

$$\Phi(\vec{x}) = \int_0^{2\pi} \frac{\lambda R d\phi'}{|\vec{x} - \vec{x}'|} \quad (1)$$

we know that

$$\frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta', \phi') \cdot Y_{lm}(\theta, \phi) \quad (2)$$

substituting it into (1) we get

$$\Phi(\vec{x}) = \int_0^{2\pi} \lambda R d\phi' 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta', \phi') \cdot Y_{lm}(\theta, \phi) \quad (3)$$

$$= 4\pi \lambda R \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} \int_0^{2\pi} Y_{lm}^*(\theta', \phi') \cdot Y_{lm}(\theta, \phi) d\phi' \quad (4)$$

Y_{lm} is defined to be

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\theta) \cdot e^{im\phi} \quad (5)$$

substituting it into (4) we get

$$\Phi(\vec{x}) = 4\pi \lambda R \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{1}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} \int_0^{2\pi} P_{lm}(\cos\theta') \cdot e^{-im\phi'} \cdot P_{lm}(\cos\theta) \cdot e^{-im\phi} d\phi' \quad (6)$$

$$= \lambda R \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{r_{<}^l}{r_{>}^{l+1}} \frac{(l-m)!}{(l+m)!} P_{lm}(\cos\theta') \cdot P_{lm}(\cos\theta) \int_0^{2\pi} e^{-im\phi'} e^{-im\phi} d\phi' \quad (7)$$

from the symmetry of the problem, we choose $\phi = 0$ and then

$$\Phi(\vec{x}) = \lambda R \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{r_{<}^l}{r_{>}^{l+1}} \frac{(l-m)!}{(l+m)!} P_{lm}(\cos\theta') \cdot P_{lm}(\cos\theta) \cdot 2\pi \delta_{m0} \quad (8)$$

$$= 2\pi \lambda R \sum_{l=0}^{\infty} \frac{r_{<}^l}{r_{>}^{l+1}} P_l(\cos\theta') \cdot P_l(\cos\theta) \quad (9)$$