Labour in the Circular Economy: a catalyst towards sustainable development

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Abstract

The transition to a circular economy has gained increasing attention for its potential to reduce negative externalities and promote sustainability. However, the literature on the effects of the circular economy on labour remains limited. This paper investigates the optimal allocation of labour between the production and circular sectors to achieve sustainable development while minimizing the impacts of waste accumulation. Using a two-sector growth model in a closed economy, we analyze the role of the circular economy in mitigating externalities related from waste stocks on labour productivity. Changes in economies are modeled due to an endogenous choice of materials and labour in the two sectors. By considering both the inclusion and exclusion of waste-related externalities, we assess the potential of the circular economy to reduce negative impacts on labour and welfarefare. The approach is innovative as it puts emphasis on the role of the labour in the circular economy, illustrating the important role of the circular sector to sustain economic development.

Keywords— Resource scarcity, labour productivity, damage function; endogenous growth, pollution, recycling

1 Introduction

The market failures related to the current linear economy, also known as the Take-Make-Dispose economic model, have led to pollution, depletion of exhaustible resources and production of waste. An increasing number of countries is dealing with negative externalities related to inadequate waste management.

Improper waste management leads to various negative externalities such as air and soil pollution and land occupation (Das, 2017; Giusti, 2009). A severe issue relates to the air pollution from waste management facilities, from methane gas emitted and from accidental blazes. Beyond health related issues impacting labour productivity, more severe accidents may occur due to improper waste management. In 2018, in Rome citizens were asked to stay inside for few days due to the toxic smokes emanating from the nearby ignited landfill[1]. Such events have negative consequences on labour productivity, even in developed cities such as Rome.

To avoid this loss in labour productivity, the circular economy could be promoted as a way to transition from a highly polluting economy to one that is more sustainable. Circular economy aims to increase resource reuse and economic performance while reducing negative externalities on the environment and on society (Wijkman & Skånberg, 2015). The circular economy model relies on concepts such as: reduce, reuse, recycle, redesign, remanufacture, recover, share. Ellen MacArthur Foundation (2018) reported that the circular economy could make "cities more liveable, reducing emissions of fine particulate matter by 50%, emissions of greenhouse gases by 23%, and traffic congestion by 47%, by 2040 [...] (and help) save businesses and households' money. Although the concept of circular economy is often mentioned by practitioners as being beneficial for the economy - mainly due to the reduction of pollution induced by inadequate waste management[2] and scholarly acknowledged in terms of resource management, only few papers (Giusti, 2009; Lusky, 1976) study the impact of inadequate resource and waste management in the linear economy on labour productivity.

This research evaluates the impact on labour productivity of, on the one hand, the negative externalities induced by waste management, and on the other hand, the circular sector, apprehended as an alternative to the linear production sector. The second section of this paper reviews the current literature linking economic growth, resource scarcity, pollution damage and labour productivity. In the third section, a two-sector Solow-type model of growth is shaped, with the first sector producing goods and waste from resources, and the second sector (the circular economy sector) producing circular resources from the stock of waste, to be used by the first sector[3]. With the increasing policy interest towards circular economy, analysing the potential of the circular economy sector to enhance labour productivity while reducing pollution, is essential for policy makers to develop adequate instruments.

The model substantiates the literature on circular economy and its impact on labour productivity by shaping a two-sector growth model in a simple closed economy. The model allows the investigation of the optimal allocation and productivity of labour between the first sector, which provides the main production, and the circular sector, to sustain economic development while limiting the negative effect of the environmental consequences of waste mismanagement on labour productivity.

The first sector is represented as a standard extraction-based production function, which exploits exhaustible resources and creates the economic outputs but also waste, which accumulates

[1] (The Independent, 2018; The Local, 2019).

[2] Practitioners are government, business and civil society.

[3] The circular resources are considered as perfect and imperfect substitutes through several scenarios.

in a stock and negatively impacting the labour productivity. In this model, exhaustible resource scarcity, pollution damage induced by the stock of waste and labour productivity are endogenous components of the overall production. The exhaustion of the resources impacts the first sector of the economy, leading to degrowth as the material input becomes scarce. The stock of waste creates various negative externalities. The first one is the use of space. Waste management facilities use space that cannot be dedicated to amenities (leisure or real estate), reducing consumer utility. The distance to the workforce from the workplace is also increased: when landfills extend or when new waste treatment facilities are built, labour force may move out (Adhvaryu et al., 2022), increasing the average commuting time to the workplace. Secondly, waste management facilities are a source of air and soil pollution, from the emissions of toxic and organic waste. Lastly, accidental events, such as blazes, affect the daily routine of labour force. Even in developed cities such as Rome, labour productivity is negatively impacted by the externalities induced by waste management.

The second sector of this model represents the circular economy sector which uses the stock of waste created by the first sector as input. The second sector provides an alternative "circular" material input to the first sector, providing an alternative to the virgin material extracted from exhaustible resources. This circular economy sector impacts labour productivity in two ways. Firstly, similar to the education sector (Benos & Zotou, 2014; Carpenter & Lawler, 2019), the circular economy offers an alternative and satisfying sector for workers, whilst taking labour force out of the market for the first sector. Secondly, by exploiting the waste generated by the first sector, the circular economy sector reduces the negative externalities affecting labour productivity. The overall impact of the circular sector on the economy is evaluated in this model.

Developing such a model allows the investigation whether sustainable development can be driven by an increase in endogenous circular economy (or put differently by a reduction in both the use of virgin material and its induced pollution on labour productivity). The model determines the optimal allocation and productivity of labour between main production and circular sector to sustain development while limiting environmental impact caused by waste on labour. The model is solved firstly when no damages from the stock of waste are considered, then when these negative externalities are accounted for.

When the model is solved accounting for no damage from the waste stock and in a centralized economy setting, the social planner's solution suggests an optimal and constant distribution of the labour supply between the main production sector and the circular sector. The model reaches a steady-state where growth is null but consumption positive. This model shows limitation to the circular economy as defined by Figge et al. (2023) but shows that under certain conditions, the economy can be sustain in the long run. Also in this setting with no damages, recycling is the main strategy of the circular economy to be fully actually activated, thus questioning on sustainable can a circular economy be. In contrast, the solutions seemed differently timed when the model is considered damages from the stock of waste. Through the comparison of the two approaches, optimal conditions can be determined: (1) a circular rate to sustain development while limiting environmental impacts by waste on labour, and (2) policy interventions to maximize social welfare. The results allows to draw policy recommendation to maximize social welfare and to foster the circular economy sector.

The outcomes of the paper supports policies promoting circular economy for labour productivity enhancement. Policy stakeholders might consider financial instruments to internalise the damages linked with waste management and to intensify the circular economy sector. The approach is innovative as it puts emphasis on the role of labour, illustrating the important role of the circular economy for sustainable economic development. The model offers a new approach to overcome externalities while improving labour productivity.

2 Literature review

The junction of four strands of the literature

This study builds on four strands of the literature. First, it relates to the literature on circular economy related to resource depletion. Authors investigate circular economy as a way to provide substitute for virgin materials extracted from exhaustible resources since mid-1970s' (Boucekkine & El Ouardighi, 2016; De Beir et al., 2010; Di Vita, 2001; Fagnart & Germain, 2011; Hoel, 1978; Hoogmartens et al., 2018; Lafforgue & Rouge, 2019; Lusky, 1976; Pittel et al., 2010; Sorensen, 2017; Zhou & Smulders, 2021). Hoogmartens et al. (2018) model the role of recycling provide alternative materials to the economy. While the authors mention the impact of extraction and recycling on employment, they do not include labour stock nor any damage function in their hotelling model. Di Vita (2001); Lafforgue & Rouge (2019); Pittel et al. (2010) model the circulation of materials, considering waste either as a flow or as a stock. Lafforgue & Rouge (2019) do not consider a stock of waste which can accumulate and thus create pollution, they introduce the circular economy sector to uniquely overcome the flow of waste. Zhou & Smulders (2021) considered also the circular economy as a way to feed the production function with non-exhaustible resources and as they consider no loss in the process, there is no need for injection of exhaustible resources. Fodha & Magris (2015) study the impact of the circular economy on economic welfare and resource exhaustion with overlapping generations. De Beir et al. (2010) concede that the potential of the circular economy is similarly not bounded by the stock of waste and produces unlimited material supply to the economy. De Beir et al. (2010) consider circular economy as dependent on the output produced in the last period to run the economy. The model presented in this paper differs from the literature as the available input for the circular economy depends on the stock of waste generated by the first sector of the economy. Also, and to make the model realistic, losses (due to entropy and technological limits) are considered in all sectors. Waste is generated through the extraction of exhaustible resources and reduced by its partial re-injection in the economy as circular material.

Second, this paper relates to the impact of pollution on the economy. In the model of Pittel et al. (2010), and to achieve a balanced growth, policy arguments are needed to overcome the negative externalities induced by waste through recycling. Similarly, the conclusions of Fagnart & Germain (2011) are pessimistic as they fail to consider pollution damage from their available flow of waste. These pessimistic results for the implementation of the circular economy can mainly be explained by the fact that the models do not account for the damages occurred by the waste accumulation. Indeed, in the above mentioned models, waste has no negative impact on the labour force. However, Hoel (1978) and Lusky (1976) provided more optimistic results when considering circular economy to control for pollution induced by exhaustible resources extraction and use (of virgin material extracted from exhaustible resources). Circular economy is considered as an alternative to cope with the pollution induced by natural resource extraction and disposal on the environment (Boucekkine & El Ouardighi, 2016; Fagnart & Germain, 2011; Hoel, 1978; Lusky, 1976; Sorensen, 2017). Boucekkine & El Ouardighi (2016); Fagnart & Germain (2011); Sorensen (2017) introduced a secondary sector, the circular sector, to overcome the harming flow of waste from virgin material. Bretschger & Pattakou (2019) modelled a damage function, representing the pollution, which impacts the capital stock needed for the main production. Following this strand, the current model presents a damage function caused by the stock of waste which accumulates from the use of goods produced from virgin and circular material through the main production function. The current flow of waste is assumed to be taken from a stock of waste. Waste need to be consider both as a flow, for the material balance constraint (i.e., the substitutability of virgin and circular material) and as the stock to account for its negative externalities. In this model developed hereafter, the circular economy is considered as an alternative to resource exhaustion, and to the pollution induced by improper waste management.

Third, this paper enriches the literature focusing on the negative impact of pollution on labour productivity. Most of the time, the pollution induced by waste, is considered to impact the environment through the reduction of capital (Boucekkine & El Ouardighi, 2016; Bretschger & Pattakou, 2019; Fagnart & Germain, 2011; Hoel, 1978; Sorensen, 2017). In the current literature on the circular economy, the role of labour is neglected. Only in Lusky (1976) labour is mentioned. Following Lusky (1976)'s approach, the model developed hereafter focuses on the impact of the inappropriate waste management on the labour productivity. The growth of the waste stock implies negative impacts on labour productivity. This impact is caused by the stock of waste which accumulates from the use of material in the economy. However, Lusky (1976) do not constrain the virgin (non-circular) material by an exhaustible resource. In the model developed hereafter, the virgin material in bounded by a finite stock of exhaustible resources. This approach builds on Williams III (2002) who shows that reducing pollution can boost labour productivity. While Williams III (2002) reduces pollution through taxation, the model presented here shows that a second sector (i.e. the circular economy sector) can reduced pollution in a way that it lifts labour productivity. Labour is considered to be devoted either to the main production function or to the circular economy sector. Like Aloi & Tournemaine (2011), the results show that introducing a new sector can provide positive effects on labour productivity.

Last but not least, the paper engages with the literature on labour productivity as a determinant factor of growth. The circular economy is modeled as a second sector impacting labour productivity, following the model developed by Rada (2007). Parallels can be drawn between the role of the circular economy and the education sector to enhance labour productivity (see for example Olley & Pakes (1996), Griliches (1997) and Acemoglu & Autor (2012)). As mentioned in the early work of Griliches (1997) on the role of education to enhance productivity, both sectors diverge a part of labour from the main production sector but increase the productivity of the labour for the economy. While for the education sector, the productivity is transferred as better qualified labour force for the production sector, in the circular economy, the productivity is improved through pollution reduction. As highlighted in the work of Lebedinski & Vandenberghe (2014), several factors (i.e. education) enhance labour productivity. The model is estimating the impact of the circular economy sector to enhance labour productivity. The circular economy sector also intervenes as a determinant of growth, through the enhancement of labour productivity and the provision of secondary material for the production sector. Similarly to the work on the contribution to labour productivity conducted by Knowles & Owen (1997) and more recently by Madsen & Murtin (2017), the model demonstrates the important role of the circular economy sector on labour productivity for economic growth.

3 Model setup

A 2 sector growth model with resource stocks' externalities

None of the studies mentioned in the previous section acknowledges the role of the circular economy sector in solving exhaustibility of resources, reducing the waste stock and the negative externalities related, and for impacting labour productivity. The main aim of this paper is to investigate the optimal labour productivity to sustain economic development while limiting environmental impacts of waste. This paper analyses the circular economy as an alternative to cope with both the exhaustibility of resources, and the pollution induced on labour productivity by improper waste management of these resources. From the model developed by Lafforgue & Rouge (2019), a new model is set up including losses in the material use and a damage function impacting labour productivity. This new model allows original results supporting policies towards the circular economy.

To model the economy, as depicted in Figure 1, a two-sector (P1 and P2) growth model is modelled in a simple closed economy which relies on assumptions about aggregate levels of material availability (m) and labour (L) and total factor of productivity[4] (A_k) of the output k. The first sector, $P1$, is a production sector, producing goods y directly consumed by consumers. The second sector, $P2$, is the circular sector, which tackles waste created by the first sector. This circular sector $P2$ is an intermediate sector and it does not produce consumer goods but provides circular substitute z to the virgin material x for the material inputs needed m for the first sector $P1$.

In our model, we assume that extraction, production, consumption, and waste production all take place simultaneously within the same time period. This is based on the premise that the duration of one time period, which is set as one year in this paper, is sufficiently long to allow for all of these activities to occur within a single time frame. Additionally, we assume that all activities resulting from the waste stock (pollution and recycling) occur instantaneously. This assumption is grounded in the belief that there is no delay in the effects of these activities, and any negative consequences are immediately realised. We consider that there is no direct costs nor labour needs, associated with the extraction of the resources. Thus both $x(t)$ and $w(t)$ are extracted from their respective costs with the only constraint being the level of the stock.

The model developed presents the circular economy sector as the net result of different circular strategies: reduce, reuse, redesign, recycle, repair and share, which fully ascertains the assumption that the circular sector reduces the waste stock, but also provides alternative materials for the production function, noted $P1$ on the Figure 1. We assume that the circular sector, which will commence at time T_r, will persist for all $t \geq T_r$. The level of circularity is defined by the exogenous coefficient β. It represents the amount of materials that is not going to the stock of waste. The circular sector and the level of circularity β is further explained in Section 3.2 and Section 3.1 respectively.

Labour and waste are the two main components of the circular sector. The waste stock is assumed to negatively impact labour. The circular sector emerges in the economy only when the damages induced by the stock of waste of the labour reach a certain threshold (i.e.when the damage is so important that the economy needs to find a palliative). Exhaustible resource, pollution damage, virgin and circular materials used and the labour productivity are endogenous components in the overall production. The changes in the economy are due to an endogenous choice of labour supply. This paper investigates under which conditions the circular sector can generate endogenous welfare improvement. The model relies on assumptions about aggregate levels

[4]In this paper, the total factor of productivity is interchangeably also noted TFP.

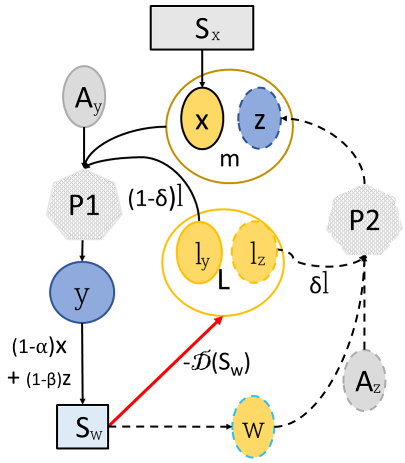


Figure 1: Representation of the overall model

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner to reach pareto optima in the centralised economy, while the output endogenous variables are blue. The damage function is represented by a red arrow. The dash lines represent the components of the recycling sector.

of material use and loss, and externalities linked with waste and impacting labour. The time t may run until infinity and is continuous.

Notations To facilitate the reading of this paper, certain notations have been employed. The growth rate of a variable k is represented by $g_k = \frac{\dot{g}_k(t)}{g_k(t)}$. The Hamiltonian and Lagrangian are denoted by $\mathcal{H}$ and $\mathcal{L}$, respectively, while the co-state variables associated with the state variable k are represented by λ_k. When time-dependant variables are unambiguous, the time index t is omitted for simplicity. In this paper, we make the standard economic assumption that all production functions exhibit increasing and concave returns to scale with respect to each input variable. Furthermore, we assume that both labour and materials inputs are essential complements for the production functions. Specifically, if either input is null, the corresponding function output will be null as well $f(M, 0) = f(0, L) = 0$. At the present stage of our analysis, we consider all three total factor productivity measures to be exogenous, constant over time, and thus with a growth null, such that$\forall t, A_y(t) = A_y0, A_z(t) = A_z0, A_m(t) = A_m0$. We adopt this simplification to facilitate our initial investigation. However, we acknowledge that in future potential extensions, this assumption may be relaxed to incorporate dynamic changes in TFP over time.

3.1 Consumption, production sector and resource stocks

The economy consists of representative consumers maximizing their utility. The utility function of the consumer is of the Constant Relative Risk-Aversion Utility (CRRA) form. Consumption is considered within a standard life time without constraints. The consumer in this economy owns physical capital (labour) supplied to the two sectors (P1 and P2), and consumes the goods (y) produced by the first sector (P1). Consumption[5] is a function of the output from the main production function: $c(t) = y(t)$. In the current model, the production of input y(t) is fully utilized and generates immediate consumer utility, denoted as u(c(t)). This utility function satisfies

[5]Pollution is not present in the consumption as it is assumed to mainly impact labour productivity.

standard economic properties, including increasing, concave, and Inada conditions. Moreover, the consumer utility is subject to a discount rate, denoted as rho, which is exogenous, constant, and positive over time.

$$u\left(c\left(t\right)\right) \quad = \quad \frac{c\left(t\right)^{1-\sigma}-1}{1-\sigma} \tag{1}$$

The above expression holds for $\sigma \neq 1$. The utility for $\sigma = 1$ writes as follow:

$$u\left(c\left(t\right)\right) \quad = \quad ln(c(t)) \tag{2}$$

The production sector P1 produces final goods (y) directly consumed by consumers. This sector is driven by labour (l_y), material (m) and an exogeneous factor of productivity (A_y). As illustrated by Figure 1, the production sector (P1) consumes material (m), that can be either virgin (x) and thus extracted from an exhaustible stock of resources (S_x), or circular (z) and thus supplied by the second (i.e. circular) sector P2. For calculation needs, the production function of the sector P1, f_{P1}, which illustrated what happens within the first sector (P1), is considered to be a Cobb-Douglas function such as in equation 3, with $\theta \in [0;1]$.

$$\begin{aligned} P1: y\left(t\right) &= f_{P1}\left(A_y, m\left(t\right), l_y\left(t\right)\right) \\ P1: y\left(t\right) &= A_y m(t)^{\theta} l_y(t)^{1-\theta} \end{aligned} \tag{3}$$

Both exhaustible and circular materials can be used as a perfect substitute (as in Di Vita (2001); Lafforgue & Rouge (2019); Pittel et al. (2010)) for material in the production function such as in equation 4

$$m(t) \quad = \quad A_m\left(x\left(t\right)+z\left(t\right)\right) \tag{4}$$

A_y is the exogenous total factor productivity (TFP) and captures energy and technology supply. It is considered to be time invariant and is not the main focus of this research paper. $l_y\left(t\right)$ is the labour supply to the production sector and is further discussed in section 3.2. $m\left(t\right)$ captures the material inputs used for the production of goods. A_m captures the exogenous energy and technology (R&D) needed to transform the material (either virgin or circular) into materials available to input in the main production function. It is considered to be time invariant and is not the main focus of this research paper.[6]

$x\left(t\right)$, represents the virgin material, which is extracted without cost, from a finite stock of exhaustible resource $S_x\left(t\right) \geq 0$ such as in equation 5

$$\begin{aligned} \dot{S}_x\left(t\right) &= -x\left(t\right) \leq 0 \\ S_x(0) &= \overline{S_{x0}} \textit{ (Initial condition)} \end{aligned} \tag{5}$$

$z\left(t\right)$, represents the circular material, created from the second production function P2, which is discussed in the section 3.2.

The consumers consume the final goods (y) produce by P1 and dispose waste which accumulates into a stock of waste (S_w), which damages the pool of labour (L) from which all healthy workers are involved in one of the two sectors P1 and P2. $S_w\left(t\right)$ is the stock of waste which accumulates.

[6]In a later stage, we could consider two different productivity for x and z. A different relation between Ax (t) & Az (t) could mean that technical characteristics of virgin and circular material can be different, or that circular material is favoured over virgin (e.g. for aluminium) or that exhaustible material is taxed or circular material is subsidized.

This stock of waste is constituted from the virgin material $x(t)$ and from the circular material $z(t)$. The waste stock dynamics follow

$$\begin{aligned}\dot{S}_w(t) &= (1-\alpha)x(t) + (1-\beta)z(t) - w(t) \\ S_x(t) &\geq S_w(t) \geq 0 \\ S_w(0) &= 0 \text{ \textit{(Initial condition)}}\end{aligned} \tag{6}$$

The stock of waste is limited by the stock of exhaustible resources. Before any virgin material is being extracted, the stock of waste is empty.[7] α captures the losses of the economy with regard of the virgin materials. β represented the level of circularity of the material. In a full circular economy, as defined for example in Figge et al. (2023), the materials do not end-up in the waste stock but they keep being used by consumers. We thus assume that $\alpha \geq \beta$ as materials from the circular sector are thought not to end up in the waste stock but rather to have their usage and lifetime extended. This assumption also leads to less outputs produced with the circular materials than the one produced with virgin materials can end up in the stock of waste. Furthermore, it is assumed that in any case, not all materials used can be recovered nor stayed with the consumers, thus $1 > \alpha > 0$ (respectively $1 > \beta > 0$). In the current stage of the paper, α and β are assumed exogeneous and constant over time. This assumption may be relaxed in possible future extensions.

3.2 Circular sector, waste damage and labour

The waste stock leads to pollution leakage into the water, soil and atmosphere, which impacts the labour (i.e., through health, limited amenities etc), as mentioned in Adhvaryu et al. (2022). The damage function of the stock of waste $\mathcal{D}(S_w(t))$ captures those negative externalities. The damage function is considered to only impact the stock of labour $L(t)$. The damage function is continuously differentiable and concave for $0 \leq S_w(t) \leq L^{1/\pi}$, $\pi \geq 1$ capturing the pollution. It is assumed that there exists a tipping point $L_0^{1/\pi}$ where the stock of waste has reached such a amount that no more labour can actually work in the economy. Hopefully, the circular economy sector will avoid this tipping point to be reached. This tipping point is assumed to be reached only if the stock of virgin (scarce) resources S_x is exhausted, such that $\forall t \quad S_x(t) \geq 0$ then $\mathcal{D}(t) \geq 0$. This assumption may be released in a later stage.

$$\begin{aligned}\mathcal{D}(S_w(t)) &= (S_w(t)^{\pi}/L_0) \\ \mathcal{D}(0) &= 0 \text{ \textit{(Initial condition)}} \\ \mathcal{D}(L_0^{1/\pi}) &= 1\end{aligned} \tag{7}$$

L captures the maximal amount of labour available per period. The maximal amount of labour L_0 is assumed constant over time. However, $L(t)$ is the stock healthy labour available in the economy at time t and it is negatively affected by the waste stock. Thus, $L(t) \leq L_0$ as the amount of healthy labour $L(t)$ depends on the amount of the stock of waste $S_w(t)$. The larger is the stock of waste, the less there is healthy labour available to work. The labour function reads:

$$L(t) = L_0 * (1 - \mathcal{D}(S_w(t))) \tag{8}$$

$$\begin{aligned}\dot{L}(t) &= -\dot{\mathcal{D}}(S_w(t)) \\ &= -\frac{\pi}{S_w(t)}\mathcal{D}(S_w(t))\end{aligned} \tag{9}$$

[7] This implies that the actual constraint is on the stock of virgin (exhaustible) material, as the stock of waste is constraint by the initial stock of virgin material available.

$$
\begin{aligned}
L(t) &= l_y(t) + l_z(t) \\
L(0) &= L_0 \textit{ (Initial condition)}
\end{aligned}
$$

$l_z(t)$ is the flow of labour input in the circular sector. As detailed in the introduction, labour is the crucial source of waste management and $l_y(t)$ is used for the production sector to produce economic output and $l_z(t)$ used in the circular sector to produce circular materials. $\delta(t)$ is the share of labour dedicated to circular economy.

$$
l_y(t) = (1-\delta_t)(1-\mathcal{D}(S_w(t))) \quad (10)
$$

$$
l_z(t) = \delta_t(1-\mathcal{D}(S_w(t))) \quad (11)
$$

When there is no circular sector $\delta_t = 0$, else it is assumed that δ_t is positive strictly inferior to 1, therefore we have: $0 \leq \delta_t < 1$.

$$
\delta(t) = \frac{l_z(t)}{l_z(t) + l_y(t)} \quad (12)
$$

When the damage induced by the waste stock S_w reaches a certain level, the economy can start a second (circular) sector $P2$ to tackle the negative effects of the waste stock. This production function $P2$ represents the circular sector. We assume that once the circular sector is activated, which is stated at $t = T_r$, the circular sector will keep on existing $\forall t \geq T_r$. A flow $w(t)$ of waste is extracted from the stock of waste $S_W(t)$ to provide input to the circular sector. It provides $z(t)$, the circular material, to the economy. This output $z(t)$ is used as substitute for the virgin material x in the material $m(t)$ input of the first production function $P1$. The circular sector is a function of labour (lz), waste available (w) and an exogenous total factor of productivity (Az). The production function $P2$ can be considered a Cobb-Douglas function such as in equation 13, where $\gamma \in [0, 1]$.

$$
\begin{aligned}
P2 : z(t) &= h_{P2}(A_z, w(t), l_z(t)) \quad (13) \\
P2 : z(t) &= A_z w(t)^{\gamma} l_z(t)^{1-\gamma}
\end{aligned}
$$

A_z is the exogenous total factor productivity (TFP) of P2 and is capturing the unlimited access to renewable energy and technology supply. It is considered to be time invariant and is not the main focus of this research paper. it is assumed that research and development in this sector is negligible. Thus, the circular sector is assumed not to be technology intensive. Most of the technology needed to recycle waste already exists and there is no need for additional capital to be invest in technology to recycle. $w(t)$ is the flow of waste used to produce circular materials. $w(t)$ is extracted from the stocks of waste $S_w(t)$, see Equation 6.

3.3 Social planner

In a centralised economy, the social planner maximizes consumer's inter-temporal utility subject to the production functions of the two sectors ($y(t)$ and $z(t)$), the stocks of exhaustible resources $S_x(t)$ and the stock of waste $S_w(t)$ as well as the share of labour $\delta(t)$ to go to the second sector. The evolutions of the resources (virgin and recycled) and of the share of labour in each sector need to be determined based on stock constraints to maximize the consumer utility over time.

The social planner needs to make a trade off between the labour productivity (e.g. pollution negatively impacts the labour) and the amount of labour supply $\delta(t)$ to dedicate to the circular sector (labour has to be diverted from the first sector P1 to serve the circular sector P2) with respect to their definition as detailed in the previous section.

$$
\max_{\delta(t), x(t), w(t)} \int_t u(c(t)) e^{-\rho t} dt \quad (14)
$$

$$\begin{aligned}
\dot{L}(t) &= -\frac{\pi}{S_w(t)}\mathcal{D}(S_w(t)) \\
\delta\,(t) &= \frac{l_z(t)}{L(t)} = \frac{1-l_y(t)}{L(t)} \\
L(0) &= L_0 \text{ \textit{(Initial condition)}}
\end{aligned}$$

$$\begin{aligned}
\dot{S}_x\,(t) &= -x\,(t) \\
S_x\,(t) &\geq 0 \\
S_x(0) &= \overline{S_{x0}} \text{ \textit{(Initial condition)}}
\end{aligned}$$

$$\begin{aligned}
\dot{S}_w\,(t) &= (1-\alpha)x\,(t) + (1-\beta)z\,(t) - w\,(t) \\
S_w\,(t) &\geq 0 \\
S_w(0) &= 0 \text{ \textit{(Initial condition)}}
\end{aligned}$$

The current-value Hamiltonian with co-state variables $\lambda_{Sx}\,(t)$ and $\lambda_{Sw}\,(t)$ is:

$$\begin{aligned}
\mathcal{H}\,(t) &= u\,(c\,(t)) - \lambda_l(t)\frac{\pi}{S_w(t)}\mathcal{D}(S_w(t)) - \lambda_{Sx}\,(t)\,(x\,(t)) \\
&+ \lambda_{Sw}\,(t)\,((1-\alpha)x\,(t) + (1-\beta)z\,(t) - w\,(t))
\end{aligned}$$

Lagrangian with multipliers $\mu_\delta(t)$, $\mu_x\,(t)$, $\mu_{Sx}\,(t)$, $\mu_w\,(t)$, and $\mu_{Sw}\,(t)$

$$\max_{\delta(t),x(t),w(t)} \mathcal{L} = \mathcal{H} + \mu_\delta\delta + \mu_L L + \mu_x x + \mu_{Sx}S_x + \mu_w w + \mu_{Sw}S_w$$

This model is solved theoretically to provide theoretical predictions on the dynamics of the economy.

4 Model without damages

An endogenous growth model without damage function

Assumptions In this section, the pollution is not internalized. The model is analyzed without accounting for the damage function, as Figure 2 illustrates. The stock of labour $L(t)$ over the full period is constant, and we assume $\forall t \in [0; \inf], L(t) = 1$. We also consider $\forall t \geq 0$ that no growth of the exogenous productivity of sectors g_{A_y} and g_{A_z} and the productivity of the material g_{A_m} all being constant and null over time thus equal to zero.

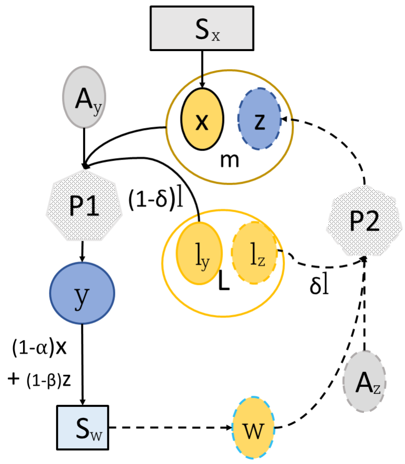


Figure 2: Hypothetical stage without damage when there would be both extraction of virgin material and recycling by the circular sector

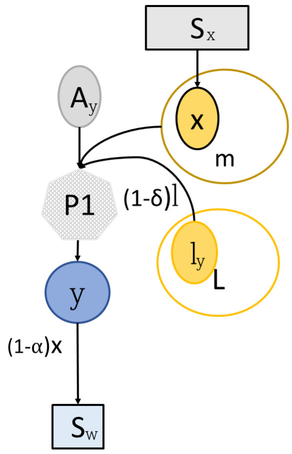
Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner to reach pareto optima in the centralised economy, while the endogenous output variables are blue. The dashed lines represent the components of the recycling sector.

Such a model provides insights on the actual economic balance happening when no policy favoring circular economy prevails. When waste leads to no damages, the only incentive to have a circular sector is the scarcity of the resources. Thus, the only reason to dedicate labour to the recycling sector is to overcome the reduced availability of x(t).

Lemma 1 There is no apparent incentive to use both exhaustible resource and circular material as the same time. The proof of lemma 1 is in Appendix A.1. This Lemma 1 induces that there exists a time $T_r > 0$ such that we have $\forall 0 < t < T_r, m(t) = A_m x(t)$, and $\forall t \geq T_r, m(t) = A_m z(t)$.

Lemma 2 There is no economic reasons for not extracting all the exhaustible resources before to start recycling the stock of waste. Only once the stock of exhaustible resources is fully depleted, the circular economy sector is introduced as a substitute to the virgin material. The proof of this Lemma 2 can be found in Appendix A.2. This lemma brings us the expression of a unique T_r that needs to satisfy $\forall 0 < t < T_r, S_x(t) > 0$ and $\forall t > 0 \in [T_r; +\infty[, S_x(t) = 0$.

In the case when no damages are accounted for, the economy has thus only the two stages represented in Figure 3 and in Figure 4, there is thus no t such as represented by Figure 2. This is an interesting results as it differs from the results from both Hoogmartens et al. (2018) and Lafforgue & Rouge (2019), where they both have simultaneously recycling and virgin resources. In the model presented later in this paper, in Section 5, such a simultaneity is existed and analysed.



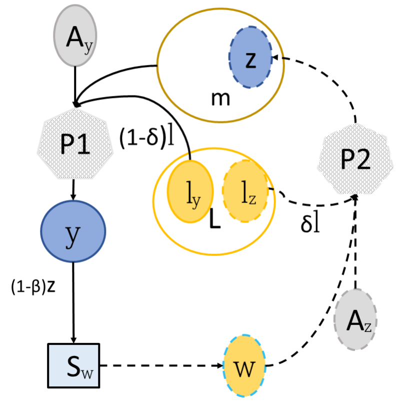


Figure 3: Economy at time $t_0 \leq t < T_r$ Figure 4: Economy at time $t \geq T_r$

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner to reach pareto optima in the centralised economy, while the endogenous output variables are blue. The dashed lines represent the components of the recycling sector. There is no damage function in this figure as it is not considered in this first step of the model.

4.1 Pre-recycling stage

In the case, illustrated by Figure 3 where there is no damage function and only the production function, $\forall t \in [0; T_r[$ we have $\delta(t) = w(t) = z(t) = 0$. The social planner's optimization problem summarizes in choosing the flow of virgin materials $x(t)$ to be extracted from its related stock $S_x(t)$. The optimization problem becomes:

$$\max_{x(t)} \int_{0;T_r} u\left(c\left(t\right)\right) e^{-\rho t} dt \tag{15}$$

$$\begin{aligned}
\dot{S}_x\left(t\right) &= -x\left(t\right) \\
S_x(T_r) &= 0 \\
S_x(0) &= \overline{S_{x0}} \text{ (Initial condition)}
\end{aligned}$$

$$\begin{aligned}
\dot{S}_w\left(t\right) &= (1-\alpha)x\left(t\right) \\
S_w(T_r) &= (1-\alpha)\overline{S_{x0}} \\
S_w(0) &= 0 \text{ (Initial condition)}
\end{aligned}$$

In this stage, when there is no recycling, the Hamiltonian writes as follow:

$$\begin{aligned}
\mathcal{H}(t) &= u\Big(A_y(A_m x(t))^\theta\Big) \\
&- \lambda_{Sx}(t)x(t) \\
&+ \lambda_{Sw}(t)\Big((1-\alpha)x(t)\Big)
\end{aligned} \tag{16}$$

The F.O.C gives:

$$\frac{\partial u((c(t))}{\partial c(t)} * \frac{\partial c(x(t))}{\partial x(t)} = \lambda_{Sx}\left(t\right) - \lambda_{Sw}(t)(1-\alpha) \tag{17}$$

$$\begin{aligned}
\frac{\dot{\lambda}_{S_x}(t)}{\lambda_{S_x}(t)} &= \rho & (18)\\
\frac{\dot{\lambda}_{S_w}(t)}{\lambda_{S_w}(t)} &= (1-\alpha)\rho & (19)
\end{aligned}$$

Equation (16) informs us that the marginal utility gain of having one more unit of material should equal the corresponding marginal cost of the stock of the scarce resource, and a share $(1-\alpha)$ of the stock of waste resource.

The Equation (17) can be rewritten such as:

$$\begin{aligned}
\frac{\partial u(t)}{\partial c(t)}c(t) + \frac{\partial c(t)}{\partial x(t)}u(t) &= \lambda_{Sx}(t) - \lambda_{Sw}(t)(1-\alpha)\\
\frac{\partial u(c(t))}{\partial c(t)}\frac{\theta}{x}c(t) &= \lambda_{Sx}(t) - \lambda_{Sw}(t)(1-\alpha) & (20)
\end{aligned}$$

The transversality conditions are:

$$\lim_{t\to T_r} \lambda_{Sx}(t)S_x(t) = 0 \quad (21)$$

$$\lim_{t\to T_r} \lambda_{Sw}(t)S_w(t) = 0 \quad (22)$$

Using the Hamiltonian equations, the equations of motion are:

$$\begin{aligned}
\dot{x}(t) &= -\frac{\partial\mathcal{H}(t)}{\partial\lambda_{Sx}(t)} = \lambda_{Sx}(t) & (23)\\
\dot{\lambda}_{Sx}(t) &= -\frac{\partial\mathcal{H}(t)}{\partial x(t)} = \frac{\partial u(c(t))}{\partial c(t)}\frac{\theta}{x}c(t) + \lambda_{Sw}(t)(1-\alpha) & (24)\\
\dot{\lambda}_{Sw}(t) &= -\frac{\partial\mathcal{H}(t)}{\partial x(t)} = (1-\alpha)\lambda_{Sx}(t) & (25)
\end{aligned}$$

Using the Hamiltonian and the corresponding equations of motion, we can determine how the state variable $x(t)$ evolves over time, from t to T_r. Similarly is for the stock of virgin resources $S_x(t)$ and of waste $S_w(t)$. The evolution of the consumption $c(t) = y(t)$ and of its related utility $u(c(t) = y(t))$ can thus be compiled from t to T_r. The solving details can be found in Appendix A.3. $\forall t < T_r$, we have that :

$$\begin{aligned}
S_x(t) &= \frac{x_0}{g_x}(e^{-g_x t} - 1) + \overline{S_{x0}} & (26)\\
S_w(t) &= (1-\alpha)\frac{x_0}{g_x}(1 - e^{-g_x t}) & (27)\\
x(t) &= x_0 e^{-g_x t} = x_0 e^{\frac{-\rho}{1-\theta(1-\sigma)}t} & (28)\\
c(t) &= c_0 e^{-g_c t} = A_z A_m^\theta x_0 e^{-\theta g_x t} = A_z A_m^\theta x_0 e^{\frac{-\theta\rho}{1-\theta(1-\sigma)}t} & (29)
\end{aligned}$$

with $g_x = \frac{\rho}{1-\theta(1-\sigma)} \in [0;1]$. From the above expressions of the expressions, and because the stocks need to be continued, a link between the initial level of extraction $x0$ and the time when the stock of virgin material is exhausted T_r can be analytically expressed as follow

$$\begin{aligned}
S_w(T_r) &= 0 & (30)\\
T_r &= \frac{1}{g_x}ln(1 + \frac{\overline{S_{x0}}g_x}{x_0}) & (31)
\end{aligned}$$

T_r is the beginning of the circular period.

4.2 Circular stage

Under such model conditions, as illustrated by Figure 4, the social planner needs to choose the flow of waste $w(t)$ to extract from the stock of waste $S_w(t)$ and the share of the labour $\delta(t)$ to be dedicated to the circular sector. The social planner optimization problem becomes:

$$\max_{w(t),\delta(t)} \int_{T_r;t} u\left(c\left(t\right)\right) e^{-\rho t} dt \tag{32}$$

$$\tag{33}$$

$$\begin{aligned} \dot{S}_w\left(t\right) &= (1-\beta)z\left(t\right) - w(t) \\ S_w(T_r) &= (1-\alpha)\overline{S_{x0}} \end{aligned}$$

In the circular stage, the Hamiltonian writes as follow:

$$\begin{aligned} \mathcal{H}(t) &= u\Big(A_y(A_m z(t))^\theta\Big) \\ &+ \lambda_{Sw}(t)\Big((1-\beta)z(t) - w(t)\Big) \end{aligned} \tag{34}$$

The continuity of Hamiltonians is proven in Appendix A.5.
The F.O.C gives:

$$c(t)^{1-\sigma}\left(\frac{\theta(1-\gamma)}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{Sw}(t)(1-\beta)\frac{(1-\gamma)z(t)}{\delta(t)} = 0 \tag{35}$$

$$c(t)^{1-\sigma}\theta\gamma\frac{1}{w(t)} + \lambda_{Sw}(t)\left((1-\beta)\gamma\frac{z(t)}{w(t)} - 1\right) = 0 \tag{36}$$

$$\frac{\dot{\lambda_{Sw}}\left(t\right)}{\lambda_{Sw}\left(t\right)} = \rho \tag{37}$$

Using the Hamiltonian and the corresponding equations of motion (see Appendix A.4), we can determine how the state variables $w(t)$ and $\delta(t)$, and the stock of waste $S_w(t)$ evolve over time.

4.3 Steady State - full circular economy

From the optimisation, the economy can be finite and end at T when $S_w(T) = 0$. There also exist a steady state such that $\dot{S}_w(t) = 0$ with $S_w(t) > 0$. In such steady state, which is reachable and non null, except if $\beta = 1$, $\forall t \in]Tr; +\infty[$ the following conditions apply :

$$w^*(t) = ((1-\beta)A_z)^{\frac{1}{1-\gamma}}\delta^*(t) \tag{38}$$

$$z^*(t) = A_z {w^*}^\gamma \delta^*(t)^{1-\gamma} \tag{39}$$

$$c^*(t) = A_y A_m (A_z {w^*}^\gamma \delta^*(t)^{1-\gamma})^\theta (1-\delta^*(t))^{1-\theta} \tag{40}$$

In that steady state, we have an analytical expression of $\delta^*(t)$ which is also invariant.

$$\delta^* = \frac{2(1-\gamma)\theta}{1-\theta+2(1-\gamma)\theta} \tag{41}$$

And which implies that $g_w = 0$ and $\gamma^* = \frac{1-\beta}{2-\beta}$, and that all our variables are constant in the steady state. This also implies that $\forall t \in]t^*; +\infty[, \ \theta \leq \gamma$.

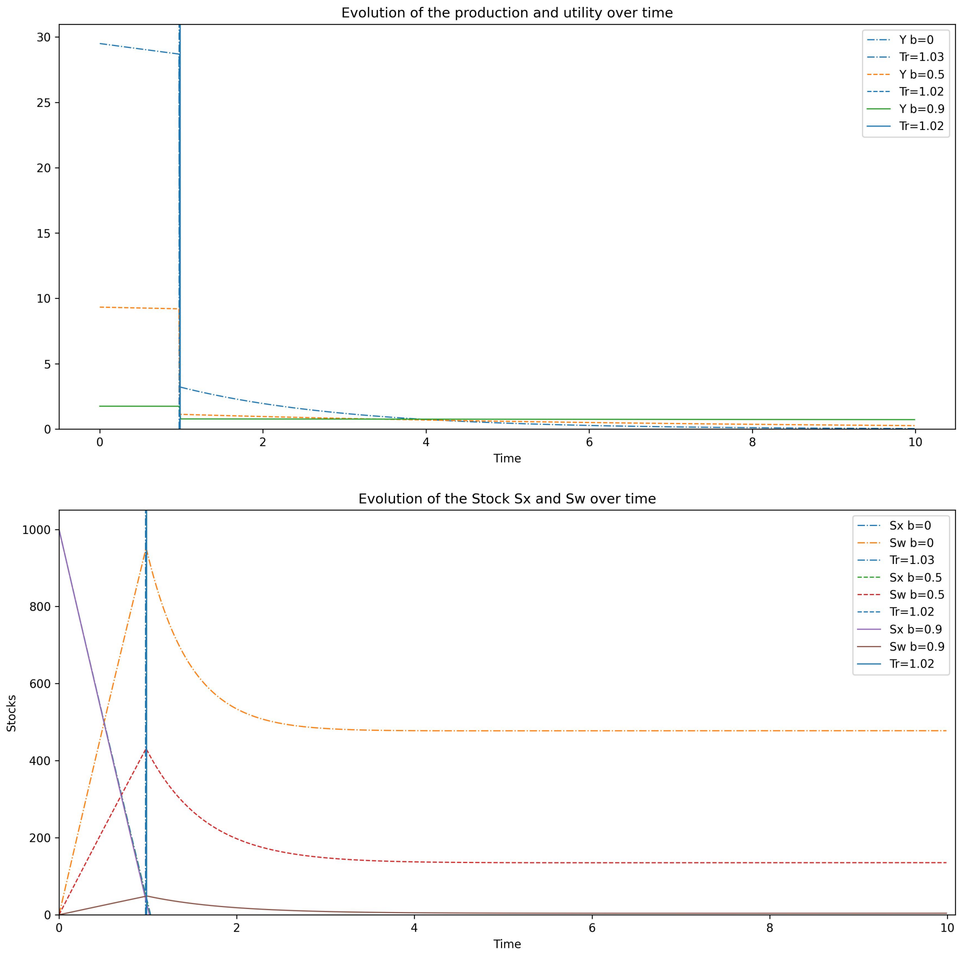


Figure 5: Evolution of the Production function and of the stocks

4.4 Trajectories

The optimal trajectories of each variables for this model are given as follow and hold for all parameters and variables of the model, except for the steady state, noted by a * that holds only if $\beta \neq 1$. The evolution of the variables are also given in Figure 5 and Figure 6, where we compute the different evolution of the variables depending on the initial choice of β. The details of the value of the parameters can be found in Appendix A.6 in the Table 1. While in these figures the steady state is closed to zero, it is reached in all the cases. Also, note that the scale was adapted to ease the reading of the pictures. Simulations were done up to $t = 10^5$ years. The results were hold constant. The consumption path is given by:

$$c(t) = \begin{cases} c_0 e^{g_c t} = c_0 e^{\frac{-\theta\rho}{1-\theta(1-\sigma)} t} & , t < T_r \\ c_{T_r} e^{g_c^*(t-T_r)} & , T_r \leq t < t^* \\ c^* = A_y A_m (A_z {w^*}^{\gamma} {\delta^*}^{1-\gamma})^{\theta} (1-\delta^*)^{1-\theta} & , t \geq t^* \end{cases}$$

where $c_0 = c(0) = A_y \left(A_m x(0)\right)^{\theta} L^{1-\theta}$ and $c_{T_r} = c(T_r) = A_y \left(A_m z(T_r)\right)^{\theta} \left((1-\delta(T_r))L\right)^{1-\theta} = A_y \left(A_m \frac{w_{T_r}}{(2-\beta)\gamma}\right)^{\theta} \left(\frac{1-\theta}{1-\theta+2(1-\gamma)\theta} L\right)^{1-\theta}$.

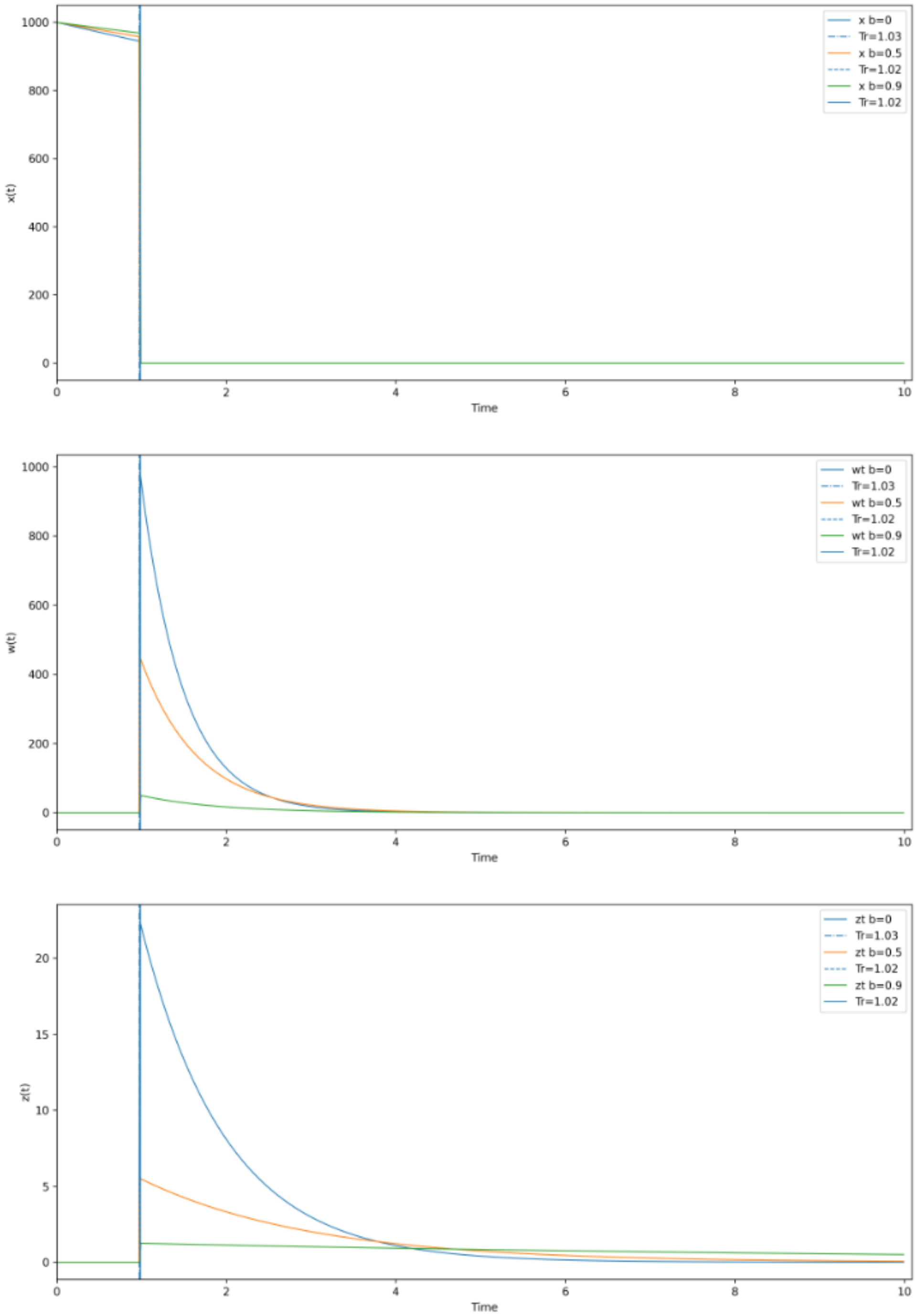


Figure 6: Evolution of the flow of virgin, waste and circular materials

The share of labour directed to the recycling sector is given by:

$$\delta(t) = \begin{cases} 0 & ,t < T_r \\ \delta^* = \frac{2(1-\gamma)\theta}{1-\theta+2(1-\gamma)\theta} & ,t \geq T_r \end{cases}$$

The virgin material used is given by:

$$x(t) = \begin{cases} x_0 e^{-g_x t} = x_0 e^{\frac{-\rho}{1-\theta(1-\sigma)}t} & ,t < T_r \\ 0 & ,t \geq T_r \end{cases}$$

The stock of exhaustible resource is given by:

$$S_x(t) = \begin{cases} \frac{x_0}{g_x}(e^{-g_x t} - 1) + \overline{S_{x0}} & \\ & , t \leq T_r \\ 0 & , t \geq T_r \end{cases}$$

The waste flow is given by:

$$w(t) = \begin{cases} 0 & , t < T_r \\ w(T_r)e^{g_w(t-T_r)} & , T_r \leq t < t^* \\ w^* = ((1-\beta)A_z)^{\frac{1}{1-\gamma}}\delta^* & , t \geq t^* \end{cases}$$

The flow of circular material is given by:

$$z(t) = \begin{cases} 0 & , t < T_r \\ \frac{w_{T_r}}{(2-\beta)\gamma}e^{g_w(t-T_r)} & , T_r \leq t < t^* \\ z^* = A_z {w^*}^{\gamma} {\delta^*}^{1-\gamma} & , t \geq t^* \end{cases}$$

The material flow is given by:

$$m(t) = \begin{cases} A_m x_0 e^{-g_x t} & , t < T_r \\ A_m z_{T_r} e^{g_w(t-T_r)} & , T_r \leq t < t^* \\ A_m A_z {w^*}^{\gamma} {\delta^*}^{1-\gamma} & , t \geq t^* \end{cases}$$

The stock of waste is given by:

$$S_w(t) = \begin{cases} (1-\alpha)\frac{x_0}{g_x}(e^{g_x t} - 1) = (1-\alpha)(\overline{s}_0) & , t < T_r \\ \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\gamma} - 1\right)}{g_w} e^{-g_w T_r}\left(e^{g_w(t-T_r)} - 1\right) + (1-\alpha)(\overline{S_{x0}}) & , T_r \leq t < t^* \\ (1-\alpha)(\overline{S_{x0}}) - S_w^* & , t \geq t^* \end{cases}$$

The co-state variables reads:

$$\lambda_{Sw}(t) = \begin{cases} \lambda_{Sw}(0)e^{\rho t} = \mu_w(t) & , t < T_r \\ \lambda_{Sw}(T_r)e^{\rho(t-T_r)} & , t \geq T_r \end{cases}$$

with $\lambda_{Sw}(T_r) = \frac{{c_{T_r}}^{1-\sigma}\theta(2-\beta)\gamma}{w(T_r)}$ and $\lambda_{Sw}(0) = \frac{{c_{T_r}}^{1-\sigma}\theta(2-\beta)\gamma}{w(T_r)}e^{-\rho T_r}$

$$\lambda_{Sx}(t) = \begin{cases} \theta\frac{{c_0}^{1-\sigma}}{x_0}e^{g_x T_r} + \lambda_{Sw}(0)e^{\rho t}(1-\alpha) = \lambda_{Sx}(0)e^{\rho t} & , t < T_r \\ 0 = {c_{T_r}}^{1-\sigma}e^{(\theta)(1-\sigma)(t-T_r)}\theta\frac{(2-\beta)\gamma}{w(T_r)} + \lambda_{Sw}(T_r)e^{\rho(t-T_r)}(1-\alpha) + \mu_x(t) & , t \geq T_r \end{cases}$$

$$\lambda_{Sw}(T_r) = \frac{{c_{T_r}}^{1-\sigma}\theta\frac{(2-\beta)\gamma}{w(T_r)} + \mu_x(t)}{\alpha - 1}$$

with $\mu_x = \theta\left(\frac{{c_0}^{1-\sigma}}{x_0}e^{g_x T_r} - {c_{T_r}}^{1-\sigma}\frac{(2-\beta)\gamma}{w(T_r)}\right)$ and $\gamma = \frac{1-\beta}{2-\beta}$

There is no continuity condition on λ_{Sx}, it is continued for $t = [0; T_r[$ and equals to 0 after T_r.

4.5 Discussion of the results when no damage function is considered

The model described above does not incorporate any damage function, which limits its applicability in real-world settings where negative externalities associated with resource use are a prevalent concern. Nevertheless, this model without damage function could apply to countries that have a minimal negative impact from their stock of waste.

In the initial period, when there is no recycling (no circular sector), the virgin resources are consumed very quickly and at a negative growth rate. The economy experiences a decline in economic growth rate due to the diminishing stock of virgin resources. This first period is very short compare to the second period where the circular sector is activated. This scenario highlights the negative consequences of consumption patterns that do not take into account the finite nature of natural resources. Nevertheless, when the circular sector is activated the economy after a degrowth, can reach a sustaining path.

The findings from this specific model demonstrate that the economy can achieve a steady state, which can result in sustaining the growth of the economy indefinitely under certain specific settings. The parameter β, which captures the level of circularity in the economy, plays a pivotal role in determining several critical parameters in the model. This includes the allocation of labour in the circular sector, the upper limit on the elasticity of substitution between materials and labour in the production sector and in the circular sector, and the flow of waste extracted from the stock, which ultimately determines consumption levels.

From the results, and as illustrated in Figure 12 we observe that the share of labour dedicated to the circular sector is determined by the β, being negatively dependent on β. Thus, the outputs from the production function are is circular, the more there is material available for the circular sector to recycle, the less their is the need to have labour substituting for material inputs in the production function. This results shows that a circular sector, dedicated to a full circular economy, where little products ends up in the waste stock (thus for a large β), as defined for example by Figge et al. (2023), is more labour dependent than a sector dedicated to recycling, where the weight of the production is placed in the materials.

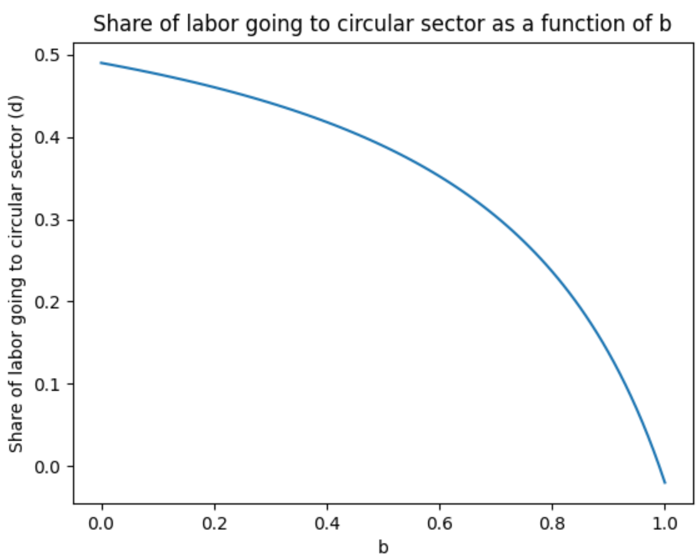


Figure 7: Evolution of δ depending on the values of β

Notes: The letter "b" in the x-axis, refers to values for β, while the letter "d" in the y-axis refers to the values for δ

When we compute the different evolution of the variables depending on the initial choice of β, in Figure 5 and Figure 6, the importance of the size of β. The figures provide an overview of the influence on the evolution of the variables based on the initial choice of β. It is evident from the results that the economy has an incentive to minimize β, indicating that while the economy can undergo an infinite path with a steady state of zero growth, it can achieve constant consumption through recycling, rather than through circularity.

In this case, the economy encourages the stock of waste to remain in circulation, thus leading to a preference for a smaller β value to enable the economy to use more resources. Furthermore, the results suggest that principles such as reuse, share, and eco-design may not be suitable in this specific scenario where the negative externalities associated with waste management are not considered. These results from the model, indicating that the smaller the β the larger the benefits for the economy, are not conducive to achieving circularity as define for example in Figge & Thorpe (2023). Indeed, recycling is often considered as the lower circular strategy Figge et al. (2023).

5 Model with damages

A two sector endogenous growth model with damage function

When a damage function is considered, the model is the one described in Section 3. As a reminder, the optimization problem of the social planner writes:

$$\max_{\delta(t),x(t),w(t)} \int_t u\left(c\left(t\right)\right)e^{-\rho t}dt$$

and it is subjected to the following constraints:

$$\begin{aligned}
\dot{L}(t) &= -\frac{\pi}{S_w(t)}\mathcal{D}(S_w(t)) \\
\delta\left(t\right) &= \frac{l_z(t)}{L(t)} = \frac{1-l_y(t)}{L(t)} \\
L(0) &= L_0 \textit{ (Initial condition)} \\
\dot{S}_x\left(t\right) &= -x\left(t\right) \\
S_x\left(t\right) &\geq 0 \\
S_x(0) &= \overline{S_{x0}} \textit{ (Initial condition)} \\
\dot{S}_w\left(t\right) &= (1-\alpha)x\left(t\right) + (1-\beta)z\left(t\right) - w\left(t\right) \\
S_w\left(t\right) &\geq 0 \\
S_w(0) &= 0 \textit{ (Initial condition)}
\end{aligned}$$

Different mechanisms are expected[8] to evolve and appear at different periods of time. There are supposedly five different periods that could be observed, depending on the labour allocation, see Figure 8 to Figure 12. Crucial determinants of these periods are the marginal cost of labour of the stock of waste S_w, or as per say, the damages induced by $\mathcal{D}(S_w)$ on the labour productivity compared to the benefits (i.e. partial removal of the stock of waste) brought with circular sector. These periods are detailed in the following sections.

5.1 Pre-recycling stage

At time $t = t_0$, before any changes have occurred, the stock of exhaustible resource S_x equals its maximum $\overline{s}_x$. This stock might be deplete up to a limit of 0. We are not sure yet that the economy will exhaust all the exhaustible resources available in this stock - This will depend on the different parameters. The initial stock of labour L equals 1.

At time $t = t_0 + \epsilon$, exhaustible (i.e., virgin) resources $x(t)$ are extracted from our stock S_x and input in the main production function. At this point in time, illustrated by figure 8, the waste stock S_w already starts to grow and to negatively impact the productivity of labour. The labour available starts to reduce because of the negative externalities due to the waste stock. There is not yet any recycling. This first period, illustrated by Figure 8, is the traditional production model with damage function. As soon as the waste damage reduces the labour productivity under a certain threshold, it is more interesting for the labour and for the economy to diverge a part of the labour supply to the circular sector, leading to the model presented in Figure 9. Thus, the second sector should commence before the stock of labour L has been exhausted.

[8] The current paper presents intuitive as the model shows restrictions to be solved analytically, thus simulations tools are being developed to understand and investigate further the model's dynamics and optimization paths. Details of the analytical resolution can be found in Appendix B. The simulations' code being developed by the author can be found in the following GitHub repository.

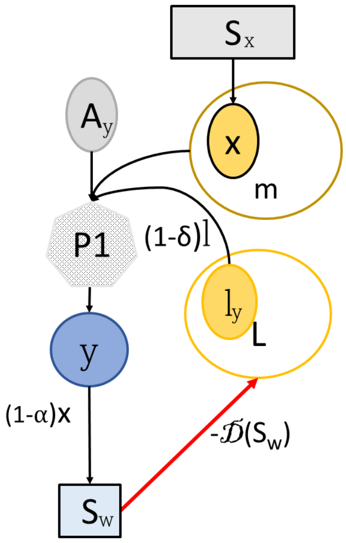


Figure 8: Economy at time $t_0 + \epsilon \leq t < T_{r+x}$

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner to reach pareto optima in the centralised economy, while the endogenous output variables are blue. The dashed lines represent the components of the recycling sector.

The exact threshold for the circular economy to emerge, is to be determined by the model. This thresholds will depend on the damage function and on the parameters of the circular sector, respectively, the total factor of productivity A_z of $P2$, and the elasticity of substitution between materials and labour in the production function, named θ. If we assumed that this elasticity is such that $\theta > 0.5$, thus during this stage, the the optimal level of virgin resource extraction will most probably increase to compensate for the losses of the labour stock.

5.2 Circular stages

By considering the damage function, the social planner should allow the circular sector to materialize before the exhaustion of the stock of labour. However, there is no certainty that the circular sector will commence before the exhaustion of the stock of virgin resources. This is a second threshold that will depend on the parameters of the model, and that could be interpreted as policy interventions to constraint the social planner to commence the circular sector before the depletion of the resources. Once the circular sector has commenced, there are then four possibles circular stages that can take place. Each of there existence needs to be verified.

Depending on the exhaustion of the stock of virgin materials S_x at the start of the circular sector, two possible stages may arise, as depicted in Figures 9 and 10. In each stage, a steady state could potentially be achieved by balancing the level of material and labour stocks with the substituability of labour in the production function, as well as the level of damages. If virgin material is injected into the system, as depicted in Figure 9, the steady state will only be temporary since the stock of virgin resources S_x will eventually become depleted, leading to the economy transitioning into the stage represented by Figure 10. If a steady state cannot be achieved, the economy will enter the stage depicted in Figure 12.

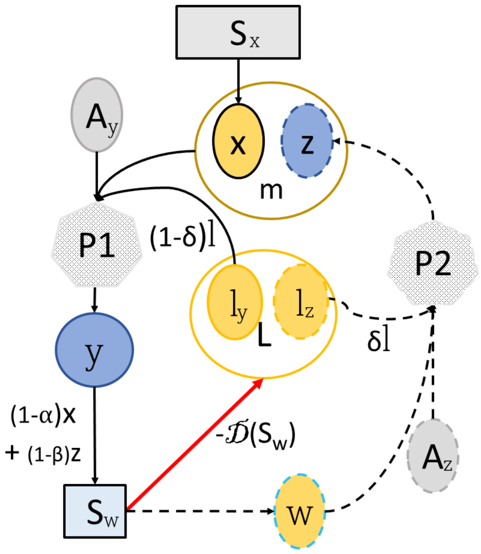


Figure 9: Economy at time $T_{r+x} \leq t < T_r$

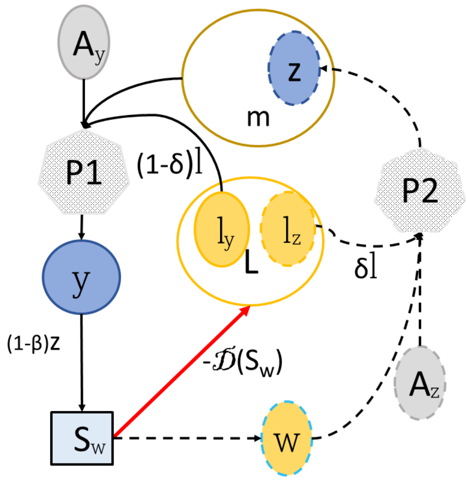


Figure 10: Economy at time $T_r \leq t < T_z$

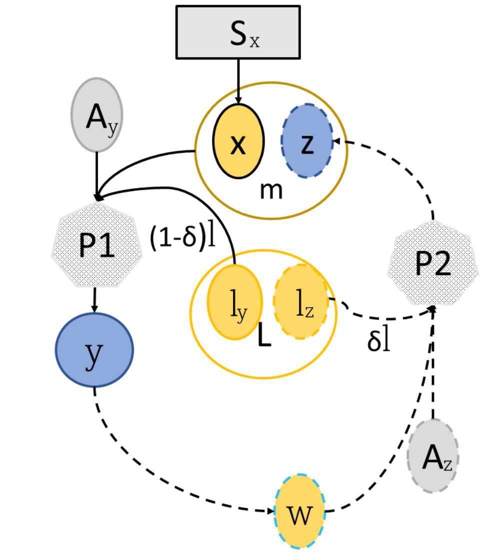


Figure 11: Economy at time $T_{r+x} \leq t < T_z$

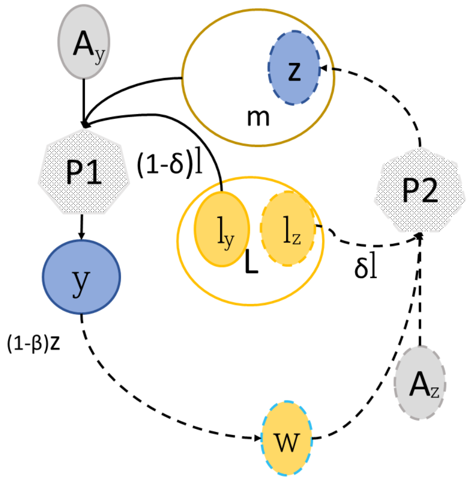


Figure 12: Economy at time $t \geq T_z$

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner to reach pareto optima in the centralised economy, while the endogenous output variables are blue. The damage function is represented by a red arrow. The dashed lines represent the components of the recycling sector.

Once the stock of waste is fully depleted through the use of waste in the circular production function, there will be no further damage to the stock of labour. However, with no available stock of resources (assuming both the stock of virgin material and of waste are depleted), the economy will inevitably degrow due to the presence of losses β, as represented in Figure 12. At the point when the stock of virgin materials is depleted, illustrated by Figure 10, the economy will have no choice but to degrow, as the availability of labour in each period will continue to decline. Policy interventions should be enacted to prevent this scenario from occurring.

Alternatively, if the stock of virgin material is not exhausted before the economy is able to reduce the stock of waste, as depicted in Figure 11, a steady state could be achieved for a certain amount of time. Such a steady state would induce that the amount of virgin material extracted compensates for the system's losses, such that $(1-\alpha)x(t)+(1-\beta)z(t)=w(t)$, $x(t)=\frac{\left((\beta-1)A_z(\frac{w(t)}{\delta(t)L(t)})^{\gamma-1}-1\right)w(t)}{1-\alpha}$. As $\gamma<1$, the larger the share of labour in the circular economy, and the larger the stock of labour available,the larger the amount of virgin material to be extracted. Nevertheless, the economy will eventually fall back into an economy without stocks but with losses (i.e., β), as shown in Figure 12, and reach a degrowth path until its end.

5.3 Discussion of the intuitions when a damage function is considered

The model above demonstrates that when a damage function is taken into account, there are two possible outcomes: either the stock of labour is affected by damages resulting from the mismanagement of the stock of waste, or degrowth occurs. This implies that in an economy where economic growth is prioritized, there is a tendency to maintain a certain level of waste stock regardless of its negative impact on labour, as it ensures economic growth and sustainability. This highlights the current situation where economic stakeholders seems to follow genuine incentives to continue generating waste instead of prioritizing proper circular strategies.

The inclusion of a damage function that accounts for negative externalities resulting from mismanagement of the stock of waste on labour shows interesting differences compared to the model without a damage function. Specifically, the time until resource exhaustion is significantly extended when a damage function linked to the waste stock is taken into account. These findings highlight the importance of internalizing the negative externalities associated with the accumulation of waste.

Furthermore, the potential existence, sustainability and overall characteristics, of steady states depend strongly on the presence (or absence) of a damage function. In the absence of a damage function, the steady state extends indefinitely. However, when a damage function is introduced, the reach of a steady state is limited to a finite period in the long run. This dynamic presents an economic incentive against the internalisation of the damage linked with waste accumulation, contradicting previous findings.

These observations highlight the importance of policy interventions aiming at finding a balance between economic growth and societal welfare. Such interventions become crucial in order to address the trade-off between achieving sustainable development and mitigating the negative impacts of waste accumulation on labour and overall well-being.

To illustrate and internalize the impact of waste mismanagement on well-being, this model could be adapted, following the work of Bretschger & Pattakou (2019), incorporating the damage function into the utility function. To address this issue, policy interventions should be designed to integrate objectives beyond purely economic ones.

Alternatively, the model can be adapted to reflect other possible realities. For example, one could include a growing total factor productivity. Following Schumpeterian models and as in Zhou & Smulders (2021), innovation and technology could be considered as ways to compensate for losses in the system and maintain constant growth without relying on any stock of resources (virgin or waste).

6 Conclusion

6.1 Circular economy should not be (only) recycling

The importance of addressing labour-related concerns in the context of the circular economy is becoming increasingly important. While practitioners often discuss the implications of a circular economy transition on labour, scholar acknowledgment is lagging behind. An expending strand of the literature is ascertaining the positive effects of the circular economy on the reduction of negative externalities (e.g., resource depletion and pollution). In the current technological (e.g., raise of artificial intelligence, internet-of-things) and climatic contexts (e.g., migrations, pandemics), it is important to promptly address issues regarding the future of labour (namely, the productivity and the role of labour in the industry of tomorrow). This study aims to fill this gap by examining the optimal allocation of labour to the circular sector to sustain development while limit environmental impacts of waste.

A theoretical model of a closed economy with two sectors, considering the improper management of waste and its negative effect on labour productivity, is developed. The waste stock is assumed to be inappropriately managed, and to negatively affect labour productivity. (e.g. January 2019, in Rome, Italy, a landfill burned forcing workers to stay home for several days (The Independent, 2019). This study analyses the circular economy as an alternative to cope with both the exhaustibility of resource, and the pollution induced by improper waste management. The theoretical model, by being analysed both when the externalities linked to waste stock are included or excluded, assesses the potential role of the circular economy to reduce waste linked externalities on labour. By comparing the model dynamics with and without waste-related externalities, we investigate the potential role of the circular economy in reducing these negative impacts on labour.

This paper investigates the optimal allocation of labour between the production and the circular sector to sustain development while limiting environmental impact caused by waste on labour. The changes in economies are assumed to be due to an endogenous choice of labour in the supply and they are analysed under which conditions the circular economy can generate endogenous growth. This approach is innovative as it puts emphasis on the role of the labour in the circular economy, illustrating the important role of the circular sector to sustain economic development. The results lead to a full circular economy model for the management of resources. The model provides different thresholds in terms of optimal labour supply, optimal amount of materials (both circular and exhaustible) to supply to the economy. From these optimal conditions, the study evaluates the labour performance and timeline of the different periods of the economy (namely, when there is only the production sector, when the circular sector is an asset for the economy, when there are both supply of virgin and circular material or only one of both etc.).

This study provides valuable insights into the role of the circularity level (β) in achieving circularity in the economy. The findings from the model without a damage function reveal that under certain specific settings, the economy can achieve a steady state, sustaining the economy indefinitely. The level of circularity, captured by the parameter β, plays a crucial role in determining various parameters in the model, such as labour allocation, elasticity of substitution, and waste flow. However, the findings indicate that when no damage function is considered, minimizing β may not always be the optimal solution to achieving circularity, and policymakers need to consider a broader range of factors that impact the economy's sustainability. The share of labour dedicated to the circular sector is negatively dependent on β, indicating that a lower level of circularity leads to a reduced need for labour to substitute material inputs in production. This suggests that a full circular economy, where minimal products end up in the waste stock, is more reliant on labour compared to a recycling-focused sector where the emphasis is on material inputs.

On the other hand, when a damage function is introduced, the results are different. The model incorporating the damage function shows that there are two possible outcomes: damage to the stock of labour due to mismanagement of waste or degrowth. In economies prioritizing economic growth, there is a tendency to maintain a certain level of waste stock despite its negative impact on labour, as it ensures economic growth and sustainability. The inclusion of the damage function significantly extends the time until resource exhaustion, highlighting the importance of internalizing the negative externalities associated with waste accumulation.

These contrasting findings highlight the trade-off between economic growth and the well-being of labour and society. The model suggests that policy interventions are necessary to obtain a balance between growth and welfare. Incorporating the damage function into the utility function can provide a better understanding of the impact of waste mismanagement on well-being. These findings emphasize the need for a holistic approach in tackling the issues arising from the accumulation of waste and fostering circular economy towards sustainable development. Further research could focus on expanding the scope of the model to account for more complex economic scenarios.

This paper contributes to the ongoing debate on the role of circularity in achieving economic growth and sustainability. The study establishes a new strand of the literature related to the circular economy and allows further research on the importance of the circular economy in the field of labour economics. This research provides predictions on the impact of the circular economy for labour productivity. The findings demonstrate that a circular sector, dedicated to a full circular economy, is more labour dependent than a sector dedicated to only recycling, and that policy interventions should be designed to integrate objectives beyond purely economic ones. Furthermore, the model highlights the negative consequences of the (mis-)management of the stock of waste, and the need to consider a damage function in the utility function. Policy makers are invited to embrace the results of this research and to develop instruments favoring fully circular sectors rather than recycling.

6.2 Limitations

The model presented in this study faces several limitations that should be taken into account when interpreting the results. Firstly, the analytical solution of the model is analytically difficult and may require simulations to obtain meaningful insights. The complexity of the model's equations and interactions between variables make it challenging to derive explicit analytical solutions. Therefore, numerical simulations are necessary to explore the dynamics and outcomes of the model.

Another limitation is the assumption of constant total factor productivity (TFP) throughout the analysis. While assuming constant TFP simplifies the model and facilitates mathematical calculations, it does not fully capture the dynamic nature of the economy. In reality, TFP for the circular sector is expected to evolve over time due to technological progress, innovation, and learning-by-doing effects. By incorporating these factors into the model, a more realistic analysis of the long-term sustainability and growth potential of the economy could be achieved.

Furthermore, the model assumes no costs associated with material extraction. This assumption overlooks the potential environmental and economic costs associated with extracting and depleting natural resources. In practice, material extraction often involves various costs, including environmental degradation, depletion of finite resources, and the need for waste management. Incorporating these costs into the model would provide a more comprehensive understanding of the trade-offs and challenges associated with resource consumption and circularity.

Lastly, the model assumes the same TFP for recycling and virgin material in the circular sector. While this assumption may be valid under certain circumstances, it does not account for potential differences in the efficiency and productivity of recycling processes compared to the extraction of virgin materials. In reality, recycling processes may require additional investments, infrastructure, and technological advancements, which can influence their productivity and TFP. Considering heterogeneous TFP across different sectors and activities would enhance the accuracy and realism of the model.

All the above mentioned limitations are essential to acknowledge. Addressing these limitations and refining the model's assumptions would contribute to a more robust analysis of the economic and environmental dynamics associated with resource consumption and circularity.

6.3 Next steps

There are several needed next steps to advance the analysis of the model presented in this paper. One of this step involves refining the simulations to allow the model to be fully computed when the damage function is included. This entails enhancing the numerical methods employed in the simulations, fine-tuning the model parameters, and conducting sensitivity analyses to assess the robustness of the results. By refining the simulations, a more detailed understanding of the dynamics and outcomes of the system can be attained.

In order to provide a more comprehensive analysis, the model can be extended to incorporate wage differentials between the linear (production) and circular sectors. This extension would allow for the consideration of varying skill requirements and productivity levels associated with circularity and linearity, thereby shedding light on labour market dynamics and distributional effects within the economy.

Furthermore, incorporating costs associated with material extraction and waste management into the model would enhance its realism. By accounting for the economic costs of these activities, the model could capture the trade-offs between economic growth, circularity, and environmental sustainability. Additionally, exploring the implications of decentralized decision-making in resource allocation and waste management would provide insights into the efficiency and effectiveness of different governance structures.

To better capture the impact of waste mismanagement on consumer welfare, the utility function could be modified to incorporate the damage function. This modification would enable the quantification of negative externalities resulting from waste mismanagement and facilitate an evaluation of the welfare implications for individuals and society. Such an analysis would support policymakers in making informed decisions regarding waste management strategies and prioritising sustainable and circular practices.

The shape of the damage function is another aspect that would be worth exploring further. Exploring different functional forms and accounting for specific characteristics of the damage function would allow for an assessment of the optimal design of the damage function. This includes investigating non-linear relationships between waste stock and damage, capturing threshold effects, and considering the dynamic interactions between waste accumulation and damages.

Finally, varying the shares of α and β parameters would enable an examination of the allocation of labour and resources between the linear and circular sectors. By exploring different (endogeneous) rate of linearity and circularity in material losses, the model could assess the implications of these parameters' dynamics on variables such as economic growth, resource depletion, and waste accumulation. This analysis would provide policymakers with insights into the optimal circular policy on recycling and circular design to promote sustainable and circular economic development.

In summary and to conclude, the next steps involve refining the simulations, incorporating wage differentials, considering costs and decentralization, modifying the utility function to include damage, exploring different shapes of the damage function, and adding dynamics for the shares of α and β. These enhancements would contribute to a more comprehensive and policy-relevant analysis of the economic and environmental dynamics related to resource consumption and circularity.

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Appendix A Model without damage function

Appendix A.1 Proof of Lemma 1

When there is no damage function considered, $x(t)$ and $z(t)$ are never used at the same time. The following section prove this hypothesis.

Let's assume that there is a time $t = T_{r+x}$ for which the second sector P2 appears to overcome the scarcity of the stock of exhaustible resources, while the stock of exhaustible resources S_x is not yet exhausted. This potential scenario is illustrated by Figure 13.

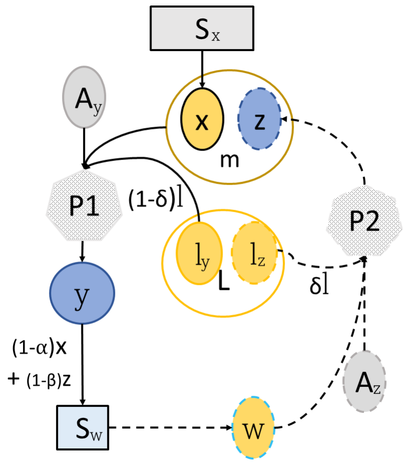


Figure 13: Hypothetical stage without damage when there would be both extraction of virgin material and recycling by the circular sector

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner to reach pareto optima in the centralised economy, while the endogenous output variables are blue. The dashed lines represent the components of the recycling sector.

P2 will also reduce the growth of the waste stock S_w. This second sector P2 is labour intensive and uses waste w extracted from the stock of waste S_w to produce circular material z to input in the production sector of the economy P1. Figure 2 illustrates the economy from T_{r+x} to T_r. Material used as inputs in the main production function are both virgin x extracted from our stock of exhaustible S_x and circular z converted from waste w extracted from the stock of waste S_w. In that case we have:

$$
\begin{aligned}
\delta(t) \neq 0 \quad &: \quad u'(c(t))\Big(A_m(t) f'_{m(t)} h'_{\delta(t)} + f'_{1-\delta(t)}\Big) + \lambda_{Sw}(t)(1-\beta) h'_{\delta(t)} = 0 \\
&\iff c(t)^{1-\sigma}\left(A_m(t)\frac{\theta}{m(t)}\frac{(1-\gamma)z(t)}{\delta(t)} - \frac{(1-\theta)}{1-\delta(t)}\right) + \lambda_{Sw}(t)(1-\beta)\frac{(1-\gamma)z(t)}{\delta(t)} = 0 \qquad (42) \\
x(t) \neq 0 \quad &: \quad u'(c(t)) A_m(t) f'_{m(t)} + \lambda_{Sw}(t)(1-\alpha) = \lambda_{Sx}(t) \\
&\iff c(t)^{1-\sigma}\left(A_m(t)\frac{\theta}{m(t)}\right) + \lambda_{Sw}(t)(1-\alpha) = \lambda_{Sx}(t) \qquad (43) \\
w(t) \neq 0 \quad &: \quad u'(c(t)) A_m(t) f'_{m(t)} h'_{w(t)} + \lambda_{Sw}(t)\big((1-\beta)h'_{w(t)} - 1\big) = 0 \\
&\iff c(t)^{1-\sigma} A_m(t)\frac{\theta}{m(t)}\frac{\gamma z(t)}{w(t)} + \lambda_{Sw}(t)\left((1-\beta)\frac{\gamma z(t)}{w(t)} - 1\right) = 0 \qquad (44) \\
S_x(t) \neq 0 \quad &: \quad \frac{\dot{\lambda}_{Sx}(t)}{\lambda_{Sx}(t)} = \rho \qquad (45)
\end{aligned}
$$

$$S_w(t) \neq 0 \quad : \quad \frac{\dot{\lambda_{Sw}}(t)}{\lambda_{Sw}(t)} = \rho \tag{46}$$

From (48) we have:

$$\lambda_{Sw}(t) = c(t)^{1-\sigma}\theta\frac{A_m(t)}{m(t)}\left(\frac{h'_w(t)}{1-(1-\beta)h'_w(t)}\right) \tag{47}$$

but from (46) we have:

$$\lambda_{Sw}(t) = c(t)^{1-\sigma}\theta\frac{A_m(t)}{m(t)}\frac{1}{(1-\beta)}\left(1-\frac{\delta(t)}{1-\delta(t)}\frac{m(t)(1-\theta)}{(1-\gamma)\theta A_m(t)z(t)}\right) \tag{48}$$

Thus that gives:

$$\left(\frac{h'_w(t)}{1-(1-\beta)h'_w(t)}\right) = \frac{1}{(1-\beta)}\left(1-\frac{\delta(t)}{1-\delta(t)}\frac{m(t)(1-\theta)}{(1-\gamma)\theta A_m(t)z(t)}\right) \tag{49}$$

Additionally, from equations (46), (47) and (48) we obtain:

$$\lambda_{Sx}(t) = c(t)^{1-\sigma}\frac{\theta}{m(t)}\left(A_m(t)+(1-\alpha)A_m(t)\left(\frac{h'_w(t)}{1-(1-\beta)h'_w(t)}\right)\right) \tag{50}$$

$$\delta(t) = \left(\frac{u'(c(t))(1-\theta)c(t)}{(1-\gamma)z(t)\left(u'(c(t))A_m(t)\theta\frac{c(t)}{m(t)}+\lambda_{Sw}(t)(1-\beta)\right)}+1\right)^{-1} \tag{51}$$

$$\iff \delta(t) = \left(\frac{(1-\theta)}{(1-\gamma)\theta\frac{z(t)A_m(t)}{m(t)}\left(1+\left(\frac{(1-\beta)h'_w(t)}{1-(1-\beta)h'_w(t)}\right)\right)}+1\right)^{-1} \tag{52}$$

For (47) and (48) we have

$$(1-\sigma)g_c = \rho + g_m \tag{53}$$

$$(1-\sigma)g_c = \rho + g_m + (g_z - g_w)\left(\frac{\lambda_{Sw}(t)}{c(t)^{1-\sigma}A_m(t)\frac{\theta}{m(t)}}-1\right) \tag{54}$$

Thus, we have either $g_z = g_w$ or we have $\frac{\lambda_{Sw}(t)}{c(t)^{1-\sigma}A_m(t)\frac{\theta}{m(t)}} = 1 = \frac{h'_w(t)}{1-(1-\beta)h'_w(t)}$ which both lead to the same conclusion, (51) and (55) now reads:

$$\lambda_{Sw}(t) = c(t)^{1-\sigma}A_m(t)\frac{\theta}{m(t)} \tag{55}$$

$$\lambda_{Sx}(t) = c(t)^{1-\sigma}\frac{\theta}{m(t)}\left(A_m(t)+(1-\alpha)A_m(t)\right) \tag{56}$$

and:

$$z(t) = \frac{1}{(2-\beta)\gamma}w(t) \iff g_z = g_w \tag{57}$$

$$\dot{S_w}(t) = (1-\alpha)x(t)+w(t)\left(\frac{1-\beta}{(2-\beta)\gamma}-1\right) \tag{58}$$

$$\frac{1}{(2-\beta)\gamma}w(t) = A_zL^{(1-\gamma)}\delta(t)^{(1-\gamma)}w(t)^{\gamma} \tag{59}$$

$$\delta(t)^{(1-\gamma)} = \frac{w(t)^{1-\gamma}}{(2-\beta)\gamma A_zL^{(1-\gamma)}} \iff g_\delta = \frac{g_w(1-\gamma)}{(1-\gamma)} \tag{60}$$

From (53) and (57) we get:

$$\delta(t) = \left(\frac{(1-\theta)}{(1-\gamma)\theta\beta\frac{z(t)A_m(t)}{m(t)}}+1\right)^{-1} = \left(\frac{(1-\theta)}{(1-\gamma)\theta\frac{z(t)A_m(t)}{m(t)}\left(1+\left(\frac{(1-\beta)h'_w(t)}{1-(1-\beta)h'_w(t)}\right)\right)}+1\right)^{-1} \tag{61}$$

$$\beta = \left(1+\left(\frac{(1-\beta)h'_w(t)}{1-(1-\beta)h'_w(t)}\right)\right) \tag{62}$$

As we have $1 = \frac{h'_w(t)}{1-(1-\beta)h'_w(t)}$ thus, $\beta = 1$ which is not possible in the current settings of the model. We can conclude that when there are no damages, and following the settings of this model, we can not have at the same time $x(t)$ and $z(t)$ as material input.

Appendix A.2 Proof of Lemma 2

In the special case where we consider all the stock of exhaustible resource to be consumed, then we have, according the transversality conditions:

$$\lim_{t\to\infty} \lambda_{Sx}(t)e^{-\rho t}S_x(t) = 0 \tag{63}$$

$$\lim_{t\to\infty} \lambda_{Sx}(0)e^{\rho t}e^{-\rho t}\big(\frac{-x_0}{g_x}e^{g_x t} + S_{xT}\big) = 0$$

$$\lim_{t\to\infty} \lambda_{Sx}(0)\big(\frac{-x_0}{g_x}e^{g_x t} + S_{xT}\big) = 0$$

For this condition to be valid, the only possibility is that $S_{xT} = 0$ ($g_x \leq 0$ and $\lambda_{Sx}(0) \neq 0$), thus in this particular case where we use all the natural resource stock, we have:

$$S_x(t) = \frac{-x_0}{g_x}e^{g_x t} = \frac{-x_0}{g_x}e^{\frac{(1-\sigma)g_{A_y}-\rho}{1-\theta(1-\sigma)}t}$$

$$S_x(0) = \overline{S_{x0}} = \frac{-x_0}{g_x} \tag{64}$$

$$x(0) = x_0 = -\overline{S_{x0}}g_x \tag{65}$$

Similarly, we have to satisfy

$$\lim_{t\to\infty} \lambda_{Sw}(t)e^{-\rho t}S_w(t) = 0 \tag{66}$$

We know from the initial condition that:

$$S_w(0) = 0 = (1-\alpha)(\frac{x_0}{g_x}) + S_{wT} \Longleftrightarrow S_{wT} = -(1-\alpha)\frac{x_0}{g_x} \tag{67}$$

$$S_w(t) = (1-\alpha)\frac{x_0}{g_x}(e^{g_x t} - 1) \tag{68}$$

$$\lim_{t\to\infty} \lambda_{Sw}(0)e^{-\rho t}e^{-\rho t}(1-\alpha)\frac{x_0}{g_x}(e^{g_x t} - 1) = 0$$

$$\lim_{t\to\infty} \lambda_{Sw}(0)(1-\alpha)\frac{x_0}{g_x}(e^{g_x t} - 1) = 0 \tag{69}$$

For this condition to be valid, the only possibility is that $\lambda_{Sw}(0) = 0$, thus in this case we have:

$$\lambda_{Sw}(t) = \mu_w(t) = 0 \Longleftrightarrow \lambda_{Sx}(t) = c(t)^{1-\sigma}\frac{\theta}{x(t)} = \lambda_{Sx}(0)e^{\rho t}$$

$$\lambda_{Sx}(0) = c(0)^{1-\sigma}\frac{\theta}{x(0)} = \Big(A_y\left(-A_m\overline{S_{x0}}g_x\right)^{\theta}L^{1-\theta}\Big)^{1-\sigma}\frac{\theta}{-\overline{S_{x0}}g_x} \tag{70}$$

$$\lambda_{Sx}(t) = \Big(A_y\left(-A_m\overline{S_{x0}}g_x\right)^{\theta}L^{1-\theta}\Big)^{1-\sigma}\frac{\theta}{-\overline{S_{x0}}g_x}e^{\rho t} \tag{71}$$

Appendix A.3 Details of the model for the pre-recycling stage

At time $t = t_0$, the stock of exhaustible resource S_x equals its maximum $\overline{S_{x0}}$. This stock might be deplete to a lower limit of 0. We are not sure yet that the economy will exhaust all the exhaustible resources available in this stock. We consider the stock of labour l equals to 1. All the others stocks are null as no waste has been produced yet. Following $t = t_0$, three periods are expected to occur. At time $t = t_0$, exhaustible resources are extracted from our stock S_x and input in the main production function. At this point in time, illustrated by figure 14, the waste stock S_w already starts to grow while the stock of exhaustible resource S_x diminishes. There is not yet any recycling. Thus for all $t_0 \leq t < T_r$:

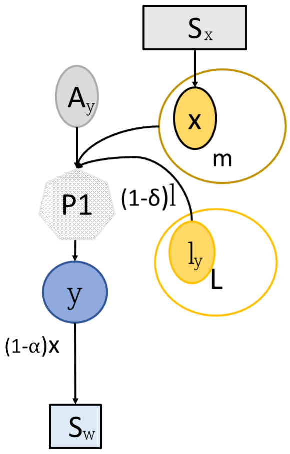


Figure 14: Economy at time $t_0 \leq t < T_r$

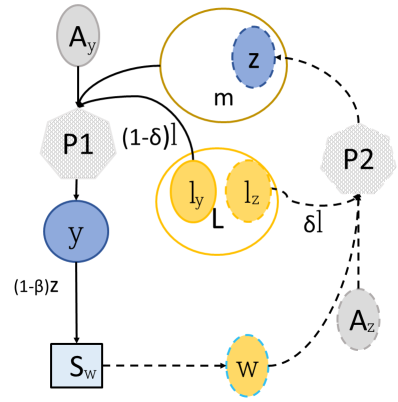


Figure 15: Economy at time $t \geq T_r$

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner to reach pareto optima in the centralised economy while the output endogenous variables are blue. The damage function is represented by a red arrow. The dash lines represent the components of the recycling sector.

$$\delta(t) = 0 \quad : \quad c(t)^{1-\sigma}(1-\theta) = \mu_\delta(t) \tag{72}$$

$$x(t) \neq 0 \quad : \quad u'(c(t))A_m(t)f'_{m(t)} + \lambda_{Sw}(t)(1-\alpha) = \lambda_{Sx}(t)$$

$$\iff \frac{c(t)^{1-\sigma}\theta}{x(t)} = \lambda_{Sx}(t) - \lambda_{Sw}(t)(1-\alpha) \tag{73}$$

$$w(t) = 0 \quad : \quad \mu_w(t) = \lambda_{Sw}(t) \tag{74}$$

$$S_x(t) \neq 0 \quad : \quad \lambda_{Sx}(t)\rho - \dot{\lambda_{Sx}}(t) = 0 \tag{75}$$

$$S_w(t) \neq 0 \quad : \quad \lambda_{Sw}(t)\rho - \dot{\lambda_{Sw}}(t) = 0 \tag{76}$$

We can rewrite the FOC for all $t_0 \leq t < T_r$ such as:

$$\delta(t) = z(t) = w(t) = 0$$

$$\mu_\delta(t) = c(t)^{1-\sigma}(1-\theta) \tag{77}$$

$$\lambda_{Sx}(t) = c(t)^{1-\sigma}\frac{\theta}{x(t)} + \lambda_{Sw}(1-\alpha) = \lambda_{Sx}(0)e^{\rho t} \tag{78}$$

$$\lambda_{Sw}(t) = \lambda_{Sw}(0)e^{\rho t} = \mu_w(t) \tag{79}$$

$$\dot{S_x}(t) = -x(t) \tag{80}$$

$$\dot{S_w}(t) = (1-\alpha)x(t) \tag{81}$$

$$c(t) = A_{y(t)}\left(A_{m(t)}x(t)\right)^\theta (L)^{1-\theta} \tag{82}$$

$$(1-\sigma)g_{c(t)} = g_{x(t)} + \rho \iff g_c(t) = \frac{-\theta\rho}{1-\theta(1-\sigma)} \tag{83}$$

$$g_{x(t)} = \frac{-\rho}{1-\theta(1-\sigma)} \tag{84}$$

With $g_{c(t)} = \frac{\dot{c(t)}}{c(t)}$ and because for $t < T_r$ we have $g_{m(t)} = g_{x(t)} = \frac{\dot{x(t)}}{x(t)}$

We can start by solving for the co-state variable $\lambda_{Sw}(t)$:

$$\dot{\lambda}_{Sw}(t) = -\lambda_{Sx}(t)(1-\alpha)$$

This is a linear first-order differential equation, which can be solved using separation of variables:

$$\begin{aligned}\int_0^t \frac{d\lambda_{Sw}(\tau)}{d\tau}d\tau &= -\int_0^t \lambda_{Sx}(\tau)(1-\alpha)d\tau \\ \lambda_{Sw}(t) &= c_1 - \alpha\int_0^t \lambda_{Sx}(\tau)d\tau\end{aligned}$$

where c_1 is a constant of integration.

Now, we can substitute this expression for $\lambda_{Sw}(t)$ into the equation for $\dot{\lambda}_{Sx}(t)$:

$$\begin{aligned}\dot{\lambda}_{Sx}(t) &= -u'(Ay(A_m x(t))^\theta)A_y A_m \theta(\lambda_{Sw}(t)(1-\alpha)) \\ &= -u'(A_y(A_m x(t))^\theta)A_y A_m \theta\left(c_1\alpha - \alpha^2\int_0^t \lambda_{Sx}(\tau)d\tau\right)\end{aligned}$$

This is a separable differential equation, which can be solved using separation of variables:

$$\begin{aligned}\int_0^t \frac{d\lambda_{Sx}(\tau)}{u'(A_y(A_m x(\tau))^\theta)A_y A_m\theta\left(c_1\alpha - \alpha^2\int_0^\tau \lambda_{Sx}(s)ds\right)} &= -1 \\ \frac{1}{u'(A_y(A_m x(t))^\theta)A_y A_m\theta\left(c_1\alpha - \alpha^2\int_0^t \lambda_{Sx}(s)ds\right)} &= -t + c_2\end{aligned}$$

For all $t_0 \leq t < T_r$, the flow variables read:

$$\delta(t) = z(t) = w(t) = 0 \tag{85}$$

$$c(t) = c_0 e^{\frac{-\theta\rho}{1-\theta(1-\sigma)}t} \tag{86}$$

$$x(t) = x_0 e^{g_x t} \tag{87}$$

$$m(t) = A_m x_0 e^{\frac{-\rho}{1-\theta(1-\sigma)}t} \tag{88}$$

where $c_0 = c(0) = A_y\left(A_m x(0)\right)^\theta L^{1-\theta}$

The stock variables read:

$$S_x(t) = \frac{-x_0}{g_x}e^{g_x t} \tag{89}$$

$$S_w(t) = (1-\alpha)(\frac{x_0}{g_x}e^{g_x t}) + S_{wT} \tag{90}$$

S_{xT} and S_{wT} being two constant to be defined.

We know from the initial condition that:

$$S_x(0) = \overline{S_{x0}} = \frac{-x_0}{g_x} \Longleftrightarrow x_0 = -g_x\overline{S_{x0}} \tag{91}$$

$$S_w(0) = 0 = (1-\alpha)(\frac{x_0}{g_x}) + S_{wT} \Longleftrightarrow S_{wT} = -(1-\alpha)\frac{x_0}{g_x} \tag{92}$$

$$S_w(t) = (1-\alpha)\frac{x_0}{g_x}(e^{g_x t} - 1) \tag{93}$$

And the co-state variables read:

$$\lambda_{Sx}(t) = c(t)^{1-\sigma}\frac{\theta}{x(t)} + \mu_w(t)(1-\alpha) = \lambda_{Sx}(0)e^{\rho t} \tag{94}$$

$$\lambda_{Sw}(t) = \mu_w(t) = \lambda_{Sw}(0)e^{\rho t} \tag{95}$$

With c2 being a second constant of integration.

Appendix A.4 Details of the model for the circular stage

After a time $t = T_r$, when no more virgin material is extracted from the exhaustible resource stock, as depicts by figure 15. Virgin material are not used as input. There is two possible cases:
(1) either the stock of exhaustible resource is depleted and $S_x = 0$

$$\delta(t) \neq 0 \quad : \quad u'(c(t))\Big(A_m(t)f'_{m(t)}h'_{\delta(t)} + f'_{1-\delta(t)}\Big) + \lambda_{Sw}(t)(1-\beta)h'_{\delta(t)} = 0$$

$$\iff c(t)^{1-\sigma}\left(\frac{\theta(1-\gamma)}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{Sw}(t)(1-\beta)\frac{(1-\gamma)z(t)}{\delta(t)} = 0 \tag{96}$$

$$x(t) = 0 \quad : \quad u'(c(t))A_m(t)f'_{m(t)} + \lambda_{Sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{Sx}(t)$$

$$\iff c(t)^{1-\sigma}\theta\frac{1}{z(t)} + \lambda_{Sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{Sx}(t) \tag{97}$$

$$w(t) \neq 0 \quad : \quad u'(c(t))A_m(t)f'_{m(t)}h'_{w(t)} + \lambda_{Sw}(t)\big((1-\beta)h'_{w(t)} - 1\big) = 0$$

$$\iff c(t)^{1-\sigma}\theta\gamma\frac{1}{w(t)} + \lambda_{Sw}(t)\left((1-\beta)\gamma\frac{z(t)}{w(t)} - 1\right) = 0 \tag{98}$$

$$S_x(t) = 0 \quad : \quad \lambda_{Sx}(t)\rho - \dot{\lambda}_{Sx}(t) = \mu_{Sx} \tag{99}$$

$$S_w(t) \neq 0 \quad : \quad \frac{\dot{\lambda}_{Sw}(t)}{\lambda_{Sw}(t)} = \rho \tag{100}$$

(2) or the stock of resources is not depleted $S_x \neq 0$, but the economic rather prefer to input circular material z instead of x for the main production sector. There is no argument in favor of virgin material which is fully substituted by the circular material.

$$\delta(t) \neq 0 \quad : \quad u'(c(t))\Big(A_m(t)f'_{m(t)}h'_{\delta(t)} + f'_{1-\delta(t)}\Big) + \lambda_{Sw}(t)(1-\beta)h'_{\delta(t)} = 0$$

$$\iff c(t)^{1-\sigma}\left(\frac{\theta(1-\gamma)}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{Sw}(t)(1-\beta)\frac{(1-\gamma)z(t)}{\delta(t)} = 0 \tag{101}$$

$$x(t) = 0 \quad : \quad u'(c(t))A_m(t)f'_{m(t)} + \lambda_{Sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{Sx}(t)$$

$$\iff c(t)^{1-\sigma}\theta\frac{A_m(t)}{A_m(t)z(t)} + \lambda_{Sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{Sx}(t) \tag{102}$$

$$w(t) \neq 0 \quad : \quad u'(c(t))A_m(t)f'_{m(t)}h'_{w(t)} + \lambda_{Sw}(t)\big((1-\beta)h'_{w(t)} - 1\big) = 0$$

$$\iff c(t)^{1-\sigma}\theta\gamma\frac{1}{w(t)} + \lambda_{Sw}(t)\left((1-\beta)\gamma\frac{z(t)}{w(t)} - 1\right) = 0 \tag{103}$$

$$S_w(t) \neq 0 \quad : \quad \frac{\dot{\lambda}_{Sw}(t)}{\lambda_{Sw}(t)} = \rho \tag{104}$$

For $t > T_r$ we have

$$x(t) = 0$$

$$z(t) = \frac{1}{(2-\beta)\gamma}w(t) \tag{105}$$

$$w(t) = w_{Tr}e^{g_w(t-Tr)} \tag{106}$$

$$\lambda_{Sw}(t) = c(t)^{1-\sigma}\frac{\theta}{z(t)} = c(t)^{1-\sigma}\frac{\theta(2-\beta)\gamma}{w(t)} = c(t)^{1-\sigma}\frac{\theta\gamma}{w(t)\frac{(\beta-1)}{2-\beta} - 1} \tag{107}$$

$$\dot{S_w}(t) = w(t)\left(\frac{1-\beta}{(2-\beta)\gamma} - 1\right) \tag{108}$$
$$\delta(t) = \left(\frac{2(1-\gamma)\theta}{1-\theta+2(1-\gamma)\theta}\right) \tag{109}$$
$$c(t) = A_{y(t)}\left(\frac{A_{m(t)}w(t)}{(2-\beta)\gamma}\right)^{\theta}\left(\frac{1-\theta}{1-\theta+2(1-\gamma)\theta}\right)^{1-\theta} \tag{110}$$
$$g_{z(t)} = g_{w(t)} = g_{m(t)} = \frac{1}{1-\gamma} \tag{111}$$
$$g_{c(t)} = \theta\frac{1}{1-\gamma} \tag{112}$$

With

$$z(t) = \frac{w_{T_r}}{(2-\beta)\gamma}e^{\frac{1}{1-\gamma}(t-T_r)} = E(t)\,(w(t)^{\gamma})\,(\delta L)^{(1-\gamma)}$$
$$\frac{w_{T_r}}{(2-\beta)\gamma}e^{\frac{1}{1-\gamma}(t-T_r)} = E(t)\left(w_{T_r}^{\gamma}e^{\gamma\frac{1}{1-\gamma}(t-T_r)}\right)(\delta L)^{(1-\gamma)}$$
$$w_{T_r} = \sqrt[1-\gamma]{(2-\beta)\gamma A_z(\delta L)^{(1-\gamma)}} \tag{113}$$
$$w_{T_r} = \sqrt[1-\gamma]{(2-\beta)\gamma A_z e(\gamma-1)*(\delta L)^{(1-\gamma)}} \tag{114}$$

The flow variables read:

$$x(t) = 0 \tag{115}$$
$$c(t) = c_{T_r}e^{(g_{A_y}+\theta\frac{1}{1-\gamma})(t-T_r)} \tag{116}$$
$$\delta(t) = \frac{2(1-\gamma)\theta}{1-\theta+2(1-\gamma)\theta} \tag{117}$$
$$z(t) = \frac{w_{T_r}}{(2-\beta)\gamma}e^{\frac{1}{1-\gamma}(t-T_r)} \tag{118}$$
$$m(t) = \frac{A_m}{(2-\beta)\gamma}w_{T_r}e^{\frac{1}{1-\gamma}(t-T_r)} \tag{119}$$
$$w(t) = w(T_r)e^{\frac{1}{1-\gamma}(t-T_r)} \tag{120}$$

where $c_{T_r} = c(T_r) = A_y\,(A_m z(T_r))^{\theta}\,((1-\delta(T_r))L)^{1-\theta} = A_y\left(A_m\frac{w_{T_r}}{(2-\beta)\gamma}\right)^{\theta}(\frac{1-\theta}{1-\theta+2(1-\gamma)\theta}L)^{1-\theta}$,
and $w(T_r) = \sqrt[1-\gamma]{(2-\beta)\gamma A_z(\delta L)^{(1-\gamma)}}$.

The stock variables read:

$$S_x(t) = S_x(T_r) \tag{121}$$
$$S_w(t) = \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\gamma}-1\right)}{g_w}e^{g_w(t-2Tr)} + S_{w2}(T) \tag{122}$$
$$= \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\gamma}-1\right)}{\frac{1}{1-\gamma}}e^{\frac{1}{1-\gamma}(t-2Tr)} + S_{w2}(T) \tag{123}$$

with $S_x(T_r)$ and $S_{w2}(T)$ endogeneous constants to be defined.

And the co-state variables read:

$$\lambda_{Sx}(t) = c(t)^{1-\sigma}\theta\frac{A_m(t)}{A_m(t)z(t)} + \lambda_{Sw}(t)(1-\alpha) + \mu_x(t) \tag{124}$$
$$= c(t)^{1-\sigma}\frac{\theta}{z(t)}\left(\frac{A_m(t)}{A_m(t)} + (1-\alpha)\right) + \mu_x(t) \tag{125}$$
$$\lambda_{Sw}(t) = c(t)^{1-\sigma}\frac{\theta}{z(t)} = \lambda_{Sw}(T_r)e^{\rho(t-T_r)} = c(Tr)^{1-\sigma}\frac{\theta}{z(Tr)}e^{\rho(t-T_r)} \tag{126}$$

$$\lambda_{Sw}(Tr) = c(Tr)^{1-\sigma}\frac{\theta}{z(Tr)} \tag{127}$$

$$= \left(A_y\left(A_m\frac{w_{T_r}}{(2-\beta)\gamma}\right)^{\theta}\left(\frac{1-\theta}{1-\theta+2(1-\gamma)\theta}L\right)^{1-\theta}\right)^{1-\sigma}\frac{\theta(2-\beta)\gamma}{w_{T_r}} \tag{128}$$

To satisfy the transversality conditions, for all $t > Tr$:

$$\lim_{t\to\infty}\lambda_{Sx}(t)e^{-\rho(t-Tr)}S_x(t) = 0 \tag{129}$$

$$\lim_{t\to\infty}\left(c(t)^{1-\sigma}\frac{\theta}{z(t)}\Big(\frac{A_m(t)}{A_m(t)}+(1-\alpha)\Big)+\mu_x(t)\right)e^{-\rho(t-Tr)}S_x(Tr) = 0$$

For this condition to be valid, there are two possibilities: **either** $S_x(Tr) = 0$, thus the economy falls into the particular case where all the natural resource stock are consumed. Because stocks need to be continued we have:

$$S_x(T_r) = 0 = \frac{x_0}{g_x}(1-e^{g_xT_r})+\overline{S_{x0}} \tag{130}$$

$$T_r = \frac{1}{g_x}ln(1+\frac{\overline{S_{x0}}g_x}{x_0}) \tag{131}$$

The only way this can be possible is for

- $x_0 > \overline{S_{x0}}g_x$
- **Or** $c(t)^{1-\sigma}\frac{\theta}{z(t)}\left(\frac{A_m(t)}{A_m(t)}+(1-\alpha)\right) = \mu_x(t)$ and thus for all $t > T_r$, $\lambda_{Sx}(t) = 0$.

At $t = T_r$ we have $S_w(t) = (1-\alpha)(\overline{S_{x0}} - S_x(T_r))$

$$(1-\alpha)(\overline{S_{x0}} - S_x(T_r)) = S_w(Tr) \tag{132}$$

$$(1-\alpha)(\overline{S_{x0}} - S_x(T_r)) = \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\gamma}-1\right)}{g_w}e^{g_w(-Tr)} + S_{w2}(T)$$

$$S_{w2}(T) = (1-\alpha)(\overline{S_{x0}} - S_x(T_r)) - \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\gamma}-1\right)}{g_w}e^{g_w(-Tr)} \tag{133}$$

$$S_w(t) = \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\gamma}-1\right)}{g_w}\left(e^{g_w(t-2T_r)}-e^{-g_wT_r}\right)+(1-\alpha)(\overline{S_{x0}} - S_x(T_r)) \tag{134}$$

$$= \frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\gamma}-1\right)}{g_w}e^{-g_wT_r}\left(e^{g_w(t-T_r)}-1\right)+(1-\alpha)(\overline{S_{x0}} - S_x(T_r))$$

$$0 = \lim_{t\to\infty}\lambda_{Sw}(t)e^{-\rho(t-Tr)}S_w(t) \tag{135}$$

$$= \lim_{t\to\infty}c(Tr)^{1-\sigma}\frac{\theta}{z(Tr)}\left(\frac{w(T_r)\left(\frac{1-\beta}{(2-\beta)\gamma}-1\right)}{g_w}e^{-g_wT_r}\left(e^{g_w(t-T_r)}-1\right)+(1-\alpha)(\overline{S_{x0}} - S_x(T_r))\right)$$

Appendix A.5 Continuity of the Hamiltonians

When waste leads to no damages, the only incentive to have a recycling function is the scarcity of the resource. The only reason to dedicate labour to the recycling sector is to overcome the reduced availability of x(t). In that case, the stock of labour $L(t)$ is constant. Skipping the time index for convenience, and taking into account the assumptions on no loss, the current-value Hamiltonian with co-state variables $\lambda_{Sx}(t)$ and $\lambda_{Sw}(t)$ is:

$$\begin{aligned}\mathcal{H} &= u\Big(f\Big(A_y, A_m x + A_m h\big(A_z, w, \delta L\big), (1-\delta)L\Big)\Big)\\ &- \lambda_{Sx} x\\ &+ \lambda_{Sw}\Big((1-\alpha)x + (1-\beta)h\big(Az, w, \delta L)\big) - w\Big)\end{aligned}$$

- Lagrangian with multipliers $\mu_\delta(t)$, $\mu_x(t)$, $\mu_{Sx}(t)$, and $\mu_w(t)$

$$\max_{\delta(t),x(t),w(t)} \mathcal{L} = \mathcal{H} + \mu_\delta \delta + \mu_x x + \mu_{Sx} S_x + \mu_w w + \mu_{Sw} S_w$$

- Optimal conditions

$$\begin{aligned}\max_{\delta(t),x(t),w(t)} \mathcal{L} &= u\Big(f\Big(A_y, A_m x + A_m h\big(A_z, w, \delta L\big), (1-\delta)L\Big)\Big)\\ &- \lambda_{Sx} x\\ &+ \lambda_{Sw}\Big((1-\alpha)x + (1-\beta)h\big(E, w, \delta L)\big) - w\Big)\\ &+ \mu_\delta \delta + \mu_x x + \mu_w w + \mu_{Sx} S_x + \mu_{Sw} S_w \quad (136)\end{aligned}$$

$$\begin{aligned}\delta(t) &: u'(c(t))\Big(A_m(t) f'_{m(t)} h'_{\delta(t)} + f'_{1-\delta(t)}\Big) + \lambda_{Sw}(t)(1-\beta)h'_{\delta(t)} + \mu_\delta(t) = 0 \quad (137)\\ x(t) &: u'(c(t))A_m(t) f'_{m(t)} + \lambda_{Sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{Sx}(t) \quad (138)\\ w(t) &: u'(c(t))A_m(t) f'_{m(t)} h'_{w(t)} + \lambda_{Sw}(t)\big((1-\beta)h'_{w(t)} - 1\big) + \mu_w(t) = 0 \quad (139)\\ S_x(t) &: \lambda_{Sx}(t)\rho - \dot{\lambda}_{Sx}(t) = \mu_{Sx} \quad (140)\\ S_w(t) &: \lambda_{Sw}(t)\rho - \dot{\lambda}_{Sw}(t) = \mu_{Sw} \quad (141)\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{L}(t)}{\partial \delta(t)} &= \delta(t) \geq 0, \delta(t) \geq 0, \delta(t)\,\delta(t) \geq 0 \text{ (complementary slackness)} \quad (142)\\ \frac{\partial \mathcal{L}(t)}{\partial \mu_x(t)} &= x(t) \geq 0, \mu_x(t) \geq 0, \mu_x(t)\,x(t) \geq 0 \text{ (complementary slackness)} \quad (143)\\ \frac{\partial \mathcal{L}(t)}{\partial \mu_{Sx}(t)} &= S_x(t) \geq 0, \mu_{Sx}(t) \geq 0, \mu_{Sx}(t)\,S_x(t) \geq 0 \text{ (complementary slackness)} \quad (144)\\ \frac{\partial \mathcal{L}(t)}{\partial \mu_w(t)} &= w(t) \geq 0, \mu_w(t) \geq 0, \mu_w(t)\,w(t) \geq 0 \text{ (complementary slackness)} \quad (145)\\ \frac{\partial \mathcal{L}(t)}{\partial \mu_{Sw}(t)} &= S_w(t) \geq 0, \mu_{Sw}(t) \geq 0, \mu_{Sw}(t)\,S_w(t) \geq 0 \text{ (complementary slackness)} \quad (146)\end{aligned}$$

The continuity of the current-value Hamiltonian gives:

$$\begin{aligned}\mathcal{H}(T_r{}^-) &= u\Big(A_y(A_m x(T_r{}^-))^\theta L^{1-\theta}\Big)\\ &- \lambda_{Sx}(T_r{}^-)x(T_r{}^-)\\ &+ \lambda_{Sw}(T_r{}^-)\Big((1-\alpha)x(T_r{}^-)\Big)\\ \mathcal{H}(T_r{}^+) &= u\Big(A_y(A_m E_{T_r{}^+} w(T_r{}^+)^\gamma \delta^{(1-\gamma)})^\theta (1-\delta)^{1-\theta} L^{1-\theta}\Big)\\ &+ \lambda_{Sw}(T_r{}^+)\Big((1-\beta)E_{T_r{}^+} w(T_r{}^+)^\gamma \delta^{(1-\gamma)} - w(T_r{}^+)\Big)\end{aligned}$$

If we consider L and A_y to be continued on T_r we have

$$\begin{aligned} x({T_r}^-)\Big(\lambda_{Sx}({T_r}^-) - \lambda_{Sw}({T_r}^-)(1-\alpha)\Big) &= u(c({T_r}^-)) - u(c({T_r}^+)) \\ &- \lambda_{Sw}({T_r}^+)\Big((1-\beta)E_{{T_r}^+}w({T_r}^+)^{\gamma}\delta^{(1-\gamma)} - w({T_r}^+)\Big) \end{aligned}$$

We know the exact value of

$$w({T_r}^+) = \sqrt[1-\gamma]{(2-\beta)\gamma E({T_r}^+)(\delta L)^{(1-\gamma)}}$$

and

$$\lambda_{Sw}({T_r}^+) = c({T_r}^+)^{1-\sigma}\frac{\theta(2-\beta)\gamma}{w({T_r}^+)}$$

with

$$c({T_r}^+) = A_{y({T_r}^+)}\left(\frac{A_{m({T_r}^+)}w({T_r}^+)}{(2-\beta)\gamma}\right)^{\theta}\left(\frac{1-\theta}{1-\theta+2(1-\gamma)\theta}\right)^{1-\theta}.$$

So $\mathcal{H}({T_r}^+)$ is fully defined and we know its value.

As we also have $\Big(\lambda_{Sx}({T_r}^-) - \lambda_{Sw}({T_r}^-)(1-\alpha)\Big) = c({T_r}^-)^{1-\sigma}\frac{\theta}{x({T_r}^-)}$

$$\begin{aligned} x({T_r}^-)\Big(c({T_r}^-)^{1-\sigma}\frac{\theta}{x({T_r}^-)}\Big) &= u(c({T_r}^-)) - \mathcal{H}({T_r}^+) \\ c({T_r}^-)^{1-\sigma}\theta &= u(c({T_r}^-)) - \mathcal{H}({T_r}^+) = \frac{c({T_r}^-)^{1-\sigma}-1}{1-\sigma} - \mathcal{H}({T_r}^+) \\ c({T_r}^-)^{1-\sigma}\left(\frac{1-\theta(1-\sigma)}{1-\sigma}\right) &= \frac{1}{1-\sigma} + \mathcal{H}({T_r}^+) \\ c({T_r}^-)^{1-\sigma} &= \frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big) \\ c({T_r}^-) &= \sqrt[1-\sigma]{\frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big)} \\ A_y\left(A_m x({T_r}^-)\right)^{\theta}L^{1-\theta} &= \sqrt[1-\sigma]{\frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big)} \\ x({T_r}^-)^{\theta} &= \frac{\sqrt[1-\sigma]{\frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big)}}{A_y A_m^{\theta} L^{1-\theta}} \\ x({T_r}^-) &= \sqrt[\theta]{\frac{\sqrt[1-\sigma]{\frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big)}}{A_y A_m^{\theta} L^{1-\theta}}} \\ \lambda_{Sx}({T_r}^-) - \lambda_{Sw}({T_r}^-)(1-\alpha) &= \frac{\frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big)\theta}{\sqrt[\theta]{\frac{\sqrt[1-\sigma]{\frac{1-\sigma}{1-\theta(1-\sigma)}\Big(\frac{1}{1-\sigma} + \mathcal{H}({T_r}^+)\Big)}}{A_y A_m^{\theta} L^{1-\theta}}}} \end{aligned}$$

Appendix A.6 Details of the simulations

Appendix A.6.1 Model specifications

Table 1: Exogeneous Parameter Values

Abbreviation	Value	Parameter
TFP of the circular production function	E	1
TFP of the main production function	Ay	1
TFP of the material function	Am	1
Discount factor	ρ	0.030
Elasticity of consumption	σ	0.10
Initial stock of virgin material	S_0	10^6
Time	t	10^4

The parameters were fixed that way to allow for the graphs to be easily readable. The exogeneous parameters were modified in order to verify the constance of the model. the results were held constants when deviating from the above mentioned parameters.

Table 2: Parameter Values - Dependent on the model specifications

Abbreviation	Value	Parameter
Share of circular output going to waste stock	β	VARIED
Share of linear output going to waste stock	α	$\beta + 0.05$
Elast. of subs. of labour & waste (circular sector)	γ	$\frac{1-\beta}{2-\beta}$
Elast. of subs. of labour & material (production sector)	θ	$\gamma - 0.01$
Growth rate of virgin materials	g_x	$\frac{\rho}{1-\theta(1-\sigma)}$
Share of labour going to the circular sector	δ	$\frac{2(1-\gamma)\theta}{1-\theta+2(1-\gamma)\theta}$

Appendix A.6.2 Python code

The Python code files developed by the authors and used to compute these models can be found in the dedicated repository on GitHub (developed by the author). These files include the necessary functions and inputs to replicate the results.

Appendix B Model with damage function

Appendix B.1 General Model set up

The first-order conditions (FOCs) are:

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial x} &= -\lambda_{Sx}(t) + u'(c(t))\frac{\partial c(t)}{\partial x} + \lambda_{Sw}(t)(1-\alpha) = 0 \\
\frac{\partial \mathcal{H}}{\partial w} &= -\lambda_{Sw}(t)(1-\beta) + u'(c(t))\frac{\partial c(t)}{\partial w} + \lambda_{Sx}(t)\frac{\partial x(t)}{\partial w} = 0 \\
\frac{\partial \mathcal{H}}{\partial c} &= u'(c(t)) - \lambda_l(t)\frac{\pi}{S_w(t)}\frac{\partial \mathcal{D}(S_w(t))}{\partial S_w(t)} = 0 \\
\frac{\partial \mathcal{H}}{\partial \delta} &= -\frac{\pi}{S_w(t)}\mathcal{D}(S_w(t)) = -\frac{\partial \lambda_l(t)}{\partial t}
\end{aligned}$$

$$\begin{aligned}
\dot{x}(t) &= -\frac{\partial \mathcal{H}(t)}{\partial \lambda_{Sx}(t)} = \lambda_{Sx}(t) \\
\dot{w}(t) &= \frac{\partial \mathcal{H}(t)}{\partial \lambda_{Sw}(t)} = (1-\alpha)x(t) + (1-\beta)h(w,\delta) - w(t) \\
\dot{\lambda} l(t) &= -\frac{\partial \mathcal{H}(t)}{\partial l(t)} = -\frac{\pi}{S_w(x,w,\delta,t)}\mathcal{D}(S_w(x,w,\delta,t)) \\
\dot{\lambda} Sx(t) &= -\frac{\partial \mathcal{H}(t)}{\partial x(t)} = -\lambda_{Sw}(t) - \frac{\partial c(x,w,\delta,t)}{\partial x} \\
\dot{\lambda} Sw(t) &= -\frac{\partial \mathcal{H}(t)}{\partial w(t)} = -\lambda Sx(t) + (1-\alpha)\lambda_{Sw}(t) - \frac{\partial g(x,w,\delta,t)}{\partial w} \\
\dot{\delta}(t) &= \frac{\partial \mathcal{H}(t)}{\partial l_y(t)} = \frac{\lambda_l(t)}{L(t)^2} + \frac{\partial c(x,w,\delta,t)}{\partial \delta} \\
\dot{L}(t) &= -\frac{\partial \mathcal{H}(t)}{\partial \lambda_l(t)} = 0 \\
\dot{S}x(t) &= -\frac{\partial \mathcal{H}(t)}{\partial x(t)} = -\lambda Sx(t) \\
\dot{S}w(t) &= -\frac{\partial \mathcal{H}(t)}{\partial w(t)} = \lambda Sw(t) - (1-\alpha)x(t) - (1-\beta)h(w,\delta) + w(t)
\end{aligned}$$

The transversality conditions are:

$$\begin{aligned}
\lim_{t\to\infty} \lambda_l(t)L(t) &= 0 \\
\lim_{t\to\infty} \lambda_{Sx}(t)S_x(t) &= 0 \\
\lim_{t\to\infty} \lambda_{Sw}(t)S_w(t) &= 0
\end{aligned}$$

$$\delta(t) \;:\; \frac{\partial u(t)}{\partial c(t)}c(t) + \frac{\partial c(t)(1-\delta(t))}{\partial \delta(t)}u(t) + \lambda_L \pi(1-\beta)\frac{\partial z(t)}{\partial \delta(t)} + \lambda_{Sw}(t)(1-\beta)\frac{\partial z(t)}{\partial \delta(t)} + \mu_\delta(t) = 0 \tag{147}$$

$$x(t) \;:\; \frac{\partial u(t)}{\partial c(t)}c(t) + \frac{\partial c(t)}{\partial x(t)}u(t) + \lambda_{Sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{Sx}(t) \tag{148}$$

$$w(t) \;:\; \frac{\partial u(t)}{\partial c(t)}c(t) + u(t)\left(\frac{\partial c(t)}{\partial z(t)}z(t) + \frac{\partial z(t)}{\partial w(t)}c(t)\right) + \lambda_{Sw}(t)\left((1-\beta)\frac{\partial z(t)}{\partial w(t)} - 1\right) = -\mu_w(t) \tag{149}$$

$$S_x(t) \;:\; \lambda_{Sx}(t)\,\rho - \dot{\lambda}_{Sx}(t) = \mu_{Sx} \tag{150}$$

$$S_w(t) \;:\; \frac{\partial u(t)}{\partial c(t)}c(t) + \left(u(t)\mathcal{D}(S_w(t))^2 - \lambda_{S_w}(t)(1-\beta)\mathcal{D}(S_w(t))\right)\frac{\partial z(t)}{\partial \mathcal{D}(S_w(t))}$$

$$+\Big(u(t)c(t)(1-\mathcal{D}(S_w(t)))+\lambda_{Sw}(t)(1-\beta)z(t)\Big)\frac{\partial\mathcal{D}(S_w(t))}{\partial\delta(t)}+\mu_{Sw}(t)$$
$$=\lambda_{Sw}(t)\,\rho-\dot{\lambda_{Sw}}(t) \quad (151)$$

Complementary slackness:

$$\frac{\partial\mathcal{L}(t)}{\partial\delta(t)} = \delta(t)\geq 0, \delta(t)\geq 0, \delta(t)\,\delta(t)\geq 0 \quad (152)$$
$$\frac{\partial\mathcal{L}(t)}{\partial\mu_x(t)} = x(t)\geq 0, \mu_x(t)\geq 0, \mu_x(t)\,x(t)\geq 0 \quad (153)$$
$$\frac{\partial\mathcal{L}(t)}{\partial\mu_{Sx}(t)} = S_x(t)\geq 0, \mu_{Sx}(t)\geq 0, \mu_{Sx}(t)\,S_x(t)\geq 0 \quad (154)$$
$$\frac{\partial\mathcal{L}(t)}{\partial\mu_w(t)} = w(t)\geq 0, \mu_w(t)\geq 0, \mu_w(t)\,w(t)\geq 0 \quad (155)$$
$$\frac{\partial\mathcal{L}(t)}{\partial\mu_{Sw}(t)} = S_w(t)\geq 0, \mu_{Sw}(t)\geq 0, \mu_{Sw}(t)\,S_w(t)\geq 0 \quad (156)$$

Appendix B.2 Intuitions

Different mechanisms are expected to evolve and appear at different periods of time. There are supposedly four different periods that could be observed, depending on the labour allocation. A crucial determinant of these periods is the marginal cost of labour of the stock of waste S_w, or as per say, $\mathcal{D}(s_w)$ compared with the recycling sector. These periods are detailed in the following section. At time $t=t_0$, before any change has occurred, the stock of exhaustible resource S_x equals its maximum $\overline{s}_x$. This stock might be deplete to a lower limit of 0. We are not sure yet that the economy will exhaust all the exhaustible resources available in this stock. The stock of labour l equals 1. All the others stocks are null as no waste has been produced yet.

Appendix B.3 Pre-recycling stage

At time $t=t_0+\epsilon$, exhaustible resources are extracted from our stock S_x and input in the main production function. At this point in time, illustrated by figure 16, the waste stock S_w already starts to grow and to negatively impact the productivity of labour. The labour available starts to reduce because of the negative externalities due to the waste stock. There is not yet any recycling. Thus for all $t_0+\epsilon\leq t<T_{r+x}$:

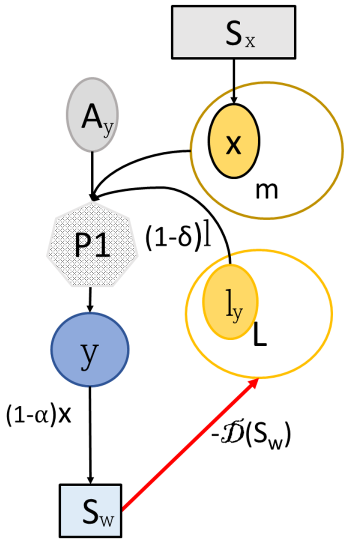
$$\delta(t)=0 \quad : \quad u'(c(t))(1-\theta)c(t)=\mu_\delta(t)\Longleftrightarrow c(t)^{1-\sigma}(1-\theta)=\mu_\delta(t) \quad (157)$$
$$x(t)\neq 0 \quad : \quad u'(c(t))A_m(t)f'_{m(t)}+\lambda_{Sw}(t)(1-\alpha)=\lambda_{Sx}(t)$$
$$\Longleftrightarrow \quad \frac{c(t)^{1-\sigma}\theta}{x(t)}=\lambda_{Sx}(t)-\lambda_{Sw}(t)(1-\alpha) \quad (158)$$
$$w(t)=0 \quad : \quad \mu_w(t)=\lambda_{Sw}(t) \quad (159)$$
$$S_x(t)\neq 0 \quad : \quad \lambda_{Sx}(t)\,\rho-\dot{\lambda_{Sx}}(t)=0 \quad (160)$$
$$S_w(t)\neq 0 \quad : \quad u'(c(t))\mathcal{D}'(s_w(t))f'_{l_y(t)}=\lambda_{Sw}(t)\,\rho-\dot{\lambda_{Sw}}(t)$$
$$\Longleftrightarrow \quad \frac{\mathcal{D}'(s_w(t))}{\mathcal{D}(s_w(t))}\Big(c(t)^{1-\sigma}(1-\theta)\Big)=\lambda_{Sw}(t)\,\rho-\dot{\lambda_{Sw}}(t) \quad (161)$$

For $t_0+\epsilon\leq t<T_{r+x}$, S_w only leads to pollution. Thus λ_{Sw} has to be negative. To match the slackness conditions, two μ_w need be such as: For $t_0+\epsilon\leq t<T_{r+x}$, we have μ_{w1}:

$$w(t)=0 \quad : \quad -\mu_{w1}(t)=\lambda_{Sw}(t) \quad (162)$$
$$(163)$$

For $t \geq T_{r+x}$, we have μ_{w2}:

$$w\,(t) = 0 \quad : \quad \mu_{w2}(t) = \lambda_{Sw}(t) \tag{164}$$



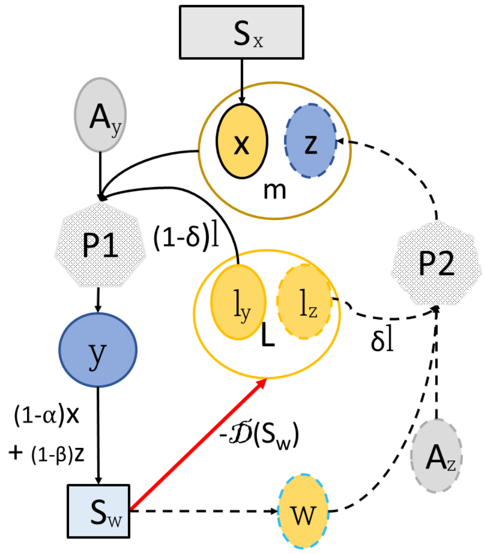


Figure 16: Economy at time $t_0+\epsilon \leq t < T_{r+x}$ Figure 17: Economy at time $T_{r+x} \leq t < T_r$

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner to reach pareto optima in the centralised economy, while the output endogenous variables are blue. The damage function is represented by a red arrow. The dash lines represent the components of the recycling sector.

Appendix B.4 Recycling stage

Appendix B.4.1 With virgin injections

At time $t = T_{r+x}$ the second sector appears to reduce the growth of the waste stock S_w and to overcome both the negative externalities linked with the stock of virgin material S_x which is depleting and with the stock of waste S_w which reduces the available labour. This second sector Z is labour intensive and uses waste w extracted from the stock of waste S_w to produce circular material z to input in the main sector of the economy P1. Figure 17 illustrates the economy from T_{r+x} to T_r. Material used as inputs in the main production function are both virgin x extracted from our stock of virgin S_x and circular z converted from waste w extracted from the stock of waste S_w. Differently from the model without damage, it is of higher interest to extract waste from the stock S_w as this stock has a negative impact on labour while the stock of exhaustible resources S_x does not impact the labour. the period without recycling detailed above is supposedly shorter than the same period when no damage function is considered. During this period, the recycling sector possibly overcomes the waste-induced negatives externalities created by the exhausted material x on the labour l. These externalities are represented by the damage function $\mathcal{D}$, a function of the stock of waste from exhaustible resource S_w. To achieve the pareto optima[9], the social planner has to decide how much labour l_z to dedicate to the recycling sector knowing that, without intervention on the stock of waste from exhaustible resource S_w, the amount of labour l available for the main production function decreases. At the same time, when labour l_z is dedicated to the recycling sector there is less labour l_y available to produce y from the main production. The social planner has to evaluate how much labour is polluted by a unit of virgin material compared with the labour needed to produce circular material to substitute the virgin material from the waste stock S_w. The social planner has to evaluate the value of δ. The next section investigates further this bargain around labour allocation. For $T_{r+x} \leq t < T_r$, the first order conditions become:

$$\delta\left(t\right) \neq 0 \quad : \quad c(t)^{1-\sigma}\left(\frac{A_m(t)}{m(t)}\theta(1-\gamma)\frac{z(t)}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{Sw}(t)(1-\beta)(1-\gamma)\frac{z(t)}{\delta(t)} = 0 \tag{165}$$

$$x\left(t\right) \neq 0 \quad : \quad c(t)^{1-\sigma}\theta\frac{A_m(t)}{m(t)} + \lambda_{Sw}(t)(1-\alpha) = \lambda_{Sx}\left(t\right) \tag{166}$$

$$w\left(t\right) \neq 0 \quad : \quad c(t)^{1-\sigma}\theta\gamma\frac{A_m(t)}{m(t)}\frac{z(t)}{w(t)} + \lambda_{Sw}(t)\left((1-\beta)\gamma\frac{z(t)}{w(t)} - 1\right) = 0 \tag{167}$$

$$S_x\left(t\right) \neq 0 \quad : \quad \lambda_{Sx}\left(t\right)\rho - \dot{\lambda}_{Sx}\left(t\right) = 0 \tag{168}$$

$$S_w\left(t\right) \neq 0 \quad : \quad \frac{\mathcal{D}'(s_w(t))}{\mathcal{D}(s_w(t))}\left(c(t)^{1-\sigma}\left(A_{m(t)}\theta(1-\gamma)\frac{z(t)}{m(t)} + (1-\theta)\right) + \lambda_{Sw}(t)(1-\beta)(1-\gamma)z(t)\right)$$
$$= \lambda_{Sw}\left(t\right)\rho - \dot{\lambda}_{Sw}\left(t\right) \tag{169}$$

from (37)

$$\delta(t) = \left(\frac{u'(c(t))(1-\theta)c(t)}{(1-\gamma)z(t)\left(u'(c(t))A_m(t)\theta\frac{c(t)}{m(t)} + \lambda_{Sw}(t)\right)} + 1\right)^{-1} \tag{170}$$

from (38), (39) and (40) we have:

$$\lambda_{Sx}(t) = c(t)^{1-\sigma}A_m(t)\frac{\theta}{m(t)}\left(\frac{A_m(t)}{A_m(t)} + \frac{\gamma\frac{z(t)}{w(t)}}{1-\gamma\frac{z(t)}{w(t)}}\right) \tag{171}$$

$$\lambda_{Sw}(t) = c(t)^{1-\sigma}A_m(t)\frac{\theta}{m(t)}\left(\frac{\gamma\frac{z(t)}{w(t)}}{1-\gamma\frac{z(t)}{w(t)}}\right) \tag{172}$$

Appendix B.5 Circular stages

For the last period, illustrated by Figure 18, no more exhaustible resources are extracted. Virgin materials are not used as input.For these two periods, when $\beta < 1$, either $A_y(t)$ or $E(t)$ shall increase after $t = T_r$ to balance for the non-injection of virgin material and for the economy to stay constant or to grow. When $\beta = 1$, and both $A_y(t)$ and $E(t)$ stay constant after $t = T_r$, then the economy declines until $t = T_z$ when it reaches a steady state for all $t > T_z$.

Appendix B.5.1 Without injection of virgin materials but with pollution from the stock of waste

There might be a time $t = T_r$, as depicted by figure 18, when the production function f_{P1} keep producing waste which accumulate in the stock of waste. This stock of waste damage the labour.

[9] In the competitive market scenario, the consumer chose to maximize the utility by compromising consumption of goods and wages.

For $T_r \leq t < T_z$ there are two possible cases: (1) either the stock of exhaustible resource is depleted and $S_x = 0$, the first order conditions become:

$$\delta(t) \neq 0 \quad : \quad c(t)^{1-\sigma}\left(\theta(1-\gamma)\frac{1}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{Sw}(t)(1-\beta)(1-\gamma)(t)\frac{z(t)}{\delta(t)} = 0 \tag{173}$$

$$x(t) = 0 \quad : \quad c(t)^{1-\sigma}\theta\frac{A_m(t)}{A_m(t)z(t)} + \lambda_{Sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{Sx}(t) \tag{174}$$

$$w(t) \neq 0 \quad : \quad c(t)^{1-\sigma}\theta\gamma\frac{1}{w(t)} + \lambda_{Sw}(t)\left((1-\beta)\gamma\frac{z(t)}{w(t)} - 1\right) = 0 \tag{175}$$

$$S_x(t) = 0 \quad : \quad \lambda_{Sx}(t)\rho - \dot{\lambda_{Sx}}(t) = \mu_{Sx} \tag{176}$$

$$S_w(t) \neq 0 \quad : \quad \frac{\mathcal{D}'(s_w(t))}{\mathcal{D}(s_w(t))}\left(c(t)^{1-\sigma}\left(\theta(1-\gamma) + (1-\theta)\right) + \lambda_{Sw}(t)(1-\beta)(1-\gamma)z(t)\right)$$
$$= \lambda_{Sw}(t)\rho - \dot{\lambda_{Sw}}(t) \tag{177}$$

(2) There is no argument in favor of virgin material which is fully substituted by the circular material. The economic rather benefits from the recycling sector than the virgin material and the first order conditions become:

$$\delta(t) \neq 0 \quad : \quad c(t)^{1-\sigma}\left(\theta(1-\gamma)\frac{1}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{Sw}(t)(1-\beta)(1-\gamma)\frac{z(t)}{\delta(t)} = 0 \tag{178}$$

$$x(t) = 0 \quad : \quad c(t)^{1-\sigma}\theta\frac{A_m(t)}{A_m(t)z(t)} + \lambda_{Sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{Sx}(t) \tag{179}$$

$$w(t) \neq 0 \quad : \quad c(t)^{1-\sigma}\theta\gamma\frac{1}{w(t)} + \lambda_{Sw}(t)\left((1-\beta)\gamma\frac{z(t)}{w(t)} - 1\right) = 0 \tag{180}$$

$$S_x(t) \neq 0 \quad : \quad \lambda_{Sx}(t)\rho - \dot{\lambda_{Sx}}(t) = 0 \tag{181}$$

$$S_w(t) \neq 0 \quad : \quad \frac{\mathcal{D}'(s_w(t))}{\mathcal{D}(s_w(t))}\left(c(t)^{1-\sigma}\left(\theta(1-\gamma) + (1-\theta)\right) + \lambda_{Sw}(t)(1-\beta)(1-\gamma)z(t)\right)$$
$$= \lambda_{Sw}(t)\rho - \dot{\lambda_{Sw}}(t) \tag{182}$$

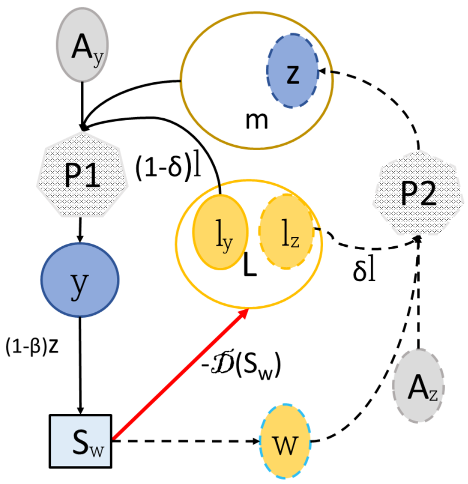


Figure 18: Economy at time $T_r \leq t < T_z$

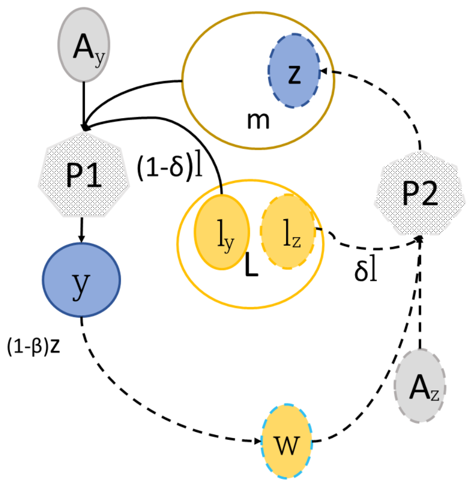


Figure 19: Economy at time $t \geq T_z$

Notes: The round shapes represent flow variables while the square shapes represent stocks. The exogenous variables are filled with grey color. Yellow is used for variables chosen by the social planner to reach pareto optima in the centralised economy, while the output endogenous variables are blue. The damage function is represented by a red arrow. The dash lines represent the components of the recycling sector.

Appendix B.5.2 Special case: without injection of virgin resources and without pollution

There might be a time $t = T_z$, depicted by Figure 19 after when all goods are recyclable by design and the stock of waste S_w is depleted. This would mean that the stock of labour l does not endure any damage and $\mathcal{D}(s_w) = L$ as depicted by Figure 19. The goods which are not used by the consumers are directly reinjected in the second sector and circular into z. For $T_r \leq t < T_z$, we have $\dot{S_w}(t) = 0 = (1-\beta)z(t) - w(t)$ or write differently $(1-\beta)z(t) = w(t)$ the first order conditions become:

$$\delta(t) \neq 0 \quad : \quad c(t)^{1-\sigma}\left(\theta(1-\gamma)\frac{1}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{Sw}(t)(1-\beta)(1-\gamma)\frac{z(t)}{\delta(t)} = 0$$

$$\iff \quad c(t)^{1-\sigma}\left(\theta(1-\gamma)\frac{1}{\delta(t)} - \frac{1-\theta}{1-\delta(t)}\right) + \lambda_{Sw}(t)(1-\gamma)\frac{w(t)}{\delta(t)} = 0 \tag{183}$$

$$x(t) = 0 \quad : \quad c(t)^{1-\sigma}\theta\frac{A_m(t)}{A_m(t)z(t)} + \lambda_{Sw}(t)(1-\alpha) + \mu_x(t) = \lambda_{Sx}(t) \tag{184}$$

$$w(t) \neq 0 \quad : \quad c(t)^{1-\sigma}\theta\gamma\frac{1}{w(t)} + \lambda_{Sw}(t)\left((1-\beta)\gamma\frac{z(t)}{w(t)} - 1\right) = 0$$

$$\iff \quad c(t)^{1-\sigma}\theta\gamma\frac{1}{w(t)} + \lambda_{Sw}(t)\left(\gamma\frac{1}{w(t)} - 1\right) = 0 \tag{185}$$

$$S_x(t) = 0 \quad : \quad \lambda_{Sx}(t)\rho - \dot{\lambda}_{Sx}(t) = \mu_{Sx} \tag{186}$$

$$S_w(t) = 0 \quad : \quad \mu_{Sw} = \lambda_{Sw}(t)\rho - \dot{\lambda}_{Sw}(t)$$

The next section evaluates the characteristics and the possible discontinuity between these periods. We investigate if at a certain point it becomes socially optimal to move from a linear economy (before $t = T_{r+x}$) to a fully circular economy (after $t = T_z$).

THE END.