|x|^2}}$

"The derivative approximates the fxn at an infinitesimally close neighborhood of the point in a linear fashion"

Now we can apply to multivariable

Consider a fxn defined on $\mathbb{R}^n$



f takes a subset of $\mathbb{R}^n$ to $\mathbb{R}$

Let's restrict this fxn with a line through c in the direction of e_1

We do that with $f(c + h \cdot e_1)$, if c is a pt. in our domain then $c + h \cdot e_1 \in$ domain if h is small enough

$$f(c + h \cdot e_1) \approx f(c) \underbrace{+ \partial_1 f(c) \cdot h e_1}_{} + h.o.t.$$

here we use ∂_1 to represent derivative opposed to f' to communicate the derivative of the function restricted to the first coordinate

We may also approximate the fn in the other direction e_2

$f(c + he_2) \approx f(c) + \partial_2 f(c) \cdot he_2 + h.o.t$,

Lets write our second approximation with term k instead of h

$$f(c + ke_2) \approx f(c) + \partial_2 f(c) \cdot ke_2 + h.o.t$$

Great so far we applied our single - var. interpretation of the derivative to multivariable, two directions e_1 & e_2. Now we get to the essence of this topic

Now consider an arbitrary sum of these two vectors

$he_1 + ke_2$, this is within our domain given h, k small enough

$$f(c + he_1 + ke_2) \approx f(c) + \underbrace{\partial_1 f(c) \cdot he_1 + \partial_2 f(c) \cdot ke_2}_{} + h.o.t$$

$$[\partial_1 f(c) \;\; \partial_2 f(c)] \begin{bmatrix} h \\ k \end{bmatrix} = \longrightarrow$$

This leads us to a new def.

Def$_n$ Let $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$ and let $c \in \mathbb{R}^n$.

f is differentiable at c iff. there exists linear transformation $D_c f : \mathbb{R}^n \to \mathbb{R}^m$ s.t. $\displaystyle\lim_{h \to 0} \frac{|f(c+h) - f(c) - (D_c f)(h)|}{|h|} = 0$

$D_c f$ is known as the differential of f at c.

If f is defined only on some open set containing c, a similar def. can be made

Using the basis above, we can write a Jacobian associated with the differential

$$D_c f = \begin{bmatrix} \langle e_1, (D_c f)(e_1) \rangle & \langle e_1, \overbrace{(D_c f) e_2}^{\text{in } \mathbb{R}^m} \rangle & \cdots \\ \langle e_2, (D_c f)(e_1) \rangle & \cdots & \\ \vdots & & \\ \langle e_m, D_c f(e_1) \rangle & & \end{bmatrix}$$

Differentiation

Q: Let $A \subset \mathbb{R}^n$, $f: A \to \mathbb{R}^n$. Show that if $f'(a,u)$ exists then $f'(a, cu)$ exists & equals $c\, f'(a;u)$

A: Show that differentiation is distributable? Or that a constant under differentiation is distributable.

If $f'(a,u)$ exists then by def of limit

$$\frac{[f(a+tu) - f(a)]}{t}$$ exists, replacing u with cu, we get

$$\frac{(f(a) + t(cu)) - f(a)}{t} \qquad c \cdot \frac{[f(a+tcu) - f(a)]}{t}$$

$= c * f'(a;u)$. Therefore, $f'(a;cu)$ exists & equals $cf'(a;u)$

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by setting $f(0) = 0$ and

$$f(x,y) = \frac{xy}{x^2+y^2} \quad \text{if } (x,y) \neq 0$$

a) for which vectors $u \neq 0$ does $f'(0,u)$ exist? Evaluate

the directional derivative of f at 0, in direction u is

$$f'(0;u) = \lim_{t \to 0} \frac{f(tu) - f(0)}{t} \quad \Leftarrow \text{remember}$$

setting $u = (u_1, u_2)$ we have

$$f(tu) = \frac{tu_1 tu_2}{(tu_1)^2 + (tu_2)^2} = \frac{u_1 u_2}{u_1^2 + u_2^2}$$

thus $f'(0;u) = \lim_{t \to 0} \frac{u_1 u_2}{u_1^2 + u_2^2} = \frac{u_1 u_2}{u_1^2 + u_2^2}$

this exists for all $u \neq 0$, thus $f'(0;u)$ exists for all $u \neq 0$

b) do $D_1 f$ & $D_2 f$ exist at 0?

$D_1 f(0) = f'(0, (1,0))$ and $D_2 f(0) = f'(0, (0,1))$

both $= 0$ & therefore exists

c) is f differentiable at 0?

~~where is this question going?~~ well, since the limit exists & both directional derivatives exist, the limit exists.

d) is f continuous at 0?

Given that $f(0) = 0$ & the numerator of $f(x,y)$ tends to zero faster than the denominator as $(x,y) \to (0,0)$

f is continuous at 0.

Continuously Differentiable Functions

1. Recall Mean Value Theorem of single variable analysis

MVT: If $\phi : [a,b] \to \mathbb{R}$ is continuous at each pt. of the closed interval $[a,b]$, and diff. at each point on the open interval (a,b), then $\exists$ a point c of (a,b) such that

$$\phi(b) - \phi(a) = \phi'(c)(b-a).$$

2. We know the mere existence of partial derivatives does not imply differentiability. By imposing a mild :) additional condition that these partial derivatives be continuous, then differentiability is assured!! ! wow

3. Introducing a new theorem

Let A be open in $\mathbb{R}^m$. Suppose partial derivatives $D_i f_i(x)$ of component functions of f exist at each point x of A & are continuous on A.

Then f is differentiable at each point of A.

Functions satisfying the hypotheses of this theorem are continuously differentiable, or of class C^1, on A.

4. Proof Let's build some theorems up to the proof.

Theorem 1 Let $A \subset \mathbb{R}^m$; let $f : A \to \mathbb{R}^n$. If f is differentiable at a, then all the directional derivatives of f at a exist, and

$$f'(a;u) = Df(a) \cdot u$$

Proof of theorem 1

Let $B = Df(a)$ & $h = tu$. by def. of differentiability

$$① \quad \frac{f(a+tu) - f(a) - B \cdot tu}{|tu|} \to 0, \text{ as } t \to 0$$

as $t \to 0$ from pos. values, we multiply ① by $|u|$ to conclude that

$$\frac{f(a+tu) - f(a)}{t} - B \cdot u \to 0 \quad \text{as } t \to 0$$

as desired. If t approaches from negative values, we may multiply by $-|u|$ to conclude the same thing.

Thus $f'(a;u) = Df(a) \cdot u$ as stated

$\square$

Munkres pg. 54 example 3, show all directional derivatives exist but the function is not differentiable at 0. Let $f(0) = 0$ and

$$\text{given } f(x,y) = \frac{x^2 y}{x^4 + y^2} \quad \text{if } (x,y) \neq 0.$$

$$u = \begin{bmatrix} h \\ k \end{bmatrix}$$

recall the def. and let $u \neq 0$,

$$\frac{f(a+tu) - f(a)}{t} = \frac{f(0+tu) - f(0)}{t} = \frac{f(tu) - f(0)}{t}$$

$$= \frac{(th)^2 k}{(th)^4 + (tk)^2} \cdot \frac{1}{t} = \frac{h^2 k}{t^2 h^4 + k^2} \quad \text{so that}$$

$$f'(0;u) = \begin{cases} h^2/k & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases} \quad \text{and thus exists for all } u \neq 0$$

But the fxn is not diff. at 0.

If a fxn $g: \mathbb{R}^2 \to \mathbb{R}$ was diff. at 0, then $Dg(0)$ has form $\begin{bmatrix} a \\ b \end{bmatrix}$ $u = \begin{bmatrix} h \\ k \end{bmatrix}$, 1×2 matrix, $g'(0;u) = ah + bk$ which is a linear fxn. This leads us to the next theorem.

Theorem 2 Let $A \subset \mathbb{R}^m$; let $f: A \to \mathbb{R}^n$. If f is diff. at a then f is cont at a.

Theorem 4 Let $A \subset \mathbb{R}^m$; let $f: A \to \mathbb{R}^n$. Suppose A contains neighborhood of a. Let $f_i: A \to \mathbb{R}$ be i^{th} component function s.t.

$$f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}$$

a) function f differentiable at a iff each component is diff. at a

b) If f diff at a, its derivative is the $n \times m$ matrix whose i^{th} row is the derivative of f_i

By theorem 4

$$Df(a) = \begin{bmatrix} Df_1(a) \\ \vdots \\ Df_n(a) \end{bmatrix}$$

where $Df(a)$ is the matrix whose entry in row i & column j is $D_j f_i(a)$

Proof

Consider $\dfrac{f(a+h) - f(a) - B \cdot h}{|h|} = F(h)$ which is defined for $0 < |h| < \varepsilon$

for some ε. Now $F(h)$ is a column matrix of size n by 1

Its i^{th} entry satisfies equation

$$F_i(h) = \frac{f_i(a+h) - f_i(a) - (\text{row } i \text{ of } B) \cdot h}{|h|}$$

The Chain Rule

Let $A \subseteq \mathbb{R}^p$, and $B \subseteq \mathbb{R}^n$. let $c \in A$. and functions

$$\mathbb{R}^m \xleftarrow{g} B \xleftarrow{f} A$$

with f diff. at c

and g diff at $f(c)$, then $g \circ f$ is diff. at c

and

$$\begin{tikzcd} & \mathbb{R}^n \arrow[dl, "D_{f(c)} g"'] & \\ \mathbb{R}^m & & \mathbb{R}^p \arrow[ul, "D_c f"'] \arrow[ll, "D_c\, g \circ f\ ?"] \end{tikzcd}$$

$$\boxed{D_c(g \circ f) = D_{f(c)} g \cdot D_c f}$$

This is the chain rule. Resulting lin. transformation equals the matrix multiplication of $D_{f(c)} g \cdot D_c f$

Think back to single var. chain rule. Interpreting the above linear operators as slopes at that point, multiplying two slopes is the same as multiplying two linear transformations

def. of derivative of $g \circ f$ at c

Intuition $\lim_{h \to 0} \dfrac{|g(f(c+h)) - g(f(c)) - D_c(g \circ f)(h)|}{|h|} = 0$

Let's approx. the first term using our results from before

$$g(f(c+h)) - g(f(c)) \approx g(f(c) + (D_c f)(h)) - g(f(c)) + h.o.t$$

$$\approx g(f(c)) + D_{f(c)} g((D_c f)(h)) - g(f(c)) + h.o.t$$

$$= (D_{f(c)} g \circ D_c f)(h)$$

Chain Rule example

$$(0,\infty)\times\mathbb{R} \xrightarrow{f} \mathbb{R}^2 \xrightarrow[i=1,2]{\pi_i} \mathbb{R}$$

$$(r,\theta) \longmapsto (\underbrace{r\cos\theta}, r\sin\theta)$$

$$f_1 = \pi_1 \circ f$$

$$id \times \cos(-1)$$

$$\begin{tikzcd}
 & (0,\infty)\times\mathbb{R} \arrow[dl, "P"'] & \\
\mathbb{R} & & (0,\infty)\times\mathbb{R} \arrow[ul] \arrow[ll, "f_1"]
\end{tikzcd}$$

$(r, \cos\theta)$ $\quad r\cos(\theta) \quad (r,\theta)$

Applying chain rule to get the commutative diagram at $(\mathbb{R}, \theta)$

$$\begin{tikzcd}
 & \mathbb{R}^2 \arrow[dl, "D_{(R,\cos\theta)}P"'] & \\
\mathbb{R} & & \mathbb{R}^2 \arrow[ul, "D_{(\mathbb{R},\theta)}(id \times \cos(-1))"'] \arrow[ll, "D_{(\mathbb{R},\theta)} f_1"]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathbb{R}^2 \arrow[dl] & \\
\mathbb{R} & & \mathbb{R}^2 \arrow[ul] \arrow[ll]
\end{tikzcd}$$

$D_R id \times D_\theta \cos = id \times \sin\theta$

$P(x,y) = xy$, $\quad D_{(a,b)}P(u,v) = bu + av$

$$\begin{tikzcd}
 & (u, \sin\theta \cdot v) \arrow[dl, "D_{(\mathbb{R},\cos\theta)}P"'] \\
\cos\theta \cdot u + \mathbb{R}\sin\theta \cdot v & (u,v) \arrow[u]
\end{tikzcd}$$

Okay, this is all pretty hard, thus we introduce the concept of partial derivatives

Chain Rule Example

$f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x,y) = (xy, 2x)$ & $g : \mathbb{R}^2 \to \mathbb{R}^3$ $g(a,b) = ((b-a), a^2, b)$

computing $g \circ f(x,y) = g(xy, 2x)$

$$= (2x - y, \; x^2y^2, \; 2x)$$

Then we can take the derivative, which by the Jacobian matrix should be a 3×2 matrix

$$D(g \circ f)(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial v_1}, & \frac{\partial f_1}{\partial v_2} \\ \frac{\partial f_2}{\partial v_1}, & \frac{\partial f_2}{\partial v_2} \\ \frac{\partial f_3}{\partial v_1} & \frac{\partial f_3}{v_2} \end{bmatrix} = \begin{bmatrix} 2-y & x \\ 2xy^2 & 2x^2y \\ 2 & 0 \end{bmatrix}$$

$$Df(x,y) = \begin{bmatrix} y & x \\ 2 & 0 \end{bmatrix} \qquad Dg(a,b) = \begin{bmatrix} -1 & 1 \\ 2a & 0 \\ 0 & 1 \end{bmatrix}$$

By chain rule

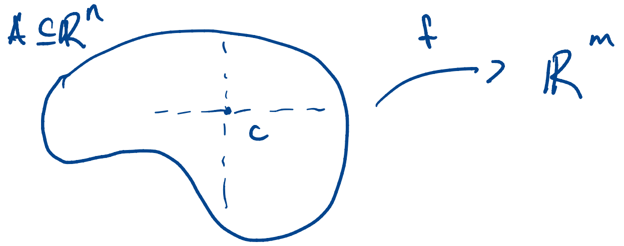
$Dg(f(x,y))\, Df(x,y)$

$$= \begin{bmatrix} -1 & 1 \\ 2xy & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y & x \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2-y & \cancel{x} \\ 2xy^2 & 2x^2y \\ 2 & 0 \end{bmatrix} = \text{same as above}$$

Proof of chain Rule in multivariable calc?

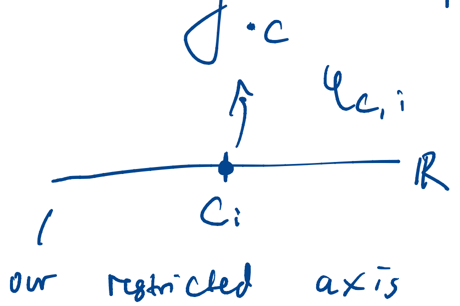
Suppose we have a function on some domain (open) $A \subseteq \mathbb{R}^n$

This fn takes values on some co-domain $\mathbb{R}^m$



- Let's look at a one-dimensional piece of f along an axis

Imagine restricting f on one of these axis, thus defining a function from $\mathbb{R}$ to $\mathbb{R}^m$



$h \in \mathbb{R} \mapsto \varphi_{c;i}(h) := (c_1, \dots c_{i-1}, h, c_{i+1}, \dots c_n)$

the function $\varphi_{c;i}$ takes an axis (a single one in $\mathbb{R}^n$) and places it on A, s.t. c is the origin

we're essentially shifting our coordinate basis

$h \in \mathbb{R} \mapsto \varphi_{c;i}(h) := (c_1, \dots c_{i-1}, h, c_{i+1}, \dots c_n)$

this function sends to the value c, except the i^{th} coordinates which is h.

and i is the direction of the axis

Varying h gives us the line through c in direction i

then composing f & $\varphi_{c;i}$, the resulting fn is a single variable input mapping into $\mathbb{R}^m$

for example, taking $m=1$, this is just an ordinary fn taking $\mathbb{R}$ to $\mathbb{R}$.

It is natural to then ask, does the derivative of $f \circ \ell_{c;i}$ exist?

If $f \circ \ell_{c;i}$ is differentiable at c_i then denote its derivative by $(\partial_i f)(c) := (f \circ \ell_{c;i})'(c_i)$

Note We've made no assumptions of the differentiability of f.

Another way to think about it

$\Rightarrow$ If we know the partial derivative exists, it doesn't necessarily mean the function is differentiable at that point

Theorem If $f: A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$ open & $c \in A$. If additionally

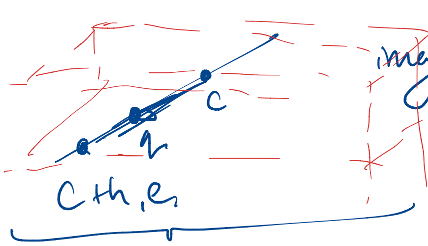
i) $\exists$ $U \subseteq \mathbb{R}^n$ open s.t. $c \in U$

ii) $\partial_i f_j$ exist and are continuous on U

then f is differentiable at c.

Proof our goal is to show $\lim_{h \to 0} \frac{|f(c+h) - f(c) - (D_c f)(h)|}{h} = 0$

we will rewrite $f(c+h) - f(c)$ using Mean Value theorem repeatedly



this is a rectangle contained in our set

imagine we restrict the fn to the interval $[c_1, c_1+h_1]$, by MVT $\exists$ some vector $\vec{q}_1 \in \mathbb{R}^n$ such that $\underbrace{f(\vec{P}_1) - f(\vec{P}_0) = h_1 \cdot \partial_1 f(\vec{q}_1)}_{MVT}$

MVT & Continuously Diff. Fxns

If $\phi : [a,b] \to \mathbb{R}$ is cont. @ each pt. of $[a,b]$, and diff. at each pt. of (a,b), then $\exists$ point c of (a,b) such that $\phi(b) - \phi(a) = \phi'(c)(b-a)$ $\square$

Theorem and Exercise

Let A be open in $\mathbb{R}^m$. Suppose partial derivatives $D_j f_i(x)$ of the component fxns of f exist at each pt x of A & are continuous on A. Then f is diff at each point of A.

PROOF Let a be a point of A. Given ϵ the parts der. $D_i f(x)$ exist & are cont. for $|x-a| < \epsilon$. We wish to show f is diff. at a

Step 1 Let h be a point on a s.t. $0 < |h| < \epsilon$. let $h_1, \ldots h_m$ be components of h. Consider the sequence of points of $\mathbb{R}^m$

$$P_0 = a$$

$$P_1 = a + h_1 e_1$$

$$P_2 = a + h_1 e_1 + h_1 e_2 \quad \ldots$$

$$\vdots$$

$$P_m = a + h_1 e_1 + \ldots h_m e_m = a + h$$

Previously, neighbourhood approximations and tangent planes

① Let $f(x,y)$ be a 2 var. fxn. The tangent plane for $f(x,y)$ at a point (a,b) is : $[z = f(x,y)]$

$$z - z_0 = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$, where $z_0 = f(a,b)$

② An alternative definition for the derivative

$f(x,y)$ is diff. at a,b if f_x, f_y exists near a,b and they are continuous

③ Chain Rule $z = (x,y)$ and $x = f(s,t)$, $y = f(s,t)$

$$\frac{dz}{dt}, \quad \frac{dz}{ds}, \quad \frac{d^2z}{d^2\cancel{t}}, \quad \frac{\partial^2 z}{d^2 s}$$

Lagrange Multiplier

Find the abs. max / min of $f(x,y) = x^2 + (y-1)^2 + 1$ on the set $\{(x,y) \mid x^2 + \frac{y^2}{4} \le 1\}$

Solution: we will present 2 possible ways to solve.

First lagrange

Set-up: the $\le$ sign means we split our constraint for analysis into two parts

① strictly less $<$ and ② equal $=$ (boundary of ellipse)

① check critical points, without boundary case

$\nabla f = 0 \Rightarrow$ critical point

$\frac{\partial f}{\partial x} = 2x = 0$, $\frac{\partial f}{\partial y} = 2(y-1) = 0$, $x=0$, $y=1$

This is a critical point inside the ellipse

$f(0,1) = 0 + 0 + 1 \boxed{=1}$ candidate for max / min (extreme val.)

② Lagrange method to find boundaries

$\nabla f = \lambda \nabla g$, $g = 1 \rightarrow g(x,y) = x^2 + y^2/4$

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} = \lambda 2x \\ \frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} = \lambda \frac{y}{2} \end{cases}$$, giving us a system of equations

$2x = \lambda 2x$ or $\lambda = 0$, $x = 0$, $y = \pm 2$

$2y - 2 = \lambda \frac{y}{2}$

$x^2 + y^2 = 1$, which gives us $\lambda = 1$, $y = 4/3$, $x = \pm\frac{\sqrt{5}}{3}$

evaluating $f\left(\pm\frac{\sqrt{5}}{3}, \frac{4}{3}\right)$ to get $\frac{15}{9}$. $f(0,2) = 2$, $f(0,-2) = 10$

$f(0,1) = 1$. thus $f(0,-2)$ is our max pt and

↳ is our minimum point (0 center).

Find min $f(x, y, z) = (x-2)^2 + y^2 + (z+3)^2$, constraint $x+2y+z=8$

solving using lagrange multiplier

$\nabla f = \lambda \nabla g$, $g = 8$ where $g = x + 2y + z$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \lambda \frac{\partial g}{\partial x} \Rightarrow & 2x &= \lambda x \\ \frac{\partial f}{\partial y} &= \lambda \frac{\partial g}{\partial y} & 2y &= \lambda 2 \\ \frac{\partial f}{\partial z} &= \lambda \frac{\partial g}{\partial z} & 2z+3 &= \lambda \\ & & x+2y+z &= 8 \end{aligned}\right\}$$

giving us a very solvable system of equations (linear) :-)

We could also solve with 2nd deriv. test.

Munkres , Miklavcic 3_B

Q Extreme values of $f(\vec{x})$ with constraint $g(\vec{x}) = k$

on $G(\vec{x}) = 0$, $G(\vec{x}) = g(\vec{x}) - k$

We must solve the following system

$$\nabla f(\vec{x}) = \lambda \nabla g(\vec{x})$$
$$g(\vec{x}) = k$$

$\Rightarrow$ solution $\{\vec{x}_i\}_{i \in I} \Rightarrow f(\vec{x}_i)_{i \in I}$ giving you the extreme values

Alt. solution version 2

$H(\vec{x}, \lambda) = f(\vec{x}) - \lambda G(\vec{x})$, some new fcn H

$\nabla H(\vec{x}, \lambda) = 0 \Rightarrow \{\vec{x}_i, \lambda_i\}_{i \in I}$ are critical points

These two solutions are equivalent

Two constraint optimization problems

ex, find max/min $f(\vec{x})$ given ① $g(\vec{x}) = k$, $h(\vec{x}) = c$

$$\underbrace{\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h, \quad g = k, \quad h = c}_{\text{solve this system}}$$

has 5 variables, $x, y, z, \lambda_1, \lambda_2$

using a computer :) , you may get $\{\vec{x}_i\}$ gives you the points & $\{f(\vec{x}_i)\}_{i \in I}$ gives you the max/min

or using v. 2 $L(\vec{x}, \lambda_1, \lambda_2) = f(\vec{x}) - \lambda_1 G(\vec{x}) - \lambda_2 H(\vec{x})$

where $G(\vec{x}) = g(x) - k$, $H(x) = h(x) - c$

$$\nabla L = 0$$

why does gradient $L = 0$ give us the sol'n

By def. $\nabla L = \nabla f - \nabla(\lambda_1 G(x)) - \nabla \lambda_2 (H(\vec{x}))$, meaning then if we take the partial derivative, we get.

$$\frac{\partial_i f}{\partial x_i} - \frac{\partial}{\partial x_i}(\lambda_1 G(\vec{x})) - \frac{\partial}{\partial x_i}(\lambda_2 H(\vec{x})) = 0 \quad \forall i$$

$$\Rightarrow \frac{\partial_i f}{\partial x_i} - \lambda_1 \frac{G(\vec{x})}{\partial x_i} - \lambda_2 \frac{H(\vec{x})}{\partial x_i} = 0,$$

Note $\frac{\partial G(\vec{x})}{\partial x_i} = \frac{\partial g(x)}{\partial x_i}$, $\frac{\partial H(\vec{x})}{\partial H_i} = \frac{dH(\vec{x})}{\partial x_i}$

$$\Rightarrow \frac{\partial_i f}{\partial x_i} = \lambda_1 \frac{G(\vec{x})}{\partial x_i} + \lambda_2 \frac{G(\vec{x})}{\partial x_i},$$

which is the same condition as

$$\nabla f = \nabla \lambda_1 g + \nabla \lambda_2 h$$

For $\frac{\partial}{\partial \lambda_1}$, by idea of critical points, on L.

$$\frac{\partial L}{\partial \lambda_1} = \frac{\partial}{} (f(\vec{x}) - \lambda_1 G(\vec{x}) - \lambda_2 H(\vec{x})),$$

we get zero on the f & H components, since they do not contain λ_1. thus we are only concerned with the effect on the second term

$$= \frac{\partial}{\partial \lambda_1}(-\lambda_1 G(\vec{x})) = -G(\vec{x}) = 0$$

$$= -g(\vec{x}) - k = 0, \Rightarrow g(\vec{x}) = k,$$

then performing $\frac{\partial}{\partial \lambda_2}$ to get $h(\vec{x}) = c$

Alright let's do an example

Find max. of $f(x,y,z) = x+2y+3z$ cut by 2

↳ this describes a surface in $\mathbb{R}^3$.

↳ other planes, $x-y+z=1$, and cylinder $x^2+y^2=1$

we are finding a resulting intersecting curve, & its max

① Let
$$\left.\begin{aligned} g(x,y,z) &= x-y+z \\ h(x,y,z) &= x^2+y^2 \end{aligned}\right\}$$ Introducing new fns., for $g=1$, $h=1$

Apply $\nabla f = \lambda_1 \nabla g + \lambda_2 \nabla h$ given $g=1$, $h=1$, solve this system

gradient equal means PDs agree

$$\left.\begin{aligned} f_x &= \lambda_1 g_x + \lambda_2 h_x \\ f_y &= \lambda_1 g_y + \lambda_2 h_y \\ f_z &= \lambda_1 g_z + \lambda_2 h_z \end{aligned}\right\} \Rightarrow \begin{aligned} 1 &= \lambda_1 + \lambda_2(2x) \quad , \frac{\partial f}{\partial x} \\ 2 &= \lambda_1(-1) + \lambda_2(2y) \quad \frac{\partial f}{\partial y} \\ 3 &= \lambda_1(1) \quad \frac{\partial f}{\partial z} \\ x-y+z &= 1 \\ x^2+y^2 &= 1 \end{aligned}$$

Then we have to solve the 5 variable system, which we won't do here as it is tedious

Now, what about n constraints?

$\{g_1(\vec{x}) \ldots g_n(\vec{x}) = k_n\}$ constraints, giving us

$$\nabla L(\vec{x}, \lambda_1, \lambda_2, \ldots \lambda_n) = 0 \Leftrightarrow \frac{\partial f}{\partial x_i} = \frac{\partial}{x_i}(\lambda_1 g_1 + \lambda g_2 + \ldots \lambda_n g_n)$$

$g_i = k_i$, $\forall i \in [1, n]$. this is a very big system!

Week 7 - Local linearity & Differential, Ch 14.3

A note on studying, review learning objectives. when studying, watching lectures, doing tests, etc, continuously relate back your learning objectives with the material you're working on

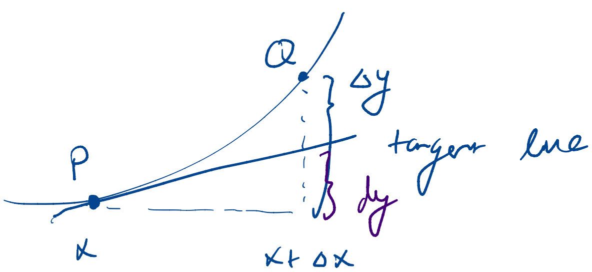
Learning objectives

14.3 A. - use P.D. to find tangent plane to a surface.
↳ what makes a good approximation?

B. - use linear localization to estimate M.V.F nearby given point

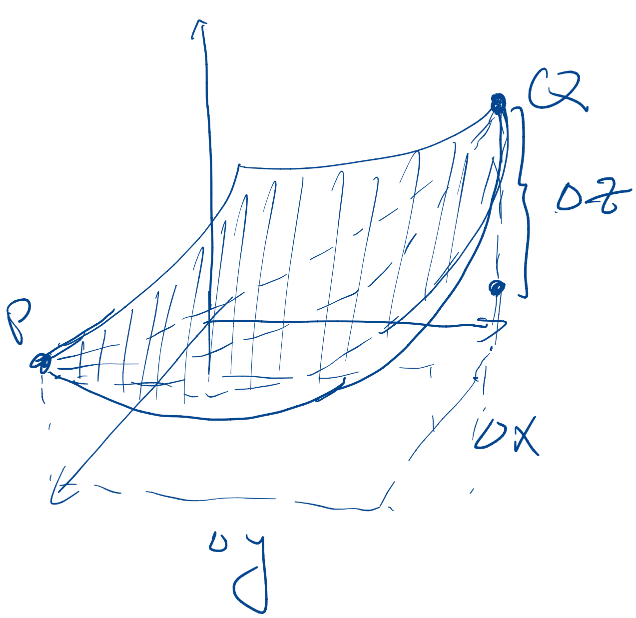
C - find differentials associated with MVF

consider 2D for now - definitions, derivative & differential



the function gives us Q, given dy we may estimate y for Q if we are sufficiently close to Q

$$\boxed{\text{As } \Delta x \to 0, \ \Delta y \simeq dy}$$



- thus for a dz, we may approximate Δz if P is close to Q, though now we are moving through x & y

Consider now in 3D, we may obtain Δz with $\Delta z_x + \Delta z_y$.

If we move along the x direction we are parallel to the xz plane

$$\Delta z \approx \underbrace{dz_x + dz_y}_{dz} = f_x dx + f_y dy$$

Ex Diff of $z = ye^{x^2 - y^2}$

$$dz \approx dz_x + dz_y = f_x dx + f_y dy$$

$$z_x = ye^{x^2-y^2} 2x = 2xye^{x^2-y^2}$$

$$z_y = e^{x^2-y^2} + ye^{x^2-y^2}(-2y) = (1-2y^2)e^{x^2-y^2}$$

$$dz = 2xye^{x^2-y^2} + (1-2y^2)e^{x^2-y^2}$$

uses a tangent plane to approximate a change in height.

Ex $\ln(2x-y) + e^{2xz}$

$$F = F_x + F_y + F_z = \frac{2}{2x-y} + 2ze^{2xz} - \frac{1}{(2x-y)}$$

$$F_x = \frac{2}{2x-y} + 2ze^{2xz}$$

$$F_y = \frac{-1}{2x-y}$$

$$F_z = 2xe^{xz}$$

The Chain Rule for Multivariable Functions

Given Δt, "x" & "y" responds by changing also (Δx & Δy) & both Δx & Δy cause a change in w, Δw

Δw = how much x changes w + how much y changes w

= R.O.C$_x \cdot \Delta x$ + R.O.C in $y \cdot \Delta y$

$$\Delta w = \frac{\partial w}{\partial x} \cdot \Delta x + \frac{\partial w}{\partial y} \cdot \Delta y$$

$$\frac{\Delta w}{\Delta t} = \frac{\partial w}{\partial x} \cdot \frac{\Delta x}{\Delta t} + \frac{\partial w}{\partial y} \cdot \frac{\Delta y}{\Delta t} \quad , \Delta t \to 0$$

$$\lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \lim_{\Delta t \to 0} \frac{\partial w}{\partial w} \cdot \frac{\Delta x}{\Delta t} + \lim_{\Delta t \to 0} \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

recall $\lim_{\Delta b \to 0} \frac{\Delta a}{\Delta b} = \frac{da}{db}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} \quad , \text{chain rule}$$

Some proofs may use h

$f(g(x))$, derivative $= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$

Example $w = x^2 y - xy^3$, $x = \cos t$, $y = e^t$, Find $\left.\frac{dw}{dt}\right|_{t=0}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt}$$

$$\frac{dw}{dt} = (y2x - y^3)(-\sin t) + (x^2 - 3xy^2)\, e^t$$

@ $t=0$, $x=1$, $y=1$

$$\left.\frac{dw}{dt}\right|_{\substack{t=0\\x=1\\y=1}} (1(2)(1) - 1)(-\sin 0) + (1^2 - 3(1)(1)^2) \cdot e^0 = -2$$

Further Extending, more PDE questions

① Euler's PDE: $xf_x + yf_y = f$, $z = f(x,y)$. Which fns f any satisfy this equation

a) x^2y^2 c) $2x+3y+1$

b) $2x+3y$ d)

Functions between subsets of Euclidean spaces

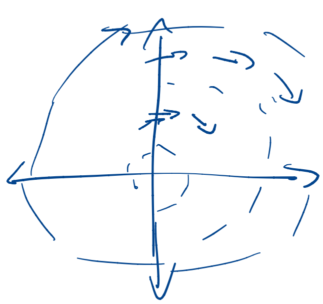
Given

$$\begin{array}{ccc} & B \subseteq \mathbb{R}^n & \\ \swarrow f & & \nwarrow g \\ \mathbb{R}^m & \xleftarrow{f \circ g} & A \subseteq \mathbb{R}^p \end{array}$$

For the diagram on the left the domain of g is A and the codomain of g is B

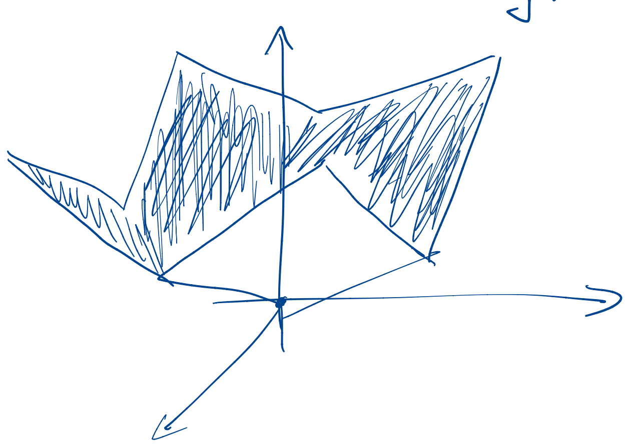
Suppose we have a vector field graph given by

$\boxed{\begin{matrix} u = \{y\} \\ v = \{-x\} \end{matrix}}$ what would such a graph look like?



where the length of a vector at a point depends on the xy values

What about $f(x,y) = |x| + |y|$



We've established functions mapping subsets of different dimensions to different dimensions.

We eventually want to discuss the continuity of such functions.

Now we discuss the limit of functions

Def Let $A \subseteq \mathbb{R}^n$, c be a limit point of A and $f: A \to \mathbb{R}^m$.
A limit of f as x approaches c is a vector $L \in \mathbb{R}^m$ satisfying: for any $\varepsilon > 0$ $\exists$ a $\delta > 0$ s.t. $f(x) \in V_\varepsilon(L)$
$\forall x \in V_\delta(c) \cap A$

Def of the derivative

Given Capacity of signal channel

$$C = k \cdot \log(1 + S/N) \quad \text{and} \quad s(t) = 4 + \cos(4\pi t)$$
$$N(t) = 4 + \sin(2\pi t)$$

find
$$\frac{dC}{dt} = \frac{dC}{dS} \cdot \frac{dS}{dt} + \frac{dC}{dN} \cdot \frac{dN}{dt}$$

$$\frac{dC}{dS} = k \cdot \left(\frac{1}{1 + S/N}\right)\left(\frac{1}{N}\right) \qquad \frac{dC}{dN} = k\left(\frac{1}{1+S/N}\right)\left(\frac{-S}{N^2}\right)$$

$$\frac{dS}{dt} = -4\pi \sin(4\pi t) \qquad \frac{dN}{dt} = 2\pi \cos(2\pi t)$$

$$\frac{dC}{dt} = k\left(\frac{1}{1+\frac{S}{N}}\right)\left(\frac{1}{N}\right)\left(-4\pi \sin(4\pi t)\right) + k\left(\frac{1}{1+\frac{S}{N}}\right)\left(\frac{-S}{N^2}\right) 2\pi \cos(2\pi t)$$

Practice

~~Let~~ $f(x,y) = x^2 + xy$. suppose $x(t) = ts$ & $y(t) = 2t+1$

compute $\frac{df}{dt}$.

$$\frac{df}{dx} = 2x + y \qquad \frac{df}{dy} = x \qquad \frac{dx}{dt} = s \qquad \frac{dy}{dt} = 2$$

$$(2x+y)s + x(2) = (2(ts) + 2t+1)s + 2(ts)$$
$$= 2ts^2 + 2ts + s + 2ts = 4ts^2 + 4ts + s$$

Practice 2

$f(x,y) = x\sin(x^2+5y^2)$ suppose $x(s,t) = t^2 s + s$ & $y(s,t) = te^{st}$

evaluate $\frac{df}{ds}$ at $(s,t) = (0,1)$

$$\frac{df}{ds} = \frac{df}{dx}\frac{dx}{ds} + \frac{df}{dy}\frac{dy}{ds}$$

$x(0,1) = 0 + 0$

$y(0,1) = 1e^{0 \cdot 1} = 1$

$= \left(\sin(x^2+5y^2) \cos(x^2+5y^2)\, 2x\right)(t^2) + 1 + \cos(x^2+5y^2)\, 10y$

$= \left(\sin(x^2+5y^2) + x\cos(x^2+5y^2)\, 2x\right)(t^2+1) + x\cos(x^2+5y^2)\, 10y\,(te^{st})t$

$= (\sin(0+5) + 0)2 + 0 = \sin(5)$

$w = x^3 + y^3 \quad , \quad x = u^2 + v^2 \quad , \quad y = 2uv$

$\frac{dw}{du} = \frac{dw}{dx} \cdot \frac{dx}{du} + \frac{dw}{dy} \cdot \frac{dy}{du} \quad , \quad \frac{dw}{dv} = \frac{dw}{dx}\frac{dx}{dv} + \frac{dw}{dy} \cdot \frac{dy}{dv}$

$= (3x^2)(2u) + (3y^2)(2v) \qquad\qquad = (3x^2)(2v) + (3y^2)(2u)$

① $w = x \tan^{-1}(yz) \quad , \quad x = u^{1/2} \quad , \quad y = e^{-2v}$

② $w = x^2 y + y^2 z^3 \quad , \quad x = r\cos(s) \quad , \quad y = r\sin(s) \quad , \quad z = re^s$

Find $\left.\frac{dw}{ds}\right|_{r=1,\ s=0} \qquad x=1 \quad , \quad y=0 \quad , \quad z=1$

Implicit Differentiation

$\boxed{\frac{dy}{dx} = \frac{-f_x}{f_y}}$, example : $2x^2 + 3(xy)^{1/2} - 2y - 4 = 0$, $y = g(x)$

find $\frac{dy}{dx}$

$f_x = 4x + \frac{3}{2}(xy)^{-1/2} y$, $f_y = \frac{3}{2}(xy)^{-1/2} x - 2$

More on the Derivative

$f: \mathbb{R}^n \to \mathbb{R}^m$, simple case $f: \mathbb{R}^1 \to \mathbb{R}^1$,

① Linear transformations in $\mathbb{R} \to \mathbb{R}$ are $x \mapsto$

$f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x_0 \in \mathbb{R}^n$ if

$$\lim_{x \to x_0} \frac{\| f(x) - f(x_0) - T(x - x_0) \|}{\| x - x_0 \|} = 0$$

where the double bars

Directional Derivatives

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{\underbrace{\|(\Delta x, \Delta y)\|}}$$

$$\longrightarrow \sqrt{h^2 u_1^2 + h^2 u^2} = \sqrt{h^2(u_1^2 + u_2^2)}$$
$$= h \cdot \sqrt{u_1^2 + u_2^2}$$
$$= h \cdot 1$$

$$\lim_{h \to 0} \frac{f(x + h u_1, y + h u_2) - f(x,y)}{h}$$

$D_{\hat{u}} f(x,y) = f_x u_1 + f_y u_2$, PDE multiplied by directional vector where u is a unit vector.

Example Find derivative of $f(x,y) = x^3 - 2x^2 + y^3$ @ $P(1,2)$ in the direction of the vector that makes an angle of $\theta = \pi/6$ with the x-axis

Finding the unit vector : given a positive angle from the x-axis, our equation for $\hat{u}$ is given by $\hat{u} = \cos\theta \vec{i} + \sin\theta \vec{j}$

$\hat{u} = \cos(\pi/6)\vec{i} + \sin(\pi/6)\vec{j} \; \left(= \frac{\sqrt{3}}{2}\vec{i} + \frac{1}{2}\vec{j} \right)$

$D_x f = 3x^2 - 4x$, $D_y f = 3y^2$

$$D_{\hat{u}} f(x,y) = (3x^2 - 4x)\frac{\sqrt{3}}{2}\vec{i} + 3y^2\left(\frac{1}{2}\vec{j}\right)$$

$\Rightarrow$ slope of the tangent line to the surface $f(x,y)$ in the direction of $\hat{u}$

plug in $P(1,2)$ to find slope at P.

Given $D_{\hat{u}} f(x,y) = f_x u_1 + f_y u_2 = \underbrace{(f_x \hat{i} + f_y \hat{j})}_{\text{gradient vector.}} \cdot (u_1 \hat{i} + u_2 \hat{j})$

∇ : gradient or del of f, $\nabla f(x,y) = f_x \hat{i} + f_y \vec{j}$, $D_u f(x,y) = \nabla f \cdot \hat{u}$

$D_{\hat{u}}$ is a slope, scalar value, while ∇f is a vector & is a component in the dot product that gives us the directional derivative

it gives us the grade, or "climb" of the surface.

properties of ∇f

- unique $\hat{u}$ & points give unique grades

#1 if $\nabla f = \vec{0}$, then $D_{\hat{u}} f = 0$ for any

#2 $D_{\hat{u}} f(x,y)$ has its max value of $\|\nabla f(x,y)\|$
and this happens when $\vec{u} = c \cdot \nabla f$ where c is a scalar

otherwise stated, " the $D_{\hat{u}}$ is highest when the gradient & unit vector are in the same direction.

We can also rewrite, using the definition of the dot product

$\|\nabla f\| \cdot \|\vec{u}\| \cos\theta = D_{\hat{u}} f(x,y) = \|\nabla f\| \cos\theta$.

Tangent Planes and Normal Lines

Knowledge Test #2 Everything in Ch. 14 Hughes

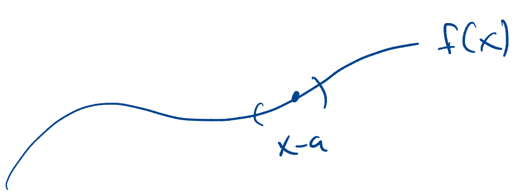
- Taylor Series

$f(x,y) : \mathbb{R}^2 \to \mathbb{R}$

$\frac{d^2 f}{d x d y} = \frac{d^2 f}{d y d x}$ if $f(x,y)$ is smooth, basically mixed partials give the same answer no matter the order derivatives are taken, if $f(x,y)$ is smooth

when $f(x,y) = \begin{cases} F(x,y) & , (x,y) \neq (0,0) \\ a & a = (0,0) \end{cases}$ however, this is debateable.

Recall Taylor Expansion for single var.



- I can approximate the behaviour of f around x using $P_n(x)$ in form

$P_n(x) = a_0 + a_1 x^1 + a_2 x^2 + \dots a_n x^n$

around $x=a$, $f(x)\big|_{x=a} \approx P_n(x)$

our job is to figure out the coefficients

Case 1 if $x=0$, case 2: substitution when $x \neq 0$

at $x=0$, $f(0) = a_0 + a_1 x + a_2 x^2 \dots = a_0 = f(0)$, $x=0$

$f'(0) \approx a_1 + 2a_2 x + 3a_3 x^2 + \dots n a_n x^{n-1} = a_1 \Rightarrow a_1 = f'(0)$

$f''(0) \approx 2a_2 + 6a_3 x + \dots n(n-1) a_n x^{n-2} = 2a_2 \Rightarrow a_2 = \frac{f''(0)}{2}$

$f'''(0) \approx 6a_3$, how do we generalize this?

$f^n(0) = n! \cdot a_n \Rightarrow a_n = \frac{f^n(0)}{n!}$

the expansion

$$f(x) \approx \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n \to x=0$$

what if $x=a$, $a \neq 0$, $t = x-a$

$$f(t) = \sum_{n=0}^{\infty} \frac{f^n(t=0)}{n!} t^n \Rightarrow \underbrace{f(x) = \sum_{n=0}^{\infty} \frac{f^n(x=a)}{n!} (x-a)^n}$$

Example e^{x^2} at $x=0$, we could use, or we have to recall $e^x : \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then $t = x^2$

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \Rightarrow t=x^2 \quad \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

On tests you may only have to compute 2-power

How does all this work in 2-var? $f(x,y) = ?$

$f(x,y) = ?$ in linear term approximation

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(x-b)$$

what about 2^{nd} term? this is where things get hard since there are 4 possibilities for 2^{nd} derivatives f_{xx} f_{xy} f_{yx} f

3. Difference quotients

Q: $e^{-x}\sin(y)$ at point $(1,3)$, estimate f_x, f_y with $0.01\ \Delta$.

$f_x = e^{-x}(-1)\sin(y)$ $\quad -e^{-1.01}\sin(3) - (-e$

$$f_x \approx = \frac{e^{-1.01}\sin(3) - e^{-1}\sin(3)}{0.01} = \frac{\sin 3\,(e^{-1.01} - e^{-1})}{0.01}$$

$f_y \approx$

Q: $f_x(5,3)$ & $f_y(5,3)$ given by $f(x,y) = \frac{x^2}{y^2+2}$

$$\frac{f(5.01,3) - f(5,3)}{0.01} = \frac{\frac{25.1001}{11} - \frac{25}{11}}{0.01} \boxed{= 0.91}$$

$$\frac{f(5,3.01) - f(5,3)}{0.01} = \frac{\frac{25}{9.0601+2} - \frac{25}{11}}{0.01} = \text{negative, use calculator}$$

Q: $z = ye^{x/y}$, equation of tangent plane.

$F_z|_{8,8,8e}$, $\quad F_y = e^{x/y} + ye^{x/y}(-x/y^2)$

$F_x = ye^{x/y}\frac{1}{y} = e^{x/y}$

$F = 2e^{x/y} + ye^{x/y}(-\frac{x}{y^2})$.

Evaluating at $(8,8)$ $\quad F = e$, $F_x = e$ $\quad F_y = 0$

therefore the equation of the tangent plane at $(8,8,8e)$ is $z = ex + 0y$ $= \boxed{z = ex}$

Q: Local Linearity & Differentials

find df, $f(x,y) = 7\sin(xy)$

$df = 7(\cos(xy)\,y) + 7(\cos(xy)\,x)$

Q 2: $f(x,y) = \sqrt{x^2+y^3}$ at $p(1,2)$

$f_x = \frac{1}{2}(x^2+y^3)^{-1/2}\,2x$, $f_y = \frac{1}{2}(x^2+y^3)^{-1/2}\,3y^2$

plugging in $(1,2)$

14.5 Gradients & Directional Derivatives in Space

$f(x,y,z) = 4x^5$

$\nabla f = (20x^4, 0, 0)$

vector