AB2.6: Stokes's Theorem

The Stokes's theorem transforms line integrals into surface integrals and generalizes Green's theorem.

Recall the definition of the curl of a vector function. Let x, y, z be right-handed Cartesian coordinates and

$$\mathbf{v}(x,y,z) = v_1(x,y,z)\mathbf{i} + v_2(x,y,z)\mathbf{j} + v_3(x,y,z)\mathbf{k}$$

be a differentiable vector function. Then the function

$$\operatorname{curl}\mathbf{v} = \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} =$$

$$\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

is called the **curl** of $\mathbf{v}$.

THEOREM *9.9.*1

Let S be a piecewise smooth oriented surface in space and let the boundary of S be a piecewise smooth simple closed curve C. Let $\mathbf{F}$ *be a continuous vector function having continuous first partial derivatives in a domain in space containing S. Then*

$$\int_S\int (\operatorname{curl}\mathbf{F}) \cdot \mathbf{n} dA = \int_C \mathbf{F} \cdot \mathbf{r}'(s) ds$$

where $\mathbf{n}$ *is the unit normal vector of S and depending on* $\mathbf{n}$ *the integration around C is taken in the sense shown in the figure.* $\mathbf{r}'(s)$ *is the unit tangent vector and s is the arc lenght of C.*

In components, the formula of the Stokes's theorem takes the form

$$\int_R\int \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)N_3\right] dudv$$

$$\int_{\tilde{C}} (F_1 dx + F_2 dy + F_3 dz),$$

where R is the region with the boundary curve $\tilde{C}$ in the uv-plane corresponding to S represented by $\mathbf{r}(u,v)$, and $\mathbf{N}(u,v) = [N_1, N_2, N_3] = \mathbf{r}_u \times \mathbf{r}_v$.

EXAMPLE 1 **Verification of the Stokes's theorem**

Verify the Stokes's theorem for

$$\mathbf{F} = [y, z, x] = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$

and S

$$z = f(x, y) = 1 - (x^2 + y^2), \quad z \geq 0$$

the paraboloid of revolution.

Solution. The curve C is the circle $\mathbf{r}(s) = [\cos s, \sin s, 0]$. The unit tangent vector is $\mathbf{r}'(s) = [-\sin s, \cos s, 0]$. Thus the line integral is simply

$$\int_C \mathbf{F} \cdot \mathbf{r}'(s) ds = \int_0^{2\pi} [\sin s(-\sin s) + 0 + 0] ds = -\pi.$$

On the other hand,

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} =$$

$$\left(\frac{\partial x}{\partial y} - \frac{\partial z}{\partial z} \right) \mathbf{i} + \left(\frac{\partial y}{\partial z} - \frac{\partial x}{\partial x} \right) \mathbf{j} + \left(\frac{\partial z}{\partial x} - \frac{\partial y}{\partial y} \right) \mathbf{k} =$$

$$-\mathbf{i} - \mathbf{j} - \mathbf{k} = [-1, -1, -1].$$

and

$$\mathbf{N} = \operatorname{grad} (z - f(x, y)) = [2x, 2y, 1].$$

Next,

$$\operatorname{curl} \mathbf{F} \cdot \mathbf{N} = -2x - 2y - 1$$

Thus, setting $x = r \cos \theta$ and $y = r \sin \theta$ in the polar coordinates, we have

$$\int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dA = \int_R \int (-2x - 2y - 1) dx dy = \int_{\tilde{R}} \int (-2r \cos \theta - 2r \sin \theta - 1) r dr d\theta,$$

where $\tilde{R}$ is the circle $r \leq 1,\ 0 \leq \theta \leq 2\pi$. We have

$$\int_{\tilde{R}} \int (-2r \cos \theta - 2r \sin \theta - 1) r dr d\theta =$$

$$-2 \int_0^{2\pi} \cos \theta d\theta \int_0^1 r dr - 2 \int_0^{2\pi} \sin \theta d\theta \int_0^1 r dr - \int_0^{2\pi} d\theta \int_0^1 r dr = 0 + 0 - \pi = -\pi.$$

EXAMPLE 2 **Evaluation of a line integral by the Stokes's theorem**

Evaluate the line integral

$$\int_C \mathbf{F} \cdot \mathbf{r}'(s) ds$$

by the Stokes's theorem for

$$\mathbf{F} = [y, xz^3, -zy^3] = y\mathbf{i} + xz^3\mathbf{j} - zy^3\mathbf{k},$$

and C is the circle

$$x^2 + y^2 = 4, \quad z = -3.$$

Solution. As a surface S bounded by C we may take the plane circular disk $x^2 + y^2 \leq 4$ in the plane $z = -3$. Then the normal vector $\mathbf{n}$ in the Stokes's theorem points in the positive z-direction, so that $\mathbf{n} = \mathbf{k}$. Hence

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz^3 & -zy^3 \end{vmatrix} =$$

$$\left(\frac{\partial(-zy^3)}{\partial y} - \frac{\partial(xz^3)}{\partial z}\right)\mathbf{i} + \left(\frac{\partial y}{\partial z} - \frac{\partial(-zy^3)}{\partial x}\right)\mathbf{j} + \left(\frac{\partial(xz^3)}{\partial x} - \frac{\partial y}{\partial y}\right)\mathbf{k}$$

$$-3z(y^2 + xz)\mathbf{i} + (z^3 - 1)\mathbf{k} = [-3z(y^2 + xz), 0, z^3 - 1].$$

$$\operatorname{curl} \mathbf{F} \cdot \mathbf{N} = \operatorname{curl} \mathbf{F} \cdot \mathbf{k} = z^3 - 1$$

and

$$\operatorname{curl} \mathbf{F} \cdot \mathbf{N}\big|_{z=-3} = -3^3 - 1 = -28.$$

Thus,

$$\int_S \int (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n} dA = \int_{x^2+y^2 \leq 4 \ z=-3} \int (-28) dx dy = -28\pi 2^2 = -112\pi.$$

PROBLEM *9.9.*1

Compute the surface integral

$$\int_S\int (\operatorname{curl}\mathbf{F})\cdot\mathbf{n}dA$$

for $\mathbf{F}=[z^2,5x,0]$ and S being the square

$$0\le x\le 1,\;\;0\le y\le 1,\;\;z=1.$$

Solution. We have

$$\mathbf{r}(u,v)=[u,v,1],\quad 0\le u\le 1,\;\;0\le v\le 1,$$

$$\mathbf{r}_u=[1,0,0],\quad \mathbf{r}_v=[0,1,0];$$

the unit normal vector

$$\mathbf{N}=\mathbf{r}_u\times\mathbf{r}_v=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\0&1&0\end{vmatrix}=\mathbf{k}=[0,0,1].$$

On S

$$\mathbf{F}(\mathbf{r}(u,v))=\mathbf{F}(S)=[1,5u,0]=\mathbf{i}+5u\mathbf{j}.$$

Hence

$$\operatorname{curl}\mathbf{F}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\1&5x&0\end{vmatrix}=$$

$$\left(\frac{\partial 0}{\partial y}-5\frac{\partial x}{\partial z}\right)\mathbf{i}+\left(\frac{\partial 1}{\partial z}-\frac{\partial 0}{\partial x}\right)\mathbf{j}+5\left(\frac{\partial x}{\partial x}-\frac{\partial 1}{\partial y}\right)\mathbf{k}=5\mathbf{k}$$

$$\operatorname{curl}\mathbf{F}\cdot\mathbf{n}=\mathbf{5}$$

The parameters u,v vary in the rectangle $R:\;0\le u\le 1,\;0\le v\le 1$. Now we can write and calculate the surface integral:

$$\int_S\int \operatorname{curl}\mathbf{F}\cdot\mathbf{n}dA=\int_R\int 5dudv=5.$$

On the other hand,

$$\int_C \mathbf{F}\cdot\mathbf{r}'(s)ds=\int_C(F_1dx+F_2dy+F_3dz)=$$

$$\int_0^1 F_1|_{y=0}\,dx-\int_0^1 F_1|_{y=1}\,dx+\int_0^1 F_2|_{x=1}\,dy-\int_0^1 F_2|_{x=0}\,dy=$$

$$\int_0^1 1\cdot dx-\int_0^1 1\cdot dx+\int_0^1 5\cdot dy-\int_0^1 0\cdot dy=1-1+5-0=5.$$

Note that integration is performed for $z=1$.

PROBLEM *9.9.3*

Compute the surface integral

$$\int_S\int (\text{curl}\,\mathbf{F})\cdot\mathbf{n}dA$$

for $\mathbf{F} = [e^z, e^z\sin y, e^z\cos y]$ and S the cylindrical paraboloid

$$z = y^2,\ \ 0\le x\le 4\ \ 0\le y\le 2.$$

Solution. Setting $x = u$, $y = v$, and correspondingly $z = y^2 = v^2$, we obtain the parametric rerpresentation of the cylindrical paraboloid

$$\mathbf{r}(u,v) = [u, v, v^2],\quad 0\le u\le 4\ \ 0\le v\le 2.$$

Then

$$\mathbf{r}_u = [1,0,0],\quad \mathbf{r}_v = [0,1,2v],$$

and the normal vector

$$\mathbf{N} = \mathbf{r}_u\times\mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 2v \end{vmatrix} = -2v\mathbf{j}+\mathbf{k} = [0,-2v,1].$$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^z & e^z\sin y & e^z\cos y \end{vmatrix} =$$

$$\left(e^z\frac{\partial\cos y}{\partial y} - \frac{\partial e^z\sin y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial e^z}{\partial z} - \frac{\partial e^z\cos y}{\partial x}\right)\mathbf{j} + \left(\frac{\partial e^z\sin y}{\partial x} - \frac{\partial e^z}{\partial y}\right)\mathbf{k} =$$

$$-2e^z\sin y\mathbf{i} + e^z\mathbf{j}.$$

On S

$$\text{curl}\,\mathbf{F} = \mathbf{F}(S) = e^{v^2}[-2\sin v, 1, 0] = e^{v^2}(-2\sin v\mathbf{i}+\mathbf{j}).$$

Hence

$$\text{curl}\,\mathbf{F}\cdot\mathbf{N} = -2ve^{v^2}.$$

The parameters u, v vary in the rectangle $R:\ 0\le u\le 4\ \ 0\le v\le 2$. Now we can write and calculate the surface integral:

$$\int_S\int \text{curl}\,\mathbf{F}\cdot\mathbf{n}dA = \int_R\int\left(-2ve^{v^2}\right)dudv =$$

$$-\int_0^4 du\int_0^2 e^{v^2}dv^2 = -4\int_0^4 e^t dt = -4(e^4-1).$$

On the other hand,

$$\int_C \mathbf{F}\cdot\mathbf{r}'(s)ds = \int_C (F_1dx + F_2dy + F_3dz) =$$

$$\int_0^4 F_1|_{z=0}\,dx - \int_0^4 F_1|_{z=4}\,dx + \int_0^2 (F_2 + F_3z'(y))|_{x=4}\,dy - \int_0^2 (F_2 + F_3z'(y))|_{x=0}\,dy =$$

$$\int_0^4 1\cdot dx - \int_0^4 e^4\cdot dx + \int_0^2 (e^{y^2}\sin y + 2ye^{y^2}\cos y)dy -$$
$$\int_0^2 e^{y^2}\sin y + 2ye^{y^2}\cos y dy = 4 - 4e^4 + I - I = -4(e^4-1).$$

PROBLEM *9.9.7*

Compute the line integral

$$\int_C \mathbf{F}\cdot\mathbf{r}'(s)ds = \int_C (F_1dx + F_2dy + F_3dz)$$

for $\mathbf{F} = [-5y, 4x, z]$ and C the circle

$$x^2 + y^2 = 4, \quad z = 1.$$

Solution. The circle lies in the plane $z = 1$; therefore, the unit normal vector is simply $\mathbf{n} = \mathbf{k} = [0, 0, 1]$.

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -5y & 4x & z \end{vmatrix} =$$

$$\left(\frac{\partial z}{\partial y} - 4\frac{\partial x}{\partial z}\right)\mathbf{i} + \left(\frac{\partial(-5y)}{\partial z} - \frac{\partial z}{\partial x}\right)\mathbf{j} + 4\left(\frac{\partial x}{\partial x} - \frac{\partial(-5y)}{\partial y}\right)\mathbf{k} = 9\mathbf{k}.$$

Now

$$\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = \mathbf{9}$$

and the surface integral in the Stokes's theorem gives the required value of the line integral:

$$\int_S\int \operatorname{curl}\mathbf{F}\cdot\mathbf{n}dA = 9\int_R\int dudv = 9\pi 2^2 = 36\pi.$$

PROBLEM *9.9.9*

Compute the line integral

$$\int_C \mathbf{F}\cdot\mathbf{r}'(s)ds = \int_C (F_1dx + F_2dy + F_3dz)$$

for $\mathbf{F} = [4z, -2x, 2x]$ and C the ellipse

$$x^2 + y^2 = 1, \quad z = y + 1.$$

Solution. The ellipse lies in the plane $z = y + 1$; therefore, from geometrical inspection, we deduce that a normal vector is simply $\mathbf{N} = -\mathbf{j} + \mathbf{k} = [0, -1, 1]$. Indeed,

$$\mathbf{r}(u, v) = [u, v, v + 1].$$

Then

$$\mathbf{r}_u = [1, 0, 0], \quad \mathbf{r}_v = [0, 1, 1],$$

and a normal vector

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = -\mathbf{j} + \mathbf{k} = [0, -1, 1].$$

Next,

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4z & -2x & 2x \end{vmatrix} =$$

$$\left(2\frac{\partial x}{\partial y} + 2\frac{\partial x}{\partial z}\right)\mathbf{i} + \left(4\frac{\partial z}{\partial z} - 2\frac{\partial x}{\partial x}\right)\mathbf{j} - 2\left(\frac{\partial x}{\partial x} - 4\frac{\partial z}{\partial y}\right)\mathbf{k} = 2(\mathbf{j} - \mathbf{k}).$$

Now

$$\operatorname{curl} \mathbf{F} \cdot \mathbf{N} = [0, -1, 1] \cdot [0, 2, -2] = -4,$$

and the surface integral in the Stokes's theorem taken over the circle $R:\ u^2 + v^2 \leq 1$ of the variation of parameters gives the required value of the line integral:

$$\int_S\!\!\int \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dA = -4 \int_R\!\!\int dudv = -4\pi.$$

On the other hand, taking the unit normal vector of the ellipse

$$x^2 + \frac{(y')^2}{2} = 1, \quad y' = \sqrt{2}y,$$

situated in the plane $S:\ z = y + 1$,

$$\mathbf{n} = \frac{1}{\sqrt{2}}[0, -1, 1]$$

and equating the integral over the ellipse in S to the area of the ellipse we have the same result

$$\int_S\!\!\int \operatorname{curl} \mathbf{F} \cdot \mathbf{n} dA = -4\frac{1}{\sqrt{2}} \int_S\!\!\int dA = -4\frac{1}{\sqrt{2}}\pi\sqrt{2} = -4\pi.$$