# CLASSIFICATION

## 2001 - 2010

### Paper 1 (P1) Topics

### ~Pure Mathematics~

**Topic 1** **Quadratics**

J01/P1/Q2
J02/P1/Q3(i)
N05/P1/Q9 -3
J08/P1/Q8

J01/P1/Q3(i)
N02/P1/Q4(ii)
J2007/P1/Q1
J09/P1/Q2

N01/P1/Q1
N02/P1/Q11(i)(ii)(iii)
J2007/P1/Q4
N09/P1/Q10

N01/P1/Q2(i)
J03/P1/Q11(i)(ii)(iii)
N2007/P1/Q1
N10/P1/Q6

J02/P1/Q1
N03/P1/Q1(i)(ii)(iii)
J08/P1/Q4

**Topic 2** **Functions and Graphs**

J01/P1/Q3(ii)(iii)
J2003/P1/Q5
N2004/P1/Q9
N2006/P1/Q10
J09/P1/Q10
N10/P1/Q7

N01/P1/Q2(ii)
J2003/P1/Q11(iv)(v)
J2005/P1/Q7
J2007/P1/Q11
N09/P1/Q4

J02/P1/Q6
N2003/P1/Q10
J2005/P1/Q10
N2007/P1/Q11
N09/P1/Q8

J02/P1/Q10
J2004/P1/Q10
N2005/P1/Q8(ii)
J08/P1/Q6
J10/P1/Q3

N02/P1/Q11(iv)(v)
N2004/P1/Q6(iii)
J2006/P1/Q11
N08/P1/Q10
J10/P1/Q11

**Topic 3** **Coordinate geometry**

J2001/P1/Q7
J2004/P1/Q6
N2006/P1/Q5
N08/P1/Q8

N2001/P1/Q6
N2004/P1/Q5
J2007/P1/Q6
J09/P1/Q8

N2002/P1/Q9
J2005/P1/Q5
N2007/P1/Q6
N09/P1/Q9

J2003/P1/Q7
N2005/P1/Q7
N2007/P1/Q9
J10/P1/Q4

N2003/P1/Q5
J2006/P1/Q5
J08/P1/Q11
N10/P1/Q8

**Topic 4** **Circular measure**

J2001/P1/Q8
N2003/P1/Q6
J2006/P1/Q7
N08/P1/Q6

N2001/P1/Q4
J2004/P1/Q5
N2006/P1/Q3
J09/P1/Q5

J2002/P1/Q7
N2004/P1/Q3
J2007/P1/Q5
N09/P1/Q7
(ii)

N2002/P1/Q3
J2005/P1/Q8
N2007/P1/Q7
N10/P1/Q4
(ii)

J2003/P1/Q9
N2005/P1/Q2
J08/P1/Q5

**Topic 5** **Trigonometry**

J2001/P1/Q1
N2002/P1/Q5
J2004/P1/Q3
N2005/P1/Q3
J2007/P1/Q8
N08/P1/Q5
N10/P1/Q2

J2001/P1/Q4
N2002/P1/Q6
N2004/P1/Q4
J2006/P1/Q2
N2007/P1/Q5
J09/P1/Q1

N2001/P1/Q3
J2003/P1/Q2
N2004/P1/Q6(i)(ii)
J2006/P1/Q6
J08/P1/Q1
J09/P1/Q4

N2001/P1/Q7
J2003/P1/Q6
J2005/P1/Q3
N2006/P1/Q2
J08/P1/Q2
N09/P1/Q5

J2002/P1/Q2
N2003/P1/Q2
N2005/P1/Q1
J2007/P1/Q3
N08/P1/Q2
J10/P1/Q1

---

**Topic 6**

J2001/P1/Q1
N2003/P1/Q7
J2006/P1/Q8
N08/P1/Q4

**Topic 7**

**Arithmetic**

J2001/P1/Q9
N2003/P1/Q3
J2006/P1/Q3
N08/P1/Q3

**Binomial T**

J2001/P1/Q5
J2005/P1/Q4
N08/P1/Q1

**Topic 8**

J2001/P1/Q
J2002/P1/Q
J2003/P1/Q
J2004/P1/Q
N2005/P1/Q
J2006/P1/Q
N2007/P1/Q
N10/P1/Q3

**Topic 9**

J2001/P1/Q
N2002/P1/Q
N2003/P1/Q
J2005/P1/Q
N2008/P1/Q
N08/P1/Q9
N10/P1/Q11

## Topic 6 Vectors

J2001/P1/Q11
N2001/P1/Q10
J2002/P1/Q5
N2002/P1/Q7
J2003/P1/Q8
N2003/P1/Q7
J2004/P1/Q9
N2004/P1/Q8
J2005/P1/Q11
N2005/P1/Q4
J2006/P1/Q8
N2006/P1/Q4
J2007/P1/Q9
N2007/P1/Q10
J08/P1/Q10
N08/P1/Q4
J09/P1/Q6
N09/P1/Q6
J10/P1/Q5
N10/P1/Q9

## Topic 7 Series

### Arithmetic and Geometric Progressions:

J2001/P1/Q9
N2001/P1/Q8
J2002/P1/Q4
N2002/P1/Q2
J2003/P1/Q4
N2003/P1/Q3
J2004/P1/Q1
N2004/P1/Q2
J2005/P1/Q6
N2005/P1/Q6
J2006/P1/Q3
N2006/P1/Q6
J2007/P1/Q7
N2007/P1/Q4
J08/P1/Q7
N08/P1/Q3
J09/P1/Q7
N09/P1/Q3
J10/P1/Q7
N10/P1/Q5

### Binomial Theorem:

J2001/P1/Q5
N2002/P1/Q1
J2003/P1/Q1
J2004/P1/Q4
N2004/P1/Q1
J2005/P1/Q4
J2006/P1/Q4
N2006/P1/Q1
N2007/P1/Q3
J08/P1/Q3
N08/P1/Q1
J09/P1/Q3
N09/P1/Q2
J10/P1/Q6
N10/P1/Q1

## Topic 8 Differentiation

J2001/P1/Q6(i)
J2001/P1/Q10
N2001/P1/Q5
N2001/P1/Q9(ii)
N2001/P1/Q11(i)(ii)
J2002/P1/Q8
J2002/P1/Q9(i)(iii)
N2002/P1/Q8
N2002/P1/Q10(i)
J2003/P1/Q3(a)
J2003/P1/Q10(i)(ii)
N2003/P1/Q4(ii)
N2003/P1/Q8
N2003/P1/Q9(i)
J2004/P1/Q7(i)
J2004/P1/Q8
N2004/P1/Q7(i)
N2004/P1/Q10(i)(ii)
J2005/P1/Q2
J2005/P1/Q9(i)
N2005/P1/Q5
N2005/P1/Q8(i)
N2005/P1/Q10(ii)
J2006/P1/Q1
J2006/P1/Q9(i)
J2006/P1/Q10(i)(ii)(iii)
N2006/P1/Q7(i)
N2006/P1/Q8(i)(ii)
N2006/P1/Q9
J2007/P1/Q10
N2007/P1/Q8
N08/P1/Q7
J09/P1/Q11
J10/P1/Q8
J10/P1/Q10
N10/P1/Q3
N10/P1/Q10

## Topic 9 Integration

J2001/P1/Q6(ii)(iii)
N2001/P1/Q9(i)
N2001/P1/Q11(iii)
J2002/P1/Q3(ii)
J2002/P1/Q9(ii)
N2002/P1/Q4(i)
N2002/P1/Q10(ii)
J2003/P1/Q3(b)
J2003/P1/Q10(iii)
N2003/P1/Q4(i)
N2003/P1/Q9(ii)
J2004/P1/Q2
J2004/P1/Q7(ii)
N2004/P1/Q7(ii)
N2004/P1/Q10(iii)
J2005/P1/Q1
J2005/P1/Q9(ii)
N2005/P1/Q10(i)(iii)
J2006/P1/Q9(ii)
J2006/P1/Q10(iv)
N2006/P1/Q7(ii)
N2006/P1/Q8(iii)
J07/P1/Q2
N2007/P1/Q2
J08/P1/Q9
N08/P1/Q9
J09/P1/Q9
N09/P1/Q1
J10/P1/Q2
J10/P1/Q9
N10/P1/Q11

[To be finished by
1st March.]

## Paper 3 (P3) Topics

## ~Pure Mathematics~

### Topic 1 Algebra

**Inequalities:**

N02/P3/Q1
J06/P3/Q2
J03/P3/Q3
N06/P3/Q1
N03/P3/Q1
J08/P3/Q1
J04/P3/Q2
N10/P3/Q1
N05/P3/Q1

**Remainder and Factor Theorem:**

J02/P3/Q3
N07/P3/Q2
J03/P3/Q4
N08/P3/Q5
N04/P3/Q3
N09/P3/Q5
J05/P3/Q5
J10/P3/Q5
J07/P3/Q2

**Partial Fractions:**

J02/P3/Q6(i)
N04/P3/Q8(a)
J08/P3/Q7(i)
N02/P3/Q6(i)
N05/P3/Q9(i)
J09/P3/Q8(i)
J03/P3/Q6(i)
J06/P3/Q9(i)
N09/P3/Q8(i)
N03/P3/Q8(i)
N06/P3/Q8(i)
J10/P3/Q10(i)
J04/P3/Q9(i)
N07/P3/Q9(i)
N10/P3/Q8(i)

**Binomial Expansions:**

J02/P3/Q2
N04/P3/Q1
J07/P3/Q1
N10/P3/Q8(ii)
N02/P3/Q6(ii)
J05/P3/Q1
N07/P3/Q9(ii)
J03/P3/Q6(ii)
N05/P3/Q9(ii)
N08/P3/Q2
N03/P3/Q2
J06/P3/Q9(ii)
J09/P3/Q5
J04/P3/Q9(ii)
N06/P3/Q5
N09/P3/Q8(ii)

### Topic 2 Logarithmic and Exponential Functions

N02/P3/Q3
J07/P3/Q4
J10/P3/Q1
J04/P3/Q4
J08/P3/Q2
N10/P3/Q2
N04/P3/Q2
N08/P3/Q1
N05/P3/Q2
J09/P3/Q1
J06/P3/Q1
N09/P3/Q1

### Topic 3 Trigonometry

**Trigonometrical Identities:**

J02/P3/Q1
J03/P3/Q10(i)
J04/P3/Q5(i)
J05/P3/Q6(i)
J09/P3/Q3

**Trigonometrical Functions:**

N02/P3/Q5
J05/P3/Q6(ii)
N07/P3/Q5
J10/P3/Q3
J03/P3/Q1
N05/P3/Q5
J08/P1/Q1
N10/P3/Q3
N03/P3/Q3
J06/P3/Q4
J08/P3/Q4
J04/P3/Q1
N06/P3/Q2
N08/P3/Q6
N04/P3/Q4
J07/P3/Q5
N09/P3/Q4

*Differentiation*

| $\mathrm{f}(x)$ | $\mathrm{f}'(x)$ |
|---|---|
| $x^n$ | $nx^{n-1}$ |
| $\ln x$ | $\dfrac{1}{x}$ |
| $\mathrm{e}^x$ | $\mathrm{e}^x$ |
| $\sin x$ | $\cos x$ |
| $\cos x$ | $-\sin x$ |
| $\tan x$ | $\sec^2 x$ |
| $uv$ | $u\dfrac{\mathrm{d}v}{\mathrm{d}x}+v\dfrac{\mathrm{d}u}{\mathrm{d}x}$ |
| $\dfrac{u}{v}$ | $\dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x}-u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$ |

If $x=\mathrm{f}(t)$ and $y=\mathrm{g}(t)$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}t}\div\dfrac{\mathrm{d}x}{\mathrm{d}t}$

*Integration*

| $\mathrm{f}(x)$ | $\int \mathrm{f}(x)\,\mathrm{d}x$ |
|---|---|
| $x^n$ | $\dfrac{x^{n+1}}{n+1}+c \quad (n \neq -1)$ |
| $\dfrac{1}{x}$ | $\ln\lvert x\rvert+c$ |
| $\mathrm{e}^x$ | $\mathrm{e}^x+c$ |
| $\sin x$ | $-\cos x+c$ |
| $\cos x$ | $\sin x+c$ |
| $\sec^2 x$ | $\tan x+c$ |

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x=uv-\int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x=\ln\lvert\mathrm{f}(x)\rvert+c$$

*Vectors*

If $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ then

$$\mathbf{a.b}=a_1b_1+a_2b_2+a_3b_3=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\theta$$

*Numerical integration*

Trapezium rule:

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x \approx \tfrac{1}{2}h\{y_0+2(y_1+y_2+\cdots+y_{n-1})+y_n\}\text{, where } h=\frac{b-a}{n}$$

# REVISED SYLLABUS

## A & AS LEVEL MATHEMATICS (9709)

The 7 units in the scheme cover the following subject areas:

- Pure Mathematics (units P1, P2 and P3);
- Mechanics (units M1 and M2);
- Probability and Statistics (units S1 and S2).

Centres and candidates may:

- take all four Advanced (A) Level components in the same examination session for the full A Level.
- follow a staged assessment route to the A Level by taking two Advanced Subsidiary (AS) papers (P1 & M1 **or** P1 & S1) in an earlier examination session;
- take the Advanced Subsidiary (AS) qualification only.

### AS Level candidates take:

| Paper 1: Pure Mathematics 1 (P1) |
|---|
| 1¾ hours<br>About 10 shorter and longer questions<br>75 marks weighted at 60% of total |

plus **one** of the following papers:

| Paper 2: Pure Mathematics 2 (P2) | Paper 4: Mechanics 1 (M1) | Paper 6: Probability and Statistics (S1) |
|---|---|---|
| 1¼ hours<br>About 7 shorter and longer questions<br>50 marks weighted at 40% of total | 1¼ hours<br>About 7 shorter and longer questions<br>50 marks weighted at 40% of total | 1¼ hours<br>About 7 shorter and longer questions<br>50 marks weighted at 40% of total |

## A Level candidates take:

| Paper 1: Pure Mathematics 1 (P1) | Paper 3 Pure Mathematics 3 (P3) |
|---|---|
| 1¾ hours<br>About 10 shorter and longer questions<br>75 marks weighted at 30% of total | 1¾ hours<br>About 10 shorter and longer questions<br>75 marks weighted at 30% of total |

plus **one** of the following combinations of two papers:

| Paper 4: Mechanics 1 (M1) | Paper 6: Probability and Statistics 1 (S1) |
|---|---|
| 1¼ hours<br>About 7 shorter and longer questions<br>50 marks weighted at 20% of total | 1¼ hours<br>About 7 shorter and longer questions<br>50 marks weighted at 20% of total |

or

| Paper 4: Mechanics 1 (M1) | Paper 5: Mechanics 2 (M2) |
|---|---|
| 1¼ hours<br>About 7 shorter and longer questions<br>50 marks weighted at 20% of total | 1¼ hours<br>About 7 shorter and longer questions<br>50 marks weighted at 20% of total |

or

| Paper 6: Probability and Statistics 1 (S1) | Paper 7: Probability and Statistics 2 (S2) |
|---|---|
| 1¼ hours<br>About 7 shorter and longer questions<br>50 marks weighted at 20% of total | 1¼ hours<br>About 7 shorter and longer questions<br>50 marks weighted at 20% of total |

### Question papers

There is no choice of questions in any of the question papers and questions will be arranged approximately in order of increasing mark allocations.

It is expected that candidates will have a calculator with standard 'scientific' functions available for use for all papers in the examination. Computers, and calculators capable of algebraic manipulation, are not permitted.

A list of formulae and tables of the normal distribution (MF9) is supplied for the use of candidates in the examination. Details of the items in this list are given for reference in Section 6.

### Relationships between units

Units P2, M2, S2 are sequential to units P1, M1, S1 respectively, and the later unit in each subject area may not be used for certification unless the corresponding earlier unit is being (or has already been) used.

Unit P3 is also sequential to unit P1, and may not be used for certification unless P1 is being (or has already been) used. The subject content of unit P2 is a subset of the subject content of unit P3; otherwise, the subject content for different units does not overlap, although later units in each subject area assume knowledge of the earlier units.

## Aims

The aims of the syllabus are the same for all students. These are set out below and describe the educational purposes of any course based on the Mathematics units for the AS and A Level examinations. The aims are not listed in order of priority.

The aims are to enable candidates to:

- develop their mathematical knowledge and skills in a way which encourages confidence and provides satisfaction and enjoyment;
- develop an understanding of mathematical principles and an appreciation of mathematics as a logical and coherent subject;
- acquire a range of mathematical skills, particularly those which will enable them to use applications of mathematics in the context of everyday situations and of other subjects they may be studying;
- develop the ability to analyse problems logically, recognise when and how a situation may be represented mathematically, identify and interpret relevant factors and, where necessary, select an appropriate mathematical method to solve the problem;
- use mathematics as a means of communication with emphasis on the use of clear expression;
- acquire the mathematical background necessary for further study in this or related subjects.

## Assessment Objectives

The abilities assessed in the examinations cover a single area: **technique with application.**

The examination will test the ability of candidates to:

- understand relevant mathematical concepts, terminology and notation;
- recall accurately and use successfully appropriate manipulative techniques;
- recognise the appropriate mathematical procedure for a given situation;
- apply combinations of mathematical skills and techniques in solving problems;
- present mathematical work, and communicate conclusions, in a clear and logical way.

## Curriculum content

The mathematical content for each unit in the scheme is detailed below. The order in which topics are listed is not intended to imply anything about the order in which they might be taught.

As well as demonstrating skill in the appropriate techniques, candidates will be expected to apply their knowledge in the solution of problems. Individual questions set may involve ideas and methods from more than one section of the relevant content list.

For all units, knowledge of the content of O Level/IGCSE Mathematics is assumed. Candidates will be expected to be familiar with scientific notation for the expression of compound units, e.g. 5 m $s^{-1}$ for 5 metres per second.

e educa-
ions. The

nd provides

s a logical

lications of
g;
be
elect an

on;

s are

y their
from more

will be
or 5

es

al

of

re

| **Unit P1: Pure Mathematics 1 (Paper 1)** | |
|---|---|
| | *Candidates should be able to:* |
| **1. Quadratics** | • carry out the process of completing the square for a quadratic polynomial $ax^2 + bx + c$, and use this form, e.g. to locate the vertex of the graph of $y = ax^2 + bx + c$ or to sketch the graph;<br>• find the discriminant of a quadratic polynomial $ax^2 + bx + c$ and use the discriminant, e.g. to determine the number of real roots of the equation $ax^2 + bx + c = 0$;<br>• solve quadratic equations, and linear and quadratic inequalities, in one unknown;<br>• solve by substitution a pair of simultaneous equations of which one is linear and one is quadratic;<br>• recognise and solve equations in $x$ which are quadratic in some function of $x$, e.g. $x^4 - 5x^2 + 4 = 0$. |
| **2. Functions** | • understand the terms function, domain, range, one-one function, inverse function and composition of functions;<br>• identify the range of a given function in simple cases, and find the composition of two given functions;<br>• determine whether or not a given function is one-one, and find the inverse of a one-one function in simple cases;<br>• illustrate in graphical terms the relation between a one-one function and its inverse. |
| **3. Coordinate geometry** | • find the length, gradient and mid-point of a line segment, given the coordinates of the end-points;<br>• find the equation of a straight line given sufficient information (e.g. the coordinates of two points on it, or one point on it and its gradient);<br>• understand and use the relationships between the gradients of parallel and perpendicular lines;<br>• interpret and use linear equations, particularly the forms $y = mx + c$ and $y - y_1 = m(x - x_1)$;<br>• understand the relationship between a graph and its associated algebraic equation, and use the relationship between points of intersection of graphs and solutions of equations (including, in simple cases, the correspondence between a line being tangent to a curve and a repeated root of an equation). |
| **4. Circular measure** | • understand the definition of a radian, and use the relationship between radians and degrees;<br>• use the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ in solving problems concerning the arc length and sector area of a circle. |
| **5. Trigonometry** | • sketch and use graphs of the sine, cosine and tangent functions (for angles of any size, and using either degrees or radians);<br>• use the exact values of the sine, cosine and tangent of 30°, 45°, 60°, and related angles, e.g. $\cos 150° = -\frac{1}{2}\sqrt{3}$;<br>• use the notations $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ to denote the principal values of the inverse trigonometric relations;<br>• use the identities $\frac{\sin\theta}{\cos\theta} \equiv \tan\theta$ and $\sin^2\theta + \cos^2\theta \equiv 1$;<br>• find all the solutions of simple trigonometrical equations lying in a specified interval (general forms of solution are not included). |

| | |
|---|---|
| **6. Vectors** | • use standard notations for vectors, i.e. $\begin{pmatrix}x\\y\end{pmatrix}$, $x\mathbf{i}+y\mathbf{j}$, $\begin{pmatrix}x\\y\\z\end{pmatrix}$, $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, $\overrightarrow{AB}$, $\mathbf{a}$;<br>• carry out addition and subtraction of vectors and multiplication of a vector by a scalar, and interpret these operations in geometrical terms;<br>• use unit vectors, displacement vectors and position vectors;<br>• calculate the magnitude of a vector and the scalar product of two vectors;<br>• use the scalar product to determine the angle between two directions and to solve problems concerning perpendicularity of vectors. |
| **7. Series** | • use the expansion of $(a+b)^n$, where $n$ is a positive integer (knowledge of the greatest term and properties of the coefficients are not required, but the notations $\binom{n}{r}$ and $n!$ should be known);<br>• recognise arithmetic and geometric progressions;<br>• use the formulae for the $n$th term and for the sum of the first $n$ terms to solve problems involving arithmetic or geometric progressions;<br>• use the condition for the convergence of a geometric progression, and the formula for the sum to infinity of a convergent geometric progression. |
| **8. Differentiation** | • understand the idea of the gradient of a curve, and use the notations $f'(x)$, $f''(x)$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ (the technique of differentiation from first principles is not required);<br>• use the derivative of $x^n$ (for any rational $n$), together with constant multiples, sums, differences of functions, and of composite functions using the chain rule;<br>• apply differentiation to gradients, tangents and normals, increasing and decreasing functions and rates of change (including connected rates of change);<br>• locate stationary points, and use information about stationary points in sketching graphs (the ability to distinguish between maximum points and minimum points is required, but identification of points of inflexion is not included). |
| **9. Integration** | • understand integration as the reverse process of differentiation, and integrate $(ax+b)^n$ (for any rational $n$ except $-1$), together with constant multiples, sums and differences;<br>• solve problems involving the evaluation of a constant of integration, e.g. to find the equation of the curve through $(1, -2)$ for which $\frac{dy}{dx} = 2x+1$;<br>• evaluate definite integrals (including simple cases of 'improper' integrals, such as $\int_0^1 x^{-\frac{1}{2}}\,dx$ and $\int_1^{\infty} x^{-2}\,dx$ );<br>• use definite integration to find<br>the area of a region bounded by a curve and lines parallel to the axes, or between two curves,<br>a volume of revolution about one of the axes. |

Revised Syllabu

**Unit P2: Pu**
**Knowledge**
**demonstrat**

1. Algebra

2. Logarith
and exp
function

3. Trigono

4. Differe

5. Integra

| **Unit P2: Pure Mathematics 2 (Paper 2)**<br>**Knowledge of the content of unit P1 is assumed, and candidates may be required to demonstrate such knowledge in answering questions.** | |
|---|---|
| | *Candidates should be able to:* |
| **1. Algebra** | • understand the meaning of $\|x\|$, and use relations such as $\|a\| = \|b\| \Leftrightarrow a^2 = b^2$ and $\|x-a\| < b \Leftrightarrow a-b < x < a+b$ in the course of solving equations and inequalities;<br>• divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero);<br>• use the factor theorem and the remainder theorem, e.g. to find factors, solve polynomial equations or evaluate unknown coefficients. |
| **2. Logarithmic and exponential functions** | • understand the relationship between logarithms and indices, and use the laws of logarithms (excluding change of base);<br>• understand the definition and properties of $e^x$ and $\ln x$, including their relationship as inverse functions and their graphs;<br>• use logarithms to solve equations of the form $a^x = b$, and similar inequalities;<br>• use logarithms to transform a given relationship to linear form, and hence determine unknown constants by considering the gradient and/ or intercept. |
| **3. Trigonometry** | • understand the relationship of the secant, cosecant and cotangent functions to cosine, sine and tangent, and use properties and graphs of all six trigonometric functions for angles of any magnitude;<br>• use trigonometrical identities for the simplification and exact evaluation of expressions and in the course of solving equations, and select an identity or identities appropriate to the context, showing familiarity in particular with the use of<br>$\sec^2\theta \equiv 1+\tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1+\cot^2\theta$,<br>the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$,<br>the formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$,<br>the expressions of $a\sin\theta + b\cos\theta$ in the forms $R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$ |
| **4. Differentiation** | • use the derivatives of $e^x$, $\ln x$, $\sin x$, $\cos x$, $\tan x$, together with constant multiples, sums, differences and composites;<br>• differentiate products and quotients;<br>• find and use the first derivative of a function which is defined parametrically or implicitly. |
| **5. Integration** | • extend the idea of 'reverse differentiation' to include the integration of $e^{ax+b}$, $\frac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$ and $\sec^2(ax+b)$ (knowledge of the general method of integration by substitution is not required);<br>• use trigonometrical relationships (such as double-angle formulae) to facilitate the integration of functions such as $\cos^2 x$;<br>• use the trapezium rule to estimate the value of a definite integral, and use sketch graphs in simple cases to determine whether the trapezium rule gives an over-estimate or an under-estimate. |

| | |
|---|---|
| 6. Numerical solution of equations | • locate approximately a root of an equation, by means of graphical considerations and/or searching for a sign change;<br>• understand the idea of, and use the notation for, a sequence of approximations which converges to a root of an equation;<br>• understand how a given simple iterative formula of the form $x_{n+1} = \mathrm{F}(x_n)$ relates to the equation being solved, and use a given iteration, or an iteration based on a given rearrangement of an equation, to determine a root to a prescribed degree of accuracy (knowledge of the condition for convergence is not included, but candidates should understand that an iteration may fail to converge). |

## Unit P3: Pure Mathematics 3 (Paper 3)

**Knowledge of the content of unit P1 is assumed, and candidates may be required to demonstrate such knowledge in answering questions.**

| | *Candidates should be able to:* |
|---|---|
| 1. Algebra | • understand the meaning of $\lvert x \rvert$, and use relations such as $\lvert a \rvert = \lvert b \rvert \Leftrightarrow a^2 = b^2$ and $\lvert x - a \rvert < b \Leftrightarrow a - b < x < a + b$ in the course of solving equations and inequalities;<br>• divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero);<br>• use the factor theorem and the remainder theorem, e.g. to find factors, solve polynomial equations or evaluate unknown coefficients.<br>• recall an appropriate form for expressing rational functions in partial fractions, and carry out the decomposition, in cases where the denominator is no more complicated than $(ax+b)(cx+d)(ex+f)$, $(ax+b)(cx+d)^2$, $(ax+b)(x^2+c^2)$, and where the degree of the numerator does not exceed that of the denominator;<br>• use the expansion of $(1+x)^n$, where $n$ is a rational number and $\lvert x \rvert < 1$ (finding a general term is not included, but adapting the standard series to expand, e.g. $(2 - \frac{1}{2}x)^{-1}$ is included). |
| 2. Logarithmic and exponential functions | • understand the relationship between logarithms and indices, and use the laws of logarithms (excluding change of base);<br>• understand the definition and properties of $e^x$ and $\ln x$, including their relationship as inverse functions and their graphs;<br>• use logarithms to solve equations of the form $a^x = b$, and similar inequalities;<br>• use logarithms to transform a given relationship to linear form, and hence determine unknown constants by considering the gradient and/or intercept. |

| | |
|---|---|
| **3. Trigonometry** | • understand the relationship of the secant, cosecant and cotangent functions to cosine, sine and tangent, and use properties and graphs of all six trigonometric functions for angles of any magnitude;<br>• use trigonometrical identities for the simplification and exact evaluation of expressions and in the course of solving equations, and select an identity or identities appropriate to the context, showing familiarity in particular with the use of<br>$\sec^2\theta \equiv 1+\tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1+\cot^2\theta$,<br>the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$,<br>the formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$,<br>the expressions of $a\sin\theta + b\cos\theta$ in the forms $R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$ |
| **4. Differentiation** | • use the derivatives of $e^x$, $\ln x$, $\sin x$, $\cos x$, $\tan x$, together with constant multiples, sums, differences and composites;<br>• differentiate products and quotients;<br>• find and use the first derivative of a function which is defined parametrically or implicitly. |
| **5. Integration** | • extend the idea of 'reverse differentiation' to include the integration of $e^{ax+b}$, $\frac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$ and $\sec^2(ax+b)$ ;<br>• use trigonometrical relationships (such as double-angle formulae) to facilitate the integration of functions such as $\cos^2 x$;<br>• integrate rational functions by means of decomposition into partial fractions (restricted to the types of partial fractions specified in paragraph 1 above);<br>• recognise an integrand of the form $\frac{k\mathrm{f}'(x)}{\mathrm{f}(x)}$ and integrate, for example, $\frac{x}{x^2+1}$ or $\tan x$ ;<br>• recognise when an integrand can usefully be regarded as a product, and use integration by parts to integrate, for example, $x\sin 2x$, $x^2e^x$ or $\ln x$ ;<br>• use a given substitution to simplify and evaluate either a definite or an indefinite integral;<br>• use the trapezium rule to estimate the value of a definite integral, and use sketch graphs in simple cases to determine whether the trapezium rule gives an over-estimate or an under-estimate. |
| **6. Numerical solution of equations** | • locate approximately a root of an equation, by means of graphical considerations and/or searching for a sign change;<br>• understand the idea of, and use the notation for, a sequence of approximations which converges to a root of an equation;<br>• understand how a given simple iterative formula of the form $x_{n+1} = \mathrm{F}(x_n)$ relates to the equation being solved, and use a given iteration, or an iteration based on a given rearrangement of an equation, to determine a root to a prescribed degree of accuracy (knowledge of the condition for convergence is not included, but candidates should understand that an iteration may fail to converge). |

| | |
|---|---|
| 7. **Vectors** | • understand the significance of all the symbols used when the equation of a straight line is expressed in the form **r** = **a** + $t$**b**;<br>• determine whether two lines are parallel, intersect or are skew;<br>• find the angle between two lines, and the point of intersection of two lines when it exists;<br>• understand the significance of all the symbols used when the equation of a plane is expressed in either of the forms $ax + by + cz = d$ or (**r** – **a**).**n** = 0 ;<br>• use equations of lines and planes to solve problems concerning distances, angles and intersections, and in particular<br>find the equation of a line or a plane, given sufficient information,<br>determine whether a line lies in a plane, is parallel to a plane, or intersects a plane, and find the point of intersection of a line and a plane when it exists,<br>find the line of intersection of two non-parallel planes,<br>find the perpendicular distance from a point to a plane, and from a point to a line,<br>find the angle between two planes, and the angle between a line and a plane. |
| 8. **Differential equations** | • formulate a simple statement involving a rate of change as a differential equation, including the introduction if necessary of a constant of proportionality;<br>• find by integration a general form of solution for a first order differential equation in which the variables are separable;<br>• use an initial condition to find a particular solution;<br>• interpret the solution of a differential equation in the context of a problem being modelled by the equation. |
| 9. **Complex numbers** | • understand the idea of a complex number, recall the meaning of the terms real part, imaginary part, modulus, argument, conjugate, and use the fact that two complex numbers are equal if and only if both real and imaginary parts are equal;<br>• carry out operations of addition, subtraction, multiplication and division of two complex numbers expressed in cartesian form $x + iy$ ;<br>• use the result that, for a polynomial equation with real coefficients, any non-real roots occur in conjugate pairs;<br>• represent complex numbers geometrically by means of an Argand diagram;<br>• carry out operations of multiplication and division of two complex numbers expressed in polar form $r(\cos\theta + i\sin\theta) \equiv r\,e^{i\theta}$;<br>• find the two square roots of a complex number;<br>• understand in simple terms the geometrical effects of conjugating a complex number and of adding, subtracting, multiplying and dividing two complex numbers;<br>• illustrate simple equations and inequalities involving complex numbers by means of loci in an Argand diagram, e.g. $\|z-a\| < k$, $\|z-a\| = \|z-b\|$, $\arg(z-a) = \alpha$. |

## Unit M1: Mechanics 1 (Paper 4)

Questions set will be mainly numerical, and will aim to test mechanical principles without involving difficult algebra or trigonometry. However, candidates should be familiar in particular with the following trigonometrical results: $\sin(90° - \theta) \equiv \cos\theta$, $\cos(90° - \theta) \equiv \sin\theta$, $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$, $\sin^2\theta + \cos^2\theta \equiv 1$.

Vector notation will not be used in the question papers, but candidates may use vector methods in their solutions if they wish.

In the following content list, reference to the equilibrium or motion of a 'particle' is not intended to exclude questions that involve extended bodies in a 'realistic' context; however, it is to be understood that any such bodies are to be treated as particles for the purposes of the question.

| **Unit M1: Mechanics 1 (Paper 4)** | |
|---|---|
| | *Candidates should be able to:* |
| 1. **Forces and equilibrium** | • identify the forces acting in a given situation;<br>• understand the vector nature of force, and find and use components and resultants;<br>• use the principle that, when a particle is in equilibrium, the vector sum of the forces acting is zero, or equivalently, that the sum of the components in any direction is zero;<br>• understand that a contact force between two surfaces can be represented by two components, the normal component and the frictional component;<br>• use the model of a 'smooth' contact, and understand the limitations of this model;<br>• understand the concepts of limiting friction and limiting equilibrium; recall the definition of coefficient of friction, and use the relationship $F = \mu R$ or $F \le \mu R$, as appropriate;<br>• use Newton's third law. |
| 2. **Kinematics of motion in a straight line** | • understand the concepts of distance and speed as scalar quantities, and of displacement, velocity and acceleration as vector quantities (in one dimension only);<br>• sketch and interpret displacement-time graphs and velocity-time graphs, and in particular appreciate that<br>the area under a velocity-time graph represents displacement,<br>the gradient of a displacement-time graph represents velocity,<br>the gradient of a velocity-time graph represents acceleration;<br>• use differentiation and integration with respect to time to solve simple problems concerning displacement, velocity and acceleration (restricted to calculus within the scope of unit P1);<br>• use appropriate formulae for motion with constant acceleration in a straight line. |
| 3. **Newton's laws of motion** | • apply Newton's laws of motion to the linear motion of a particle of constant mass moving under the action of constant forces, which may include friction;<br>• use the relationship between mass and weight;<br>• solve simple problems which may be modelled as the motion of a particle moving vertically or on an inclined plane with constant acceleration;<br>• solve simple problems which may be modelled as the motion of two particles, connected by a light inextensible string which may pass over a fixed smooth peg or light pulley. |

| | |
|---|---|
| 4. Energy, work and power | • understand the concept of the work done by a force, and calculate the work done by a constant force when its point of application undergoes a displacement not necessarily parallel to the force (use of the scalar product is not required);<br>• understand the concepts of gravitational potential energy and kinetic energy, and use appropriate formulae;<br>• understand and use the relationship between the change in energy of a system and the work done by the external forces, and use in appropriate cases the principle of conservation of energy;<br>• use the definition of power as the rate at which a force does work, and use the relationship between power, force and velocity for a force acting in the direction of motion;<br>• solve problems involving, for example, the instantaneous acceleration of a car moving on a hill with resistance. |

## Unit S1: Probability & Statistics 1 (Paper 6)

| | *Candidates should be able to:* |
|---|---|
| 1. Representation of data | • select a suitable way of presenting raw statistical data, and discuss advantages and/or disadvantages that particular representations may have;<br>• construct and interpret stem-and-leaf diagrams, box-and-whisker plots, histograms and cumulative frequency graphs;<br>• understand and use different measures of central tendency (mean, median, mode) and variation (range, interquartile range, standard deviation), e.g. in comparing and contrasting sets of data;<br>• use a cumulative frequency graph to estimate the median value, the quartiles and the interquartile range of a set of data;<br>• calculate the mean and standard deviation of a set of data (including grouped data) either from the data itself or from given totals such as $\sum x$ and $\sum x$ and $\sum x^2$, or $\sum(x-a)$ and $\sum(x-a)^2$. |
| 2. Permutations and combinations | • understand the terms permutation and combination, and solve simple problems involving selections;<br>• solve problems about arrangements of objects in a line, including those involving:<br>repetition (e.g. the number of ways of arranging the letters of the word 'NEEDLESS'),<br>restriction (e.g. the number of ways several people can stand in a line if 2 particular people must — or must not — stand next to each other). |
| 3. Probability | • evaluate probabilities in simple cases by means of enumeration of equiprobable elementary events (e.g. for the total score when two fair dice are thrown), or by calculation using permutations or combinations;<br>• use addition and multiplication of probabilities, as appropriate, in simple cases;<br>• understand the meaning of exclusive and independent events, and calculate and use conditional probabilities in simple cases, e.g. situations that can be represented by means of a tree diagram. |

| | |
|---|---|
| **4. Discrete random variables** | • construct a probability distribution table relating to a given situation involving a discrete random variable variable $X$, and calculate $E(X)$ and $Var(X)$;<br>• use formulae for probabilities for the binomial distribution, and recognise practical situations where the binomial distribution is a suitable model (the notation $B(n, p)$ is included);<br>• use formulae for the expectation and variance of the binomial distribution. |
| **5. The normal distribution** | • understand the use of a normal distribution to model a continuous random variable, and use normal distribution tables;<br>• solve problems concerning a variable $X$, where $X \sim N(\mu, \sigma^2)$, including<br>finding the value of $P(X > x_1)$, or a related probability, given the values of $x_1$, $\mu$, $\sigma$,<br>finding a relationship between $x_1$, $\mu$ and $\sigma$ given the value of $P(X > x_1)$ or a related probability;<br>• recall conditions under which the normal distribution can be used as an approximation to the binomial distribution ($n$ large enough to ensure that $np > 5$ and $nq > 5$), and use this approximation, with a continuity correction, in solving problems. |

# June 2001 Paper 1

## *Pure Mathematics (P1)*

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1 Find all the values of $x$ in the interval $0^\circ \le x \le 180^\circ$ which satisfy the equation $5\cos 2x = 2$ [4]

***Suggested Solution:***

Given range is $0^\circ \le x \le 180^\circ$

for angle $2x$ range is $0^\circ \le 2x \le 360^\circ$

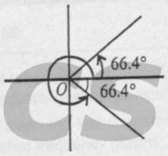
$5\cos 2x = 20 \quad \Rightarrow \quad \cos 2x = 0.4$

$\Rightarrow \quad 2x = \cos^{-1}(0.4)$

(basic angle $\alpha = 66.4^\circ$)

$\therefore \quad 2x = 66.4^\circ,\ 293.6^\circ$

or $\quad x = 33.2^\circ, 146.8^\circ$ (Ans).



Note that:
cos is positive in 1st and 4th quadrant.

2 Find the coordinates of the points of intersection of the line $y + 2x = 9$ and the curve $y^2 = 6x + 1$. [4]

***Suggested Solution:***

$y + 2x = 9 \cdots\cdots\cdots \text{(i)}$ and $y^2 = 6x + 1 \cdots\cdots\cdots \text{(ii)}$

from eq.(i) $y = 9 - 2x$, putting in eq.(ii) and solving simultaneously

$(9-2x)^2 = 6x + 1$

$\Rightarrow \quad 4x^2 - 36x + 81 = 6x + 1 \quad \Rightarrow \quad 4x^2 - 42x + 80 = 0 \quad \Rightarrow \quad 2x^2 - 21x + 40 = 0$

$\Rightarrow \quad (x-8)(2x-5) = 0 \quad \Rightarrow \quad x = 8 \text{ or } x = \frac{5}{2}$

$\therefore$ when $x = 8$ $\qquad$ when $x = \frac{5}{2}$

$y = 9 - 2(8) \qquad y = 9 - 2\left(\frac{5}{2}\right)$

$y = -7 \qquad y = 4$

find value of $y$ from linear equation.

$\therefore x = 8,\ y = -7$ and $x = \frac{5}{2},\ y = 4$

Hence points of intersection are $(8, -7)$ and $(\frac{5}{2}, 4)$ (Ans)

**3.** The function $f : x \mapsto x^2 + 5x + 8$ is defined for the domain $x \geq a$, where $a$ is a constant.

(i) Express $x^2 + 5x + 8$ in the form $(x+p)^2 + q$. [2]

(ii) Find the smallest value of $a$ for which f has an inverse. [1]

(iii) Find the domain of $f^{-1}$ corresponding to this value of $a$. [2]

***Suggested Solution:***

(i) $x^2 + 5x + 8$

$$= x^2 + 5x + \left(\frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 8 = \left(x + \frac{5}{2}\right)^2 + 8 - \frac{25}{4} = \left(x + \frac{5}{2}\right)^2 + \frac{7}{4} \quad \textbf{(Ans)}$$

$$\therefore \; p = \frac{5}{2} \text{ and } q = \frac{7}{4}$$

(ii) From part (i), we have $f : x \mapsto \left(x + \frac{5}{2}\right)^2 + \frac{7}{4}$

hence the smallest value of $a = -\frac{5}{2}$ **(Ans)**

(iii) Domain of $f^{-1}(x)$ is $x \geq \frac{7}{4}$ **(Ans)**

**Horizontal line test:**

**(ii)** If any horizontal line intersects the graph at only one point, that graph is the graph of 1–1 function.

Note that a function f($x$) has an inverse only if it is a 1-1 function.

**(iii)** Domain of f($x$) is the range of $f^{-1}(x)$ and vice versa.

**4.** Prove the identity. $(1 + \sin\theta)\left(\frac{1}{\cos\theta} - \tan\theta\right) \equiv \cos\theta$ [4]

***Suggested Solution:***

$$\text{L.H.S.} = (1 + \sin\theta)\left(\frac{1}{\cos\theta} - \tan\theta\right)$$

$$= (1 + \sin\theta)\left(\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\right)$$

$$= (1 + \sin\theta)\left(\frac{1 - \sin\theta}{\cos\theta}\right)$$

$$= \frac{1 - \sin^2\theta}{\cos\theta} = \frac{\cos^2\theta}{\cos\theta} = \cos\theta = \text{R.H.S.} \quad \textbf{(Proved)}$$

5. (i) Find the first three terms in the expansion, in ascending powers of $x$, of $(2-x)^6$. [3]

(ii) Find the value of the constant $a$ for which the coefficient of $x$ in the expansion of $(1+ax)(2-x)^6$ is 64. [3]

Binomial theorem:

$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + .... + {}^nC_r a^{n-r}b^r + .... + {}^nC_n b^n$

***Suggested Solution:***

(i) $(2-x)^6 = {}^6C_0\,(2)^6(-x)^0 + {}^6C_1\,(2)^5(-x)^1 + {}^6C_2\,(2)^4(-x)^2$

$= 64 - 192x + 240x^2$ **(Ans)**

(ii) using the result from part (i)

$$(1+ax)(2-x)^6 = (1+ax)\left(64-192x+240x^2\right)$$

collecting the terms of $x$ only, we have

$$-192x + 64ax = (-192+64a)\,x$$

as the coefficient of $x$ is 64

$\therefore\quad -192+64a = 64 \;\Rightarrow\; 64a = 256 \;\Rightarrow\; a = 4$ (Ans)

6. The equation of a curve is $y = (3x+1)^{\frac{3}{2}} + 1$

(i) Find the gradient of the curve at the point where $x = 1$ [3]

(ii) Find $y = \int [(3x+1)^{\frac{3}{2}} + 1]\,dx$ [3]

(iii) Hence find the area of the region enclosed by the curve, the $x$-axis and the lines $x = 0$ and $x = 1$. [2]

***Suggested Solution:***

Gradient of a curve at any point $(x, y)$, is given by $\frac{dy}{dx}$

(i) $y = (3x+1)^{\frac{3}{2}} + 1$

$$\frac{dy}{dx} = \frac{3}{2}(3x+1)^{\frac{1}{2}}(3+0) = \frac{9}{2}(3x+1)^{\frac{1}{2}}$$

at $x = 1$

$$\frac{dy}{dx} = \frac{9}{2}[3(1)+1]^{\frac{1}{2}} = \frac{9}{2}(4)^{\frac{1}{2}} = 9$$

$\therefore$ Gradient of the curve at $x = 1$, is 9 **(Ans)**

(ii) $y = \int [(3x+1)^{\frac{3}{2}} + 1]\,dx = \dfrac{(3x+1)^{\frac{5}{2}}}{(3)(\frac{5}{2})} + x + C$

$= \frac{2}{15}(3x+1)^{\frac{5}{2}} + x + C$ (where C is constant of integration) **(Ans)**

(iii) $\text{Area} = \int_0^1 y\,dx = \int_0^1 [(3x+1)^{\frac{3}{2}} + 1]\,dx$

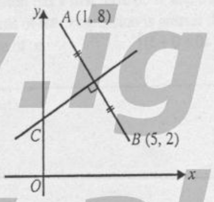
using the result of part (ii), we have

$$\text{Area} = \int_0^1 [(3x+1)^{\frac{3}{2}} + 1]\,dx = \left[\frac{(3x+1)^{\frac{5}{2}}}{\frac{5}{2}(3)} + x\right]_0^1 = \left[\frac{2}{15}(3x+1)^{\frac{5}{2}} + x\right]_0^1$$

$$= \left(\frac{2}{15}(3(1)+1)^{\frac{5}{2}} + (1)\right) - \left(\frac{2}{15}(3(0)+1)^{\frac{5}{2}} + (0)\right)$$

$$= \left(\frac{2}{15}(4)^{\frac{5}{2}} + 1\right) - \left(\frac{2}{15}(1)^{\frac{5}{2}} + 0\right) = \left(\frac{64}{15} + 1\right) - \left(\frac{2}{15}\right) = \frac{77}{15} = 5\frac{2}{15} \text{ sq. units} \quad \text{(Ans)}$$

7.



The diagram shows two points $A(1, 8)$ and $B(5, 2)$. The perpendicular bisector of $AB$ cuts the $y$-axis at $C$. Find.

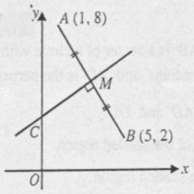
(i) the equation of the perpendicular bisector of $AB$. [4]

(ii) the area of triangle $ABC$. [4]

*Suggested Solution:*

(i) Let $M$ be the mid point of $AB$,

$\therefore$ Coordinates of $M$ are $\left(\frac{1+5}{2}, \frac{8+2}{2}\right)$ or $M(3, 5)$



Gradient of $AB = \frac{8-2}{1-5} = -\frac{6}{4} = -\frac{3}{2}$

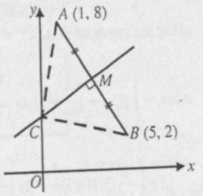
$\therefore$ Gradient of $MC = +\frac{2}{3}$

If two lines are perpendicular than product of their gradient is equal to −1.

perpendicular bisector of $AB$ is $MC$

$\therefore$ using $y - y_1 = m(x - x_1)$ equation of $MC$ is:

$$y - 5 = \frac{2}{3}(x-3) \Rightarrow 3y - 15 = 2x - 6 \Rightarrow 2x - 3y + 9 = 0 \quad \text{(Ans)}$$



(ii) Coordinates of $C$ are $y$-intercept of equation of $MC$ in part (i)

putting $x = 0$ in the equation of $M\dot{C}$

$2(0) - 3y + 9 = 0 \;\Rightarrow\; 3y = 9 \;\Rightarrow\; y = 3$

$\therefore$ coordinates of $C$ are $(0, 3)$

Now $|AB| = \sqrt{(1-5)^2 + (8-2)^2} \quad = \sqrt{16+36} \;= \sqrt{52} \;= 2\sqrt{13}$

and $|CM| = \sqrt{(0-3)^2 + (3-5)^2} \quad = \sqrt{9+4} \;= \sqrt{13}$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}|AB| \times |CM| = \frac{1}{2} \times 2\sqrt{13} \times \sqrt{13} \;= 13$ sq.units. **(Ans)**

*Alternative Method*:

(ii) Area of $\Delta\, ABC$ with $A\,(1, 8)$, $B\,(5, 2)$ and $C\,(0, 3)$

$$= \frac{1}{2}\begin{vmatrix} 1 & \diagdown & 5 & \diagdown & 0 & \diagdown & 1 \\ 8 & & 2 & & 3 & & 8 \end{vmatrix} = \frac{1}{2}\left|(2+15+0)-(40+0+3)\right|$$

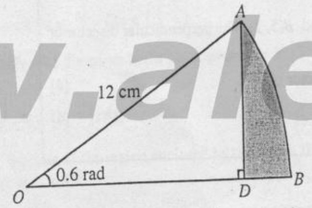
$$= \frac{1}{2}|17-43| = \frac{26}{2} = 13 \text{ sq. units} \quad \textbf{(Ans)}$$

Area of a triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given as

$$\text{Area} = \frac{1}{2}\begin{vmatrix} x_1 & \diagdown & x_2 & \diagdown & x_3 & \diagdown & x_1 \\ y_1 & & y_2 & & y_3 & & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} (x_1 y_2 + x_2 y_3 + x_3 y_1) \\ -(x_2 y_1 + x_3 y_2 + x_1 y_3) \end{bmatrix}$$

**8.**



In the diagram, $OAB$ is a sector of a circle with center $O$ and radius 12 cm. Angle $AOB = 0.6$ radians and $AD$ is the perpendicular from $A$ to $OB$. Find.

(i) the lengths of $AD$ and $OD$. [2]

(ii) the perimeter of the shaded region. [3]

(iii) the area of the shaded region. [3]

*Suggested Solution:*

(i) By considering triangle $AOD$, we have

Angle is in radians. Change your calculator to radian mode

$\sin 0.6 = \frac{AD}{12} \;\Rightarrow\; AD = 12\sin 0.6 \;\Rightarrow\; AD = 6.78$ (3 s.f) **(Ans)**

$\cos 0.6 = \frac{OD}{12} \;\Rightarrow\; OD = 12\cos 0.6 \;\Rightarrow\; OD = 9.90$ (3 s.f) **(Ans)**

(ii) $BD = OB - OD \;\Rightarrow\; BD = 12 - 9.90 \;\Rightarrow\; BD = 2.1$ cm

using $S = r\theta$, arc length $AB = 12 \times 0.6 = 7.2$ cm

peremeter of the shaded region $= \overline{AD} + \overline{BD} + \overset{\frown}{AB}$

$$= 6.78 + 2.1 + 7.2 = 16.08 \approx 16.1 \text{ cm} \quad \textbf{(Ans)}$$

(iii) Area of shaded region = area of sector $\overset{\frown}{OAB}$ − area of $\Delta OAD$

$$= \frac{1}{2}\,(12)^2\,(0.6) - \frac{1}{2}\,(9.90)\,(6.78)$$

$$= 43.2 - 33.561 = 9.639 \;\; = 9.64 \text{ cm}^2 \;\; (\text{ to } 3 \text{ s.f}) \;\; \textbf{(Ans).}$$

9. (a) Find the sum to infinity of the geometric progression whose second term is 162 and whose fifth term is 48. [4]

(b) An arithmetic progression has a first term of 6 and a fifth term of 18. The sum of the first $n$ terms of the progression is greater than 2000. Find the smallest possible value of $n$. [5]

*Suggested Solution:*

General G.P $= a, a\,r, a\,r^2 \ldots\ldots a\,r^{n-1}$

nth term of a G.P is
$T_n = a\,r^{n-1}$

(a) $T_2 = ar \;\Rightarrow\; 162 = ar \cdots\cdots\cdots(\text{i})$,

and $T_5 = ar^4 \;\Rightarrow\; 48 = ar^4 \cdots\cdots\cdots(\text{ii})$

dividing eq. (ii) by eq. (i)

$$\frac{ar^4}{ar} = \frac{48}{162} \;\Rightarrow\; r^3 = \frac{8}{27} \;\Rightarrow\; r = \frac{2}{3}$$

putting value of $r$ in eq. (i)

$$162 = a\left(\frac{2}{3}\right) \;\Rightarrow\; a = 243$$

Now, using $S_\infty = \dfrac{a}{1-r}$, $\;S_\infty = \dfrac{243}{1-\frac{2}{3}} = 243 \times 3 = 729$ **(Ans)**

(b) Given: $a = 6$ and $T_5 = 18$

$$T_5 = a + (5-1)\,d \;\Rightarrow\; 18 = 6 + (5-1)\,d \;\Rightarrow\; d = 3$$

using $S_n = \dfrac{n}{2}[2a + (n-1)d]$, as $S_n > 2000$

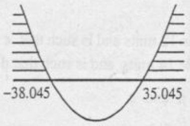
$\Rightarrow \dfrac{n}{2}[2(6) + (n-1)(3)] > 2000 \;\Rightarrow\; \dfrac{n}{2}[3n + 9] > 2000$

$\Rightarrow 3n^2 + 9n > 4000 \;\Rightarrow\; \underline{3n^2 + 9n - 4000 > 0}$

$\Rightarrow (n - \underline{35.045})(n + \underline{38.045}) > 0$

$\Rightarrow n < -38.045$ (ignored) or $n > 35.045$

$\therefore$ smallest value of $n = 36$ **(Ans)**



Solving $3n^2 + 9n - 4000 = 0$ by using quadratic formula gives $n = 35.045$, and $n = -38.045$

$\therefore (n - 35.045)$ and $(n + 38.045)$ are factors.

**10.** The equation of a curve is $y = 2x + \frac{24}{x}$, where $x > 0$.

(i) Obtain expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of $x$. [4]

(ii) Show that the $x$-coordinate of the stationary point on the curve is $2\sqrt{3}$ [3]

(iii) Determine the nature of the stationary point. [2]

*Suggested Solution:*

(i) $\frac{dy}{dx} = 2 - \frac{24}{x^2}$ **(Ans)**

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(2 - \frac{24}{x^2}\right) = \frac{d}{dx}(2 - 24x^{-2}) = 0 + 48x^{-3} = \frac{48}{x^3} \quad \textbf{(Ans)}$$

(ii) For stationary values put $\frac{dy}{dx} = 0$

$$2 - \frac{24}{x^2} = 0 \Rightarrow 2 = \frac{24}{x^2} \Rightarrow x^2 = 12 \Rightarrow x = \pm\sqrt{12}$$

as $x > 0$, $\therefore x = +\sqrt{12} = 2\sqrt{3}$ **(Shown)**

Nature:

if $\left.\frac{d^2y}{dx^2}\right|_{x = x_1} > 0$, $y$ is min

if $\left.\frac{d^2y}{dx^2}\right|_{x = x_1} < 0$, $y$ is max

(iii) as $\frac{d^2y}{dx^2} = \frac{48}{x^3}$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=2\sqrt{3}} = \frac{48}{(2\sqrt{3})^3} = 4.518 > 0$$

$\therefore$ $y$ is minimum at $x = 2\sqrt{3}$

$$y = 2x + \frac{24}{x} \Rightarrow y = 2(2\sqrt{3}) + \frac{24}{(2\sqrt{3})} \Rightarrow y = \frac{24+24}{(2\sqrt{3})} = \frac{24}{\sqrt{3}} = 8\sqrt{3}$$

$\therefore$ stationary point $\left(2\sqrt{3}, 8\sqrt{3}\right)$ is a minimum point. **(Ans)**

**11.** The points $A$ and $B$ have position vectors **a** and **b** respectively, relative to an origin $O$. The vectors **a** and **b** are given by

$$\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{b} = 6\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}.$$

(i) Obtain the value of $\mathbf{a} \cdot \mathbf{b}$ and hence find angle $AOB$, correct to the nearest degree. [5]

Vector **c** has magnitude 12 units and is such that $\mathbf{c} = p\mathbf{a}$, where $p$ is a constant. Vector **d** has magnitude 14 units and is such that $\mathbf{d} = q\mathbf{b}$, where $q$ is a constant.

(ii) Find the values of $p$ and $q$. [2]

(iii) Find the magnitude of $\mathbf{d} - \mathbf{c}$. [3]

*Suggested Solution:*

If
$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$
$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then
$\mathbf{a}.\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
and magnitude of a is
$|\mathbf{a}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$

(i) We have $\mathbf{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$

$$\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} = 12 + 4 + 3 = 19$$

now $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos A\hat{O}B \Rightarrow 19 = \left(\sqrt{4+4+1}\right)\left(\sqrt{36+4+9}\right) \cos A\hat{O}B$

$\Rightarrow 19 = (3)(7) \cos A\hat{O}B \Rightarrow \cos A\hat{O}B = \frac{19}{21} \Rightarrow A\hat{O}B = 25.2^\circ \approx 25^\circ$ (Ans)

(ii) $\mathbf{c} = p\mathbf{a} \Rightarrow \mathbf{c} = p\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \mathbf{c} = \begin{pmatrix} 2p \\ -2p \\ p \end{pmatrix}$

given that $|\mathbf{c}| = 12$

$\Rightarrow \sqrt{(2p)^2 + (-2p)^2 + (p)^2} = 12 \Rightarrow \sqrt{9p^2} = 12 \Rightarrow p = 4$ (Ans)

also $\mathbf{d} = q\mathbf{b} \Rightarrow \mathbf{d} = q\begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} \Rightarrow \mathbf{d} = \begin{pmatrix} 6q \\ -2q \\ 3q \end{pmatrix}$

as $|\mathbf{d}| = 14$

$\Rightarrow \sqrt{36q^2 + 4q^2 + 9q^2} = 14 \Rightarrow \sqrt{49q^2} = 14 \Rightarrow q = 2$ (Ans)

(iii) $\mathbf{c} = p\mathbf{a} \Rightarrow \mathbf{c} = 4\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \mathbf{c} = \begin{pmatrix} 8 \\ -8 \\ 4 \end{pmatrix}$

$\mathbf{d} = q\mathbf{b} \Rightarrow \mathbf{d} = 2\begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} \Rightarrow \mathbf{d} = \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix}$

$$\mathbf{d} - \mathbf{c} = \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix} - \begin{pmatrix} 8 \\ -8 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix}$$

$\therefore \ |\mathbf{d} - \mathbf{c}| = \sqrt{4^2 + 4^2 + 2^2} = \sqrt{36} = 6$ (Ans)

# November 2001 Paper 1

## *Pure Mathematics (P1)*

Answer all the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Find the value of the constant $k$ for which the line $y+2x=k$ is a tangent to the curve $y=x^2-6x+14$. [4]

*Suggested Solution:*

Let $y=x^2-6x+14$ ………(i)

and $y+2x=k \Rightarrow y=k-2x$ ………(ii)

Eliminate '*y*' from the given linear and quadratic equation.

solving eq. (i) and eq.(ii) simultaneously, we have

$$x^2-6x+14=k-2x$$

$$\Rightarrow x^2+2x-6x+14-k=0 \Rightarrow x^2-4x+(14-k)=0$$

as line is tangent to the curve,

$\therefore$ discriminant$=0$

$$\Rightarrow b^2-4ac=0 \Rightarrow (-4)^2-4(1)(14-k)=0$$

$$\Rightarrow 16-56+4k=0 \Rightarrow 4k=40 \Rightarrow k=10 \quad \text{(Ans).}$$

When a line is tangent to the curve then the quadratic equation emerging from their simultaneous solution has equal roots.

**2.** (i) Express $2x^2-12x+11$ in the form $a(x+b)^2+c$. [3]

(ii) Given that $f: x \mapsto 2x^2-12x+11$, for the domain $x \geq 0$, find the range of f. [2]

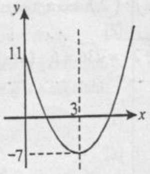
*Suggested Solution:*

(i) $2x^2-12x+11$

$$\Rightarrow 2(x^2-6x)+11 \Rightarrow 2[x^2-6x+(3)^2-(3)^2]+11$$

$$\Rightarrow 2[(x-3)^2-9]+11 \Rightarrow 2(x-3)^2-18+11$$

$$\Rightarrow 2(x-3)^2-7 \quad \text{(Ans)}$$



If $y=ax^2+bx+c$ is written in form $y=a(x-h)^2+k$, than co-ordinates of turning points are $(h, k)$.

(ii) from part (i), we have $f(x)=2(x-3)^2-7$

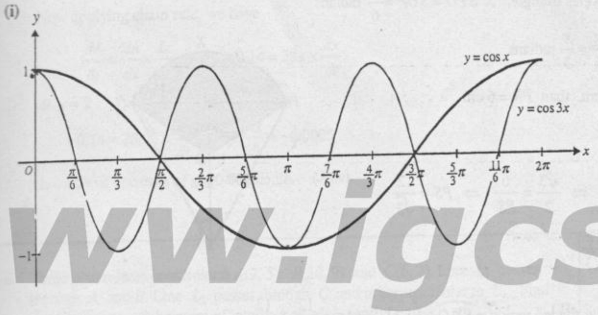
$\therefore$ for domain $x \geq 0$, range of f is: $f(x) \geq -7$ (Ans).

3. (i) Sketch and label, on the same diagram, the graphs of $y = \cos x$ and $y = \cos 3x$ for the interval $0 \le x \le 2\pi$. [3]

(ii) Given that $f : x \mapsto \cos x$, for the domain $0 \le x \le k$, find the largest value of $k$ for which f has an inverse. [2]

*Suggested Solution:*

(i)



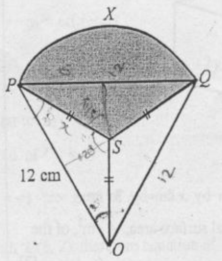
Note that $y = \cos 3x$ means that there should be three complete cycles of $\cos x$ in the range $0 \le x \le 2\pi$

A graph of a function represents a (1-1) function if it passes a horizontal line test. i.e any horizontal line has only one point of intersection with the graph.

(ii) The largest value of $k$ for which $f : x \mapsto \cos x$ has an inverse is $k = \pi$ (Ans).

A function $f(x)$ has an inverse only if it is a 1-1 function

4.



The diagram shows an equilateral triangle $OPQ$ of side 12 cm, and the point $S$ such that $OS = PS = QS$. The arc $PXQ$ has centre $O$ and radius 12 cm. Find the perimeter of the shaded region giving your answer in terms of $\pi$ and $\sqrt{3}$ [6]

*Suggested Solution:*

As $OPQ$ is an equilateral triangle, therefore $P\hat{O}Q = \frac{\pi}{3}$ radians (or 60°)

Arc length $PQ = 12 \times \frac{\pi}{3} = 4\pi$ cm

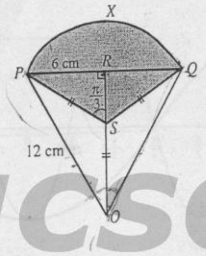
Length of arc is , S = $r\theta$ where $\theta$ is in radians

Extend $OS$ to meet chord $PQ$ at $R$

$\Delta PSO$ is an isossceles triangle, $\therefore S\hat{P}O = S\hat{O}P = \frac{\pi}{6}$ radians

$\Rightarrow P\hat{S}R = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$ radians

also as $PQ = 12$ cm, than $PR = 6$ cm



In right $\Delta$ $PSR$

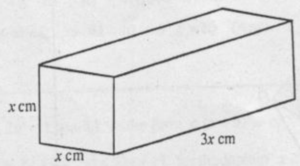
$$\sin\frac{\pi}{3} = \frac{PR}{PS} \Rightarrow \frac{\sqrt{3}}{2} = \frac{6}{PS} \Rightarrow PS = \frac{12}{\sqrt{3}}$$

$$QS = PS = \frac{12}{\sqrt{3}}$$

$\therefore$ perimeter of the shaded region $= \overset{\frown}{PXQ} + PS + QS = 4\pi + \frac{12}{\sqrt{3}} + \frac{12}{\sqrt{3}}$

$$= 4\pi + \frac{24}{\sqrt{3}} = 4\pi + 8\sqrt{3} = 4(\pi + 2\sqrt{3}) \text{ (Ans).}$$

**5.**



The diagram shows a rectangular block of ice, $x$ cm by $x$ cm by $3x$ cm.

(i) Obtain an expression, in terms of $x$, for the total surface area, $A$ cm$^2$, of the block and write down an expression for $\frac{dA}{dx}$. [3]

(ii) Given that the ice is melting in such a way that $A$ is decreasing at a constant rate of 0.14 cm$^2$ s$^{-1}$, calculate the rate of decrease of $x$ at the instant when $x = 2$. [3]

*Suggested Solution:*

(i) Total surface area $A = 2[(x \times x) + (x \times 3x) + (x \times 3x)]$

$$= 2[x^2 + 3x^2 + 3x^2] = 14x^2$$

$$\frac{dA}{dx}\left(14x^2\right) = 28x \text{ (Ans)}$$

Total surface area *A*, of a cuboid is given as:

$$A = 2\begin{bmatrix}(L \times W) + (W \times H) \\ +(L \times H)\end{bmatrix}$$

where *L*, *W* and *H* are length, width and height respectively.

(ii) Given that $\frac{dA}{dt} = -0.14 \text{ cm}^2\text{s}^{-1}$, when $x = 2$

Note that negative sign indicates the decrease

Now applying chain rule, we have

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt} \Rightarrow -0.14 = 28x \times \frac{dx}{dt}$$

at $x = 2$

$$-0.14 = 28(2) \times \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -0.0025$$

$\therefore$ rate of decrease of $x = 0.0025$ cm/s (Ans)

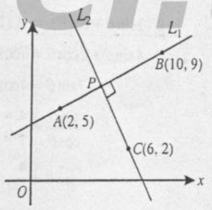
Chain Rule:

$$\frac{dA}{dx} = \frac{\frac{dA}{dt}}{\frac{dx}{dt}},$$

or $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$

6. Three points have coordinates $A(2, 5)$, $B(10, 9)$ and $C(6, 2)$. Line $L_1$ passes through $A$ and $B$. Line $L_2$ passes through $C$ and is perpendicular to $L_1$. Find the coordinates of the point of intersection of $L_1$ and $L_2$. [7]

*Suggested Solution:*



Given that line $L_1$ passes through $A(2, 5)$ and $B(10, 9)$

Gradient of $L_1 = \frac{9-5}{10-2} = \frac{1}{2}$

equation of $L_1$ is: $y - y_1 = m(x - x_1)$

$$\Rightarrow y - 5 = \frac{1}{2}(x-2) \Rightarrow 2y - 10 = x - 2 \Rightarrow 2y - x = 8 \cdots\cdots\text{(i)}$$

as $L_2$ is perpendicular to $L_1$

$\therefore$ (grad. of $L_1$)(grad. of $L_2$) $= -1$

$$\Rightarrow \frac{1}{2}(\text{grad. of } L_2) = -1 \Rightarrow \text{grad. of } L_2 = -2$$

Since lines $L_1$ and $L_2$ are perpendicular to each other, so (slope of $L_1$) (slope of $L_2$) $= -1$

i.e. $m_1 m_2 = -1$.

$L_2$ also passes through C(6, 2), therefore equation of $L_2$ is

$$y - 2 = -2(x-6) \Rightarrow y = -2x + 14 \cdots\cdots\text{(ii)}$$

solving eq. (i) and eq. (ii) simultaneously we have

$$2(-2x+14) - x = 8 \Rightarrow -5x = -20 \Rightarrow x = 4$$

Equation of a straight line with gradient *m* and passing through $(x_1, y_1)$ is $(y - y_1) = m(x - x_1)$.

putting value of $x$ in eq. (ii)

$$y = -2(4) + 14 \Rightarrow y = 6$$

$\therefore$ point of intersection of $L_1$ and $L_2$ are (4, 6). (Ans)

7. It is given that $a = 2\sin\theta + \cos\theta$ and $b = 2\cos\theta - \sin\theta$, where $0° \le \theta \le 360°$.

(i) Show that $a^2 + b^2$ is constant for all values of $\theta$. [3]

(ii) Given that $2a = 3b$, show that $\tan\theta = \frac{4}{7}$ and find the corresponding values of $\theta$. [4]

***Suggested Solution:***

(i) Given that:

$a = 2\sin\theta + \cos\theta$; and

$b = 2\cos\theta - \sin\theta$

now

$$a^2 + b^2 = (2\sin\theta + \cos\theta)^2 + (2\cos\theta - \sin\theta)^2$$

$$= (4\sin^2\theta + 4\sin\theta\cos\theta + \cos^2\theta) + (4\cos^2\theta - 4\sin\theta\cos\theta + \sin^2\theta)$$

$$= 5\sin^2\theta + 5\cos^2\theta$$

$$= 5(\sin^2\theta + \cos^2\theta) = 5 \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

$\therefore\ a^2 + b^2 = 5$, which is a constant. **(Ans).**

(ii) Given that $2a = 3b$

$$2(2\sin\theta + \cos\theta) = 3(2\cos\theta - \sin\theta)$$

$$4\sin\theta + 2\cos\theta = 6\cos\theta - 3\sin\theta$$

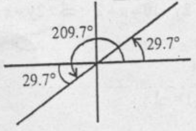
$$7\sin\theta = 4\cos\theta$$

$$\frac{\sin\theta}{\cos\theta} = \frac{4}{7}$$

$$\tan\theta = \frac{4}{7} \quad \textbf{(shown)}$$

basic angle $\alpha = 29.7°$

$\therefore\ \theta = 29.7°, 209.7°$ **(Ans).**



tan $\theta$ is positive in I and III quadrant

$\therefore$ in the Ist quadrant $\theta = \alpha$

and in the III quadrant

$\theta = 180 + \alpha$

8 A precious metal is extracted from a mine. In the first year of operation, 2000 kg of the metal was extracted. In each succeeding year, the amount extracted was 90% of the previous year's amount. Find

(i) the amount of metal extracted in the 10th year of operation, [2]

(ii) the total amount of metal extracted in the first 20 years of operation, [2]

(iii) the total amount of metal that would be extracted over a very long period of time. [3]

*Suggested Solution:*

(i) Let the amount of metal extracted in 1st year is $P = 2000$ kg

$\therefore$ amount of metal extracted in 2nd year $= 90\%$ of $P = \frac{90}{100}P = 0.9\,P$

amount of metal extracted in 3rd year $= 90\%$ of $0.9\,P = \frac{90}{100}(0.9)\,P$

$= (0.9)(0.9)P = (0.9)^2 P$

Now the amount of metal extracted in the 10th year $= (0.9)^9 P = (0.9)^9 (2000)$

$= 774.84$ kg $= 775$ kg (to 3sf) **(Ans)**

A General G.P is:

a, a r, a r$^2$ ...... a r$^{n-1}$

(ii) Total amount extracted in the first 20 years is:

$$P + 0.9P + (0.9)^2 P + (0.9)^3 P \;\ldots\ldots\;\ldots\ldots\; + (0.9)^{19} P$$

which is a G.P. with $a = P$, $r = 0.9$ and $n = 20$

using $S_n = \frac{a(1-r^n)}{1-r}$, we have

$$S_{21} = \frac{2000(1-(0.9)^{20})}{1-0.9} = \frac{2000 - 2000(0.9)^{20}}{0.1} = 17568.467 = 17600 \text{ (3sf)} \textbf{ (Ans)}$$

Sum of $n$ terms of a G.P is

$S_n = \frac{a(1-r^n)}{1-r}$, $|r|<1$

(iii) Using $S_\infty = \frac{a}{1-r}$, as $|r| < 1$

$$S_\infty = \frac{2000}{1-0.9} = 20000\text{kg} \quad \textbf{(Ans)}.$$

In a G.P, sum to infinity, $S_\infty$ is possible only if $|r|<1$

9. A curve is such that $\frac{dy}{dx} = \frac{24}{x^3} - 3$.

   (i) Given that the curve passes through the point (1, 16), find the equation of the curve. [4]

   (ii) Find the coordinates of the stationary point on the curve. [4]

*Suggested Solution:*

(i) $\frac{dy}{dx} = \frac{24}{x^3} - 3 \;\Rightarrow\; dy = \left(\frac{24}{x^3} - 3\right)dx \;\Rightarrow\; dy = \left(24x^{-3} - 3\right)dx$

integrating both sides with respect to $x$.

$$\int dy = \int\left(24x^{-3} - 3\right)dx \;\Rightarrow\; y = 24\left(\frac{x^{-2}}{-2}\right) - 3x + K \;\Rightarrow\; y = -\frac{12}{x^2} - 3x + K$$

as the curve passes through the point (1, 16)

$$\Rightarrow 16 = -\frac{12}{(1)^2} - 3(1) + K \;\Rightarrow\; 16 = -12 - 3 + K \;\Rightarrow\; K = 31$$

$\therefore$ required equation of the curve is: $y = -\frac{12}{x^2} - 3x + 31$ **(Ans)**

When $\frac{dy}{dx}$ (or gradient) of the curve is given, always integrate both sides of the given expression to find the equation of a curve.

(ii) Given that $\frac{dy}{dx} = \frac{24}{x^3} - 3$

for stationary point $\frac{dy}{dx} = 0$

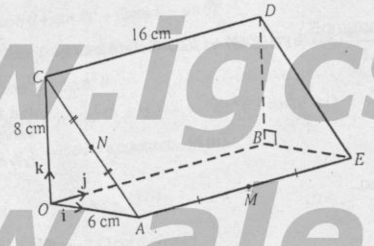
$\Rightarrow \frac{24}{x^3} - 3 = 0 \Rightarrow \frac{24}{x^3} = 3 \Rightarrow x^3 = 8 \Rightarrow x = 2$ (the only real value of $x$)

putting the value of $x$ in equation of curve $y = -\frac{12}{x^2} - 3x + 31$

$\Rightarrow y = -12\frac{1}{(2)^2} - 3(2) + 31 \Rightarrow -3 - 6 + 31 \Rightarrow y = 22$

$\therefore$ coordinates of stationary point are; (2, 22) **(Ans).**

**10.**



The diagram shows a prism with cross-section in the shape of a right-angled triangle $OAC$ where $OA = 6$ cm and $OC = 8$ cm. The cross-section through $E$ is the triangle $BED$. The length of the prism is 16 cm. $M$ is the mid-point of $AE$ and $N$ is the mid-point of $AC$.

Unit vectors **i**, **j** and **k** are parallel to $OA$, $OB$ and $OC$ respectively as shown.

(i) Express each of the vectors $\overrightarrow{MN}$ and $\overrightarrow{MD}$ in terms of **i**, **j** and **k**. [4]

(ii) Evaluate $\overrightarrow{MN} \cdot \overrightarrow{MD}$ and hence find the value of angle $NMD$, giving your answer to the nearest degree. [5]

***Suggested Solution:***

(i) Given that $\overrightarrow{OA} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 16 \\ 0 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}$

$$\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 16 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 16 \\ 0 \end{pmatrix}$$

$$\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD} = \begin{pmatrix} 0 \\ 16 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} = \begin{pmatrix} 0 \\ 16 \\ 8 \end{pmatrix}$$

Note that:
$\overrightarrow{AE} = \overrightarrow{OB}$ and $\overrightarrow{BD} = \overrightarrow{OC}$

**Learning corner**

If p.v. of $A = \vec{a}$
and p.v. of $B = \vec{b}$
then p.v. of midpoint $M$
of $\overrightarrow{AB} = \dfrac{\vec{a}+\vec{b}}{2}$
same rule is applicable if the position vectors are given in three dimensions

as $M$ is the mid point of $AE$,

$$\therefore \quad \overrightarrow{OM} = \frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OE}\right) = \frac{1}{2}\left[\begin{pmatrix}6\\0\\0\end{pmatrix}+\begin{pmatrix}6\\16\\0\end{pmatrix}\right] = \begin{pmatrix}6\\8\\0\end{pmatrix} \quad \text{or} \quad \overrightarrow{OM} = 6\mathrm{i}+8\mathrm{j}$$

also $N$ is the mid point of $AC$,

$$\therefore \quad \overrightarrow{ON} = \frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OC}\right) = \frac{1}{2}\left[\begin{pmatrix}6\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\8\end{pmatrix}\right] = \begin{pmatrix}3\\0\\4\end{pmatrix} \quad \text{or} \quad \overrightarrow{ON} = 3\mathrm{i}+4\mathrm{k}$$

now,

$$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} \quad \Rightarrow \overrightarrow{MN} = \begin{pmatrix}3\\0\\4\end{pmatrix} - \begin{pmatrix}6\\8\\0\end{pmatrix} \quad \Rightarrow \overrightarrow{MN} = \begin{pmatrix}-3\\-8\\4\end{pmatrix}$$

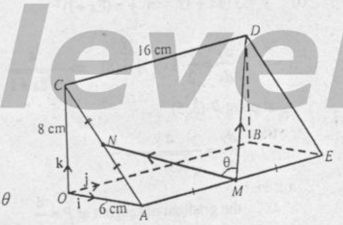
$$\therefore \quad \overrightarrow{MN} = -3\mathrm{i}-8\mathrm{j}+4\mathrm{k} \quad \text{(Ans)}$$

and

$$\overrightarrow{MD} = \overrightarrow{OD} - \overrightarrow{OM} = \begin{pmatrix}0\\16\\8\end{pmatrix} - \begin{pmatrix}6\\8\\0\end{pmatrix} \quad \Rightarrow \overrightarrow{MD} = \begin{pmatrix}-6\\8\\8\end{pmatrix}$$

$$\therefore \quad \overrightarrow{MD} = -6\mathrm{i}+8\mathrm{j}+8\mathrm{k} \quad \text{(Ans)}$$

(ii) $\overrightarrow{MN} \cdot \overrightarrow{MD} = \begin{pmatrix}-3\\-8\\4\end{pmatrix} \cdot \begin{pmatrix}-6\\8\\8\end{pmatrix} = 18-64+32 = -14 \quad \text{(Ans)}$



Let $N\hat{M}D = \theta$

$$\therefore \quad \overrightarrow{MN} \cdot \overrightarrow{MD} = \left|\overrightarrow{MN}\right| \left|\overrightarrow{MD}\right| \cos\theta$$

$$\Rightarrow \quad -14 = \left(\sqrt{(-3)^2+(-8)^2+(4)^2}\right)\left(\sqrt{(-6)^2+(8)^2+(8)^2}\right)\cos\theta$$

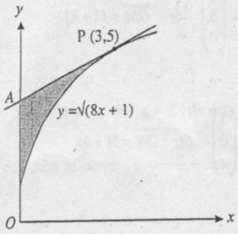
$$\Rightarrow \quad -14 = \left(\sqrt{89}\right)\left(\sqrt{164}\right)\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{-14}{\sqrt{14596}} \quad \Rightarrow \quad \theta = 96.654°$$

$$\therefore \quad N\hat{M}D = 97° \quad \text{(Ans)}$$

If
$\mathbf{a} = a_1\mathrm{i} + a_2\mathrm{j} + a_3\mathrm{k}$
$\mathbf{b} = b_1\mathrm{i} + b_2\mathrm{j} + b_3\mathrm{k}$, then
$\mathbf{a}.\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
and magnitude of $\mathbf{a}$ is
$\left|\mathbf{a}\right| = \sqrt{(a_1)^2+(a_2)^2+(a_3)^2}$

11



The diagram shows the curve $y=\sqrt{(8x+1)}$ and the tangent at the point $P(3, 5)$ on the curve. This tangent meets the $y$-axis at $A$. Find

(i) the equation of the tangent at $P$, [4]

(ii) the coordinates of $A$, [1]

(iii) the area of the shaded region. [6]

*Suggested Solution:*

(i) $y=\sqrt{(8x+1)} \Rightarrow y=(8x+1)^{\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx}=\frac{1}{2}(8x+1)^{-\frac{1}{2}}(8) \Rightarrow \frac{4}{\sqrt{8x+1}}$

at point $P$ (3, 5)

$\frac{dy}{dx}=\frac{4}{\sqrt{8(3)+1}}=\frac{4}{5}$

$\therefore$ the gradient of tangent at $P=\frac{4}{5}$

the tangent also passes through $P(3, 5)$

$\therefore$ equation of tangent is:

$y-5=\frac{4}{5}(x-3) \Rightarrow 5y-25=4x-12 \Rightarrow 5y-4x=13$ (Ans)

(ii) $A$ is the $y$-intercept of the tangent

$\therefore$ putting $x=0$ in the equation of tangent, we have

$5y-4(0)=13 \Rightarrow 5y=13 \Rightarrow y=\frac{13}{5}$

$\therefore$ coordinates of $A$ are $\left(0, \frac{13}{5}\right)$ **(Ans)**

(iii) From part (i), equation of tangent is:

$$5y-4x=13 \quad\Rightarrow\quad 5y=4x+13 \quad\Rightarrow\quad y=\frac{4}{5}x+\frac{13}{5}$$

Area of shaded region = area under the tangent $AP$ – area under the curve

$$\text{Area}=\int_0^3\left(\frac{4}{5}x+\frac{13}{5}\right)dx-\int_0^3\sqrt{8x+1}\,dx$$

$$=\int_0^3\left(\frac{4}{5}x+\frac{13}{5}\right)dx-\frac{1}{8}\int_0^3 8\sqrt{8x+1}\,dx$$

For definite integral, do not put the integration constant

$$=\left[(\frac{4}{5}\times\frac{x^2}{2})+\frac{13}{5}x\right]_0^3-\frac{1}{8}\left[\frac{(8x+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3$$

$$=\left[\frac{2}{5}x^2+\frac{13}{5}x\right]_0^3-\left[\frac{(8x+1)^{\frac{3}{2}}}{12}\right]_0^3$$

$$=\left[\left(\frac{2}{5}(3)^2+\frac{13}{5}(3)\right)-\left(\frac{2(0)^2}{5}+\frac{13}{5}(0)\right)\right]-\left[\left(\frac{(8(3)+1)^{\frac{3}{2}}}{12}\right)-\left(\frac{(8(0)+1)^{\frac{3}{2}}}{12}\right)\right]$$

$$=\left[\left(\frac{18}{5}+\frac{39}{5}\right)-(0)\right]-\left[\left(\frac{125}{12}\right)-\left(\frac{1}{12}\right)\right]$$

$$=\left(\frac{57}{5}\right)-\left(\frac{124}{12}\right)$$

$$=\frac{16}{15}=1\frac{1}{15}\ \text{unit}^2\quad\textbf{(Ans)}$$

# June 2002 Paper 1

## Pure Mathematics (P1)

Answer all the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** The line $x + 2y = 9$ intersects the curve $xy + 18 = 0$ at the points $A$ and $B$. Find the coordinates of $A$ and $B$. [4]

***Suggested Solution:***

Let $x + 2y = 9$ ·········(i) and $xy + 18 = 0$ ·········(ii)

from equation (i) $x = 9 - 2y$ putting it in equation (ii)

$$(9-2y)y + 18 = 0$$
$$9y - 2y^2 + 18 = 0$$
$$2y^2 - 9y - 18 = 0$$
$$2y^2 - 12y + 3y - 18 = 0$$
$$2y(y-6) + 3(y-6) = 0$$
$$(y-6)(2y+3) = 0$$

$\Rightarrow 2y + 3 = 0$ or $y - 6 = 0$

$y = -\frac{3}{2}$ $\quad y = 6$

when $y = -\frac{3}{2}$, $x = 9 - 2\left(-\frac{3}{2}\right) \Rightarrow x = 12$

when $y = 6$, $x = 9 - 2(6) \Rightarrow x = -3$

$\therefore A(12, -\frac{3}{2})$ and $B(-3, 6)$ Ans

**2.** (i) Show that $\sin x \tan x$ may be written as $\dfrac{1-\cos^2 x}{\cos x}$. [1]

(ii) Hence solve the equation $2\sin x \tan x = 3$, for $0° \le x \le 360°$. [4]

***Suggested Solution:***

(i) $\sin x \tan x = \sin x \times \dfrac{\sin x}{\cos x} = \dfrac{\sin^2 x}{\cos x} = \dfrac{1-\cos^2 x}{\cos x}$ (Shown)

$\sin^2 x + \cos^2 x = 1$
$\Rightarrow \sin^2 x = 1 - \cos^2 x$

(ii) $2\sin x\ \tan x = 3$

using the result of part (i), we have

$$2\left(\frac{1-\cos^2 x}{\cos x}\right) = 3$$

$$2(1-\cos^2 x) = 3\cos x$$

$$2-2\cos^2 x = 3\cos x$$

$$2\cos^2 x + 3\cos x - 2 = 0$$

$$2\cos^2 x + 4\cos x - \cos x - 2 = 0$$

$$2\cos x(\cos x + 2) - 1(\cos x + 2) = 0$$

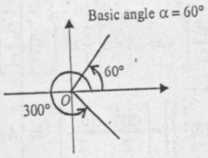
$$(\cos x + 2)(2\cos x - 1) = 0$$

$\cos x + 2 = 0$ or $2\cos x - 1 = 0$

$\cos x = -2$ (impossible) $\quad \cos x = \dfrac{1}{2}$

$\therefore\ x = 60°,\ 360° - 60°$

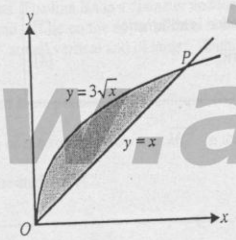
$x = 60°,\ 300°$ **Ans**



$\cos\theta$ is positive in I and IV quadrant

$\therefore$ in the Ist quadrant $\theta = \alpha$ and in the IV quadrant $\theta = 360 - \alpha$

3.



The diagram shows the curve $y = 3\sqrt{x}$ and the line $y = x$ intersecting at $O$ and $P$.

Find

(i) the coordinates of $P$, [1]

(ii) the area of the shaded region. [5]

*Suggested Solution:*

(i) Let $y = x$ ·········(i) and $y = 3\sqrt{x}$ ·········(ii)

putting eq. (i) into eq. (ii) and solving simultaneously

$x = 3\sqrt{x} \Rightarrow x^2 = (3\sqrt{x})^2 \Rightarrow x^2 = 9x \Rightarrow x^2 - 9x = 0$

$\Rightarrow x(x-9) = 0 \Rightarrow x = 0$ or $x = 9$

when $x = 0,\ y = 0 \Rightarrow (0, 0)$ is one point of intersection.

when $x = 9,\ y = 9 \Rightarrow (9, 9)$ is other point of intersection.

$\therefore$ coordinates of $P(9, 9)$ **Ans**

(ii) Let $y_1 = 3\sqrt{x}$ and $y_2 = x$

then, required shaded area, $A = \int_{0}^{9} (y_1 - y_2)\,dx$

$$\Rightarrow A = \int_0^9 (3\sqrt{x} - x)\,dx \quad \Rightarrow A = \int_0^9 3x^{\frac{1}{2}}\,dx - \int_0^9 x\,dx$$

$$\Rightarrow A = 3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^9 - \left[\frac{x^2}{2}\right]_0^9 \quad \Rightarrow A = 3\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^9 - \left[\frac{x^2}{2}\right]_0^9$$

$$\Rightarrow A = 3\left[\frac{2}{3}\left((9)^{\frac{3}{2}} - (0)^{\frac{3}{2}}\right)\right] - \left[\frac{(9)^2}{2} - \frac{(0)^2}{2}\right] \quad \Rightarrow A = 3\left(\frac{2}{3}(3)^3\right) - \left(\frac{(9)^2}{2}\right)$$

$$\Rightarrow A = 54 - \frac{81}{2} \quad \Rightarrow A = \frac{27}{2} = 13\frac{1}{2} \text{ sq. units} \quad \textbf{(Ans)}$$

4. A progression has a first term of 12 and a fifth term of 18.
   (i) Find the sum of the first 25 terms if the progression is arithmetic. [3]
   (ii) Find the 13th term if the progression is geometric. [4]

*Suggested Solution:*

(i) Given that

First term $a = 12$, 5th term $T_5 = 18$

As the progression is an A.P

$$\therefore T_5 = a + (5-1)d \quad \Rightarrow 18 = 12 + 4d \quad \Rightarrow d = \frac{6}{4} = \frac{3}{2}$$

Sum of $n$ terms of an A.P. is
$S_n = \frac{n}{2}[2a + (n-1)d]$

Now

$$S_{25} = \frac{25}{2}[2a + (25-1)d] = \frac{25}{2}\left[2(12) + 24(\frac{3}{2})\right] = \frac{25}{2}[24 + 36] = \frac{25}{2}(60) = 750$$

$\therefore$ sum of first 25 terms $= 750$ **(Ans)**

(ii) Given that the progression is a G.P

$$\therefore T_5 = ar^{5-1} \quad \Rightarrow 18 = 12r^4 \quad \Rightarrow r^4 = \frac{3}{2} \quad \Rightarrow r = \left(\frac{3}{2}\right)^{\frac{1}{4}}$$

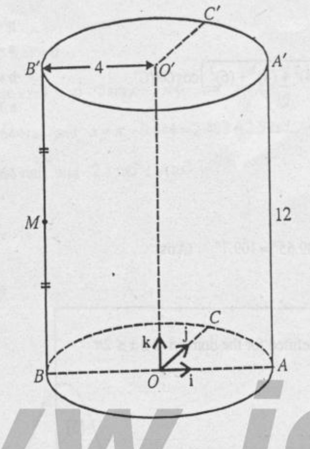
nth term of a G.P is
$T_n = a\,r^{n-1}$

Now

$$T_{13} = ar^{13-1} \quad \Rightarrow T_{13} = 12\left[\left(\frac{3}{2}\right)^{\frac{1}{4}}\right]^{12} \quad \Rightarrow T_{13} = 12\left(\frac{3}{2}\right)^3 \quad \Rightarrow T_{13} = 12\left(\frac{27}{8}\right)$$

$$\Rightarrow T_{13} = \frac{81}{2} = 40\frac{1}{2} \quad \textbf{(Ans)}$$

5.



The diagram shows a solid cylinder standing on a horizontal circular base, centre $O$ and radius 4 units. The line $BA$ is a diameter and the radius $OC$ is at 90° to $OA$. Points $O'$, $A'$, $B'$ and $C'$ lie on the upper surface of the cylinder such that $OO'$, $AA'$, $BB'$ and $CC'$ are all vertical and of length 12 units. The mid-point of $BB'$ is $M$.

Unit vectors **i**, **j** and **k** are parallel to $OA$, $OC$ and $OO'$ respectively.

(i) Express each of the vectors $\overrightarrow{MO}$ and $\overrightarrow{MC'}$ in terms of **i**, **j** and **k**. [3]

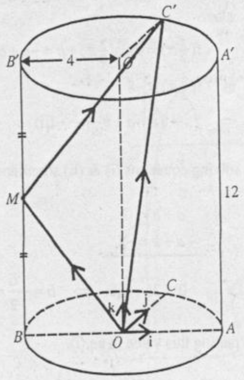
(ii) Hence find the angle $OMC'$. [4]

***Suggested Solution:***

(i) we have

$$\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM} = -4\mathbf{i} + 6\mathbf{k} = \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix}$$

$$\overrightarrow{OC'} = \overrightarrow{OO'} + \overrightarrow{O'C'} = 12\mathbf{k} + 4\mathbf{j} = 4\mathbf{j} + 12\mathbf{k} = \begin{pmatrix} 0 \\ 4 \\ 12 \end{pmatrix}$$



Now

$$\overrightarrow{MO} = -\overrightarrow{OM} = \begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix} \text{ (Ans)}$$

$$\overrightarrow{MC'} = \overrightarrow{MO} + \overrightarrow{OC'} = \begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 12 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix} \quad \text{(Ans)}$$

(ii) $\overline{MO}\cdot\overline{MC'}=\left|\overline{MO}\right|\left|\overline{MC'}\right|\cos O\hat{M}C'$

$$\begin{pmatrix}4\\0\\-6\end{pmatrix}\cdot\begin{pmatrix}4\\4\\6\end{pmatrix}=\left(\sqrt{(4)^2+(0)^2+(-6)^2}\right)\times\left(\sqrt{(4)^2+(4)^2+(6)^2}\right)\cos O\hat{M}C'$$

$$16+0-36=\left(\sqrt{52}\right)\left(\sqrt{68}\right)\cos O\hat{M}C'$$

$$\cos O\hat{M}C'=-\frac{20}{\sqrt{3536}}$$

$\cos O\hat{M}C'=-0.336336 \quad \Rightarrow \quad O\hat{M}C'=109.65°\approx 109.7°$ (Ans)

If
$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, and
$\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, then
$\mathbf{a}.\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$

**6.** The function f, where $f(x)=a\sin x+b$, is defined for the domain $0\le x\le 2\pi$.

Given that $f(\frac{1}{2}\pi)=2$ and that $f(\frac{3}{2}\pi)=-8$,

(i) find the values of $a$ and $b$, [3]

(ii) find the values of $x$ for which $f(x)=0$, giving your answers in radians correct to 2 decimal places, [2]

(iii) sketch the graph of $y=f(x)$. [2]

***Suggested Solution:***

(i) Given that $f(x)=a\sin x+b$

$$f(\tfrac{1}{2}\pi)=a\sin(\tfrac{1}{2}\pi)+b=a+b$$

as $f(\frac{1}{2}\pi)=2$

$\Rightarrow \quad 2=a+b\cdots\cdots\cdots$(i)

also

$$f(\tfrac{3}{2}\pi)=a\sin(\tfrac{3}{2}\pi)+b=-a+b$$

as $f(\frac{3}{2}\pi)=-8$

$\Rightarrow \quad -8=-a+b\cdots\cdots\cdots$(ii)

solving equations (i) & (ii) simultaneously

$$\begin{array}{r} a+b=\ 2\\ \underline{-a+b=-8}\\ 0+2b=-6 \end{array} \quad \Rightarrow \quad b=-\frac{6}{2}=-3$$

putting this value in eq.(i)

$2=a+(-3) \quad \Rightarrow \quad a=5$

$\therefore \quad a=5,\ b=-3$ (Ans)

Note that

$\sin(\frac{1}{2}\pi)=\sin(90°)=1$

$\sin(\frac{3}{2}\pi)=\sin(270°)=-1$

(ii) With above values of $a$ and $b$, $f(x)$ becomes

$f(x) = 5\sin x - 3$

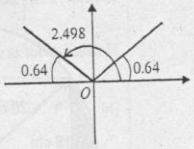
Given that $f(x) = 0$

$\Rightarrow \quad 0 = 5\sin x - 3 \quad \Rightarrow \quad 5\sin x - 3 = 0 \quad \Rightarrow \quad \sin x = \frac{3}{5}$

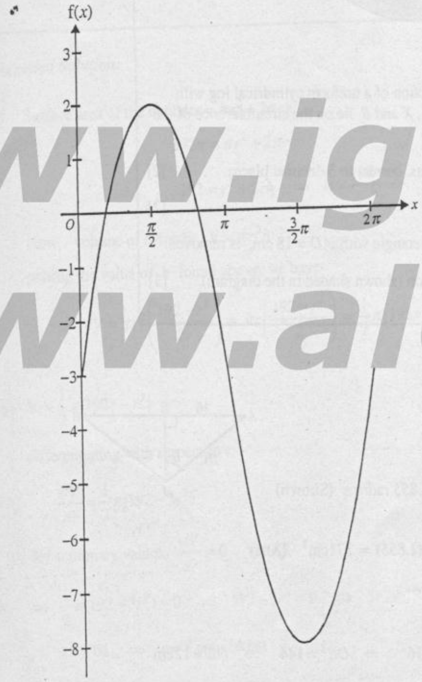
$\Rightarrow \quad x = 0.644\text{ rad} \text{ and } x = \pi - 0.644 = 2.498 \approx 2.5\text{ rad}$

$\therefore \quad x = 0.64\text{ rad, and } 2.5\text{ rad}$ **(Ans)**

Basic angle $\alpha = 0.64$ radians

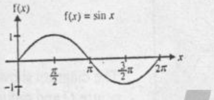


(iii) $f(x) = 5\sin x - 3$

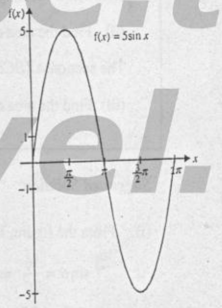


To sketch the curve we usually perform the following steps.

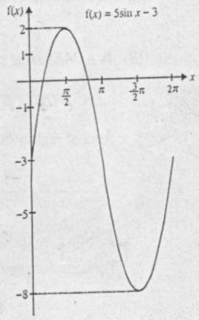
Step 1.



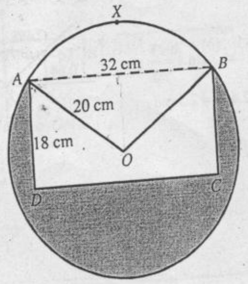
Step 2.



Step 3.



**7.**



The diagram shows the circular cross-section of a uniform cylindrical log with centre $O$ and radius 20 cm. The points $A$, $X$ and $B$ lie on the circumference of the cross-section and $AB = 32$ cm.

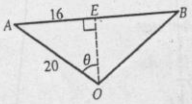
(i) Show that angle $AOB = 1.855$ radians, correct to 3 decimal places. [2]

(ii) Find the area of the sector $AXBO$. [2]

The section $AXBCD$, where $ABCD$ is a rectangle with $AD = 18$ cm, is removed.

(iii) Find the area of the new cross-section (shown shaded in the diagram) [3]

***Suggested Solution:***



(i) From the figure, in $\Delta OAE$

$$\sin\theta = \frac{16}{20} \Rightarrow \theta = 0.9273 \text{ radians}$$

$$\therefore \quad \text{angle } AOB = 2(0.9273) = 1.8546 \approx 1.855 \text{ radians} \quad \textbf{(Shown)}$$

(ii) Area of sector $AXBO = \frac{1}{2}r^2\theta = \frac{1}{2}(20)^2(1.855) = 371\text{cm}^2$ **(Ans)**

(iii) In $\Delta OAE$, using pythagorus theorem

$$OA^2 = OE^2 + EA^2 \quad \Rightarrow \quad 20^2 = OE^2 + 16^2 \quad \Rightarrow \quad OE^2 = 144 \quad \Rightarrow \quad OE = 12\text{cm}$$

Area of region $AOBCD =$ area of rectangle $ABCD -$ area of triangle $AOB$

$$= (18 \times 32) - \left(\frac{1}{2} \times 32 \times 12\right)$$

$$= 576 - 192 = 384 \text{ cm}^2$$

Total area of region AX$BCD =$ area of region $AOBCD +$ area of sector $AXBO$

$$= 384 + 371 = 755 \text{ cm}^2$$

Area of the required shaded region = area of circle − area of $AXBCD$

$$= \pi(20)^2 - 755 = 501.8 \approx 502 \text{ cm}^2 \quad \textbf{(Ans)}$$

**8.** A hollow circular cylinder, open at one end, is constructed of thin sheet metal. The total external surface area of the cylinder is $192\pi$ cm$^2$. The cylinder has a radius of $r$ cm and a height of $h$ cm.

(i) Express $h$ in terms of $r$ and show that the volume, $V$ cm$^3$, of the cylinder is given by

$$V = \frac{1}{2}\pi(192r - r^3) \qquad [4]$$

Given that $r$ can vary,

(ii) find the value of $r$ for which $V$ has a stationary value, [3]

(iii) find this stationary value and determine whether it is a maximum or a minimum. [3]

***Suggested Solution:***

(i) Surface area of the cylinder $= \pi r^2 + 2\pi rh$

$$192\pi = \pi r^2 + 2\pi rh$$

$$192 = r^2 + 2rh \quad \Rightarrow \quad h = \frac{192 - r^2}{2r}$$

Now, volume of cylinder $V = \pi r^2 h$

putting the value of $h$ found above, we have

$$V = \pi r^2\left(\frac{192 - r^2}{2r}\right) = \pi r\left(\frac{192 - r^2}{2}\right) = \frac{1}{2}\pi\left(192r - r^3\right) \quad \text{(Shown)}$$

(ii) $V = \frac{1}{2}\pi(192r - r^3)$

differentiating with respect to $r$

$$\frac{dV}{dr} = \frac{1}{2}\pi(192 - 3r^2)$$

for stationary values, $\frac{dV}{dr} = 0$

$\Rightarrow \frac{1}{2}\pi(192 - 3r^2) = 0 \quad \Rightarrow \quad 192 - 3r^2 = 0 \quad \Rightarrow \quad 3r^2 = 192$

$\Rightarrow r^2 = 64 \quad \Rightarrow \quad r = 8$ cm (Ans)

(iii) $V = \frac{1}{2}\pi(192r - r^3)$

at $r = 8$

$$V = \frac{1}{2}\pi[192(8) - (8)^3] = \frac{1}{2}\pi(1536 - 512) = \frac{1}{2}\pi(1024) = 512\pi$$

$\therefore$ stationary value $= 512\pi$ cm$^3$ (Ans)

Now

$$\frac{dV}{dr} = \frac{1}{2}\pi(192 - 3r^2)$$

$$\Rightarrow \quad \frac{d^2V}{dr^2} = \frac{1}{2}\pi(-6r) = -3\pi r$$

at $r = 8$

$$\frac{d^2V}{dr^2} = -3\pi(8) = -24\pi < 0$$

$\therefore$ $V$ is maximum at $r = 8$ **(Ans)**

Nature:

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} > 0$, $y$ is min

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} < 0$, $y$ is max

**9.** A curve is such that $\frac{dy}{dx} = \frac{12}{(2x+1)^2}$ and $P\,(1, 5)$ is a point on the curve.

(i) The normal to the curve at $P$ crosses the $x$-axis at $Q$. Find the coordinates of $Q$. [4]

**(ii)** Find the equation of the curve. [4]

(iii) A point is moving along the curve in such a way that the $x$-coordinate is increasing at a constant rate of 0.3 units per second. Find the rate of increase of the $y$-coordinate when $x = 1$. [3]

*Suggested Solution:*

(i) Gradient, $\frac{dy}{dx} = \frac{12}{(2x+1)^2}$

at $P(1, 5)$

$$\frac{dy}{dx} = \frac{12}{(2(1)+1)^2} = \frac{12}{9} = \frac{4}{3}$$

$\therefore$ gradient of tangent at $P(1, 5) = \frac{4}{3}$

$\Rightarrow$ gradient of normal $= -\frac{3}{4}$

equation of the normal at $P(1, 5)$ is given by

$$y - 5 = -\frac{3}{4}(x - 1) \quad \Rightarrow \quad 4y - 20 = -3x + 3 \quad \Rightarrow \quad 4y + 3x = 23$$

As the normal crosses the $x$-axis at $Q$, therefore put $y = 0$ in the above equation

$$4(0) + 3x = 23 \quad \Rightarrow \quad x = \frac{23}{3}$$

$\therefore$ coordinates of $Q = (\frac{23}{3}, 0)$ **(Ans)**

Tangent and normal to the curve are perpendicular to each other, and for perpendicular lines:
(grad. of tangent) × (grad. of normal) = −1

Equation of a straight line with gradient $m$ and passing through $(x_1, y_1)$ is $(y - y_1) = m\,(x - x_1)$.

(ii) Given that

$$\frac{dy}{dx}=\frac{12}{(2x+1)^2} \Rightarrow dy=\frac{12}{(2x+1)^2}\,dx \Rightarrow dy=12(2x+1)^{-2}\,dx$$

integrating both sides w.r.t. $x$

$$\int dy=12\int (2x+1)^{-2}\,dx$$

$$\Rightarrow y=12\frac{(2x+1)^{-1}}{(-1)(2)}+C \Rightarrow y=-\frac{6}{2x+1}+C$$

as the curve passes through point $P(1, 5)$

$$\therefore\quad 5=-\frac{6}{2(1)+1}+C \Rightarrow 5=-2+C \Rightarrow C=7$$

$\therefore$ required equation of the curve is

$$y=-\frac{6}{2x+1}+7 \quad \textbf{(Ans)}$$

When $\frac{dy}{dx}$ (or gradient) of the curve is given, always integrate both sides of the given expression to find the equation of a curve.

(iii) Given that

$$\frac{dx}{dt}=0.3 \text{ units/sec} \quad \text{and} \quad \frac{dy}{dx}=\frac{12}{(2x+1)^2}$$

now

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dt}=\frac{12}{(2x+1)^2}\times 0.3$$

at $x=1$

$$\frac{dy}{dt}=\frac{12}{(2(1)+1)^2}\times 0.3=\frac{12}{(3)^2}\times 0.3=0.4$$

$\therefore$ rate of increase of the $y$-coordinate $=0.4$ units/seconds **(Ans)**

Chain Rule:

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}},$$

or $\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$

**10.** The functions f and g are defined by

$$\mathrm{f}: x \mapsto 3x+2, \quad x\in\mathbb{R},$$

$$\mathrm{g}: x \mapsto \frac{6}{2x+3}, \quad x\in\mathbb{R},\ x\neq -1.5$$

(i) Find the value of $x$ for which $\mathrm{fg}(x)=3$. [3]

(ii) Sketch, in a single diagram, the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$, making clear the relationship between the two graphs. [3]

(iii) Express each of $\mathrm{f}^{-1}(x)$ and $\mathrm{g}^{-1}(x)$ in terms of $x$, and solve the equation $\mathrm{f}^{-1}(x)=\mathrm{g}^{-1}(x)$ [5]

***Suggested Solution:***

(i) Given that $f(x)=3x+2$ and $g(x)=\dfrac{6}{2x+3}$

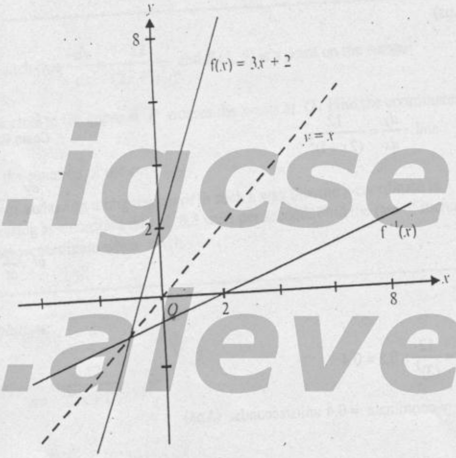
$$fg(x)=f\left(\frac{6}{2x+3}\right)=3\left(\frac{6}{2x+3}\right)+2=\frac{18}{2x+3}+2$$

as $fg(x)=3$

$$\Rightarrow\ 3=\frac{18}{2x+3}+2\ \Rightarrow\ 1=\frac{18}{2x+3}\ \Rightarrow\ 2x+3=18$$

$$\Rightarrow\ 2x=15\ \Rightarrow\ x=7.5\ \textbf{(Ans)}$$

(ii)



(iii) $f(x)=3x+2$

Let $f(x)=y\ \Rightarrow\ y=3x+2$

making $x$ the subject

$$y=3x+2\ \Rightarrow\ x=\frac{y-2}{3}$$

as $f(x)=y\ \Rightarrow\ f^{-1}(y)=x$

$$\therefore\ f^{-1}(y)=\frac{y-2}{3}\ \Rightarrow\ f^{-1}(x)=\frac{x-2}{3}\ \textbf{(Ans)}$$

$g(x) = \frac{6}{2x+3}$

let $g(x) = z \Rightarrow z = \frac{6}{2x+3}$

making $x$ the subject

$$z = \frac{6}{2x+3} \Rightarrow 2xz + 3z = 6 \Rightarrow 2xz = 6 - 3z \Rightarrow x = \frac{6-3z}{2z}$$

as $g(x) = z \Rightarrow g^{-1}(z) = x$

$\therefore\ g^{-1}(z) = \frac{6-3z}{2z} \Rightarrow g^{-1}(x) = \frac{6-3x}{2x}$ (Ans)

Given that $f^{-1}(x) = g^{-1}(x)$

$$\Rightarrow \frac{x-2}{3} = \frac{6-3x}{2x}$$

$$2x(x-2) = 3(6-3x)$$

$$2x^2 - 4x = 18 - 9x$$

$$2x^2 + 5x - 18 = 0$$

$$2x^2 + 9x - 4x - 18 = 0$$

$$x(2x+9) - 2(2x+9) = 0$$

$$(2x+9)(x-2) = 0$$

$\Rightarrow 2x + 9 = 0$ or $x - 2 = 0$

$x = -\frac{9}{2}$ or $x = 2$

$\therefore\ x = -\frac{9}{2}$ or 2 (Ans)

# June 2002 Paper 3

## Pure Mathematics (P3)

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Prove the identity. $\cot\theta - \tan\theta \equiv 2\cot 2\theta$ [3]

***Suggested Solution:***

$$\text{L.H.S} = \cot\theta - \tan\theta$$

$$= \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{\cos 2\theta}{\sin\theta\cos\theta}$$

multiplying and dividing by 2

$$= \frac{2\cos 2\theta}{2\sin\theta\cos\theta}$$

$$= \frac{2\cos 2\theta}{\sin 2\theta} = 2\times\frac{\cos 2\theta}{\sin 2\theta} = 2\cot 2\theta = \text{R.H.S} \quad \textbf{(Proved)}$$

Note:

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$2\sin\theta\cos\theta = \sin 2\theta$

**2.** Expand $(1-3x)^{-\frac{1}{3}}$ in ascending powers of $x$, up to and including the term in $x^3$, simplifying the coefficients. [4]

***Suggested Solution:***

$$(1-3x)^{-\frac{1}{3}} = 1 + \left(-\frac{1}{3}\right)(-3x) + \frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)}{2!}(-3x)^2 + \frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)\left(-\frac{1}{3}-2\right)}{3!}(-3x)^3 \ldots.$$

$$= 1 + x + \frac{-\frac{1}{3}\left(-\frac{4}{3}\right)}{2.1}\left(9x^2\right) + \frac{-\frac{1}{3}\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3.2.1}\left(-27x^3\right)\ldots.$$

$$= 1 + x + \frac{\frac{4}{9}}{2}\left(9x^2\right) + \frac{-\frac{28}{27}}{6}\left(-27x^3\right)\ldots.$$

$$= 1 + x + \frac{4}{9(2)}\left(9x^2\right) + \frac{28}{27(6)}\left(27x^3\right)\ldots.$$

$$= 1 + x + 2x^2 + \frac{14}{3}x^3 \quad \textbf{(Ans)}$$

Note:

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2$
$+ \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$

3. The polynomial $x^4 + 4x^2 + x + a$ is denoted by $p(x)$. It is given that $(x^2 + x + 2)$ is a factor of $p(x)$.

Find the value of $a$ and the other quadratic factor of $p(x)$. [4]

***Suggested Solution:***

Given that $x^2 + x + 2$ is a factor of $p(x) = x^4 + 4x^2 + x + a$

using long division, we have

$$\begin{array}{r l}
 & x^2 - x + 3 \\
\cline{2-2}
x^2 + x + 2 \big) & x^4 + 4x^2 + x + a \\
 & x^4 + 2x^2 \quad + x^3 \\
 & - \quad - \qquad\quad - \\
\cline{2-2}
 & -x^3 + 2x^2 + x + a \\
 & -x^3 - x^2 - 2x \\
 & + \quad + \quad + \\
\cline{2-2}
 & \qquad 3x^2 + 3x + a \\
 & \qquad 3x^2 + 3x + 6 \\
 & \qquad - \quad - \quad - \\
\cline{2-2}
 & \qquad\qquad a - 6
\end{array}$$

applying factor theorem

remainder, $a - 6 = 0 \Rightarrow a = 6$ (Ans)

the other quadratic factor of $p(x) = x^2 - x + 3$ (Ans)

Factor Theorem:
When a polynomial p(x) is divided by another polynomial q(x) of lesser degree and the remainder is zero, then q(x) is a factor of p(x).

The other factor is the quotient

4. The sequence of values given by the iterative formula

$$x_{n+1} = \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right),$$

with initial value $x_1 = 1$, converges to $\alpha$.

(i) Use this formula to find $\alpha$ correct to 2 decimal places, showing the result of each iteration. [3]

(ii) State an equation satisfied by $\alpha$, and hence find the exact value of $\alpha$. [2]

*Suggested Solution:*

(i) $x_{n+1} = \frac{2}{3}\left(x_n + \frac{1}{x_n^2}\right)$, with $x_1 = 1$

$\therefore \quad x_2 = \frac{2}{3}\left(x_1 + \frac{1}{x_1^2}\right) = \frac{2}{3}\left(1 + \frac{1}{1}\right) = \frac{4}{3} = 1.33333$

$x_3 = \frac{2}{3}\left((1.33333) + \frac{1}{(1.33333)^2}\right) = 1.26389$

$x_4 = \frac{2}{3}\left((1.26389) + \frac{1}{(1.26389)^2}\right) = 1.25993$

$x_5 = \frac{2}{3}\left((1.25993) + \frac{1}{(1.25993)^2}\right) = 1.25992$

$x_6 = \frac{2}{3}\left((1.25992) + \frac{1}{(1.25992)^2}\right) = 1.25992$

$\therefore \quad \alpha = 1.26$ (2 dec.pl) **(Ans)**

When the question did not specify the number of iterations to use, it is *incorrect* to use only one or two iterations of the formula, in fact, we need to continue until we get the same answer twice for the required degree of accuracy.

(ii) Removing subscripts from the iterative formula, we have

$$x = \frac{2}{3}\left(x + \frac{1}{x^2}\right) \Rightarrow 3x = 2x + \frac{2}{x^2} \Rightarrow x = \frac{2}{x^2} \Rightarrow x^3 = 2$$

$\therefore$ equation satisfied by $\alpha$ is: $x^3 = 2$ **(Ans)**

exact value of $\alpha$ is:

$x^3 = 2 \Rightarrow x = \sqrt[3]{2}$ **(Ans)**

**5.** The equation of a curve is $y = 2\cos x + \sin 2x$. Find the $x$-coordinates of the stationary points on the curve for which $0 < x < \pi$, and determine the nature of each of these stationary points. [7]

*Suggested Solution:*

Given $y = 2\cos x + \sin 2x$

$$\frac{dy}{dx} = -2\sin x + 2\cos 2x$$

for stationary values, $\frac{dy}{dx} = 0$

$\Rightarrow \quad -2\sin x + 2\cos 2x = 0 \quad \Rightarrow \quad -\sin x + \cos 2x = 0$

$\Rightarrow \quad -\sin x + (1 - 2\sin^2 x) = 0 \quad \Rightarrow \quad 2\sin^2 x + \sin x - 1 = 0$

$\Rightarrow \quad 2\sin^2 x + 2\sin x - \sin x - 1 = 0 \quad \Rightarrow \quad 2\sin x(\sin x + 1) - 1(\sin x + 1) = 0$

$\Rightarrow \quad (\sin x + 1)(2\sin x - 1) = 0$

Note that $\sin\theta$ is +ve in 1st and 2nd quadrant

$\Rightarrow \quad \sin x + 1 = 0$ or $2\sin x - 1 = 0$

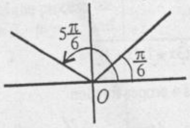
$\Rightarrow \quad \sin x = -1$ $\qquad \sin x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{3}{2}\pi$ $\qquad x = \frac{\pi}{6}, 5\frac{\pi}{6}$

as $\frac{3}{2}\pi$ is not in the required range

$\therefore$ for $0 < x < \pi$, $x = \frac{\pi}{6}, 5\frac{\pi}{6}$ (Ans)

Basic angle $\alpha = \frac{\pi}{6}$



Now

$$\frac{dy}{dx} = -2\sin x + 2\cos 2x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -2\cos x - 4\sin 2x$$

when $x = \frac{\pi}{6}$

$$\frac{d^2y}{dx^2} = -2\cos\left(\frac{\pi}{6}\right) - 4\sin 2\left(\frac{\pi}{6}\right) = -2\cos\left(\frac{\pi}{6}\right) - 4\sin\left(\frac{\pi}{3}\right)$$

$$= -2\left(\frac{\sqrt{3}}{2}\right) - 4\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3} - 2\sqrt{3} = -3\sqrt{3} < 0$$

$\therefore$ when $x = \frac{\pi}{6}$, the stationary point is maximum (Ans)

Remember:

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} > 0$, $y$ is min

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} < 0$, $y$ is max

when $x = \frac{5}{6}\pi$

$$\frac{d^2y}{dx^2} = -2\cos\left(5\frac{\pi}{6}\right) - 4\sin 2\left(5\frac{\pi}{6}\right) = -2\cos\left(5\frac{\pi}{6}\right) - 4\sin\left(5\frac{\pi}{3}\right)$$

$$= -2\left(-\frac{\sqrt{3}}{2}\right) - 4\left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3} + 2\sqrt{3} = 3\sqrt{3} > 0$$

$\therefore$ when $x = \frac{5}{6}\pi$, the stationary point is minimum (Ans)

6. Let $f(x) = \frac{4x}{(3x+1)(x+1)^2}$

   (i) Express $f(x)$ in partial fractions. [5]

   (ii) Hence show that $\int_0^1 f(x)\,dx = 1 - \ln 2$ [5]

*Suggested Solution:*

(i) $f(x) = \dfrac{4x}{(3x+1)(x+1)^2}$

as this is a proper fraction

$$\therefore \quad \frac{4x}{(3x+1)(x+1)^2} \equiv \frac{A}{3x+1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Remember that identity is an algebraic open sentence which is true for all values of the variable

$$\Rightarrow \quad 4x \equiv A(x+1)^2 + B(3x+1)(x+1) + C(3x+1)$$

for $x = -1$

$$4(-1) = A(-1+1)^2 + B(3(-1)+1)(-1+1) + C(3(-1)+1)$$

$$\Rightarrow \quad -4 = 0+0+C(-3+1) \quad \Rightarrow \quad -4 = -2C \quad \Rightarrow \quad C = 2$$

for $x = -\frac{1}{3}$

$$4(-\tfrac{1}{3}) = A(-\tfrac{1}{3}+1)^2 + B(3(-\tfrac{1}{3})+1)(-\tfrac{1}{3}+1) + C(3(-\tfrac{1}{3})+1)$$

$$\Rightarrow \quad -\frac{4}{3} = A\left(\frac{2}{3}\right)^2 + 0 + 0 \quad \Rightarrow \quad -\frac{4}{3} = A\left(\frac{4}{9}\right) \quad \Rightarrow \quad A = -3$$

for $x = 0$

$$4(0) = A(0+1)^2 + B(3(0)+1)(0+1) + C(3(0)+1)$$

$$\Rightarrow \quad 0 = A(1)^2 + B(1)(1) + C(1) \quad \Rightarrow \quad 0 = A+B+C$$

putting values of $C$ and $A$

$$0 = -3+B+2 \quad \Rightarrow \quad B = 1$$

$$\therefore \quad \frac{4x}{(3x+1)(x+1)^2} \equiv \frac{-3}{3x+1} + \frac{1}{x+1} + \frac{2}{(x+1)^2} \quad \text{(Ans)}$$

(ii) $\text{L.H.S} = \int_0^1 f(x)\,dx$

$$\Rightarrow \quad \int_0^1 \left( \frac{-3}{3x+1} + \frac{1}{x+1} + \frac{2}{(x+1)^2} \right) dx$$

$$= \quad \int_0^1 \frac{-3}{3x+1}\,dx + \int_0^1 \frac{1}{x+1}\,dx + \int_0^1 \frac{2}{(x+1)^2}\,dx$$

$$= \quad \int_0^1 \frac{-3}{3x+1}\,dx + \int_0^1 \frac{1}{x+1}\,dx + 2\int_0^1 (x+1)^{-2}\,dx$$

$$= \quad [-\ln(3x+1)]_0^1 + [\ln(x+1)]_0^1 + 2\left[\frac{(x+1)^{-1}}{-1}\right]_0^1$$

$$= \quad [-\ln(3x+1)]_0^1 + [\ln(x+1)]_0^1 - 2\left[\frac{1}{x+1}\right]_0^1$$

$$= \quad [-\ln 4 - (-\ln 1)] + [\ln 2 - \ln 1] - 2\left[\frac{1}{2} - 1\right] = -\ln 4 + \ln 2 - 2\left[-\frac{1}{2}\right]$$

$$= \quad -\ln 2^2 + \ln 2 + 1 = -2\ln 2 + \ln 2 + 1 = 1 - \ln 2 \quad \text{(Shown)}$$

7. In a certain chemical process a substance is being formed, and $t$ minutes after the start of the process there are $m$ grams of the substance present. In the process the rate of increase of $m$ is proportional to $(50-m)^2$. When $t=0$, $m=0$ and $\frac{dm}{dt}=5$.

(i) Show that $m$ satisfies the differential equation

$$\frac{dm}{dt}=0.002(50-m)^2$$ [2]

(ii) Solve the differential equation, and show that the solution can be expressed in the form

$$m=50-\frac{500}{t+10}$$ [5]

(iii) Calculate the mass of the substance when $t=10$, and find the time taken for the mass to increase from 0 to 45 grams. [2]

(iv) State what happens to the mass of the substance as $t$ becomes very large. [1]

***Suggested Solution:***

(i) Given that

$$\frac{dm}{dt}\propto(50-m)^2 \Rightarrow \frac{dm}{dt}=k(50-m)^2\cdots\cdots\text{(i)} \quad \text{where } k \text{ is a constant}$$

when $m=0$, $\frac{dm}{dt}=5$. Putting these values in eq. (i)

$$5=k(50-0)^2 \Rightarrow 5=2500k \Rightarrow k=\frac{5}{2500}=0.002$$

$\therefore$ eq. (i) becomes

$$\frac{dm}{dt}=0.002(50-m)^2 \quad \textbf{(Shown)}$$

(ii) $\frac{dm}{dt}=0.002(50-m)^2 \Rightarrow \frac{1}{(50-m)^2}\,dm=0.002\,dt$

integrating both sides

$$\int\frac{1}{(50-m)^2}dm=\int 0.002\,dt \Rightarrow \int(50-m)^{-2}\,dm=0.002\int 1\,dt$$

$$\Rightarrow -\int-(50-m)^{-2}\,dm=0.002\int 1\,dt \Rightarrow -\frac{(50-m)^{-1}}{-1}=0.002t+C$$

$$\Rightarrow \frac{1}{50-m}=0.002t+C\cdots\cdots\text{(ii)}$$

when $t=0$, $m=0$. Putting these values in eq. (ii)

$$\frac{1}{50-0}=0.002(0)+C \Rightarrow C=\frac{1}{50}$$

∴ eq. (ii) becomes

$$\frac{1}{50-m}=0.002t+\frac{1}{50}$$

$$\Rightarrow \frac{1}{50-m}=\frac{2}{1000}t+\frac{1}{50} \Rightarrow \frac{1}{50-m}=\frac{t}{500}+\frac{1}{50} \Rightarrow \frac{1}{50-m}=\frac{t+10}{500}$$

taking reciprocal

$$\frac{50-m}{1}=\frac{500}{t+10} \Rightarrow 50-m=\frac{500}{t+10} \Rightarrow m=50-\frac{500}{t+10} \quad \textbf{(Shown)}$$

(iii) When $t = 10$

$$m=50-\frac{500}{10+10} = 50-\frac{500}{20} = 50-25 = 25 \text{ grams} \quad \textbf{(Ans)}$$

It is given that when $m=0,\ t=0$

Therefore at $m=45$

$$45=50-\frac{500}{t+10} \Rightarrow -5=-\frac{500}{t+10} \Rightarrow t+10=-\frac{500}{-5}$$

$$\Rightarrow t+10=100 \Rightarrow t=90 \text{ minutes}$$

∴ It takes 90 minutes to increase the mass from 0 to 45 grams **(Ans)**

(iv) $m=50-\dfrac{500}{t+10}$

As $t \to \infty \Rightarrow \dfrac{500}{t+10} \to 0$

$\Rightarrow m \to 50$

∴ $m$ tends to increases to 50 grams as $t$ becomes very large. **(Ans)**

8. The straight line $l$ passes through the points $A$ and $B$ whose position vectors are $\mathbf{i}+\mathbf{k}$ and $4\mathbf{i}-\mathbf{j}+3\mathbf{k}$ respectively. The plane $p$ has equation $x+3y-2z=3$.

   (i) Given that $l$ intersects $p$, find the position vector of the point of intersection. [4]

   (ii) Find the equation of the plane which contains $l$ and is perpendicular to $p$, giving your answer in the form $ax+by+cz=1$. [6]

***Suggested Solution:***

(i) Given that $\overrightarrow{OA}=\begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad \overrightarrow{OB}=\begin{pmatrix}4\\-1\\3\end{pmatrix}$

Equation of line $l$ passing through $A$ and $B$ is:

$$\mathbf{r}=\overrightarrow{OA}+\lambda(\overrightarrow{AB}) \Rightarrow \mathbf{r}=\overrightarrow{OA}+\lambda(\overrightarrow{OB}-\overrightarrow{OA})$$

$$\Rightarrow \mathbf{r}=\begin{pmatrix}1\\0\\1\end{pmatrix}+\lambda\left[\begin{pmatrix}4\\-1\\3\end{pmatrix}-\begin{pmatrix}1\\0\\1\end{pmatrix}\right] \Rightarrow \mathbf{r}=\begin{pmatrix}1\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\\2\end{pmatrix} \cdots\cdots\cdots(\text{i})$$

position vector of general point on the line $l$ is

$$\mathbf{r}=\begin{pmatrix}1+3\lambda\\-\lambda\\1+2\lambda\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1+3\lambda\\-\lambda\\1+2\lambda\end{pmatrix}\cdots\cdots\cdots\text{(ii)}$$

Equation of the plane $p$ is

$$x+3y-2z=3\cdots\cdots\cdots\text{(iii)}$$

putting the values of $x$, $y$ and $z$ from eq. (ii) into eq. (iii), and solving simultaneously

$$1+3\lambda+3(-\lambda)-2(1+2\lambda)=3$$

$$\Rightarrow\ 1+3\lambda-3\lambda-2-4\lambda=3 \Rightarrow -4\lambda=4 \Rightarrow \lambda=-1$$

putting $\lambda=-1$ in equation (ii) we have

$$\mathbf{r}=\begin{pmatrix}1+3\lambda\\-\lambda\\1+2\lambda\end{pmatrix} \Rightarrow \mathbf{r}=\begin{pmatrix}1+3(-1)\\-(-1)\\1+2(-1)\end{pmatrix}\quad \mathbf{r}=\begin{pmatrix}-2\\1\\-1\end{pmatrix}$$

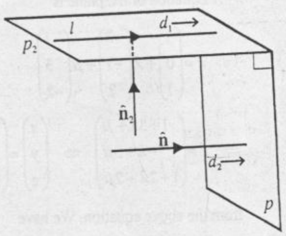
$\therefore$ the p.v of the point of intersection is $\begin{pmatrix}-2\\1\\-1\end{pmatrix}$ or $-2\mathbf{i}+\mathbf{j}-\mathbf{k}$ (Ans)

(ii) Let $d_1$ be the direction vector of line $l$.

$$\therefore\ d_1=\begin{pmatrix}3\\-1\\2\end{pmatrix}$$

Let $d_2$ be the direction vector of the normal $\mathbf{n}$ of the plane $p$

$$\therefore\ d_2=\begin{pmatrix}1\\3\\-2\end{pmatrix}$$



Let $p_2$ be the required plane with normal $\mathbf{n}_2$

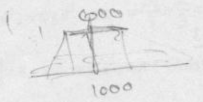
To find $\mathbf{n}_2$, we apply cross product

$$\mathbf{n}_2=d_1\times d_2=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&-1&2\\1&3&-2\end{vmatrix}$$

$$=\mathbf{i}(2-6)-\mathbf{j}(-6-2)+\mathbf{k}[9-(-1)]=-4\mathbf{i}+8\mathbf{j}+10\mathbf{k}=-2(2\mathbf{i}-4\mathbf{j}-5\mathbf{k})$$

$$\therefore\ \mathbf{n}_2=2\mathbf{i}-4\mathbf{j}-5\mathbf{k}$$

Equation of the plane $p_2$ in scalar product form is: $\mathbf{r}\cdot\begin{pmatrix}2\\-4\\-5\end{pmatrix}=D\cdots\cdots\cdots\text{(i)}$



point (1, 0, 1) lies on line $l$ and also on plane $p_2$

$\therefore$ it will satisfy the above equation

$$\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-4\\-5\end{pmatrix}=D \Rightarrow D=2+0-5 \Rightarrow D=-3$$

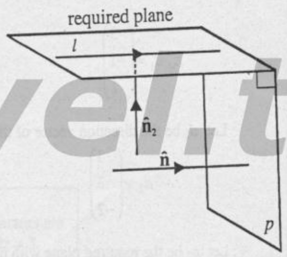
$\therefore$ equation of the required plane $p_2$ is

$$\mathbf{r}\cdot\begin{pmatrix}2\\-4\\-5\end{pmatrix}=-3 \quad\text{or}\quad 2x-4y-5z=-3 \quad\text{or}\quad -2x+4y+5z=3 \quad \textbf{(Ans)}$$

***Alternative Solution to part (ii):***

The direction vector of line $l$ is $\begin{pmatrix}3\\-1\\2\end{pmatrix}$

The direction vector of the normal $\mathbf{n}$ of the plane $p$ is $\begin{pmatrix}1\\3\\-2\end{pmatrix}$



One given point on the line $l$ is $\begin{pmatrix}1\\0\\1\end{pmatrix}$

$\therefore$ Equation of the plane is

$$r=\begin{pmatrix}1\\0\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}+\mu\begin{pmatrix}1\\3\\-2\end{pmatrix}$$

$$\Rightarrow\ r=\begin{pmatrix}1+3\lambda+\mu\\-\lambda+3\mu\\1+2\lambda-2\mu\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1+3\lambda+\mu\\-\lambda+3\mu\\1+2\lambda-2\mu\end{pmatrix}$$

from the above equation, We have

$x=1+3\lambda+\mu \cdots\cdots\cdots(a)$

$y=-\lambda+3\mu \cdots\cdots\cdots(b)$

$z=1+2\lambda-2\mu \cdots\cdots\cdots(c)$

solving equations $(a)$ and $(b)$ simultaneously for $\lambda$ and $\mu$

$$\begin{array}{ll}[\text{eq.}(a)\times 3] & 3x=3+9\lambda+3\mu\\ & \phantom{3}y=\ -\ \lambda+3\mu\\ & -\quad\ +\quad -\\ \hline & 3x-y=3+10\lambda \quad\Rightarrow\quad \lambda=\dfrac{3x-y-3}{10}\end{array}$$

also

$$x = 1 + 3\lambda + \mu$$

[eq.$(b)\times 3$] $$3y = -3\lambda + 9\mu$$

$$x + 3y = 1 + 10\mu \;\Rightarrow\; \mu = \frac{x+3y-1}{10}$$

putting these values of $\lambda$ and $\mu$ in equation $(c)$

$$z = 1 + 2\left(\frac{3x-y-3}{10}\right) - 2\left(\frac{x+3y-1}{10}\right)$$

$$\Rightarrow\; 10z = 10 + 2(3x-y-3) - 2(x+3y-1) \;\Rightarrow\; 10z = 10 + 6x - 2y - 6 - 2x - 6y + 2$$

$$\Rightarrow\; 10z = 4x - 8y + 6 \;\Rightarrow\; 4x - 8y - 10z = -6 \;\Rightarrow\; 2x - 4y - 5z = -3$$

$$\Rightarrow\; -2x + 4y + 5z = 3$$

$\therefore$ equation of the required plane is: $-2x + 4y + 5z = 3$ (Ans)

9. The complex number $1 + i\sqrt{3}$ is denoted by $u$.

(i) Express $u$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$. Hence, or otherwise, find the modulus and argument of $u^2$ and $u^3$. [5]

(ii) Show that $u$ is a root of the equation $z^2 - 2z + 4 = 0$, and state the other root of this equation. [2]

(iii) Sketch an Argand diagram showing the points representing the complex numbers i and $u$. Shade the region whose points represent every complex number $z$ satisfying both the inequalities

$$|z - i| \le 1 \quad \text{and} \quad \arg z \ge \arg u$$ [4]

***Suggested Solution:***

(i) Given that $u = 1 + i\sqrt{3}$

$$|u| = \sqrt{(1)^2 + (\sqrt{3})^2} \;\Rightarrow\; |u| = \sqrt{1+3} \;\Rightarrow\; |u| = 2$$

$$\arg u = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) \;\Rightarrow\; \arg u = \frac{\pi}{3}$$

$$\therefore\; u = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \quad \text{(Ans)}$$

Now $$|u^2| = |u||u| = 2\times 2 = 4$$

$$\arg(u^2) = \arg u + \arg u = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\therefore\; |u^2| = 4, \text{ and } \arg(u^2) = \frac{2\pi}{3} \quad \text{(Ans)}$$

$$|u^3| = |u^2||u| = 4\times 2 = 8$$

$$\arg(u^3) = \arg u^2 + \arg u = \frac{2\pi}{3} + \frac{\pi}{3} = \frac{3\pi}{3} = \pi$$

$$\therefore\; |u^3| = 8, \text{ and } \arg(u^3) = \pi \quad \text{(Ans)}$$

(ii) $z^2 - 2z + 4 = 0$

applying quadratic formula

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{-3}}{2} = \frac{2 \pm 2\sqrt{-1 \times 3}}{2}$$

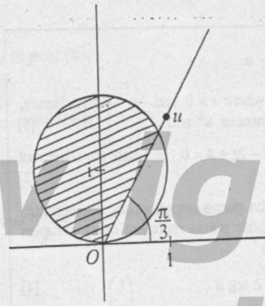
$$= \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

$\Rightarrow \quad z = 1 + i\sqrt{3} \quad$ or $\quad z = 1 - i\sqrt{3}$

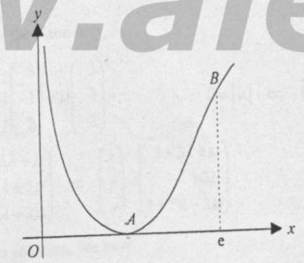
$\therefore \quad u = 1 + i\sqrt{3}$ is a root **(Ans)**

other root is $z = 1 - i\sqrt{3}$ **(Ans)**

(iii)



**10.**



The function f is defined by $f(x) = (\ln x)^2$ for $x > 0$. The diagram shows a sketch of the graph of $y = f(x)$. The minimum point of the graph is $A$. The point $B$ has $x$-coordinate e.

(i) State the $x$-coordinate of $A$. [1]

(ii) Show that $f''(x) = 0$ at $B$. [4]

(iii) Use the substitution $x = e^u$ to show that the area of the region bounded by the $x$-axis, the line $x = e$, and the part of the curve between $A$ and $B$ is given by

$$\int_0^1 u^2 e^u \, du$$ [3]

(iv) Hence, or otherwise, find the exact value of this area. [3]

**Suggested Solution:**

(i) $f(x) = (\ln x)^2$

differentiating w.r.t. $x$

$\Rightarrow \quad f'(x) = 2(\ln x) \times \frac{1}{x}$

for stationary values $f'(x) = 0$

$\therefore \quad 2(\ln x) \times \frac{1}{x} = 0 \;\Rightarrow\; 2\ln x = 0 \;\Rightarrow\; \ln x = 0 \;\Rightarrow\; \ln x = \ln 1 \;\Rightarrow\; x = 1$

$\therefore$ $x$-coordinate of $A = 1$ (Ans)

Note that
$f'(x) = \frac{dy}{dx}$, and
$f''(x) = \frac{d^2y}{dx^2}$

(ii) $f'(x) = \frac{2\ln x}{x}$

differentiating $w.r.t.\,x$

$\Rightarrow \quad f''(x) = \frac{x\left(2 \times \frac{1}{x}\right) - 2\ln x(1)}{x^2} = \frac{2 - 2\ln x}{x^2}$

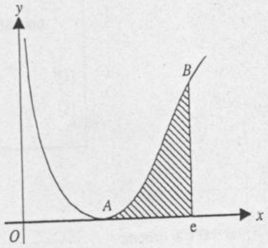
at $B$, $x = \mathrm{e}$

$\Rightarrow \quad f''(\mathrm{e}) = \frac{2 - 2\ln \mathrm{e}}{(\mathrm{e})^2} = \frac{2-2}{\mathrm{e}^2} = 0$

$\therefore \quad f''(x) = 0$ at $B$ **(Shown)**

(iii) The required shaded area is given by

$$A = \int_1^{\mathrm{e}} y\,dx \;\Rightarrow\; A = \int_1^{\mathrm{e}} f(x)\,dx \;\Rightarrow\; A = \int_1^{\mathrm{e}} (\ln x)^2\,dx$$



given substitution is

61

$x = \mathrm{e}^u$

differentiating $w.r.t.\,x$

$\Rightarrow \quad \frac{d}{dx}(x) = \frac{d}{dx}(\mathrm{e}^u) \;\Rightarrow\; 1 = \mathrm{e}^u \times \frac{du}{dx} \;\Rightarrow\; dx = \mathrm{e}^u\,du$

for limits

when $x = \mathrm{e} \;\Rightarrow\; \mathrm{e} = \mathrm{e}^u \;\Rightarrow\; u = 1$

when $x = 1 \;\Rightarrow\; 1 = \mathrm{e}^u \;\Rightarrow\; \mathrm{e}^0 = \mathrm{e}^u \;\Rightarrow\; u = 0$

Now, substituting the new limits, and values of $x$ and $dx$, the area becomes

$$A = \int_0^1 (\ln \mathrm{e}^u)^2\,\mathrm{e}^u\,du \;\Rightarrow\; A = \int_0^1 (u\ln \mathrm{e})^2\,\mathrm{e}^u\,du \;\Rightarrow\; A = \int_0^1 u^2\,\mathrm{e}^u\,du \quad \textbf{(Shown)}$$

(iv) $A = \int_0^1 u^2\, e^u\, du$

using integrating by parts

$$A = \left[u^2 e^u\right]_0^1 - \int_0^1 2u\, e^u\, du$$

$$\Rightarrow \quad A = \left[u^2 e^u\right]_0^1 - 2\int_0^1 u\, e^u\, du$$

$$\Rightarrow \quad A = \left[u^2 e^u\right]_0^1 - 2\left[\left[ue^u\right]_0^1 - \int_0^1 e^u \times 1\, du\right]$$

$$\Rightarrow \quad A = \left[u^2 e^u\right]_0^1 - 2\left[\left[ue^u\right]_0^1 - \left[e^u\right]_0^1\right]$$

$$\Rightarrow \quad A = \left((1)^2 e^1 - (0)^2 e^0\right) - 2\left[\left((1)e^1 - (0)e^0\right) - \left(e^1 - e^0\right)\right]$$

$$\Rightarrow \quad A = (e) - 2\left[(e) - (e-1)\right]$$

$$\Rightarrow \quad A = (e) - 2[e - e + 1]$$

$$\Rightarrow \quad A = e - 2$$

$\therefore$ Area $= e - 2$ unit$^2$ (Ans)

# November 2002 Paper 1

## Pure Mathematics (P1)

Answer all the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. Find the value of the term which is independent of $x$ in the expansion of

$$\left(x+\frac{3}{x}\right)^4$$ [3]

*Suggested Solution:*

Using $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$, we have

$$T_{r+1} = {}^4C_r\; x^{4-r}\left(\frac{3}{x}\right)^r = {}^4C_r\; x^{4-r}\left(3^r x^{-r}\right) = {}^4C_r\;(3^r x^{4-2r})$$

collecting powers of $x$ only, $4-2r=0 \Rightarrow r=2$

$\therefore$ term independent of $x = {}^4C_2\; x^{4-(2)}\left(\frac{3}{x}\right)^2 = 6x^2\left(\frac{9}{x^2}\right) = 54$ (Ans)

2. A geometric progression, for which the common ratio is positive, has a second term of 18 and a fourth term of 8. Find
   (i) the first term and the common ratio of the progression, [3]
   (ii) the sum to infinity of the progression. [2]

*Suggested Solution:*

(i) Given that $T_2 = 18$, $T_4 = 8$, and $r > 0$

nth term of a G.P is
$T_n = a\,r^{n-1}$

$\therefore \quad T_2 = a\,r^{2-1} \Rightarrow 18 = ar$ ………(i)

and $T_4 = a\,r^{4-1} \Rightarrow 8 = a\,r^3$ ………(ii)

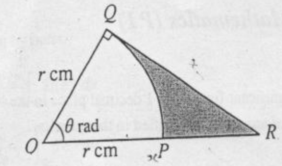
dividing eq.(ii) by eq.(i), $\frac{8}{18} = \frac{ar^3}{ar} \Rightarrow r^2 = \frac{4}{9} \Rightarrow r = \frac{2}{3}$ (Ans)

putting value of $r$ in eq. (i)

$$ar = 18 \Rightarrow a\left(\frac{2}{3}\right) = 18 \Rightarrow a = 27 \text{ (Ans)}$$

(ii) $S_\infty = \frac{a}{1-r} \Rightarrow S_\infty = \frac{27}{1-\frac{2}{3}} \Rightarrow S_\infty = \frac{27}{\frac{1}{3}} \Rightarrow S_\infty = 81$ (Ans)

**3.**



In the diagram, $OPQ$ is a sector of a circle, centre $O$ and radius $r$ cm. Angle $QOP = \theta$ radians. The tangent to the circle at $Q$ meets $OP$ extended at $R$.

(i) Show that the area, $A$ cm$^2$, of the shaded region is given by

$$A = \frac{1}{2}r^2(\tan\theta - \theta)$$ [2]

(ii) In the case where $\theta = 0.8$ and $r = 15$, evaluate the length of the perimeter of the shaded region. [4]

*Suggested Solution:*

(i) In $\Delta OQR$, $\tan\theta = \dfrac{QR}{OQ} \Rightarrow QR = OQ\tan\theta \Rightarrow QR = r\tan\theta$

Shaded area = Area of the $\Delta OQR$ − area of the sector $OPQ$

$$= \frac{1}{2}\times OQ\times QR - \frac{1}{2}\times r^2\theta$$

$$= \frac{1}{2}\times r\times r\tan\theta - \frac{1}{2}\times r^2\theta$$

$$= \frac{1}{2}\times r^2\tan\theta - \frac{1}{2}\times r^2\theta = \frac{1}{2}r^2(\tan\theta - \theta) \quad \textbf{(Shown)}$$

If angle is in radians than

area of sector $= \frac{1}{2}r^2\theta$

arc length, $S = r\theta$

(ii) $QR = r\tan\theta \Rightarrow QR = 15\tan 0.8 \Rightarrow QR = 15.445$ cm

using Pythagoras theorem

$$OR^2 = OQ^2 + QR^2$$

$$\Rightarrow OR^2 = 15^2 + 15.445^2 \Rightarrow OR = \sqrt{463.548} \Rightarrow OR = 21.53 \text{ cm}$$

now $PR = OR - OP \Rightarrow PR = 21.53 - 15 \Rightarrow PR = 6.53$ cm

arc length $\widehat{PQ} = r\theta = 15\times 0.8 = 12$ cm

$\therefore$ perimeter of shaded region $= \widehat{PQ} + PR + QR = 12 + 6.53 + 15.445$

$= 33.975 = 34.0$ cm (3s.f.) **(Ans)**

4. The gradient at any point $(x, y)$ on a curve is $\sqrt{(1+2x)}$. The curve passes through the point (4, 11). Find

(i) the equation of the curve, [4]

(ii) the point at which the curve intersects the $y$-axis. [2]

*Suggested Solution:*

(i) $\frac{dy}{dx}=\sqrt{(1+2x)} \Rightarrow dy=\sqrt{(1+2x)}\ dx$

Integrating both sides

$$\int dy=\int (1+2x)^{\frac{1}{2}}\,dx \Rightarrow y=\frac{(1+2x)^{\frac{3}{2}}}{2\left(\frac{3}{2}\right)}+K \Rightarrow y=\frac{(1+2x)^{\frac{3}{2}}}{3}+K$$

as curve passes through (4, 11)

$$\therefore\ 11=\frac{(1+2(4))^{\frac{3}{2}}}{3}+K \Rightarrow 11=\frac{(9)^{\frac{3}{2}}}{3}+K \Rightarrow K=2$$

required equation of the curve: $y=\frac{1}{3}(1+2x)^{\frac{3}{2}}+2$ (Ans)

Gradient at any point $(x, y)$ on a curve is $\frac{dy}{dx}$

When $\frac{dy}{dx}$ (or gradient) of the curve is given, always integrate both sides of the given expression to find the equation of a curve.

(ii) $y=\frac{1}{3}(1+2x)^{\frac{3}{2}}+2$

for $y$-intercept, put $x=0$

$$\therefore\ y=\frac{1}{3}(1+2(0))^{\frac{3}{2}}+2 \Rightarrow y=\frac{1}{3}(1)+2 \Rightarrow y=\frac{7}{3}$$

required point is $\left(0, \frac{7}{3}\right)$ (Ans)

5. (i) Show that the equation $3\tan\theta = 2\cos\theta$ can be expressed as

$2\sin^2\theta + 3\sin\theta - 2 = 0.$ [3]

(ii) Hence solve the equation $3\tan\theta = 2\cos\theta$, for $0^\circ \le \theta \le 360^\circ$ [3]

*Suggested Solution:*

(i) $3\tan\theta=2\cos\theta \Rightarrow 3\left(\frac{\sin\theta}{\cos\theta}\right)=2\cos\theta \Rightarrow 3\sin\theta=2\cos^2\theta$

$\Rightarrow 3\sin\theta=2(1-\sin^2\theta) \Rightarrow 3\sin\theta=2-2\sin^2\theta$

$\Rightarrow 2\sin^2\theta+3\sin\theta-2=0$ (shown)

(ii) From part (i), we see that $3\tan\theta = 2\cos\theta$ can be written as

$$2\sin^2\theta + 3\sin\theta - 2 = 0$$

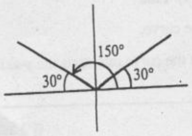
$$\Rightarrow\ 2\sin^2\theta + 4\sin\theta - \sin\theta - 2 = 0$$

$$\Rightarrow\ (\sin\theta + 2)(2\sin\theta - 1) = 0$$

$$\Rightarrow\ \sin\theta = -2 \text{ (invalid), or } \sin\theta = \frac{1}{2}$$

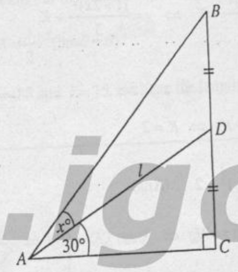
basic angle = 30°

$\therefore\ x = 30°,\ 150°$ **(Ans)**



Note that $\sin\theta$ is +ve in 1st and 2nd quadrant

**6.**



In the diagram, triangle $ABC$ is right angled and $D$ is the mid point of $BC$. Angle $DAC = 30°$ and angle $BAD = x°$. Denoting the length of $AD$ by $l$,

(i) Express each of $AC$ and $BC$ exactly in terms of $l$, and show that

$$AB = \frac{1}{2}l\sqrt{7}$$ [4]

(ii) show that $x = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) - 30$. [2]

*Suggested Solution:*

(i) In $\Delta ACD$

$$\cos 30° = \frac{AC}{AD} \Rightarrow AC = AD\cos 30 \Rightarrow AC = \frac{\sqrt{3}}{2}l \text{ (Ans)}$$

$$\sin 30° = \frac{CD}{AD} \Rightarrow CD = AD\sin 30 \Rightarrow CD = l\left(\frac{1}{2}\right) \Rightarrow CD = \frac{1}{2}l$$

$$\text{as } BC = 2CD \Rightarrow BC = 2\left(\frac{l}{2}\right) \Rightarrow BC = l \text{ (Ans)}$$

Now using pythagoras theorem

$$AB = \sqrt{AC^2 + BC^2} \Rightarrow AB = \sqrt{\left(\frac{\sqrt{3}}{2}l\right)^2 + l^2} \Rightarrow AB = \sqrt{\frac{3l^2}{4} + l^2}$$

$$\Rightarrow AB = \sqrt{\frac{7l^2}{4}} \Rightarrow AB = \frac{1}{2}l\sqrt{7} \text{ (Shown)}$$

(ii) In $\Delta ABC$

$$\tan(x+30^\circ)=\frac{BC}{AC} \Rightarrow \tan(x+30^\circ)=\frac{l}{\frac{\sqrt{3}}{2}l} \Rightarrow \tan(x+30^\circ)=\frac{2}{\sqrt{3}}$$

$$\Rightarrow (x+30^\circ)=\tan^{-1}\left(\frac{2}{\sqrt{3}}\right) \Rightarrow x=\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)-30^\circ \quad \textbf{(Shown)}$$

7. Given that $\mathbf{a}=\begin{pmatrix}2\\-2\\1\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}2\\6\\3\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}p\\p\\p+1\end{pmatrix}$ find

(i) the angle between the directions of $\mathbf{a}$ and $\mathbf{b}$, [4]

(ii) the value of $p$ for which $\mathbf{b}$ and $\mathbf{c}$ are perpendicular. [3]

***Suggested Solution:***

(i) Let $\theta$ be the angle between $\mathbf{a}$ and $\mathbf{b}$

$$\therefore \quad \mathbf{a}\bullet\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta$$

$$\Rightarrow \begin{pmatrix}2\\-2\\1\end{pmatrix}\bullet\begin{pmatrix}2\\6\\3\end{pmatrix}=\left(\sqrt{2^2+2^2+1^2}\right)\left(\sqrt{2^2+6^2+3^2}\right)\cos\theta$$

$$\Rightarrow 4-12+3=\left(\sqrt{9}\right)\left(\sqrt{49}\right)\cos\theta$$

$$\Rightarrow -5=21\cos\theta \Rightarrow \cos\theta=-\frac{5}{21} \Rightarrow \theta=103.8^\circ \quad \text{(Ans)}$$

If
$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, and
$\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, then
$\mathbf{a}.\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$
and magnitude of $\mathbf{a}$ is
$|\mathbf{a}|=\sqrt{(a_1)^2+(a_2)^2+(a_3)^2}$

(ii) If $\mathbf{b}$ and $\mathbf{c}$ are perpendicular then $\mathbf{b}\bullet\mathbf{c}=0$

$$\Rightarrow \begin{pmatrix}2\\6\\3\end{pmatrix}\bullet\begin{pmatrix}p\\p\\p+1\end{pmatrix}=0$$

$$\Rightarrow 2p+6p+3p+3=0 \Rightarrow 11p+3=0 \Rightarrow p=-\frac{3}{11} \quad \text{(Ans).}$$

8. A curve has equation $y=x^3+3x^2-9x+k$, where $k$ is a constant.

(i) write down an expression for $\dfrac{dy}{dx}$. [2]

(ii) Find the $x$-coordinates of the two stationary points on the curve. [2]

(iii) Hence find the two values of $k$ for which the curve has a stationary point on the $x$-axis. [3]

*Suggested Solution:*

(i) $\frac{dy}{dx} = 3x^2 + 6x - 9$ (Ans)

(ii) For stationary point, $\frac{dy}{dx} = 0$

$\Rightarrow 3x^2 + 6x - 9 = 0 \Rightarrow x^2 + 2x - 3 = 0 \Rightarrow x^2 + 3x - x - 3 = 0$

$\Rightarrow (x+3)(x-1) = 0 \Rightarrow x = 1,$ or $x = -3$ (Ans)

(iii) For stationary points on the $x$-axis, put $y = 0$

$\Rightarrow x^3 + 3x^2 - 9x + k = 0$

when $x = 1$

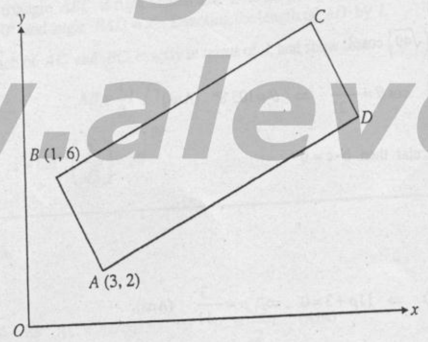
$(1)^3 + 3(1)^2 - 9(1) + k = 0 \Rightarrow -5 + k = 0 \Rightarrow k = 5$

when $x = -3$

$(-3)^3 + 3(-3)^2 - 9(-3) + k = 0 \Rightarrow 27 + k = 0 \Rightarrow k = -27$

$\therefore k = 5, -27$ (Ans)

**9.**



The diagram shows a rectangle $ABCD$, where $A$ is (3, 2) and $B$ is (1, 6).

(i) Find the equation of $BC$. [4]

Given that the equation of $AC$ is $y = x - 1$, find

(ii) the coordinates of $C$, [2]

(iii) the perimeter of the rectangle $ABCD$. [3]

*Suggested Solution:*

(i) gradient of $AB = \frac{6-2}{1-3} = \frac{4}{-2} = -2$

as $AB$ is ⊥ to $BC$ $\Rightarrow$ gradient of $BC = \frac{1}{2}$

$\therefore$ equation of $BC$ passing through $B(1, 6)$ is:

$y-6=\frac{1}{2}(x-1) \Rightarrow 2y-12=x-1 \Rightarrow 2y-x=11$ (Ans)

For perpendicular lines

$m_1 \times m_2 = -1$

Equation of a line passing through $P(x_1, y_1)$ and having gradient $m$ has equation:

$y - y_1 = m(x - x_1)$

(ii) Given that

equation of $AC$: $y=x-1 \cdots\cdots\cdots$(i)

and equation of $BC$ from part (i): $2y-x=11 \cdots\cdots\cdots$(ii)

Solving eq.(i) and eq.(ii) simultaneously;

$2(x-1)-x=11 \Rightarrow x=13$

putting value of $x$ in eq. (i)

$y=13-1 \Rightarrow y=12$

$\therefore$ coordinates of $C(13, 12)$ (Ans)

(iii) $|AB|=\sqrt{(3-1)^2+(2-6)^2}=\sqrt{4+16}=\sqrt{20}=2\sqrt{5}$

$|BC|=\sqrt{(13-1)^2+(12-6)^2}=\sqrt{144+36}=\sqrt{180}=6\sqrt{5}$

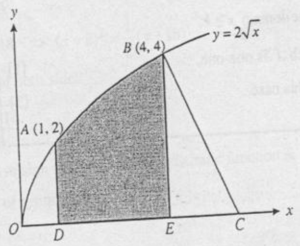
$\therefore$ perimeter of $ABCD=2(AB+BC)=2(2\sqrt{5}+6\sqrt{5})=16\sqrt{5}$

$= 35.8$ units (Ans)

Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is:

$|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

10.



The diagram shows the points $A(1, 2)$ and $B(4, 4)$ on the curve $y=2\sqrt{x}$. The line $BC$ is the normal to the curve at $B$, and $C$ lies on the $x$-axis. Lines $AD$ and $BE$ are perpendicular to the $x$-axis.

(i) Find the equation of the normal $BC$. [4]

(ii) Find the area of the shaded region. [4]

*Suggested Solution:*

(i) $y = 2\sqrt{x}$

$$\frac{dy}{dx} = 2\times\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}}$$

gradient of the tangent at $B\,(4,4)$ : $\frac{dy}{dx} = \frac{1}{\sqrt{4}} = \frac{1}{2} = \frac{1}{2}$

$\therefore$ gradient of the normal $= -2$

now equation of normal $BC$ is:

$y-4=-2(x-4) \;\Rightarrow\; y+2x=12$ (Ans)

Tangent and normal to the curve are perpendicular to each other, and for perpendicular lines

$m_1 \times m_2 = -1$

Equation of a line in point-slope form is:

$y - y_1 = m\,(x - x_1)$

(ii) Area of the shaded region, $A = \int_1^4 y\,dx$

$$\Rightarrow A = \int_1^4 2\sqrt{x}\,dx = 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4 = \frac{4}{3}\left[x^{\frac{3}{2}}\right]_1^4 = \frac{4}{3}\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{4}{3}(7) = \frac{28}{3} = 9\frac{1}{3}$$

$\therefore$ area of shaded region $= 9\frac{1}{3}$ sq.units. (Ans)

11. (i) Express $2x^2+8x-10$ in the form $a\,(x+b)^2+c$. [3]

(ii) For the curve $y=2x^2+8x-10$, state the least value of $y$ and the corresponding value of $x$. [2]

(iii) Find the set of values of $x$ for which $y \geq 14$. [3]

Given that $f : x \mapsto 2x^2+8x-10$ for the domain $x \geq k$,

(iv) find the least value of $k$ for which f is one-one, [1]

(v) express $f^{-1}(x)$ in terms of $x$ in this case. [3]

*Suggested Solution:*

(i) $y = 2x^2 + 8x - 10$

$= 2(x^2+4x) - 10$

$= 2\left[(x)^2 + 2(x)(2) + (2)^2 - (2)^2\right] - 10$

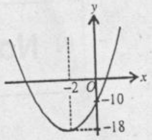
$= 2\left[(x+2)^2 - 4\right] - 10$

$= 2(x+2)^2 - 8 - 10$

$= 2(x+2)^2 - 18$ (Ans)

(ii) from part (i), equation of the curve can be given as $2(x+2)^2-18$

$\therefore$ least value of $y=-18$, and it occures when $x=-2$ (Ans)



If $y=ax^2+bx+c$ is expressed as $y=a(x-h)^2+k$ then, $(h, k)$ are the coordinates of the turning point.

(iii) $y \geq 14$

$\Rightarrow 2(x+2)^2-18 \geq 14$

$\Rightarrow 2(x+2)^2-32 \geq 0$

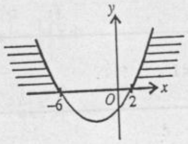
$\Rightarrow (x+2)^2-16 \geq 0$

$\Rightarrow (x+2)^2-(4)^2 \geq 0$

$\Rightarrow (x+2+4)(x+2-4) \geq 0$

$\Rightarrow (x+6)(x-2) \geq 0$

$\therefore\ x \leq -6,\ x \geq 2$ (Ans)



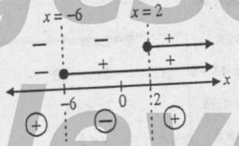
*Alternative Method:*

$(x+6)(x-2) \geq 0$

critical values are $x=-6$ and $x=2$

Hence positive sections of the line are

$\therefore x \leq -6$ or $x \geq 2$.



(iv) Least value of $k=-2$ for which f is one-one. (Ans)

(v) Let $\mathrm{f}(x)=y \Rightarrow y=2(x+2)^2-18$

making $x$ as subject

$$2(x+2)^2=y+18 \Rightarrow (x+2)^2=\frac{1}{2}(y+18)$$

taking square root of both sides

$$x+2=\pm\sqrt{\frac{y+18}{2}}$$

retaining the positive sign as the restricted quadratic function is to the right of axis of symmetry.

$$x=-2+\sqrt{\frac{y+18}{2}}$$

as $\mathrm{f}(x)=y \Rightarrow \mathrm{f}^{-1}(y)=x$

$$\therefore\ \mathrm{f}^{-1}(y)=-2+\sqrt{\frac{y+18}{2}}$$

$$\Rightarrow \mathrm{f}^{-1}(x)=-2+\sqrt{\frac{x+18}{2}} \quad \text{(Ans)}$$

Any quadratic function $y=ax^2+bx+c$ does not have inverse function as it is not a 1–1 function. However if we restrict the quadratic function for the domain which is either to the right of axis of symmetry or to the left there off, the restricted quadratic function thus obtained is a 1–1 function and has an inverse.

# November 2002 Paper 3

## Pure Mathematics (P3)

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. Solve the inequality. $|9-2x|<1$. [3]

***Suggested Solution:***

Using definition

When $|x|<a$

then $x<a$ and $x>-a$

$$|9-2x|<1$$

$$\Rightarrow (9-2x)<1 \quad , \quad 9-2x>-1$$

$$-2x<1-9 \quad , \quad -2x>-10$$

$$-2x<-8 \quad , \quad 2x<10$$

$$2x>8 \quad , \quad x<5$$

$$x>4$$

$$\therefore \quad 4<x<5 \quad \text{(Ans)}$$

***Alternative Solution:***

$$|9-2x|<1$$

squaring both sides

$$(9-2x)^2<1$$

$$81-36x+4x^2<1$$

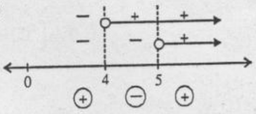
$$4x^2-36x+80<0$$

$$x^2-9x+20<0$$

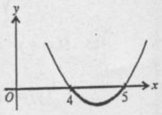
$$x^2-5x-4x+20<0$$

$$x(x-5)-4(x-5)<0$$

$$(x-5)(x-4)<0$$



Alternately we can find the set of values of $x$ by drawing the curve as follows



$$\therefore \quad 4<x<5 \quad \text{(Ans)}$$

2. Find the exact value of $\int_1^2 x\ln x\,dx$ [4]

***Suggested Solution:***

If integration by parts contain lnx, then always take lnx as the 1st function

$$\int_1^2 x\ln x\,dx$$

using integration by parts

$$\left[\ln x\left(\frac{x^2}{2}\right)\right]_1^2-\int_1^2\frac{x^2}{2}\times\frac{1}{x}\,dx=\left[\left(\frac{x^2}{2}\right)\ln x\right]_1^2-\frac{1}{2}\int_1^2 x\,dx$$

$$=\left[\left(\frac{x^2}{2}\right)\ln x\right]_1^2-\frac{1}{2}\left[\frac{x^2}{2}\right]_1^2=\left[\left(\frac{x^2}{2}\right)\ln x\right]_1^2-\frac{1}{4}\left[x^2\right]_1^2$$

$$=\left[\left(\frac{2^2}{2}\ln 2\right)-\left(\frac{1^2}{2}\ln 1\right)\right]-\frac{1}{4}\left[2^2-1^2\right]$$

$$=(2\ln 2-0)-\frac{1}{4}(3)=2\ln 2-\frac{3}{4}\quad\text{(Ans)}$$

3. (i) Show that the equation

$$\log_{10}(x+5)=2-\log_{10}x$$

may be written as a quadratic equation in $x$. [3]

(ii) Hence find the value of $x$ satisfying the equation

$$\log_{10}(x+5)=2-\log_{10}x$$ [2]

***Suggested Solution:***

(i) $\log_{10}(x+5)=2-\log_{10}x$

$\Rightarrow\ \log_{10}(x+5)+\log_{10}x=2$

$\Rightarrow\ \log_{10}x(x+5)=2$

If $\log_a x = y$
then $x = a^y$

getting into exponential form

$\Rightarrow\ x(x+5)=10^2$

$\Rightarrow\ x^2+5x-100=0$ (Ans)

(ii) $x^2+5x-100=0$

applying quadratic formula

$$x=\frac{-5\pm\sqrt{5^2-4(1)(-100)}}{2(1)}=\frac{-5\pm\sqrt{25+400}}{2}=\frac{-5\pm\sqrt{425}}{2}$$

Ignore the negative value of $x$ as log of a negative number is undefined.

$$\Rightarrow\ x=\frac{-5+\sqrt{425}}{2}\quad\text{or}\quad x=\frac{-5-\sqrt{425}}{2}\ \left(\text{ignore negative value of } x\right)$$

$$\therefore\ x=\frac{-5+\sqrt{425}}{2}=\frac{-5+5\sqrt{17}}{2}=\frac{5}{2}\left(-1+\sqrt{17}\right)\quad\text{(Ans)}$$

4. The curve $y = e^x + 4e^{-2x}$ has one stationary point.

(i) Find the $x$-coordinate of this point. [4]

(ii) Determine whether the stationary point is a maximum or a minimum point. [2]

***Suggested Solution:***

(i) $y = e^x + 4e^{-2x}$

$$\frac{dy}{dx} = e^x + 4e^{-2x}(-2) = e^x - 8e^{-2x}$$

for stationary values $\frac{dy}{dx} = 0$

$$\Rightarrow e^x - 8e^{-2x} = 0 \Rightarrow e^x = 8e^{-2x} \Rightarrow e^x = \frac{8}{e^{2x}}$$

$$\Rightarrow e^{3x} = 8 \Rightarrow (e^x)^3 = (2)^3 \Rightarrow e^x = 2$$

taking ln on both sides

$$\ln e^x = \ln 2 \Rightarrow x \ln e = \ln 2 \Rightarrow x = \ln 2$$

$\therefore$ $x$-coordinate is, $x = \ln 2$ **(Ans)**

(ii) From part (i), we have

$$\frac{dy}{dx} = e^x - 8e^{-2x}$$

$$\therefore \frac{d^2y}{dx^2} = e^x - 8e^{-2x}(-2) = e^x + 16e^{-2x} = e^x + \frac{16}{e^{2x}}$$

at $x = \ln 2$

$$\frac{d^2y}{dx^2} = e^{\ln 2} + \frac{16}{e^{2(\ln 2)}} = 2 + \frac{16}{4} = 6 > 0$$

$\therefore$ the stationary point is a minimum point. **(Ans)**

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} > 0$, $y$ is min

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} < 0$, $y$ is max

5. (i) Express $4\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0^0 < \alpha < 90^0$, stating the value of $\alpha$ correct to 2 decimal places. [3]

Hence

(ii) solve the equation [4]

$$4\sin\theta - 3\cos\theta = 2,$$

giving all values of $\theta$ such that $0^\circ < \theta < 360^\circ$

(iii) write down the greatest value of $\frac{1}{4\sin\theta - 3\cos\theta + 6}$. [1]

*Suggested Solution:*

(i) $R\sin(\theta-\alpha) \equiv 4\sin\theta - 3\cos\theta$

$R\sin\theta\cos\alpha - R\cos\theta\sin\alpha \equiv 4\sin\theta - 3\cos\theta$

Formula used here is:
$\sin(a+b)$
$= \sin a\cos b - \cos a\sin b$

comparing the coefficients of $\cos\theta$ and $\sin\theta$, we have

$R\cos\alpha = 4\cdots\cdots$(i) and $R\sin\alpha = 3\cdots\cdots$(ii)

eq. (ii) ÷ eq.(i) gives

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{3}{4} \Rightarrow \tan\alpha = \frac{3}{4} \Rightarrow \alpha = 36.87°$

squaring and adding eq. (i) & eq. (ii) gives

$(R\sin\alpha)^2 + (R\cos\alpha)^2 = 4^2 + 3^2 \Rightarrow R^2(\sin^2\alpha + \cos^2) = 25 \Rightarrow R^2 = 25 \Rightarrow R = 5$

$\therefore \quad 4\sin\theta - 3\cos\theta = 5\sin(\theta - 36.87°)$ **(Ans)**

(ii) $4\sin\theta - 3\cos\theta = 2,$

using the result of part (i), the above equation can be written as

$5\sin(\theta - 36.87°) = 2$

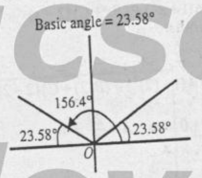
$\Rightarrow \sin(\theta - 36.87°) = 0.4$

$\Rightarrow (\theta - 36.87°) = 23.58°$ or $156.4°$

$\Rightarrow (\theta - 36.87°) = 23.58° \qquad (\theta - 36.87°) = 156.4°$

$\Rightarrow \theta = 60.5° \qquad \theta = 193.3°$

$\therefore \quad \theta = 60.5°, 193.3°$ **(Ans)**



(iii) We know that

$-1 \le \sin\theta \le 1$

$\Rightarrow -5 \le 5\sin(\theta - 36.87°) \le 5$

adding 6 throughout, we have

$-5 + 6 \le 5\sin(\theta - 36.87°) + 6 \le 5 + 6$

$\Rightarrow 1 \le 5\sin(\theta - 36.87°) + 6 \le 11$

taking reciprocal

Note that the inequality sign changes when reciprocals are taken

$1 \ge \frac{1}{5\sin(\theta - 36.87°) + 6} \ge \frac{1}{11}$

i.e. $\frac{1}{5\sin(\theta - 36.87°) + 6}$ is always less than or equal to 1 and always greater then $\frac{1}{11}$

$\therefore$ greatest value = 1 **(Ans)**

**6.** Let $f(x)=\dfrac{6+7x}{(2-x)(1+x^2)}$.

(i) Express $f(x)$ in partial fractions. [4]

(ii) Show that, when $x$ is sufficiently small for $x^4$ and higher powers to be neglected, [5]

$$f(x)=3+5x-\frac{1}{2}x^2-\frac{15}{4}x^3.$$

*Suggested Solution:*

(i) $f(x)=\dfrac{6+7x}{(2-x)(1+x^2)}$

$\therefore \quad \dfrac{6+7x}{(2-x)(1+x^2)}\equiv\dfrac{A}{2-x}+\dfrac{Bx+C}{1+x^2}$

$\Rightarrow \quad 6+7x\equiv A(1+x^2)+(Bx+C)(2-x)$

for $x=2$

$6+7(2)=A(1+(2)^2)+(B(2)+C)(2-2)$

$\Rightarrow \quad 20=A(5)+0 \quad\Rightarrow\quad A=\dfrac{20}{5}=4$

for $x=0$

$6+7(0)=A(1+(0))+(B(0)+C)(2-(0))$

$\Rightarrow \quad 6=A+2C$

As $\quad A=4$

$\Rightarrow \quad 6=4+2C \quad\Rightarrow\quad 2=2C \quad\Rightarrow\quad C=1$

for $x=-1$

$6+7(-1)=A(1+(-1)^2)+(B(-1)+C)(2-(-1))$

$\Rightarrow \quad -1=2A+(-B+C)(3) \quad\Rightarrow\quad -1=2A-3B+3C$

As $\quad A=4,$ and $C=1$

$\Rightarrow \quad -1=2(4)-3B+3(1) \quad\Rightarrow\quad -1=8-3B+3 \quad\Rightarrow\quad -12=-3B \quad\Rightarrow\quad B=4$

$\therefore \dfrac{6+7x}{(2-x)(1+x^2)}\equiv\dfrac{4}{2-x}+\dfrac{4x+1}{1+x^2}$ (Ans)

(ii) Using the result of part (i), we have

$$f(x)=\frac{4}{2-x}+\frac{4x+1}{1+x^2}$$

$$=\frac{4}{2\left(1-\frac{x}{2}\right)}+\frac{4x+1}{1+x^2}$$

$$=2\left(1-\frac{x}{2}\right)^{-1}+(4x+1)(1+x^2)^{-1}$$

applying binomial theorem

$$2\left[1+(-1)\left(-\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(-\frac{x}{2}\right)^3+....\right]+(4x+1)\left[1+(-1)\left(x^2\right)+......\right]$$

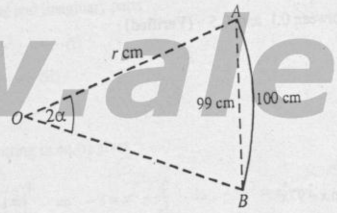
$$=\quad 2\left[1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}\right]+(4x+1)\left[1-x^2\right]$$

$$=\quad \left(2+x+\frac{x^2}{2}+\frac{x^3}{4}\right)+(4x+1-4x^3-x^2)$$

$$=\quad 2+1+x+4x+\frac{x^2}{2}-x^2+\frac{x^3}{4}-4x^3=3+5x-\frac{x^2}{2}-\frac{15x^3}{4}$$

$$\therefore\quad 3+5x-\frac{1}{2}x^2-\frac{15}{4}x^3$$ (Shown)

7.



The diagram shows a curved rod $AB$ of length 100 cm which forms an arc of a circle. The end points $A$ and $B$ of the rod are 99 cm apart. The circle has radius $r$ cm and the arc $AB$ subtends an angle of $2\alpha$ radians at $O$, the centre of the circle.

(i) Show that $\alpha$ satisfies the equation $\frac{99}{100}x=\sin x$. [3]

(ii) Given that this equation has exactly one root in the interval $0<x<\frac{1}{2}\pi$, verify by calculation that this root lies between 0.1 and 0.5. [2]

(iii) Show that if the sequence of values given by the iterative formula

$$x_{n+1}=50\sin x_n-48.5x_n$$

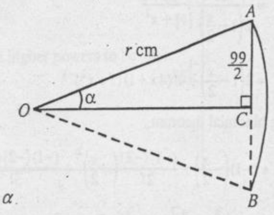
converges, then it converges to a root of the equation in part (i). [2]

(iv) Use this iterative formula, with initial value $x_1=0.25$, to find $\alpha$ correct to 3 decimal places, showing the result of each iteration. [2]

*Suggested Solution:*

(i) Using $s = r\theta$

$$100 = r(2\alpha) \;\Rightarrow\; r = \frac{100}{2\alpha} = \frac{50}{\alpha} \;\cdots\cdots\cdots\text{(i)}$$



In $\Delta OAC$

$$\sin\alpha = \frac{\frac{99}{2}}{r} \;\Rightarrow\; r = \frac{99}{2\sin\alpha} \;\cdots\cdots\cdots\text{(ii)}$$

from equations (i) and (ii)

$$\frac{50}{\alpha} = \frac{99}{2\sin\alpha} \;\Rightarrow\; \frac{100}{\alpha} = \frac{99}{\sin\alpha} \;\Rightarrow\; \frac{99}{100}\alpha = \sin\alpha$$

$\therefore$ $\alpha$ satisfies the equation $\frac{99}{100}x = \sin x$ **(Shown)**

(ii) Let $f(x) = \sin x - \frac{99}{100}x$

Now

$$f(0.1) = \sin(0.1) - \frac{99}{100}(0.1) = 0.00083 \;\; > 0$$

$$f(0.5) = \sin(0.5) - \frac{99}{100}(0.5) = -0.01557 \;\; < 0$$

$\therefore$ the equation has exactly one root between 0.1 and 0.5 **(Verified)**

(iii) $x_{n+1} = 50\sin x_n - 48.5x_n$

Removing the subscripts, we have

$$x = 50\sin x - 48.5x$$

$$\Rightarrow\; x = 50\sin x - \frac{97}{2}x \;\Rightarrow\; 2x = 100\sin x - 97x$$

$$\Rightarrow\; 2x + 97x = 100\sin x \;\Rightarrow\; 99x = 100\sin x \;\Rightarrow\; \frac{99}{100}x = \sin x$$

which is the same equation as in part (i) **(Ans)**

(iv) $x_{n+1} = 50\sin x_n - 48.5x_n$

Given that $x_1 = 0.25$

$$\therefore\; x_2 = 50\sin x_1 - 48.5x_1 = 50\sin(0.25) - 48.5(0.25) = 0.245198$$

$$x_3 = 50\sin(0.245198) - 48.5(0.245198) = 0.245317$$

$$x_4 = 50\sin(0.245317) - 48.5(0.245317) = 0.245318$$

$$x_5 = 50\sin(0.245318) - 48.5(0.245318) = 0.245318$$

$\therefore$ $\alpha = 0.245$ (3 s.f) **(Ans)**

Note:

Change your calculator to the radian mode.

8 (a) Find the two square roots of the complex number $-3+4i$, giving your answers in the form $x+iy$, where $x$ and $y$ are real. [5]

(b) The complex number $z$ is given by

$$z=\frac{-1+3i}{2+i}.$$

(i) Express $z$ in the form $x+iy$, where $x$ and $y$ are real. [2]

(ii) Show on a sketch of an Argand diagram, with origin $O$, the points $A$, $B$ and $C$ representing the complex numbers $-1+3i$, $2+i$ and $z$ respectively. [1]

(iii) State an equation relating the lengths $OA$, $OB$ and $OC$. [1]

***Suggested Solution:***

(a) Let $\sqrt{-3+4i}=x+iy, \quad x, y\in\mathbb{R}$

squaring both sides

$$\left(\sqrt{-3+4i}\right)^2=(x+iy)^2$$

$$-3+4i=x^2+2ixy+(iy)^2$$

$$-3+4i=x^2+2ixy-y^2$$

equating the real and imaginary parts

$$-3=x^2-y^2 \cdots\cdots\cdots\text{(i)}$$

$$4=2xy \cdots\cdots\cdots\text{(ii)}$$

from eq. (ii)

$y=\frac{2}{x}$ putting in eq.(i) gives

$$-3=x^2-\left(\frac{2}{x}\right)^2 \Rightarrow -3=x^2-\frac{4}{x^2} \Rightarrow -3x^2=x^4-4$$

$$\Rightarrow x^4+3x^2-4=0 \Rightarrow x^4+4x^2-x^2-4=0 \Rightarrow x^2(x^2+4)-1(x^2+4)=0$$

$$\Rightarrow (x^2+4)(x^2-1)=0$$

$x^2-1=0$ or $x^2+4=0$ (no real solution)

$\Rightarrow x^2=1$

$\Rightarrow x=\pm 1$

when $x=1,\ y=\frac{2}{1}=2$

$\Rightarrow x+iy=1+2i$

when $x=-1,\ y=\frac{2}{-1}=-2$

$\Rightarrow x+iy=-1-2i$

$\therefore$ the two square roots of $\sqrt{-3+4i}$ are $\pm(1+2i)$ **(Ans)**

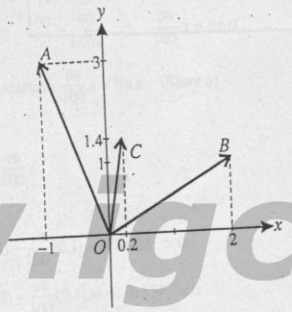
(b) (i) $z = \frac{-1+3i}{2+i}$

realising the denominator

$$z = \frac{-1+3i}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{-2+i+6i-3i^2}{(2)^2-(i)^2} = \frac{-2+7i+3}{4+1} = \frac{1+7i}{5} = \frac{1}{5} + \frac{7}{5}i \quad \textbf{(Ans)}$$

(ii)



(iii) $|OA| = \sqrt{(-1)^2+(3)^2} = \sqrt{1+9} = \sqrt{10}$

$|OB| = \sqrt{(2)^2+(1)^2} = \sqrt{4+1} = \sqrt{5}$

$|OC| = \sqrt{\left(\frac{1}{5}\right)^2+\left(\frac{7}{5}\right)^2} = \sqrt{\frac{1}{25}+\frac{49}{25}} = \sqrt{\frac{50}{25}} = \sqrt{2}$

$\therefore \ |OA| = |OB| \times |OC|$ **(Ans)**

**9.** In an experiment to study the spread of a soil disease, an area of $10\ \text{m}^2$ of soil was exposed to infection. In a simple model, it is assumed that the infected area grows at a rate which is proportional to the product of the infected area and the uninfected area. Initially, $5\ \text{m}^2$ was infected and the rate of growth of the infected area was $0.1\ \text{m}^2$ per day. At time $t$ days after the start of the experiment, an area $a\ \text{m}^2$ is infected and an area $(10-a)\ \text{m}^2$ is uninfected.

(i) Show that $\frac{da}{dt} = 0.004a(10-a)$. [2]

(ii) By first expressing $\frac{1}{a(10-a)}$ in partial fractions, solve this differential equation, obtaining an expression for $t$ in terms of $a$. [6]

(iii) Find the time taken for 90% of the soil area to become infected, according to this model. [2]

*Suggested Solution:*

(i) Given that

$$\frac{da}{dt} \propto a(10-a) \quad \Rightarrow \quad \frac{da}{dt} = k\,a(10-a) \cdots\cdots\cdots \text{(i)}$$

at $\quad t=0,\ a=5,\ \frac{da}{dt} = 0.1\ \text{m}^2/\text{day}$

$\Rightarrow \quad 0.1 = k(5)(10-5) \quad \Rightarrow \quad 0.1 = 25k \quad \Rightarrow \quad k = 0.004$

$\therefore$ equation (i) becomes

$$\frac{da}{dt} = 0.004\,a(10-a) \quad \textbf{(Shown)}$$

(ii) $\dfrac{1}{a(10-a)} \equiv \dfrac{A}{a} + \dfrac{B}{10-a}$

$\Rightarrow \quad 1 = A(10-a) + Ba$

for $a=0$

$$1 = A(10-0) + B(0) \quad \Rightarrow \quad 1 = 10A \quad \Rightarrow \quad A = \frac{1}{10}$$

for $a=10$

$$1 = A(10-10) + B(10) \quad \Rightarrow \quad 1 = 10B \quad \Rightarrow \quad B = \frac{1}{10}$$

$$\therefore \quad \frac{1}{a(10-a)} \equiv \frac{1}{10a} + \frac{1}{10(10-a)} = \frac{1}{10}\left(\frac{1}{a} + \frac{1}{10-a}\right) \cdots\cdots\cdots \text{(i)} \quad \textbf{(Ans)}$$

Now

$$\frac{da}{dt} = 0.004\,a(10-a) \quad \Rightarrow \quad \frac{1}{a(10-a)}\,da = 0.004\,dt$$

integrating both sides

$$\int \frac{1}{a(10-a)}\,da = 0.004\int dt$$

using equation (i)

$$\frac{1}{10}\int\left(\frac{1}{a} + \frac{1}{10-a}\right)da = 0.004\int dt \quad \Rightarrow \quad \frac{1}{10}\int\left(\frac{1}{a} - \frac{-1}{10-a}\right)da = 0.004\int dt$$

$$\Rightarrow \quad \frac{1}{10}[\ln a - \ln(10-a)] = 0.004t + C$$

$$\Rightarrow \quad \frac{1}{10}\ln\left|\frac{a}{10-a}\right| = 0.004t + C$$

when $\quad a=5,\ t=0$

$$\Rightarrow \quad \frac{1}{10}\ln\left|\frac{5}{10-5}\right| = 0.004(0) + C \quad \Rightarrow \quad \frac{1}{10}\ln(1) = 0 + C \quad \Rightarrow \quad C = 0$$

$$\therefore \quad \frac{1}{10}\ln\left|\frac{a}{10-a}\right| = 0.004t \quad \Rightarrow \quad t = \frac{1}{0.04}\ln\left|\frac{a}{10-a}\right|$$

$$\Rightarrow \quad t = 25\ln\left|\frac{a}{10-a}\right| \quad \textbf{(Ans)}$$

(iii) Infected soil area = 90% of $a = \frac{90}{100} \times 10 = 9\text{ m}^2$

$$\therefore \quad t = 25\ln\left|\frac{a}{10-a}\right| = 25\ln\left|\frac{9}{10-9}\right| = 25\ln|9| = 54.931 \approx 55 \text{ days} \quad \textbf{(Ans)}$$

**10.** With respect to the origin $O$, the points $A, B, C, D$ have position vectors given by

$$\overrightarrow{OA} = 4\mathbf{i} + \mathbf{k}, \quad \overrightarrow{OB} = 5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}, \quad \overrightarrow{OC} = \mathbf{i} + \mathbf{j}, \quad \overrightarrow{OD} = -\mathbf{i} - 4\mathbf{k}.$$

(i) Calculate the acute angle between the lines $AB$ and $CD$. [4]

(ii) Prove that the lines $AB$ and $CD$ intersect. [4]

(iii) The point $P$ has position vector $\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$. Show that the perpendicular distance from $P$ to the line $AB$ is equal to $\sqrt{3}$ [4]

*Suggested Solution:*

(i) Direction vectors of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}5\\-2\\-2\end{pmatrix} - \begin{pmatrix}4\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\-3\end{pmatrix} \qquad \overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = \begin{pmatrix}-1\\0\\-4\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}-2\\-1\\-4\end{pmatrix}$$

applying scalar product

$$\overrightarrow{AB} \cdot \overrightarrow{CD} = \left|\overrightarrow{AB}\right| \left|\overrightarrow{CD}\right| \cos\theta$$

$$\Rightarrow \quad \begin{pmatrix}1\\-2\\-3\end{pmatrix} \cdot \begin{pmatrix}-2\\-1\\-4\end{pmatrix} = \left(\sqrt{(1)^2 + (-2)^2 + (-3)^2}\right)\left(\sqrt{(-2)^2 + (-1)^2 + (-4)^2}\right)\cos\theta$$

$$\Rightarrow \quad -2 + 2 + 12 = \left(\sqrt{1+4+9}\right)\left(\sqrt{4+1+16}\right)\cos\theta$$

$$\Rightarrow \quad 12 = \left(\sqrt{14}\right)\left(\sqrt{21}\right)\cos\theta$$

$$\Rightarrow \quad \frac{12}{\sqrt{294}} = \cos\theta \quad \Rightarrow \quad \theta = 45.585 \approx 45.6° \quad \textbf{(Ans)}$$

(ii) Let equation of line $\overline{AB}$ and $\overline{CD}$ be $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively

$$\therefore \quad \mathbf{r}_1 = \overline{OA} + \lambda\overline{AB} \quad \Rightarrow \quad \mathbf{r}_1 = \begin{pmatrix}4\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\-3\end{pmatrix} \quad \Rightarrow \quad \mathbf{r}_1 = \begin{pmatrix}4+\lambda\\-2\lambda\\1-3\lambda\end{pmatrix}$$

$$\text{and } \mathbf{r}_2 = \overline{OC} + \mu\overline{CD} \quad \Rightarrow \quad \mathbf{r}_2 = \begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}-2\\-1\\-4\end{pmatrix} \quad \Rightarrow \quad \mathbf{r}_2 = \begin{pmatrix}1-2\mu\\1-\mu\\-4\mu\end{pmatrix}$$

if the lines intersect, then $\mathbf{r}_1 = \mathbf{r}_2$

$$\Rightarrow \begin{pmatrix} 4+\lambda \\ -2\lambda \\ 1-3\lambda \end{pmatrix} = \begin{pmatrix} 1-2\mu \\ 1-\mu \\ -4\mu \end{pmatrix}$$

equating coefficients of **i** and **j**, we have

$$4+\lambda = 1-2\mu \Rightarrow \lambda = -3-2\mu \cdots\cdots\cdots \text{(i)}$$

$$-2\lambda = 1-\mu \cdots\cdots\cdots \text{(ii)}$$

putting eq. (i) into eq. (ii)

$$-2(-3-2\mu) = 1-\mu \Rightarrow 6+4\mu = 1-\mu \Rightarrow 5\mu = -5 \Rightarrow \mu = -1$$

$$\Leftrightarrow \lambda = -3-2(-1) \Rightarrow \lambda = -1$$

putting these values of $\lambda$ and $\mu$ in coefficients of **k**

line $\overrightarrow{AB}$ $1-3\lambda = 1-3(-1) = 4$

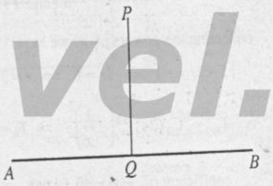
line $\overrightarrow{CD}$ $-4\mu = -4(-1) = 4$

same values, therefore the two lines intersect **(Proved)**

(iii) Given that $\overrightarrow{OP} = \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$

Let $Q$ be the foot of the perpendicular from $P$ to the line $AB$.

Taking point $Q$ as a general point on line $AB$, we have



$$\overrightarrow{OQ} = \begin{pmatrix} 4+\lambda \\ -2\lambda \\ 1-3\lambda \end{pmatrix}$$

$$\therefore \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 4+\lambda \\ -2\lambda \\ 1-3\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 3+\lambda \\ -5-2\lambda \\ -5-3\lambda \end{pmatrix} \cdots\cdots\cdots \text{(i)}$$

as $\overrightarrow{PQ}$ is perpendicular to $\overrightarrow{AB}$, therefore $\overrightarrow{PQ} \cdot \overrightarrow{AB} = 0$

$$\Rightarrow \begin{pmatrix} 3+\lambda \\ -5-2\lambda \\ -5-3\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} = 0 \Rightarrow (3+\lambda)+(-2)(-5-2\lambda)+(-3)(-5-3\lambda) = 0$$

$$\Rightarrow 3+\lambda+10+4\lambda+15+9\lambda = 0 \Rightarrow 14\lambda = -28 \Rightarrow \lambda = -2$$

putting this value of $\lambda$ in eq. (i), we have

$$\overrightarrow{PQ} = \begin{pmatrix} 3-2 \\ -5+4 \\ -5+6 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

now

$$\left|\overrightarrow{PQ}\right| = \sqrt{(1)^2+(-1)^2+(1)^2} = \sqrt{1+1+1} = \sqrt{3} \quad \textbf{(Proved)}$$

# June 2003 Paper 1

## *Pure Mathematics (P1)*

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Find the value of the coefficient of $\frac{1}{x}$ in the expansion of $\left(2x-\frac{1}{x}\right)^5$. [3]

***Suggested Solution:***

using $T_{r+1} = {}^nC_r\,(a)^{n-r}(b)^r$

$$T_{r+1} = {}^5C_r(2x)^{5-r}\left(-\frac{1}{x}\right)^r = {}^5C_r\,2^{5-r}\,x^{5-r}(-x^{-r}) = -{}^5C_r\,2^{5-r}\,x^{5-2r}$$

Expression for general term is:

$T_{r+1} = {}^nC_r\,(a)^{n-r}\,(b)^r$

$\frac{1}{x} = x^{-1}$, therefore $-1$ is the required power of $x$.

collecting the powers of $x$, we have

$$5-2r=-1 \Rightarrow 2r=6 \Rightarrow r=3$$

You need to collect powers of $x$ from general term and equate them to the power of $x$ required.

$$\therefore\ T_{3+1} = {}^5C_3\,(2x)^{5-3}\left(-\frac{1}{x}\right)^3 \Rightarrow T_4 = 10(4x^2)\left(-\frac{1}{x}\right)^3 \Rightarrow T_4 = -40\left(\frac{1}{x}\right)$$

$\therefore$ coefficient of $\frac{1}{x} = -40$ **(Ans).**

**2.** Find all the values of $x$ in the interval $0° \le x \le 180°$ which satisfy the equation [4]

$\sin 3x + 2\cos 3x = 0$ .

***Suggested Solution:***

$$\sin 3x + 2\cos 3x = 0 \ \Rightarrow\ \sin 3x = -2\cos 3x \ \Rightarrow\ \frac{\sin 3x}{\cos 3x} = -2 \ \Rightarrow\ \tan 3x = -2$$

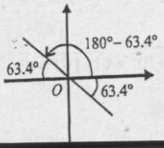
basic angle $\alpha = 63.4°$

given interval for $x$ is $0° \le x \le 180°$, therefore interval for $3x$ is $0° \le 3x \le 540°$.

$\Rightarrow\ 3x = 180° - 63.4°,\ 360° - 63.4°,\ 540° - 63.4°$

$\Rightarrow\ 3x = 116.6°,\ 296.6°,\ 476.6°$

$\Rightarrow\ x = 38.9°, 98.9°, 158.9°$ **(Ans).**



Note that tan is −ve in (II) and (IV) quadrant.

3. (a) Differentiate $4x+\frac{6}{x^2}$ with respect to $x$. [2]

(b) Find $\int\left(4x+\frac{6}{x^2}\right)dx$. [3]

*Suggested Solution:*

(a) Let $y=4x+\frac{6}{x^2} \Rightarrow y=4x+6x^{-2}$

differentiating w.r.t. $x$

$$\frac{dy}{dx}=4-12x^{-3} \Rightarrow \frac{dy}{dx}=4-\frac{12}{x^3} \text{ (Ans).}$$

(b) $\int\left(4x+\frac{6}{x^2}\right)dx$

$$\Rightarrow \int\left(4x+6x^{-2}\right)dx \Rightarrow 4\left(\frac{x^2}{2}\right)+6\left(\frac{x^{-1}}{-1}\right)+K \Rightarrow 2x^2-\frac{6}{x}+K \text{ (Ans).}$$

Note that:

$\frac{d}{dx}x^n=nx^{n-1}$

$\int x^n dx=\frac{x^{n+1}}{n+1}+C$

4. In an arithmetic progression, the first term is $-10$, the 15th term is $11$ and the last term is $41$. Find the sum of all the terms in the progression. [5]

*Suggested Solution:*

Given: $a=-10,\ T_{15}=11,$ last term $T_n=41$

let $d$ be the common difference,

$$\therefore T_{15}=a+(15-1)d \Rightarrow 11=-10+14d \Rightarrow d=\frac{21}{14} \Rightarrow d=\frac{3}{2}$$

to find $n$

$$T_n=a+(n-1)d \Rightarrow 41=-10+(n-1)\left(\frac{3}{2}\right) \Rightarrow 51=\frac{3}{2}(n-1)$$

$$\Rightarrow 102=3n-3 \Rightarrow 3n=105 \Rightarrow n=35$$

now using $S_n=\frac{n}{2}[a+l]$

$$S_{35}=\frac{35}{2}[-10+41] \Rightarrow S_{35}=542\frac{1}{2} \text{ or } 542.5 \text{ (Ans).}$$

In an A.P

$T_n=a+(n-1)d$

where

$a$ = first term
$n$ = no. of terms.
$d$ = common difference
$T_n$ = $n$th term.

Sum of $n$ terms of an A.P is

$S_n=\frac{n}{2}[2a+(n-1)d]$ and

$S_n=\frac{n}{2}[a+l]$

where $l$ is the last term.

*Alternative method:*

$$S_{35}=\frac{n}{2}[2a+(n-1)d] \Rightarrow S_{35}=\frac{35}{2}\left[2(-10)+(35-1)\left(\frac{3}{2}\right)\right]$$

$$\Rightarrow S_{35}=\frac{35}{2}[-20+51] \Rightarrow S_{35}=\frac{35}{2}(31)=542.5 \text{ (Ans).}$$

5. The function $f$ is defined by $f : x \mapsto ax + b$, for $x \in \mathbb{R}$, where $a$ and $b$ are constants. It is given that $f(2) = 1$ and $f(5) = 7$

(i) Find the values of $a$ and $b$. [2]

(ii) Solve the equation $ff(x) = 0$. [3]

***Suggested Solution:***

(i) $f(2) = 2a + b \Rightarrow 1 = 2a + b$ ……… (i)

$f(5) = 5a + b \Rightarrow 7 = 5a + b$ ……… (ii)

from equation (i), $b = 1 - 2a$, substituting in equation (ii)

$7 = 5a + (1 - 2a) \Rightarrow 5a - 2a = 6 \Rightarrow a = 2$

$\Rightarrow b = 1 - 2a \Rightarrow b = 1 - 2(2) \Rightarrow b = -3$

$\therefore a = 2$ and $b = -3$ **(Ans).**

(ii) from part (i), $f(x) = 2x - 3$

$ff(x) = 0$

$\Rightarrow f(2x - 3) = 0 \Rightarrow 2(2x - 3) - 3 = 0 \Rightarrow 4x - 9 = 0 \Rightarrow x = \frac{9}{4}$ **(Ans).**

**6.** (i) Sketch the graph of the curve $y = 3 \sin x$, for $-\pi \le x \le \pi$. [2]

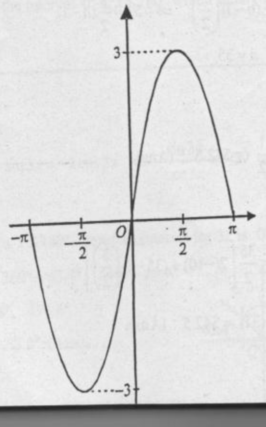
The straight line $y = kx$, where $k$ is a constant, passes through the maximum point of this curve for $-\pi \le x \le \pi$.

(ii) Find the value of $k$ in terms of $\pi$. [2]

(iii) State the coordinates of the other point, apart from the origin, where the line and the curve intersect. [1]

***Suggested Solution:***

(i)



(ii) coordinates of maximum point on curve is $\left(\frac{\pi}{2}, 3\right)$.

Line $y = kx$ passes through the maximum point of the curve, therefore maximum point satisfies the equation $y = kx$.

$\Rightarrow 3 = k\left(\frac{\pi}{2}\right) \Rightarrow k = \frac{6}{\pi}$ **(Ans).**

(iii) Coordinates of the other point apart from the origin is the minimum point of this curve

$\therefore$ required point is $\left(-\frac{\pi}{2}, -3\right)$. **(Ans).**

7. The line $L_1$ has equation $2x + y = 8$. The line $L_2$ passes through the point $A(7, 4)$ and is perpendicular to $L_1$.
   - (i) Find the equation of $L_2$. [4]
   - (ii) Given that the lines $L_1$ and $L_2$ intersect at the point $B$, find the length of $AB$. [4]

*Suggested Solution:*

(i) eq. of $L_1$: $2x + y = 8 \Rightarrow y = -2x + 8$ ……… (i)

$\therefore$ gradient of $L_1 = -2$

as $L_2$ is perpendicular to $L_1$ and passing through $A(7, 4)$

$\Rightarrow$ gradient of $L_2 = \frac{1}{2}$

eq. of $L_2$: $y - 4 = \frac{1}{2}(x - 7) \Rightarrow 2y - 8 = x - 7 \Rightarrow 2y - x - 1 = 0$ **(Ans).**

Since lines $L_1$ and $L_2$ are perpendicular to each other. So (slope of $L_1$) (slope of $L_2$) $= -1$

i.e. $m_1 \times m_2 = -1$.

Equation of a straight line with gradient $m$ and passing through $(x_1, y_1)$ is $(y - y_1) = m(x - x_1)$.

(ii) from part (i) Equation of $L_2$: $2y - x - 1 = 0$ ……… (ii)

from eq.(i), $y = -2x + 8$, put in eq. (ii)

$2(-2x + 8) - x - 1 = 0 \Rightarrow -5x + 15 = 0 \Rightarrow x = 3$

$\Rightarrow y = -2x + 8 \Rightarrow y = -2(3) + 8 \Rightarrow y = 2$

$\therefore$ point $B$ is $(3, 2)$

$\therefore$ length of $AB$: $|AB| = \sqrt{(3-7)^2 + (2-4)^2} = \sqrt{16+4} = \sqrt{20} = 4.47$ **(Ans).**

Distance between two points, $A(x_1, y_1)$ and $B(x_2, y_2)$ is:

$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

8. The points $A$, $B$, $C$ and $D$ have position vectors $3\mathbf{i} + 2\mathbf{k}$, $2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $2\mathbf{j} + 7\mathbf{k}$ and $-2\mathbf{i} + 10\mathbf{j} + 7\mathbf{k}$ respectively.
   - (i) Use a scalar product to show that $BA$ and $BC$ are perpendicular. [4]
   - (ii) Show that $BC$ and $AD$ are parallel and find the ratio of the length of $BC$ to the length of $AD$. [4]

*Suggested Solution:*

(i) Given: $\overrightarrow{OA}=\begin{pmatrix}3\\0\\2\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}2\\-2\\5\end{pmatrix}$, $\overrightarrow{OC}=\begin{pmatrix}0\\2\\7\end{pmatrix}$, and $\overrightarrow{OD}=\begin{pmatrix}-2\\10\\7\end{pmatrix}$

$$\overrightarrow{BA}=\overrightarrow{OA}-\overrightarrow{OB}=\begin{pmatrix}3\\0\\2\end{pmatrix}-\begin{pmatrix}2\\-2\\5\end{pmatrix}=\begin{pmatrix}3-2\\0+2\\2-5\end{pmatrix}=\begin{pmatrix}1\\2\\-3\end{pmatrix}$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\begin{pmatrix}0\\2\\7\end{pmatrix}-\begin{pmatrix}2\\-2\\5\end{pmatrix}=\begin{pmatrix}0-2\\2+2\\7-5\end{pmatrix}=\begin{pmatrix}-2\\4\\2\end{pmatrix}$$

Condition for two vectors $\overline{AB}$ and $\overline{CD}$ to be perpendicular is that $\overline{AB}\bullet\overline{CD}=0$

now $\overrightarrow{BA}\bullet\overrightarrow{BC}=\begin{pmatrix}1\\2\\-3\end{pmatrix}\bullet\begin{pmatrix}-2\\4\\2\end{pmatrix}=1(-2)+2(4)+(-3)(2)=-2+8-6=0$

$\therefore$ $\overrightarrow{BA}\perp\overrightarrow{BC}$ **(shown).**

(ii) $\overrightarrow{BC}=\begin{pmatrix}-2\\4\\2\end{pmatrix}\Rightarrow\overrightarrow{BC}=2\begin{pmatrix}-1\\2\\1\end{pmatrix}\Rightarrow\frac{1}{2}\overrightarrow{BC}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$ ......... (i)

$$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=\begin{pmatrix}-2\\10\\7\end{pmatrix}-\begin{pmatrix}3\\0\\2\end{pmatrix}=\begin{pmatrix}-5\\10\\5\end{pmatrix}$$

$\Rightarrow\overrightarrow{AD}=5\begin{pmatrix}-1\\2\\1\end{pmatrix}\Rightarrow\frac{1}{5}\overrightarrow{AD}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$ ......... (ii)

If vector $\overline{AB}$ is parallel to vector $\overline{CD}$, then there exists a constant $k$, such that $\overline{AB}=k\overline{CD}$

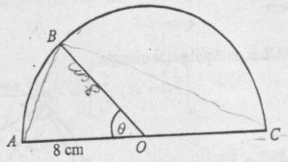
from (i) and (ii)

$\frac{1}{2}\overrightarrow{BC}=\frac{1}{5}\overrightarrow{AD}\Rightarrow\overrightarrow{BC}=\frac{2}{5}\overrightarrow{AD}\Rightarrow\overrightarrow{BC}\parallel\overrightarrow{AD}$ **(shown).**

also $\frac{BC}{AD}=\frac{2}{5}$

$\therefore$ ratio $BC:AD=2:5$ **(Ans).**

**9.**



The diagram shows a semicircle $ABC$ with centre $O$ and radius 8 cm. Angle $AOB = \theta$ radians.

(i) In the case where $\theta = 1$, calculate the area of the sector $BOC$, [3]

(ii) Find the value of $\theta$ for which the perimeter of sector $AOB$ is one half of the perimeter of sector $BOC$. [3]

(iii) In the case where $\theta = \frac{1}{3}\pi$, show that the exact length of the perimeter of triangle $ABC$ is $\left(24+8\sqrt{3}\right)$ cm. [3]

***Suggested Solution:***

(i) Using area $= \frac{1}{2}r^2\theta$

area of sector $BOC = \frac{1}{2}r^2(\pi-\theta) = \frac{1}{2}(8)^2(\pi-\theta)$

when $\theta = 1$ radian

area of sector $BOC = \frac{1}{2}(8)^2(\pi-1) = 32(\pi-1) = 68.5\text{ cm}^2$ **(Ans).**

When the angle is in radians then,

arc length $S = r\theta$

area of sector $= \frac{1}{2}r^2\theta$

(ii) arc length $AOB = r\theta = 8\theta$

arc length $BOC = r\theta = 8(\pi-\theta)$

According to given condition:

perimeter of sector $AOB = \frac{1}{2}$ perimeter of sector $BOC$

$$OA+OB+\overset{\frown}{AB} = \frac{1}{2}\left(OC+OB+\overset{\frown}{BC}\right)$$

$$(8+8+8\theta) = \frac{1}{2}(8+8+8(\pi-\theta))$$

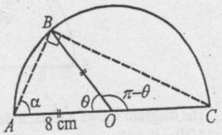
$$16+8\theta = 8+4\pi-4\theta$$

$$12\theta = 4\pi-8$$

$$\theta = \frac{4\pi-8}{12} = \frac{\pi-2}{3} = 0.381\text{ radian}$$ **(Ans).**

(iii) triangle $AOB$ is isosceles as $OA = OB = 8$ cm

when $\theta = \frac{\pi}{3}$, then $\alpha = \frac{\pi}{3}$, therefore $\Delta OAB$ is an equilateral triangle.



therefore $AB = 8$ cm

using Pythagoras theorem in triangle ABC

$$BC=\sqrt{16^2-8^2}=\sqrt{256-64}=\sqrt{192}=8\sqrt{3}$$

$\therefore$ perimeter of $\Delta ABC = 8+8\sqrt{3}+16 = (24+8\sqrt{3})$ cm. **(Shown)**

**10.** The equation of a curve is $y=\sqrt{(5x+4)}$.

(i) Calculate the gradient of the curve at the point where $x = 1$. [3]

(ii) A point with coordinates $(x, y)$ moves along the curve in such a way that the rate of increase of $x$ has the constant value 0.03 units per second. Find the rate of increase of $y$ at the instant when $x = 1$. [2]

(iii) Find the area enclosed by the curve, the $x$-axis, the $y$-axis and the line $x = 1$ [5]

***Suggested Solution:***

(i) $y=\sqrt{5x+4} \Rightarrow y=(5x+4)^{\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{1}{2}(5x+4)^{-\frac{1}{2}}\times 5 \Rightarrow \frac{dy}{dx}=\frac{5}{2\sqrt{5x+4}}$$

If $y=(ax+b)^n$, then $\frac{dy}{dx}=n(ax+b)^{n-1}(a)$

when $x=1$

$$\left.\frac{dy}{dx}\right|_{x=1}=\frac{5}{2\sqrt{5(1)+4}}=\frac{5}{2\sqrt{9}}=\frac{5}{6} \quad \textbf{(Ans).}$$

(ii) Given: $\frac{dx}{dt}=0.03$

from part (i), gradient of the curve when $x = 1$ is $\frac{dy}{dx}=\frac{5}{6}$

applying chain rule, we have

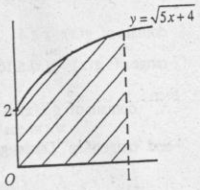
$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt} \Rightarrow \frac{dy}{dt}=\frac{5}{6}\times 0.03 \Rightarrow \frac{dy}{dt}=0.025 \text{ units/sec} \quad \textbf{(Ans).}$$

(iii) area $A = \int_0^1 y\,dx = \int_0^1 (5x+4)^{\frac{1}{2}}\,dx$

$$= \left[\frac{(5x+4)^{\frac{3}{2}}}{5\left(\frac{3}{2}\right)}\right]_0^1 = \left[\frac{2}{15}\ (5x+4)^{\frac{3}{2}}\right]_0^1$$

$$= \left[\frac{2}{15}\ (5(1)+4)^{\frac{3}{2}}\right] - \left[\frac{2}{15}(5(0)+4)^{\frac{3}{2}}\right] = \left[\frac{2}{15}(9)^{\frac{3}{2}}\right] - \left[\frac{2}{15}(4)^{\frac{3}{2}}\right]$$

$$= \frac{54}{15} - \frac{16}{15} = \frac{38}{15} = 2.53 \text{ unit}^2 \quad \textbf{(Ans).}$$



11. The equation of a curve is $y = 8x - x^2$.

(i) Express $8x - x^2$ in the form $a-(x+b)^2$, stating the numerical values of $a$ and $b$. [3]

(ii) Hence, or otherwise, find the coordinates of the stationary point of the curve. [2]

(iii) Find the set of values of $x$ for which $y \geq -20$. [3]

The function g is defined by $g : x \mapsto 8x - x^2$, for $x \geq 4$.

(iv) State the domain and range of $g^{-1}$. [2]

(v) Find an expression, in terms of $x$, for $g^{-1}(x)$. [3]

*Suggested Solution:*

If $y = ax^2 + bx + c$ is written in form

$y = a(x-h)^2 + k$,

than co-ordinates of turning points are $(h, k)$.

(i) $y = 8x - x^2 \Rightarrow y = -(x^2 - 8x) \Rightarrow y = -[x^2 - 8x + (4)^2 - (4)^2]$

$\Rightarrow y = -\left[(x-4)^2 - 16\right] \Rightarrow y = 16 - (x-4)^2$

$\therefore\ a = 16$, and $b = -4$ **(Ans).**

(ii) From part **(i)**, we see that the equation of curve can be written as:

$y = 16 - (x-4)^2$

$\Rightarrow$ stationary value exists at $x = 4$ and corresponding value of $y$ is 16.

hence the stationary point is (4, 16). **(Ans).**

(iii) $y \geq -20$

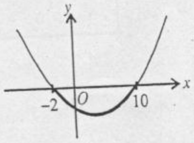
$\Rightarrow 16 - (x-4)^2 \geq -20 \Rightarrow -(x-4)^2 \geq -36 \Rightarrow (x-4)^2 \leq 36$

$\Rightarrow (x-4)^2 - 36 \leq 0 \Rightarrow (x-4)^2 - (6)^2 \leq 0$

$\Rightarrow [(x-4)+6]\,[(x-4)-6] \leq 0 \Rightarrow (x+2)(x-10) \leq 0$

using sketch method

$x \geq -2$ and $x \leq 10$, or $-2 \leq x \leq 10$ **(Ans)**



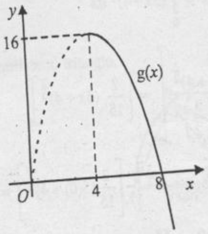
(iv) $g : x \mapsto 8x - x^2$, for $x \geq 4$

domain of $g(x)$: $x \geq 4$

range of $g(x)$: $g(x) \leq 16$

$\therefore$ domain of $g^{-1}(x)$ is $x \leq 16$ (Ans).

and rangeof $g^{-1}(x)$ is $g^{-1}(x) \geq 4$ (Ans).



Domain of a function is the range of the inverse of the function and vice versa.

(v) Let $g(x) = y \Rightarrow y = 8x - x^2 \Rightarrow y = 16 - (x-4)^2$

$\Rightarrow (x-4)^2 = 16 - y \Rightarrow x - 4 = \pm\sqrt{16-y}$

keeping the +ve sign because the given function $g(x)$ is to the right hand side of axis of symmetry

$x = 4 + \sqrt{16-y}$

as $g(x) = y \Rightarrow g^{-1}(y) = x$

$\therefore\ g^{-1}(y) = 4 + \sqrt{16-y}$

$\Rightarrow\ g^{-1}(x) = 4 + \sqrt{16-x}$ (Ans).

# June 2003 Paper 3

## *Pure Mathematics (P3)*

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** (i) Show that the equation

$$\sin(x-60°)-\cos(30°-x) = 1$$

can be written in the form $\cos x = k$, where $k$ is a constant. [2]

(ii) Hence solve the equation, for $0^0 < x < 180^0$. [2]

***Suggested Solution:***

(i) $\sin(x-60°)-\cos(30°-x) = 1$

$\Rightarrow (\sin x\cos 60-\cos x\sin 60)-(\cos 30\cos x+\sin 30\sin x)=1$

$\Rightarrow \left(\frac{1}{2}\sin x-\frac{\sqrt{3}}{2}\cos x\right)-\left(\frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x\right)=1$

$\Rightarrow \frac{1}{2}\sin x-\frac{\sqrt{3}}{2}\cos x-\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x=1$

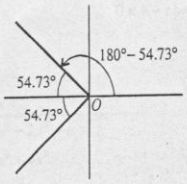
$\Rightarrow -2\frac{\sqrt{3}}{2}\cos x=1 \Rightarrow -\sqrt{3}\cos x=1 \Rightarrow \cos x=-\frac{1}{\sqrt{3}}$ (Ans)

(ii) From part (i), $\cos x=-\frac{1}{\sqrt{3}}$

Basic angle $\alpha = 54.73°$

As given range is $0° < x < 180°$

$\therefore \quad x = 180°-54.73° = 125.27° \approx 125.3°$ (Ans)



Note that:
cos$x$ is negative in 2nd and 3rd quadrant but as the given range is 0<x<180, therefore angle in the 2nd quadrant is considered.

**2.** Find the exact value of $\int_0^1 xe^{2x}\,dx$ [4]

***Suggested Solution:***

$$\int_0^1 x e^{2x}\,dx$$

using integration by parts

$$= \left[x\frac{e^{2x}}{2}\right]_0^1 - \int_0^1 \frac{e^{2x}}{2}\times\frac{d}{dx}(x)\,dx \;=\; \frac{1}{2}\left[xe^{2x}\right]_0^1 - \frac{1}{2}\int_0^1 e^{2x}\,dx$$

$$= \frac{1}{2}\left[xe^{2x}\right]_0^1 - \frac{1}{2}\left[\frac{e^{2x}}{2}\right]_0^1 \;=\; \frac{1}{2}\left[xe^{2x}\right]_0^1 - \frac{1}{4}\left[e^{2x}\right]_0^1$$

$$= \frac{1}{2}\left[\left((1)e^{2(1)}\right)-\left((0)e^{2(0)}\right)\right] - \frac{1}{4}\left[\left(e^{2(1)}\right)-\left(e^{2(0)}\right)\right]$$

$$= \frac{1}{2}\left(e^2-0\right)-\frac{1}{4}\left(e^2-1\right) \;=\; \frac{1}{2}e^2-\frac{1}{4}e^2+\frac{1}{4} \;=\; \frac{1}{4}e^2+\frac{1}{4}$$

$$= \frac{1}{4}(e^2+1) \quad \textbf{(Ans)}$$

The question requires the answer in exact form. So do not give your answer in decimals

3 Solve the inequality $|x-2|<3-2x$ [4]

***Suggested Solution :***

$$|x-2|<3-2x$$

squarring both sides

$$(|x-2|)^2<(3-2x)^2$$

$$x^2-4x+4<9-12x+4x^2$$

$$9-12x+4x^2-x^2+4x-4>0$$

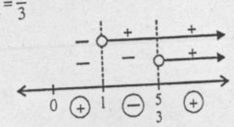
$$3x^2-8x+5>0$$

$$3x^2-3x-5x+5>0$$

$$3x(x-1)-5(x-1)>0$$

$$(x-1)(3x-5)>0$$

critical values are $x=1, \quad x=\frac{5}{3}$



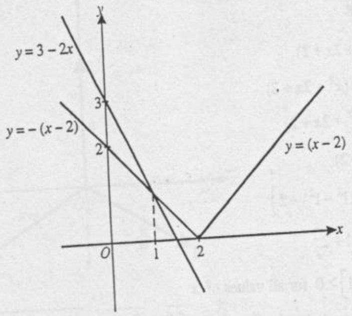
$\therefore \quad x<1, \quad x>\frac{5}{3}$

In the process of squaring both sides of the equation or inequalities, we are likely to get extra roots which are not the actual roots of the equation or inequality. The process of verification therefore is important.

On verification we find that $x>\frac{5}{3}$ does not satify the inequality.

$\therefore \quad x<1$ **(Ans)**

*Alternative Solution :*



By sketching the graphs of $y=|x-2|$ and $y=3-2x$, we see that there is only one point of intersection between the lines $y=-(x-2)$ and $y=3-2x$

$\therefore\quad -(x-2)=3-2x$

$\Rightarrow\quad -x+2=3-2x \quad\Rightarrow\quad x=1$

we see that graph of $y=|x-2|$ is below the graph of $y=3-2x$ for those values of $x$ which are less than 1

$\therefore$ required solution is: $x<1$ **(Ans)**

4. The polynomial $x^4-2x^3-2x^2+a$ is denoted by $f(x)$. It is given that $f(x)$ is divisible by $x^2-4x+4$.
   (i) Find the value of $a$. [3]
   (ii) When $a$ has this value, show that $f(x)$ is never negative. [4]

***Suggested Solution:***

(i) $f(x)=x^4-2x^3-2x^2+a$

$$\begin{array}{r|l}
 & x^2+2x+2 \\ \hline
x^2-4x+4 & x^4-2x^3-2x^2+a \\
 & x^4-4x^3+4x^2 \\
 & - \quad + \quad - \\ \cline{2-2}
 & \quad 2x^3-6x^2+a \\
 & \quad 2x^3-8x^2 \quad +8x \\
 & \quad - \quad + \quad\quad - \\ \cline{2-2}
 & \quad\quad 2x^2-8x+a \\
 & \quad\quad 2x^2-8x+8 \\
 & \quad\quad - \quad + \quad - \\ \cline{2-2}
 & \quad\quad\quad a-8
\end{array}$$

given that $(x^2-4x+4)$ is a factor of $f(x)$

$\therefore$ Remainder $=0$

$\Rightarrow\quad a-8=0 \quad\Rightarrow\quad a=8$ (Ans)

(ii) $f(x)=x^4-2x^3-2x^2+8$

$$=(x^2-4x+4)(x^2+2x+2)$$

$$=(x^2-2x-2x+4)(x^2+2x+2)$$

$$=\left[(x-2)(x-2)\right](x^2+2x+2)$$

$$=(x-2)^2(x^2+2x+2)$$

$$=(x-2)^2\left[x^2+2x+1^2-1^2)+2\right]$$

$$=(x-2)^2\left[(x+1)^2-1+2\right]$$

$$=(x-2)^2\left[(x+1)^2+1\right]\geq 0 \text{ for all values of } x$$

$\therefore$ $f(x)$ is never negative **(Ans)**

**5.** The complex number $2i$ is denoted by $u$. The complex number with modulus 1 and argument $\frac{2}{3}\pi$ is denoted by $w$.

(i) Find in the form $x+iy$, where $x$ and $y$ are real, the complex numbers $w$, $uw$ and $\frac{u}{w}$. [4]

(ii) Sketch an Argand diagram showing the points $U$, $A$ and $B$ representing the complex numbers $u$, $uw$ and $\frac{u}{w}$ respectively. [2]

(iii) Prove that triangle $UAB$ is equilateral. [2]

***Suggested Solution:***

(i) Given that

$$|w|=1 \quad \text{and} \quad \arg(w)=\frac{2}{3}\pi$$

$$\therefore \quad w=1\left[\cos\left(\frac{2}{3}\pi\right)+i\sin\left(\frac{2}{3}\pi\right)\right]=-\frac{1}{2}+i\frac{\sqrt{3}}{2} \quad \textbf{(Ans)}$$

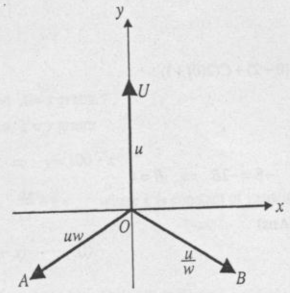
Any complex Number $z=x+iy$ can be expressed in its polar form $z=r(\cos\theta+i\sin\theta)$, where $|z|=r$, $\arg z=\theta$ where $-\pi<\theta\leq\pi$

$$uw=(2i)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-i+i^2\sqrt{3}=-\sqrt{3}-i \quad \textbf{(Ans)}$$

$$\frac{u}{w}=\frac{2i}{-\frac{1}{2}+i\frac{\sqrt{3}}{2}}=\frac{4i}{-1+i\sqrt{3}}\times\frac{(-1-i\sqrt{3})}{(-1-i\sqrt{3})}=\frac{-4i-4i^2\sqrt{3}}{(-1)^2-(i\sqrt{3})^2}=\frac{-4i+4\sqrt{3}}{1+3}$$

$$=\frac{4\sqrt{3}-4i}{4}=\sqrt{3}-i \quad \textbf{(Ans)}$$

(ii)



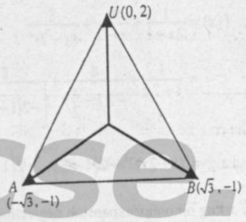
(iii) $|UA| = \sqrt{(-\sqrt{3}-0)^2 + (-1-2)^2} = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$

$|AB| = \sqrt{(\sqrt{3}+\sqrt{3})^2 + (-1+1)^2} = \sqrt{(2\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$

$|UB| = \sqrt{(\sqrt{3}-0)^2 + (-1-2)^2} = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$

$\Rightarrow \quad |UA| = |AB| = |UB|$

$\therefore \quad UAB$ is an equilateral triangle **(Proved)**



6. Let $f(x) = \dfrac{9x^2+4}{(2x+1)(x-2)^2}$.

(i) Express $f(x)$ in partial fractions. [5]

(ii) Show that, when $x$ is sufficiently small for $x^3$ and higher powers to be neglected,

$$f(x) = 1 - x + 5x^2$$ [4]

*Suggested Solution:*

(i) $f(x) = \dfrac{9x^2+4}{(2x+1)(x-2)^2} \equiv \dfrac{A}{2x+1} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2}$

$\Rightarrow \quad 9x^2 + 4 \equiv A(x-2)^2 + B(2x+1)(x-2) + C(2x+1)$

for $x = 2$

$9(2)^2 + 4 = A(2-2)^2 + B(2(2)+1)(2-2) + C(2(2)+1)$

$\Rightarrow \quad 40 = 5C \quad \Rightarrow \quad C = 8$

for $x = -\frac{1}{2}$

$9(-\frac{1}{2})^2 + 4 = A(-\frac{1}{2}-2)^2 + B(2(-\frac{1}{2})+1)(-\frac{1}{2}-2) + C(2(-\frac{1}{2})+1)$

$\Rightarrow \quad \frac{9}{4} + 4 = A(-\frac{5}{2})^2 \quad \Rightarrow \quad \frac{25}{4} = A(\frac{25}{4}) \quad \Rightarrow \quad A = 1$

for $x=0$

$$9(0)^2+4=A(0-2)^2+B(2(0)+1)(0-2)+C(2(0)+1)$$

$$\Rightarrow \quad 4=4A-2B+C$$

putting values of $A$ and $C$, we have

$$4=4(1)-2B+8 \;\Rightarrow\; 4=12-2B \;\Rightarrow\; -8=-2B \;\Rightarrow\; B=4$$

$$\therefore \quad f(x)=\frac{1}{2x+1}+\frac{4}{x-2}+\frac{8}{(x-2)^2} \quad \textbf{(Ans)}$$

(ii) We have

$$f(x)=\frac{1}{2x+1}+\frac{4}{x-2}+\frac{8}{(x-2)^2}$$

$$=\frac{1}{1+2x}+\frac{4}{-2(1-\frac{x}{2})}+\frac{8}{\left[-2(1-\frac{x}{2})\right]^2}=\frac{1}{1+2x}-\frac{2}{1-\frac{x}{2}}+\frac{2}{(1-\frac{x}{2})^2}$$

$$=(1+2x)^{-1}-2\left(1-\frac{x}{2}\right)^{-1}+2\left(1-\frac{x}{2}\right)^{-2}$$

using binomial expansion neglecting $x^3$ and higher powers

$$=\left[1-2x+\frac{(-1)(-2)}{2!}(2x)^2\right]-2\left[1+\frac{x}{2}+\frac{(-1)(-2)}{2!}(-\frac{x}{2})^2\right]$$
$$+2\left[1+2(\frac{x}{2})+\frac{(-2)(-3)}{2!}(-\frac{x}{2})^2\right]$$

Binomial Expansion:

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

$$=\left(1-2x+4x^2\right)-2\left(1+\frac{x}{2}+\frac{x^2}{4}\right)+2\left(1+x+\frac{3}{4}x^2\right)$$

$$=1-2x+4x^2-2-x-\frac{1}{2}x^2+2+2x+\frac{3}{2}x^2$$

$$=1-2x+4x^2-2-x-\frac{1}{2}x^2+2+2x+\frac{3}{2}x^2=1-x+5x^2$$

$$\therefore \quad f(x)=1-x+5x^2 \quad \textbf{(Shown)}$$

**7.** In a chemical reaction a compound $X$ is formed from a compound $Y$. The masses in grams of $X$ and $Y$ present at time $t$ seconds after the start of the reaction are $x$ and $y$ respectively. The sum of the two masses is equal to 100 grams throughout the reaction. At any time, the rate of formation of $X$ is proportional to the mass of $Y$ at that time. When $t=0$, $x=5$ and $\frac{dx}{dt}=1.9$.

(i) Show that $x$ satisfies the differential equation.

$$\frac{dx}{dt}=0.02(100-x).$$ [2]

(ii) Solve this differential equation, obtaining an expression for $x$ in terms of $t$. [6]

(iii) State what happens to the value of $x$ as $t$ becomes very large. [1]

*Suggested Solution:*

(i) Given that

mass of compound $X = x$ grams
mass of compound $Y = y$ grams

$$x + y = 100 \quad \Rightarrow \quad y = 100 - x$$

$$\frac{dx}{dt} \propto y \quad \Rightarrow \quad \frac{dx}{dt} = ky, \quad \text{where } k \text{ is constant of variation}$$

$$\Rightarrow \quad \frac{dx}{dt} = k(100 - x) \cdots\cdots\cdots \text{(i)}$$

when $x = 5$, $\frac{dx}{dt} = 1.9$, putting in equation (i)

$$1.9 = k(100 - 5) \quad \Rightarrow \quad 1.9 = 95k \quad \Rightarrow \quad k = 0.02$$

$\therefore$ equation (i) becomes

$$\frac{dx}{dt} = 0.02(100 - x) \quad \textbf{(Shown)}$$

(ii) $\frac{dx}{dt} = 0.02(100 - x) \quad \Rightarrow \quad \frac{1}{100 - x}\,dx = 0.02\,dt$

integrating both sides

$$\int \frac{1}{100 - x}\,dx = \int 0.02\,dt \quad \Rightarrow \quad -\ln|100 - x| = 0.02t + C$$

when $t = 0$, $x = 5$

$$\therefore \quad -\ln|100 - 5| = 0.02(0) + C \quad \Rightarrow \quad C = -\ln 95$$

$$\Rightarrow \quad -\ln|100 - x| = 0.02t - \ln 95 \quad \Rightarrow \quad \ln 95 - \ln|100 - x| = 0.02t$$

$$\Rightarrow \quad \ln\left|\frac{95}{100 - x}\right| = 0.02t \quad \Rightarrow \quad \ln\left|\frac{95}{100 - x}\right| = (0.02t)\ln e$$

$$\Rightarrow \quad \ln\left|\frac{95}{100 - x}\right| = \ln e^{0.02t} \quad \Rightarrow \quad \frac{95}{100 - x} = e^{0.02t} \quad \Rightarrow \quad \frac{95}{e^{0.02t}} = 100 - x$$

$$\Rightarrow \quad x = 100 - 95e^{-0.02t} \quad \textbf{(Ans)}$$

$(100 - x)$ may be negative for some values of $x$. Therefore use of modulus sign is important because log of negative number is not defined.

(iii) From part **(ii)**, we have

$$x = 100 - 95e^{-0.02t}$$

When $t$ becomes very large, then

$$t \to \infty$$

$$\Rightarrow \quad 95e^{-0.02t} \to 0$$

$$\therefore \quad x \to 100$$

i.e. when $t$ becomes large, $x$ approaches 100 grams.

This means the whole mass of compound $Y$ is transformed into mass of $X$.

8. The equation of a curve is $y = \ln x + \frac{2}{x}$, where $x > 0$.

(i) Find the coordinates of the stationary point of the curve and determine whether it is a maximum or a minimum point. [5]

(ii) The sequence of values given by the iterative formula.

$$x_{n+1} = \frac{2}{3 - \ln x_n},$$

with initial value $x_1 = 1$, converges to $\alpha$. State an equation satisfied by $\alpha$, and hence show that $\alpha$ is the $x$-coordinate of a point on the curve where $y = 3$. [2]

(iii) Use this iterative formula to find $\alpha$ correct to 2 decimal places, showing the result of each iteration. [3]

***Suggested Solution:***

(i) $y = \ln x + \frac{2}{x} \Rightarrow y = \ln x + 2x^{-1}$

differentiating w.r.t. $x$

$$\frac{dy}{dx} = \frac{1}{x} + (-2x^{-2}) \Rightarrow \frac{dy}{dx} = \frac{1}{x} - \frac{2}{x^2}$$

for stationary values $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{1}{x} - \frac{2}{x^2} = 0 \Rightarrow \frac{1}{x} = \frac{2}{x^2} \Rightarrow x^2 = 2x \Rightarrow x^2 - 2x = 0$$

$$\Rightarrow x(x-2) = 0 \Rightarrow x = 0, \quad x = 2$$

when $x = 0$, $y$ is undefined. Therefore no stationary point exists.

when $x = 2$,

$$y = \ln 2 + \frac{2}{2} \Rightarrow y = \ln 2 + 1$$

$\therefore$ $(2, \ln 2 + 1)$ is a stationary point. **(Ans)**

Now

$$\frac{dy}{dx} = \frac{1}{x} - \frac{2}{x^2} \Rightarrow \frac{dy}{dx} = x^{-1} - 2x^{-2}$$

$$\therefore \frac{d^2y}{dx^2} = -x^{-2} + 4x^{-3} \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{x^2} + \frac{4}{x^3}$$

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} > 0$, $y$ is min

at $x = 2$

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} < 0$, $y$ is max

$$\frac{d^2y}{dx^2} = -\frac{1}{2^2} + \frac{4}{2^3} = -\frac{1}{4} + \frac{4}{8} = \frac{1}{4} > 0$$

$\therefore$ $(2, \ln 2 + 1)$ is a minimum point. **(Ans)**

(ii) $x_{n+1} = \dfrac{2}{3 - \ln x_n}$,

removing the subscripts, we have

$$x = \frac{2}{3 - \ln x}$$

$$\Rightarrow \quad 3 - \ln x = \frac{2}{x}$$

$$\Rightarrow \quad 3 = \ln x + \frac{2}{x}$$

which is the given equation of the curve with $y$ replaced by 3. **(Ans)**

(iii) $x_{n+1} = \dfrac{2}{3 - \ln x_n}$

Given that initial value $x_1 = 1$

$$\therefore \quad x_2 = \frac{2}{3 - \ln x_1} = \frac{2}{3 - \ln(1)} = \frac{2}{3} = 0.666666$$

$$x_3 = \frac{2}{3 - \ln x_2} = \frac{2}{3 - \ln(0.666666)} = 0.587291$$

$$x_4 = \frac{2}{3 - \ln x_3} = \frac{2}{3 - \ln(0.587291)} = 0.566214$$

$$x_5 = \frac{2}{3 - \ln x_4} = \frac{2}{3 - \ln(0.566214)} = 0.560415$$

$$x_6 = \frac{2}{3 - \ln x_5} = \frac{2}{3 - \ln(0.560415)} = 0.558803$$

$$x_7 = \frac{2}{3 - \ln x_6} = \frac{2}{3 - \ln(0.558803)} = 0.558354$$

$\therefore \quad \alpha = 0.56$ (2 dec.pl) **(Ans)**

**9.** Two planes have equation $x + 2y - 2z = 2$ and $2x - 3y + 6z = 3$. The planes intersect in the straight line $l$.

(i) Calculate the acute angle between the two planes. [4]

(ii) Find a vector equation for the line $l$. [6]

*Suggested Solution:*

(i) Plane I: $x+2y-2z=2 \Rightarrow \mathbf{r}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}=2,$ normal vector to I: $\mathbf{n}_1=\begin{pmatrix}1\\2\\-2\end{pmatrix}$

Plane II: $2x-3y+6z=3 \Rightarrow \mathbf{r}\cdot\begin{pmatrix}2\\-3\\6\end{pmatrix}=3,$ normal vector to II: $\mathbf{n}_2=\begin{pmatrix}2\\-3\\6\end{pmatrix}$

Angle between two planes is the angle between their respective normal vectors.

$$\therefore \quad \cos\theta=\hat{\mathbf{n}}_1.\hat{\mathbf{n}}_2 \Rightarrow \cos\theta=\frac{\mathbf{n}_1}{|\mathbf{n}_1|}\cdot\frac{\mathbf{n}_2}{|\mathbf{n}_2|} \Rightarrow |\mathbf{n}_1||\mathbf{n}_2|\cos\theta=\mathbf{n}_1.\mathbf{n}_2$$

$$\Rightarrow \left|\sqrt{(1)^2+(2)^2+(-2)^2}\right|\left|\sqrt{(2)^2+(-3)^2+(6)^2}\right|\cos\theta=\begin{pmatrix}1\\2\\-2\end{pmatrix}.\begin{pmatrix}2\\-3\\6\end{pmatrix}$$

$$\Rightarrow \left|\sqrt{1+4+4}\right|\left|\sqrt{4+9+36}\right|\cos\theta=2-6-12$$

$$\Rightarrow \left|\sqrt{9}\right|\left|\sqrt{49}\right|\cos\theta=-16 \Rightarrow (3)(7)\cos\theta=-16 \Rightarrow \cos\theta=-\frac{16}{21}$$

for acute angle, ignore the negative sign

$$\therefore \quad \cos\theta=\frac{16}{21} \Rightarrow \theta=40.367\approx 40.4^\circ \quad \text{(Ans)}$$

For any vector **v**, the unit vector $\hat{\mathbf{v}}$, is given as:

$$\hat{\mathbf{v}}=\frac{\mathbf{v}}{|\mathbf{v}|}$$

If

$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, and

$\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, then

$\mathbf{a}.\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$

and magnitude of **a** is

$|\mathbf{a}|=\sqrt{(a_1)^2+(a_2)^2+(a_3)^2}$

**(ii)** We have

$x+2y-2z=2 \cdots\cdots\cdots\text{(i)}$

$2x-3y+6z=3 \cdots\cdots\cdots\text{(ii)}$

eq. (i)×3 and adding (i) & (ii)

$$\begin{array}{l} 3x+6y-6z=6 \\ \underline{2x-3y+6z=3} \\ 5x+3y\ =9 \quad \Rightarrow \quad x=\dfrac{9-3y}{5}\cdots\cdots\cdots\text{(iii)} \end{array}$$

eq. (i)×3, eq. (ii)×2 and adding (i) & (ii)

$$\begin{array}{l} 3x+6y-6z=6 \\ \underline{4x-6y+12z=6} \\ 7x+6z\ =12 \quad \Rightarrow \quad x=\dfrac{12-6z}{7}\cdots\cdots\cdots\text{(iv)} \end{array}$$

making the common variable $x$ as subject from equations (iii) & (iv)

$$x=\frac{9-3y}{5}=\frac{12-6z}{7} \quad\Rightarrow\quad x=\frac{-3y+9}{5}=\frac{-6z+12}{7}$$

$$\Rightarrow\quad x=\frac{-3(y-3)}{5}=\frac{-6(z-2)}{7}$$

$$\Rightarrow\quad \frac{x-0}{1}=\frac{y-3}{-\frac{5}{3}}=\frac{z-2}{-\frac{7}{6}}$$

which is the Cartesian equation of a line passing through (0, 3, 2) and having direction ratios $(1,-\frac{5}{3},-\frac{7}{6})$

multiplying the direction ratios by $-6$ and simplifying

$$1:-\frac{5}{3}:-\frac{7}{6}=-6:+10:+7$$

Note that direction ratios can always be simplified in order to avoid fractions but the fixed point cannot be simplified.

$\therefore$ the vector equation of line is $\mathbf{r}=\begin{pmatrix}0\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}-6\\10\\7\end{pmatrix}$

or $\mathbf{r}=3\mathbf{j}+2\mathbf{k}+\lambda(-6\mathbf{i}+10\mathbf{j}+7\mathbf{k})$ **(Ans)**

*Alternative solution to part (ii):*

Let us take any two points on the line of intersection of two planes by giving any values to $z$ and then solving equations in $x$ and $y$ simultaneously.

$\therefore$ for $z=1$, we have

eq. (i) : $x+2y-2(1)=2 \;\Rightarrow\; x+2y=4 \cdots\cdots\cdots$(A)

eq. (ii) : $2x-3y+6(1)=3 \;\Rightarrow\; 2x-3y=-3 \cdots\cdots\cdots$(B)

Solving equations (A) and (B) simultaneously gives

$x=\frac{6}{7}$, and $y=\frac{11}{7}$

$\therefore$ $(\frac{6}{7},\frac{11}{7},1)$ is one point on the line of intersection

for $z=0$, we have

eq. (i) : $x+2y-2(0)=2 \;\Rightarrow\; x+2y=2 \cdots\cdots\cdots$(C)

eq. (ii) : $2x-3y+6(0)=3 \;\Rightarrow\; 2x-3y=3 \cdots\cdots\cdots$(D)

Solving equations (C) and (D) simultaneously gives

$x=\frac{12}{7}$, and $y=\frac{1}{7}$

$\therefore$ $(\frac{12}{7},\frac{1}{7},0)$ is another point on the line of intersection now, direction vector $\vec{d}$ of the line passing through $(\frac{6}{7}, \frac{11}{7}, 1)$ and $(\frac{12}{7}, \frac{1}{7}, 0)$ is

$$\vec{d} = \begin{pmatrix} \frac{12}{7}-\frac{6}{7} \\ \frac{1}{7}-\frac{11}{7} \\ 0-1 \end{pmatrix} = \begin{pmatrix} \frac{6}{7} \\ -\frac{10}{7} \\ -1 \end{pmatrix}$$

Note that $(\frac{6}{7}, \frac{11}{7}, 1)$ is only one of the infinite points on the line.

Therefore in the vector equation of a line, fixed point could be different but direction vector always remains same.

$\therefore$ equation of line is:

$$\mathbf{r} = \begin{pmatrix} \frac{6}{7} \\ \frac{11}{7} \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} \frac{6}{7} \\ -\frac{10}{7} \\ -1 \end{pmatrix} \Rightarrow \mathbf{r} = \begin{pmatrix} \frac{6}{7} \\ \frac{11}{7} \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -6 \\ 10 \\ 7 \end{pmatrix}$$

or $\mathbf{r} = \frac{6}{7}\mathbf{i} + \frac{11}{7}\mathbf{j} + \mathbf{k} + \lambda(-6\mathbf{i} + 10\mathbf{j} + 7\mathbf{k})$ **(Ans)**

**10.** (i) Prove the identity

$$\cot x - \cot 2x \equiv \operatorname{cosec} 2x$$ [3]

(ii) Show that $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \cot x \, dx = \frac{1}{2}\ln 2$. [3]

(iii) Find the exact value of $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \operatorname{cosec} 2x \, dx$, giving your answer in the form $a \ln b$ [4]

***Suggested Solution:***

(i) $\text{L.H.S.} = \cot x - \cot 2x$

$$= \frac{\cos x}{\sin x} - \frac{\cos 2x}{\sin 2x}$$

$$= \frac{\sin 2x \cos x - \cos 2x \sin x}{\sin x \sin 2x}$$

$$= \frac{\sin(2x - x)}{\sin x \sin 2x} = \frac{\sin x}{\sin x \sin 2x} = \frac{1}{\sin 2x} = \operatorname{cosec} 2x = \text{R.H.S.}$$ **(Proved)**

Note that:

$\sin(A-B)$
$= \sin A \cos B - \cos A \sin B$

(ii) $\text{L.H.S.} = \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \cot x \, dx$

$$= \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \frac{\cos x}{\sin x} dx = \Big[ \ln|\sin x| \Big]_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} = \Big[\ln \sin(\tfrac{1}{4}\pi)\Big] - \Big[\ln \sin(\tfrac{1}{6}\pi)\Big]$$

$\ln(\frac{a}{b}) = \ln a - \ln b$

$$= \ln(\frac{1}{\sqrt{2}}) - \ln(\frac{1}{2}) = (\ln 1 - \ln\sqrt{2}) - (\ln 1 - \ln 2)$$

$$= (-\ln 2^{\frac{1}{2}}) - (-\ln 2) = -\frac{1}{2}\ln 2 + \ln 2 = \frac{1}{2}\ln 2$$ **(Shown)**

(iii) $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \operatorname{cosec} 2x \, dx$

using the result of part (i)

$$\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \operatorname{cosec} 2x \, dx = \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} (\cot x - \cot 2x) \, dx$$

$$= \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \cot x \, dx - \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \cot 2x \, dx \cdots\cdots (1)$$

from part (ii) $\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \cot x \, dx = \frac{1}{2} \ln 2$

$\therefore$ equation (1) becomes

$$\frac{1}{2}\ln 2 - \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \cot 2x \, dx$$

$$= \frac{1}{2}\ln 2 - \int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi} \frac{\cos 2x}{\sin 2x} \, dx$$

$$= \frac{1}{2}\ln 2 - \left[\frac{\ln|\sin 2x|}{2}\right]_{\frac{1}{6}\pi}^{\frac{1}{4}\pi}$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\left[\ln\left(\sin 2(\tfrac{1}{4}\pi)\right) - \ln\left(\sin 2(\tfrac{1}{6}\pi)\right)\right]$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\left[\ln\left(\sin\frac{\pi}{2}\right) - \ln\left(\sin\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\left[\ln(1) - \ln\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{1}{2}\ln 2 + \frac{1}{2}\ln\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}\ln 2 + \frac{1}{2}\left(\ln\sqrt{3} - \ln 2\right)$$

$$= \frac{1}{2}\ln 2 + \frac{1}{2}\left(\frac{1}{2}\ln 3 - \ln 2\right)$$

$$= \frac{1}{2}\ln 2 + \frac{1}{4}\ln 3 - \frac{1}{2}\ln 2$$

$= \frac{1}{4}\ln 3$ **(Ans)**

# November 2003 Paper 1

## Pure Mathematics (P1)

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Find the coordinates of the points of intersection of the line $y+2x=11$ and the curve $xy=12$. [4]

***Suggested Solution:***

Let $y+2x=11 \Rightarrow y=11-2x \cdots\cdots\cdots$(i)

and $xy = 12 \cdots\cdots\cdots$(ii)

putting value of $y$ from eq. (i) into eq. (ii)

$\therefore \quad x(11-2x)-12=0$

$\Rightarrow 11x-2x^2-12=0 \Rightarrow 2x^2-11x+12=0 \Rightarrow 2x^2-8x-3x+12=0$

$\Rightarrow 2x(x-4)-3(x-4)=0 \Rightarrow (x-4)(2x-3)=0$

$\Rightarrow x-4=0$ or $2x-3=0$

$\Rightarrow x=4$ and $x=\frac{3}{2}$

Find the value of $y$ from the linear equation

when $x=4, \quad y=11-2(4) \Rightarrow y=3$

when $x=\frac{3}{2}, \quad y=11-2(\frac{3}{2}) \Rightarrow y=8$

$\therefore$ coordinates are: $(4, 3)$ and $\left(\frac{3}{2}, 8\right)$ **(Ans)**

**2.** (i) Show that the equation $4\sin^4\theta+5=7\cos^2\theta$ may be written in the form $4x^2+7x-2=0$, where $x=\sin^2\theta$. [1]

(ii) Hence solve the equation $4\sin^4\theta+5=7\cos^2\theta$, for $0° \le \theta \le 360°$. [4]

***Suggested Solution:***

Remember that $\sin^2\theta+\cos^2\theta=1$

(i) $4\sin^4\theta+5=7\cos^2\theta$

$\Rightarrow 4\sin^4\theta+5=7(1-\sin^2\theta) \Rightarrow. 4\sin^4\theta+5=7-7\sin^2\theta$

$\Rightarrow 4\sin^4\theta+7\sin^2\theta-2=0$

putting $\sin^2\theta=x$, the equation becomes

$4x^2+7x-2=0$ (shown)

(ii) Using result of part (i), the equation $4\sin^4\theta + 5 = 7\cos^2\theta$ can be written as

$$4x^2 + 7x - 2 = 0$$

$$\Rightarrow\ 4x^2 + 8x - x - 2 = 0 \quad \Rightarrow\ 4x(x+2) - 1(x+2) = 0$$

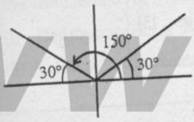
$$\Rightarrow\ (x+2)(4x-1) = 0$$

$$\Rightarrow\ x = -2 \text{ and } x = \frac{1}{4}$$

i.e $\sin^2\theta = -2$ (invalid) and $\sin^2\theta = \frac{1}{4} \Rightarrow \sin\theta = \pm\frac{1}{2}$

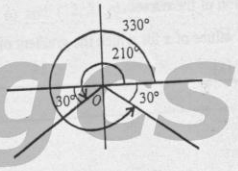
when $\sin\theta = \frac{1}{2}$

basic angle $= 30°$,



when $\sin\theta = -\frac{1}{2}$

Basic angle $= 30°$



Note that $\sin\theta$ is +ve in 1st and 2nd quadrant

Note that $\sin\theta$ is –ve in 3rd and 4th quadrant

$\therefore\ \theta = 30°, 150°, 210°, 330°.$ **(Ans).**

3. (a) A debt of \$3726 is repaid by weekly payments which are in arithmetic progression.

The first payment is \$60 and the debt is fully repaid after 48 weeks. Find the third payment. [3]

(b) Find the sum to infinity of the geometric progression whose first term is 6 and whose second term is 4. [3]

***Suggested Solution:***

(a) It is given that $S_n = \$3726$, $a = \$60$, $n = 48$

let $d$ be the common difference of the A.P.

using $S_n = \frac{n}{2}[2a + (n-1)d]$

$$\Rightarrow\ 3726 = \frac{48}{2}[2(60) + (48-1)d] \quad \Rightarrow\ 3726 = 24(120 + 47d)$$

$$\Rightarrow\ 3726 = 2880 + 1128d \quad \Rightarrow\ d = \frac{3}{4} = 0.75$$

$\therefore$ third payment: $T_3 = a + (3-1)d = 60 + (2)\frac{3}{4} = 61\frac{1}{2} = \$\,61.50$ **(Ans).**

(b) given: $a = 6,\ T_2 = 4,\ S_\infty = ?$

using $T_n = ar^{n-1}$, we have

$$T_2 = ar^{2-1} \Rightarrow 4 = 6r \Rightarrow r = \frac{4}{6} = \frac{2}{3}$$

$$\therefore\ S_\infty = \frac{a}{1-r} \Rightarrow S_\infty = \frac{6}{1-\frac{2}{3}} \Rightarrow S_\infty = 18\ \text{(Ans).}$$

4. A curve is such that $\frac{dy}{dx} = 3x^2 - 4x + 1$. The curve passes through the point (1, 5).

(i) Find the equation of the curve. [3]

(ii) Find the set of values of $x$ for which the gradient of the curve is positive. [3]

***Suggested Solution:***

(i) $\frac{dy}{dx} = 3x^2 - 4x + 1 \Rightarrow dy = (3x^2 - 4x + 1)\,dx$

integrating both sides

$$\int dy = \int (3x^2 - 4x + 1)\,dx$$

$$\Rightarrow\quad y = 3(\frac{x^3}{3}) - 4(\frac{x^2}{2}) + x + C \Rightarrow y = x^3 - 2x^2 + x + C$$

as the curve passes through (1, 5), therefore coordinates will satisfy the above equation.

$$5 = (1)^3 - 2(1)^2 + (1) + C \Rightarrow 5 = 1 - 2 + 1 + C \Rightarrow C = 5$$

$\therefore$ equation of the curve is: $y = x^3 - 2x^2 + x + 5$ **(Ans).**

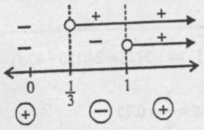
When $\frac{dy}{dx}$ (or gradient) of the curve is given, always integrate both sides of the given expression to find the equation of a curve.

(ii) as the gradient is positive, therefore $\frac{dy}{dx} > 0$

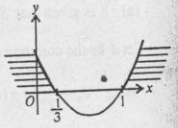
$$\Rightarrow 3x^2 - 4x + 1 > 0$$
$$\Rightarrow 3x^2 - 3x - x + 1 > 0$$
$$\Rightarrow (x-1)(3x-1) > 0$$

critical values are $x = 1$ and $x = \frac{1}{3}$

$\therefore\ x < \frac{1}{3}$ or $x > 1$ **(Ans).**



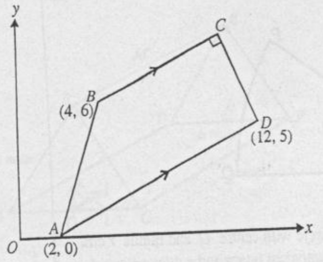
Alternatively we can find the set of values of $x$ by drawing the curve as follows.



Required values are:

$x < \frac{1}{3}$ and $x > 1$

**5.**



The diagram shows a trapezium $ABCD$ in which $BC$ is parallel to $AD$ and angle $BCD = 90°$

The coordinates of $A$, $B$ and $D$ are $(2, 0)$, $(4, 6)$ and $(12, 5)$ respectively.

(i) Find the equations of $BC$ and $CD$. [5]

(ii) Calculate the coordinates of $C$. [2]

*Suggested Solution:*

(i) Gradient of $AD = \frac{5-0}{12-2} = \frac{1}{2}$

AD and BC are parallel and parallel lines have same gradient

as $AD \parallel BC$, therefore gradient of $BC = \frac{1}{2}$

equation of $BC$: $y-6 = \frac{1}{2}(x-4) \Rightarrow 2y-12 = x-4 \Rightarrow 2y-x=8$ **(Ans).**

$CD \perp BC \Rightarrow$ gradient of $CD = -2$

equation of $CD$: $y-5 = -2(x-12) \Rightarrow y+2x = 29$ **(Ans).**

Since lines BC and $CD$ are perpendicular to each other. so (slope of $BC$) × (slope of $CD$) = − 1

i.e. $m_1 \times m_2 = -1$.

Equation of a straight line in slope-point form, with gradient $m$ and passing through $(x_1, y_1)$ is

$(y - y_1) = m(x - x_1)$.

(ii) Solving equations of BC and CD simultaneously

eq. of $BC$: $2y-x=8 \Rightarrow y = \frac{8+x}{2}$ ………(i)

eq. of $CD$: $2x+y=29$ ………(ii)

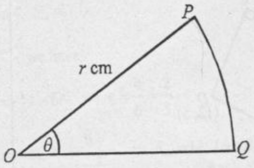
putting value of $y$ from (i) into (ii)

$$2x + \left(\frac{8+x}{2}\right) = 29 \Rightarrow 4x+8+x=58 \Rightarrow 5x=50 \Rightarrow x=10$$

$$y = \frac{8+x}{2} \Rightarrow y = \frac{8+10}{2} \Rightarrow y=9$$

$\therefore$ coordinates of $C$ are $(10, 9)$ **(Ans).**

**6.**



The diagram shows the sector $OPQ$ of a circle with centre $O$ and radius $r$ cm.

The angle $POQ$ is $\theta$ radians and the perimeter of the sector is 20 cm.

(i) Show that $\theta = \frac{20}{r} - 2$. [2]

(ii) Hence express the area of the sector in terms of $r$. [2]

(iii) In the case where $r = 8$, find the length of the chord $PQ$. [3]

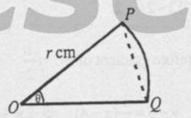
***Suggested Solution:***

(i) given perimeter = 20

from fig. perimeter of the sector $= r + r +$ arc length $PQ$

$\Rightarrow r + r + r\theta = 20 \Rightarrow 2r + r\theta = 20 \Rightarrow r\theta = 20 - 2r$

$\Rightarrow \theta = \frac{20}{r} - 2$ **(Shown)**



(ii) Area of sector $OPQ = \frac{1}{2}r^2\theta$. using the value of $\theta$ from part (i), we have

Area of sector $OPQ = \frac{1}{2}r^2\left(\frac{20}{r} - 2\right) = 10r - r^2$

$\therefore$ Area $= (10r - r^2)$ cm$^2$ **(Ans).**

(iii) When $r = 8$, $\theta = \frac{20}{8} - 2 \Rightarrow \theta = 0.5$ rad

$\therefore$ using cosine rule

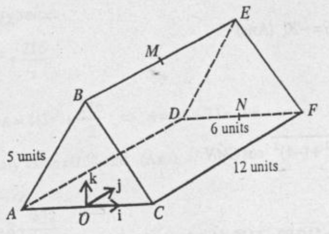
$PQ^2 = (8)^2 + (8)^2 - 2(8)(8)\cos(0.5)$

$\Rightarrow PQ^2 = 64 + 64 - 112.330$

$\Rightarrow PQ^2 = 15.669 \Rightarrow PQ = 3.96$ cm **(Ans).**

Note that angle is in radians. You must change your calculator to radian mode.

7.



The diagram shows a triangular prism with a horizontal rectangular base *ADFC*, where *CF* = 12 units and *DF* = 6 units. The vertical ends *ABC* and *DEF* are isosceles triangles with *AB* = *BC* = 5 units.

The mid-points of *BE* and *DF* are *M* and *N* respectively. The origin *O* is at the mid-point of *AC*.

Unit vectors **i**, **j** and **k** are parallel to *OC*, *ON* and *OB* respectively.

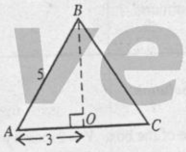
(i) Find the length of *OB*. [1]

(ii) Express each of the vectors $\overrightarrow{MC}$ and $\overrightarrow{MN}$ in terms of **i**, **j** and **k**. [3]

(iii) Evaluate $\overrightarrow{MC}\cdot\overrightarrow{MN}$ and hence find angle *CMN*, giving your answer correct to the nearest degree. [4]

***Suggested Solution:***

(i) Consider $\Delta ABC$ from the main figure using pythagoras theorem



$$OA^2+OB^2=AB^2 \;\Rightarrow\; 3^2+OB^2=5^2 \;\Rightarrow\; OB^2=16$$

$$\Rightarrow\; OB=4 \text{ units} \quad \text{(Ans)}$$

|||

(ii) We have

$$\overrightarrow{OC}=\begin{pmatrix}3\\0\\0\end{pmatrix}, \qquad \overrightarrow{OM}=\overrightarrow{OB}+\overrightarrow{BM}=\begin{pmatrix}0\\0\\4\end{pmatrix}+\begin{pmatrix}0\\6\\0\end{pmatrix} \;\Rightarrow\; \overrightarrow{OM}=\begin{pmatrix}0\\6\\4\end{pmatrix}$$

$$\overrightarrow{MC}=\overrightarrow{OC}-\overrightarrow{OM}=\begin{pmatrix}3\\0\\0\end{pmatrix}-\begin{pmatrix}0\\6\\4\end{pmatrix}=\begin{pmatrix}3\\-6\\-4\end{pmatrix} \qquad \therefore\; \overrightarrow{MC}=3\text{i}-6\text{j}-4\text{k} \;\; \text{(Ans)}$$

also $\overrightarrow{ON}=\begin{pmatrix}0\\12\\0\end{pmatrix}$

$$\overrightarrow{MN}=\overrightarrow{ON}-\overrightarrow{OM}=\begin{pmatrix}0\\12\\0\end{pmatrix}-\begin{pmatrix}0\\6\\4\end{pmatrix}=\begin{pmatrix}0\\6\\-4\end{pmatrix} \qquad \therefore\; \overrightarrow{MN}=6\text{j}-4\text{k} \;\; \text{(Ans).}$$

(iii) $\overrightarrow{MC} \cdot \overrightarrow{MN} = \begin{pmatrix} 3 \\ -6 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 6 \\ -4 \end{pmatrix} = 0 - 36 + 16 = -20$ **(Ans)**

now $\overrightarrow{MC} \cdot \overrightarrow{MN} = \left|\overrightarrow{MC}\right| \left|\overrightarrow{MN}\right| \cos C\hat{M}N$

$\Rightarrow \quad -20 = \sqrt{3^2 + (-6)^2 + (-4)^2} \times \sqrt{6^2 + (-4)^2} \cos C\hat{M}N$

$\Rightarrow \quad -20 = \sqrt{61} \times \sqrt{52} \cos C\hat{M}N$

$\Rightarrow \quad \cos C\hat{M}N \doteq \dfrac{-20}{56.3205} \Rightarrow C\hat{M}N = 110.8° \approx 111°$ **(Ans)**

If
$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, and
$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then
$\mathbf{a}.\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
and magnitude of $\mathbf{a}$ is
$|\mathbf{a}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$

**8.** A solid rectangular block has a base which measures $2x$ cm by $x$ cm. The height of the block is $y$ cm and the volume of the block is $72 \text{ cm}^3$.

(i) Express $y$ in terms of $x$ and show that the total surface area, $A \text{ cm}^2$, of the block is given by

$$A = 4x^2 + \frac{216}{x}.$$

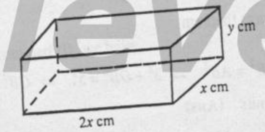
[3]

Given that $x$ can vary,

(ii) find the value of $x$ for which $A$ has a stationary value, [3]

(iii) find this stationary value and determine whether it is a maximum or a minimum. [3]

*Suggested Solution:*



(i) Volume of the box, $V = 2x \times x \times y$

given volume $V = 72 \text{ cm}^3$

$\Rightarrow \quad 72 = 2x \times x \times y \Rightarrow 72 = 2x^2 y \Rightarrow y = \dfrac{36}{x^2}$ .........(i)

surface area $A = 2[(2x)(x) + (2x)(y) + (x)(y)] = 4x^2 + 6xy$

putting value of $y$ from eq. (i)

$$A = 4x^2 + 6x\left(\frac{36}{x^2}\right) \Rightarrow A = 4x^2 + \frac{216}{x} \quad \textbf{(shown)}$$

Total surface area $A$, of a cuboid is given as:

$$A = 2\begin{bmatrix} (L \times W) + (W \times H) \\ +(L \times H) \end{bmatrix}$$

where $L$, $W$ and $H$ are length, width and height respectively.

(ii) $A = 4x^2 + \dfrac{216}{x} \Rightarrow \dfrac{dA}{dx} = 8x - \dfrac{216}{x^2}$

for stationary values $\dfrac{dA}{dx} = 0$

$\Rightarrow \quad 8x - \dfrac{216}{x^2} = 0 \Rightarrow 8x^3 - 216 = 0 \Rightarrow 8x^3 = 216 \Rightarrow x^3 = 27$

$\Rightarrow \quad x = 3$ **(Ans)**

(iii) For stationary value

$$A = 4x^2 + \frac{216}{x}.$$

at $x = 3$, $A = 4(3)^2 + \frac{216}{3} \Rightarrow A = 36 + 72 = 108$

$\therefore$ stationary value $= 108$ cm$^2$ **(Ans)**

$$\frac{d^2A}{dx^2} = 8 + \frac{432}{x^3}$$

at $x = 3$

$$\frac{d^2A}{dx^2} = 8 + \frac{432}{(3)^3} = 24 > 0$$

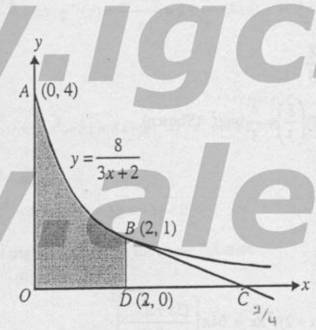
$\therefore$ stationary value is a minimum value **(Ans).**

Nature:

If $\frac{d^2A}{dx^2} > 0$ A is min.

If $\frac{d^2A}{dx^2} < 0$ A is max.

**9.**



2/4

The diagram shows points $A\ (0, 4)$ and $B\ (2, 1)$ on the curve $y = \frac{8}{3x+2}$. The tangent to the curve at $B$ crosses the $x$-axis at $C$. The point $D$ has coordinates $(2, 0)$.

(i) Find the equation of the tangent to the curve at $B$ and hence show that the area of triangle $BDC$ is $\frac{4}{3}$. [6]

(ii) Show that the volume of the solid formed when the shaded region $ODBA$ is rotated completely about the $x$-axis is $8\pi$. [5]

*Suggested Solution:*

(i) $y = \frac{8}{3x+2} \Rightarrow y = 8(3x+2)^{-1}$

$\frac{dy}{dx} = -8(3x+2)^{-2}(3) \Rightarrow \frac{dy}{dx} = \frac{-24}{(3x+2)^2}$

at point $B(2, 1)$, $\frac{dy}{dx} = \frac{-24}{[3(2)+2]^2} = \frac{-24}{8^2} = \frac{-24}{64} = -\frac{3}{8}$

$\therefore$ equation of the tangent at $B(2, 1)$:

$y - 1 = -\frac{3}{8}(x-2) \Rightarrow 8y - 8 = -3x + 6 \Rightarrow 3x + 8y = 14$ **(Ans).**

The tangent meets $x$-axis at $C$. therefore putting $y = 0$, we have

$3x + 8(0) = 14 \Rightarrow 3x = 14 \Rightarrow x = \frac{14}{3}$

$\therefore$ coordinates of $C\left(\frac{14}{3}, 0\right)$

in $\Delta BDC$, $BD = 1$, and $CD = \frac{14}{3} - 2 = \frac{8}{3}$

$\therefore$ area of $\Delta BDC = \frac{1}{2}(BD)(DC) = \frac{1}{2}(1)\left(\frac{8}{3}\right) = \frac{4}{3}$ unit$^2$ **(Shown).**

(ii) $y = \frac{8}{3x+2} \Rightarrow y = 8(3x+2)^{-1}$

Volume of revolution, $V = \pi\int_0^2 y^2 dx$

$$\Rightarrow V = \pi\int_0^2 \left(8(3x+2)^{-1}\right)^2 dx = 64\pi\int_0^2 (3x+2)^{-2} dx = 64\pi\left[\frac{(3x+2)^{-1}}{(-1)(3)}\right]_0^2$$

$$= 64\pi\left[\left(\frac{(3(2)+2)^{-1}}{-3}\right) - \left(\frac{(3(0)+2)^{-1}}{-3}\right)\right] = 64\pi\left[\frac{8^{-1}}{-3} - \frac{2^{-1}}{-3}\right]$$

$$= 64\pi\left[-\frac{1}{24} + \frac{1}{6}\right] = 64\pi\left(\frac{1}{8}\right) = 8\pi \text{ (Shown).}$$

**10.** Functions f and g are defined by

$f: x \mapsto 2x - 5, \quad x \in \mathbb{R}$

$g: x \mapsto \frac{4}{2-x}, \quad x \in \mathbb{R}, \ x \neq 2.$

(i) Find the value of $x$ for which $fg(x) = 7$. [3]

(ii) Express each of $f^{-1}(x)$ and $g^{-1}(x)$ in terms of $x$. [3]

(iii) Show that the equation $f^{-1}(x) = g^{-1}(x)$ has no real roots. [3]

(iv) Sketch, on a single diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between these two graphs. [3]

*Suggested Solution:*

(i) $\mathrm{fg}(x) = \mathrm{f}\left(\frac{4}{2-x}\right) = 2\left(\frac{4}{2-x}\right) - 5 \Rightarrow \mathrm{fg}(x) = \frac{8}{2-x} - 5 \Rightarrow \mathrm{fg}(x) = \frac{5x-2}{2-x}$

as $\mathrm{fg}(x) = 7$

$\therefore\ 7 = \frac{5x-2}{2-x} \Rightarrow 14 - 7x = 5x - 2 \Rightarrow 12x = 16 \Rightarrow x = \frac{16}{12} = \frac{4}{3}$ **(Ans).**

(ii) Let $\mathrm{f}(x) = y \Rightarrow \mathrm{f}^{-1}(y) = x$

$\therefore\ y = 2x - 5 \Rightarrow x = \frac{y+5}{2}$

or $\mathrm{f}^{-1}(y) = \frac{y+5}{2} \Rightarrow \mathrm{f}^{-1}(x) = \frac{x+5}{2}$ **(Ans)**

Let $\mathrm{g}(x) = z \Rightarrow \mathrm{g}^{-1}(z) = x$

$\therefore\ z = \frac{4}{2-x} \Rightarrow 2z - zx = 4 \Rightarrow zx = 2z - 4 \Rightarrow x = \frac{2z-4}{z}$

or $\mathrm{g}^{-1}(z) = \frac{2z-4}{z} \Rightarrow \mathrm{g}^{-1}(x) = \frac{2x-4}{x} = 2 - \frac{4}{x}$ **(Ans)**

(iii) $\mathrm{f}^{-1}(x) = \mathrm{g}^{-1}(x)$

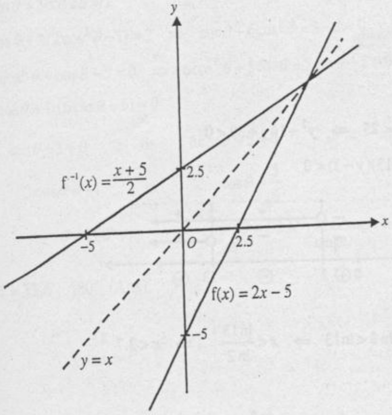
$\Rightarrow \frac{x+5}{2} = 2 - \frac{4}{x} \Rightarrow x^2 + 5x = 4x - 8 \Rightarrow x^2 + x + 8 = 0$

discriminant $= b^2 - 4ac$

$= (1)^2 - 4(1)(8) = 1 - 32 = -31 < 0$

as discriminent is negative, therefore the above equation has no real roots. **(Shown)**

(iv)



$\mathrm{f}^{-1}(x)$ is a reflection of the graph of $\mathrm{f}(x)$ in the line $y = x$.

# November 2003 Paper 3

## Pure Mathematics (P3)

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Solve the inequality $\left|2^x-8\right|<5$. [4]

*Suggested Solution:*

Note:
When $|x|<a$
then $x<a$ and $x>-a$

$\left|2^x-8\right|<5$

$\Rightarrow \quad 2^x-8<5$ or $2^x-8>-5$

$\Rightarrow \quad 2^x<13$

$\Rightarrow \quad \ln(2^x)<\ln(13)$

$\Rightarrow \quad x\ln 2<\ln 13$

$\Rightarrow \quad x<\dfrac{\ln 13}{\ln 2}$

$\Rightarrow \quad x<3.7$

$\Rightarrow \quad 2^x>-5+8$

$\Rightarrow \quad 2^x>3$

$\Rightarrow \quad \ln(2^x)>\ln 3$

$\Rightarrow \quad x\ln 2>\ln 3$

$\Rightarrow \quad x>\dfrac{\ln 3}{\ln 2} \Rightarrow x>1.58$

$\therefore \quad 1.58<x<3.7$ (Ans)

*Alternative Solution:*

$\left|2^x-8\right|<5$

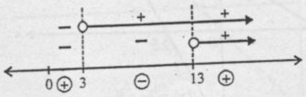
Let $2^x=y \Rightarrow |y-8|<5$

squaring both sides

$\Rightarrow (y-8)^2<5^2 \Rightarrow y^2-16y+64<25 \Rightarrow y^2-16y+39<0$

$\Rightarrow y^2-13y-3y+39<0 \Rightarrow (y-13)(y-3)<0$

critical values are $y=13, \quad y=3$



$\therefore \quad y<13, \quad y>3$

Now when $y<13$

$\Rightarrow 2^x<13 \Rightarrow \ln 2^x<\ln 13 \Rightarrow x\ln 2<\ln 13 \Rightarrow x<\dfrac{\ln 13}{\ln 2} \Rightarrow x<3.7$

and when $y>3$

$\Rightarrow 2^x>3 \Rightarrow \ln 2^x>\ln 3 \Rightarrow x\ln 2>\ln 3 \Rightarrow x>\dfrac{\ln 3}{\ln 2} \Rightarrow x>1.58$

$\therefore \quad 1.58<x<3.7$ (Ans)

2. Expand $\left(2+x^2\right)^{-2}$ in ascending powers of $x$, up to and including the term in $x^4$, simplifying the coefficients. [4]

*Suggested Solution:*

$$\left(2+x^2\right)^{-2}=\left[2\left(1+\frac{x^2}{2}\right)\right]^{-2}=2^{-2}\left(1+\frac{x^2}{2}\right)^{-2}=\frac{1}{4}\left(1+\frac{x^2}{2}\right)^{-2}$$

applying binomial theorem upto term $x^4$

$$=\frac{1}{4}\left[1+(-2)\left(\frac{x^2}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x^2}{2}\right)^2\right]$$

$$=\frac{1}{4}\left[1-x^2+\frac{3}{4}x^4\right]$$

$$=\frac{1}{4}-\frac{1}{4}x^2+\frac{3}{16}x^4 \quad \text{(Ans)}$$

Binomial Expansion:

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

3. Solve the equation

$$\cos\theta+3\cos 2\theta=2,$$

giving all solutions in the interval $0^\circ \le \theta \le 180^\circ$. [5]

*Suggested Solution:*

$$\cos\theta+3\cos 2\theta=2$$

$$\Rightarrow \cos\theta+3(2\cos^2\theta-1)=2 \quad \Rightarrow \quad \cos\theta+6\cos^2\theta-3-2=0$$

$$\Rightarrow 6\cos^2\theta+\cos\theta-5=0 \quad \Rightarrow \quad 6\cos^2\theta+6\cos\theta-5\cos\theta-5=0$$

$$\Rightarrow (\cos\theta+1)(6\cos\theta-5)=0$$

either $\cos\theta+1=0$ or $6\cos\theta-5=0$

$\cos\theta=-1$ or $\cos\theta=\dfrac{5}{6}$

$\theta=180^\circ$ $\quad$ $\theta=33.6^\circ$, other angle is out of given range.

$\therefore \quad \theta=33.6^\circ,\ 180^\circ$ (Ans)

Remember:

$$\cos 2\theta=2\cos^2\theta-1$$

4. The equation of a curve is

$$\sqrt{x}+\sqrt{y}=\sqrt{a}\,.$$

where a is a positive constant.

(i) Express $\frac{dy}{dx}$ in terms of $x$ and $y$. [3]

(ii) The straight line with equation $y=x$ intersects the curve at the point $P$. Find the equation of the tangent to the curve at $P$. [3]

***Suggested Solution:***

(i) $\sqrt{x}+\sqrt{y}=\sqrt{a} \Rightarrow x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$

differentiating w.r.t. $x$

$$\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}y^{-\frac{1}{2}}\times\frac{d}{dx}(y)=0 \Rightarrow \frac{1}{2}\left(x^{-\frac{1}{2}}+y^{-\frac{1}{2}}\left(\frac{dy}{dx}\right)\right)=0$$

$$\Rightarrow x^{-\frac{1}{2}}+y^{-\frac{1}{2}}\left(\frac{dy}{dx}\right)=0 \Rightarrow y^{-\frac{1}{2}}\left(\frac{dy}{dx}\right)=-x^{-\frac{1}{2}}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{2}}} \Rightarrow \frac{dy}{dx}=-\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}} \Rightarrow \frac{dy}{dx}=-\sqrt{\frac{y}{x}} \quad \text{(Ans)}$$

(ii) Let $y=x$ ………(i) and $\sqrt{x}+\sqrt{y}=\sqrt{a}$ ………(ii)

putting eq. (i) into eq. (ii) gives

$$\sqrt{x}+\sqrt{x}=\sqrt{a} \Rightarrow 2\sqrt{x}=\sqrt{a} \Rightarrow 4x=a \Rightarrow x=\frac{a}{4}$$

putting $x$ into eq. (i) gives $y=\frac{a}{4}$

$\therefore$ point $P$ is $=\left(\frac{a}{4},\frac{a}{4}\right)$

Now

From part (i), we have $\frac{dy}{dx}=-\sqrt{\frac{y}{x}}$

at point $P$ $\quad \frac{dy}{dx}=-\sqrt{\frac{\frac{a}{4}}{\frac{a}{4}}}=-1$

$\therefore$ Equation of tangent at $P$ is obtained using

$$y-y_1=m(x-x_1)$$

$$\Rightarrow y-\frac{a}{4}=-1\left(x-\frac{a}{4}\right) \Rightarrow y-\frac{a}{4}=-x+\frac{a}{4} \Rightarrow x+y=\frac{a}{2}$$

$$\Rightarrow 2x+2y=a \quad \text{(Ans)}$$

**5.** (i) By sketching suitable graphs, show that the equation.

$$\sec x = 3 - x^2$$

has exactly one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Show that, if a sequence of values given by the iterative formula.

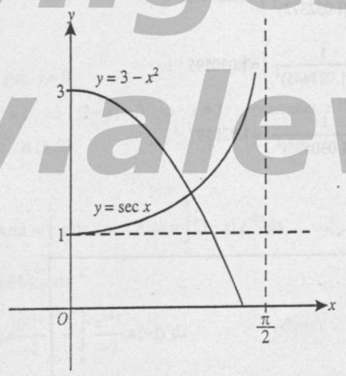
$$x_{n+1} = \cos^{-1}\left(\frac{1}{3 - x_n^2}\right)$$

converges, then it converges to a root of the equation given in part (i). [2]

(iii) Use this iterative formula, with initial value $x_1 = 1$, to determine the root in the interval $0 < x < \frac{1}{2}\pi$ correct to 2 decimal places, showing the result of each iteration. [3]

***Suggested Solution:***

(i) Sketching $y = \sec x$, and $y = 3 - x^2$ for given interval $0 < x < \frac{1}{2}\pi$,



We see that there is only one point of intersection for the given interval.

$\therefore$ $\sec x = 3 - x^2$ has exactly one root in the interval $0 < x < \frac{1}{2}\pi$ (Shown)

(ii) $x_{n+1} = \cos^{-1}\left(\frac{1}{3 - x_x^2}\right)$

removing the subscripts, we have

$$x = \cos^{-1}\left(\frac{1}{3 - x_2}\right) \Rightarrow \cos x = \frac{1}{3 - x^2} \Rightarrow 3 - x^2 = \frac{1}{\cos x} \Rightarrow 3 - x^2 = \sec x$$

which is the same equation as given in part (i) (Shown)

(iii) $x_{n+1} = \cos^{-1}\left(\dfrac{1}{3-x_n^2}\right)$

Given that initial value $x_1 = 1$

$\therefore \quad x_2 = \cos^{-1}\left(\dfrac{1}{3-x_1^2}\right) = \cos^{-1}\left(\dfrac{1}{3-(1)^2}\right) = 1.047198$

$x_3 = \cos^{-1}\left(\dfrac{1}{3-x_2^2}\right) = \cos^{-1}\left(\dfrac{1}{3-(1.047198)^2}\right) = 1.017632$

$x_4 = \cos^{-1}\left(\dfrac{1}{3-x_3^2}\right) = \cos^{-1}\left(\dfrac{1}{3-(1.017632)^2}\right) = 1.036710$

$x_5 = \cos^{-1}\left(\dfrac{1}{3-x_4^2}\right) = \cos^{-1}\left(\dfrac{1}{3-(1.036710)^2}\right) = 1.024627$

$x_6 = \cos^{-1}\left(\dfrac{1}{3-x_5^2}\right) = \cos^{-1}\left(\dfrac{1}{3-(1.024627)^2}\right) = 1.032372$

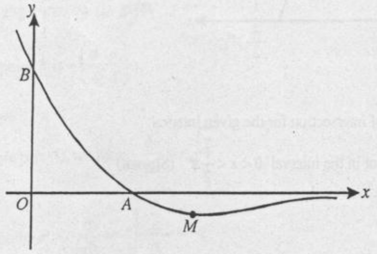
$x_7 = \cos^{-1}\left(\dfrac{1}{3-x_6^2}\right) = \cos^{-1}\left(\dfrac{1}{3-(1.032372)^2}\right) = 1.027445$

$x_8 = \cos^{-1}\left(\dfrac{1}{3-x_7^2}\right) = \cos^{-1}\left(\dfrac{1}{3-(1.027445)^2}\right) = 1.030595$

$x_9 = \cos^{-1}\left(\dfrac{1}{3-x_8^2}\right) = \cos^{-1}\left(\dfrac{1}{3-(1.030595)^2}\right) = 1.028587$

$\therefore \quad x = 1.03$ (2 dec. pl) **(Ans)**

**6.**



The diagram shows the curve $y = (3-x)e^{-2x}$ and its minimum point $M$. The curve intersects the $x$-axis at $A$ and the $y$-axis at $B$.

(i) Calculate the $x$-coordinate of $M$. [4]

(ii) Find the area of the region bounded by $OA$, $OB$ and the curve, giving your answer in terms of e. [5]

*Suggested Solution:*

(i) $y=(3-x)e^{-2x}$

differentiating w.r.t. $x$, using using $u.v$ form

$$\frac{dy}{dx}=e^{-2x}\frac{d}{dx}(3-x)+(3-x)\frac{d}{dx}(e^{-2x})$$

$$\Rightarrow \frac{dy}{dx}=e^{-2x}(-1)+(3-x)(e^{-2x})(-2)$$

$$=-e^{-2x}-6e^{-2x}+2xe^{-2x}=2xe^{-2x}-7e^{-2x}=(2x-7)e^{-2x}$$

for maxima or minima $\frac{dy}{dx}=0$

$\Rightarrow (2x-7)e^{-2x}=0$

$\Rightarrow (2x-7)=0$ or $e^{-2x}=0$ (not possible as $e^{-2x}\neq 0$)

$\Rightarrow x=\frac{7}{2}$

$\therefore$ $x$-coordinate of $M=\frac{7}{2}=3\frac{1}{2}$ (Ans)

(ii) $y=(3-x)e^{-2x}$

for $x$-intercept put $y=0$

$\therefore (3-x)e^{-2x}=0 \Rightarrow (3-x)=0 \Rightarrow x=3$ (remember $e^{-2x}\neq 0$)

$\therefore$ $x$-intercept $A\,(3, 0)$

Now

$$\text{bounded Area}=\int_0^3 y\,dx \Rightarrow \text{Area}=\int_0^3 (3-x)\,e^{-2x}\,dx$$

using integration by parts

$$\text{Area}=\left[(3-x)\times\frac{e^{-2x}}{-2}\right]_0^3-\int_0^3\frac{e^{-2x}}{-2}\times(-1)\,dx$$

$$=-\frac{1}{2}\left[\frac{3-x}{e^{2x}}\right]_0^3-\frac{1}{2}\int_0^3 e^{-2x}\,dx$$

$$=-\frac{1}{2}\left[\frac{3-x}{e^{2x}}\right]_0^3-\frac{1}{2}\left[\frac{e^{-2x}}{-2}\right]_0^3=-\frac{1}{2}\left[\frac{3-x}{e^{2x}}\right]_0^3+\frac{1}{4}\left[e^{-2x}\right]_0^3$$

$$=-\frac{1}{2}\left[\left(\frac{3-(3)}{e^{2(3)}}\right)-\left(\frac{3-(0)}{e^{2(0)}}\right)\right]+\frac{1}{4}\left[\left(e^{-2(3)}\right)-\left(e^{-2(0)}\right)\right]$$

$$=-\frac{1}{2}\left[0-3\right]+\frac{1}{4}\left[e^{-6}-1\right]=\frac{3}{2}+\frac{1}{4}e^{-6}-\frac{1}{4}$$

$$=\frac{1}{4}e^{-6}+\frac{5}{4}=\frac{1}{4}(e^{-6}+5) \quad \text{(Ans)}$$

7. The complex number $u$ is given by $u = \frac{7+4i}{3-2i}$.

(i) Express $u$ in the form $x + iy$, where $x$ and $y$ are real. [3]

(ii) Sketch an Argand diagram showing the point representing the complex number $u$. Show on the same diagram the locus of the complex number $z$ such that $|z-u| = 2$ [3]

(iii) Find the greatest value of $\arg z$ for points on this locus. [3]

*Suggested Solution:*

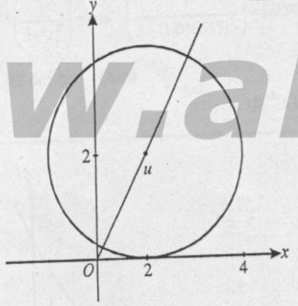
(i) $u = \frac{7+4i}{2-2i}$

realizing the denominator

$$u = \frac{7+4i}{3-2i} \times \frac{3+2i}{3+2i}$$

$$= \frac{21+14i+12i+8i^2}{(3)^2-(2i)^2} = \frac{21+26i-8}{9+4} = \frac{13+26i}{13} = \frac{13}{13} + \frac{26}{13}i = 1+2i \quad \text{(Ans)}$$

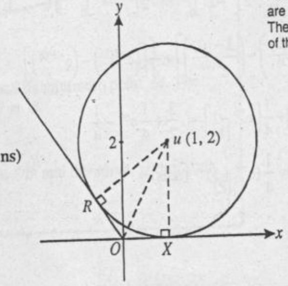
(ii) $|z-u| = 2$ is a circle with centre at $u$ and radius 2units



(iii) Greatest value of $\arg z$ for the points on the locus is the obtuse angle $R\hat{O}X$ shown in the diagram

$$\tan u\hat{O}X = \frac{2}{1} \Rightarrow u\hat{O}X = 1.107$$

$$\therefore \quad R\hat{O}X = 2(u\hat{O}X) = 2(1.107) = 2.21 \text{ radians} \quad \text{(Ans)}$$



Note that lines *OR* and *OX* are tangents to the circle. Therefore angle *uOX* is half of the angle *ROX*.

8. Let $f(x)=\dfrac{x^3-x-2}{(x-1)(x^2+1)}$.

(i) Express $f(x)$ in the form

$$A+\frac{B}{x-1}+\frac{Cx+D}{x^2+1},$$

where $A, B, C$ and $D$ are constants. [5]

(ii) Hence show that $\int_2^3 f(x)\,dx=1$ [4]

***Suggested Solution:***

(i) $f(x)=\dfrac{x^3-x-2}{(x-1)(x^2+1)} \Rightarrow f(x)=\dfrac{x^3-x-2}{x^3-x^2+x-1}$ which is an improper fraction

$\therefore$ using long division

$$\begin{array}{r|l} & \quad 1 \\ \hline x^3-x^2+x-1 & x^3-x-2 \\ & x^3+x-1\ \ -x^2 \\ & -\quad -\quad +\quad + \\ \hline & x^2-2x-1 \end{array}$$

$\therefore\ f(x)=1+\dfrac{x^2-2x-1}{(x-1)(x^2+1)} \quad \therefore\ A\equiv1$

Now

$$\frac{x^2-2x-1}{(x-1)(x^2+1)}\equiv\frac{B}{x-1}+\frac{Cx+D}{x_2+1}$$

$\Rightarrow x^2-2x-1\equiv B(x^2+1)+(Cx+D)(x-1)$

for $x=1$

$(1)^2-2(1)-1=B((1)^2+1)+(C(1)+D)((1)-1)$

$\Rightarrow 1-2-1=2B+0 \Rightarrow 2B=-2 \Rightarrow B=-1$

for $x=0$

$(0)^2-2(0)-1=B((0)^2+1)+(C(0)+D)((0)-1)$

$\Rightarrow -1=B+D(-1) \Rightarrow -1=B-D \Rightarrow -1=-1-D \Rightarrow D=0$

for $x=-1$

$(-1)^2-2(-1)-1=B((-1)^2+1)+(C(-1)+D)((-1)-1)$

$\Rightarrow 1+2-1=2B+(-C+D)(-2) \Rightarrow 2=2B+2C-2D$

$\Rightarrow 2=2(-1)+2C-2(0) \Rightarrow 2=-2+2C \Rightarrow 4=2C \Rightarrow C=2$

$\therefore\ f(x)=1+\dfrac{-1}{x-1}+\dfrac{2x+0}{x^2+1} \Rightarrow f(x)=1-\dfrac{1}{x-1}+\dfrac{2x}{x^2+1}$ (Ans)

(ii) $\int_2^3 \mathrm{f}(x)\,\mathrm{d}x = 1$

$$\text{L.H.S.} = \int_2^3 \mathrm{f}(x)\,\mathrm{d}x$$

$$= \int_2^3\left(1 - \frac{1}{x-1} + \frac{2x}{x^2+1}\right)\mathrm{d}x = \int_2^3 1\,\mathrm{d}x - \int_2^3 \frac{1}{x-1}\,\mathrm{d}x + \int_2^3 \frac{2x}{x^2+1}\,\mathrm{d}x$$

$$= \left[x\right]_2^3 - \left[\ln(x-1)\right]_2^3 + \left[\ln(x^2+1)\right]_2^3$$

$$= \left[3-2\right] - \left[(\ln(3-1) - \ln(2-1)\right] + \left[\ln((3)^2+1) - \ln((2)^2+1)\right]$$

$$= 1 - \ln 2 + \ln 1 + \ln 10 - \ln 5$$

$$= 1 + \ln 10 - (\ln 2 + \ln 5)$$

$$= 1 + \ln 10 - (\ln(2\times 5))$$

$$= 1 + \ln 10 - \ln 10$$

$$= 1$$

$\therefore \int_2^3 \mathrm{f}(x)\,\mathrm{d}x = 1$ **(Shown)**

Note that:

$\ln 1 = 0$

$\ln(a\times b) = \ln a + \ln b$

**9.** Compressed air is escaping from a container. The pressure of the air in the container at time $t$ is $P$, and the constant atmospheric pressure of the air outside the container is $A$. The rate of decrease of $P$ is proportional to the square root of the pressure difference $(P-A)$. Thus the differential equation connecting $P$ and $t$ is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = -k\sqrt{(P-A)},$$

where $k$ is a positive constant.

(i) Find, in any form, the general solution of this differential equation. [3]

(ii) Given that $P = 5A$ when $t = 0$ and that $P = 2A$ when $t = 2$, show that $k = \sqrt{A}$. [4]

(iii) Find the value of $t$ when $P = A$. [2]

(iv) Obtain an expression for $P$ in terms of $A$ and $t$. [2]

***Suggested Solution:***

(i) Given that

$$\frac{dP}{dt}=-k\sqrt{(P-A)} \Rightarrow \frac{1}{\sqrt{(P-A)}}\,dP=-k\,dt \Rightarrow (P-A)^{-\frac{1}{2}}\,dP=-k\,dt$$

integrating both sides

$$\int(P-A)^{-\frac{1}{2}}\,dP=\int -k\,dt \Rightarrow \frac{(P-A)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}=-kt+C$$

$$\Rightarrow \frac{(P-A)^{\frac{1}{2}}}{\frac{1}{2}}=-kt+C \Rightarrow 2\sqrt{(P-A)}=-kt+C \quad \text{(Ans)}$$

(ii) Let $2\sqrt{(P-A)}=-kt+C \cdots\cdots\cdots\text{(i)}$

Given that $P=5A$, when $t=0$

$$\Rightarrow 2\sqrt{(5A-A)}=-k(0)+C \Rightarrow 2\sqrt{4A}=C \Rightarrow C=4\sqrt{A}$$

$\therefore$ eq. (i) becomes

$$2\sqrt{P-A}=-kt+4\sqrt{A}$$

also given that $P=2A$ when $t=2$

$$\therefore\ 2\sqrt{(2A)-A}=-k(2)+4\sqrt{A} \Rightarrow 2\sqrt{A}=-2k+4\sqrt{A}$$

$$\Rightarrow 2k=4\sqrt{A}-2\sqrt{A} \Rightarrow 2k=2\sqrt{A} \Rightarrow k=\sqrt{A} \quad \text{(Shown)}$$

(iii) Equation (i) from part **(ii)**, with values of $k$ and $C$ found, can now be written as

$$2\sqrt{(P-A)}=-\sqrt{A}\,t+4\sqrt{A}$$

when $P=A$

$$2\sqrt{(A-A)}=-\sqrt{A}\,t+4\sqrt{A} \Rightarrow 2(0)=-\sqrt{A}\,t+4\sqrt{A}$$

$$\Rightarrow \sqrt{A}\,t=4\sqrt{A} \Rightarrow t=4 \quad \text{(Ans)}$$

(iv) We have

$$2\sqrt{(P-A)}=-\sqrt{A}\,t+4\sqrt{A}$$

$$\Rightarrow 2\sqrt{(P-A)}=\sqrt{A}(4-t)$$

squaring both sides

$$\Rightarrow \left(2\sqrt{(P-A)}\right)^2=\left(\sqrt{A}(4-t)\right)^2 \Rightarrow 4(P-A)=A(4-t)^2$$

$$\Rightarrow P-A=\frac{1}{4}A(4-t)^2 \Rightarrow P=\frac{1}{4}A(4-t)^2+A \quad \text{(Ans)}$$

10. The lines $l$ and $m$ have vector equations.

$$\mathbf{r} = \mathbf{i} - 2\mathbf{k} + s(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \quad \text{and} \quad \mathbf{r} = 6\mathbf{i} - 5\mathbf{j} + 4\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$$

respectively.

(i) Show that $l$ and $m$ intersect, and find the position vector of their point of intersection. [5]

(ii) Find the equation of the plane containing $l$ and $m$, giving your answer in the form $ax + by + cz = d$. [6]

***Suggested Solution:***

(i) Let line $l$ be $\mathbf{r}_1$ and line $m$ be $\mathbf{r}_2$

$$\therefore \quad \mathbf{r}_1 = \mathbf{i} - 2\mathbf{k} + s(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \Rightarrow \mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \Rightarrow \mathbf{r}_1 = \begin{pmatrix} 1+2s \\ s \\ -2+3s \end{pmatrix}$$

$$\text{and} \quad \mathbf{r}_2 = 6\mathbf{i} - 5\mathbf{j} + 4\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \Rightarrow \mathbf{r}_2 = \begin{pmatrix} 6 \\ -5 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \mathbf{r}_2 = \begin{pmatrix} 6+t \\ -5-2t \\ 4+t \end{pmatrix}$$

Now, if line $l$ and $m$ intersect

Then $\mathbf{r}_1 = \mathbf{r}_2$

$$\Rightarrow \begin{pmatrix} 1+2s \\ s \\ -2+3s \end{pmatrix} = \begin{pmatrix} 6+t \\ -5-2t \\ 4+t \end{pmatrix}$$

equating coefficients of i and j, we have

$$1 + 2s = 6 + t \Rightarrow 2s - t = 5 \cdots\cdots\cdots(\text{i})$$

$$s = -5 - 2t \cdots\cdots\cdots(\text{ii})$$

putting eq. (ii) into eq. (i)

$$2(-5-2t) - t = 5 \Rightarrow -10 - 4t - t = 5 \Rightarrow -5t = 15 \Rightarrow t = -3$$

$$\therefore \quad s = -5 - 2(-3) \Rightarrow s = 1$$

putting these values of s and $t$ in coefficients of **k**

line $l$: $-2 + 3s = -2 + 3(1) = 1$

line $m$: $4 + t = 4 + (-3) = 1$

As the **k** coordinates on both lines at the assumed point of intersection are same, therefore the two lines $l$ and $m$ intersect. **(Shown)**

For point of intersection, put value of $s$ in equation of line $l$

$$\therefore \begin{pmatrix} 1+2s \\ s \\ -2+3s \end{pmatrix} = \begin{pmatrix} 1+2(1) \\ 1 \\ -2+3(1) \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \text{ or } 3\mathbf{i} + \mathbf{j} + \mathbf{k} \quad \textbf{(Ans)}$$

We can also put value of $t$ in the equation of line $m$ and obtain the point of intersection.

ii) Equation of the plane containing lines $l$ and $m$ passing through point (3, 1, 1) is given by:

$$\mathbf{r}=\begin{pmatrix}3\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\3\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\1\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3+2\lambda+\mu\\1+\lambda-2\mu\\1+3\lambda+\mu\end{pmatrix}$$

$$\Rightarrow x=3+2\lambda+\mu \cdots\cdots\text{(i)}$$
$$y=1+\lambda-2\mu\cdots\cdots\text{(ii)}$$
$$z=1+3\lambda+\mu\cdots\cdots\text{(iii)}$$

from eq. (i), $\mu=x-2\lambda-3$, put in eq. (ii)

$$\Rightarrow y=1+\lambda-2(x-2\lambda-3) \Rightarrow y=1+\lambda-2x+4\lambda+6 \Rightarrow y=7+5\lambda-2x$$

$$\Rightarrow \lambda=\frac{y+2x-7}{5}$$

$$\therefore \mu=x-2\lambda-3 \Rightarrow \mu=x-2\left(\frac{y+2x-7}{5}\right)-3 \Rightarrow \mu=\frac{5x-2y-4x+14-15}{5}$$

$$\Rightarrow \mu=\frac{x-2y-1}{5}$$

putting the values of $\mu$ and $\lambda$ in eq. (iii)

$$z=1+3\lambda+\mu$$

$$\Rightarrow z=1+3\left(\frac{y+2x-7}{5}\right)+\left(\frac{x-2y-1}{5}\right)$$

$$\Rightarrow 5z=5+3y+6x-21+x-2y-1 \Rightarrow 5z=7x+y-17$$

$\therefore$ required equation of plane is: $7x+y-5z=17$ (Ans)

***Alternative Solution to part (ii):***

Using direction vectors of lines $l$ and $m$, we can find the normal vector $\mathbf{n}$

$$\therefore \mathbf{n}=\vec{d}_1\times\vec{d}_2$$

i.e. $$\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&3\\1&-2&1\end{vmatrix} \Rightarrow \mathbf{n}=\mathbf{i}(1+6)-\mathbf{j}(2-3)+\mathbf{k}(-4-1) \Rightarrow \mathbf{n}=7\mathbf{i}+\mathbf{j}-5\mathbf{k}$$

$\therefore$ equation of plane in scalar product form is given by

$$\mathbf{r}\cdot(7\mathbf{i}+\mathbf{j}-5\mathbf{k})=D$$

as this plane passes through the point of intersection (3, 1, 1)

$\therefore$ using $\mathbf{r}\cdot\mathbf{n}=\mathbf{a}\cdot\mathbf{n}$, we have

$$\mathbf{r}\cdot(7\mathbf{i}+\mathbf{j}-5\mathbf{k})=\begin{pmatrix}3\\1\\1\end{pmatrix}\cdot\begin{pmatrix}7\\1\\-5\end{pmatrix} \Rightarrow \mathbf{r}\cdot(7\mathbf{i}+\mathbf{j}-5\mathbf{k})=21+1-5 \Rightarrow \mathbf{r}\cdot(7\mathbf{i}+\mathbf{j}-5\mathbf{k})=17$$

replacing $\mathbf{r}$ by $(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$ we have, $7x+y-5z=17$ (Ans)

# June 2004 Paper 1

## Pure Mathematics (P1)

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** A geometric progression has first term 64 and sum to infinity 256. Find

(i) the common ratio, [2]

(ii) the sum of the first ten terms. [2]

*Suggested Solution:*

(i) Given that

$a = 64, \quad S_\infty = 256$

let $r$ be the common ratio

using $S_\infty = \dfrac{a}{1-r}$

$$256 = \frac{64}{1-r} \Rightarrow 1-r = \frac{64}{256} \Rightarrow 1-r = \frac{1}{4} \Rightarrow r = 1-\frac{1}{4} \Rightarrow r = \frac{3}{4}$$

$\therefore$ the common ratio $= \dfrac{3}{4}$ **(Ans)**

(ii) Using $S_n = \dfrac{a(1-r^n)}{1-r}$ we have

$$S_{10} = \frac{64(1-(0.75)^{10})}{1-0.75} = \frac{64(1-(0.75)^{10})}{0.25} = 256(1-(0.75)^{10})$$

$$= 256(1-0.056314) = 256(0.94369) = 241.58 \approx 242 \quad \textbf{(Ans)}$$

**2.** Evaluate $\int_0^1 \sqrt{(3x+1)}\, dx$. [4]

*Suggested Solution:*

$$\int_0^1 \sqrt{(3x+1)}\,dx = \int_0^1 (3x+1)^{\frac{1}{2}}\,dx = \left[\frac{(3x+1)^{\frac{3}{2}}}{3\times\frac{3}{2}}\right]_0^1 = \left[\frac{(3x+1)^{\frac{3}{2}}}{\frac{9}{2}}\right]_0^1$$

$$= \frac{2}{9}\left[(3x+1)^{\frac{3}{2}}\right]_0^1 = \frac{2}{9}\left[(3(1)+1)^{\frac{3}{2}} - (3(0)+1)^{\frac{3}{2}}\right]$$

$$= \frac{2}{9}\left[(4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] = \frac{2}{9}[8-1] = \frac{14}{9} = 1\frac{5}{9} \quad \text{(Ans)}$$

Remember:

For definite integral, do not put the integration constant

3. (i) Show that the equation $\sin^2\theta + 3\sin\theta\cos\theta = 4\cos^2\theta$ can be written as a quadratic equation in $\tan\theta$. [2]

(ii) Hence, or otherwise, solve the equation in part (i) for $0° \le \theta \le 180°$. [3]

*Suggested Solution:*

(i) $\sin^2\theta + 3\sin\theta\cos\theta = 4\cos^2\theta$

dividing throughout by $\cos^2\theta$, we have

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{3\sin\theta\cos\theta}{\cos^2\theta} = \frac{4\cos^2\theta}{\cos^2\theta}$$

$$\Rightarrow \quad \frac{\sin^2\theta}{\cos^2\theta} + \frac{3\sin\theta}{\cos\theta} = 4$$

$$\Rightarrow \quad \tan^2\theta + 3\tan\theta = 4$$

$$\Rightarrow \quad \tan^2\theta + 3\tan\theta - 4 = 0 \quad \text{(Ans)}$$

(ii) Using the result of part (i), we have

$$\tan^2\theta + 3\tan\theta - 4 = 0$$

$$\tan^2\theta + 4\tan\theta - \tan\theta - 4 = 0$$

$$\tan\theta(\tan\theta + 4) - 1(\tan\theta + 4) = 0$$

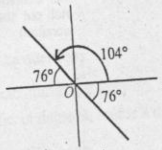
$$(\tan\theta + 4)(\tan\theta - 1) = 0$$

$$\Rightarrow \quad \tan\theta + 4 = 0 \quad \text{or} \quad \tan\theta - 1 = 0$$

$$\Rightarrow \quad \tan\theta = -4 \quad \text{or} \quad \tan\theta = 1$$

when $\tan\theta = -4$

basic angle $= 76°$

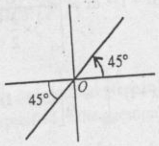


req. angle for given range

$\theta = 180° - 76° = 104°$

when $\tan\theta = 1$

basic angle $= 45°$



req. angle for given range

$\theta = 45°$

Note that $\tan\theta$ is positive in 1st and 3rd quadrant and negative in 2nd and 4th quadrant.

$\therefore\ \theta = 45°,\ 104°$ **(Ans)**

**4.** Find the coefficient of $x^3$ in the expansion of

(i) $(1+2x)^6$, [3]

(ii) $(1-3x)(1+2x)^6$. [3]

*Suggested Solution:*

Expression for general term is: $T_{r+1} = {}^nC_r\,(a)^{n-r}(b)^r$

(i) $(1+2x)^6$

$T_{r+1} = {}^6C_r\,(1)^{6-r}(2x)^r = {}^6C_r\,(2x)^r$

for term containing $x^3$, put $r = 3$

$\therefore\ T_{3+1} = {}^6C_3(2x)^3 = 20(8x^3) = 160x^3$

$\therefore$ coefficient of $x^3 = 160$ **(Ans)**

**(ii)** $(1-3x)(1+2x)^6$

Binomial theorem:

$(a+b)^n = {}^nC_0\,a^n + {}^nC_1\,a^{n-1}b + {}^nC_2\,a^{n-2}b^2 + .... + {}^nC_r\,a^{n-r}b^r + .... + {}^nC_n\,b^n$

$(1-3x)(1+2x)^6$

$\Rightarrow\ (1-3x)[1+6(2x)+{}^6C_2\,(2x)^2+{}^6C_3\,(2x)^3]$

$\Rightarrow\ (1-3x)[1+12x+15\,(4x^2)+20\,(8x^3)]$

$\Rightarrow\ (1-3x)[1+12x+60x^2+160x^3]$

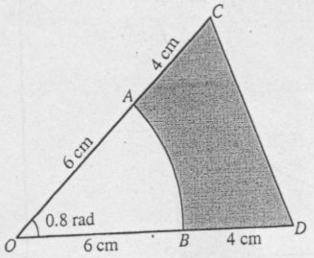
$\Rightarrow\ 1+12x+60x^2+160x^3-3x-36x^2-180x^3-480x^4$

Terms containing $x^3$ are

$160x^3 - 180x^3 = -20x^3$

$\therefore$ coefficient of $x^3 = -20$ **(Ans)**

5.



In the diagram, $OCD$ is an isosceles triangle with $OC = OD = 10$ cm and angle $COD = 0.8$ radians. The points $A$ and $B$, on $OC$ and $OD$ respectively, are joined by an arc of a circle with centre $O$ and radius 6 cm. Find

(i) the area of the shaded region, [3]

(ii) the perimeter of the shaded region. [4]

*Suggested Solution:*

(i) Shaded area = area of $\Delta OCD$ − area of sector $OAB$

Angle is in radians. Change your calculator to radian mode

$$= \frac{1}{2}(10)(10)\sin(0.8) - \frac{1}{2}(6)^2(0.8)$$

$$= 35.8678 - 14.4 = 21.467 \approx 21.5 \text{ cm}^2 \quad \text{(Ans)}$$

(ii) Applying cosine rule in $\Delta OCD$

$$CD^2 = (10)^2 + (10)^2 - 2(10)(10)\cos(0.8)$$

$$= 100 + 100 - 139.341$$

$$= 60.6587$$

or $CD = 7.788$

Applying $S = r\theta$ for arc length $AB$,

$$\widehat{AB} = 6(0.8) = 4.8$$

$\therefore$ perimeter of the shaded region $= 4.8 + 4 + 7.788 + 4 = 20.588 \approx 20.6$ cm (Ans)

6. The curve $y = 9 - \frac{6}{x}$ and the line $y + x = 8$ intersect at two points. Find

(i) the coordinates of the two points, [4]

(ii) the equation of the perpendicular bisector of the line joining the two points. [4]

*Suggested Solution:*

(i) Given that $y+x=8 \Rightarrow y=8-x$

putting in $y=9-\frac{6}{x}$

$$8-x=9-\frac{6}{x}$$

$\Rightarrow \quad 8x-x^2=9x-6$

$\Rightarrow \quad x^2+9x-8x-6=0$

$\Rightarrow \quad x^2+x-6=0$

$\Rightarrow \quad x^2+3x-2x-6=0$

$\Rightarrow \quad x(x+3)-2(x+3)=0$

$\Rightarrow \quad (x+3)(x-2)=0$

$\Rightarrow \quad x=-3 \quad$ or $\quad x=2$

find the value of $y$ from linear equation

when $x=-3, \quad y=8-(-3) \Rightarrow y=11$

$\therefore$ (−3, 11) is a point of intersection **(Ans)**

when $x=2, \quad y=8-(2) \Rightarrow y=6$

$\therefore$ (2, 6) is other point of intersection **(Ans)**

(ii) Let points of intersections be $A(-3, 11)$ and $B(2, 6)$

If two lines are perpendicular than product of their gradients is equal to −1.

$\therefore$ coordinates of mid point of $AB=\left(\frac{-3+2}{2}, \frac{11+6}{2}\right)=\left(-\frac{1}{2}, \frac{17}{2}\right)$

gradient of $AB=\frac{6-11}{2-(-3)}=\frac{-5}{5}=-1$

132

$\Rightarrow$ gradient of perpendicular bisector = +1

$\therefore$ equation of perpendicular bisector passing through $\left(-\frac{1}{2}, \frac{17}{2}\right)$ with gradient = 1 is:

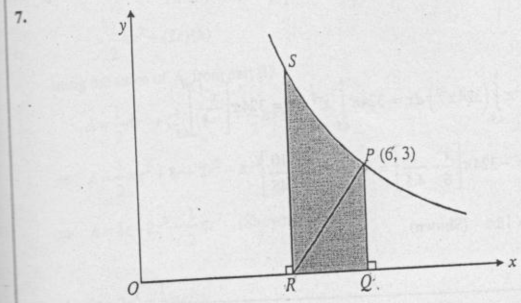
Equation of a line in point-slope form is: $y-y_1=m(x-x_1)$

$$y-\frac{17}{2}=1(x-(-\frac{1}{2}))$$

$\Rightarrow \quad y-\frac{17}{2}=x+\frac{1}{2}$

$\Rightarrow \quad 2y-17=2x+1 \quad \Rightarrow \quad 2y-2x=18 \quad$ or $\quad y-x=9$ **(Ans)**

7.



The diagram shows part of the graph of $y = \frac{18}{x}$ and the normal to the curve at $P$ (6, 3). This normal meets the $x$-axis at $R$. The point $Q$ on the $x$-axis and the point $S$ on the curve are such that $PQ$ and $SR$ are parallel to the $y$-axis.

(i) Find the equation of the normal at $P$ and show that $R$ is the point $\left(4\frac{1}{2}, 0\right)$ [5]

(ii) Show that the volume of the solid obtained when the shaded region $PQRS$ is rotated through $360^0$ about the $x$-axis is $18\pi$. [4]

*Suggested Solution:*

(i) $y = \frac{18}{x} \Rightarrow y = 18x^{-1}$

Gradient of a curve at any point ($x$, $y$), is given by $\frac{dy}{dx}$

differentiating w. r. t. $x$

$$\frac{dy}{dx} = 18(-x^{-2}) = -\frac{18}{x^2}$$

at point $P(6, 3)$

$$\frac{dy}{dx} = -\frac{18}{(6)^2} = -\frac{1}{2}$$

Tangent and normal to the curve are perpendicular to each other, and for perpendicular lines

$m_1 \times m_2 = -1$

$\therefore$ gradient of the tangent at $P = -\frac{1}{2}$

$\Rightarrow$ gradient of the normal at $P = 2$

$\therefore$ equation of the normal at $P(6, 3)$ is

$y - 3 = 2(x - 6) \Rightarrow y - 3 = 2x - 12 \Rightarrow y = 2x - 9$ **(Ans)**

Equation of a line in point-slope form is:
$y - y_1 = m(x - x_1)$

the normal meets the $x$-axis at $R$, therefore putting $y = 0$ in the above equation

$$0 = 2x - 9 \Rightarrow 2x = 9 \Rightarrow x = \frac{9}{2} = 4\frac{1}{2}$$

$\therefore$ coordinates of $R\left(4\frac{1}{2}, 0\right)$ **(Shown)**

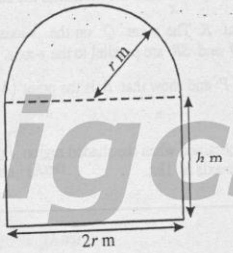
(ii) Volume $V = \pi \int_{4.5}^{6} y^2\, dx$

$$\Rightarrow \quad V = \pi \int_{4.5}^{6} \left(\frac{18}{x}\right)^2 dx = \pi \int_{4.5}^{6} \left(324x^{-2}\right) dx = 324\pi \int_{4.5}^{6} x^{-2}\, dx = 324\pi \left[\frac{x^{-1}}{-1}\right]_{4.5}^{6}$$

$$= -324\pi \left[\frac{1}{x}\right]_{4.5}^{6} = -324\pi \left[\frac{1}{6} - \frac{1}{4.5}\right] = -324\pi \left[\frac{1}{6} - \frac{10}{45}\right]$$

$$= -324\pi \left[-\frac{1}{18}\right] = 18\pi \quad \text{(Shown)}$$

**8.**



The diagram shows a glass window consisting of a rectangle of height $h$ m and width $2r$ m and a semicircle of radius $r$ m. The perimeter of the window is 8 m.

(i) Express $h$ in terms of $r$. [2]

(ii) Show that the area of the window, $A$ m$^2$, is given by

$$A = 8r - 2r^2 - \frac{1}{2}\pi r^2 \qquad [2]$$

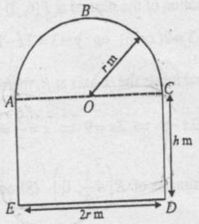
Given that $r$ can vary,

(iii) find the value of $r$ for which $A$ has a stationary value, [4]

(iv) determine whether this stationary value is a maximum or a minimum. [2]

***Suggested Solution:***

(i) Let $O, A, B, C, D, E$ be the points as shown in the figure.



perimeter of semicircle $ABC = \frac{1}{2}(2\pi r) = \pi r$

perimeter of the window $= 8$ (given)

$\Rightarrow \quad \widehat{ABC} + CD + DE + EA = 8$

$\Rightarrow \quad \pi r + h + 2r + h = 8$

$\Rightarrow \quad \pi r + 2h + 2r = 8$

$\Rightarrow \quad 2h = 8 - 2r - \pi r$

$\Rightarrow \quad h = \frac{1}{2}(8 - 2r - \pi r) = 4 - r - \frac{1}{2}\pi r$ (Ans)

(ii) Area $A$ = area of semicircle $ABC$ + area of rectangle $ACDE$

$$=\frac{1}{2}\pi r^2+(2r)(h)$$

using the value of $h$ from part (i)

$$A=\frac{1}{2}\pi r^2+(2r)(4-r-\frac{1}{2}\pi r)$$

$$\Rightarrow\quad A=\frac{1}{2}\pi r^2+8r-2r^2-\pi r^2$$

$$\Rightarrow\quad A=8r-2r^2-\frac{1}{2}\pi r^2 \quad \text{(Shown)}$$

(iii) $A=8r-2r^2-\frac{1}{2}\pi r^2$

$$\frac{dA}{dr}=8-2(2r)-\frac{1}{2}\pi(2r)=8-4r-\pi r$$

for stationary value $\frac{dA}{dr}=0$

$$\Rightarrow\ 8-4r-\pi r=0\ \Rightarrow\ 4r+\pi r=8\ \Rightarrow\ r(4+\pi)=8\ \Rightarrow\ r=\frac{8}{4+\pi}\quad\text{(Ans)}$$

(iv) $\frac{dA}{dr}=8-4r-\pi r$

$$\therefore\ \frac{d^2A}{dr^2}=-4-\pi=-(4+\pi)\ <0$$

$$\Rightarrow\ A \text{ is maximum for } r=\frac{8}{4+\pi}\quad\text{(Ans)}$$

**9.** Relative to an origin $O$, the position vectors of the points $A$, $B$, $C$ and $D$ are given by

$$\overline{OA}=\begin{pmatrix}1\\3\\-1\end{pmatrix},\quad \overline{OB}=\begin{pmatrix}3\\-1\\3\end{pmatrix},\quad \overline{OC}=\begin{pmatrix}4\\2\\p\end{pmatrix}\ \text{and}\ \overline{OD}=\begin{pmatrix}-1\\0\\q\end{pmatrix},$$

Where $p$ and $q$ are constants. Find

(i) the unit vector in the direction of $\overline{AB}$ [3]

(ii) the value of $p$ for which angle $AOC = 90°$, [3]

(iii) the values of $q$ for which the length of $\overrightarrow{AD}$ is 7 units. [4]

*Suggested Solution:*

(i) $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA} \Rightarrow \overrightarrow{AB}=\begin{pmatrix}3\\-1\\3\end{pmatrix}-\begin{pmatrix}1\\3\\-1\end{pmatrix} \Rightarrow \overrightarrow{AB}=\begin{pmatrix}2\\-4\\4\end{pmatrix}$

$$\left|\overrightarrow{AB}\right|=\sqrt{(2)^2+(-4)^2+(4)^2}=\sqrt{36}=6$$

If
$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, then
magnitude of $\mathbf{a}$ is
$|\mathbf{a}|=\sqrt{(a_1)^2+(a_2)^2+(a_3)^2}$

unit vector in the direction of $\overrightarrow{AB}=\dfrac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}=\dfrac{1}{6}\begin{pmatrix}2\\-4\\4\end{pmatrix}=\begin{pmatrix}\frac{1}{3}\\-\frac{2}{3}\\\frac{2}{3}\end{pmatrix}$ (Ans)

For any vector $\mathbf{v}$, the unit vector $\hat{\mathbf{v}}$, is given as:
$\hat{\mathbf{v}}=\dfrac{\mathbf{v}}{|\mathbf{v}|}$

(ii) Given that $A\hat{O}C=90°$

$\Rightarrow \overrightarrow{OA}\bullet\overrightarrow{OC}=0$

$\Rightarrow \begin{pmatrix}1\\3\\-1\end{pmatrix}\bullet\begin{pmatrix}4\\2\\p\end{pmatrix}=0 \Rightarrow 4+6-p=0 \Rightarrow p=10$ (Ans)

If
$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, and
$\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, then
$\mathbf{a}.\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$

(iii) $\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=\begin{pmatrix}-1\\0\\q\end{pmatrix}-\begin{pmatrix}1\\3\\-1\end{pmatrix}=\begin{pmatrix}-2\\-3\\q+1\end{pmatrix}$

Given that $\left|\overrightarrow{AD}\right|=7$

$\Rightarrow \sqrt{(-2)^2+(-3)^2+(q+1)^2}=7$

$\Rightarrow \sqrt{4+9+(q+1)^2}=7$

$\Rightarrow 4+9+(q+1)^2=49$

$\Rightarrow (q+1)^2=49-13$

$\Rightarrow (q+1)^2=36 \Rightarrow q+1=\pm 6$

$\Rightarrow q+1=6$ or $q+1=-6$

$\therefore q=5$ or $q=-7$ (Ans)

**10.** The functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto x^2-2x, \quad x\in\mathbb{R}$$
$$\mathrm{g}: x \mapsto 2x+3, \quad x\in\mathbb{R}$$

(i) Find the set of values of $x$ for which $\mathrm{f}(x)>15$. [3]

(ii) Find the range of f and state, with a reason, whether f has an inverse. [4]

(iii) Show that the equation $\mathrm{gf}(x)=0$ has no real solutions. [3]

(iv) Sketch, in a single diagram, the graphs of $y=\mathrm{g}(x)$ and $y=\mathrm{g}^{-1}(x)$, making clear the relationship between the graphs. [2]

*Suggested Solution:*

(i) $f(x) > 15$

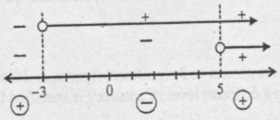
$\Rightarrow \quad x^2 - 2x > 15$

$\Rightarrow \quad x^2 - 2x - 15 > 0$

$\Rightarrow \quad x^2 + 3x - 5x - 15 > 0$

$\Rightarrow \quad x(x+3) - 5(x+3) > 0$

$\Rightarrow \quad (x+3)(x-5) > 0$

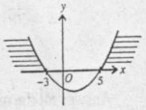


critical values are $x = -3$, and $x = 5$

hence positive sections of the line are

$\Rightarrow \quad x < -3, \quad x > 5$ **(Ans)**

Alternatively



We can also find the set of values of $x$ by drawing a curve for $(x+3)(x-5) > 0$ as shown above.

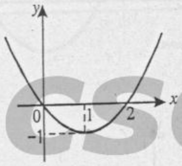
(ii) $f(x) = x^2 - 2x \quad \Rightarrow \quad f(x) = x(x-2)$

critical values are $x = 0$, and $x = 2$

A quick sketch shows that minimum value exists at $x = 1$

$\therefore \quad f(1) = (1)^2 - 2(1) = 1 - 2 = -1$

$\Rightarrow$ range of $f(x)$ is $f(x) \geq -1$ **(Ans)**



$f(x)$ is not a 1-1 function and therefore does not have an inverse. **(Ans)**

(iii) $gf(x) = g(x^2 - 2x)$

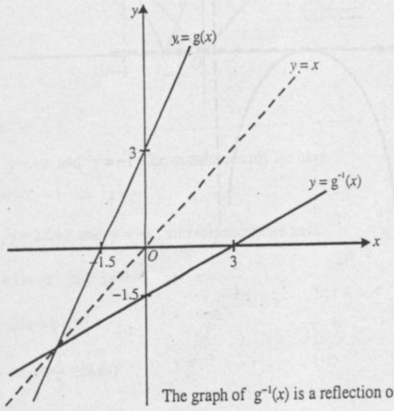
$= 2(x^2 - 2x) + 3 = 2x^2 - 4x + 3$

$a = 2, \quad b = -4, \quad c = 3$

Discriminant $= b^2 - 4ac = (-4)^2 - 4(2)(3) = 16 - 24 = -8 < 0$

$\Rightarrow$ $gf(x)$ has no real roots **(Ans)**

(iv)



The graph of $g^{-1}(x)$ is a reflection of graph of $g(x)$ in the line $y = x$

# June 2004 Paper 3

## Pure Mathematics (P3)

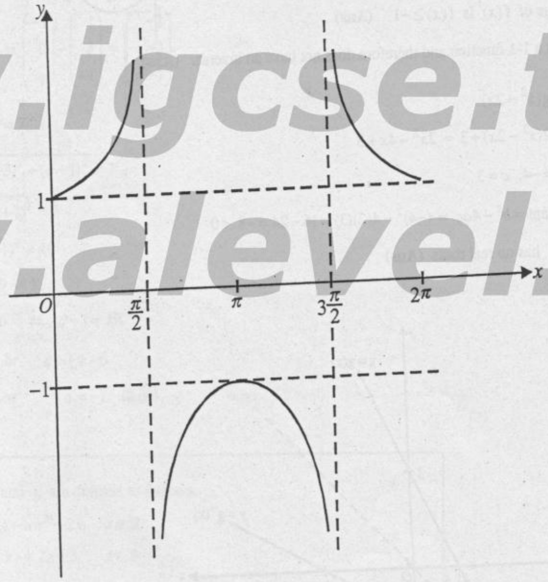
Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Sketch the graph of $y = \sec x$, for $0 \le x \le 2\pi$. [3]

*Suggested Solution:*



2. Solve the inequality $|2x+1|<|x|$. [4]

***Suggested Solution:***

$$|2x+1|<|x|$$

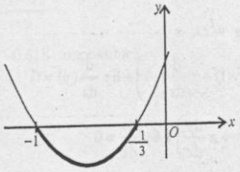
squaring both sides

$$\left(|2x+1|\right)^2<\left(|x|\right)^2 \Rightarrow 4x^2+4x+1<x^2 \Rightarrow 3x^2+4x+1<0$$

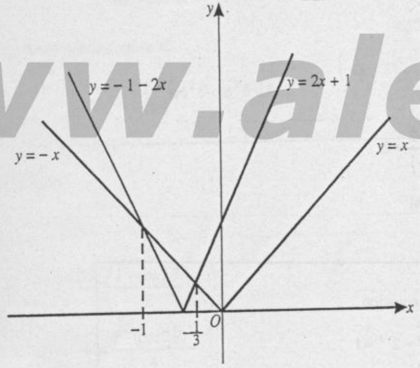
$$\Rightarrow 3x^2+3x+x+1<0 \Rightarrow (x+1)(3x+1)<0$$

critical values are $x=-\frac{1}{3},\quad x=-1$

$\therefore\quad -1<x<-\frac{1}{3}$ (Ans)



***Alternative Solution (Using Sketch method):***



Solving $y=-x$ and $y=-1-2x$ simultaneously we have

$$-x=-2x-1 \Rightarrow x=-1$$

Solving $y=2x+1$ and $y=-x$ simultaneously we have

$$2x+1=-x \Rightarrow 3x=-1 \Rightarrow x=-\frac{1}{3}$$

$\therefore$ for $|2x+1|<|x|$

$-1<x<-\frac{1}{3}$ (Ans)

3. Find the gradient of the curve with equation

$$2x^2 - 4xy + 3y^2 = 3,$$

at the point (2, 1). [4]

***Suggested Solution:***

$$2x^2 - 4xy + 3y^2 = 3$$

differentiating w.r.t. $x$

$$4x - 4\left(y(1) + x\frac{\mathrm{d}}{\mathrm{d}x}(y)\right) + 6y\frac{\mathrm{d}}{\mathrm{d}x}(y) = 0$$

$$\Rightarrow \quad 4x - 4\left(y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 6y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\Rightarrow \quad 4x - 4y - 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 6y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}(6y - 4x) = 4y - 4x$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4y - 4x}{6y - 4x} = \frac{2(2y - 2x)}{2(3y - 2x)}$$

$$\therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y - 2x}{3y - 2x}$$

at the point (2, 1)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2(1) - 2(2)}{3(1) - 2(2)} = \frac{2 - 4}{3 - 4} = \frac{-2}{-1} = 2$$

$\therefore$ gradient of curve $= 2$ **(Ans)**

4. (i) Show that if $y = 2^x$, then the equation

$$2^x - 2^{-x} = 1$$

can be written as a quadratic equation in $y$. [2]

(ii) Hence solve the equation

$$2^x - 2^{-x} = 1.$$ [4]

***Suggested Solution:***

(i) $2^x - 2^{-x} = 1 \Rightarrow 2^x - \frac{1}{2^x} = 1$

given that $y = 2^x$

$\Rightarrow y - \frac{1}{y} = 1 \Rightarrow y^2 - 1 = y \Rightarrow y^2 - y - 1 = 0$ (req. quad. eq. in $y$) **(Ans)**

(iii) Using the result of part (ii), we have

$$y^2 - y - 1 = 0$$

using quadratic formula

$$y = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

either $y = \frac{1+\sqrt{5}}{2}$ or $y = \frac{1-\sqrt{5}}{2}$

$\Rightarrow \quad 2^x = \frac{1+\sqrt{5}}{2}$ or $\Rightarrow \quad 2^x = \frac{1-\sqrt{5}}{2}$

$\Rightarrow \quad 2^x = 1.618$ or $\Rightarrow \quad 2^x = -0.618$ (impossible)

$\Rightarrow \quad \ln(2^x) = \ln(1.618)$

$\Rightarrow \quad x \ln 2 = \ln(1.618)$

$\Rightarrow \quad x = \frac{\ln(1.618)}{\ln 2} \quad \Rightarrow \quad x = 0.694$ (Ans)

5. (i) Prove the identity

$$\sin^2\theta\cos^2\theta \equiv \frac{1}{8}(1-\cos 4\theta). \qquad [3]$$

(ii) Hence find the exact value of

$$\int_0^{\frac{1}{3}\pi} \sin^2\theta\cos^2\theta \, d\theta. \qquad (4) \qquad [3]$$

***Suggested Solution:***

(i) $\text{L.H.S} = \sin^2\theta\cos^2\theta$

$$= \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)$$

$$= \frac{1-\cos^2 2\theta}{4} = \frac{1-(\cos 2\theta)^2}{4}$$

$$= \frac{1-\frac{(1+\cos 4\theta)}{2}}{4}$$

$$= \frac{\frac{2-1-\cos 4\theta)}{2}}{4}$$

$$= \frac{1-\cos 4\theta}{8}$$

$$= \frac{1}{8}(1-\cos 4\theta) = \text{R.H.S} \quad \textbf{(Proved)}$$

Note that:

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

also

$$\cos^2(2\theta) = \frac{1+\cos 2(2\theta)}{2}$$

$$= \frac{1+\cos 4\theta}{2}$$

(ii) $\int_0^{\frac{1}{3}\pi} \sin^2\theta\cos^2\theta\ \mathrm{d}\theta$

Using the result of pat (i), the above equation can be written as

$$\int_0^{\frac{1}{3}\pi} \frac{1}{8}(1-\cos 4\theta)\ \mathrm{d}\theta$$

$$= \frac{1}{8}\int_0^{\frac{1}{3}\pi}(1-\cos 4\theta)\ \mathrm{d}\theta$$

$$= \frac{1}{8}\int_0^{\frac{1}{3}\pi} 1\ \mathrm{d}\theta - \frac{1}{8}\int_0^{\frac{1}{3}\pi}\cos 4\theta\ \mathrm{d}\theta$$

$$= \frac{1}{8}\left[\theta\right]_0^{\frac{\pi}{3}} - \frac{1}{8}\left[\frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{3}}$$

Remember:

For definite integral, do not put the constant of integration.

$$= \frac{1}{8}\left[\frac{\pi}{3}-0\right] - \frac{1}{8}\left[\frac{\sin 4(\frac{\pi}{3})}{4} - \frac{\sin 4(0)}{4}\right]$$

$$= \frac{1}{24}\pi - \frac{1}{8}\left[\frac{\sin\frac{4}{3}\pi}{4}\right]$$

$$= \frac{1}{24}\pi - \frac{1}{32}\left[\sin\frac{4}{3}\pi\right] = \frac{1}{24}\pi - \frac{1}{32}\left[-\frac{\sqrt{3}}{2}\right]$$

$$= \frac{1}{24}\pi + \frac{\sqrt{3}}{64} = \frac{1}{8}\left(\frac{1}{3}\pi + \frac{\sqrt{3}}{8}\right) \quad \textbf{(Ans)}$$

**6.** Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^3+1}{y^2},$$

obtaining an expression for $y$ in terms of $x$. [6]

***Suggested Solution:***

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^3+1}{y^2} \Rightarrow \left(\frac{y^2}{y^3+1}\right)\mathrm{d}y = \mathrm{d}x$$

integrating both sides

$$\int \frac{y^2}{y^3+1}\ \mathrm{d}y = \int 1\ \mathrm{d}x$$

multiplying and dividing the left hand side by 3

$$\frac{1}{3}\int\frac{3y^2}{y^3+1}\,\mathrm{d}y=\int 1\,\mathrm{d}x$$

$$\Rightarrow \quad \frac{1}{3}\ln\left(y^3+1\right)=x+C$$

given that $y=1$ when $x=0$

$$\Rightarrow \quad \frac{1}{3}\ln\left((1)^3+1\right)=0+C \quad \Rightarrow \quad \frac{1}{3}\ln(2)=C \quad \Rightarrow \quad C=\frac{1}{3}\ln 2$$

$$\therefore \quad \frac{1}{3}\ln\left(y^3+1\right)=x+\frac{1}{3}\ln 2 \quad \Rightarrow \quad \frac{1}{3}\ln\left(y^3+1\right)-\frac{1}{3}\ln 2=x$$

$$\Rightarrow \quad \frac{1}{3}\left(\ln(y^3+1)-\ln 2\right)=x \quad \Rightarrow \quad \frac{1}{3}\ln\left(\frac{y^3+1}{2}\right)=x$$

$$\Rightarrow \quad \ln\left(\frac{y^3+1}{2}\right)=3x \quad \Rightarrow \quad \ln\left(\frac{y^3+1}{2}\right)=(3x)\ln e$$

$$\Rightarrow \quad \ln\left(\frac{y^3+1}{2}\right)=\ln e^{3x} \quad \Rightarrow \quad \frac{y^3+1}{2}=e^{3x} \quad \Rightarrow \quad y^3+1=2e^{3x}$$

$$\Rightarrow \quad y^3=2e^{3x}-1 \quad \Rightarrow \quad y=\left(2e^{3x}-1\right)^{\frac{1}{3}} \quad \text{(Ans)}$$

7\. (i) The equation $x^3+x+1=0$ has one real root. Show by calculation that this root lies between $-1$ and $0$. [2]

(ii) Show that, if a sequence of values given by the iterative formula.

$$x_{n+1}=\frac{2x_n^3-1}{3x_n^2+1}$$

Converges, then it converges to the root of the equation given in part (i). [2]

(iii) Use this iterative formula, with initial value $x_1=-0.5$, to determine the root correct to 2 decimal places, showing the result of each iteration. [3]

*Suggested Solution:*

(i) Let $\mathrm{f}(x)=x^3+x+1$

$\therefore$ $\mathrm{f}(-1)=(-1)^3+(-1)+1=-1-1+1=-1 \quad <0$

and $\mathrm{f}(0)=(0)^3+(0)+1=1 \quad >0$

$\therefore$ there is a root between $-1$ and $0$ **(Shown)**

(ii) Given that

$$x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 + 1}$$

removing the subscripts, we have

$$x = \frac{2x^3 - 1}{3x^2 + 1} \Rightarrow 3x^3 + x = 2x^3 - 1 \Rightarrow x^3 + x + 1 = 0$$

which is the same equation as in part (i) **(Ans)**

(iii) $x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 + 1}$

Given that initial value $x_1 = -0.5$

$$\therefore \quad x_2 = \frac{2x_1^3 - 1}{3x_1^2 + 1} = \frac{2(-0.5)^3 - 1}{3(-0.5)^2 + 1} = \frac{-1.25}{1.75} = -0.714286$$

$$x_3 = \frac{2x_2^3 - 1}{3x_2^2 + 1} = \frac{2(-0.714286)^3 - 1}{3(-0.714286)^2 + 1} = \frac{-1.728864}{2.530613} = -0.683180$$

$$x_4 = \frac{2x_3^3 - 1}{3x_3^2 + 1} = \frac{2(-0.683180)^3 - 1}{3(-0.683180)^2 + 1} = \frac{-1.637728}{2.400205} = -0.682328$$

$\therefore$ the required root $\alpha = -0.68$ (to 2 decimal places) **(Ans)**

8. (i) Find the roots of the equation $z^2 - z + 1 = 0$, giving your answers in the form $x + iy$, where $x$ and $y$ are real. [2]

(ii) Obtain the modulus and argument of each root. [3]

(iii) Show that each root also satisfies the equation $z^3 = -1$. [2]

*Suggested Solution:*

(i) $z^2 - z + 1 = 0$,

applying quadratic formula

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} \Rightarrow z = \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow z = \frac{1 \pm \sqrt{-3}}{2}$$

$$\Rightarrow z = \frac{1+\sqrt{-3}}{2} \quad \text{or} \quad z = \frac{1-\sqrt{-3}}{2}$$

$$\Rightarrow z = \frac{1}{2} + \frac{\sqrt{3 \times -1}}{2}, \quad z = \frac{1}{2} - \frac{\sqrt{3 \times -1}}{2}$$

$$\Rightarrow z = \frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad z = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Note that,
$i = \sqrt{-1}$

$\therefore$ the roots are $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ or $\frac{1}{2} - i\frac{\sqrt{3}}{2}$ **(Ans)**

(ii) First root: $z=\frac{1}{2}+\mathrm{i}\frac{\sqrt{3}}{2}$

Modulus: $|z|=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{\frac{4}{4}}=1$ (Ans)

Argument: $\arg(z)=\tan^{-1}\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\tan^{-1}\sqrt{3}=\frac{\pi}{3}$ (Ans)

Remember:
The argument of a complex number must be in radians.

Second root: $z=\frac{1}{2}-\mathrm{i}\frac{\sqrt{3}}{2}$

Modulus: $|z|=\sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=\sqrt{\frac{4}{4}}=1$ (Ans)

Argument: $\arg(z)=\tan^{-1}\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\tan^{-1}(-\sqrt{3})=-\frac{\pi}{3}$ (Ans)

(iii) $z^3=-1$

$\Rightarrow \quad z^3+1=0$

$\Rightarrow \quad (z)^3+(1)^3=0$

$\Rightarrow \quad (z+1)(z^2-z+1)=0$

since $(z^2-z+1)$ is a factor of $z^3=-1$

$\therefore$ two roots obtained in part (i) are also the roots of the equation $z^3=-1$

Note:
$a^3+b^3=(a+b)(a^2-ab+b^2)$

9. Let $\mathrm{f}(x)=\frac{x^2+7x-6}{(x-1)(x-2)(x+1)}$.

(i) Express f($x$) in partial fractions. [4]

(ii) Show that, when $x$ is sufficiently small fox $x^4$ and higher powers to be neglected,

$$\mathrm{f}(x)=-3+2x-\frac{3}{2}x^2+\frac{11}{4}x^3.$$ [5]

*Suggested Solution:*

(i) $$f(x)=\frac{x^2+7x-6}{(x-1)(x-2)(x+1)}\equiv\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x+1}$$

$$\Rightarrow\quad x^2+7x-6\equiv A(x-2)(x+1)+B(x-1)(x+1)+C(x-1)(x-2)$$

for $x=1$

$$(1)^2+7(1)-6=A(1-2)(1+1)+B(1-1)(1+1)+C(1-1)(1-2)$$

$$\Rightarrow\quad 1+7-6=A(-1)(2)+B(0)+C(0)\quad\Rightarrow\quad 2=-2A\quad\Rightarrow\quad A=-1$$

for $x=2$

$$(2)^2+7(2)-6=A(2-2)(2+1)+B(2-1)(2+1)+C(2-1)(2-2)$$

$$\Rightarrow\quad 4+14-6=A(0)+B(1)(3)+C(0)\quad\Rightarrow\quad 12=3B\quad\Rightarrow\quad B=4$$

for $x=-1$

$$(-1)^2+7(-1)-6=A(-1-2)(-1+1)+B(-1-1)(-1+1)+C(-1-1)(-1-2)$$

$$\Rightarrow\quad 1-7-6=A(0)+B(0)+C(-2)(-3)\quad\Rightarrow\quad -12=6C\quad\Rightarrow\quad C=-2$$

$$\therefore\quad f(x)=-\frac{1}{x-1}+\frac{4}{x-2}-\frac{2}{x+1}\quad\text{(Ans)}$$

(ii) $$f(x)=-\frac{1}{x-1}+\frac{4}{x-2}-\frac{2}{x+1}=\frac{-1}{-1+x}+\frac{4}{-2+x}-\frac{2}{1+x}$$

$$=\frac{-1}{-(1-x)}+\frac{4}{-2\left(1-\frac{x}{2}\right)}-\frac{2}{1+x}=\frac{1}{1-x}-\frac{2}{1-\frac{x}{2}}-\frac{2}{1+x}$$

$$=(1-x)^{-1}-2\left(1-\frac{x}{2}\right)^{-1}-2(1+x)^{-1}$$

using binomial expansion neglecting $x^4$ and higher powers

Binomial expansion:

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\dots$$

$$f(x)=\left[1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+\frac{(-1)(-2)(-3)}{3!}(-x)^3\right]$$

$$-2\left[1+(-1)\left(-\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(-\frac{x}{2}\right)^3\right]$$

$$-2\left[1+(-1)(x)+\frac{(-1)(-2)}{2!}(x)^2+\frac{(-1)(-2)(-3)}{3!}(x)^3\right]$$

$$=(1+x+x^2+x^3)\quad-2\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}\right)-2(1-x+x^2-x^3)$$

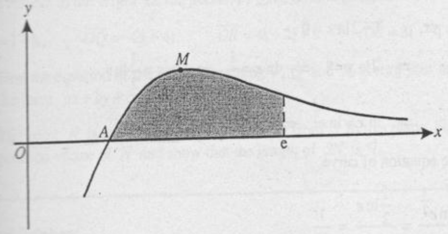
$$=1+x+x^2+x^3-2-x-\frac{x^2}{2}-\frac{x^3}{4}-2+2x-2x^2+2x^3$$

$$=1-2-2+x-x+2x+x^2-\frac{x^2}{2}-2x^2+x^3-\frac{x^3}{4}+2x^3$$

$$=-3+2x-\frac{3}{2}x^2+\frac{11}{4}x^3$$

$$\therefore\quad f(x)=-3+2x-\frac{3}{}x^2+\frac{11}{}x^3\quad\text{(Shown)}$$

**10.**



The diagram shows the curve $y=\frac{\ln x}{x^2}$ and its maximum point $M$. The curve cuts the $x$–axis at $A$.

(i) Write down the $x$-coordinate of $A$. [1]

(ii) Find the exact coordinates of $M$. [5]

(iii) Use integration by parts to find the exact area of the shaded region enclosed by the curve, the $x$-axis and the line $x=\text{e}$. [5]

*Suggested Solution:*

(i) Given that

$$y=\frac{\ln x}{x^2}$$

point $A$ is the $x$-intercept of the curve, $\therefore$ put $y=0$

$$\Rightarrow \quad \frac{\ln x}{x^2}=0 \;\Rightarrow\; \ln x=0 \;\Rightarrow\; \ln x=\ln 1 \;\Rightarrow\; x=1$$

$\therefore$ $x$-coordinate of $A$ is $x=1$ **(Ans)**

(ii) $y=\frac{\ln x}{x^2}$

differentiating w.r.t. $x$

$$\frac{\text{d}}{\text{d}x}(y)=\frac{\text{d}}{\text{d}x}\left(\frac{\ln x}{x^2}\right) \;\Rightarrow\; \frac{\text{d}y}{\text{d}x}=\frac{x^2\frac{\text{d}}{\text{d}x}(\ln x)-\ln x\frac{\text{d}}{\text{d}x}(x^2)}{(x^2)^2}$$

$$\Rightarrow \quad \frac{\text{d}y}{\text{d}x}=\frac{x^2\left(\frac{1}{x}\right)-\ln x(2x)}{x^4} \;\Rightarrow\; \frac{\text{d}y}{\text{d}x}=\frac{x-2x\ln x}{x^4}$$

for stationary value $\frac{\text{d}y}{\text{d}x}=0$

$$\Rightarrow \quad \frac{x-2x\ln x}{x_4}=0 \;\Rightarrow\; x-2x\ln x=0 \;\Rightarrow\; x(1-2\ln x)=0$$

either

$$x = 0 \text{ (not possible)} \quad \text{or} \quad 1 - 2\ln x = 0$$

$$\Rightarrow 2\ln x = 1 \Rightarrow \ln x = \frac{1}{2} \Rightarrow \ln x = \frac{1}{2}\ln e$$

$$\Rightarrow \ln x = \ln e^{\frac{1}{2}} \Rightarrow x = e^{\frac{1}{2}} \text{ or } x = \sqrt{e}$$

putting the value of $x$ in the equation of curve

$$y = \frac{\ln x}{x^2} = \frac{\ln\sqrt{e}}{\left(\sqrt{e}\right)^2} = \frac{\ln e^{\frac{1}{2}}}{e} = \frac{\frac{1}{2}\ln e}{e} = \frac{1}{2e}$$

$\therefore$ coordinates of $M\left(\sqrt{e}, \frac{1}{2e}\right)$ **(Ans)**

**(iii)** Shaded area, $A = \int_1^e y \, dx$

$$\Rightarrow A = \int_1^e \frac{\ln x}{x^2} \, dx \Rightarrow A = \int_1^e (\ln x)x^{-2} \, dx$$

integrating by parts, we have

$$A = \left[\ln x\left(\frac{x^{-1}}{-1}\right)\right]_1^e - \int_1^e \left(\frac{x^{-1}}{-1}\right) \times \frac{d}{dx}(\ln x) \, dx$$

$$\Rightarrow A = \left[-\frac{\ln x}{x}\right]_1^e + \int_1^e \left(\frac{1}{x}\right) \times \frac{1}{x} \, dx \Rightarrow A = \left[-\frac{\ln x}{x}\right]_1^e + \int_1^e x^{-2} \, dx$$

$$\Rightarrow A = \left[-\frac{\ln x}{x}\right]_1^e + \left[\frac{x^{-1}}{-1}\right]_1^e \Rightarrow A = \left[-\frac{\ln x}{x}\right]_1^e - \left[\frac{1}{x}\right]_1^e$$

$$\Rightarrow A = \left[\left(-\frac{\ln e}{e}\right) - \left(-\frac{\ln 1}{1}\right)\right] - \left[\frac{1}{e} - \frac{1}{1}\right]$$

$$\Rightarrow A = \left[\left(-\frac{1}{e}\right) - 0\right] - \left[\frac{1}{e} - 1\right] \Rightarrow A = -\frac{1}{e} - \frac{1}{e} + 1$$

$$\Rightarrow A = 1 - \frac{2}{e} \text{ sq. units} \quad \textbf{(Ans)}$$

11. With respect to the origin $O$, the points $P, Q, R, S$ have position vectors given by

$$\overrightarrow{OP} = \mathbf{i} - \mathbf{k}, \quad \overrightarrow{OQ} = -2\mathbf{i} + 4\mathbf{j}, \quad \overrightarrow{OR} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \overrightarrow{OS} = 3\mathbf{i} + 5\mathbf{j} - 6\mathbf{k}.$$

(i) Find the equation of the plane containing $P$, $Q$ and $R$, giving your answer in the form $ax + by + cz = d$. [6]

(ii) The point $N$ is the foot of the perpendicular from $S$ to this plane. Find the position vector of $N$ and show that the length of $SN$ is 7. [6]

*Suggested Solution:*

(i) Given that $\overrightarrow{OP} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \overrightarrow{OQ} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}, \quad \overrightarrow{OR} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$

Let $\overline{d_1} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$

and $\overline{d_2} = \overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$

Now, the normal vector $\mathbf{n}$ which is perpendicular to the two direction vectors $\overline{d_1}$ and $\overline{d_2}$ is given by

$$\mathbf{n} = \overline{d_1} \times \overline{d_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 3 & 2 & 2 \end{vmatrix} = \mathbf{i}(8-2) - \mathbf{j}(-6-3) + \mathbf{k}(-6-12) = 6\mathbf{i} + 9\mathbf{j} - 18\mathbf{k}$$

$\therefore \quad \mathbf{n} = 6\mathbf{i} + 9\mathbf{j} - 18\mathbf{k} \Rightarrow \mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ or $\begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$

Now, equation of the plane in scalar product form is

$$\mathbf{r} \cdot \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} = D$$

as point $P(1, 0, -1)$ lies on the plane

$$\therefore \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} = D \Rightarrow D = 2 + 0 + 6 \Rightarrow D = 8$$

$\therefore$ required equation of the plane is

$$\mathbf{r} \cdot \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} = 8 \quad \text{or} \quad 2x + 3y - 6z = 8 \quad \text{(Ans)}$$

*Alternative Solution to part* (i):

we have $\overline{PQ}=\begin{pmatrix}-3\\4\\1\end{pmatrix}$ and $\overline{PR}=\begin{pmatrix}3\\2\\2\end{pmatrix}$

equation of the plane passing through $P(1, 0, -1)$ having two directions $\overline{PQ}$ and $\overline{PR}$ is given by

$$\mathbf{r}=\begin{pmatrix}1\\0\\-1\end{pmatrix}+\lambda\begin{pmatrix}-3\\4\\1\end{pmatrix}+\mu\begin{pmatrix}3\\2\\2\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1-3\lambda+3\mu\\0+4\lambda+2\mu\\-1+\lambda+2\mu\end{pmatrix}$$

$$\Rightarrow \quad x=1-3\lambda+3\mu \cdots\cdots\cdots(\text{i})$$

$$y=4\lambda+2\mu \cdots\cdots\cdots(\text{ii})$$

$$z=-1+\lambda+2\mu \cdots\cdots\cdots(\text{iii})$$

to eliminate $\lambda$ and $\mu$, equation (ii) can be written as

$$y=4\lambda+2\mu \;\Rightarrow\; \lambda=\frac{1}{4}(y-2\mu) \text{ put in eq. (i)}$$

$$\Rightarrow \quad x=1-3\left(\frac{1}{4}(y-2\mu)\right)+3\mu \;\Rightarrow\; 4x=4-3y+6\mu+12\mu$$

$$\Rightarrow \quad 4x=4-3y+18\mu \;\Rightarrow\; \mu=\frac{4x+3y-4}{18} \cdots\cdots\cdots(\text{iv})$$

$$\therefore \quad \lambda=\frac{1}{4}(y-2\mu) \;\Rightarrow\; \lambda=\frac{1}{4}\left[y-2\left(\frac{4x+3y-4}{18}\right)\right]$$

$$\Rightarrow \quad \lambda=\frac{1}{4}\left[y-\frac{4x+3y-4}{9}\right] \;\Rightarrow\; \lambda=\frac{1}{4}y-\frac{4x+3y-4}{36}$$

$$\Rightarrow \quad \lambda=\frac{9y-4x-3y+4}{36} \;\Rightarrow\; \lambda=\frac{6y-4x+4}{36} \cdots\cdots\cdots(\text{v})$$

putting (iv) and (v) in equation (iii)

$$z=-1+\lambda+2\mu \;\Rightarrow\; z=-1+\frac{6y-4x+4}{36}+2\left(\frac{4x+3y-4}{18}\right)$$

$$\Rightarrow \quad 36z=-36+6y-4x+4+4(4x+3y-4)$$

$$\Rightarrow \quad 36z=-36+6y-4x+4+16x+12y-16$$

$$\Rightarrow \quad 36z=-48+18y+12x \;\Rightarrow\; 12x+18y-36z=48$$

dividing by 6 gives

$$2x+3y-6z=8 \quad (\text{Ans})$$

(ii) From part (i), equation of the plane is

$2x+3y-6z=8$ or $\mathbf{r}\cdot(2\mathbf{i}+3\mathbf{j}-6\mathbf{k})=8$ ………(A)

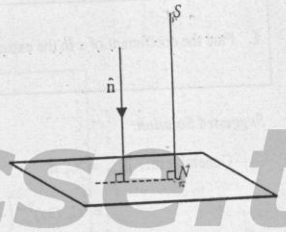
$\therefore$ normal $\mathbf{n}$ to the plane is $2\mathbf{i}+3\mathbf{j}-6\mathbf{k}$

now, $SN$ is perpendicular to the plane and therefore parallel to normal $\mathbf{n}$

$\therefore$ equation of the line $SN$ passing through $S$ and parallel to $\mathbf{n}$ is given by

$$\mathbf{r}=\begin{pmatrix}3\\5\\-6\end{pmatrix}+t\begin{pmatrix}2\\3\\-6\end{pmatrix} \Rightarrow \mathbf{r}=\begin{pmatrix}3+2t\\5+3t\\-6-6t\end{pmatrix} \text{………(B)}$$

as the line $SN$ intersects the plane, therefore putting eq. (B) into eq. (A), we get



$$\begin{pmatrix}3+2t\\5+3t\\-6-6t\end{pmatrix}\cdot\begin{pmatrix}2\\3\\-6\end{pmatrix}=8 \Rightarrow 2(3+2t)+3(5+3t)-6(-6-6t)=8$$

$$\Rightarrow 6+4t+15+9t+36+36t=8 \Rightarrow 57+49t=8$$

$$\Rightarrow 49t=-49 \Rightarrow t=-1$$

putting the value of $t$ in equation (B)

$$\mathbf{r}=\begin{pmatrix}3+2(-1)\\5+3(-1)\\-6-6(-1)\end{pmatrix}=\begin{pmatrix}1\\2\\0\end{pmatrix}$$

$\therefore$ position vector of $N$ is $=\begin{pmatrix}1\\2\\0\end{pmatrix}$

now

$$\overline{SN}=\overline{ON}-\overline{OS}=\begin{pmatrix}1\\2\\0\end{pmatrix}-\begin{pmatrix}3\\5\\-6\end{pmatrix}=\begin{pmatrix}-2\\-3\\6\end{pmatrix}$$

$$\therefore \left|\overline{SN}\right|=\sqrt{(-2)^2+(-3)^2+(6)^2}=\sqrt{4+9+36}=\sqrt{49}=7 \text{ units} \quad \textbf{(Shown)}$$

# November 2004 Paper 1

## Pure Mathematics (P1)

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. Find the coefficient of $x$ in the expansion of $\left(3x-\frac{2}{x}\right)^5$ [4]

***Suggested Solution:***

General term $T_{r+1}={}^{n}C_r\,a^{n-r}\,b^r$

$$={}^{5}C_r\,(3x)^{5-r}\left(-\frac{2}{x}\right)^r$$

$$={}^{5}C_r\,(3)^{5-r}(x)^{5-r}(-2)^r(x)^{-r}={}^{5}C_r\,(3)^{5-r}(-2)^r(x)^{5-2r}$$

required power of $x$ is 1

collecting the powers of $x$ only, we have

$$5-2r=1 \;\Rightarrow\; 2r=4 \;\Rightarrow\; r=2$$

$$\Rightarrow\; T_{2+1}={}^{5}C_2\,(3)^{5-2}(-2)^2(x)^{5-2(2)}=10\,(27)\,(4)\,(x)^1=1080x$$

$\therefore$ coefficient of $x=1080$ (Ans)

2. Find

(i) the sum of the first ten terms of the geometric progression 81, 54, 36,…… , [3]

(ii) the sum of all the terms in the arithmetic progression 180, 175, 170, ……, 25. [3]

***Suggested Solution:***

(i) Given that $a=81,\quad r=\frac{54}{81}=\frac{2}{3}$

using $S_n=\frac{a(1-r^n)}{1-r}$, we have

$$S_{10}=\frac{81[1-(\frac{2}{3})^{10}]}{1-\frac{2}{3}}=\frac{81\left(1-\frac{2^{10}}{3^{10}}\right)}{\frac{1}{3}}=81(3)\left(\frac{3^{10}-2^{10}}{3^{10}}\right)=243\left(\frac{59049-1024}{59049}\right)$$

$$=243\left(\frac{58025}{59049}\right)=238.786\approx 239 \quad \text{(Ans)}$$

② Given that $a = 180, \quad d = -5, \quad T_n = l = 25$

using $T_n = a + (n-1)d$, we have

$$25 = 180 + (n-1)(-5)$$
$$25 = 180 - 5n + 5$$
$$5n = 180 - 25 + 5$$
$$5n = 160 \quad \Rightarrow \quad n = 32$$

using $S_n = \frac{n}{2}(a + l)$

$$S_{32} = \frac{32}{2}(180 + 25) = 16(205) = 3280$$

$\therefore$ sum of all the terms $= 3280$ **(Ans)**

3.

In the diagram, $AC$ is an arc of a circle, centre $O$ and radius 6 cm. The line $BC$ is perpendicular to $OC$ and $OAB$ is a straight line. Angle $AOC = \frac{1}{3}\pi$ radians. Find the area of the shaded region, giving your answer in terms of $\pi$ and $\sqrt{3}$. [5]

*Suggested Solution:*

In $\Delta OBC$

$$\tan(\tfrac{\pi}{3}) = \frac{BC}{OC}$$

$$\Rightarrow \quad \sqrt{3} = \frac{BC}{6} \quad \Rightarrow \quad BC = 6\sqrt{3} \text{ cm}$$

Note that
$\tan(\frac{\pi}{3}) = \tan 60° = \sqrt{3}$

Shaded area = area of $\Delta OBC$ − area of sector $OAC$

$$= \left(\frac{1}{2} \times 6 \times 6\sqrt{3}\right) - \left(\frac{1}{2} \times 6^2 \times \frac{1}{3}\pi\right)$$

$$= 18\sqrt{3} - 6\pi = 6\left(3\sqrt{3} - \pi\right) \text{ cm}^2 \quad \textbf{(Ans)}$$

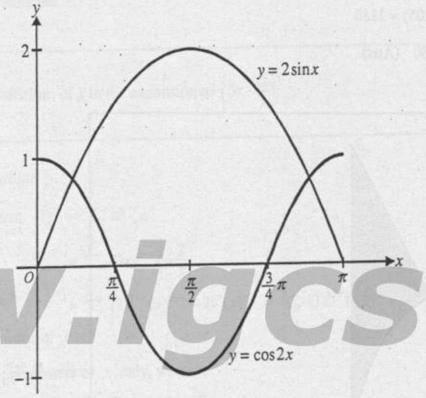
Remember
area of sector $= \frac{1}{2}r^2\theta$

4. (i) Sketch and label, on the same diagram, the graphs of $y = 2\sin x$ and $y = \cos 2x$, for the interval $0 \le x \le \pi$. [4]

(ii) Hence state the number of solutions of the equation $2\sin x = \cos 2x$ in the interval $0 \le x \le \pi$. [1]

***Suggested Solution:***

(i)



Note that $y = \cos 2x$ means that there should be two complete cycles of $\cos x$ in the range.
$0 \le x \le 2\pi$

The range given here is $0 \le x \le \pi$, therefore there should be one complete cycle of $\cos x$

Similarly in $y = 2\sin x$, the amplitude should be from −2 to +2, but the cycle remains unchanged. As the given range is $0 \le x \le \pi$, therefore half curve of $\sin x$ is needed.

(ii) Number of solutions are the number of points of intersection of the two graphs.

$\Rightarrow$ number of solution = 2 **(Ans)**

5. The equation of a curve is $y = x^2 - 4x + 7$ and the equation of a line is $y + 3x = 9$. The curve and the line intersect at the points $A$ and $B$.

(i) The mid-point of $AB$ is $M$. Show that the coordinates of $M$ are $\left(\frac{1}{2}, 7\frac{1}{2}\right)$ [4]

(ii) Find the coordinates of the point $Q$ on the curve at which the tangent is parallel to the line $y + 3x = 9$. [3]

(iii) Find the distance $MQ$. [1]

*Suggested Solution:*

(i) Let $y = x^2 - 4x + 7 \cdots\cdots\cdots \text{(i)}$

$y + 3x = 9 \quad \Rightarrow \quad y = 9 - 3x \cdots\cdots\cdots \text{(ii)}$

putting eq.(ii) into eq.(i) and solving simultaneously

$$9 - 3x = x^2 - 4x + 7$$

$$x^2 - 4x + 3x + 7 - 9 = 0$$

$$x^2 - x - 2 = 0$$

$$x^2 - 2x + x - 2 = 0$$

$$x(x-2) + 1(x-2) = 0$$

$$(x-2)(x+1) = 0$$

$$x - 2 = 0 \quad , \quad x + 1 = 0$$

$$x = 2 \quad , \quad x = -1$$

when $x = -1, \quad y = 9 - 3(-1) \Rightarrow y = 12$

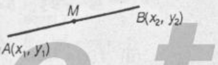
$\therefore$ point $A$ is $(-1, 12)$

when $x = 2, \quad y = 9 - 3(2) \Rightarrow y = 3$

$\therefore$ point $B$ is $(2, 3)$

$M$ is the mid-point of $A\,(-1,\ 12)$ and $B\,(2,\ 3)$

$\therefore$ coordinates of $M = \left(\frac{-1+2}{2}, \frac{12+3}{2}\right) = \left(\frac{1}{2}, \frac{15}{2}\right) = \left(\frac{1}{2}, 7\frac{1}{2}\right)$ **(Shown)**



If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on a straight line, then coordinates of mid point $M$ of $AB$ is:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

(ii) Gradient of the line $y = 9 - 3x$ is: $\frac{dy}{dx} = -3$

Gradient of the curve $y = x^2 - 4x + 7$ is: $\frac{dy}{dx} = 2x - 4$

Given that

gradient of curve at $Q$ = gradient of line

$$\Rightarrow \qquad 2x - 4 = -3$$

$$\Rightarrow \qquad 2x = 1 \quad \Rightarrow \quad x = \frac{1}{2}$$

putting this value of $x$ in the equation of curve, we have

$$y = \left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + 7 = \frac{1}{4} - 2 + 7 = \frac{21}{4}$$

$\therefore$ coordinates of $Q = \left(\frac{1}{2}, \frac{21}{4}\right)$ **(Ans)**

(iii) Distance $|MQ| = \sqrt{\left(\frac{1}{2} - \frac{1}{2}\right)^2 + \left(\frac{15}{2} - \frac{21}{4}\right)^2} = \sqrt{0 + \left(\frac{15}{2} - \frac{21}{4}\right)^2}$

$$= \sqrt{\left(\frac{30-21}{4}\right)^2} = \sqrt{\left(\frac{9}{4}\right)^2} = \frac{9}{4} = 2\frac{1}{4} \text{ units} \quad \textbf{(Ans)}$$

Distance between $A$ and $B$ is given by:

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

6. The function $f : x \mapsto 5\sin^2 x + 3\cos^2 x$ is defined for the domain $0 \le x \le \pi$.

(i) Express $f(x)$ in the form $a + b\sin^2 x$, stating the values of $a$ and $b$. [2]

(ii) Hence find the values of $x$ for which $f(x) = 7\sin x$. [3]

(iii) State the range of f. [2]

*Suggested Solution:*

(i) $f(x) = 5\sin^2 x + 3\cos^2 x$

$= 5\sin^2 x + 3(1 - \sin^2 x)$

$= 5\sin^2 x + 3 - 3\sin^2 x$

$= 2\sin^2 x + 3 = 3 + 2\sin^2 x$

$\Rightarrow a = 3, \quad b = 2$ **(Ans)**

(ii) $f(x) = 7\sin x$

$\Rightarrow 3 + 2\sin^2 x = 7\sin x$

$\Rightarrow 2\sin^2 x - 7\sin x + 3 = 0$

$\Rightarrow 2\sin^2 x - 6\sin x - \sin x + 3 = 0$

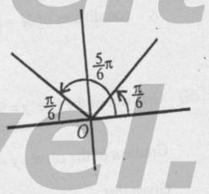
$\Rightarrow 2\sin x(\sin x - 3) - 1(\sin x - 3) = 0$

$\Rightarrow (\sin x - 3)(2\sin x - 1) = 0$

$\Rightarrow \sin x - 3 = 0, \quad 2\sin x - 1 = 0$

$\Rightarrow \sin x = 3$ (not possible), or $\sin x = \frac{1}{2}$ (basic angle $= \frac{\pi}{6}$)

$\therefore \quad x = \frac{1}{6}\pi, \ \frac{5}{6}\pi$ **(Ans)**



(iii) From part (i), we see that $f(x)$ can be written as

$f(x) = 3 + 2\sin^2 x$

Now, we know that

$-1 \le \sin x \le 1$

taking square

$\Rightarrow 0 \le \sin^2 x \le 1$

multiplying by 2

$\Rightarrow 0 \le 2\sin^2 x \le 2$

adding 3

$\Rightarrow 3 + 0 \le 3 + 2\sin^2 x \le 3 + 2$

$\Rightarrow 3 \le 3 + 2\sin^2 x \le 5 \quad \Rightarrow 3 \le f(x) \le 5$

$\therefore$ range of $f(x)$ is $3 \le f(x) \le 5$ **(Ans)**

We know that the range of sinx is from −1 to +1.

But when we square the function, the range will then be from 0 to +1

This is because the least square number is 0. Therefore the least value of $x^2$ is 0. This means that the least value of $(\sin^2 x)$ is 0.

7. A curve is such that $\frac{dy}{dx} = \frac{6}{\sqrt{(4x-3)}}$ and $P(3, 3)$ is a point on the curve.

(i) Find the equation of the normal to the curve at $P$, giving your answer in the form $ax + by = c$. [3]

(ii) Find the equation of the curve. [4]

*Suggested Solution:*

(i) $\frac{dy}{dx} = \frac{6}{\sqrt{(4x-3)}}$

gradient of tangent at $P(3,\ 3)$

$$\frac{dy}{dx} = \frac{6}{\sqrt{(4(3)-3)}} = \frac{6}{\sqrt{9}} = \frac{6}{3} = 2$$

$\Rightarrow$ gradient of normal at $P = -\frac{1}{2}$

$\therefore$ equation of normal at $P(3,\ 3)$ is given by

$$y - 3 = -\frac{1}{2}(x-3) \Rightarrow 2y - 6 = -x + 3 \Rightarrow 2y + x = 9 \quad \text{(Ans)}$$

Gradient at any point $(x,\ y)$ on a curve is given by $\frac{dy}{dx}$

Tangent and normal to the curve are perpendicular to each other, and the product of the gradients of perpendicular lines is equal to −1.

(i.e. $m_1 \times m_2 = -1$)

Equation of a line in point-slope form is:
$y - y_1 = m\,(x - x_1)$

(ii) $\frac{dy}{dx} = \frac{6}{\sqrt{(4x-3)}}$

$$\Rightarrow \quad dy = \frac{6}{\sqrt{(4x-3)}}\,dx \Rightarrow dy = 6(4x-3)^{-\frac{1}{2}}\,dx$$

integrating both sides

$$\Rightarrow \quad \int dy = \int 6(4x-3)^{-\frac{1}{2}}\,dx$$

$$\Rightarrow \quad \int dy = 6\int (4x-3)^{-\frac{1}{2}}\,dx$$

$$\Rightarrow \quad y = 6\left[\frac{(4x-3)^{\frac{1}{2}}}{\frac{1}{2}\times 4}\right] + C$$

$$\Rightarrow \quad y = 6\left[\frac{(4x-3)^{\frac{1}{2}}}{2}\right] + C$$

$$\Rightarrow \quad y = 3(4x-3)^{\frac{1}{2}} + C$$

as the curve passes through $P(3,\ 3)$

$$\therefore \quad 3 = 3[4(3)-3]^{\frac{1}{2}} + C \Rightarrow 3 = 3(9)^{\frac{1}{2}} + C \Rightarrow 3 = 9 + C \Rightarrow C = -6$$

$\therefore$ equation of the curve is

$$y = 3(4x-3)^{\frac{1}{2}} - 6 \quad \text{or} \quad y = 3\sqrt{(4x-3)} - 6 \quad \text{(Ans)}$$

When $\frac{dy}{dx}$ (or gradient) of the curve is given, always integrate both sides of the given expression to find the equation of the curve.

8. The points $A$ and $B$ have position vectors $\mathbf{i}+7\mathbf{j}+2\mathbf{k}$ and $-5\mathbf{i}+5\mathbf{j}+6\mathbf{k}$ respectively, relative to an origin $O$.

(i) Use a scalar product to calculate angle $AOB$, giving your answer in radians correct to 3 significant figures. [4]

(ii) The point $C$ is such that $\overrightarrow{AB}=2\overrightarrow{BC}$. Find the unit vector in the direction of $\overrightarrow{OC}$. [4]

*Suggested Solution:*

(i) Applying scalar product

$$\overline{OA}\cdot\overline{OB}=\left|\overline{OA}\right|\left|\overline{OB}\right|\cos A\hat{O}B$$

$$\begin{pmatrix}1\\7\\2\end{pmatrix}\cdot\begin{pmatrix}-5\\5\\6\end{pmatrix}=\left(\sqrt{(1)^2+(7)^2+(2)^2}\right)\left(\sqrt{(-5)^2+(5)^2+(6)^2}\right)\cos A\hat{O}B$$

$$-5+35+12=\left(\sqrt{1+49+4}\right)\left(\sqrt{25+25+36}\right)\cos A\hat{O}B$$

$$42=\left(\sqrt{54}\right)\left(\sqrt{86}\right)\cos A\hat{O}B$$

$$42=\left(\sqrt{4644}\right)\cos A\hat{O}B$$

$$\cos A\hat{O}B=\frac{42}{\sqrt{4644}}=0.61632$$

$$A\hat{O}B=0.9067\approx 0.907 \text{ radians} \quad \text{(Ans)}$$

If
$\mathbf{a}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, and
$\mathbf{b}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then
$\mathbf{a.b}=ax+by+cz$
and magnitude of $\mathbf{a}$ is
$|\mathbf{a}|=\sqrt{(a)^2+(b)^2+(c)^2}$

(ii) Given that

$$\overline{AB}=2\overline{BC}$$

$$\Rightarrow \quad \overline{OB}-\overline{OA}=2\left(\overline{OC}-\overline{OB}\right)$$

$$\Rightarrow \quad \begin{pmatrix}-5\\5\\6\end{pmatrix}-\begin{pmatrix}1\\7\\2\end{pmatrix}=2\left[\overline{OC}-\begin{pmatrix}-5\\5\\6\end{pmatrix}\right]$$

$$\Rightarrow \quad \begin{pmatrix}-6\\-2\\4\end{pmatrix}=2\left[\overline{OC}-\begin{pmatrix}-5\\5\\6\end{pmatrix}\right]$$

$$\Rightarrow \quad \frac{1}{2}\begin{pmatrix}-6\\-2\\4\end{pmatrix}=\overline{OC}-\begin{pmatrix}-5\\5\\6\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}-3\\-1\\2\end{pmatrix}=\overline{OC}-\begin{pmatrix}-5\\5\\6\end{pmatrix} \quad \Rightarrow \quad \overline{OC}=\begin{pmatrix}-3\\-1\\2\end{pmatrix}+\begin{pmatrix}-5\\5\\6\end{pmatrix}=\begin{pmatrix}-8\\4\\8\end{pmatrix}$$

$$\therefore \quad \overline{OC}=-8\mathbf{i}+4\mathbf{j}+8\mathbf{k}$$

unit vector in the direction of $\overline{OC} = \dfrac{\overline{OC}}{|\overline{OC}|}$

$$= \frac{-8i+4j+8k}{\sqrt{(-8)^2+(4)^2+(8)^2}} = \frac{-8i+4j+8k}{\sqrt{64+16+64}}$$

$$= \frac{-8i+4j+8k}{\sqrt{144}} = \frac{-8i+4j+8k}{12}$$

$$= -\frac{8}{12}i + \frac{4}{12}j + \frac{8}{12}k = -\frac{2}{3}i + \frac{1}{3}j + \frac{2}{3}k$$

$$= \frac{1}{3}(-2i+j+2k) \quad \text{(Ans)}$$

For any vector v, the unit vector $\hat{v}$, is given as:

$$\hat{v} = \frac{v}{|v|}$$

8. The function $f : x \mapsto 2x - a$, where $a$ is a constant, is defined for all real $x$.

(i) In the case where $a = 3$, solve the equation $ff(x) = 11$ [3]

The function $g : x \mapsto x^2 - 6x$ is defined for all real $x$.

(ii) Find the value of $a$ for which the equation $f(x) = g(x)$ has exactly one real solution. [3]

The function $h : x \mapsto x^2 - 6x$ is defined for the domain $x \geq 3$.

(iii) Express $x^2 - 6x$ in the form $(x-p)^2 - q$, where $p$ and $q$ are constants. [2]

(iv) Find an expression for $h^{-1}(x)$ and state the domain of $h^{-1}$. [4]

*Suggested Solution:*

(i) When $a = 3$, $f(x) = 2x - 3$

now, $ff(x) = f(2x-3)$

$$= 2(2x-3) - 3 = 4x - 6 - 3 = 4x - 9$$

since $ff(x) = 11$

$\Rightarrow \quad 4x - 9 = 11$

$\Rightarrow \quad 4x = 20 \Rightarrow x = 5$ (Ans)

(ii) Given that $f(x) = g(x)$

$\Rightarrow \quad 2x - a = x^2 - 6x \Rightarrow x^2 - 8x + a = 0$

given that the equation has exactly one real solution

$\therefore \quad x^2 - 8x + a = 0$ has equal roots

$\Rightarrow$ discriminant $= 0$

$\Rightarrow \quad b^2 - 4ac = 0$

$\Rightarrow \quad (-8)^2 - 4(1)(a) = 0 \Rightarrow 64 - 4a = 0 \Rightarrow 4a = 64 \Rightarrow a = 16$ (Ans)

(iii) $x^2 - 6x$

$= (x)^2 - 2(x)(3) + (3)^2 - (3)^2 = (x-3)^2 - 9$ (Ans)

(iv) From part (iii), we see that $\mathrm{h}(x)$ can be written as

$$\mathrm{h}(x)=(x-3)^2-9$$

let $y=\mathrm{h}(x)$

$$\Rightarrow\ y=(x-3)^2-9$$

$$\Rightarrow\ (x-3)^2=y+9$$

$$\Rightarrow\ (x-3)=\sqrt{y+9}$$

$$\Rightarrow\ x=\sqrt{y+9}+3$$

as $y=\mathrm{h}(x)\ \Rightarrow\ x=\mathrm{h}^{-1}(y)$

$\therefore\ \ \mathrm{h}^{-1}(y)=\sqrt{y+9}+3\ \Rightarrow\ \mathrm{h}^{-1}(x)=\sqrt{x+9}+3$ **(Ans)**

We see that the minimum point of the function h(x) is (3, −9).

So the range of the function is h(x) ≥ −9 .

Now, remember that the range of h(x) is equal to the domain of $\mathrm{h}^{-1}(x)$ and vice versa.

$\mathrm{h}(x)=(x-3)^2-9$, this shows that the range of $\mathrm{h}(x)$ is: $\mathrm{h}(x)\geq -9$

$\therefore$ Domain of $\mathrm{h}^{-1}(x)$ is: $x\geq -9$ **(Ans)**

**10.** A curve has equation $y=x^2+\dfrac{2}{x}$

(i) Write down expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ [3]

(ii) Find the coordinates of the stationary point on the curve and determine its nature. [4]

(iii) Find the volume of the solid formed when the region enclosed by the curve, the $x$-axis and the lines $x=1$ and $x=2$ is rotated completely about the $x$-axis. [6]

***Suggested Solution:***

(i) $y=x^2+\dfrac{2}{x}\ \Rightarrow\ y=x^2+2x^{-1}$

$\dfrac{dy}{dx}=2x+2(-x^{-2})=2x-\dfrac{2}{x^2}$ **(Ans)**

$\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(2x-\dfrac{2}{x^2}\right)=\dfrac{d}{dx}\left(2x-2x^{-2}\right)=2-2(-2x^{-3})=2+\dfrac{4}{x^3}$ **(Ans)**

(ii) $\dfrac{dy}{dx}=2x-\dfrac{2}{x^2}$

for stationary values, $\dfrac{dy}{dx}=0$

$$\Rightarrow\ 2x-\frac{2}{x^2}=0\ \Rightarrow\ 2x=\frac{2}{x^2}\ \Rightarrow\ 2x^3=2\ \Rightarrow\ x^3=1\ \Rightarrow\ x=1$$

putting this value of $x$ in eq. of curve $y=x^2+\dfrac{2}{x}$

$$y=(1)^2+\frac{2}{1}\ \Rightarrow\ y=1+2\ \Rightarrow\ y=3$$

$\therefore$ coordinates of stationary point = (1, 3) **(Ans)** now, $\frac{d^2y}{dx^2}=2+\frac{4}{x^3}$

at point (1, 3)

$$\frac{d^2y}{dx^2}=2+\frac{4}{(1)^3}=6>0$$

∴ point (1, 3) is a minimum point (Ans)

Nature:

If $\frac{d^2y}{dx^2}>0$ point is min.

If $\frac{d^2y}{dx^2}<0$ point is max.

(iii) Volume, $V=\pi\int_1^2 y^2\,dx$

$$\Rightarrow V=\pi\int_1^2\left(x^2+\frac{2}{x}\right)^2 dx$$

$$\Rightarrow V=\pi\int_1^2\left(x^4+4x+\frac{4}{x^2}\right)dx$$

$$\Rightarrow V=\pi\int_1^2\left(x^4+4x+4x^{-2}\right)dx$$

$$\Rightarrow V=\pi\left[\frac{x^5}{5}+4\left(\frac{x^2}{2}\right)+4\left(\frac{x^{-1}}{-1}\right)\right]_1^2$$

$$\Rightarrow V=\pi\left[\frac{x^5}{5}+2x^2-\frac{4}{x}\right]_1^2$$

$$\Rightarrow V=\pi\left[\left(\frac{(2)^5}{5}+2(2)^2-\frac{4}{2}\right)-\left(\frac{(1)^5}{5}+2(1)^2-\frac{4}{1}\right)\right]$$

$$\Rightarrow V=\pi\left[\left(\frac{32}{5}+8-2\right)-\left(\frac{1}{5}+2-4\right)\right]$$

$$\Rightarrow V=\pi\left[\left(\frac{32}{5}+6\right)-\left(\frac{1}{5}-2\right)\right]$$

$$\Rightarrow V=\pi\left[\left(\frac{62}{5}\right)-\left(-\frac{9}{5}\right)\right]=\pi\left[\frac{62}{5}+\frac{9}{5}\right]=\frac{71}{5}\pi$$

∴ volume $=\frac{71}{5}\pi=14\frac{1}{5}\pi$ unit$^3$ (Ans)

Remember:
For definite integral, do not put the integration constant

# November 2004 Paper 3

## *Pure Mathematics (P3)*

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Expand $\frac{1}{(2+x)^3}$ in ascending powers of $x$, up to and including the term in $x^2$, simplifying the coefficients. [4]

***Suggested Solution:***

$$\frac{1}{(2+x)^3}=\frac{1}{\left[2\left(1+\frac{x}{2}\right)\right]^3}=\frac{1}{2^3\left(1+\frac{x}{2}\right)^3}=\frac{1}{8}\left(1+\frac{x}{2}\right)^{-3}$$

using binomial expansion

$$\frac{1}{8}\left(1+\frac{x}{2}\right)^{-3}=\frac{1}{8}\left[1+(-3)\left(\frac{x}{2}\right)+\frac{(-3)(-3-1)}{2!}\left(\frac{x}{2}\right)^2+.......\right]$$

$$=\frac{1}{8}\left[1-\frac{3}{2}x+\frac{3}{2}x^2\right]$$ ignoring higher powers of $x$

$$=\frac{1}{8}-\frac{3}{16}x+\frac{3}{16}x^2 \text{ (Ans)}$$

Binomial Expansion:

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+...$$

**2.** Solve the equation.

$$\ln(1+x)=1+\ln x,$$

giving your answer correct to 2 significant figures. [4]

***Suggested Solution:***

$$\ln(1+x)=1+\ln x$$

$$\Rightarrow \ln(1+x)-\ln x=1 \Rightarrow \ln\left(\frac{1+x}{x}\right)=1 \Rightarrow \ln\left(\frac{1+x}{x}\right)=\ln e$$

$$\Rightarrow \frac{1+x}{x}=e \Rightarrow 1+x=xe \Rightarrow xe-x=1 \Rightarrow x(e-1)=1$$

$$\Rightarrow x=\frac{1}{e-1}=0.58197\approx 0.58 \text{ (2 dec.pl) (Ans)}$$

Note that:

$\lg_a m-\lg_a n=\lg_a\left(\frac{m}{n}\right)$'

$\ln e=1$

3. The polynomial $2x^3+ax^2-4$ is denoted by $p(x)$. It is given that $(x-2)$ is a factor of $p(x)$.

(i) Find the value of $a$. [2]

When $a$ has this value,

(ii) factorise $p(x)$, [2]

(iii) solve the inequality $p(x)>0$, justifying your answer. [2]

*Suggested Solution:*

(i) $p(x)=2x^3+ax^2-4$

given that $(x-2)$ is a factor of $p(x)$

$\therefore$ Remainder $=0$

$\Rightarrow$ $p(2)=2(2)^3+a(2)^2-4=0$

$\Rightarrow$ $16+4a-4=0 \Rightarrow 4a=-2 \Rightarrow a=-3$ (Ans)

(ii) $(x-2)$ is a factor of $p(x)=2x^3-3x^2-4$. Using long division, we have

$$\begin{array}{r|l}
 & 2x^2+x+2 \\ \hline
x-2 & 2x^3-3x^2-4 \\
 & 2x^3-4x^2 \\
 & \underline{-\quad +\quad} \\
 & x^2-4 \\
 & x^2\quad -2x \\
 & \underline{-\quad +} \\
 & 2x-4 \\
 & 2x-4 \\
 & \underline{-\quad +} \\
 & 0
\end{array}$$

Quotient, $Q(x)=2x^2+x+2$

$\therefore$ Factors of $p(x)=(x-2)(2x^2+x+2)$ (Ans)

***Alternative solution to part (ii)***

$(x-2)$ is a factor of $p(x)=2x^3-3x^2-4$. Therefore using synthetic division method, we have

$$\begin{array}{c|cccc}
 & x^3 & x^2 & x^1 & x^0 \\
 & 2 & -3 & 0 & -4 \\
2 & & 4 & 2 & 4 \quad \text{adding} \\ \hline
 & 2 & 1 & 2 & \multicolumn{1}{|c}{0}
\end{array}$$

$\Rightarrow$ Quotient, $Q(x)=2x^2+x+2$

$\therefore$ Factors of $p(x)=(x-2)(2x^2+x+2)$ (Ans)

(iii) Given that $p(x) > 0$

$$\Rightarrow\ 2x^3 - 3x^2 - 4 > 0\ \Rightarrow\ (x-2)(2x^2+x+2) > 0$$

consider $2x^2 + x + 2$

$$2x^2+x+2 = 2\left(x^2+\frac{1}{2}x+1\right) = 2\left[\left(x^2+\frac{1}{2}x+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2\right)+1\right]$$

$$=2\left[\left(x+\frac{1}{4}\right)^2-\frac{1}{16}+1\right] = 2\left[\left(x+\frac{1}{4}\right)^2+\frac{15}{16}\right] = 2\left(x+\frac{1}{4}\right)^2+\frac{15}{8}$$

$\therefore$ the inequality becomes

$$(x-2)\left[2\left(x+\frac{1}{4}\right)^2+\frac{15}{8}\right]>0$$

But $2\left(x+\frac{1}{4}\right)^2+\frac{15}{8}>0$ for all values of $x$

$\therefore$ $(x-2)\left[2\left(x+\frac{1}{4}\right)^2+\frac{15}{8}\right]>0$ only if $(x-2)>0$ $\Rightarrow$ $x>2$ **(Ans)**

**4.** (i) Show that the equation

$$\tan(45^\circ + x) = 2\tan(45^\circ - x)$$

can be written in the form

$$\tan^2 x - 6\tan x + 1 = 0.$$ [4]

(ii) Hence solve the equation $\tan(45^\circ + x) = 2\tan(45^\circ - x)$, for $0^\circ < x < 90^\circ$. [3]

***Suggested Solution:***

Formula used:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\ \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\ \tan B}$$

(i) $\tan(45^\circ + x) = 2\tan(45^\circ - x)$

$$\Rightarrow\ \frac{\tan 45^\circ + \tan x}{1 - \tan 45^\circ \tan x} = 2\left(\frac{\tan 45^\circ - \tan x}{1 + \tan 45^\circ \tan x}\right)$$

$$\Rightarrow\ \frac{1+\tan x}{1-\tan x} = 2\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$\Rightarrow\ (1+\tan x)^2 = 2(1-\tan x)^2$$

$$\Rightarrow 1+\tan^2 x + 2\tan x = 2(1+\tan^2 x - 2\tan x)$$

$$\Rightarrow\ 1+\tan^2 x+2\tan x = 2+2\tan^2 x - 4\tan x$$

$$\Rightarrow \tan^2 x - 6\tan x + 1 = 0$$ **(Shown)**

(ii) Using the result of part **(i)**

$$\tan^2 x - 6\tan x + 1 = 0$$

using quadratic formula, we have

$$\tan x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(1)}}{2(1)} = \frac{6 \pm \sqrt{36-4}}{2} = \frac{6 \pm \sqrt{32}}{2}$$

$$\Rightarrow \quad \tan x = \frac{6 \pm 5.6568}{2}$$

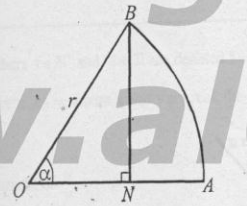
$$\therefore \quad \tan x = \frac{6+5.6568}{2} = 5.8284 \quad \text{or} \quad \tan x = \frac{6-5.6568}{2} = 0.1716$$

basic angle $\alpha$=80.3°　　　　basic angle $\alpha = 9.7°$

given range is $0° < x < 90°$

$\therefore \quad x = 80.3°, 9.7°$ **(Ans)**

5.



The diagram shows a sector $OAB$ of a circle with centre $O$ and radius $r$. The angle $AOB$ is $\alpha$ radians, where $0 < \alpha < \frac{1}{2}\pi$. The point $N$ on $OA$ is such that $BN$ is perpendicular to $OA$. The area of the triangle $ONB$ is half the area of the sector $OAB$.

(i) Show that $\alpha$ satisfies the equation $\sin 2x = x$. [3]

(ii) By sketching a suitable pair of graphs, show that this equation has exactly one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(iii) Use the iterative formula

$$x_{n+1} = \sin(2x_n),$$

with initial value $x_1 = 1$, to find $\alpha$ correct to 2 decimal places, showing the result of each iteration. [3]

**Suggested Solution:**

(i) In $\Delta ONB$

$$\cos\alpha = \frac{ON}{r} \Rightarrow ON = r\cos\alpha$$

$$\sin\alpha = \frac{BN}{r} \Rightarrow BN = r\sin\alpha$$

Given that

area of $\Delta ONB$ = half the area of sector $OAB$

$$\Rightarrow \frac{1}{2}(ON \times BN) = \frac{1}{2}\left(\frac{1}{2}r^2\alpha\right)$$

$$\Rightarrow \frac{1}{2}(r\cos\alpha)(r\sin\alpha) = \frac{1}{4}r^2\alpha$$

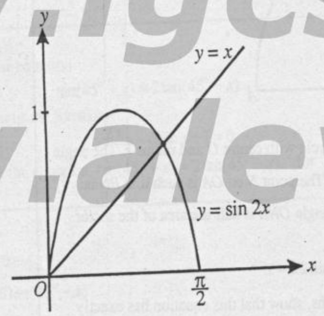
$$\Rightarrow r^2\sin\alpha\cos\alpha = \frac{1}{2}r^2\alpha$$

$$\Rightarrow 2\sin\alpha\cos\alpha = \alpha$$

$$\Rightarrow \sin 2\alpha = \alpha$$

$\therefore$ $\alpha$ satisfies the equation $\sin 2x = x$

(ii)



Sketching the graphs of $y = \sin 2x$ for $0 < x < \frac{\pi}{2}$ and $y = x$ we find that there is only one point of intersection of the two graphs between 0 and $\frac{\pi}{2}$. Therefore there is only one root in the interval $0 < x < \frac{\pi}{2}$. **(Ans)**

(iii) $x_{n+1} = \sin(2x_n)$,

Given that $x_1 = 1$

$\therefore \quad x_2 = \sin(2x_1) = \sin(2(1)) = 0.909297$

$x_3 = \sin(2x_2) = \sin 2(0.909297) = 0.969455$

$x_4 = \sin(2x_3) = \sin 2(0.969455) = 0.933008$

$x_5 = \sin(2x_4) = \sin 2(0.933008) = 0.956738$

$x_6 = \sin(2x_5) = \sin 2(0.956738) = 0.941858$

$x_7 = \sin(2x_6) = \sin 2(0.941858) = 0.951439$

$x_8 = \sin(2x_7) = \sin 2(0.951439) = 0.945366$

$x_9 = \sin(2x_8) = \sin 2(0.945366) = 0.949256$

$x_{10} = \sin(2x_9) = \sin 2(0.949256) = 0.946780$

$\therefore \quad \alpha = 0.95$ (2 dec.pl) **(Ans)**

Note:

Keep your calculator on radian mode.

6. The complex numbers $1+3i$ and $4+2i$ are denoted by $u$ and $v$ respectively.

(i) Find, in the form $x+iy$, where x and y are real, the complex numbers $u-v$ and $\frac{u}{v}$. [3]

(ii) State the argument of $\frac{u}{v}$. [1]

In an Argand diagram, with origin $O$, the points $A$, $B$ and $C$ represent the numbers $u$, $v$ and $u-v$ respectively.

(iii) State fully the geometrical relationship between $OC$ and $BA$. [2]

(iv) Prove that angle $AOB = \frac{1}{4}\pi$ radians. [2]

*Suggested Solution:*

(i) Given that

$u = 1+3i$ and $v = 4+2i$

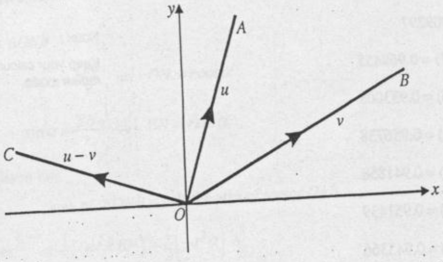
$u - v = (1+3i) - (4+2i) = 1 + 3i - 4 - 2i = -3 + i$ **(Ans)**

$$\frac{u}{v} = \frac{1+3i}{4+2i} \times \frac{4-2i}{4-2i} = \frac{(1+3i)(4-2i)}{(4)^2-(2i)^2} = \frac{4+10i-6i^2}{16+4} = \frac{4+10i-6(-1)}{16+4}$$

$$= \frac{10+10i}{20} = \frac{1}{2} + \frac{1}{2}i \quad \textbf{(Ans)}$$

(ii) Argument $(\frac{u}{v}) = \arctan\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right) = \arctan(1) = \frac{\pi}{4}$ **(Ans)**

(iii)



$OC$ represents a complex number $-3+i$ with column vector $\begin{pmatrix}-3\\1\end{pmatrix}$

$$\overline{BA}=\overline{OA}-\overline{OB}=\begin{pmatrix}1\\3\end{pmatrix}-\begin{pmatrix}4\\2\end{pmatrix}=\begin{pmatrix}-3\\1\end{pmatrix}$$

$\Rightarrow$ $BA$ also represents a complex number with column vector $\begin{pmatrix}-3\\1\end{pmatrix}$

$\therefore$ $OC$ and $BA$ are parallel

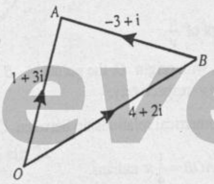
$OC$ and $BA$ are equal in length. **(Ans)**

Remember:

If $z=a+ib$, then

$|z|=\sqrt{a^2+b^2}$

(iv) Consider triangle $OAB$



$$|OA|=\sqrt{(1)^2+(3)^2}=\sqrt{1+9}=\sqrt{10}$$

$$|OB|=\sqrt{(4)^2+(2)^2}=\sqrt{16+4}=\sqrt{20}$$

$$|BA|=\sqrt{(-3)^2+(1)^2}=\sqrt{9+1}=\sqrt{10}$$

applying cosine rule

$$(BA)^2=(OB)^2+(OA)^2-2(OB)(OA)\cos A\hat{O}B$$

$$\Rightarrow\ \left(\sqrt{10}\right)^2=\left(\sqrt{20}\right)^2+\left(\sqrt{10}\right)^2-2\left(\sqrt{20}\right)\left(\sqrt{10}\right)\cos A\hat{O}B$$

$$\Rightarrow\ 10=20+10-2\sqrt{200}\cos A\hat{O}B$$

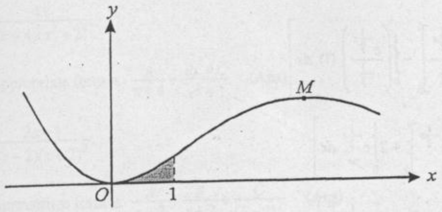
$$\Rightarrow\ 2\sqrt{200}\cos A\hat{O}B=20$$

$$\Rightarrow\ 2\left(10\sqrt{2}\right)\cos A\hat{O}B=20$$

$$\Rightarrow\ \cos A\hat{O}B=\frac{20}{20\sqrt{2}}=\frac{1}{\sqrt{2}}$$

$$\Rightarrow\ A\hat{O}B=\frac{1}{4}\pi \text{ radians}$$ **(Proved)**

7.



The diagram shows the curve $y = x^2 e^{-\frac{1}{2}x}$.

(i) Find the $x$-coordinate of $M$, the maximum point of the curve. [4]

(ii) Find the area of the shaded region enclosed by the curve, the $x$-axis and the line $x = 1$, giving your answer in terms of e. [5]

*Suggested Solution:*

(i) $y = x^2 e^{-\frac{1}{2}x}$

$$\frac{dy}{dx} = e^{-\frac{1}{2}x}(2x) + x^2\left(e^{-\frac{1}{2}x}\right)\left(-\frac{1}{2}\right) = e^{-\frac{1}{2}x}\left(2x - \frac{1}{2}x^2\right)$$

for stationary values $\frac{dy}{dx} = 0$

$\Rightarrow\ e^{-\frac{1}{2}x}\left(2x - \frac{1}{2}x^2\right) = 0$

$\Rightarrow\ e^{-\frac{1}{2}x} = 0$, (not possible) or $2x - \frac{1}{2}x^2 = 0$

$\Rightarrow\ x\left(2 - \frac{1}{2}x\right) = 0$

$\Rightarrow\ x = 0$ or $2 - \frac{1}{2}x = 0$

$\Rightarrow\ x = 0$ or $\underline{x = 4}$

$\therefore$ $x$-coordinate of $M = 4$ (Ans)

(ii) Area of the shaded region is given by

$$A = \int_0^1 y\,dx \Rightarrow A = \int_0^1 x^2 e^{-\frac{1}{2}x}\,dx$$

Using integration by parts

$$\Rightarrow A = \left[x^2 \frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right]_0^1 - \int_0^1 \left(\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right)(2x)\,dx \Rightarrow A = \left[-2x^2 e^{-\frac{1}{2}x}\right]_0^1 + 4\int_0^1 x\,e^{-\frac{1}{2}}\,dx$$

integrating by parts again

$$A=\left[-2x^2\mathrm{e}^{-\frac{1}{2}x}\right]_0^1+4\left[\left[x\times\frac{\mathrm{e}^{-\frac{1}{2}x}}{-\frac{1}{2}}\right]_0^1-\int_0^1\left(\frac{\mathrm{e}^{-\frac{1}{2}x}}{-\frac{1}{2}}\right)(1)\;dx\right]$$

$$=\left[-2x^2\mathrm{e}^{-\frac{1}{2}x}\right]_0^1+4\left[\left[-2x\times\mathrm{e}^{-\frac{1}{2}x}\right]_0^1+2\int_0^1\mathrm{e}^{-\frac{1}{2}x}\;dx\right]$$

$$=\left[-2x^2\mathrm{e}^{-\frac{1}{2}x}\right]_0^1+4\left[\left[-2x\times\mathrm{e}^{-\frac{1}{2}x}\right]_0^1+2\left[\frac{\mathrm{e}^{-\frac{1}{2}x}}{-\frac{1}{2}}\right]_0^1\right]$$

$$=\left[-2x^2\mathrm{e}^{-\frac{1}{2}x}\right]_0^1+4\left[-2x\times\mathrm{e}^{-\frac{1}{2}x}\right]_0^1-16\left[\mathrm{e}^{-\frac{1}{2}x}\right]_0^1$$

$$=\left[\left(-2(1)^2\mathrm{e}^{-\frac{1}{2}(1)}\right)-\left(-2(0)^2\mathrm{e}^{-\frac{1}{2}(0)}\right)\right]+4\left[\left(-2(1)\times\mathrm{e}^{-\frac{1}{2}(1)}\right)-\left(-2(0)\times\mathrm{e}^{-\frac{1}{2}(0)}\right)\right]$$
$$-16\left[\left(e^{-\frac{1}{2}(1)}\right)-\left(e^{-\frac{1}{2}(0)}\right)\right]$$

$$=\left[\left(-2\mathrm{e}^{-\frac{1}{2}}\right)-0\right]+4\left[\left(-2\times\mathrm{e}^{-\frac{1}{2}}\right)-0\right]-16\left[\left(\mathrm{e}^{-\frac{1}{2}}\right)-1\right]$$

$$=-2\mathrm{e}^{-\frac{1}{2}}-8\mathrm{e}^{-\frac{1}{2}}-16\mathrm{e}^{-\frac{1}{2}}+16=-26\mathrm{e}^{-\frac{1}{2}}+16=16-\frac{26}{\sqrt{\mathrm{e}}}$$

$\therefore$ area of shaded region $=16-\frac{26}{\sqrt{\mathrm{e}}}$ square units **(Ans)**

**8.** An appropriate form for expressing $\frac{3x}{(x+1)(x-2)}$ in partial fractions is

$$\frac{A}{x+1}+\frac{B}{x-2},$$

where $A$ and $B$ are constants.

(a) Without evaluating any constants, state appropriate forms for expressing the following in partial fractions:

(i) $\frac{4x}{(x+4)(x^2+3)}$, [1]

(ii) $\frac{2x+1}{(x-2)(x+2)^2}$. [2]

(b) Show that $\int_3^4 \frac{3x}{(x+1)(x-2)}\,dx=\ln 5$. [6]

*Suggested Solution:*

(a) (i) $\dfrac{4x}{(x+4)(x^2+3)}$,

Appropriate form is $\dfrac{A}{x+4}+\dfrac{Bx+C}{x^2+3}$ **(Ans)**

(ii) $\dfrac{2x+1}{(x-2)(x+2)^2}$.

Appropriate form is $\dfrac{A}{x-2}+\dfrac{B}{x+2}+\dfrac{C}{(x+2)^2}$ **(Ans)**

(b) L.H.S. $=\displaystyle\int_3^4 \frac{3x}{(x+1)(x-2)}\,dx$

Resolving $\dfrac{3x}{(x+1)(x-2)}$ into partial fractions

$$\frac{3x}{(x+1)(x-2)} \equiv \frac{A}{x+1}+\frac{B}{x-2}$$

$\Rightarrow \quad 3x \equiv A(x-2)+B(x+1)$

as the above sentence is an identity

∴ for $x=-1$

$3(-1)=A(-1-2)+B(-1+1) \quad \Rightarrow \quad -3=-3A \quad \Rightarrow \quad A=1$

and for $x=2$

$3(2)=A(2-2)+B(2+1) \quad \Rightarrow \quad 6=3B \quad \Rightarrow \quad B=2$

∴ $\dfrac{3x}{(x+1)(x-2)} \equiv \dfrac{1}{x+1}+\dfrac{2}{x-2}$

now

$$\int_3^4 \frac{3x}{(x+1)(x-2)}\,dx=\int_3^4\left(\frac{1}{x+1}+\frac{2}{x-2}\right)dx$$

$=\left[\ln(x+1)+2\ln(x-2)\right]_3^4$

$=\left(\ln(4+1)+2\ln(4-2)\right)-\left(\ln(3+1)+2\ln(3-2)\right)$

$=\left(\ln 5+2\ln 2\right)-\left(\ln 4+2\ln 1\right)$

$=\ln 5+2\ln 2-\ln 4$

$=\ln 5+\ln 2^2-\ln 4$

$=\ln 5+\ln 4-\ln 4$

$=\ln 5$ **(Shown)**

Note that an algebraic open sentence is an equation if it is true for some values of the unknown, and an algebraic open sentence is an identity if it is true for all values of the unknown.

9. The lines $l$ and $m$ have vector equations

$$\mathbf{r} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} + s(\mathbf{i} + \mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(-2\mathbf{i} + \mathbf{j} + \mathbf{k})$$

respectively.

(i) Show that $l$ and $m$ do not intersect. [4]

The point $P$ lies on $l$ and the point $Q$ has position vector $2\mathbf{i} - \mathbf{k}$.

(ii) Given that the line $PQ$ is perpendicular to $l$, find the position vector of $P$. [4]

(iii) Verify that $Q$ lies on $m$ and that $PQ$ is perpendicular to $m$. [2]

***Suggested Solution:***

(i) Let equation of $l$: $\mathbf{r}_1 = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k} + s(\mathbf{i} + \mathbf{j} - \mathbf{k}) \Rightarrow \mathbf{r}_1 = \begin{pmatrix} 2+s \\ -1+s \\ 4-s \end{pmatrix}$

and equation of $m$: $\mathbf{r}_2 = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(-2\mathbf{i} + \mathbf{j} + \mathbf{k}) \Rightarrow \mathbf{r}_2 = \begin{pmatrix} -2-2t \\ 2+t \\ 1+t \end{pmatrix}$

Let us assume that the two lines intersect

then $\mathbf{r}_1 = \mathbf{r}_2$

$$\Rightarrow \begin{pmatrix} 2+s \\ -1+s \\ 4-s \end{pmatrix} = \begin{pmatrix} -2-2t \\ 2+t \\ 1+t \end{pmatrix}$$

172

equating coefficients of **i** and **j**, we have

$$2 + s = -2 - 2t \Rightarrow s + 2t = -4 \cdots\cdots\cdots \text{(i)}$$

$$-1 + s = 2 + t \Rightarrow s = 3 + t \cdots\cdots\cdots \text{(ii)}$$

putting eq. (ii) into eq. (i)

$$(3+t) + 2t = -4 \Rightarrow 3t = -7 \Rightarrow t = -\frac{7}{3}$$

$$\therefore \quad s = 3 + \left(-\frac{7}{3}\right) \Rightarrow s = \frac{2}{3}$$

putting these values of s and $t$ in coefficients of **k**

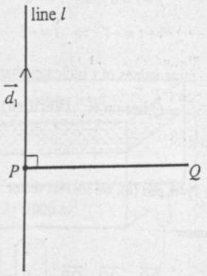
line $l$: $4 - s = 4 - \frac{2}{3} = \frac{10}{3}$

line $m$: $1 + t = 1 - \frac{7}{3} = -\frac{4}{3}$

As the z coordinates on both lines at the assumed point of intersection are different, therefore the two lines $l$ and $m$ do not intersect. **(Shown)**

(ii) Given that $\overline{OQ} = 2\mathbf{i} - \mathbf{k} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

Taking point $P$ as a general point on line $l$, we have



$$\overline{OP} = \begin{pmatrix} 2+s \\ -1+s \\ 4-s \end{pmatrix} \cdots\cdots\cdots(\text{i})$$

$$\therefore \quad \overline{PQ} = \overline{OQ} - \overline{OP} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 2+s \\ -1+s \\ 4-s \end{pmatrix} = \begin{pmatrix} -s \\ 1-s \\ -5+s \end{pmatrix}$$

as $\overline{PQ}$ is perpendicular to line $l$

$\therefore \quad \overline{PQ} \cdot \vec{d_1} = 0$ (where $\vec{d_1}$ is the direction vector of line $l$)

$$\Rightarrow \begin{pmatrix} -s \\ 1-s \\ -5+s \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 \Rightarrow (-s)+(1-s)+(-1)(-5+s)=0$$

$$\Rightarrow -s+1-s+5-s=0 \Rightarrow -3s=-6 \Rightarrow s=2$$

putting this value of $s$ in eq. (i), we have

$$\overline{OP} = \begin{pmatrix} 2+2 \\ -1+2 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

$\therefore$ position vector of $P = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$ or $4\mathbf{i}+\mathbf{j}+2\mathbf{k}$ (Ans)

(iii) Equation of $m$: $\mathbf{r} = -2\mathbf{i}+2\mathbf{j}+\mathbf{k}+t(-2\mathbf{i}+\mathbf{j}+\mathbf{k}) \Rightarrow \mathbf{r} = \begin{pmatrix} -2-2t \\ 2+t \\ 1+t \end{pmatrix}$

$\therefore$ General point on $m$ is represented by $\begin{pmatrix} -2-2t \\ 2+t \\ 1+t \end{pmatrix}$

p.v. of given point $Q$ is $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

Now, if $Q$ lies on $m$, then

$$\begin{pmatrix} -2-2t \\ 2+t \\ 1+t \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

equating the coefficients of **i**, **j** and **k**, we have

$$-2-2t=2 \Rightarrow -2t=4 \Rightarrow t=-2$$

$$2+t=0 \Rightarrow t=-2$$

$$1+t=-1 \Rightarrow t=-2$$

same values of $t$ indicates that $Q$ satisfies the equation of the line $m$.

$\therefore$ $Q$ lies on $m$ **(Verified)**

From part (ii), we see that vector $\overline{OP}$ is given by: $\overline{OP}=\begin{pmatrix}4\\1\\2\end{pmatrix}$

$$\therefore \quad \overline{PQ}=\overline{OQ}-\overline{OP}=\begin{pmatrix}2\\0\\-1\end{pmatrix}-\begin{pmatrix}4\\1\\2\end{pmatrix}=\begin{pmatrix}-2\\-1\\-3\end{pmatrix}$$

let $\overline{d_2}$ be the direction vector of line $m$ i.e. $\overline{d_2}=\begin{pmatrix}-2\\1\\1\end{pmatrix}$

$\therefore$ using scalar product

$$\overline{PQ}\bullet\overline{d_2}=\begin{pmatrix}-2\\-1\\-3\end{pmatrix}\bullet\begin{pmatrix}-2\\1\\1\end{pmatrix}=4-1-3=0$$

$\therefore$ $\overline{PQ}$ is $\perp$ to $m$ **(Verified)**

**10.** A rectangular reservoir has a horizontal base of area 1000 $m^2$. At time $t=0$, it is empty and water begins to flow into it at a constant rate of 30 $m^3s^{-1}$. At the same time, water begins to flow out at a rate proportional to $\sqrt{h}$, where $h$ m is the depth of the water at time $t$ s. When $h=1$, $\frac{dh}{dt}=0.02$.

(i) Show that $h$ satisfies the differential equation

$$\frac{dh}{dt}=0.01\left(3-\sqrt{h}\right).$$ [3]

It is given that, after making the substitution $x=3-\sqrt{h}$, the equation in part (i) becomes

$$(x-3)\frac{dx}{dt}=0.005x$$

(ii) Using the fact that $x=3$ when $t=0$, solve this differential equation, obtaining an expression for $t$ in terms of $x$. [5]

(iii) Find the time at which the depth of water reaches 4 m. [2]

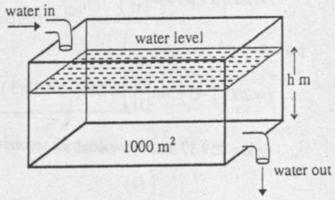
*Suggested Solution:*

(i) Given that

$\frac{dV}{dt}=30\cdots\cdots\cdots(\text{i})$ and also $\frac{dV}{dt}\propto-\sqrt{h} \Rightarrow \frac{dV}{dt}=-k\sqrt{h}\cdots\cdots\cdots(\text{ii})$

combining equations (i) & (ii), we have

$$\frac{dV}{dt}=30-k\sqrt{h}\cdots\cdots\cdots(\text{iii})$$



Now

volume of tank = base area × height

$\therefore \quad V=1000h$

differentiating w.r.t. $t$

$$\frac{dV}{dt}=1000\frac{dh}{dt} \quad \text{putting it in eq (iii)}$$

$$1000\frac{dh}{dt}=30-k\sqrt{h}\cdots\cdots\cdots(\text{iv})$$

when $h=1,\ \frac{dh}{dt}=0.02$

$\Rightarrow\ 1000(0.02)=30-k\sqrt{1} \Rightarrow 20=30-k \Rightarrow k=10$

$\therefore$ eq. (iv) becomes

$$1000\frac{dh}{dt}=30-10\sqrt{h} \Rightarrow \frac{dh}{dt}=\frac{10}{1000}(3-\sqrt{h})$$

$\Rightarrow\ \frac{dh}{dt}=0.01(3-\sqrt{h})$ **(Shown)**

(ii) $(x-3)\frac{dx}{dt}=0.005x$

$\Rightarrow\ \left(\frac{x-3}{x}\right)dx=0.005\,dt \Rightarrow \left(1-\frac{3}{x}\right)dx=0.005\,dt$

integrating both sides

$$\int\left(1-\frac{3}{x}\right)dx=\int 0.005\,dt \Rightarrow x-3\ln x=0.005t+C\cdots\cdots\cdots(\text{i})$$

given that $x=3$ when $t=0$

$\Rightarrow\ 3-3\ln 3=0.005(0)+C \Rightarrow C=3-3\ln 3$

putting value of $C$ in eq. (i)

$$x-3\ln x=0.005t+3-3\ln 3 \Rightarrow x-3+3\ln 3-3\ln x=0.005t$$

$\Rightarrow\ x-3+3(\ln 3-\ln x)=0.005t \Rightarrow x-3+3\ln\left|\frac{3}{x}\right|=0.005t$

$\Rightarrow\ t=\frac{1}{0.005}\left(x-3+3\ln\left|\frac{3}{x}\right|\right) \Rightarrow t=200\left(x-3+3\ln\left|\frac{3}{x}\right|\right)$ **(Ans)**

(iii) When $h = 4$ m

$$x = 3 - \sqrt{h} \quad \Rightarrow \quad x = 3 - \sqrt{4} \quad \Rightarrow \quad x = 1$$

using the result of part (ii), we have

$$t = 200\left(x - 3 + 3\ln\left|\frac{3}{x}\right|\right)$$

putting $x = 1$

$$t = 200\left(1 - 3 + 3\ln\left|\frac{3}{1}\right|\right) = 200(-2 + 3\ln 3) = 200(1.2958) = 259.167$$

$\therefore$ time = 259.17 seconds ≈ 4min 19 seconds (Ans)

Answer all the

Give non-exac

case of angles

Time : 1 hour

1. A curve

find the

Suggested

integ

as th

will

∴

2. Find

Sugges

# June 2005 Paper 1

## Pure Mathematics (P1)

Answer all the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. A curve is such that $\frac{dy}{dx} = 2x^2 - 5$. Given that the point (3, 8) lies on the curve, find the equation of the curve. [4]

*Suggested Solution:*

When $\frac{dy}{dx}$ (or gradient) of the curve is given, always integrate both sides of the given expression to find the equation of a curve.

$$\frac{dy}{dx} = 2x^2 - 5 \;\Rightarrow\; dy = (2x^2 - 5)\,dx$$

Integrating both sides

$$\int dy = \int (2x^2 - 5)\,dx \;\Rightarrow\; y = 2\left(\frac{x^3}{3}\right) - 5(x) + K$$

as the above curve passes through $P(3, 8)$, therefore $P(3, 8)$ will satisfy the eq. of the curve

$$(8) = \frac{2}{3}(3)^3 - 5(3) + K \;\Rightarrow\; 8 = 18 - 15 + K \;\Rightarrow\; K = 5$$

$\therefore$ equation of curve is: $y = \frac{2}{3}x^3 - 5x + 5$ (Ans)

2. Find the gradient of the curve $y = \frac{12}{x^2 - 4x}$ at the point where $x = 3$. [4]

*Suggested Solution:*

$$y = \frac{12}{x^2 - 4x} \;\Rightarrow\; y = 12(x^2 - 4x)^{-1}$$

$$\frac{dy}{dx} = 12\left[-1(x^2 - 4x)^{-2} \times (2x - 4)\right] = \frac{-24x + 48}{(x^2 - 4x)^2} = \frac{-24(x - 2)}{(x^2 - 4x)^2}$$

at $x = 3$

$$\frac{dy}{dx} = \frac{-24(3 - 2)}{[(3)^2 - 4(3)]^2} = \frac{-24}{(9 - 12)^2} = \frac{-24}{(-3)^2} = -\frac{8}{3}$$

$\therefore$ gradient $= -\frac{8}{3}$ (Ans)

3. (i) Show that the equation $\sin\theta+\cos\theta=2(\sin\theta-\cos\theta)$ can be expressed as $\tan\theta=3$ [2]

(ii) Hence solve the equation $\sin\theta+\cos\theta=2(\sin\theta-\cos\theta)$, for $0^\circ\le\theta\le360^\circ$. [2]

*Suggested solution:*

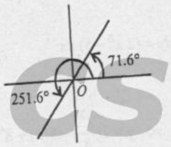
(i)
$$\sin\theta+\cos\theta=2(\sin\theta-\cos\theta)$$
$$\sin\theta+\cos\theta=2\sin\theta-2\cos\theta$$
$$\sin\theta-2\sin\theta=-2\cos\theta-\cos\theta$$
$$-\sin\theta=-3\cos\theta$$
$$\frac{\sin\theta}{\cos\theta}=3 \Rightarrow \tan\theta=3 \text{ (Shown)}$$

(ii) From part (i), the given equation can be written as

$$\tan\theta=3$$

basic angle $\alpha=71.6^\circ$

$\therefore\quad \theta=71.6^\circ, 251.6^\circ$ (Ans)



$\tan\theta$ is positive in I and III quadrant

$\therefore$ in the Ist quadrant $\theta=\alpha$ and in the III quadrant $\theta=180+\alpha$

4. (i) Find the first 3 terms in the expansion of $(2-x)^6$ in ascending powers of $x$. [3]

(ii) Find the value of $k$ for which there is no term in $x^2$ in the expansion of $(1+kx)(2-x)^6$. [2]

*Suggested Solution:*

(i) $(2-x)^6$
$$={}^6C_0\,(2)^6(-x)^0+{}^6C_1\,(2)^5(-x)^1+{}^6C_2\,(2)^4(-x)^2$$
$$=2^6-6(2)^5x+15(2)^4x^2 = 64-192x+240x^2 \quad \text{(Ans)}$$

Binomial theorem:
$(a+b)^n={}^nC_0\,a^n+{}^nC_1\,a^{n-1}b+{}^nC_2\,a^{n-2}b^2+\ldots.+{}^nC_r\,a^{n-r}b^r+\ldots.+{}^nC_n\,b^n$

(ii) no term in $x^2$ means that coefficient of $x^2$ is zero.
$$(1+kx)(2-x)^6$$
$$=\ (1+kx)(64-192x+240x^2)$$
$$=\ 64-192x+240x^2+64kx-192kx^2+240kx^3$$

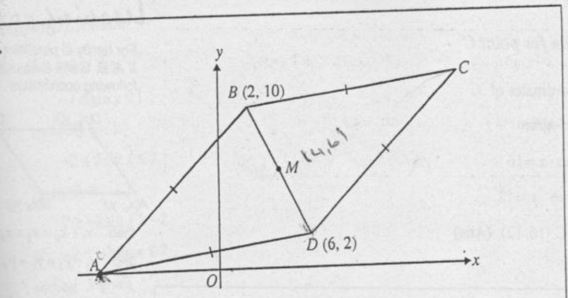
collecting terms containing $x^2$ only

$\Rightarrow\ 240x^2-192kx^2$

$\because$ coefficient of $x^2=0$

$\Rightarrow\ 240-192k=0 \ \Rightarrow\ k=\frac{5}{4}$ (Ans)

5.



The diagram shows a rhombus $ABCD$. The points $B$ and $D$ have coordinates (2, 10) and (6, 2) respectively, and $A$ lies on the $x$-axis. The mid-point of $BD$ is $M$. Find, by calculation, the coordinates of each of $M$, $A$ and $C$. [6]

***Suggested Solution:***

Coordinates of $M = \left(\frac{2+6}{2}, \frac{10+2}{2}\right) = (4,\ 6)$ **(Ans).**

Point $A$ is on the $x$-axis, therefore $y$-coordinate is zero.

Let coordinates of $A$ be $(k, 0)$

as $ABCD$ is a rhombus, therefore

$$|AB| = |AD|$$

$$\sqrt{(k-2)^2+(0-10)^2} = \sqrt{(k-6)^2+(0-2)^2}$$

$$\sqrt{k^2-4k+4+100} = \sqrt{k^2-12k+36+4}$$

squaring both sides we have

$$k^2-4k+104 = k^2-12k+40$$

$$8k = -64 \quad \Rightarrow \quad k = -8$$

$\therefore$ coordinates of $A(-8, 0)$ **(Ans).**

Let $(x, y)$ be the coordinates of $C$

$\therefore$ mid point of $AC = \left(\frac{x-8}{2}, \frac{y+0}{2}\right) = \left(\frac{x-8}{2}, \frac{y}{2}\right)$

and mid point of $BD = M = (4,\ 6)$

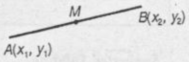
$ABCD$ is a rhombus, so the diagonals bisect each other at the midpoint $M$

$\therefore$ mid point of $AC$ = mid point of $BD$

$\Rightarrow \quad \left(\frac{x-8}{2}, \frac{y}{2}\right) = (4,\ 6)$

$\Rightarrow \quad \frac{x-8}{2} = 4 \quad \Rightarrow \quad x = 16 \qquad$ and $\qquad \frac{y}{2} = 6 \quad \Rightarrow \quad y = 12$

$\therefore$ coordinates of $C$ are (16, 12) **(Ans)**



If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points, then coordinates of mid point are:

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Distance formula:

$$|AB| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

*Alternative solution for point C*

Let $(x, y)$ be the coordinates of $C$

as $ABCD$ is a parallelogram

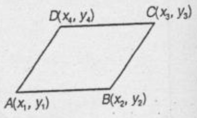
$$x-8=2+6 \;\Rightarrow\; x=16$$

$$y+0=10+2 \;\Rightarrow\; y=12$$

$\therefore$ coordinates of $C$ (16, 12) **(Ans)**

Learning corner

For family of parallelograms.
If $A$, $B$, $C$ and $D$ have the following coordinates



then $x_1+x_3=x_2+x_4$

and $y_1+y_3=y_2+y_4$

**6.** A geometric progression has 6 terms. The first term is 192 and the common ratio is 1.5. An arithmetic progression has 21 terms and common difference 1.5. Given that the sum of all the terms in the geometric progression is equal to the sum of all the terms in the arithmetic progression, find the first term and the last term of the arithmetic progression. [6]

*Suggested Solution:*

Given that in a G.P. $a=192,\; r=1.5,\; n=6$

$$\therefore\quad S_6=192\left(\frac{(1.5)^6-1}{1.5-1}\right)=192\left(\frac{11.3906-1}{0.5}\right)=192\left(20.7812\right)=3989.99$$

In the A.P. it is given that $d=1.5,\; n=21$

Let the first term $=a$

$$\therefore\quad S_{21}=\frac{21}{2}\left[2a+(21-1)(1.5)\right]=\frac{21}{2}\left[2a+30\right]=10.5(2a+30)$$

given condition is

$S_6$ of G.P. $=S_{21}$ of A.P.

$$\Rightarrow\quad 3990=10.5(2a+30)$$

$$\Rightarrow\quad 380=2a+30 \;\Rightarrow\; a=175$$

$\therefore$ first term $=175$ **(Ans)**

$T_n=a+(n-1)d \;\Rightarrow\; T_{21}=175+(21-1)1.5 \;\Rightarrow\; T_{21}=205$

$\therefore$ last term $=205$ **(Ans)**

Sum of $n$ terms of a G.P is

$$S_n=\frac{a(1-r^n)}{1-r},\text{ for } |r|<1$$

or

$$S_n=\frac{a(r^n-1)}{r-1},\text{ for } |r|>1$$

note that in this question $r>1$.

Sum to $n$ terms of an A.P. is

$$S_n=\frac{n}{2}[2a+(n-1)d]$$

**7.** A function f is defined by $f: x \mapsto 3-2\sin x$, for $0° \le x \le 360°$.

(i) Find the range of f. [2]

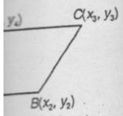
(ii) Sketch the graph of $y=f(x)$. [2]

A function g is defined by $g: x \mapsto 3-2\sin x$, for $0° \le x \le A°$, where $A$ is a constant.

(iii) State the largest value of $A$ for which g has an inverse. [1]

(iv) When $A$ has this value, obtain an expression, in terms of $x$, for $g^{-1}(x)$. [2]

of parallelograms.
nd $D$ have the
oordinates



$+x_3 = x_2 + x_4$
$+y_3 = y_2 + y_4$

of $n$ terms of a G.P

$\frac{a(1-r^n)}{1-r}$, for $|r|<1$

$\frac{a(r^n-1)}{r-1}$, for $|r|>1$

that in this question

to $n$ terms of an
is

$= \frac{n}{2}[2a+(n-1)d]$

---

*Suggested Solution:*

(i) We know that

$$-1 \le \sin x \le 1$$

multiply by 2

$$-2 \le 2\sin x \le 2$$

multiply by $-1$

$$2 \ge -2\sin x \ge -2$$

or $-2 \le -2\sin x \le 2$

adding 3 gives

$$-2+3 \le 3-2\sin x \le 2+3$$

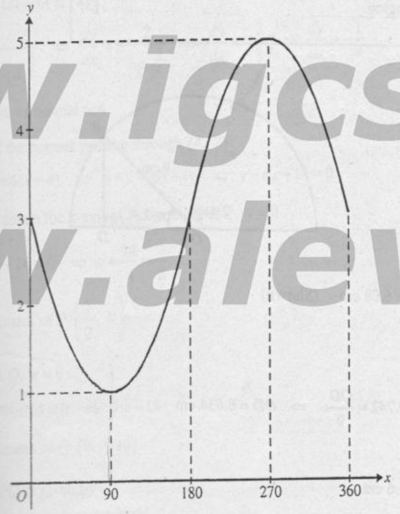
$\Rightarrow \quad 1 \le 3-2\sin x \le 5 \quad \Rightarrow 1 \le \mathrm{f}(x) \le 5$

$\therefore$ required range is $1 \le \mathrm{f}(x) \le 5$ **(Ans)**

Direction of inequality sign changes when both sides are multiplied by a negative number, i.e.

If $a > b$ then $-a < -b$

(ii)



(iii) From the graph in part **(ii)**, the function $g(x)$ is a 1-1 function for $0° \le x \le 90°$.

therefore the largest value of $A$ for which $g(x)$ has an inverse is $90°$. **(Ans)**

(iv) Let $g(x) = y \quad \Rightarrow \quad y = 3 - 2\sin x$

Making $x$ as subject

$$2\sin x = 3 - y \quad \Rightarrow \quad x = \sin^{-1}\left(\frac{3-y}{2}\right)$$

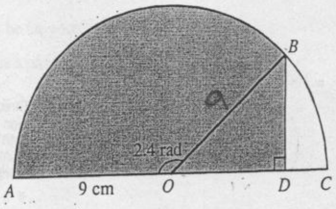
As $g(x) = y \quad \Rightarrow \quad g^{-1}(y) = x$

$\therefore \quad g^{-1}(y) = \sin^{-1}\left(\frac{3-y}{2}\right) \quad \Rightarrow \quad g^{-1}(x) = \sin^{-1}\left(\frac{3-x}{2}\right)$ **(Ans).**

A graph of a function represents a (1-1) function if it passes a horizontal line test. i.e. any horizontal line has only one point of intersection with the graph.

Note that after passing $90^{\circ}$, the graph starts to turn upwards, and now if you draw any horizontal line, the line will cross two points on the curve. So after passing $90^{\circ}$, the curve is no more a 1–1 function.

**8.**



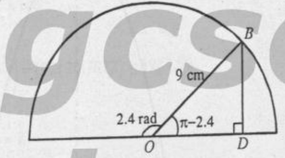
In the diagram, $ABC$ is a semi circle, centre $O$ and radius 9 cm. The line $BD$ is perpendicular to the diameter $AC$ and angle $AOB = 2.4$ radians.

(i) Show that $BD = 6.08$ cm, correct to 3 significant figures. [2]

(ii) Find the perimeter of the shaded region. [3]

(iii) Find the area of the shaded region. [3]

*Suggested Solution:*



(i) In $\Delta OBD$

$B\hat{O}D = \pi - 2.4 = 0.742\text{rad}$

radius $OB = 9$ cm (given)

$\therefore \quad \sin B\hat{O}D = \dfrac{BD}{OB}$

$\Rightarrow \quad \sin 0.742 = \dfrac{BD}{9} \quad \Rightarrow \quad BD = 6.08$ cm **(Shown)**

(ii) In $\Delta OBD$

$\cos B\hat{O}D = \dfrac{OD}{OB} \quad \Rightarrow \quad \cos 0.742 = \dfrac{OD}{9} \quad \Rightarrow \quad OD = 6.634$ cm

using $S = r\theta$

arc length $AB = 9 \times 2.4 = 21.6$ cm

$\therefore$ perimeter of the shaded region $= \overset{\frown}{AB} + BD + OD + OA$

$= 21.6 + 6.08 + 6.634 + 9$

$= 43.314 \approx 43.3$ cm **(Ans)**

(iii) Area of shaded region = area of sector $\overset{\frown}{AOB}$ + area of $\Delta OBD$

$= \dfrac{1}{2} r^2\theta + \dfrac{1}{2}(OD)(BD)$

$= \dfrac{1}{2}(9)^2(2.4) + \dfrac{1}{2}(6.63)(6.08)$

$= 97.2 + 20.155 = 117.355 \approx 117 \text{ cm}^2$ **(Ans)**

9. A curve has equation $y=\frac{4}{\sqrt{x}}$.

(i) The normal to the curve at the point $(4, 2)$ meets the $x$-axis at $P$ and the $y$-axis at $Q$. Find the length of $PQ$, correct to 3 significant figures. [6]

(ii) Find the area of the region enclosed by the curve, the $x$-axis and the lines $x=1$ and $x=4$ [4]

***Suggested Solution:***

(i) $y=\frac{4}{\sqrt{x}} \Rightarrow y=4x^{-\frac{1}{2}}$

$$\frac{dy}{dx}=4\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) \Rightarrow \frac{dy}{dx}=-2x^{-\frac{3}{2}} \Rightarrow \frac{dy}{dx}=-\frac{2}{x^{\frac{3}{2}}}$$

gradient of tangent at (4, 2)

$$\frac{dy}{dx}=-\frac{2}{(4)^{\frac{3}{2}}} \Rightarrow \frac{dy}{dx}=-\frac{2}{(2^2)^{\frac{3}{2}}} \Rightarrow \frac{dy}{dx}=-\frac{2}{8} \Rightarrow \frac{dy}{dx}=-\frac{1}{4}$$

Tangent and normal to the curve are perpendicular to each other, and for perpendicular lines (grad. of tangent) × (grad. of normal) = −1

$\therefore$ gradient of normal = 4

equation of the normal passing through (4, 2):

$$y-2=4(x-4) \Rightarrow y-2=4x-16 \Rightarrow y-4x+14=0$$

the normal meets the $x$-axis at $P$, therefore at $P$, $y=0$

$$\Rightarrow 0-4x+14=0 \Rightarrow x=\frac{14}{4}=\frac{7}{2}$$

$\therefore$ coordinates of $P\left(\frac{7}{2}, 0\right)$

similarly at $Q$, $x=0$

$$\Rightarrow y-4(0)+14=0 \Rightarrow y=-14$$

$\therefore$ coordinates of $Q$ $(0, -14)$

length of $PQ$ is given by

$$|PQ|=\sqrt{\left(\frac{7}{2}-0\right)^2+(0+14)^2} = \sqrt{\frac{49}{4}+196} = \sqrt{208.25}=14.4 \text{ units } \textbf{(Ans)}$$

Distance between two points, $A(x_1, y_1)$ and $B(x_2, y_2)$ is:

$$|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

(ii) Area $= \int_1^4 y\,dx$

$$=\int_1^4 \frac{4}{\sqrt{x}}\,dx=4\int_1^4 x^{-\frac{1}{2}}\,dx=4\left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_1^4=4\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4=8\left[x^{\frac{1}{2}}\right]_1^4$$

$$=8\left((4)^{\frac{1}{2}}-(1)^{\frac{1}{2}}\right) = 8(2-1) = 8 \text{ sq.units } \textbf{(Ans)}$$

Remember:
For definite integral, do not put the constant of integration.

**10.** The equation of a curve is $y = x^2 - 3x + 4$.

(i) Show that the whole of the curve lies above the $x$-axis. [3]

(ii) Find the set of values of $x$ for which $x^2 - 3x + 4$ is a decreasing function of $x$. [1]

The equation of a line is $y + 2x = k$, where $k$ is a constant.

(iii) In the case where $k = 6$, find the coordinates of the points of intersection of the line and the curve. [3]

(iv) Find the value of $k$ for which the line is a tangent to the curve. [3]

*Suggested Solution:*

(i) $y = x^2 - 3x + 4$

$a = +1,\ b = -3,\ c = 4$

as $a > 0$, the graph of the curve is a parabola which opens upwards

discriminant $= b^2 - 4ac = (-3)^2 - 4(1)(4) = 9 - 16 = -7 < 0$

$\therefore$ roots are imaginary, i.e. curve does not touch or intersect $x$-axis.

therefore the whole curve lies above $x$-axis. (Shown)

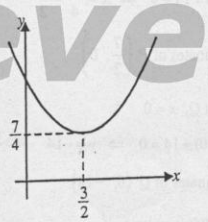
*Alternative Solution:*

By using completing the square method, we have

$$y = x^2 - 3x + 4$$

$$\Rightarrow\ y = x^2 - 2(x)\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4$$

$$\Rightarrow\ y = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 4 \ \Rightarrow\ y = \left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$$



turning point is $\left(\frac{3}{2}, \frac{7}{4}\right)$

i.e $y > 0$ for all values of $x$, therefore the whole curve lies above $x$-axis.

(ii) $y = x^2 - 3x + 4$

$\frac{dy}{dx} = 2x - 3$

for decreasing function $\frac{dy}{dx} < 0$

$\Rightarrow\ 2x - 3 < 0\ \Rightarrow\ x < \frac{3}{2}$ (Ans)

(iii) When $k = 6$, the line becomes

$$y + 2x = 6 \Rightarrow y = 6 - 2x \cdots\cdots\cdots(i)$$

equation of curve $y = x^2 - 3x + 4 \cdots\cdots\cdots(ii)$

putting value of $y$ from eq. (i) into eq. (ii) and solving simultaneously we have

$$6 - 2x = x^2 - 3x + 4 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$$

$\Rightarrow$ $x = 2$, and $x = -1$

when $x = 2$, $y = 6 - 2(2) \Rightarrow y = 2$

when $x = -1$, $y = 6 - 2(-1) \Rightarrow y = 8$

$\therefore$ points of intersection are (2, 2), and (−1, 8) **(Ans).**

(iv) Let $y + 2x = k \Rightarrow y = k - 2x \cdots\cdots\cdots(i)$

and $y = x^2 - 3x + 4 \cdots\cdots\cdots(ii)$

putting value of $y$ from eq. (i) into eq. (ii) and solving simultaneously we have

$$k - 2x = x^2 - 3x + 4 \Rightarrow x^2 - x + (4 - k) = 0$$

the line is tangent to the curve, so the above equation must have equal roots, i.e.

When a line is tangent to the curve then the quadratic equation emerging from their simultaneous solution has equal roots.

$\therefore$ Discriminant = 0

$\Rightarrow (-1)^2 - 4(1)(4 - k) = 0$

$\Rightarrow 1 - 16 + 4k = 0 \Rightarrow 4k = 15 \Rightarrow k = \frac{15}{4} = 3\frac{3}{4}$ (Ans)

11. Relative to an origin $O$, the position vectors of the points $A$ and $B$ are given by

$$\overrightarrow{OA} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}.$$

(i) Use a scalar product to find angle $AOB$, correct to the nearest degree. [4]

(ii) Find the unit vector in the direction of $\overrightarrow{AB}$. [3]

(iii) The point $C$ is such that $\overrightarrow{OC} = 6\mathbf{j} + p\mathbf{k}$, where $p$ is a constant. Given that the lengths of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are equal, find the possible values of $p$. [4]

Learning corner

*Suggested Solution:*

(i) we have $\overline{OA}=\begin{pmatrix}2\\3\\-1\end{pmatrix}$, and $\overline{OB}=\begin{pmatrix}4\\-3\\2\end{pmatrix}$

$$\overline{OA}\bullet\overline{OB}=\left|\overline{OA}\right|\left|\overline{OB}\right|\cos A\hat{O}B$$

$$\begin{pmatrix}2\\3\\-1\end{pmatrix}\bullet\begin{pmatrix}4\\-3\\2\end{pmatrix}=\left(\sqrt{(2)^2+(3)^2+(-1)^2}\right)\times\left(\sqrt{(4)^2+(-3)^2+(2)^2}\right)\cos A\hat{O}B$$

$$8-9-2=\left(\sqrt{14}\right)\left(\sqrt{29}\right)\cos A\hat{O}B$$

$$\cos A\hat{O}B=-\frac{3}{\sqrt{406}} \Rightarrow \cos A\hat{O}B=-0.14889 \Rightarrow A\hat{O}B=98.562°$$

$\therefore$ $A\hat{O}B=99°$ correct to the nearest degree **(Ans)**

If
$\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, and
$\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$, then
$\mathbf{a}.\mathbf{b}=a_1b_1+a_2b_2+a_3b_3$
and magnitude of **a** is
$|\mathbf{a}|=\sqrt{(a_1)^2+(a_2)^2+(a_3)^2}$

(ii) $\overline{AB}=\overline{OB}-\overline{OA}=\begin{pmatrix}4\\-3\\2\end{pmatrix}-\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}2\\-6\\3\end{pmatrix}$

Unit vector in the direction of $\overline{AB}$

$$=\frac{\overline{AB}}{\left|\overline{AB}\right|}=\frac{2\mathrm{i}-6\mathrm{j}+3\mathrm{k}}{\sqrt{(2)^2+(-6)^2+(3)^2}}=\frac{2\mathrm{i}-6\mathrm{j}+3\mathrm{k}}{\sqrt{4+36+9}}=\frac{2\mathrm{i}-6\mathrm{j}+3\mathrm{k}}{7}$$

$\therefore$ required unit vector is $\left(\frac{2}{7}\mathrm{i}-\frac{6}{7}\mathrm{j}+\frac{3}{7}\mathrm{k}\right)$ (Ans)

For any vector **v**, the unit vector $\hat{\mathbf{v}}$, is given as:
$\hat{\mathbf{v}}=\frac{\mathbf{v}}{|\mathbf{v}|}$

**(iii)** We have

$$\overline{OC}=\begin{pmatrix}0\\6\\p\end{pmatrix},\quad \overline{AC}=\overline{OC}-\overline{OA}=\begin{pmatrix}0\\6\\p\end{pmatrix}-\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}-2\\3\\p+1\end{pmatrix},\quad \text{and}\quad \overline{AB}=\begin{pmatrix}2\\-6\\3\end{pmatrix}$$

given condition is

$$\left|\overline{AB}\right|=\left|\overline{AC}\right|$$

$$\left|\sqrt{(2)^2+(-6)^2+(3)^2}\right|=\left|\sqrt{(-2)^2+(3)^2+(p+1)^2}\right|$$

$$\sqrt{49}=\sqrt{13+(p+1)^2}$$

squaring both sides

$$49=13+(p+1)^2$$

$$(p+1)^2=36$$

$$(p+1)=\pm 6$$

either $(p+1)=+6$ or $(p+1)=-6$

$\Rightarrow$ $p=5$, or $p=-7$

$\therefore$ possible values of $p$ are 5 or $-7$ (Ans)

# June 2005 Paper 3

## *Pure Mathematics (P3)*

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. Expand $(1+4x)^{-\frac{1}{2}}$ in ascending powers of $x$, up to and including the term in $x^3$, simplifying the coefficients. [4]

***Suggested Solution:***

Using binomial expansion upto the term in $x^3$

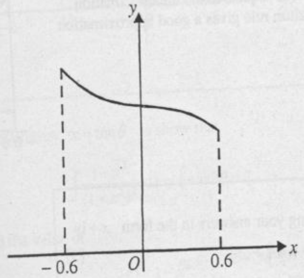
$$(1+4x)^{-\frac{1}{2}}=1+\left(-\frac{1}{2}\right)(4x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(4x)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}(4x)^3$$

$$=1-2x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2\times1}(16x^2)-\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3\times2\times1}(64x^3)$$

$$=1-2x+\frac{3}{8}(16x^2)-\frac{15}{8\times6}(64x^3)$$

$$=1-2x+6x^2-20x^3 \quad \text{(Ans)}$$

2.



The diagram shows a sketch of the curve $y=\frac{1}{1+x^3}$ for values of $x$ from –0.6 to 0.6.

(i) Use the trapezium rule, with two intervals, to estimate the value of

$$\int_{-0.6}^{0.6} \frac{1}{1+x^3}\,dx$$

giving your answer correct to 2 decimal places. [3]

(ii) Explain, with reference to the diagram, why the trapezium rule may be expected to give a good approximation to the true value of the integral in this case. [1]

***Suggested Solution:***

(i) Let $y = f(x) = \frac{1}{1+x^3}$

width of interval, $d = \frac{0.6-(-0.6)}{2} = \frac{1.2}{2} = 0.6$

$\therefore \quad y_0 = f(-0.6) = \frac{1}{1+(-0.6)^3} = 1.2755$

$y_1 = f(0) = \frac{1}{1+(0)^3} = 1$

$y_2 = f(0.6) = \frac{1}{1+(0)^3} = 0.8224$

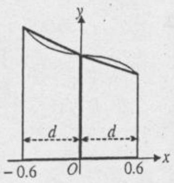
using trapezium rule

$$\int_{-0.6}^{0.6} \frac{1}{1+x^3}\,dx \approx \frac{1}{2}d\left[y_0 + 2y_1 + y_2\right]$$

$$\approx \frac{1}{2}(0.6)\left[1.2755 + 2(1) + 0.8224\right]$$

$$\approx \frac{1}{2}(0.6)\left[4.0979\right] \approx 1.22937 \approx 1.23 \text{ (to 2 dec. pl)} \quad \textbf{(Ans)}$$

(ii) The overestimation from the interval $-0.6$ to $0$ equals to the underestimation from the interval $0$ to $0.6$. Therefore trapezium rule gives a good approximation of the given integral.



3. (i) Solve the equation $z^2 - 2iz - 5 = 0$, giving your answers in the form $x + iy$ where $x$ and $y$ are real. [3]

(ii) Find the modulus and argument of each root. [3]

(iii) Sketch an Argand diagram showing the points representing the roots. [1]

*Suggested Solution:*

(i) $z^2 - 2i z - 5 = 0$

using quadratic formula

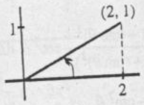
$$z = \frac{-(-2i) \pm \sqrt{(-2i)^2 - 4(1)(-5)}}{2(1)} = \frac{2i \pm \sqrt{4i^2 + 20}}{2} = \frac{2i \pm \sqrt{4(-1) + 20}}{2}$$

$$= \frac{2i + \sqrt{16}}{2} = \frac{2i \pm 4}{2} = i \pm 2 = \pm 2 + i$$

$\therefore \quad z = 2 + i$ and $-2 + i$ **(Ans)**

(ii) Modulus: $|2+i| = \sqrt{2^2 + (1)^2} = \sqrt{5}$ **(Ans)**

$\arg(2+i) = \tan^{-1}\left(\frac{1}{2}\right) = 0.464$ radians



Note that:

If $z = a + ib$, then

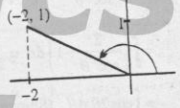
$|z| = \sqrt{a^2 + b^2}$

The range of principal argument $\alpha$ is $-\pi < \alpha \le \pi$

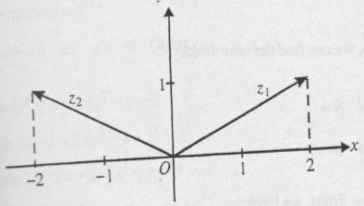
Modulus: $|-2+i| = \sqrt{(-2)^2 + (1)^2} = \sqrt{5}$ **(Ans)**

$\arg(-2+i) = \tan^{-1}\left(-\frac{1}{2}\right)$, basic angle $= 0.464$ radians

$\therefore$ required argument $= \pi - 0.464 = 2.68$ radians **(Ans)**



(iii) Let $z_1 = 2 + i$ and $z_2 = -2 + i$



4. (i) Use the substitution $x = \tan\theta$ to show that

$$\int \frac{1 - x^2}{(1 + x^2)^2}\, dx = \int \cos 2\theta \, d\theta$$ [4]

(ii) Hence find the value of

$$\int_0^1 \frac{1 - x^2}{(1 + x^2)^2}\, dx$$ [3]

*Suggested Solution:*

(i) L.H.S. $= \int \frac{1-x^2}{(1+x^2)^2}\,dx$

given substitution is

$x = \tan\theta$

differentiating w.r.t. $x$ gives

$$1 = \sec^2\theta \frac{d\theta}{dx} \Rightarrow dx = \sec^2\theta\, d\theta$$

substituting the values of $x$ and $dx$ we have

$$\int \frac{1-x^2}{(1+x^2)^2}\,dx = \int \frac{1-(\tan\theta)^2}{(1+(\tan\theta)^2)^2}\sec^2\theta\, d\theta$$

$$= \int \frac{1-\tan^2\theta}{(1+\tan^2\theta)^2}\sec^2\theta\, d\theta = \int \frac{1-(\sec^2\theta-1)}{(\sec^2\theta)^2}\sec^2\theta\, d\theta$$

$$= \int \frac{2-\sec^2\theta}{\sec^2\theta}\,d\theta = \int\left(\frac{2}{\sec^2\theta}-\frac{\sec^2\theta}{\sec^2\theta}\right)d\theta$$

$$= \int\left(\frac{2}{\sec^2\theta}-1\right)d\theta = \int (2\cos^2\theta-1)\,d\theta$$

$$= \int \cos 2\theta\, d\theta \quad \text{(Shown)}$$

Note that:

- $\frac{d}{dx}(\tan\theta) = \sec^2\theta$
- $1+\tan^2\theta = \sec^2\theta$

(ii) $\int_0^1 \frac{1-x^2}{(1+x^2)^2}\,dx$

using the substitution given in part (i), we can find the new limits

$$\text{new limits} \begin{cases} x=1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4} \\ x=1 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0 \end{cases}$$

using the result of part (i), and with new limits, we have.

$$\therefore \int_0^1 \frac{1-x^2}{(1+x^2)^2}\,dx = \int_0^{\frac{\pi}{4}} \cos 2\theta\, d\theta$$

$$= \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\sin 2\theta\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\sin 2\left(\frac{\pi}{4}\right) - \sin 2(0)\right]$$

$$= \frac{1}{2}\left[\sin\frac{\pi}{2} - \sin 0\right] = \frac{1}{2}(1-0) = \frac{1}{2} \quad \text{(Ans)}$$

Remember:
For definite integral, do not put the constant of integration

5. The polynomial $x^4+5x+a$ is denoted by p($x$). It is given that $x^2-x+3$ is a factor of p($x$).

(i) Find the value of $a$ and factorise p($x$) completely. [6]

(ii) Hence state the number of real roots of the equation p($x$) = 0, justifying your answer. [2]

***Suggested Solution:***

(i) Given that $p(x)=x^4+5x+a$

using long division, we have

$$\begin{array}{r|l} & x^2+x-2 \\ \hline x^2-x+3 & x^4+5x+a \\ & x^4 \quad -x^3+3x^2 \\ & - \qquad + \quad - \\ \hline & x^3-3x^2+5x+a \\ & x^3-\ x^2+3x \\ & - \quad + \quad - \\ \hline & -2x^2+2x+a \\ & -2x^2+2x-6 \\ & + \quad - \quad + \\ \hline & a+6 \end{array}$$

given that $(x^2-x+3)$ is a factor of p($x$)

$\therefore$ Remainder = 0

$\Rightarrow$ $a+6=0 \Rightarrow a=-6$ **(Ans)**

$\therefore$ $p(x)=(x^2+x-2)(x^2-x+3)$

$=(x^2+2x-x-2)(x^2-x+3)$

$=\left[x(x+2)-1(x+2)\right](x^2-x+3)$

$=(x+2)(x-1)(x^2-x+3)$ **(Ans)**

(ii) $p(x)=0$

$\Rightarrow$ $(x+2)(x-1)(x^2-x+3)=0$

$\Rightarrow$ $(x^2-x+3)$ has no real roots, (disc $<0$)

$\therefore$ p($x$) has only two real roots, $x=1$ and $x=-2$ **(Ans)**

6. (i) Prove the identity

$$\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3.$$ [4]

(ii) Hence solve the equation

$$\cos 4\theta + 4\cos 2\theta = 2,$$

for $0^\circ \le \theta \le 360^\circ$ [4]

***Suggested Solution:***

(i) $\text{L.H.S} = \cos 4\theta + 4\cos 2\theta$

$$= \cos 2(2\theta) + 4\cos 2\theta$$
$$= (2\cos^2 2\theta - 1) + 4(2\cos^2\theta - 1)$$
$$= 2\cos^2 2\theta - 1 + 8\cos^2\theta - 4$$
$$= 2(\cos 2\theta)^2 + 8\cos^2\theta - 5$$
$$= 2(2\cos^2\theta - 1)^2 + 8\cos^2\theta - 5$$
$$= 2(4\cos^4\theta - 4\cos^2\theta + 1) + 8\cos^2\theta - 5$$
$$= 8\cos^4\theta - 8\cos^2\theta + 2 + 8\cos^2\theta - 5$$
$$= 8\cos^4\theta - 3 = \text{R.H.S} \quad \textbf{(Proved)}$$

Note that:
$\cos 2\theta = 2\cos^2\theta - 1$
similarly
$\cos 2(2\theta) = 2\cos^2(2\theta) - 1$

(ii) $\cos 4\theta + 4\cos 2\theta = 2$

using result of part (i), the above equation can be written as

$$8\cos^4\theta - 3 = 2 \Rightarrow 8\cos^4\theta = 5 \Rightarrow \cos^4\theta = \frac{5}{8}$$

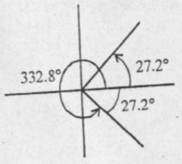
$\cos^2\theta = \sqrt{\frac{5}{8}}$ and $\cos^2\theta = -\sqrt{\frac{5}{8}}$ (ignored)

$\therefore \quad \cos^2\theta = 0.79057$

$\Rightarrow \quad \cos\theta = \pm\, 0.88914$

now, when $\cos\theta = 0.88914$

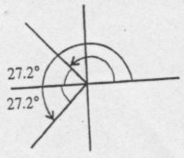
basic angle $\alpha = 27.23^\circ$



$\therefore \quad \theta = 27.2^\circ,\ 332.8^\circ$

and, when $\cos\theta = -0.88914$

basic angle $\alpha = 27.23^\circ$



Note that $\cos\theta$ is positive in 1st and 4th quadrant and negative in 2nd and 3rd quadrant.

$\therefore \quad \theta = 152.8^\circ,\ 207.2^\circ$

$\therefore$ solution: $\theta = 27.2^\circ,\ 152.8^\circ,\ 207.2^\circ,\ 332.8^\circ$ **(Ans)**

7. (i) By sketching a suitable pair of graphs, show that the equation

$$\operatorname{cosec} x = \frac{1}{2}x + 1,$$

where $x$ is in radians, has a root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify, by calculation, that this root lies between 0.5 and 1. [2]

(iii) Show that this root also satisfies the equation

$$x = \sin^{-1}\left(\frac{2}{x+2}\right). \qquad [1]$$

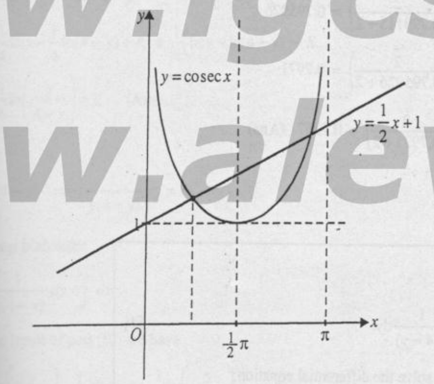
(iv) Use the iterative formula

$$x_{n+1} = \sin^{-1}\left(\frac{2}{x_n + 2}\right),$$

with initial value $x_1 = 0.75$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

*Suggested Solution:*

(i)



By drawing the graphs of $y = \operatorname{cosec} x$ and $y = \frac{1}{2}x + 1$, we see that there is one root between $0 < x < \frac{1}{2}\pi$. **(Shown)**

(ii) Let $f(x) = \operatorname{cosec} x - \frac{1}{2}x - 1$

$\therefore$ $f(0.5) = \operatorname{cosec}(0.5) - \frac{1}{2}(0.5) - 1 = 2.08583 - 0.25 - 1 = 0.836 \; > 0$

and $f(1) = \operatorname{cosec}(1) - \frac{1}{2}(1) - 1 = 1.18839 - 0.5 - 1 = -0.312 \; < 0$

$\Rightarrow$ the root lies between 0.5 and 1 **(Verified)**

Do not forget to change your calculator to radian mode.

**(iii)** $x = \sin^{-1}\left(\dfrac{2}{x+2}\right) \Rightarrow \sin x = \dfrac{2}{x+2} \Rightarrow \operatorname{cosec} x = \dfrac{x+2}{2}$

$\Rightarrow 2\operatorname{cosec} x = x + 2 \Rightarrow \operatorname{cosec} x = \dfrac{x+2}{2} \Rightarrow \operatorname{cosec} x = \dfrac{1}{2}x + 1$

which is the same equation given in part **(i)**. Therefore the root also satisfies the above equation.

**(iv)** $x_{n+1} = \sin^{-1}\left(\dfrac{2}{x_n+2}\right)$,

Given that initial value $x_1 = 0.75$

$$\therefore \quad x_2 = \sin^{-1}\left(\frac{2}{x_1+2}\right) = \sin^{-1}\left(\frac{2}{0.75+2}\right) = 0.8143399$$

$$x_3 = \sin^{-1}\left(\frac{2}{x_2+2}\right) = \sin^{-1}\left(\frac{2}{0.8143399+2}\right) = 0.790416$$

$$x_4 = \sin^{-1}\left(\frac{2}{x_3+2}\right) = \sin^{-1}\left(\frac{2}{0.790416+2}\right) = 0.799115$$

$$x_5 = \sin^{-1}\left(\frac{2}{x_4+2}\right) = \sin^{-1}\left(\frac{2}{0.799115+2}\right) = 0.795926$$

$$x_6 = \sin^{-1}\left(\frac{2}{x_5+2}\right) = \sin^{-1}\left(\frac{2}{0.795926+2}\right) = 0.7971$$

$$x_7 = \sin^{-1}\left(\frac{2}{x_6+2}\right) = \sin^{-1}\left(\frac{2}{0.7971+2}\right) = 0.7967 \quad \textbf{(Ans)}$$

$\therefore \quad \alpha = 0.80$ (2 dec.pl) **(Ans)**

**8.** (i) Using partial fractions, find

$$\int \frac{1}{y(4-y)}\,dy.$$ [4]

(ii) Given that $y = 1$ when $x = 0$, solve the differential equation

$$\frac{dy}{dx} = y(4-y),$$

obtaining an expression for $y$ in terms of $x$. [4]

(iii) State what happens to the value of $y$ if $x$ becomes very large and positive. [1]

*Suggested Solution:*

(i) $\dfrac{1}{y(4-y)} \equiv \dfrac{A}{y} + \dfrac{B}{4-y}$

$\Rightarrow \; 1 \equiv A(4-y) + By$

for $y = 0$

$1 = A(4-0) + B(0) \;\Rightarrow\; 1 = 4A \;\Rightarrow\; A = \dfrac{1}{4}$

for $y = 4$

$1 = A(4-4) + B(4) \;\Rightarrow\; 1 = 4B \;\Rightarrow\; B = \dfrac{1}{4}$

$\therefore \quad \dfrac{1}{y(4-y)} = \dfrac{1}{4y} + \dfrac{1}{4(4-y)}$

$$\Rightarrow \int \frac{1}{y(4-y)}\,dy = \int\left(\frac{1}{4y} + \frac{1}{4(4-y)}\right)dy$$

$$= \frac{1}{4}\int \frac{1}{y}\,dy + \frac{1}{4}\int \frac{1}{4-y}\,dy = \frac{1}{4}\ln y + \frac{1}{4}\left(\frac{\ln(4-y)}{(-1)}\right) + K$$

$$= \frac{1}{4}\ln y - \frac{1}{4}\ln(4-y) + K = \frac{1}{4}\left[\ln y - \ln(4-y)\right] + K$$

$$= \frac{1}{4}\ln\left(\frac{y}{4-y}\right) + K \quad \text{(Ans)}$$

(ii) $\dfrac{dy}{dx} = y(4-y) \;\Rightarrow\; \dfrac{1}{y(4-y)}\,dy = dx$

integrating both sides

$$\int \frac{1}{y(4-y)}\,dy = \int dx$$

using the result of part (i), we have

$$\frac{1}{4}\ln\left(\frac{y}{4-y}\right) = \int dx \;\Rightarrow\; \frac{1}{4}\ln\left(\frac{y}{4-y}\right) = x + C \cdots\cdots\cdots\text{(i)}$$

given that $y = 1$ when $x = 0$

$$\Rightarrow \frac{1}{4}\ln\left(\frac{1}{4-1}\right) = 0 + C \;\Rightarrow\; \frac{1}{4}\ln\left(\frac{1}{3}\right) = C$$

$\therefore$ eq. (i) becomes

$$\frac{1}{4}\ln\left(\frac{y}{4-y}\right) = x + \frac{1}{4}\ln\left(\frac{1}{3}\right) \;\Rightarrow\; \frac{1}{4}\ln\left(\frac{y}{4-y}\right) - \frac{1}{4}\ln\left(\frac{1}{3}\right) = x$$

$$\Rightarrow \frac{1}{4}\left(\ln\frac{y}{4-y}-\ln\frac{1}{3}\right)=x \Rightarrow \frac{1}{4}\left(\ln\left(\frac{\frac{y}{4-y}}{\frac{1}{3}}\right)\right)=x$$

$$\Rightarrow \frac{1}{4}\left(\ln\frac{3y}{4-y}\right)=x \Rightarrow \ln\frac{3y}{4-y}=4x \Rightarrow \ln\frac{3y}{4-y}=(4x)\ln e$$

$$\Rightarrow \ln\frac{3y}{4-y}=\ln e^{4x} \Rightarrow \frac{3y}{4-y}=e^{4x} \Rightarrow 3y=e^{4x}(4-y)$$

$$\Rightarrow 3y=4e^{4x}-e^{4x}y \Rightarrow 3y+e^{4x}y=4e^{4x} \Rightarrow y(3+e^{4x})=4e^{4x}$$

$$\Rightarrow y=\frac{4e^{4x}}{3+e^{4x}} \quad \textbf{(Ans)}$$

Note that

- $\ln e = 1$
- $\ln a^b = b\ln a$

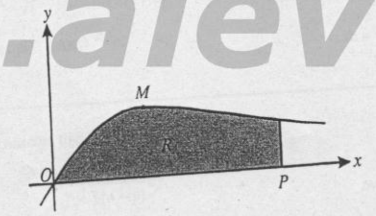
(iii) $y=\frac{4e^{4x}}{3+e^{4x}} \Rightarrow y=\frac{4e^{4x}}{e^{4x}\left(\frac{3}{e^{4x}}+1\right)} \Rightarrow y=\frac{4}{\frac{3}{e^{4x}}+1}$

Now, when

$x\to\infty$, $\quad \frac{3}{e^{4x}}\to 0 \quad$ and $\quad y\to\frac{4}{0+1}$

$\therefore$ $y$ approaches 4 as $x$ becomes very large **(Ans)**

**9.**



The diagram shows part of the curve $y=\frac{x}{x^2+1}$ and its maximum point $M$. The shaded region $R$ is bounded by the curve and by the lines $y=0$ and $x=p$.

(i) Calculate the $x$-coordinate of $M$. [4]

(ii) Find the area of $R$ in terms of $p$. [3]

(iii) Hence calculate the value of $p$ for which the area of $R$ is 1, giving your answer correct to 3 significant figures. [2]

*Suggested Solution:*

(i) $y = \dfrac{x}{x^2+1}$

differentiating w.r.t. $x$

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{x}{x^2+1}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x^2+1)\frac{d}{dx}(x)-(x)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(1)-(x)(2x)}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1-x^2}{(x^2+1)^2}$$

Formula:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{(v)\frac{d}{dx}(u)-(u)\frac{d}{dx}(v)}{v^2}$$

for maxima or minima, $\dfrac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{1-x^2}{(x^2+1)^2} = 0 \;\Rightarrow\; 1-x^2 = 0 \;\Rightarrow\; x^2 = 1 \;\Rightarrow\; x = \pm 1$$

$\therefore$ $x$-coordinate of $M$ is $x = 1$ **(Ans)**

(ii) Area of $R$, $A = \displaystyle\int_0^p y\,\mathrm{d}x$

$$\Rightarrow \quad A = \int_0^p \frac{x}{x^2+1}\,\mathrm{d}x \;\Rightarrow\; A = \frac{1}{2}\int_0^p \frac{2x}{x^2+1}\,\mathrm{d}x$$

performing integration, we have

$$A = \frac{1}{2}\Big[\ln(x^2+1)\Big]_0^p = \frac{1}{2}\Big[\ln(p^2+1) - \ln(0^2+1)\Big]$$

$$= \frac{1}{2}\Big[\ln(p^2+1) - \ln(1)\Big] = \frac{1}{2}\ln(p^2+1) \quad \textbf{(Ans)}$$

(iii) From part (ii), we have

area of $R = \dfrac{1}{2}\ln(p^2+1)$

given that, area of $R = 1$

$$\therefore \quad \frac{1}{2}\ln(p^2+1) = 1 \;\Rightarrow\; \ln(p^2+1) = 2 \;\Rightarrow\; \ln(p^2+1) = (2)\ln \mathrm{e}$$

$$\Rightarrow \quad \ln(p^2+1) = \ln \mathrm{e}^2 \;\Rightarrow\; p^2+1 = e^2 \;\Rightarrow\; p^2 = (e^2-1)$$

$$\Rightarrow \quad p = \pm\sqrt{e^2-1}$$

ignoring the negative value

$$p = \sqrt{e^2-1} \;\Rightarrow\; p = 2.5276 \approx 2.53 \quad \textbf{(Ans)}$$

**10.** With respect to the origin $O$, the points $A$ and $B$ have position vectors given by

$$\overline{OA} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} \qquad \text{and} \qquad \overline{OB} = \mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$$

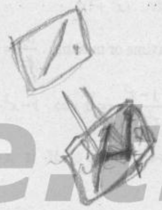
The line $l$ has vector equation $\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

(i) Prove that the line $l$ does not intersect the line through $A$ and $B$. [5]

(ii) Find the equation of the plane containing $l$ and the point $A$, giving your answer in the form $ax + by + cz = d$. [6]

***Suggested Solution:***

(i) Given that $\overline{OA} = \begin{pmatrix}2\\2\\1\end{pmatrix}$ and $\overline{OB} = \begin{pmatrix}1\\4\\3\end{pmatrix}$



$$\therefore \quad \overline{AB} = \overline{OB} - \overline{OA} = \begin{pmatrix}1\\4\\3\end{pmatrix} - \begin{pmatrix}2\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\2\\2\end{pmatrix}$$

equation of line $AB$ is:

$$\mathbf{r}_1 = \overline{OA} + \lambda\overline{AB} \;\Rightarrow\; \mathbf{r}_1 = \begin{pmatrix}2\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\2\end{pmatrix} \;\Rightarrow\; \mathbf{r}_1 = \begin{pmatrix}2-\lambda\\2+2\lambda\\1+2\lambda\end{pmatrix}$$

given equation of $l$: $\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \;\Rightarrow\; \mathbf{r} = \begin{pmatrix}4+s\\-2+2s\\2+s\end{pmatrix}$

Let us assume that the two lines intersect

Then $\mathbf{r}_1 = \mathbf{r}$

$$\Rightarrow \begin{pmatrix}2-\lambda\\2+2\lambda\\1+2\lambda\end{pmatrix} = \begin{pmatrix}4+s\\-2+2s\\2+s\end{pmatrix}$$

equating coefficients of **i** and **j**, we have

$$2 - \lambda = 4 + s \;\Rightarrow\; s + \lambda = -2 \;\Rightarrow\; s = -2 - \lambda \cdots\cdots\cdots\text{(i)}$$

$$2 + 2\lambda = -2 + 2s \;\Rightarrow\; 2\lambda - 2s = -4 \;\Rightarrow\; \lambda - s = -2 \cdots\cdots\cdots\text{(ii)}$$

putting eq. (i) into eq. (ii)

$$\lambda - (-2 - \lambda) = -2 \;\Rightarrow\; \lambda + 2 + \lambda = -2 \;\Rightarrow\; 2\lambda = -4 \;\Rightarrow\; \lambda = -2$$

$$\therefore \quad s = -2 - (-2) \;\Rightarrow\; s = -2 + 2 \;\Rightarrow\; s = 0$$

putting these values of $s$ and $\lambda$ in the coefficients of **k**

line $\mathbf{r}_1$: $1 + 2\lambda = 1 + 2(-2) = 1 - 4 = -3$

line $l$: $2 + s = 2 + 0 = 2$

As the $z$ coordinates on both lines at the assumed point of intersection are different, therefore the two lines $l$ and $\mathbf{r}_1$ do not intersect. **(Shown)**

(ii) We see that $(4, -2, 2)$ is a point on line $l$. Let this point be $C$.

$$\therefore \quad \overline{OC} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$$

$$\text{now,} \quad \overline{AC} = \overline{OC} - \overline{OA} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

let dir. vector $\overline{AC} = \vec{d}_1$ and let $\vec{d}_2$ be the dir. vector of line $l$. i.e. $\vec{d}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$\therefore$ normal vector $\mathbf{n}$ to the required plane is

$$\mathbf{n} = \vec{d}_1 \times \vec{d}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \mathbf{i}(-4-2) - \mathbf{j}(2-1) + \mathbf{k}(4+4)$$

$$\Rightarrow \quad \mathbf{n} = -6\mathbf{i} - \mathbf{j} + 8\mathbf{k} \quad \text{or} \quad \begin{pmatrix} -6 \\ -1 \\ 8 \end{pmatrix}$$

Now, equation of the plane in scalar product form is

$$\mathbf{r} \cdot \mathbf{n} = D \quad \Rightarrow \quad \mathbf{r} \cdot \begin{pmatrix} -6 \\ -1 \\ 8 \end{pmatrix} = D$$

as point $A(2, 2, 1)$ lies on the plane

$$\therefore \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ -1 \\ 8 \end{pmatrix} = D \quad \Rightarrow \quad D = -12 - 2 + 8 \quad \Rightarrow \quad D = -6$$

$\therefore$ required equation of the plane is

$$\mathbf{r} \cdot \begin{pmatrix} -6 \\ -1 \\ 8 \end{pmatrix} = -6 \quad \text{or} \quad -6x - y + 8z = -6 \quad \text{or} \quad 6x + y - 8z = 6 \quad \text{(Ans)}$$

*Alternative solution to part (ii)*

We have $\overline{OA} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $\vec{d}_1 = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$, and $\vec{d}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Equation of the plane passing through point $A$ with two direction vectors $\vec{d}_1$ and $\vec{d}_2$ is given by

$$\mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$1 \ 2 \ 1$

$2 \ -4 \ 1$

$i(-4-2)$

$-6$

$(2-1)$

$-j$

$(4+4)$

$-6i - j + 8k$

$-6x - y + 8z = d$

$-6(2) - 8(2) + 8(1) = d$

$-12 - 2 + 8$

$d = -6$

$6x + y - 8z = 6$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad x = 2 + 2\lambda + \mu \cdots\cdots\cdots \text{(i)}$$

$$y = 2 - 4\lambda + 2\mu \cdots\cdots\cdots \text{(ii)}$$

$$z = 1 + \lambda + \mu \cdots\cdots\cdots \text{(iii)}$$

from eq. (i), $\mu = x - 2 - 2\lambda$, put in eq. (ii)

$$y = 2 - 4\lambda + 2(x - 2 - 2\lambda) \;\Rightarrow\; y = 2 - 4\lambda + 2x - 4 - 4\lambda$$

$$\Rightarrow \quad y = 2x - 8\lambda - 2 \;\Rightarrow\; 8\lambda = 2x - 2 - y \;\Rightarrow\; \lambda = \frac{2x - 2 - y}{8} \cdots\cdots\cdots \text{(iv)}$$

$$\therefore \quad \mu = x - 2 - 2\left(\frac{2x - 2 - y}{8}\right) \;\Rightarrow\; \mu = x - 2 - \frac{2x - 2 - y}{4}$$

$$\Rightarrow \quad \mu = \frac{4x - 8 - (2x - 2 - y)}{4} \;\Rightarrow\; \mu = \frac{2x - 6 + y}{4} \cdots\cdots\cdots \text{(v)}$$

putting the values of $\lambda$ and $\mu$ from eq. (iv) & eq. (v) into eq. (iii)

$$z = 1 + \left(\frac{2x - 2 - y}{8}\right) + \left(\frac{2x - 6 + y}{4}\right)$$

$$\Rightarrow \quad z = \frac{8 + 2x - 2 - y + 2(2x - 6 + y)}{8} \;\Rightarrow\; z = \frac{8 + 2x - 2 - y + 4x - 12 + 2y}{8}$$

$$\Rightarrow \quad 8z = 6x + y - 6 \;\Rightarrow\; 6x + y - 8z = 6 \quad \text{(Ans)}$$

# November 2005 Paper 1

## *Pure Mathematics (P1)*

Answer all the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. Solve the equation $3\sin^2\theta - 2\cos\theta - 3 = 0$, for $0^\circ \le \theta \le 180^\circ$. [4]

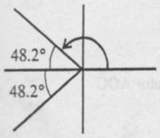
***Suggested Solution:***

$$3\sin^2\theta - 2\cos\theta - 3 = 0$$
$$3(1-\cos^2\theta) - 2\cos\theta - 3 = 0$$
$$3 - 3\cos^2\theta - 2\cos\theta - 3 = 0$$
$$-3\cos^2\theta - 2\cos\theta = 0$$
$$3\cos^2\theta + 2\cos\theta = 0$$
$$\cos\theta(3\cos\theta + 2) = 0$$

$3\cos\theta + 2 = 0$ or $\cos\theta = 0 \Rightarrow \theta = 90^\circ$

$\Rightarrow \cos\theta = -\frac{2}{3}$

basic angle $\alpha = 48.18^\circ$



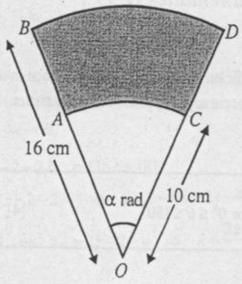
Note that $\cos\theta$ is –ve in $2^{nd}$ and $3^{rd}$ quadrant

req. angle for given range

$\theta = 180^\circ - 48.18 = 131.82^\circ$

$\therefore \theta = 90^\circ,\ 131.8^\circ$ (Ans)

**2.**



In the diagram, $OAB$ and $OCD$ are radii of a circle, centre $O$ and radius 16 cm. Angle $AOC = \alpha$ radians. $AC$ and $BD$ are arcs of circles, centre $O$ and radii 10 cm and 16 cm respectively.

(i) In the case where $\alpha = 0.8$, find the area of the shaded region. [2]

(ii) Find the value of $\alpha$ for which the perimeter of the shaded region is 28.9 cm. [3]

***Suggested Solution:***

(i) Area of the sector $\overset{\frown}{OAC} = \frac{1}{2}r^2\alpha = \frac{1}{2}(10)^2(0.8) = 40\text{ cm}^2$

similarly

Area of the sector $\overset{\frown}{OBD} = \frac{1}{2}(16)^2(0.8) = 102.4\text{ cm}^2$

$\therefore$ Area of the shaded region = Area of sector $\overset{\frown}{OBD}$ − area of sector $\overset{\frown}{AOC}$

$= 102.4 - 40 = 62.4\text{ cm}^2$ **(Ans)**

(ii) Using $S = r\theta$,

arc length $\overset{\frown}{AC} = 10\alpha$

arc length $\overset{\frown}{BD} = 16\alpha$

given that

perimeter of shaded region = 28.9

$\Rightarrow \quad 16\alpha + 10\alpha + 6 + 6 = 28.9$

$26\alpha + 12 = 28.9$

$26\alpha = 16.9$

$\alpha = 0.65$ rad **(Ans)**

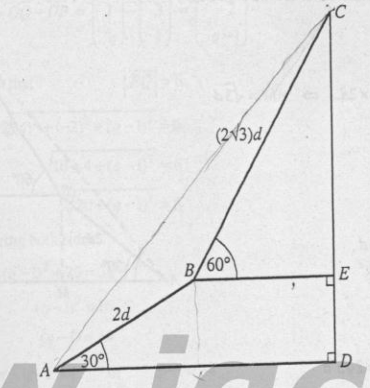
area of sector $= \frac{1}{2}r^2\theta$
arc length, $S = r\theta$
where $\theta$ is in radians

ector = $\frac{1}{2}r^2\theta$

$S = r\theta$

in radians

3.



In the diagram, $ABED$ is a trapezium with right angles at $E$ and $D$, and $CED$ is a straight line. The lengths of $AB$ and $BC$ are $2d$ and $(2\sqrt{3})d$ respectively, and angles $BAD$ and $CBE$ are 30° and 60° respectively.

(i) Find the length of $CD$ in terms of $d$. [2]

(ii) Show that angle $CAD = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$. [3]

*Suggested Solution:*

(i) Draw $BM \perp AD$

In $\Delta\, ABM$

$$\sin 30 = \frac{BM}{AB}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{BM}{2d} \quad \Rightarrow \quad BM = \frac{2d}{d} = d$$

note that $BM = ED$, $\quad \therefore\ ED = d$

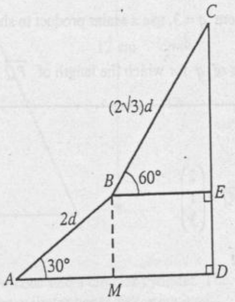
consider $\Delta\, BCE$

$$\sin 60^\circ = \frac{CE}{BC}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{CE}{2\sqrt{3}d} \quad \Rightarrow \quad CE = \frac{\sqrt{3}}{2} \times 2\sqrt{3}d \quad \Rightarrow \quad CE = 3d$$

now

$$CD = CE + ED = 3d + d = 4d \quad \text{(Ans)}$$



Note that:

$\sin 30° = \frac{1}{2}$

$\sin 60° = \frac{\sqrt{3}}{2}$

(ii) In Δ $ABM$,

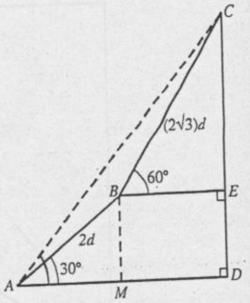
$$\cos 30^\circ = \frac{AM}{AB}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{AM}{2d} \quad \Rightarrow \quad AM = \frac{\sqrt{3}}{2} \times 2d \quad \Rightarrow \quad AM = \sqrt{3}\,d$$

In Δ $BCE$,

$$\cos 60^\circ = \frac{BE}{BC}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{BE}{2\sqrt{3}\,d} \quad \Rightarrow \quad BE = \sqrt{3}\,d$$



as $BE = MD$, $\therefore$ $MD = \sqrt{3}\,d$

now

$$AD = AM + MD = \sqrt{3}\,d + \sqrt{3}\,d = 2\sqrt{3}\,d$$

$$\therefore \quad \tan C\hat{A}D = \frac{CD}{AD} = \frac{4d}{2\sqrt{3}\,d} = \frac{2}{\sqrt{3}} \quad \text{or} \quad C\hat{A}D = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right) \quad \textbf{(Shown)}$$

**4.** Relative to an origin $O$, the position vectors of points $P$ and $Q$ are given by

$$\overrightarrow{OP} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OQ} = \begin{pmatrix} 2 \\ 1 \\ q \end{pmatrix},$$

where $q$ is a constant.

(i) In the case where $q = 3$, use a scalar product to show that $\cos POQ = \frac{1}{7}$. [3]

(ii) Find the values of $q$ for which the length of $\overrightarrow{PQ}$ is 6 units. [4]

***Suggested Solution:***

(i) When $q = 3$, $\overline{OQ} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

using scalar product

$$\overline{OP} \cdot \overline{OQ} = \left|\overline{OP}\right| \times \left|\overline{OQ}\right| \cos P\hat{O}Q$$

$$\Rightarrow \quad \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \left(\sqrt{(-2)^2 + (3)^2 + (1)^2}\right)\left(\sqrt{(2)^2 + (1)^2 + (3)^2}\right) \cos P\hat{O}Q$$

$$\Rightarrow \quad -4 + 3 + 3 = \left(\sqrt{4+9+1}\right)\left(\sqrt{4+1+9}\right) \cos P\hat{O}Q$$

$$\Rightarrow \quad 2 = \left(\sqrt{14}\right)\left(\sqrt{14}\right) \cos P\bar{O}Q$$

$$\Rightarrow \quad 2 = 14 \cos P\hat{O}Q \quad \Rightarrow \quad \cos P\hat{O}Q = \frac{2}{14} \quad \Rightarrow \quad \cos P\hat{O}Q = \frac{1}{7} \quad \textbf{(Shown)}$$

If
$\mathbf{a} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, and
$\mathbf{b} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then
$\mathbf{a}.\mathbf{b} = ax + by + cz$
and magnitude of **a** is
$|\mathbf{a}| = \sqrt{(a)^2 + (b)^2 + (c)^2}$

|+b|+ck, and
|+y|+zk , then
= ax + by + cz
magnitude of **a** is
$= \sqrt{(a)^2 + (b)^2 + (c)^2}$

(ii) $\overline{PQ} = \overline{OQ} - \overline{OP} = \begin{pmatrix} 2 \\ 1 \\ q \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ q-1 \end{pmatrix}$

given that $\left|\overline{PQ}\right| = 6$

$\Rightarrow \quad \sqrt{(4)^2 + (-2)^2 + (q-1)^2} = 6$

$\Rightarrow \quad \sqrt{16 + 4 + (q-1)^2} = 6$

$\Rightarrow \quad \sqrt{20 + (q-1)^2} = 6$

squaring both sides

$(q-1)^2 + 20 = 36$

$\Rightarrow \quad (q-1)^2 = 16$

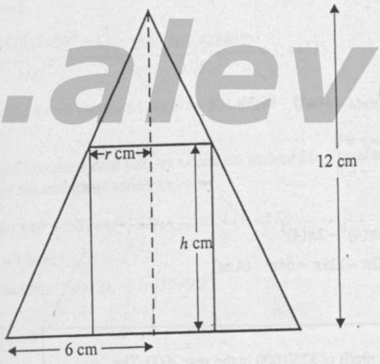
$\Rightarrow \quad (q-1) = \pm 4$

either $q - 1 = 4$ or $q - 1 = -4$

$q = 5$ or $q = -3$

$\therefore \quad q = 5$ or $-3$ **(Ans)**

5.



The diagram shows the cross-section of a hollow cone and a circular cylinder. The cone has radius 6 cm and height 12 cm, and the cylinder has radius $r$ cm and height $h$ cm. The cylinder just fits inside the cone with all of its upper edge touching the surface of the cone.

(i) Express $h$ in terms of $r$ and hence show that the volume, $V$ cm$^3$, of the cylinder is given by

$$V = 12\pi r^2 - 2\pi r^3 .$$ [3]

(ii) Given that $r$ varies, find the stationary value of $V$. [4]

***Suggested Solution:***

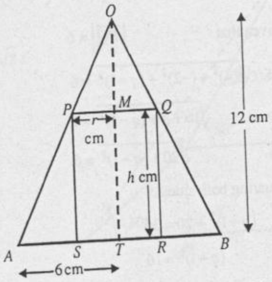
(i) We see that $\Delta OPM$ and $\Delta OAT$ are similar

$$\therefore \quad \frac{OM}{OT}=\frac{PM}{AT}$$

$$\Rightarrow \quad \frac{12-h}{12}=\frac{r}{6}$$

$$\Rightarrow \quad 12-h=12\times\frac{r}{6}$$

$$\Rightarrow \quad 12-h=2r \quad \Rightarrow \quad h=12-2r \quad \text{(Ans)}$$



volume of the cylinder is given by

$$V=\pi r^2 h$$

$$\Rightarrow \quad V=\pi(r^2)(12-2r)$$

$$\Rightarrow \quad V=12\pi r^2-2\pi r^3 \quad \text{(Shown)}$$

(ii) $V=12\pi r^2-2\pi r^3$

differentiating w.r.t. $r$

$$\frac{d}{dr}(V)=\frac{d}{dr}(12\pi r^2-2\pi r^3)$$

$$\frac{dV}{dr}=24\pi r-6\pi r^2$$

for stationary values, $\frac{dV}{dr}=0$

$$\Rightarrow \quad 24\pi r-6\pi r^2=0 \quad \Rightarrow \quad 6\pi r(4-r)=0$$

either $6\pi r=0$ or $4-r=0$

$\Rightarrow$ $r=0$ (ignored) or $r=4$

$\therefore$ stationary value of $V=12\pi(4)^2-2\pi(4)^3$

$$=192\pi-128\pi=64\pi \quad \text{(Ans)}$$

6. A small trading company made a profit of \$250 000 in the year 2000. The company considered two different plans, plan $A$ and plan $B$, for increasing its profits.

Under plan $A$, the annual profit would increase each year by 5% of its value in the preceding year. Find, for plan $A$,

(i) the profit for the year 2008, [3]

(ii) the total profit for the 10 years 2000 to 2009 inclusive. [2]

Under plan $B$, the annual profit would increase each year by a constant amount \$$D$.

(iii) Find the value of $D$ for which the total profit for the 10 years 2000 to 2009 inclusive would be the same for both plans. [3]

*Suggested Solution:*

(i) Let $P$ be the profit in year 2000

profit for the year $2001 = P + 5\%\text{ of }P = P + \frac{5}{100} \times P = P + 0.05P = (1.05)P$

profit for the year $2002 = (1.05)P + 5\% \times (1.05)P = (1.05)P + 0.05(1.05)P$

$= (1.05)P(1 + 0.05) = (1.05)^2 P$

profit for the year $2003 = (1.05)^2 P + 5\% \times (1.05)^2 P = (1.05)^2 P + 0.05(1.05)^2 P$

$= (1.05)^2 P(1 + 0.05) = (1.05)^3 P$

proceeding in the same way, the profit in the year 2008 would be $= (1.05)^8 P$

as $P = 250,000$

$\therefore$ profit in the year $2008 = (1.05)^8 \times 250000 = 369363.86 \approx 369000$ (Ans)

(ii) Total profits for 10 years is given by the geometric series

$$P + (1.05)P + (1.05)^2 P + (1.05)^3 P \text{ ..........} + (1.05)^9 P$$

for this series, we have

$a = P = 250000, \quad r = 1.05, \quad n = 10$

using $S_n = \frac{a(r^n - 1)}{r - 1}$

$$S_{10} = \frac{250000\left[(1.05)^{10} - 1\right]}{1.05 - 1} = \frac{250000(0.628895)}{0.05} = 3144475$$

$\therefore$ the total profit for 10 years $= \$3144475 \approx \$3140000$ (Ans)

(iii) As the profit would increase each year by a constant amount $\$D$. Therefore total profit is given by the arithmetic series, i.e.

$$P + (P + D) + (P + 2D) + (P + 3D) + \text{..........} + (P + 9D)$$

for this series, we have

$a = P = 250000, \quad d = D, \quad l = P + 9D$

using $S_n = \frac{n}{2}[a + l]$

$$S_{10} = \frac{10}{2}\left[P + (P + 9D)\right] = 5\left[2P + 9D\right] = 10P + 45D$$

now, given that

profit for plan $A$ = profit for plan $B$

$\Rightarrow \quad 3144475 = 10P + 45D$

$3144475 = 10(250000) + 45D$

$644475 = 45D$

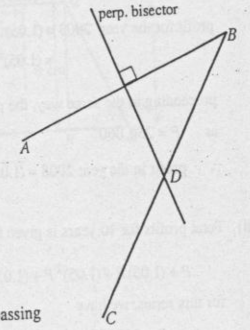
$D = 14321.667 \approx 14300$ (Ans)

7. Three points have coordinates $A\,(2,\ 6)$, $B\,(8,\ 10)$ and $C\,(6,\ 0)$. The perpendicular bisector of $AB$ meets the line $BC$ at $D$. Find

(i) the equation of the perpendicular bisector of $AB$ in the form $ax+by=c$, [4]

(ii) the coordinates of $D$. [4]

***Suggested Solution:***



(i) Given that $A(2,\ 6)$, $B(8,\ 10)$, $C(6,\ 0)$

coordinates of the mid point, $M$ of $AB = \left(\frac{2+8}{2},\ \frac{6+10}{2}\right) = (5,\ 8)$

$\therefore\quad M=(5,\ 8)$

gradient of $AB = \frac{10-6}{8-2} = \frac{4}{6} = \frac{2}{3}$

$\Rightarrow$ gradient of perpendicular to $AB = -\frac{3}{2}$

$\therefore$ equation of perpendicular bisector of $AB$ with gradient $-\frac{3}{2}$ and passing through (5, 8) is

$$y-8=-\frac{3}{2}(x-5) \quad\Rightarrow\quad 2y-16=-3x+15 \quad\Rightarrow\quad 3x+2y=31 \quad \text{(Ans)}$$

Coordinates of mid point

$$M=\left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}\right)$$

(ii) Gradient of $BC = \frac{0-10}{6-8} = \frac{-10}{-2} = 5$

equation of $BC$:

$$y-0=5(x-6) \quad\Rightarrow\quad y=5x-30 \cdots\cdots\cdots \text{(i)}$$

from part (i) the equation of perp. bisector of $AB = 3x+2y=31 \cdots\cdots\cdots$(ii)

this perpendicular bisector of $AB$ meets line $BC$ at $D$, therefore solving eq.(i) and eq.(ii) simultaneously

$$3x+2(5x-30)=31 \quad\Rightarrow\quad 3x+10x-60=31 \quad\Rightarrow\quad 13x=91 \quad\Rightarrow\quad x=7$$

putting $x$ in eq.(i), $y=5(7)-30 \quad\Rightarrow\quad y=5$

$\therefore$ coordinates of $D=(7,\ 5)$ **(Ans)**

Tangent and normal to the curve are perpendicular to each other, and for perpendicular lines $m_1 \times m_2 = -1$

8. A function f is defined by $f : x \mapsto (2x-3)^3-8$, for $2 \le x \le 4$.

(i) Find an expression, in terms of $x$, for $f'(x)$ and show that f is an increasing function. [4]

(ii) Find an expression, in terms of $x$, for $f^{-1}(x)$ and find the domain of $f^{-1}$. [4]

*Suggested Solution:*

(i) Given that $f(x) = (2x-3)^3 - 8$

$f'(x) = 3(2x-3)^2(2) = 6(2x-3)^2$ **(Ans)**

since $(2x-3)^2$ gives +ve value for all values of $x$

$\Rightarrow$ $f'(x) > 0$ for all values of $x$

$\therefore$ $f(x)$ is an increasing function. **(Shown)**

(ii) $f(x) = (2x-3)^3 - 8$

Let $f(x) = y \Rightarrow y = (2x-3)^3 - 8$

making $x$ the subject

$$(2x-3)^3 = y+8$$
$$2x-3 = (y+8)^{\frac{1}{3}}$$
$$2x = (y+8)^{\frac{1}{3}} + 3$$
$$x = \frac{1}{2}(y+8)^{\frac{1}{3}} + \frac{3}{2}$$

as $f(x) = y \Rightarrow f^{-1}(y) = x$

$$f^{-1}(y) = \frac{1}{2}(y+8)^{\frac{1}{3}} + \frac{3}{2}$$

$\Rightarrow$ $f^{-1}(x) = \frac{1}{2}(x+8)^{\frac{1}{3}} + \frac{3}{2}$ **(Ans)**

now, range of $f(x)$ is the domain of $f^{-1}(x)$

for the given range $2 \le x \le 4$

$f(2) = (2(2)-3)^3 - 8 = (1)^3 - 8 = -7$

and $f(4) = (2(4)-3)^3 - 8 = (5)^3 - 8 = 117$

$\therefore$ Domain of $f^{-1}(x)$ is $-7 \le x \le 117$ **(Ans)**

*Learning* corner

Note that

$f'(x) = \frac{dy}{dx}$

A squared quantity is always positive or $\ge 0$

Increasing function means 'positive gradient'.

You should realise that for the given range $2 \le x \le 4$, the gradient of the curve is +ve.

The domain of $f^{-1}(x)$ is same as range of $f(x)$

The equation of a curve is $xy = 12$ and the equation of a line $l$ is $2x + y = k$, where $k$ is a constant.

(i) In the case where $k = 11$, find the coordinates of the points of intersection of $l$ and the curve. [3]

(ii) Find the set of values of $k$ for which $l$ does not intersect the curve. [4]

(iii) In the case where $k = 10$, one of the points of intersection is $P(2, 6)$. Find the angle, in degrees correct to 1 decimal place, between $l$ and the tangent to the curve at $P$. [4]

*Suggested Solution:*

(i) When $k = 11$, the equation of the line $l$ is

$$2x + y = 11 \Rightarrow y = 11 - 2x$$

putting the value of $y$ in the equation of curve and solving simultaneously

$$x(11 - 2x) = 12$$
$$11x - 2x^2 - 12 = 0$$
$$2x^2 - 11x + 12 = 0$$
$$2x^2 - 8x - 3x + 12 = 0$$
$$2x(x - 4) - 3(x - 4) = 0$$
$$(x - 4)(2x - 3) = 0$$

either $x - 4 = 0$ or $2x - 3 = 0$

$\Rightarrow$ $x = 4$ or $x = \frac{3}{2}$

when $x = 4$, $y = 11 - 2(4) \Rightarrow y = 11 - 8 \Rightarrow y = 3$

$\therefore$ (4, 3) is one point of intersection

when $x = \frac{3}{2}$, $y = 11 - 2\left(\frac{3}{2}\right) \Rightarrow y = 11 - 3 \Rightarrow y = 8$

$\therefore$ $\left(\frac{3}{2}, 8\right)$ is the second point of intersection

$\therefore$ points of intersection of line $l$ and the curve are (4, 3), and $\left(\frac{3}{2}, 8\right)$ **(Ans)**

(ii) Let $2x + y = k \Rightarrow y = k - 2x$ ………(i)

and $xy = 12$ ………(ii)

substituting eq.(i) into eq.(ii)

$\Rightarrow$ $x(k - 2x) = 12$

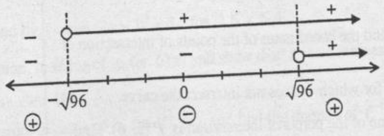
$\Rightarrow$ $kx - 2x^2 = 12$

$\Rightarrow$ $2x^2 - kx + 12 = 0$

to find the values of $k$ for which $l$ does not intersect the curve

Discriminant, $b^2 - 4ac < 0$

$\Rightarrow$ $(-k)^2 - 4(2)(12) < 0 \Rightarrow k^2 - 96 < 0 \Rightarrow (k + \sqrt{96})(k - \sqrt{96}) < 0$



$\therefore$ $-\sqrt{96} < k < \sqrt{96}$ **(Ans)**

(iii) When $k = 10$, the equation of line $l$ becomes

$$2x + y = 10 \quad \Rightarrow \quad y = 10 - 2x$$

gradient of line, $\frac{dy}{dx} = -2$

$\therefore \quad \tan\alpha = -2$

basic angle $= 63.4^\circ$

$\Rightarrow \quad \alpha = 180 - 63.4 = 116.6^\circ$

similarly, for equation of curve

$$xy = 12 \quad \Rightarrow \quad y = \frac{12}{x} \quad \Rightarrow \quad y = 12x^{-1}$$

gradient of curve, $\frac{dy}{dx} = -12x^{-2} = -\frac{12}{x^2}$

at point $P(2, 6)$

$$\frac{dy}{dx} = -\frac{12}{(2)^2} = -3$$

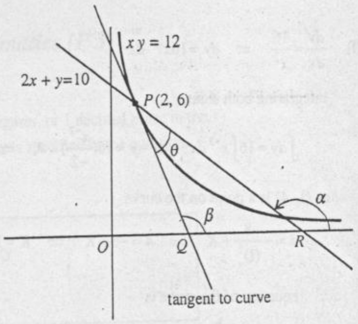
$\therefore \quad \tan\beta = -3$

basic angle $= 71.6^\circ \quad \Rightarrow \quad \beta = 180 - 71.6 = 108.4^\circ$

now, from figure, in $\Delta PQR$, we have

$$\alpha = \theta + \beta \quad \Rightarrow \quad 116.6 = \theta + 108.4 \quad \Rightarrow \quad \theta = 116.6 - 108.4 = 8.2$$

$\therefore$ required angle $\theta = 8.2^\circ$ (Ans)



Gradient of a line is also defined as tan of the angle which a line makes with the +ve direction of $x$-axis.

So if gradient is +ve, angle made is acute and if gradient is –ve, angle made is obtuse.

In this question, both the tangent and the line $l$ have negative gradients, therefore the angles ( $\alpha$ and $\beta$ ) will be obtuse angles.

10. A curve is such that $\frac{dy}{dx} = \frac{16}{x^3}$ and $(1, 4)$ is a point on the curve.

(i) Find the equation of the curve. [4]

(ii) A line with gradient $-\frac{1}{2}$ is a normal to the curve. Find the equation of this normal, giving your answer in the form $ax + by = c$. [4]

(iii) Find the area of the region enclosed by the curve, the $x$-axis and the lines $x = 1$ and $x = 2$. [4]

***Suggested Solution:***

(i) $\frac{dy}{dx}=\frac{16}{x^3} \Rightarrow dy=16x^{-3}\,dx$

integrating both sides

$$\int dy=16\int x^{-3}\,dx \Rightarrow y=16(\frac{x^{-2}}{-2})+K \Rightarrow y=-\frac{8}{x^2}+K$$

as (1, 4) is a point on the curve

$$\therefore \quad 4=-\frac{8}{(1)^2}+K \Rightarrow 4=-8+K \Rightarrow K=12$$

$\therefore$ equation of the curve is

$$y=-\frac{8}{x^2}+12 \quad \text{(Ans)}$$

(ii) gradient of the tangent to the curve is $\frac{dy}{dx}=\frac{16}{x^3}$

$\therefore$ gradient of normal is $=-\frac{x^3}{16}$

given that, gradient of normal to the curve $=-\frac{1}{2}$

$$\therefore \quad -\frac{x^3}{16}=-\frac{1}{2} \Rightarrow x^3=(\frac{1}{2})16 \Rightarrow x^3=8 \Rightarrow x=2$$

putting this value of $x$ in the equation of the curve found in part (i)

$$y=-\frac{8}{(2)^2}+12 \Rightarrow y=-2+12 \Rightarrow y=10$$

$\therefore$ (2, 10) is a point where the normal meets the curve

equation of the normal is

$$y-10=-\frac{1}{2}(x-2) \Rightarrow 2y-20=-x+2 \Rightarrow x+2y=22 \quad \text{(Ans)}$$

(iii) Required area, $A=\int_1^2 y\,dx$

$$=\int_1^2\left(-\frac{8}{x^2}+12\right)dx = \int_1^2(-8x^{-2}+12)\,dx$$

$$=\left[-(\frac{8x^{-1}}{-1})+12x\right]_1^2 = \left[\frac{8}{x}+12x\right]_1^2$$

$$=\left(\frac{8}{2}+12(2)\right)-\left(\frac{8}{1}+12(1)\right) = (4+24)-(8+12)$$

$$=28-20 = 8 \text{ sq. units} \quad \text{(Ans)}$$

# November 2005 Paper 3

## Pure Mathematics (P 3)

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. Given that $a$ is a positive constant, solve the inequality

$$|x-3a|>|x-a|.$$

[4]

*Suggested Solution:*

$$|x-3a|>|x-a|$$

squaring both sides

$$(|x-3a|)^2>(|x-a|)^2$$

$$x^2-6ax+9a^2>x^2-2ax+a^2$$

$$-6ax+2ax>a^2-9a^2$$

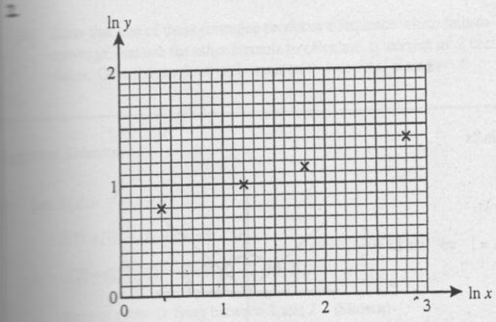
$$-4ax>-8a^2$$

dividing by $-4a$

$x<2a$ (Ans)

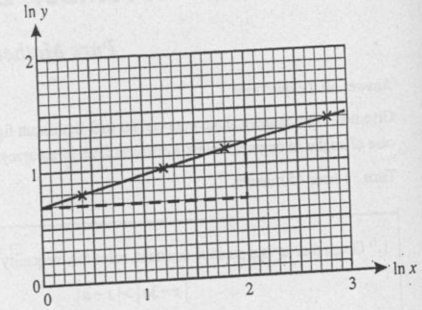
Note that dividing by a negative number changes the inequality sign.

2.



Two variable quantities $x$ and $y$ are related by the equation $y=Ax^n$, where $A$ and $n$ are constants. The diagram shows the result of plotting $\ln y$ against $\ln x$ for four pairs of values of $x$ and $y$. Use the diagram to estimate the values of $A$ and $n$. [5]

*Suggested Solution:*



$y = Ax^n$

taking ln on both sides

$\ln(y) = \ln(Ax^n)$

$\Rightarrow \ \ln y = \ln A + \ln x^n$

$\Rightarrow \ \ln y = \ln A + n\ln x$

or $\ \ln y = n\ln x + \ln A$

by drawing the best fit line we find that

$y$-intercept $= 0.7$

$\therefore \ \ln A = 0.7$

$\Rightarrow \ \ln A = 0.7\ln e \ \Rightarrow \ \ln A = \ln e^{0.7} \ \Rightarrow \ A = e^{0.7}$

$\Rightarrow \ A = 2.014 \approx 2.01$ (3 dec.pl) **(Ans)**

Note that in the equation
$\ln y = n\ln x + \ln A$,
$\ln y$ is the $y$-intercept and
$n$ is the gradient.

also gradient of the line is

$\text{gradient} = \frac{0.5}{2} = 0.25$

$\therefore \ n = 0.25$ **(Ans)**

**3.** The equation of a curve is $y = x + \cos 2x$. Find the $x$-coordinates of the stationary points of the curve for which $0 \le x \le \pi$, and determine the nature of each of these stationary points. [7]

*Suggested Solution:*

$y = x + \cos 2x$

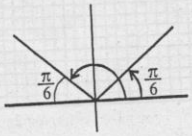
differentiating w.r.t. $x$

$\frac{d}{dx}(y) = \frac{d}{dx}(x + \cos 2x)$

$\frac{dy}{dx} = 1 - \sin 2x \times (2) = 1 - 2\sin 2x$

for stationary value, $\frac{dy}{dx} = 0$

$\Rightarrow \ 1 - 2\sin 2x = 0 \ \Rightarrow \ 2\sin 2x = 1 \ \Rightarrow \ \sin 2x = \frac{1}{2}$



basic angle $\alpha = \frac{\pi}{6}$

$\therefore \ 2x = \frac{\pi}{6}, \ (\pi - \frac{\pi}{6}) \ \Rightarrow \ 2x = \frac{\pi}{6}, \ \frac{5\pi}{6}$

$\Rightarrow \ x = \frac{\pi}{12}, \ \frac{5\pi}{12}$ **(Ans)**

e equation
ln A,
tercept and
ient.

---

Now

$$\frac{dy}{dx}=1-2\sin 2x$$

$$\therefore \quad \frac{d^2y}{dx^2}=-4\cos 2x$$

when $\quad x=\frac{\pi}{12}$

$$\frac{d^2y}{dx^2}=-4\cos 2(\frac{\pi}{12}) = -4\cos(\frac{\pi}{6}) = -4(\frac{\sqrt{3}}{2}) = -2\sqrt{3} \ <0$$

$\therefore$ at $x=\frac{1}{12}\pi$, the curve has a maximum point **(Ans)**

when $\quad x=\frac{5\pi}{12}$

$$\frac{d^2y}{dx^2}=-4\cos 2(\frac{5\pi}{12}) = -4\cos(\frac{5\pi}{6}) = -4(-\frac{\sqrt{3}}{2}) = +2\sqrt{3} \ >0$$

$\therefore$ at $x=\frac{5}{12}\pi$, the curve has a minimum point **(Ans)**

Remember:

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} >0$, $y$ is min

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} <0$, $y$ is max

4. The equation $x^3-x-3=0$ has one real root, $\alpha$.

(i) Show that $\alpha$ lies between 1 and 2. [2]

Two iterative formulae derived from this equation are as follows:

$$x_{n+1}=x_n^3-3 \qquad (A)$$

$$x_{n+1}=(x_n+3)^{\frac{1}{3}} \qquad (B)$$

Each formula is used with initial value $x_1 = 1.5$

(ii) Show that one of these formulae produces a sequence which fails to converge, and use the other formula to calculate $\alpha$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [5]

*Suggested Solution:*

(i) Let $f(x)=x^3-x-3$

$$f(1)=(1)^3-(1)-3=-3 \ <0$$

$$f(2)=(2)^3-(2)-3=8-2-3=3 \ >0$$

$\therefore$ there is a root $\alpha$ lying between 1 and 2 **(Shown)**

(ii) Equation $A$:

$$x_{n+1} = x_n^3 - 3$$
$$x_1 = 1.5$$
$$x_2 = x_1^3 - 3 = (1.5)^3 - 3 = 0.375$$
$$x_3 = x_2^3 - 3 = (0.375)^3 - 3 = -2.9473$$
$$x_4 = x_3^3 - 3 = (-2.9473)^3 - 3 = -28.601$$
$$x_5 = x_4^3 - 3 = (-28.601)^3 - 3 = -23399.11$$

$\therefore$ the formula does not converge

Equation $B$:

$$x_{n+1} = (x_n + 3)^{\frac{1}{3}}$$
$$x_1 = 1.5$$
$$x_2 = (x_1 + 3)^{\frac{1}{3}} = (1.5 + 3)^{\frac{1}{3}} = 1.6510$$
$$x_3 = (x_2 + 3)^{\frac{1}{3}} = (1.6510 + 3)^{\frac{1}{3}} = 1.6692$$
$$x_4 = (x_3 + 3)^{\frac{1}{3}} = (1.6692 + 3)^{\frac{1}{3}} = 1.6714$$
$$x_5 = (x_4 + 3)^{\frac{1}{3}} = (1.6714 + 3)^{\frac{1}{3}} = 1.6717$$
$$x_6 = (x_5 + 3)^{\frac{1}{3}} = (1.6717 + 3)^{\frac{1}{3}} = 1.6717$$

the formula converges to $\alpha$

$\therefore \quad \alpha = 1.67$ (2 dec.pl) **(Ans)**

5. By expressing $8 \sin \theta - 6 \cos \theta$ in the form $R \sin(\theta - \alpha)$, solve the equation

$$8 \sin \theta - 6 \cos \theta = 7,$$

for $0° \leq \theta \leq 360°$. [7]

*Suggested Solution:*

Let $8\sin\theta - 6\cos\theta \equiv R\sin(\theta - \alpha)$

$\Rightarrow \quad 8\sin\theta - 6\cos\theta \equiv R[\sin\theta\cos\alpha - \cos\theta\sin\alpha]$

$\Rightarrow \quad 8\sin\theta - 6\cos\theta \equiv R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

comparing the coefficients of $\cos\theta$ and $\sin\theta$, we have

$\Rightarrow \quad R\cos\alpha = 8 \cdots\cdots\cdots(i) \qquad R\sin\alpha = 6 \cdots\cdots\cdots(ii)$

eq. (ii) ÷ eq.(i) gives

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{6}{8} \Rightarrow \tan\alpha = \frac{3}{4} \Rightarrow \alpha = 36.87° \approx 36.9°$$

squaring and adding eq. (i) & eq. (ii) gives

$$(R\sin\alpha)^2 + (R\cos\alpha)^2 = 6^2 + 8^2 \Rightarrow R^2\sin^2\alpha + R^2\cos^2\alpha = 36 + 64$$

$$\Rightarrow \quad R^2(\sin^2\alpha + \cos^2) = 100 \Rightarrow R^2 = 100 \Rightarrow R = 10$$

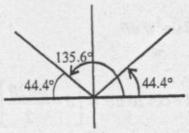
$\therefore \quad 8\sin\theta - 6\cos\theta = 10\sin(\theta - 36.9°)$ **(Ans)**

Now consider $8\sin\theta - 6\cos\theta = 7$

using the above result, the equation can be written as

$$10\sin(\theta - 36.9^\circ) = 7$$

$$\Rightarrow \sin(\theta - 36.9^\circ) = \frac{7}{10}$$



$$\therefore (\theta - 36.9^\circ) = 44.4,\ 135.6^\circ$$

$$\Rightarrow \theta = (44.4 + 36.9),\ (135.6+36.9)$$

$$\Rightarrow \theta = 81.3^\circ,\ 172.5^\circ \quad \textbf{(Ans)}$$

6. (i) Use the substitution $x = \sin^2\theta$ to show that

$$\int \sqrt{\left(\frac{x}{1-x}\right)}\,dx = \int 2\sin^2\theta\,d\theta \qquad [4]$$

(ii) Hence find the exact value of

$$\int_0^{\frac{1}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\,dx \qquad [4]$$

*Suggested Solution:*

(i) Given substitution: $x = \sin^2\theta$

differentiating w.r.t. $x$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\sin^2\theta) \Rightarrow 1 = 2\sin\theta\cos\theta \times \frac{d\theta}{dx} \Rightarrow dx = 2\sin\theta\cos\theta\,d\theta$$

$$\therefore \int \sqrt{\left(\frac{x}{1-x}\right)}\,dx = \int \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}} \times 2\sin\theta\cos\theta\,d\theta$$

$$= \int \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \times 2\sin\theta\cos\theta\,d\theta$$

$$= \int \frac{\sin\theta}{\cos\theta} \times 2\sin\theta\cos\theta\,d\theta$$

$$= \int 2\sin^2\theta\,d\theta \quad \textbf{(Shown)}$$

(ii) $\int_0^{\frac{1}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\,dx$

using the substitution given in part (i), we can find the new limits

new limits $\begin{cases} x = \frac{1}{4} \Rightarrow \sin^2\theta = \frac{1}{4} \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \\ x = 0 \Rightarrow \sin^2\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0 \end{cases}$

using the result of part (i), and with new limits, we have.

$$\int_0^{\frac{1}{4}} \sqrt{\left(\frac{x}{1-x}\right)}\, dx = \int_0^{\frac{\pi}{6}} 2\sin^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} 1-\cos 2\theta\, d\theta = \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{\pi}{6} - \frac{\sin 2(\frac{\pi}{6})}{2}\right) - \left(0 - \frac{\sin 2(0)}{2}\right)$$

$$= \left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right) - 0 = \frac{\pi}{6} - \frac{1}{2}(\frac{\sqrt{3}}{2}) = \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) \quad \textbf{(Ans)}$$

Formula used:

$2\sin^2\theta = 1-\cos 2\theta$

Remember:

For definite integral, do not put the integration constant

7. The equation $2x^3 + x^2 + 25 = 0$ has one real root and two complex roots.
   (i) Verify that $1+2i$ is one of the complex roots. [3]
   (ii) Write down the other complex root of the equation. [1]
   (iii) Sketch an Argand diagram showing the point representing the complex number $1+2i$. Show on the same diagram the set of points representing the complex numbers $z$ which satisfy

$$|z| = |z-1-2i|$$ [4]

*Suggested Solution:*

(i) Let $f(x) = 2x^3 + x^2 + 25$

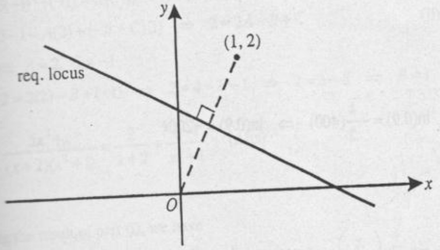
$$\begin{aligned} f(1+2i) &= 2(1+2i)^3 + (1+2i)^2 + 25 \\ &= 2\left[1+3(2i)+3(2i)^2+(2i)^3\right] + \left[1+2(2i)+(2i)^2\right] + 25 \\ &= 2(1+6i-12-8i) + (1+4i-4) + 25 \\ &= 2(-11-2i) + (-3+4i) + 25 \\ &= -22-4i-3+4i+25 \\ &= 0 \end{aligned}$$

$\therefore$ $(1+2i)$ is a complex root of the given equation. **(Verified)**

(ii) Other root is the conjugate of the first root

$\therefore$ other root $= 1-2i$ **(Ans)**

(iii) $|z| = |z-1-2i| \Rightarrow |z| = |z-(1+2i)|$



The equation can be written as:

$|z-(0, 0)| = |z-(1, 2)|$

Therefore the locus of points equidistant from (0, 0) and (1, 2) is the perpendicular bisector of the line joining the two points.

The locus is the perpendicular bisector of the line segment joining origin and the point (1, 2). Therefore $z$ lies on the locus shown in the diagram.

5. In a certain chemical reaction the amount, $x$ grams, of a substance present is decreasing. The rate of decrease of $x$ is proportional to the product of $x$ and the time, $t$ seconds, since the start of the reaction. Thus $x$ and $t$ satisfy the differential equation.

$$\frac{dx}{dt} = -kxt,$$

where $k$ is a positive constant. At the start of the reaction, when $t=0$, $x=100$.

(i) Solve this differential equation, obtaining a relation between $x, k$ and $t$. [5]

(ii) 20 seconds after the start of the reaction the amount of substance present is 90 grams. Find the time after the start of the reaction at which the amount of substance present is 50 grams. [3]

*Suggested Solution:*

(i) $\frac{dx}{dt} = -kxt \Rightarrow \frac{1}{x}\,dx = -kt\,dt$

integrating both sides

$$\int \frac{1}{x}\,dx = -k\int t\,dt \Rightarrow \ln x = -k\frac{t^2}{2} + C \cdots\cdots\cdots(i)$$

when $t=0$, $x=100$

$\Rightarrow \ln(100) = -k(\frac{0}{2}) + C \Rightarrow C = \ln 100$

$\therefore$ eq. (i) becomes

$$\ln x = -\frac{k}{2}t^2 + \ln 100 \Rightarrow \ln x - \ln 100 = -\frac{k}{2}t^2$$

$\Rightarrow \ln\left(\frac{x}{100}\right) = -\frac{k}{2}t^2$ (Ans)

(ii) From part (i), we have

$$\ln\left(\frac{x}{100}\right) = -\frac{k}{2}t^2 \cdots\cdots\cdots \text{(ii)}$$

when $t = 20, \quad x = 90$

$$\Rightarrow \ln\left(\frac{90}{100}\right) = -\frac{k}{2}(20)^2 \Rightarrow \ln(0.9) = -\frac{k}{2}(400) \Rightarrow \ln(0.9) = -200k$$

$$\Rightarrow k = 0.0005268$$

$\therefore$ eq. (ii) becomes

$$\ln\left(\frac{x}{100}\right) = -\frac{0.0005268}{2}t^2 \Rightarrow \ln\left(\frac{x}{100}\right) = -0.0002634t^2$$

now, when $x = 50$

$$\ln\left(\frac{50}{100}\right) = -0.0002634t^2 \Rightarrow -0.69315 = -0.0002634t^2$$

$$\Rightarrow t^2 = \frac{0.69315}{0.0002634} \Rightarrow t^2 = 2631.548 \Rightarrow t = 51.299$$

$\therefore$ time = 51.3 seconds **(Ans)**

**9.** (i) Express $\dfrac{3x^2+x}{(x+2)(x^2+1)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{3x^2+x}{(x+2)(x^2+1)}$ in ascending powers of $x$, up to and including the term in $x^3$. [5]

***Suggested Solution:***

(i) $$\frac{3x^2+x}{(x+2)(x^2+1)} \equiv \frac{A}{x+2} + \frac{Bx+c}{x^2+1}$$

You should always make sure that the given fraction is a proper fraction

$$\Rightarrow 3x^2+x \equiv A(x^2+1) + (Bx+C)(x+2)$$

for $x = -2$

$$3(-2)^2+(-2) = A((-2)^2+1) + (B(-2)+C)((-2)+2)$$

$$\Rightarrow 12-2 = A(5)+0 \Rightarrow 5A = 10 \Rightarrow A = 2$$

for $x = 0$

$$3(0)^2+(0) = A((0)^2+1) + (B(0)+C)(0+2)$$

$$\Rightarrow 0 = A(1)+(C)(2) \Rightarrow 0 = A+2C$$

putting $A = 2$

$$0 = 2+2C \Rightarrow C = -1$$

for $x=-1$

$$3(-1)^2+(-1)=A((-1)^2+1)+(B(-1)+C)((-1)+2)$$

$$\Rightarrow\ 3-1=A(2)+(-B+C)(1)\ \Rightarrow\ 2=2A-B+C$$

putting $A=2,\ C=-1$

$$2=2(2)-B+(-1)\ \Rightarrow\ 2=4-B-1\ \Rightarrow\ 2=3-B\ \Rightarrow\ B=1$$

r=a+tb.

$$\therefore\ \frac{3x^2+x}{(x+2)(x^2+1)}\equiv\frac{2}{x+2}+\frac{x-1}{x^2+1}\quad\text{(Ans)}$$

Ax₁+By+

(ii) Using the result of part (i), we have

$$\frac{3x^2+x}{(x+2)(x^2+1)}=\frac{2}{x+2}+\frac{x-1}{x^2+1}$$

$$=\frac{2}{2\left(1+\frac{x}{2}\right)}+(x-1)(1+x^2)^{-1}=\left(1+\frac{x}{2}\right)^{-1}+(x-1)(1+x^2)^{-1}$$

applying binomial expansion up to the term in $x^3$

$$=\left[1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{x}{2}\right)^3\right]+(x-1)\left[1+(-1)(x^2)\right]$$

$$=1-\frac{x}{2}+\frac{2}{2\times1}\left(\frac{x^2}{4}\right)+\frac{-6}{3\times2\times1}\left(\frac{x^3}{8}\right)+(x-1)(1-x^2)$$

$$=1-\frac{x}{2}+\frac{x^2}{4}-\frac{x^3}{8}+x-x^3-1+x^2$$

$$=x-\frac{x}{2}+x^2+\frac{x^2}{4}-x^3-\frac{x^3}{8}$$

$$=\frac{1}{2}x+\frac{5}{4}x^2-\frac{9}{8}x^3\quad\text{(Ans)}$$

sin θ

10. The straight line $l$ passes through the points $A$ and $B$ with position vectors.

$$2\mathbf{i}+2\mathbf{j}+\mathbf{k}\qquad\text{and}\qquad\mathbf{i}+4\mathbf{j}+2\mathbf{k}$$

respectively. This line intersects the plane $p$ with equation $x-2y+2z=6$ at the point $C$.

(i) Find the position vector of $C$. [4]

(ii) Find the acute angle between $l$ and $p$. [4]

(iii) Show that the perpendicular distance from $A$ to $p$ is equal to 2. [3]

ould always make
t the given
is a proper

*Suggested Solution:*

(i) Given that $\overline{OA}=\begin{pmatrix}2\\2\\1\end{pmatrix}$ and $\overline{OB}=\begin{pmatrix}1\\4\\2\end{pmatrix}$

$\therefore$ direction vector $\overline{AB}=\overline{OB}-\overline{OA}=\begin{pmatrix}1\\4\\2\end{pmatrix}-\begin{pmatrix}2\\2\\1\end{pmatrix}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$

equation of line $l$ is

$$\mathbf{r}=\overline{OA}+\lambda\overline{AB} \Rightarrow \mathbf{r}=\begin{pmatrix}2\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\1\end{pmatrix} \Rightarrow \mathbf{r}=\begin{pmatrix}2-\lambda\\2+2\lambda\\1+\lambda\end{pmatrix}\cdots\cdots\cdots(i)$$

Remember that **r** is the position vector of any point (x, y, z) on the line

$$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2-\lambda\\2+2\lambda\\1+\lambda\end{pmatrix} \Rightarrow \begin{matrix}x=2-\lambda\\y=2+2\lambda\\z=1+\lambda\end{matrix}$$

now, equation of the plane $p$ is

$$x-2y+2z=6\cdots\cdots\cdots(ii)$$

as the line $l$ intersects the plane $p$, therefore putting the values of $x$, $y$ and $z$ in eq. (ii)

$\therefore \quad (2-\lambda)-2(2+2\lambda)+2(1+\lambda)=6$

$\Rightarrow \quad 2-\lambda-4-4\lambda+2+2\lambda=6 \Rightarrow -3\lambda=6 \Rightarrow \lambda=-2$

putting $\lambda$ in eq. (i), we have

$$\mathbf{r}=\begin{pmatrix}2-(-2)\\2+2(-2)\\1+(-2)\end{pmatrix}=\begin{pmatrix}2+2\\2-4\\1-2\end{pmatrix}=\begin{pmatrix}4\\-2\\-1\end{pmatrix}$$

$\therefore$ position vector of point $C$ is

$$\overline{OC}=\begin{pmatrix}4\\-2\\-1\end{pmatrix} \text{ or } 4\mathbf{i}-2\mathbf{j}-\mathbf{k} \quad \textbf{(Ans)}$$

(ii) Equation of plane $p$ in scalar product form is $\mathbf{r}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}=6$

$\therefore$ the normal **n** to the plane is $\mathbf{n}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$

let **d** be the direction vector of the line $l$. $\therefore \ \mathbf{d}=\begin{pmatrix}-1\\2\\1\end{pmatrix}$

---

now, the

sin θ

⇒ (√(−

⇒ (√1

⇒ 3√6

∴ acut

(iii) We have

∴ perp

*Alternative*

Let the

∴ w

$\overline{O}$

∴ n

as this

r

now, the angle between line $l$ and plane $p$ is given by

$$\sin\theta = \hat{\mathbf{d}}\bullet\hat{\mathbf{n}} \Rightarrow \sin\theta = \frac{\mathbf{d}\bullet\mathbf{n}}{|\mathbf{d}|\,|\mathbf{n}|} \Rightarrow |\mathbf{d}|\,|\mathbf{n}|\sin\theta = \mathbf{d}\bullet\mathbf{n}$$

$$\Rightarrow \left(\sqrt{(-1)^2+(2)^2+(1)^2}\right)\left(\sqrt{(1)^2+(-2)^2+(2)^2}\right)\sin\theta = \begin{pmatrix}-1\\2\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-2\\2\end{pmatrix}$$

$$\Rightarrow \left(\sqrt{1+4+1}\right)\left(\sqrt{1+4+4}\right)\sin\theta = -1-4+2 \Rightarrow \left(\sqrt{6}\right)\left(\sqrt{9}\right)\sin\theta = -3$$

$$\Rightarrow 3\sqrt{6}\sin\theta = -3 \Rightarrow \sin\theta = -\frac{1}{\sqrt{6}} \Rightarrow \theta = -24.1°$$

$\therefore$ acute angle $\theta = 24.1°$ **(Ans)**

For any vector **v**, the unit vector $\hat{\mathbf{v}}$, is given as:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

(iii) We have

$$\overline{OA} = \begin{pmatrix}2\\2\\1\end{pmatrix}, \qquad \text{equation of plane: } x-2y+2z-6=0$$

$$\therefore \text{ perpendicular distance from } A \text{ to } p = \left|\frac{1(2)-2(2)+2(1)-6}{\sqrt{1^2+(-2)^2+(2)^2}}\right| = \left|\frac{2-4+2-6}{\sqrt{9}}\right|$$

$$= \left|\frac{-6}{3}\right| = |-2| = 2 \quad \textbf{(Shown)}$$

Perpendicular distance from a point $P(x_1, y_1, z_1)$ to the plane $Ax+By+Cz+D=0$, is

$$= \left|\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right|$$

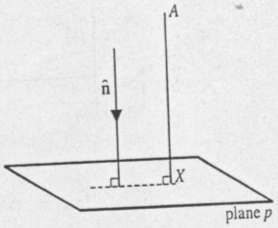
*Alternative Solution for part* ***(iii):***

Let the perpendicular from $A$ meets the plane $p$ at $X$

$\therefore$ we have,

$$\overline{OA} = \begin{pmatrix}2\\2\\1\end{pmatrix}, \qquad \text{equation of plane: } \mathbf{r}\bullet\begin{pmatrix}1\\-2\\2\end{pmatrix} = 6\cdots\cdots\cdots\text{(i)}$$

$$\therefore \text{ normal to the plane, } \mathbf{n} = \begin{pmatrix}1\\-2\\2\end{pmatrix}$$



as this normal is parallel to the line $AX$, therefore equation of $\overrightarrow{AX}$ is given by

$$\mathbf{r} = \begin{pmatrix}2\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\2\end{pmatrix} \Rightarrow \mathbf{r} = \begin{pmatrix}2+\lambda\\2-2\lambda\\1+2\lambda\end{pmatrix}\cdots\cdots\cdots\text{(ii)}$$

$\therefore$ putting eq. (ii) into eq. (i) and solving simultaneously

$$\begin{pmatrix}2+\lambda\\2-2\lambda\\1+2\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}=6 \;\Rightarrow\; (2+\lambda)-2(2-2\lambda)+2(1+2\lambda)$$

$$\Rightarrow\; 2+\lambda-4+4\lambda+2+4\lambda=6 \;\Rightarrow\; 9\lambda=6 \;\Rightarrow\; \lambda=\frac{6}{9} \;\Rightarrow\; \lambda=\frac{2}{3}$$

The perpendicular from point $A$ meets the plane $p$ at $X$. So $X$ is the point of intersection of the plane and the line $AX$, and can be taken out by solving the equations of plane and line simultaneously.

$\therefore$ putting this value of $\lambda$ in eq.(ii), gives the position vector of point $X$

$$\therefore\quad \mathbf{r}=\overline{OX}=\begin{pmatrix}2+\frac{2}{3}\\2-2(\frac{2}{3})\\1+2(\frac{2}{3})\end{pmatrix}=\begin{pmatrix}\frac{8}{3}\\\frac{2}{3}\\\frac{7}{3}\end{pmatrix}$$

Remember that **r** is the position vector of any point on the line

$\therefore$ $\perp$ distance of $A$ to plane $p$ is:

$$\left|\overline{AX}\right|=\sqrt{\left(\tfrac{8}{3}-2\right)^2+\left(\tfrac{2}{3}-2\right)^2+\left(\tfrac{7}{3}-1\right)^2}$$

$$=\sqrt{\left(\frac{2}{3}\right)^2+\left(-\frac{4}{3}\right)^2+\left(\frac{4}{3}\right)^2}$$

$$=\sqrt{\frac{4}{9}+\frac{16}{9}+\frac{16}{9}}$$

$$=\sqrt{\frac{36}{9}}=\sqrt{4}=2 \quad \text{(Shown)}$$

Distance between two points:

$A(x_1, y_1)$ and $B(x_2, y_2)$ is:

$$|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

# June 2006 Paper 1

## *Pure Mathematics (P1)*

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. A curve has equation $y = \frac{k}{x}$. Given that the gradient of the curve is −3 when $x = 2$, find the value of the constant $k$. [3]

***Suggested Solution:***

$$y = \frac{k}{x} \quad \Rightarrow \quad y = kx^{-1}$$

differentiating w.r.t. $x$

$$\frac{dy}{dx} = -kx^{-2} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{k}{x^2}$$

given that $\frac{dy}{dx} = -3$ when $x = 2$, we have

$$-3 = -\frac{k}{(2)^2} \quad \Rightarrow \quad -3 = -\frac{k}{4} \quad \Rightarrow \quad k = 12 \quad \text{(Ans)}$$

2. Solve the equation

$$\sin 2x + 3\cos 2x = 0,$$

for $0^\circ \le x \le 180^\circ$. [4]

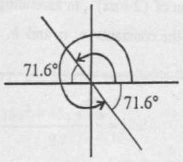
***Suggested Solution:***

$$\sin 2x + 3\cos 2x = 0$$

$$\Rightarrow \quad \sin 2x = -3\cos 2x$$

$$\Rightarrow \quad \frac{\sin 2x}{\cos 2x} = -3 \quad \Rightarrow \quad \tan 2x = -3$$

basic angle $\alpha = 71.56^\circ \approx 71.6^\circ$



$$\therefore \quad 2x = 180^\circ - 71.6^\circ, \quad \text{or} \quad 360^\circ - 71.6^\circ$$

$$\Rightarrow \quad 2x = 108.4, \quad \text{or} \quad 288.4^\circ$$

$$\Rightarrow \quad x = 54.2^\circ, \; 144.2^\circ \quad \text{(Ans)}$$

**3.** Each year a company gives a grant to a charity. The amount given each year increases by 5% of its value in the preceding year. The grant in 2001 was \$5000. Find

(i) the grant given in 2011. [3]

(ii) the total amount of money given to the charity during the years 2001 to 2011 inclusive. [2]

*Suggested Solution:*

(i) Let the grant in the year 2001 be $P$

$\therefore$ grant in 2001: $P = 5000$

grant in 2002: $P + 5\%\text{ of } P = P + \frac{5}{100}P = (1.05)P$

grant in 2003: $(1.05)P + 5\%\text{ of }(1.05)P = 1.05(1.05)P = (1.05)^2P$

grant in 2004: $(1.05)^2P + 5\%\text{ of }(1.05)^2P = 1.05(1.05)^2P = (1.05)^3P$

$\therefore$ the grant in the year $2011 = (1.05)^{10}P$

$$= (1.05)^{10} \times (5000)$$

$$= 8144.47 \approx \$8140 \text{ (3 sf)} \quad \textbf{(Ans)}$$

(ii) Total grant given to the charity from year 2001 to 2011 inclusive is

$$P + (1.05)P + (1.05)^2P + (1.05)^3P \text{ ............ } + (0.5)^{10}P$$

which is a G.P with first term $a = P = 5000$, common ratio $r = 1.05$ and $n = 11$

$\therefore$ using $S_n = \frac{a(r^n - 1)}{r - 1}$

$$S_{11} = \frac{5000[(1.05)^{11} - 1]}{1.05 - 1} = \frac{5000[0.7103394]}{0.05} = 71033.94 \approx 71034$$

$\therefore$ total amount given during the years 2001 to 2011 = \$71034 **(Ans)**

Sum to $n$ terms of a G.P is

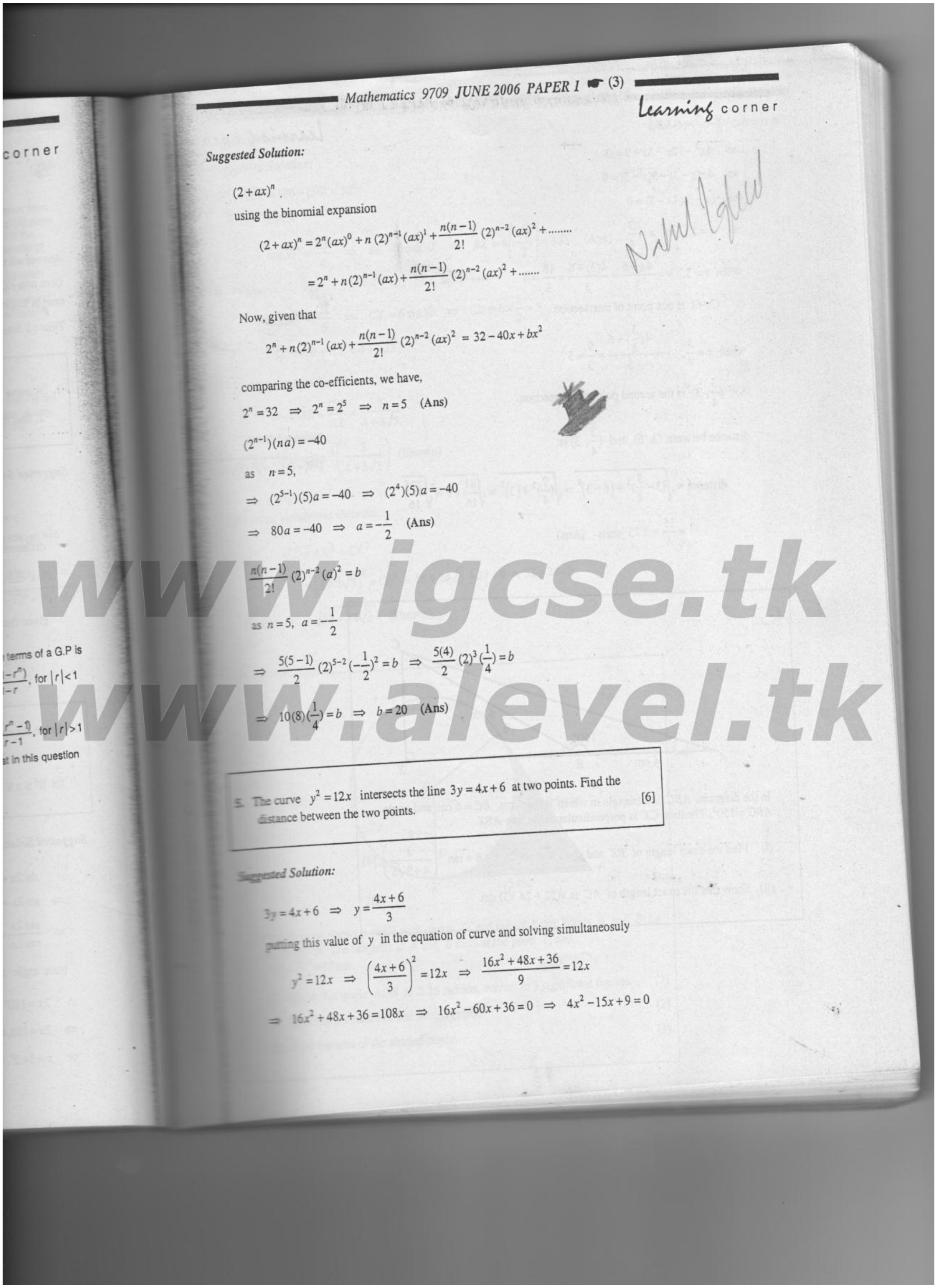
$S_n = \frac{a(1-r^n)}{1-r}$, for $|r| < 1$

or

$S_n = \frac{a(r^n-1)}{r-1}$, for $|r| > 1$

note that in this question $r > 1$.

**4.** The first three terms in the expansion of $(2 + ax)^n$, in ascending powers of $x$, are $32 - 40x + bx^2$. Find the values of the constants $n$, $a$ and $b$. [5]



terms of a G.P is

$\frac{(1-r^n)}{1-r}$, for $|r|<1$

$\frac{(r^n-1)}{r-1}$, for $|r|>1$

at in this question

*Suggested Solution:*

$(2+ax)^n$.

using the binomial expansion

$$(2+ax)^n = 2^n(ax)^0 + n\,(2)^{n-1}(ax)^1 + \frac{n(n-1)}{2!}(2)^{n-2}(ax)^2 + \ldots\ldots$$

$$= 2^n + n(2)^{n-1}(ax) + \frac{n(n-1)}{2!}(2)^{n-2}(ax)^2 + \ldots\ldots$$

Now, given that

$$2^n + n(2)^{n-1}(ax) + \frac{n(n-1)}{2!}(2)^{n-2}(ax)^2 = 32 - 40x + bx^2$$

comparing the co-efficients, we have,

$2^n = 32 \;\Rightarrow\; 2^n = 2^5 \;\Rightarrow\; n = 5$ (Ans)

$(2^{n-1})(na) = -40$

as $n = 5$,

$\Rightarrow\; (2^{5-1})(5)a = -40 \;\Rightarrow\; (2^4)(5)a = -40$

$\Rightarrow\; 80a = -40 \;\Rightarrow\; a = -\frac{1}{2}$ (Ans)

$\frac{n(n-1)}{2!}(2)^{n-2}(a)^2 = b$

as $n = 5,\; a = -\frac{1}{2}$

$\Rightarrow\; \frac{5(5-1)}{2}(2)^{5-2}(-\frac{1}{2})^2 = b \;\Rightarrow\; \frac{5(4)}{2}(2)^3(\frac{1}{4}) = b$

$\Rightarrow\; 10(8)(\frac{1}{4}) = b \;\Rightarrow\; b = 20$ (Ans)

5. The curve $y^2 = 12x$ intersects the line $3y = 4x + 6$ at two points. Find the distance between the two points. [6]

*Suggested Solution:*

$3y = 4x + 6 \;\Rightarrow\; y = \frac{4x+6}{3}$

putting this value of $y$ in the equation of curve and solving simultaneosuly

$y^2 = 12x \;\Rightarrow\; \left(\frac{4x+6}{3}\right)^2 = 12x \;\Rightarrow\; \frac{16x^2+48x+36}{9} = 12x$

$\Rightarrow\; 16x^2 + 48x + 36 = 108x \;\Rightarrow\; 16x^2 - 60x + 36 = 0 \;\Rightarrow\; 4x^2 - 15x + 9 = 0$

$\Rightarrow \quad 4x^2-12x-3x+9=0$

$\Rightarrow \quad 4x(x-3)-3(x-3)=0$

$\Rightarrow \quad (x-3)(4x-3)=0$

$\Rightarrow \quad x=3, \text{ or } x=\frac{3}{4}$

when $x=3, \quad y=\frac{4x+6}{3}=\frac{4(3)+6}{3}=\frac{18}{3}=6$

$\therefore$ (3, 6) is one point of intersection.

when $x=\frac{3}{4}, \quad y=\frac{4(\frac{3}{4})+6}{3}=\frac{9}{3}=3$

$\therefore$ $(\frac{3}{4}, 3)$ is the second point of intersection.

distance between (3, 6) and $(\frac{3}{4}, 3)$ is:

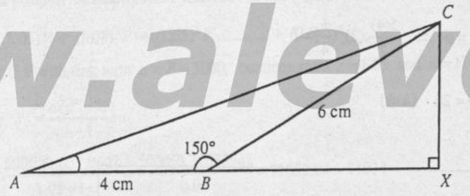
$$\text{distance}=\sqrt{(3-\frac{3}{4})^2+(6-3)^2}=\sqrt{(\frac{9}{4})^2+(3)^2}=\sqrt{\frac{81}{16}+9}=\sqrt{\frac{225}{16}}$$

$$=\frac{15}{4}=3.75 \text{ units} \quad \text{(Ans)}$$

Distance formula:

$|AB|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

**6.**



In the diagram, $ABC$ is a triangle in which $AB=4$ cm, $BC=6$ cm and angle $ABC=150°$. The line $CX$ is perpendicular to the line $ABX$.

(i) Find the exact length of $BX$ and show that angle $CAB=\tan^{-1}\left(\frac{3}{4+3\sqrt{3}}\right)$. [4]

(ii) Show that the exact length of $AC$ is $\sqrt{(52+24\sqrt{3})}$ cm. [2]

*Suggested Solution:*

(i) $C\hat{B}X = 180 - 150^\circ = 30^\circ$

In $\Delta BCX$

$$\cos 30^\circ = \frac{BX}{6} \Rightarrow BX = 6\cos 30^\circ \Rightarrow BX = 6\left(\frac{\sqrt{3}}{2}\right) = 3\sqrt{3} \quad \textbf{(Ans)}$$

now,

$$\sin 30^\circ = \frac{CX}{6} \Rightarrow CX = 6\sin 30 \Rightarrow CX = 6 \times \frac{1}{2} = 3$$

$$AX = AB + BX = 4 + 3\sqrt{3}$$

consider $\Delta ACX$

$$\tan C\hat{A}B = \frac{CX}{AX} = \frac{3}{4+3\sqrt{3}}$$

$$\Rightarrow C\hat{A}B = \tan^{-1}\left(\frac{3}{4+3\sqrt{3}}\right) \quad \textbf{(Shown)}$$

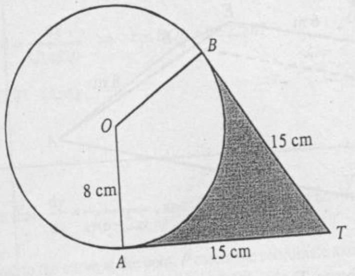
(ii) using pythagoras theorem

$$AC^2 = AX^2 + CX^2$$

$$= \left(4+3\sqrt{3}\right)^2 + (3)^2 = 16 + 24\sqrt{3} + 27 + 9 = 52 + 24\sqrt{3}$$

or $AC = \sqrt{52 + 24\sqrt{3}}$ **(Shown)**

7.



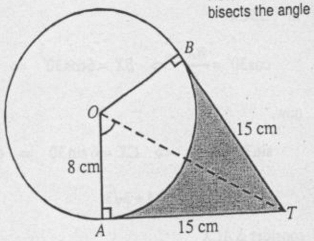
The diagram shows a circle with centre $O$ and radius 8 cm. Points $A$ and $B$ lie on the circle. The tangents at $A$ and $B$ meet at the point $T$, and $AT = BT = 15\,\text{cm}$.

(i) Show that angle $AOB$ is 2.16 radians, correct to 3 significant figures. [3]

(ii) Find the perimeter of the shaded region. [2]

(iii) Find the area of the shaded region. [3]

*Suggested Solution:*



Note that the line *OT* bisects the angle *AOB*

(i) In $\Delta OAT$

$$\tan A\hat{O}T = \frac{15}{8} \Rightarrow A\hat{O}T = 1.0808$$

$$\therefore \quad A\hat{O}B = 2(A\hat{O}T) = 2(1.0808)$$
$$= 2.1616$$
$$\approx 2.16 \text{ (to 3sf) radians} \quad \textbf{(shown)}$$

(ii) Arc length $\widehat{AB} = s = r\theta$

$$= 8(2.16) = 17.3$$

$\therefore$ Perimeter of shaded region $= 15 + 15 + 17.3 = 47.3$ cm **(Ans)**

(iii) Area of $\Delta OAT = \frac{1}{2} \times 15 \times 8 = 60 \text{ cm}^2$

Note that triangles *OAT* and *OBT* are congruent.

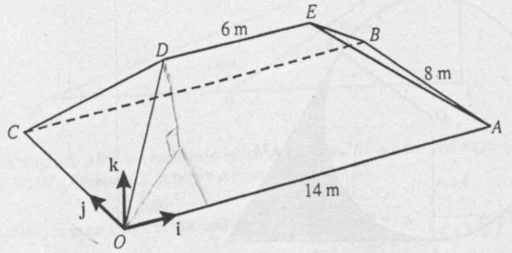
$\therefore$ Area of the kite = area of $OAT$ + area of $OBT = 60 + 60 = 120 \text{ cm}^2$

area of the sector $AOB = \frac{1}{2} r^2 \theta = \frac{1}{2}(8)^2(2.16) = 69.12 \text{ cm}^2$

$\therefore$ area of the shaded region = area of kite − area of sector

$$= 120 - 69.12 = 50.88 \approx 50.9 \text{ cm}^2 \text{ (to 3sf)} \quad \textbf{(Ans)}$$

**8.**



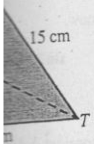
The diagram shows the roof of a house. The base of the roof, $OABC$, is rectangular and horizontal with $OA = CB = 14$ m and $OC = AB = 8$ m. The top of the roof $DE$ is 5 m above the base and $DE = 6$ m. The sloping edges $OD$, $CD$, $AE$ and $BE$ are all equal in length.

Unit vectors $\mathbf{i}$ and $\mathbf{j}$ are parallel to $OA$ and $OC$ respectively and the unit vector $\mathbf{k}$ is vertically upwards.

(i) Express the vector $\overrightarrow{OD}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, and find its magnitude. [4]

(ii) Use a scalar product to find angle $DOB$. [4]

ote that the line *OT* isects the angle *AOB*

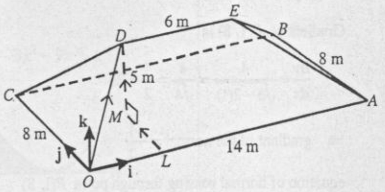


Note that triangles *OAT* and *OBT* are congruent.

---

***Suggested Solution:***

(i) We have $OA = 14\text{ m}$, $DE = 6\text{ m}$ and . $MD = 5\text{ m}$



$$\therefore \quad OL = \frac{1}{2}(OA - DE) = \frac{1}{2}(14-6) = 4\text{ m}$$

and $LM = \frac{1}{2}OC = \frac{1}{2}(8) = 4\text{ m}$

now

$$\overrightarrow{OD} = \overrightarrow{OL} + \overrightarrow{LM} + \overrightarrow{MD}$$

$$\Rightarrow \quad \overrightarrow{OD} = 4\mathbf{i} + 4\mathbf{j} + 5\mathbf{k} \quad \textbf{(Ans)}$$

magnitude of $\overrightarrow{OD}$:

$$\left|\overrightarrow{OD}\right| = \sqrt{(4)^2 + (4)^2 + (5)^2} = \sqrt{16+16+25} = \sqrt{57} \quad \textbf{(Ans)}$$

(ii) $OB = OA + AB = 14\mathbf{i} + 8\mathbf{j}$

applying scalar product

$$\overrightarrow{OD} \cdot \overrightarrow{OB} = \left|\overrightarrow{OD}\right|\left|\overrightarrow{OB}\right| \cos D\hat{O}B$$

$$\begin{pmatrix}4\\4\\5\end{pmatrix} \cdot \begin{pmatrix}14\\8\\0\end{pmatrix} = \left(\sqrt{(4)^2+(4)^2+(5)^2}\right)\left(\sqrt{(14)^2+(8)^2+(0)^2}\right) \cos D\hat{O}B$$

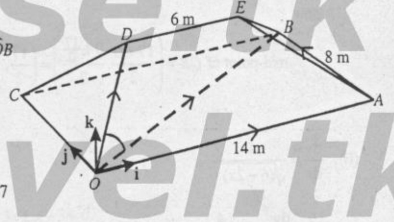
$$56+32+0 = \left(\sqrt{16+16+25}\right)\left(\sqrt{196+64}\right) \cos D\hat{O}B$$

$$88 = \left(\sqrt{57}\right)\left(\sqrt{260}\right) \cos D\hat{O}B$$

$$88 = \left(\sqrt{14820}\right) \cos D\hat{O}B$$

$$\cos D\hat{O}B = \frac{88}{\sqrt{14820}} \quad \Rightarrow \quad \cos D\hat{O}B = 0.722867$$

$$\Rightarrow \quad D\hat{O}B = 43.7° \quad \textbf{(Ans)}$$



9. A curve is such that $\dfrac{dy}{dx} = \dfrac{4}{\sqrt{(6-2x)}}$, and $P(1, 8)$ is a point on the curve.

(i) The normal to the curve at the point *P* meets the coordinate axes at *Q* and at *R*. Find the coordinates of the mid-point of *QR*. [5]

(ii) Find the equation of the curve. [4]

Learning corner

*Suggested Solution:*

(i) $\dfrac{dy}{dx}=\dfrac{4}{\sqrt{(6-2x)}}$

Gradient at $P(1, 8)$ is

$$\frac{dy}{dx}=\frac{4}{\sqrt{6-2(1)}}=\frac{4}{\sqrt{4}}=\frac{4}{2}=2$$

$\Rightarrow$ gradient of the normal $=-\dfrac{1}{2}$

equation of normal passing through point $P(1, 8)$

$$y-8=-\frac{1}{2}(x-1) \quad\Rightarrow\quad 2y-16=-x+1 \quad\Rightarrow\quad x+2y=17$$

this normal meets the $x$-axis at point $Q$. therefore point $Q$ is the $x$-intercept

putting $y=0$, we have

$$x+2(0)=17 \quad\Rightarrow\quad x=17 \qquad \therefore \text{ point } Q \text{ is, } Q(17, 0)$$

similarly the normal meets the $y$-axis at point $R$.

$\therefore$ putting $x=0$

$$0+2y=17 \quad\Rightarrow\quad y=\frac{17}{2} \qquad \therefore \text{ point } R \text{ is, } R(0, \tfrac{17}{2})$$

now

$$\text{mid-point of } QR=\left(\frac{17+0}{2}, \frac{0+\frac{17}{2}}{2}\right)=\left(\frac{17}{2}, \frac{17}{4}\right) \quad \text{(Ans)}$$

Gradient at any point $(x, y)$ on a curve is given by $\dfrac{dy}{dx}$

Tangent and normal to the curve are perpendicular to each other, and for perpendicular lines:

(grad. of tangent) × (grad. of normal) = −1

Equation of a line in point-slope form is:

$y-y_1=m(x-x_1)$

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points on a straight line, then coordinates of mid point $M$ of $AB$ is:

$M=\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

(ii) $\dfrac{dy}{dx}=\dfrac{4}{\sqrt{(6-2x)}}$

$$\frac{dy}{dx}=\frac{4}{\sqrt{(6-2x)}} \quad\Rightarrow\quad dy=4(6-2x)^{-\frac{1}{2}}\,dx$$

integrating both sides

$$\int dy=4\int(6-2x)^{-\frac{1}{2}}\,dx$$

$$\Rightarrow\quad y=4\left(\frac{(6-2x)^{-\frac{1}{2}+1}}{(-\frac{1}{2}+1)(-2)}\right)+K \quad\Rightarrow\quad y=4\left(\frac{(6-2x)^{\frac{1}{2}}}{-1}\right)+K$$

$$\Rightarrow\quad y=-4(6-2x)^{\frac{1}{2}}+K \quad\Rightarrow\quad y=-4\sqrt{6-2x}+K$$

as curve passes through point $P(1, 8)$

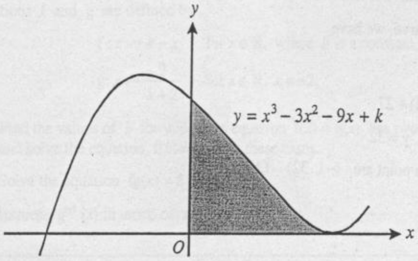
$$\therefore\quad 8=-4\sqrt{6-2(1)}+K \quad\Rightarrow\quad 8=-4\sqrt{4}+K \quad\Rightarrow\quad 8=-8+K \quad\Rightarrow\quad K=16$$

required euqation of the curve is

$$y=-4\sqrt{6-2x}+16 \quad\text{or}\quad y=16-4\sqrt{6-2x} \quad \text{(Ans)}$$

When $\dfrac{dy}{dx}$ (or gradient) of the curve is given, always integrate both sides of the given expression to find the equation of a curve.

10.



The diagram shows the curve $y = x^3 - 3x^2 - 9x + k$, where $k$ is constant. The curve has a minimum point on the $x$-axis.

(i) Find the value of $k$. [4]

(ii) Find the coordinates of the maximum point of the curve. [1]

(iii) State the set of values of $x$ for which $x^3 - 3x^2 - 9x + k$ is a decresing function of $x$. [1]

(iv) Find the area of the shaded region. [4]

*Suggested Solution:*

(i) $y = x^3 - 3x^2 - 9x + k$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

for maxima or minima, put $\frac{dy}{dx} = 0$

$\Rightarrow\ 3x^2 - 6x - 9 = 0 \ \Rightarrow\ x^2 - 2x - 3 = 0 \ \Rightarrow\ x^2 - 3x + x - 3 = 0$

$\Rightarrow\ x(x-3) + 1(x-3) = 0 \ \Rightarrow\ (x-3)(x+1) = 0$

$\Rightarrow\ x = 3, \quad \text{or} \quad x = -1$

considering the diagram and the values of $x$ found, we see that the curve has a minimum point at $x = 3$,

$\therefore$ minimum point of the curve is (3, 0) as it lies on the $x$-axis

putting (3, 0) in the equation of curve, we have

$0 = (3)^3 - 3(3)^2 - 9(3) + k$

$\Rightarrow\ 0 = 27 - 27 - 27 + k \ \Rightarrow\ k = 27 \quad$ (Ans)

(ii) From part (i), we see that the curve is maximum at $x = -1$

Putting it in the equation of curve, we have

$$y = x^3 - 3x^2 - 9x + k$$

$$\Rightarrow \quad y = (-1)^3 - 3(-1)^2 - 9(-1) + 27$$

$$\Rightarrow \quad y = -1 - 3 + 9 + 27 \quad \Rightarrow \quad y = 32$$

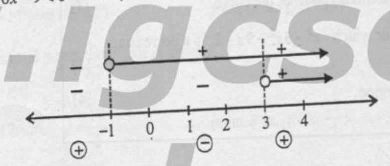
$\therefore$ coordinates of maximum point are $(-1,\ 32)$ **(Ans)**

(iii) $y = x^3 - 3x^2 - 9x + k$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

for decreasing function $\frac{dy}{dx} < 0$

$$\Rightarrow \quad 3x^2 - 6x - 9 < 0 \quad \Rightarrow \quad (x-3)(x+1) < 0$$



$\therefore$ required set of values of $x$ are: $-1 < x < 3$ (Ans) 1

(iv) Area of shaded region $= \int_0^3 y\,dx$

$$= \int_0^3 (x^3 - 3x^2 - 9x + 27)\,dx$$

$$= \left[\frac{x^4}{4} - 3\left(\frac{x^3}{3}\right) - 9\left(\frac{x^2}{2}\right) + 27x\right]_0^3$$

$$= \left[\frac{x^4}{4} - x^3 - \frac{9}{2}x^2 + 27x\right]_0^3$$

$$= \left[\frac{(3)^4}{4} - (3)^3 - \frac{9}{2}(3)^2 + 27(3)\right] - \left[\frac{(0)^4}{4} - (0)^3 - \frac{9}{2}(0)^2 + 27(0)\right]$$

$$= \left[\frac{81}{4} - 27 - \frac{81}{2} + 81\right] - [0] = \frac{135}{4} = 33\frac{3}{4} \text{ sq. units} \quad \textbf{(Ans)}$$

**11.** Functions f and g are defined by

$$f : x \mapsto k - x \quad \text{for } x \in \mathbb{R}, \text{ where } k \text{ is a constant,}$$

$$g : x \mapsto \frac{9}{x+2} \quad \text{for } x \in \mathbb{R},\ x \neq -2.$$

(i) Find the values of $k$ for which the equation $f(x) = g(x)$ has two equal roots and solve the equation $f(x) = g(x)$ in these cases. [6]

(ii) Solve the equation $fg(x) = 5$ when $k = 6$. [3]

(iii) Express $g^{-1}(x)$ in terms of $x$. [2]

***Suggested Solution:***

(i) We have $f(x) = k - x$, and $g(x) = \frac{9}{x+2}$

given that $f(x) = g(x)$

$\Rightarrow \quad k - x = \frac{9}{x+2} \quad \Rightarrow \quad (x+2)(k-x) = 9$

$\Rightarrow \quad kx - x^2 + 2k - 2x - 9 = 0 \quad \Rightarrow \quad -x^2 + kx - 2x + 2k - 9 = 0$

$\Rightarrow \quad -x^2 + (k-2)x + (2k-9) = 0$

for equal roots discriminent, $b^2 - 4ac = 0$

$\Rightarrow \quad (k-2)^2 - 4(-1)(2k-9) = 0$

$\Rightarrow \quad k^2 - 4k + 4 + 8k - 36 = 0$

$\Rightarrow \quad k^2 + 4k - 32 = 0 \quad \Rightarrow \quad k^2 + 8k - 4k - 32 = 0$

$\Rightarrow \quad k(k+8) - 4(k+8) = 0 \quad \Rightarrow \quad (k+8)(k-4) = 0$

$\Rightarrow \quad k = -8, \quad k = 4$ **(Ans)**

now, when $k = -8$

$f(x) = g(x)$

$\Rightarrow \quad -8 - x = \frac{9}{x+2} \quad \Rightarrow \quad (x+2)(-8-x) = 9 \quad \Rightarrow \quad -8x - x^2 - 16 - 2x = 9$

$\Rightarrow \quad -x^2 - 10x - 25 = 0 \quad \Rightarrow \quad x^2 + 10x + 25 = 0 \quad \Rightarrow \quad x^2 + 5x + 5x + 25 = 0$

$\Rightarrow \quad x(x+5) + 5(x+5) = 0 \quad \Rightarrow \quad (x+5)(x+5) = 0 \quad \Rightarrow \quad (x+5)^2 = 0$

$\Rightarrow \quad (x+5) = 0 \quad \Rightarrow \quad x = -5$ **(Ans)**

and, when $k = 4$

$f(x) = g(x)$

$\Rightarrow \quad 4 - x = \frac{9}{x+2} \quad \Rightarrow \quad (x+2)(4-x) = 9 \quad \Rightarrow \quad 4x - x^2 + 8 - 2x = 9$

$\Rightarrow \quad -x^2 + 2x - 1 = 0 \quad \Rightarrow \quad x^2 - 2x + 1 = 0 \quad \Rightarrow \quad (x-1)(x-1) = 0$

$\Rightarrow \quad (x-1)^2 = 0 \quad \Rightarrow \quad (x-1) = 0 \quad \Rightarrow \quad x = 1$ **(Ans)**

(ii) When $k = 6$, then

$$f(x) = 6 - x \qquad \text{and} \qquad g(x) = \frac{9}{x+2}$$

given that $fg(x) = 5$

$$\Rightarrow f\left(\frac{9}{x+2}\right) = 5 \Rightarrow 6 - \frac{9}{x+2} = 5 \Rightarrow \frac{9}{x+2} = 1$$

$$\Rightarrow x + 2 = 9 \Rightarrow x = 7 \quad \text{(Ans)}$$

(iii) $g(x) = \frac{9}{x+2}, \quad x \neq -2$

Let $g(x) = y \Rightarrow y = \frac{9}{x+2}$

making $x$ the subject

$$y(x+2) = 9 \Rightarrow xy + 2y = 9 \Rightarrow xy = 9 - 2y \Rightarrow x = \frac{9-2y}{y}$$

as $g(x) = y \Rightarrow g^{-1}(y) = x$

$$\therefore \quad g^{-1}(y) = \frac{9-2y}{y}$$

$$\Rightarrow g^{-1}(x) = \frac{9-2x}{x} \quad \text{or} \quad g^{-1}(x) = \frac{9}{x} - 2, \quad x \neq 0 \quad \text{(Ans)}$$

# June 2006 Paper 3

## *Pure Mathematics (P3)*

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Given that $x = 4(3^{-y})$, express $y$ in terms of $x$. [3]

***Suggested Solution:***

$x = 4(3^{-y})$

$\Rightarrow \quad x = 4(\frac{1}{3^y}) \quad \Rightarrow \quad 3^y = \frac{4}{x}$

taking ln on both sides

$\ln(3^y) = \ln(\frac{4}{x}) \quad \Rightarrow \quad y\ln 3 = \ln\left(\frac{4}{x}\right) \quad \Rightarrow \quad y\ln 3 = \ln 4 - \ln x$

$\Rightarrow \quad y = \frac{\ln 4 - \ln x}{\ln 3}$ (Ans)

**2.** Solve the inequality $2x > |x-1|$. [4]

***Suggested Solution:***

$2x > |x-1|$

squaring both sides

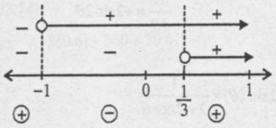
$4x^2 > x^2 - 2x + 1$

$\Rightarrow \quad 3x^2 + 2x - 1 > 0$

$\Rightarrow \quad 3x^2 + 3x - x - 1 > 0$

$\Rightarrow \quad 3x(x+1) - 1(x+1) > 0$

$\Rightarrow \quad (x+1)(3x-1) > 0$



$\therefore \quad x < -1, \quad x > \frac{1}{3}$

but $x < -1$ does not satisfy the inequality

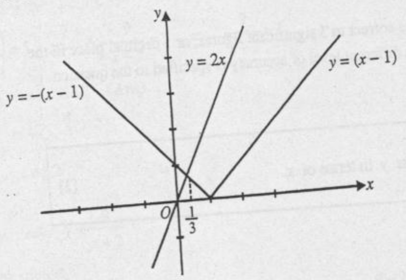
$\therefore \quad x > \frac{1}{3}$ (Ans)

In the process of squaring both sides of the equation or inequalities, we are likely to get extra roots which are not the actual roots of the equation or inequality. The process of verification therefore is important.

***Alternative Solution:***

$2x > |x-1|$

sketching the graphs of $y = 2x$ and $y = |x-1|$,



We see that the only point of intersection is between lines $y = 2x$ and $y = -(x-1)$.

solving the two equations simultaneously

$$2x = -(x-1) \Rightarrow 2x = -x+1 \Rightarrow 3x = 1 \Rightarrow x = \frac{1}{3}$$

$\therefore$ for $2x > |x-1|$, $x > \frac{1}{3}$ **(Ans)**

**3.** The parametric equations of a curve are

$$x = 2\theta + \sin 2\theta, \qquad y = 1 - \cos 2\theta.$$

Show that $\frac{dy}{dx} = \tan\theta$. [5]

***Suggested Solution:***

$x = 2\theta + \sin 2\theta$ $\qquad$ $y = 1 - \cos 2\theta$

differentiating w.r.t. $\theta$

$\frac{dx}{d\theta} = 2 + 2\cos 2\theta$ $\qquad$ $\frac{dy}{d\theta} = +2\sin 2\theta$

now

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = (2\sin 2\theta) \times \frac{1}{2 + 2\cos\theta}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin 2\theta}{2(1+\cos 2\theta)} = \frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{2\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$\therefore \frac{dy}{dx} = \tan\theta$ **(shown)**

4. (i) Express $7\cos\theta + 24\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of $R$ and the value of $\alpha$ correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$7\cos\theta + 24\sin\theta = 15,$$

giving all solutions in the interval $0° \le \theta \le 360°$. [4]

***Suggested Solution:***

(i) Let $7\cos\theta + 24\sin\theta \equiv R\cos(\theta - \alpha)$

$\Rightarrow \quad 7\cos\theta + 24\sin\theta \equiv R[\cos\theta\cos\alpha + R\sin\theta\sin\alpha]$

$\Rightarrow \quad 7\cos\theta + 24\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

comparing the coefficients of $\cos\theta$ and $\sin\theta$, we have

$\Rightarrow \quad R\cos\alpha = 7 \cdots\cdots\cdots(i) \qquad R\sin\alpha = 24 \cdots\cdots\cdots(ii)$

eq. (ii) ÷ eq.(i) gives

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{24}{7} \Rightarrow \tan\alpha = \frac{24}{7} \Rightarrow \alpha = 73.74° \quad \text{(Ans)}$$

squaring and adding eq. (i) & eq. (ii) gives

$$(R\sin\alpha)^2 + (R\cos\alpha)^2 = 24^2 + 7^2 \Rightarrow R^2\sin^2\alpha + R^2\cos^2\alpha = 576 + 49$$

$$\Rightarrow R^2(\sin^2\alpha + \cos^2) = 625 \Rightarrow R^2 = 625 \Rightarrow R = 25$$

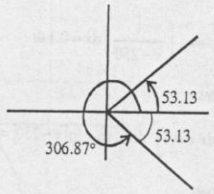
$$\therefore \quad 7\cos\theta + 24\sin\theta \equiv 25\cos(\theta - 73.74°) \quad \text{(Ans)}$$

(ii) $7\cos\theta + 24\sin\theta = 15$

using the result of part (i), we have

$25\cos(\theta - 73.74) = 15$

$\Rightarrow \quad \cos(\theta - 73.74) = \dfrac{15}{25}$



$\therefore \quad (\theta - 73.74) = 53.13, \quad 306.87°$

$\Rightarrow \quad \theta = (53.13 + 73.74), \quad (306.87 + 73.74)$

$\Rightarrow \quad \theta = 126.87°, \quad 380.61°$(out of range, $\therefore$ $380.61 - 360 = 20.61$)

$\Rightarrow \quad \theta = 126.87°, \quad 20.61° \quad$ (Ans)

**5.** In a certain industrial process, a substance is being produced in a container. The mass of the substance in the container $t$ minutes after the start of the process is $x$ grams. At any time, the rate of formation of the substance is proportional to its mass. Also, throughout the process, the substance is removed from the container at a constant rate of 25 grams per minute. When $t = 0$, $x = 1000$ and $\frac{dx}{dt} = 75$.

(i) Show that $x$ and $t$ satisfy the differential equation

$$\frac{dx}{dt} = 0.1(x - 250).$$ [2]

(ii) Solve this differential equation, obtaining an expression for $x$ in terms of $t$. [6]

***Suggested Solution:***

(i) Given that

$$\frac{dx}{dt} \propto x \quad \Rightarrow \quad \frac{dx}{dt} = kx$$

the substance removed at constant rate of 25 grams/min

$$\Rightarrow \quad \frac{dx}{dt} = kx - 25 \cdots\cdots\cdots \text{(i)}$$

given that $x = 1000$, $\frac{dx}{dt} = 75$, putting in equation (i) we have

$$\Rightarrow \quad 75 = 1000k - 25 \quad \Rightarrow \quad 1000k = 100 \quad \Rightarrow \quad k = 0.1$$

$\therefore$ equation (i) becomes

$$\frac{dx}{dt} = 0.1x - 25 \quad \Rightarrow \quad \frac{dx}{dt} = 0.1(x - 250) \quad \textbf{(shown)}$$

(ii) $$\frac{dx}{dt} = 0.1(x - 250) \quad \Rightarrow \quad \left(\frac{1}{x - 250}\right) dx = 0.1\, dt$$

integrating both sides

$$\Rightarrow \quad \int \left(\frac{1}{x - 250}\right) dx = 0.1 \int dt \quad \Rightarrow \quad \ln|x - 250| = 0.1t + C$$

when $t = 0$, $x = 1000$

$$\Rightarrow \quad \ln|1000 - 250| = 0.1(0) + C \quad \Rightarrow \quad C = \ln 750$$

$$\therefore \quad \ln|x - 250| = 0.1t + \ln 750 \quad \Rightarrow \quad \ln|x - 250| - \ln 750 = 0.1t$$

$$\Rightarrow \quad \ln\left|\frac{x - 250}{750}\right| = 0.1t \quad \Rightarrow \quad \ln\left|\frac{x - 250}{750}\right| = 0.1t(\ln e) \quad \Rightarrow \quad \ln\left|\frac{x - 250}{750}\right| = \ln e^{0.1t}$$

$$\Rightarrow \quad \frac{x - 250}{750} = e^{0.1t} \quad \Rightarrow \quad x - 250 = 750e^{0.1t}$$

$$\Rightarrow \quad x = 250 + 750e^{0.1t} \quad \text{(Ans)}$$

Note that:

- $\ln e = 1$
- $\ln a - \ln be = \ln(\frac{a}{b})$
- $b \ln a = \ln a^b$

**6.** (i) By sketching a suitable pair of graphs, show that the equation

$$2\cot x = 1 + e^x,$$

where $x$ is in radians, has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.5 and 1.0. [2]

(iii) Show that this root also satisfies the equation

$$x = \tan^{-1}\left(\frac{2}{1+e^x}\right).$$ [1]

(iv) Use the iterative formula

$$x_{n+1} = \tan^{-1}\left(\frac{2}{1+e^{x_n}}\right)$$

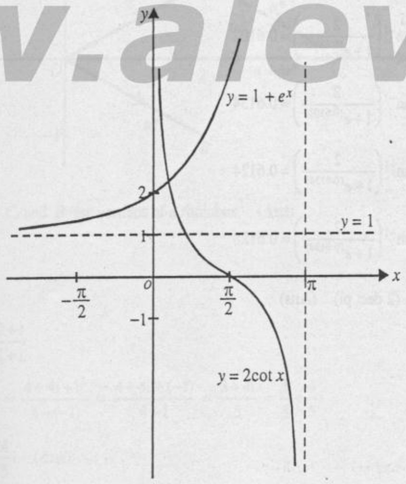
with initial value $x_1 = 0.7$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

*Suggested Solution:*

(i) $2\cot x = 1 + e^x$

this equation can be written as $y = 2\cot x$ and $y = 1 + e^x$

sketching the two graphs on same coordinate axis, we have



As the two graphs meet only at one point in the interval $0 < x < \frac{1}{2}\pi$, therefore $x$ has only one root in this interval.

(ii) Let $f(x) = 2\cot x - 1 - e^x$

$$f(0.5) = 2\cot(0.5) - 1 - e^{0.5} = 3.66 - 1 - 1.65 = 1.01 > 0 \text{ (or } +\text{ve)}$$

$$f(1) = 2\cot(1) - 1 - e^1 = 1.284 - 1 - 2.718 = -2.43 < 0 \text{ (or } -\text{ve)}$$

the change of sign indicates that the root lies between 0.5 and 1.0 **(verified)**

Remember to change your calculator to radian mode.

(iii) We have

$$2\cot x = 1 + e^x$$

$$\Rightarrow \frac{2}{\tan x} = 1 + e^x \Rightarrow \frac{2}{1+e^x} = \tan x \Rightarrow x = \tan^{-1}\left(\frac{2}{1+e^x}\right)$$

$\therefore$ the root also satisfies the above equation **(shown)**

(iv) $x_{n+1} = \tan^{-1}\left(\frac{2}{1+e^{x_n}}\right)$

Given that initial value $x_1 = 0.7$

$$\therefore \quad x_2 = \tan^{-1}\left(\frac{2}{1+e^{x_1}}\right) = \tan^{-1}\left(\frac{2}{1+e^{(0.7)}}\right) = 0.5859$$

$$x_3 = \tan^{-1}\left(\frac{2}{1+e^{x_2}}\right) = \tan^{-1}\left(\frac{2}{1+e^{(0.5859)}}\right) = 0.6208$$

$$x_4 = \tan^{-1}\left(\frac{2}{1+e^{x_3}}\right) = \tan^{-1}\left(\frac{2}{1+e^{(0.6208)}}\right) = 0.6102$$

$$x_5 = \tan^{-1}\left(\frac{2}{1+e^{x_4}}\right) = \tan^{-1}\left(\frac{2}{1+e^{(0.6102)}}\right) = 0.6134$$

$$x_6 = \tan^{-1}\left(\frac{2}{1+e^{x_5}}\right) = \tan^{-1}\left(\frac{2}{1+e^{(0.6134)}}\right) = 0.6124$$

$$x_7 = \tan^{-1}\left(\frac{2}{1+e^{x_6}}\right) = \tan^{-1}\left(\frac{2}{1+e^{(0.6124)}}\right) = 0.6128$$

$\therefore$ required root, $x = 0.61$ (2 dec. pl) **(Ans)**

You must change your calculator to radian mode to get the correct value of the root.

corner

change
r to radian

change your
o radian mode
correct value of

7. The complex number $2+\mathrm{i}$ is denoted by $u$. Its complex conjugate is denoted by $u^*$.

(i) Show, on a sketch of an Argand diagram with origin $O$, the points $A$, $B$ and $C$ representing the complex numbers $u$, $u^*$ and $u+u^*$ respectively. Describe in geometrical terms the relationship between the four points $O$, $A$, $B$ and $C$. [4]

(ii) Express $\dfrac{u}{u^*}$ in the form $x+\mathrm{i}y$, where $x$ and $y$ are real. [3]

(iii) By considering the argument of $\dfrac{u}{u^*}$ or otherwise, prove that

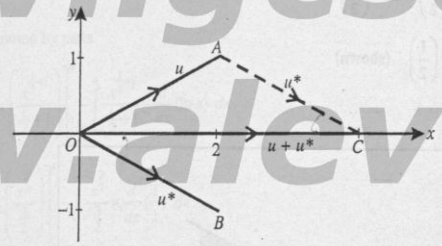
$$\tan^{-1}\left(\frac{4}{3}\right)=2\tan^{-1}\left(\frac{1}{2}\right).$$ [2]

***Suggested Solution:***

(i) Given that

$u=2+\mathrm{i} \quad \Rightarrow \quad$ pt. $A(2,\ 1)$

$u^*=2-\mathrm{i} \quad \Rightarrow \quad$ pt. $B(2,\ -1)$



Points $O$, $A$, $C$, and $B$ are vertices of a rhombus. **(Ans)**

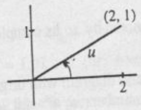
(ii) $\dfrac{u}{u^*}=\dfrac{2+\mathrm{i}}{2-\mathrm{i}}$

$=\dfrac{2+\mathrm{i}}{2-\mathrm{i}}\times\dfrac{2+\mathrm{i}}{2+\mathrm{i}}$

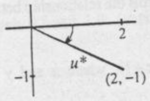
$=\dfrac{(2+\mathrm{i})^2}{2^2-\mathrm{i}^2}=\dfrac{4+4\mathrm{i}+\mathrm{i}^2}{4-(-1)}=\dfrac{4+4\mathrm{i}+(-1)}{4+1}=\dfrac{3+4\mathrm{i}}{5}=\dfrac{3}{5}+\dfrac{4}{5}\mathrm{i}$

$\Rightarrow \dfrac{u}{u^*}=\dfrac{3}{5}+\dfrac{4}{5}\mathrm{i}$ **(Ans)**

**(iii)** $\arg(u) = \tan^{-1}\left(\frac{1}{2}\right)$



$\arg(u^*) = \tan^{-1}\left(-\frac{1}{2}\right)$



using the result of part (i)

$$\arg\left(\frac{u}{u^*}\right) = \tan^{-1}\left(\frac{\frac{4}{5}}{\frac{3}{5}}\right) = \tan^{-1}\left(\frac{4}{3}\right)$$

now, we know that

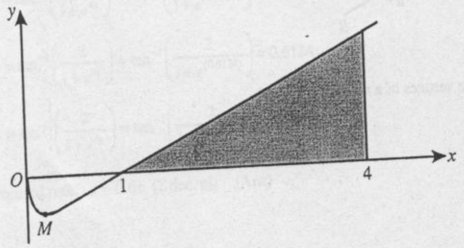
$$\arg\left(\frac{u}{u^*}\right) = \arg u - \arg u^*$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(-\frac{1}{2}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{4}{3}\right) = 2\tan^{-1}\left(\frac{1}{2}\right) \quad \text{(shown)}$$

**8.**



The diagram shows a sketch of the curve $y = x^{\frac{1}{2}}\ln x$ and its minimum point $M$. The curve cuts the $x$-axis at the point (1, 0).

**(i)** Find the exact value of the $x$-coordinate of $M$. [4]

**(ii)** Use integration by parts to find the area of the shaded region enclosed by the curve, the $x$-axis and the line $x = 4$. Give your answer correct to 2 decimal places. [5]

***Suggested Solution:***

(i) $y = x^{\frac{1}{2}} \ln x$

differentiating w.r.t. $x$

$$\frac{dy}{dx} = \ln x \frac{d}{dx}(x^{\frac{1}{2}}) + x^{\frac{1}{2}} \frac{dy}{dx}(\ln x)$$

$$= \ln x(\frac{1}{2} x^{-\frac{1}{2}}) + x^{\frac{1}{2}}(\frac{1}{x}) = \frac{1}{2\sqrt{x}} \ln x + \frac{1}{\sqrt{x}}$$

Note that:

$$\frac{d}{dx}(uv) = v\,\frac{d}{dx}(u) + u\,\frac{d}{dx}(v)$$

for stationary values, $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{1}{2\sqrt{x}} \ln x + \frac{1}{\sqrt{x}} = 0 \quad \Rightarrow \quad \frac{\ln x + 2}{2\sqrt{x}} = 0 \quad \Rightarrow \quad \ln x + 2 = 0 \quad \Rightarrow \quad \ln x = -2$$

$$\Rightarrow \ln x = -2(\ln e) \quad \Rightarrow \quad \ln x = \ln e^{-2} \quad \Rightarrow \quad x = e^{-2} \quad \Rightarrow \quad x = \frac{1}{e^2}$$

$\therefore$ $x$ – coordinate of $M$ is: $x = \frac{1}{e^2}$ **(Ans)**

(ii) Shaded area, $A = \int_1^4 y\, dx \quad \Rightarrow \quad A = \int_1^4 x^{\frac{1}{2}} \ln x\, dx$

using integration by parts

$$A = \left[\ln x\left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)\right]_1^4 - \int_1^4 \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \times \frac{d}{dx}(\ln x)\, dx$$

$$= \left[\ln x\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right)\right]_1^4 - \int_1^4 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} \times \frac{d}{dx}(\frac{1}{x})\, dx$$

$$= \left[\frac{2}{3} x^{\frac{3}{2}} \ln x\right]_1^4 - \frac{2}{3}\int_1^4 x^{\frac{1}{2}}\, dx$$

$$= \left[\frac{2}{3} x^{\frac{3}{2}} \ln x\right]_1^4 - \frac{2}{3}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4 = \left[\frac{2}{3} x^{\frac{3}{2}} \ln x\right]_1^4 - \frac{4}{9}\left[x^{\frac{3}{2}}\right]_1^4$$

$$= \left[\left(\frac{2}{3}(4)^{\frac{3}{2}} \ln(4)\right) - \left(\frac{2}{3}(1)^{\frac{3}{2}} \ln(1)\right)\right] - \frac{4}{9}\left[\left((4)^{\frac{3}{2}}\right) - \left((1)^{\frac{3}{2}}\right)\right]$$

$$= \left(\frac{2}{3}(8) \ln(4) - 0\right) - \frac{4}{9}(8-1) = \frac{16}{3} \ln 4 - \frac{4}{9}(7) = \frac{16}{3} \ln 4 - \frac{28}{9}$$

$$= 7.3936 - 3.1111 = 4.2825 \approx 4.28$$

$\therefore$ area of shaded region, $A = 4.28$ sq.units (to 2 dec.pl) **(Ans)**

9. (i) Express $\frac{10}{(2-x)(1+x^2)}$ in partial fractions. [5]

(ii) Hence, given that $|x|<1$, obtain the expansion of $\frac{10}{(2-x)(1+x^2)}$ in ascending powers of $x$, up to and including the term in $x^3$, simplifying the coefficients. [5]

*Suggested Solution:*

(i) $\frac{10}{(2-x)(1+x^2)} \equiv \frac{A}{2-x}+\frac{Bx+c}{1+x^2}$

$\Rightarrow \; 10 \equiv A(1+x^2)+(Bx+C)(2-x)$

for $x=2$

$10 = A(1+(2)^2)+(B(2)+C)(2-(2))$

$\Rightarrow \; 10=5A+0 \;\Rightarrow\; 10=5A \;\Rightarrow\; A=2$

for $x=0$

$10 = A(1+(0)^2)+(B(0)+C)(2-(0))$

$\Rightarrow \; 10=A+2C \;\Rightarrow\; 10=2+2C \;\Rightarrow\; 8=2C \;\Rightarrow\; C=4$

for $x=1$

$10 = A(1+(1)^2)+(B(1)+C)(2-(1))$

$\Rightarrow \; 10=2A+(B+C) \;\Rightarrow\; 10=2(2)+B+4 \;\Rightarrow\; 10=8+B \;\Rightarrow\; B=2$

$\therefore \; \frac{10}{(2-x)(1+x^2)} = \frac{2}{2-x}+\frac{2x+4}{1+x^2}$ (Ans)

(ii) Using the result of part (i), we have

$$\frac{10}{(2-x)(1+x^2)} = \frac{2}{2-x}+\frac{2x+4}{1+x^2}$$

$$=\frac{2}{2\left(1-\frac{x}{2}\right)}+\frac{2x+4}{1+x^2} = \left(1-\frac{x}{2}\right)^{-1}+(2x+4)(1+x^2)^{-1}$$

applying binomial theorem up to the term in $x^3$

$$=\left[1+(-1)\left(-\frac{x}{2}\right)+\frac{-1(-1-1)}{2!}\left(-\frac{x}{2}\right)^2+\frac{-1(-1-1)(-1-2)}{3!}\left(-\frac{x}{2}\right)^3\right]+(2x+4)\left(1-x^2\right)$$

$$=\left[1+\frac{1}{2}x+\frac{2}{2}\left(\frac{x^2}{4}\right)+\frac{+6}{3\times2\times1}\left(\frac{x^3}{8}\right)\right]+2x-2x^3+4-4x^2$$

$$=\left(1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3\right)+2x-2x^3+4-4x^2$$

$$= 1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+2x-2x^3+4-4x^2$$

$$= 1+4+\frac{1}{2}x+2x+\frac{1}{4}x^2-4x^2+\frac{1}{8}x^3-2x^3$$

$$= 5+\frac{5}{2}x-\frac{15}{4}x^2-\frac{15}{8}x^3 \quad \textbf{(Ans)}$$

**10.** The point $A$ and $B$ have position vectors, relative to the origin $O$, given by

$$\overrightarrow{OA}=\begin{pmatrix}-1\\3\\5\end{pmatrix} \quad \text{and} \quad \overrightarrow{OB}=\begin{pmatrix}3\\-1\\-4\end{pmatrix}.$$

The line $l$ passes through $A$ and is parallel to $OB$. The point $N$ is the foot of the perpendicular from B to $l$.

(i) State a vector equation for the line $l$. [1]

(ii) Find the position vector of $N$ and show that $BN = 3$. [6]

(iii) Find the equation of the plane containing $A$, $B$ and $N$, giving your answer in the form $ax+by+cz=d$. [5]

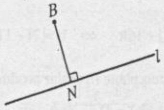
*Suggested Solution:*

(i) Equation of line $l$ is:

$$r=\overline{OA}+\lambda\overline{OB} \;\Rightarrow\; r=\begin{pmatrix}-1\\3\\5\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\\-4\end{pmatrix} \quad \text{(Ans)}$$

(ii) Taking point $N$ as a general point on line $l$, we have

$$\overline{ON}=\begin{pmatrix}-1+3\lambda\\3-\lambda\\5-4\lambda\end{pmatrix} \cdots\cdots\cdots\text{(i)}$$



$$\therefore \quad \overline{BN}=\overline{ON}-\overline{OB}=\begin{pmatrix}-1+3\lambda\\3-\lambda\\5-4\lambda\end{pmatrix}-\begin{pmatrix}3\\-1\\-4\end{pmatrix}=\begin{pmatrix}-4+3\lambda\\4-\lambda\\9-4\lambda\end{pmatrix}$$

as $\overline{BN}$ is perpendicular to line $l$

$\therefore \quad \overline{BN}\cdot\vec{d}=0$ (where $\vec{d}$ is the direction vector of line $l$)

$$\Rightarrow \begin{pmatrix}-4+3\lambda\\4-\lambda\\9-4\lambda\end{pmatrix}\cdot\begin{pmatrix}3\\-1\\-4\end{pmatrix}=0 \;\Rightarrow\; (-12+9\lambda)+(-4+\lambda)+(-36+16\lambda)=0$$

$$\Rightarrow -12-4-36+9\lambda+\lambda+16\lambda=0 \;\Rightarrow\; 26\lambda=52 \quad \lambda=2$$

putting this value of $\lambda$ in eq. (i), we have

$$\overline{ON} = \begin{pmatrix} -1+3(2) \\ 3-(2) \\ 5-4(2) \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$$

$\therefore$ position vector of $N = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$ or $5\mathbf{i}+\mathbf{j}-3\mathbf{k}$ **(Ans)**

now, putting the value of $\lambda$ in $\overline{BN}$, we have

$$\overline{BN} = \begin{pmatrix} -4+3(2) \\ 4-(2) \\ 9-4(2) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

length of $\overline{BN}$: $|\overline{BN}| = \sqrt{(2)^2+(2)^2+(1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$

$\therefore$ $BN = 3$ **(Shown)**

(iii) Consider two direction vectors $\overline{BN}$ and $\overline{BA}$

$$\overline{BN} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \quad \overline{BA} = \overline{OA} - \overline{OB} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ 4 \\ 9 \end{pmatrix}$$

let, $\overline{BN} = \vec{d}_1$ and $\overline{BA} = \vec{d}_2$

using direction vectors of lines $\overline{BN}$ and $\overline{BA}$, we can find the normal vector **n**

$\therefore$ $\mathbf{n} = \vec{d}_1 \times \vec{d}_2$

i.e. $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ -4 & 4 & 9 \end{vmatrix} \Rightarrow \mathbf{n} = \mathbf{i}(18-4) - \mathbf{j}(18+4) + \mathbf{k}(8+8)$

$\Rightarrow$ $\mathbf{n} = 14\mathbf{i} - 22\mathbf{j} + 16\mathbf{k} \Rightarrow \mathbf{n} = 7\mathbf{i} - 11\mathbf{j} + 8\mathbf{k}$

$\therefore$ equation of req.plane in scalar product form is given by

$\mathbf{r} \cdot (7\mathbf{i} - 11\mathbf{j} + 8\mathbf{k}) = D$

as this plane passes through the point $B(3, -1, -4)$

$\therefore$ using $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$, we have

$$\mathbf{r} \cdot (7\mathbf{i} - 11\mathbf{j} + 8\mathbf{k}) = \begin{pmatrix} 3 \\ -1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -11 \\ 8 \end{pmatrix} \Rightarrow \mathbf{r} \cdot (7\mathbf{i} - 11\mathbf{j} + 8\mathbf{k}) = 21+11-32$$

$\Rightarrow$ $\mathbf{r} \cdot (7\mathbf{i} - 11\mathbf{j} + 8\mathbf{k}) = 0$

required equation of plane is, $\mathbf{r} \cdot \begin{pmatrix} 7 \\ -11 \\ 8 \end{pmatrix} = 0$ or $7x - 11y + 8z = 0$ **(Ans)**

***Alternative solution to part (iii):***

Equation of the plane passing through $B\begin{pmatrix}3\\-1\\-4\end{pmatrix}$, with two directions $\overrightarrow{BN}=\begin{pmatrix}2\\2\\1\end{pmatrix}$ and

$\overrightarrow{BA}=\begin{pmatrix}-4\\4\\9\end{pmatrix}$ is given by

$$\mathrm{r}=\begin{pmatrix}3\\-1\\-4\end{pmatrix}+\lambda\begin{pmatrix}2\\2\\1\end{pmatrix}+\mu\begin{pmatrix}-4\\4\\9\end{pmatrix} \Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3+2\lambda-4\mu\\-1+2\lambda+4\mu\\-4+\lambda+9\mu\end{pmatrix}$$

$\Rightarrow \quad x=3+2\lambda-4\mu \cdots\cdots\cdots\text{(i)}$

$y=-1+2\lambda+4\mu\cdots\cdots\cdots\text{(ii)}$

$z=-4+\lambda+9\mu\cdots\cdots\cdots\text{(iii)}$

from eq. (ii), $\lambda=\dfrac{y+1-4\mu}{2}$

from eq. (iii), $\lambda=z+4-9\mu$

$\Rightarrow \dfrac{y+1-4\mu}{2}=z+4-9\mu \Rightarrow y+1-4\mu=2z+8-18\mu \Rightarrow 14\mu=2z-y+7$

$\Rightarrow \mu=\dfrac{2z-y+7}{14}$

putting the value of $\mu$ in equation (iii)

$\therefore \quad \lambda=z+4-9\left(\dfrac{2z-y+7}{14}\right) \Rightarrow \lambda=\dfrac{14z+56-18z+9y-63}{14} \Rightarrow \lambda=\dfrac{9y-4z-7}{14}$

putting the values of $\mu$ and $\lambda$ in eq. (i)

$x=3+2\lambda-4\mu$

$\Rightarrow \quad x=3+2\left(\dfrac{9y-4z-7}{14}\right)-4\left(\dfrac{2z-y+7}{14}\right)$

$\Rightarrow \quad x=\dfrac{42+18y-8z-14-8z+4y-28}{14} \Rightarrow x=\dfrac{22y-16z}{14} \Rightarrow x=\dfrac{11y-8z}{7}$

$\Rightarrow \quad 7x=11y-8z \Rightarrow 7x-11y+8z=0$

$\therefore$ required equation of plane is: $7x-11y+8z=0$ (Ans)

# November 2006 Paper 1

## Pure Mathematics (P1)

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Find the coefficient of $x^2$ in the expansion of $\left(x+\frac{2}{x}\right)^6$. [3]

*Suggested Solution:*

General term, $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$= {}^6C_r\,(x)^{6-r}\left(\frac{2}{x}\right)^r$$

$$= {}^6C_r\,(x)^{6-r}\left(2x^{-1}\right)^r = {}^6C_r\,(x)^{6-r}(2)^r(x)^{-r} = {}^6C_r\,(2)^r(x)^{6-2r}$$

we choose the value of $r$ for which $x^{6-2r} = x^2$

$\Rightarrow\ 6-2r=2 \ \Rightarrow\ 2r=4 \ \Rightarrow\ r=2$

$\Rightarrow\ \mathrm{T}_{2+1} = {}^6C_2\,(2)^2(x)^{6-2(2)} = 15(4)(x)^2 = 60x^2$

$\therefore$ coefficient of $x^2 = 60$ (Ans)

**2.** Given that $x=\sin^{-1}\left(\frac{2}{5}\right)$, find the exact value of

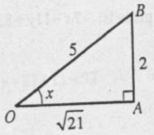
(i) $\cos^2 x$, [2]

(ii) $\tan^2 x$. [2]

*Suggested Solution:*

$x=\sin^{-1}\left(\frac{2}{5}\right) \ \Rightarrow\ \sin x = \frac{2}{5}$

by considering the $\Delta OAB$, we have

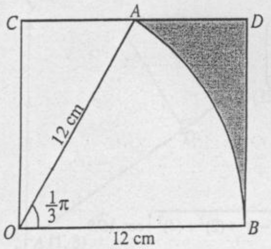


Apply Pythagoras theorem to obtain the base *OA* of the triangle *OAB*.

(i) $\cos x = \frac{\sqrt{21}}{5} \ \Rightarrow\ \cos^2 x = \frac{21}{25}$ (Ans).

(ii) $\tan x = \frac{2}{\sqrt{21}} \ \Rightarrow\ \tan^2 x = \frac{4}{21}$ (Ans).

**3.**



In the diagram, $AOB$ is a sector of a circle with centre $O$ and radius 12 cm. The point A lies on the side $CD$ of the rectangle $OCDB$. Angle $AOB = \frac{1}{3}\pi$ radians. Express the area of the shaded region in the form $a(\sqrt{3}) - b\pi$, stating the values of the integers $a$ and $b$. [6]

***Suggested Solution:***

Consider the $\Delta OAC$

$$A\hat{O}C = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

$$\sin(\frac{\pi}{6}) = \frac{AC}{12} \Rightarrow AC = 12\sin(\frac{\pi}{6}) = 12 \times \frac{1}{2} = 6\text{cm}$$

$$\therefore \quad AC = AD = 6\text{cm}$$

also $$\cos(\frac{\pi}{6}) = \frac{OC}{12} \Rightarrow OC = 12\cos(\frac{\pi}{6}) = 12 \times \frac{\sqrt{3}}{2} = 6\sqrt{3} \text{ cm}$$

$$\therefore \quad OC = BD = 6\sqrt{3} \text{ cm}$$

now, shaded area = area of trapezium $OBDA$ − area of sector $OAB$

$$= \frac{1}{2} \times 6\sqrt{3}\ (12+6) - \frac{1}{2}(12)^2(\frac{\pi}{3})$$

$$= 54\sqrt{3} - 24\pi$$

$\therefore \quad a = 54, \quad b = 24$ **(Ans)**

Note that :

$\sin(\frac{1}{6}\pi) = \sin(30°) = \frac{1}{2}$

$\cos(\frac{1}{6}\pi) = \cos(30°) = \frac{\sqrt{3}}{2}$

**4.** The position vectors of points $A$ and $B$ are $\begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$ respectively, relative to an origin $O$.

(i) Calculate angle $AOB$. [3]

(ii) The point $C$ is such that $\overline{AC} = 3\overline{AB}$. Find the unit vector in the direction of $\overline{OC}$. [4]

*Suggested Solution:*

(i) Given that $\overline{OA} = \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}$, $\overline{OB} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$

applying scalar product

$$\overline{OA} \cdot \overline{OB} = \left|\overline{OA}\right| \left|\overline{OB}\right| \cos A\hat{O}B$$

$$\begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = \left(\sqrt{(-3)^2+(6)^2+(3)^2}\right)\left(\sqrt{(-1)^2+(2)^2+(4)^2}\right) \cos A\hat{O}B$$

$$3+12+12 = \left(\sqrt{9+36+9}\right)\left(\sqrt{1+4+16}\right) \cos A\hat{O}B$$

$$27 = \left(\sqrt{54}\right)\left(\sqrt{21}\right) \cos A\hat{O}B$$

$$\cos A\hat{O}B = \frac{27}{(\sqrt{54})\,(\sqrt{21})} = 0.801784$$

$$A\hat{O}B = 36.699 \approx 36.7° \quad \textbf{(Ans)}$$

If
$\mathbf{a} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, and
$\mathbf{b} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then
$\mathbf{a}.\mathbf{b} = ax + by + cz$
and magnitude of $\mathbf{a}$ is
$|\mathbf{a}| = \sqrt{(a)^2 + (b)^2 + (c)^2}$

(ii) Given that

$$\overline{AC} = 3\overline{AB}$$

$$\Rightarrow \quad \overline{OC} - \overline{OA} = 3\left(\overline{OB} - \overline{OA}\right)$$

$$\Rightarrow \quad \overline{OC} - \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix} = 3\left[\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}\right]$$

$$\Rightarrow \quad \overline{OC} - \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad \overline{OC} = \begin{pmatrix} 6 \\ -12 \\ 3 \end{pmatrix} + \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix}$$

$$\therefore \quad \overline{OC} = 3\mathrm{i} - 6\mathrm{j} + 6\mathrm{k}$$

unit vector in the direction of $\overline{OC} = \dfrac{\overline{OC}}{\left|\overline{OC}\right|}$

$$= \frac{3\mathrm{i}-6\mathrm{j}+6\mathrm{k}}{\sqrt{(3)^2+(-6)^2+(6)^2}} = \frac{3\mathrm{i}-6\mathrm{j}+6\mathrm{k}}{\sqrt{9+36+36}}$$

$$= \frac{3\mathrm{i}-6\mathrm{j}+6\mathrm{k}}{\sqrt{81}} = \frac{3\mathrm{i}-6\mathrm{j}+6\mathrm{k}}{9}$$

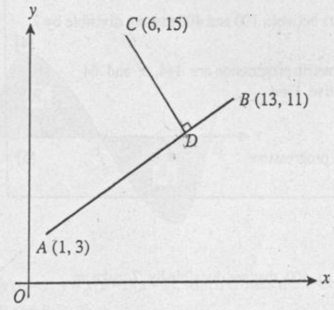
$$= \frac{3}{9}\mathrm{i} - \frac{6}{9}\mathrm{j} + \frac{6}{9}\mathrm{k} = \frac{1}{3}\mathrm{i} - \frac{2}{3}\mathrm{j} + \frac{2}{3}\mathrm{k}$$

$$= \frac{1}{3}(\mathrm{i} - 2\mathrm{j} + 2\mathrm{k}) \quad \textbf{(Ans)}$$

For any vector $\mathbf{v}$, the unit vector $\hat{\mathbf{v}}$, is given as:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

**5.**



The three points $A(1, 3)$, $B(13, 11)$ and $C(6, 15)$ are shown in the diagram. The perpendicular from $C$ to $AB$ meets $AB$ at the point $D$. Find

(i) the equation of $CD$, [3]

(ii) the coordinates of $D$. [4]

***Suggested Solution:***

(i) gradient of $AB = \frac{11-3}{13-1} = \frac{8}{12} = \frac{2}{3}$

$\Rightarrow$ gradient of $CD = -\frac{3}{2}$

$\therefore$ Equation of $CD$ passing through $C(6, 15)$ is given by

$$y-15 = -\frac{3}{2}(x-6) \Rightarrow 2y-30 = -3x+18 \Rightarrow 3x+2y=48 \quad \text{(Ans)}$$

If two lines are perpendicular than product of their gradient is equal to −1

i.e. $m_1 \times m_2 = -1$

Equation of a straight line in slope-point form, with gradient $m$ and passing through $(x_1, y_1)$ is $(y-y_1) = m(x-x_1)$.

(ii) from part (i), gradient of $AB = \frac{2}{3}$

and equation of $CD$ is: $3x+2y=48 \Rightarrow y = \frac{48-3x}{2}$ ………(i)

now, equation of $AB$ passing through $A$ is:

$$y-3 = \frac{2}{3}(x-1) \Rightarrow 3y-9 = 2x-2 \Rightarrow 3y-2x = 7 \text{ ………(ii)}$$

$CD$ meets $AB$ at $D$, therefore solving eq.(i) and eq.(ii) simultaneously

$$3(\frac{48-3x}{2}) - 2x = 7 \Rightarrow \frac{144-9x}{2} - 2x = 7 \Rightarrow 144-9x-4x = 14$$

$\Rightarrow 13x = 130 \Rightarrow x = 10$

putting the value of $x$ in eq.(i)

$$y = \frac{48-3(10)}{2} = \frac{18}{2} = 9$$

$\therefore$ coordinates of $D$ are (10, 9) (Ans)

**6.** (a) Find the sum of all the integers between 100 and 400 that are divisible by 7. [4]

(b) The first three terms in a geometric progression are 144, $x$ and 64 respectively, where $x$ is positive. Find

(i) the value of $x$,

(ii) the sum to infinity of the progression. [5]

***Suggested Solution:***

(a) Sum of all integers between 100 to 400 that are divisible by 7 make an arithmetic series

$$105 + 112 + 119 + \ldots\ldots\ldots\ldots\ldots + 399$$

$\therefore \quad a = 105, \quad d = 7, \quad T_n = 399, \quad n = ?$

using, $T_n = a + (n-1)d$

$$399 = 105 + (n-1)(7) \Rightarrow 294 = 7n - 7 \Rightarrow 7n = 301 \Rightarrow n = 43$$

using, $S_n = \frac{n}{2}(a+l)$

$$S_{43} = \frac{43}{2}(105+399) = \frac{43}{2}(504) = 10836 \quad \text{(Ans)}$$

(b) (i) Given first three terms in a G.P:

$$144, \; x, \; 64$$

common ratio is $\frac{x}{144}$ and $\frac{64}{x}$

$$\Rightarrow \frac{x}{144} = \frac{64}{x} \Rightarrow x^2 = 144 \times 64 \Rightarrow x^2 = 9216 \Rightarrow x = \pm 96$$

$\therefore$ taking positive value only, $x = 96$ (Ans)

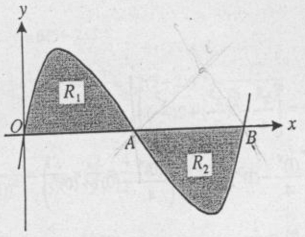
(ii) We have

common ratio, $r = \frac{96}{144} = \frac{2}{3}, \quad a = 144$

using $S_\infty = \frac{a}{1-r}$

$$S_\infty = \frac{144}{1-\frac{2}{3}} = \frac{144}{\frac{1}{3}} = 144 \times \frac{3}{1} = 432 \quad \text{(Ans)}$$

7.



The diagram shows the curve $y = x(x-1)(x-2)$, which crosses the $x$-axis at the points $O(0,\ 0)$, $A(1,\ 0)$ and $B(2,\ 0)$.

(i) The tangents to the curve at the points $A$ and $B$ meet at the point $C$. Find the $x$-coordinate of $C$. [5]

(ii) Show by integration that the area of the shaded region $R_1$ is the same as the area of the shaded region $R_2$. [4]

*Suggested Solution:*

(i) Given $y = x(x-1)(x-2)$

$$\Rightarrow \quad y = x(x^2 - 2x - x + 2) \quad \Rightarrow \quad y = x(x^2 - 3x + 2) \quad \Rightarrow \quad y = x^3 - 3x^2 + 2x$$

differentiating w.r.t $x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x + 2$$

gradient of tangent at $A(1,\ 0)$, $\quad \frac{\mathrm{d}y}{\mathrm{d}x} = 3(1)^2 - 6(1) + 2 = 3 - 6 + 2 = -1$

$\therefore$ equation of tangent at $A$ is:

$$y - 0 = -1(x-1) \quad \Rightarrow \quad y = -x + 1 \cdots\cdots\cdots \text{(i)}$$

similarly,

gradient of tangent at $B(2,\ 0)$, $\quad \frac{\mathrm{d}y}{\mathrm{d}x} = 3(2)^2 - 6(2) + 2 = 12 - 12 + 2 = 2$

$\therefore$ equation of tangent at $B$ is:

$$y - 0 = 2(x-2) \quad \Rightarrow \quad y = 2x - 4 \cdots\cdots\cdots \text{(ii)}$$

solving (i) and (ii) simultaneously

$$-x + 1 = 2x - 4 \quad \Rightarrow \quad 3x = 5 \quad \Rightarrow \quad x = \frac{5}{3}$$

$\therefore$ $x$-coordinate of $C$ is $\frac{5}{3}$ (Ans)

(ii) $y = x(x-1)(x-2) = x^3 - 3x^2 + 2x$

area of region $R_1 = \int_0^1 y\,dx$

$$= \int_0^1 (x^3 - 3x^2 + 2x)\,dx = \left[\frac{x^4}{4} - 3(\frac{x^3}{3}) + 2(\frac{x^2}{2})\right]_0^1$$

$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \left(\frac{(1)^4}{4} - (1)^3 + (1)^2\right) - \left(\frac{(0)^4}{4} - (0)^3 + (0)^2\right)$$

$$= \left(\frac{1}{4} - 1 + 1\right) - (0 - 0 + 0) = \frac{1}{4}$$

area of region $R_2 = \int_1^2 y\,dx$

$$= \int_1^2 (x^3 - 3x^2 + 2x)\,dx = \left[\frac{x^4}{4} - 3(\frac{x^3}{3}) + 2(\frac{x^2}{2})\right]_1^2$$

$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2 = \left(\frac{(2)^4}{4} - (2)^3 + (2)^2\right) - \left(\frac{(1)^4}{4} - (1)^3 + (1)^2\right)$$

$$= \left(\frac{16}{4} - 8 + 4\right) - \left(\frac{1}{4} - 1 + 1\right) = (4 - 8 + 4) - \left(\frac{1}{4}\right) = -\frac{1}{4}$$

Note that area of $R_2$ is negative as it lies below x-axis. Absolute value of the area is always positive.

$\therefore$ area of region $R_2 = \frac{1}{4}$

$\Rightarrow$ both regions $R_1$ and $R_2$ have the same area **(Shown)**

8. The equation of a curve is $y = \frac{6}{5-2x}$.

(i) Calculate the gradient of the curve at the point where $x = 1$. [3]

(ii) A point with coordinates $(x, y)$ moves along the curve in such a way that the rate of increase of $y$ has a constant value of 0.02 units per second. Find the rate of increase of $x$ when $x = 1$. [2]

(iii) The region between the curve, the $x$–axis and the lines $x = 0$ and $x = 1$ is rotated through 360° about the $x$–axis. Show that the volume obtained is $\frac{12}{5}\pi$. [5]

orner

rea of $R_2$ is
it lies below
olute value of
is always

***Suggested Solution:***

(i) $y=\dfrac{6}{5-2x} \Rightarrow y=6(5-2x)^{-1}$

$$\frac{dy}{dx}=6\left[-1(5-2x)^{-2}(-2)\right]=12(5-2x)^{-2}=\frac{12}{(5-2x)^2}$$

at $x=1$

$$\frac{dy}{dx}=\frac{12}{(5-2(1))^2}=\frac{12}{(3)^2}=\frac{12}{9}=\frac{4}{3}$$

$\therefore$ required gradient $=\dfrac{4}{3}$ **(Ans)**

Gradient at any point $(x, y)$ on a curve is given by $\dfrac{dy}{dx}$

(ii) Given that $\dfrac{dy}{dt}=0.02$ units/sec

from part (i), the gradient at $x=1$ is: $\dfrac{dy}{dx}=\dfrac{4}{3}$

using chain rule, we have

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$$

$$\Rightarrow 0.02=\frac{4}{3}\times\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt}=0.02\times\frac{3}{4}=0.015$$

$\therefore$ rate of increase of $x=0.015$ units/sec **(Ans)**

Chain Rule:

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}},$$

or $\dfrac{dy}{dt}=\dfrac{dy}{dx}\times\dfrac{dx}{dt}$

(iii) Volume, $V=\pi\displaystyle\int_0^1 y^2\,dx$

$$\Rightarrow V=\pi\int_0^1\left(\frac{6}{5-2x}\right)^2 dx$$

$$=\pi\int_0^1 36(5-2x)^{-2}\,dx=36\pi\int_0^1(5-2x)^{-2}\,dx$$

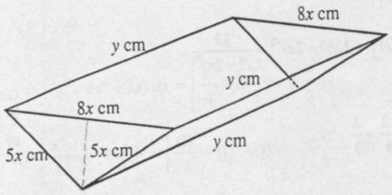
$$=36\pi\left[\frac{(5-2x)^{-1}}{(-1)(-2)}\right]_0^1=18\pi\left[\frac{1}{5-2x}\right]_0^1$$

$$=18\pi\left(\frac{1}{5-2(1)}-\frac{1}{5-2(0)}\right)=18\pi\left(\frac{1}{3}-\frac{1}{5}\right)=18\pi\left(\frac{2}{15}\right)=\frac{12}{5}\pi$$

$\therefore$ volume, $V=\dfrac{12}{5}\pi$ units$^3$ **(Shown)**

For definite integral, do not put the constant of integration.

**9.**



The diagram shows an open container constructed out of 200 cm$^2$ of cardboard. The two vertical end pieces are isosceles triangles with sides $5x$ cm, $5x$ cm and $8x$ cm, and the two side pieces are rectangles of length $y$ cm and width $5x$ cm, as shown. The open top is a horizontal rectangle.

**(i)** Show that $y = \dfrac{200-24x^2}{10x}$ [3]

**(ii)** Show that the volume, $V$ cm$^3$, of the container is given by

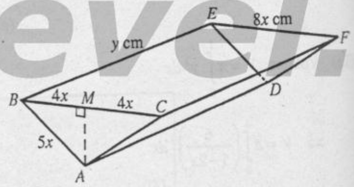
$$V = 240x - 28.8x^3.$$ [2]

Given that $x$ can vary,

**(iii)** find the value of $x$ for which $V$ has a stationary value, [3]

**(iv)** determine whether it is a maximum or a minimum stationary value. [2]

***Suggested Solution:***



(i) Consider the opposite figure

applying phythagoras theorem in $\Delta AMB$

$AM^2 + BM^2 = AB^2$

$\Rightarrow \quad AM^2 + (4x)^2 = (5x)^2 \quad \Rightarrow \quad AM^2 = 25x^2 - 16x^2$

$\Rightarrow \quad AM^2 = 9x^2 \quad \Rightarrow \quad AM = 3x$

$\Delta ABC$ and $\Delta DEF$ are congruent.

$\therefore \quad$ total area of two triangles $(ABC \ \& \ DEF) = 2\left(\dfrac{1}{2}(8x)(3x)\right) = 2\left(12x^2\right) = 24x^2$

also, total area of two rectangles $(ADFC \ \& \ ADEB) = 2\left(y \times 5x\right) = 10xy$

$\therefore \quad$ total surface area $= 24x^2 + 10xy$

given that total surface area $= 200$ cm$^2$

$\Rightarrow \quad 24x^2 + 10xy = 200$

$\Rightarrow \quad 10xy = 200 - 24x^2 \quad \Rightarrow \quad y = \dfrac{200-24x^2}{10x}$ (shown)

(ii) Volume, $V =$ base area × height = (area of $\Delta ABC$) × $BE$

$$\Rightarrow \quad V = \left(\frac{1}{2}(8x)(3x)\right)(y) = (12x^2)(y)$$

putting the value of $y$ from part (i)

$$V = (12x^2)\left(\frac{200-24x^2}{10x}\right) = (6x)\left(\frac{200-24x^2}{5}\right) = \left(\frac{1200x-144x^3}{5}\right)$$

$$= \frac{1200x}{5} - \frac{144x^3}{5} = 240x - 28.8x^3 \quad \text{(shown)}$$

(iii) $V = 240x - 28.8x^3$

$$\frac{dV}{dx} = 240 - 86.4x^2$$

for stationary values, put $\frac{dV}{dx} = 0$

$$\Rightarrow \quad 240 - 86.4x^2 = 0 \quad \Rightarrow \quad 86.4x^2 = 240 \quad \Rightarrow \quad x^2 = \frac{240}{86.4}$$

$\Rightarrow \quad x^2 = 2.7778 \quad \Rightarrow \quad x = 1.67$ (taking +ve value only) (Ans)

(iv) $\frac{dV}{dx} = 240 - 86.4x^2$

$$\Rightarrow \quad \frac{d^2V}{dx^2} = -172.8x$$

at $x = 1.667$

$$\frac{d^2V}{dx^2} = -172.8(1.667) = -288.06 < 0$$

∴ It is a maximum stationary value (Ans)

The function f is defined by $f : x \mapsto x^2 - 3x$ for $x \in \mathbb{R}$.

(i) Find the set of values of $x$ for which $f(x) > 4$. [3]

(ii) Express $f(x)$ in the form $(x-a)^2 - b$, stating the values of $a$ and $b$. [2]

(iii) Write down the range of f. [1]

(iv) State, with a reason, whether f has an inverse. [1]

The function g is defined by $g : x \mapsto x - 3\sqrt{x}$ for $x \geq 0$.

(v) Solve the equation $g(x) = 10$. [3]

*Suggested Solution:*

(i) $f(x)=x^2-3x$

given that $f(x)>4$

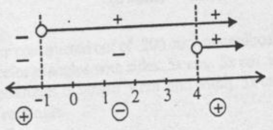
$\Rightarrow\ x^2-3x>4\ \Rightarrow\ x^2-3x-4>0\ \Rightarrow\ x^2-4x+x-4>0$

$\Rightarrow\ x(x-4)+1(x-4)>0\ \Rightarrow\ (x-4)(x+1)>0$

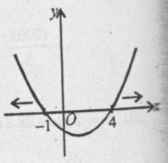
critical values are $x=-1$ and $x=4$

using line method, the set of required values of $x$ are:

$x<-1$ and $x>4$ (Ans)



Alternatively, using sketch method, we can draw a quick sketch of the curve $y=(x-4)(x+1)$ and find the set of values of $x$ as follows.



Required values are:

$x<-1$ and $x>4$

(ii) $f(x)=x^2-3x$

$$=x^2-3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2=\left(x-\frac{3}{2}\right)^2-\frac{9}{4}$$

$\therefore\ a=\frac{3}{2},\ b=\frac{9}{4}$ (Ans)

(iii) From part (ii), we see that f(x) can be written as:

$$f(x)=\left(x-\frac{3}{2}\right)^2-\frac{9}{4}$$

$\therefore$ range of f is: $f(x)\geq-\frac{9}{4}$ (Ans)

(iv) f does not have an inverse because f is not a $1-1$ function. (Ans)

(v) $g(x)=x-3\sqrt{x},\ x\geq0$

given that $g(x)=10$

$\Rightarrow\ x-3\sqrt{x}=10\ \Rightarrow\ x-10=3\sqrt{x}$

squaring both sides

$(x-10)^2=(3\sqrt{x})^2\ \Rightarrow\ x^2-20x+100=9x\ \Rightarrow\ x^2-29x+100=0$

$\Rightarrow\ x^2-25x-4x+100=0\ \Rightarrow\ x(x-25)-4(x-25)=0$

$\Rightarrow\ (x-25)(x-4)=0$

either $x-25=0$ or $x-4=0$

$\Rightarrow\ x=25,$ or $x=4$

on verification, we find that $x=4$ does not satisfy the original equation

$\therefore\ x=25$ (Ans)

Equation involving $\sqrt{}$ sign are called radical equations

In the process of squaring in radical equations, the degree of the equation is raised and there is likelyhood of getting the root which may not be the root of the original equation. Such roots are called extraneous roots.

Therefore, the process of verification is very important while solving radical equations.

# November 2006 Paper 3

## *Pure Mathematics (P3)*

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. Find the set of values of $x$ satisfying the inequality $|3^x - 8| < 0.5$, giving 3 significant figures in your answer. [4]

*Suggested Solution:*

$$|3^x - 8| < 0.5$$

$\Rightarrow \quad 3^x - 8 < 0.5$ or $3^x - 8 > -0.5$

$\Rightarrow \quad 3^x < 8 + 0.5 \qquad \Rightarrow \quad 3^x > -0.5 + 8$

$\Rightarrow \quad 3^x < 8.5 \qquad \Rightarrow \quad 3^x > 7.5$

taking ln on both sides on both inequalities

$\ln 3^x < \ln 8.5 \qquad \ln 3^x > \ln 7.5$

$\Rightarrow \quad x\ln 3 < \ln 8.5 \qquad \Rightarrow \quad x\ln 3 > \ln 7.5$

$\Rightarrow \quad x < \dfrac{\ln 8.5}{\ln 3} \qquad \Rightarrow \quad x > \dfrac{\ln 7.5}{\ln 3}$

$x < 1.95 \qquad \Rightarrow \quad x > 1.83$

$\therefore \quad 1.83 < x < 1.95$ **(Ans)**

2. Solve the equation

$$\tan x \tan 2x = 1,$$

giving all solutions in the interval $0^\circ < x < 180^\circ$. [4]

*Suggested Solution:*

$$\tan x \tan 2x = 1$$

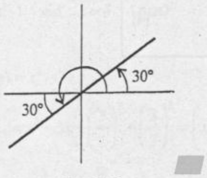
$$\Rightarrow \quad \tan x\left(\frac{2\tan x}{1-\tan^2 x}\right) = 1$$

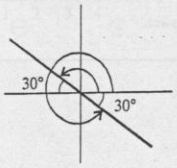
$$\Rightarrow \quad 2\tan^2 x = 1 - \tan^2 x$$

$$\Rightarrow \quad 3\tan^2 x = 1 \quad \Rightarrow \quad \tan^2 x = \frac{1}{3} \quad \Rightarrow \quad \tan x = \pm\frac{1}{\sqrt{3}}$$

for $\tan x = \frac{1}{\sqrt{3}}$ and for $\tan x = -\frac{1}{\sqrt{3}}$

basic angle $= 30^\circ$ basic angle $= 30^\circ$





Note that tan*x* is +ve in 1st and 3rd quadrant, and –ve in 2nd and 4th quadrant

$\therefore \quad x = 30^\circ, 210^\circ$ $\therefore \quad x = 150^\circ, 330^\circ$

now, for $0^\circ < x < 180^\circ$, $x = 30^\circ, 150^\circ$ **(Ans)**

**3.** The curve with equation $y = 6e^x - e^{3x}$ has one stationary point.

**(i)** Find the *x*-coordinate of this point. [4]

**(ii)** Determine whether this point is a maximum or a minimum point. [2]

*Suggested Solution:*

(i) $y = 6e^x - e^{3x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6e^x - 3e^{3x}$$

for stationary values, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$$\Rightarrow \quad 6e^x - 3e^{3x} = 0 \quad \Rightarrow \quad 3e^x(2 - e^{2x}) = 0$$

$\Rightarrow \quad 3e^x = 0$ or $2 - e^{2x} = 0$

(impossible) $e^{2x} = 2$

taking ln on both sides

$$\ln e^{2x} = \ln 2$$

$$2x\ln e = \ln 2$$

$$2x = \ln 2$$

$$x = \frac{1}{2}\ln 2 \quad \Rightarrow \quad x = \ln\sqrt{2} \quad \textbf{(Ans)}$$

(ii) $y = 6e^x - e^{3x} \quad \Rightarrow \quad \frac{dy}{dx} = 6e^x - 3e^{3x}$

taking the second derivative, we have

$$\frac{d^2y}{dx^2} = 6e^x - 3e^{3x}(3) = 6e^x - 9e^{3x}$$

at $x = \ln\sqrt{2}$

$$\frac{d^2y}{dx^2} = 6e^{\ln\sqrt{2}} - 9e^{3(\ln\sqrt{2})}$$

$$= 6e^{\ln\sqrt{2}} - 9e^{\ln 2^{\frac{3}{2}}} = 6\sqrt{2} - 9(2)^{\frac{3}{2}} = 8.485 - 25.456 = -16.97 < 0$$

$\therefore$ this point is a maximum point (Ans)

- if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} > 0$, $y$ is min
- if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} < 0$, $y$ is max

4. Given that $y = 2$ when $x = 0$, solve the differential equation

$$y\frac{dy}{dx} = 1 + y^2,$$

obtaining an expression for $y^2$ in terms of $x$. [6]

*Suggested Solution:*

$$y\frac{dy}{dx} = 1 + y^2 \quad \Rightarrow \quad \frac{y}{1+y^2}\,dy = dx$$

integrating both sides

$$\int \frac{y}{1+y^2}\,dy = \int 1\,dx$$

multiply and divide by 2 on the L.H.S.

$$\frac{1}{2}\int \frac{2y}{1+y^2}\,dy = \int 1\,dx$$

$\Rightarrow \frac{1}{2}\ln(1+y^2) = x + K \cdots\cdots\cdots(i),$ ($K$ is the constant of integration)

given that $y = 2$ when $x = 0$

$\Rightarrow \frac{1}{2}\ln(1+(2)^2) = 0 + K \quad \Rightarrow \quad \frac{1}{2}\ln(1+4) = K \quad \Rightarrow \quad K = \frac{1}{2}\ln 5$

$\therefore$ eq. (i) becomes

$$\frac{1}{2}\ln(1+y^2) = x + \frac{1}{2}\ln 5$$

$\Rightarrow \frac{1}{2}\ln(1+y^2) - \frac{1}{2}\ln 5 = x \quad \Rightarrow \quad \frac{1}{2}\left(\ln(1+y^2) - \ln 5\right) = x \quad \Rightarrow \quad \frac{1}{2}\ln\left(\frac{1+y^2}{5}\right) = x$

$\Rightarrow \ln\left(\frac{1+y^2}{5}\right) = 2x \quad \Rightarrow \quad \ln\left(\frac{1+y^2}{5}\right) = 2x\ln e \quad \Rightarrow \quad \ln\left(\frac{1+y^2}{5}\right) = \ln e^{2x}$

$\Rightarrow \frac{1+y^2}{5} = e^{2x} \quad \Rightarrow \quad 1 + y^2 = 5e^{2x} \quad \Rightarrow \quad y^2 = 5e^{2x} - 1$ (Ans)

- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
- $\ln a^b = b\ln a$

**5.** (i) Simplify $\left(\sqrt{(1+x)}+\sqrt{(1-x)}\right)\left(\sqrt{(1+x)}-\sqrt{(1-x)}\right)$, showing your working, and deduce that

$$\frac{1}{\sqrt{(1+x)}+\sqrt{(1-x)}}=\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{2x}$$ [2]

(ii) Using this result, or otherwise, obtain the expansion of

$$\frac{1}{\sqrt{(1+x)}+\sqrt{(1-x)}}$$

in ascending powers of $x$, up to and including the term in $x^2$. [4]

*Suggested Solution:*

(i) $\left(\sqrt{(1+x)}+\sqrt{(1-x)}\right)\left(\sqrt{(1+x)}-\sqrt{(1-x)}\right)$

$= \left(\sqrt{(1+x)}\right)^2-\left(\sqrt{(1-x)}\right)^2 = (1+x)-(1-x) = 1+x-1+x = 2x$ **(Ans)**

now, L.H.S. $= \dfrac{1}{\sqrt{1+x}+\sqrt{1-x}}$

rationalising the denominator

$$\frac{1}{\sqrt{1+x}+\sqrt{1-x}}\times\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$$

$$=\frac{\sqrt{1+x}-\sqrt{1-x}}{\left(\sqrt{(1+x)}\right)^2-\left(\sqrt{(1-x)}\right)^2}=\frac{\sqrt{1+x}-\sqrt{1-x}}{(1+x)-(1-x)}=\frac{\sqrt{1+x}-\sqrt{1-x}}{2x}=\text{R.H.S.}$$

**(ii)** Using the result of part **(i)**, we have

$$\frac{1}{\sqrt{(1+x)}+\sqrt{(1-x)}}=\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{2x}$$

$$=\frac{1}{2x}\left[(1+x)^{\frac{1}{2}}-(1-x)^{\frac{1}{2}}\right]$$

$$=\frac{1}{2x}\left[\left(1+\frac{1}{2}(x)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(x)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(x)^3\right)-\left(1+\frac{1}{2}(-x)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(-x)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3\right)\right]$$

$$=\frac{1}{2x}\left[\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3\right)-\left(1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3\right)\right]$$

$$=\frac{1}{2x}\left[1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-1+\frac{1}{2}x+\frac{1}{8}x^2+\frac{1}{16}x^3\right]$$

$$=\frac{1}{2x}\left[\frac{1}{2}x+\frac{1}{2}x+\frac{1}{16}x^3+\frac{1}{16}x^3\right]=\frac{1}{2x}\left[x+\frac{1}{8}x^3\right]=\frac{1}{2}+\frac{1}{16}x^2$$ **(Ans)**

**6.** The equation of a curve is $x^3 + 2y^3 = 3xy$.

(i) Show that $\frac{dy}{dx} = \frac{y - x^2}{2y^2 - x}$ [4]

(ii) Find the coordinates of the point, other than the origin, where the curve has a tangent which is parallel to the $x$-axis. [5]

*Suggested Solution:*

(i) $x^3 + 2y^3 = 3xy$

differentiating w.r.t $x$

$$3x^2 + 6y^2 \frac{d}{dx}(y) = 3\left[y\frac{d}{dx}(x) + x\frac{d}{dx}(y)\right]$$

$$\Rightarrow \quad 3x^2 + 6y^2 \frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

$$\Rightarrow \quad 6y^2 \frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 3x^2$$

$$\Rightarrow \quad (6y^2 - 3x)\frac{dy}{dx} = 3y - 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3y - 3x^2}{6y^2 - 3x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3(y - x^2)}{3(2y^2 - x)} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y - x^2}{2y^2 - x} \quad \textbf{(shown)}$$

(ii) $\frac{dy}{dx} = \frac{y - x^2}{2y^2 - x}$

we know that tangent is parallel to $x$–axis at the points where gradient $\frac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{y - x^2}{2y^2 - x} = 0 \quad \Rightarrow \quad y - x^2 = 0 \quad \Rightarrow \quad y = x^2$$

putting this value in the original equation of the curve, we have

$$x^3 + 2(x^2)^3 = 3x(x^2)$$

$$\Rightarrow \quad x^3 + 2x^6 = 3x^3 \quad \Rightarrow \quad 2x^6 - 2x^3 = 0 \quad \Rightarrow \quad 2x^3[x^3 - 1] = 0$$

$$\Rightarrow \quad 2x^3 = 0 \quad \text{or} \quad x^3 - 1 = 0$$

$$\Rightarrow \quad x = 0 \qquad \Rightarrow \quad (x-1)(x^2 + x + 1) = 0$$

$$\Rightarrow \quad x - 1 = 0 \quad \text{or} \quad (x^2 + x + 1) = 0$$

$$\Rightarrow \quad x = 1 \qquad \text{imaginary roots}$$

now, when $x = 0$, $y = 0$

and when $x = 1$, $y = (1)^2 \Rightarrow y = 1$

∴ (1, 1) is the point other than origin **(Ans)**

7. The line $l$ has equation $\mathbf{r} = \mathbf{j} + \mathbf{k} + s(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$. The plane $p$ has equation $x + 2y + 3z = 5$.

(i) Show that the line $l$ lies in the plane $p$. [3]

(ii) A second plane is perpendicular to the plane $p$, parallel to the line $l$ and contains the point with position vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$. Find the equation of this plane, giving your answer in the form $ax + by + cz = d$. [6]

*Suggested Solution:*

(i) Given that line $l$ has equation: $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \mathbf{r} = \begin{pmatrix} s \\ 1-2s \\ 1+s \end{pmatrix}$

equation of given plane $p$ is: $x + 2y + 3z = 5 \Rightarrow \mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5$

putting the general point of the line in the equation of plane $p$

$$\begin{pmatrix} s \\ 1-2s \\ 1+s \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5$$

$$s + 2(1-2s) + 3(1+s) = 5$$

$$s + 2 - 4s + 3 + 3s = 5$$

$$5 = 5$$

$\therefore$ the line $l$ lies in the plane $p$ **(Shown)**

(ii) The two direction vectors of the second plane are given by the direction vector of the line $(\overrightarrow{d_1})$, and the normal vector of the plane $p$ $(\overrightarrow{d_2})$.

$$\therefore \quad \overrightarrow{d_1} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \overrightarrow{d_2} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

now, the normal vector $\mathbf{n}_2$ which is perpendicular to the second plane is given by

$$\mathbf{n}_2 = \overrightarrow{d_1} \times \overrightarrow{d_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \mathbf{i}(-6-2) - \mathbf{j}(3-1) + \mathbf{k}(2+2) = -8\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

or $\mathbf{n}_2 = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$\therefore$ equation of the plane in scalar product form is

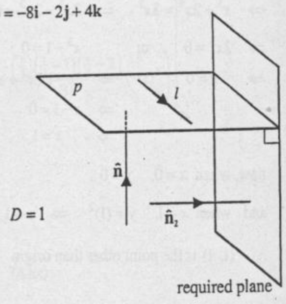
$\mathbf{r} \cdot (4\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = D$

as this plane contains the point $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$

$\therefore$ $(2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \cdot (4\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = D \Rightarrow 8 + 1 - 8 = D \Rightarrow D = 1$

$\therefore$ required equation of the plane is

$\mathbf{r} \cdot (4\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = 1$ or $4x + y - 2z = 1$ (Ans)



***Alternative Solution to part (ii):***

We have $\vec{d}_1 = \begin{pmatrix}1\\-2\\1\end{pmatrix}$, $\vec{d}_2 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, and given point $a = \begin{pmatrix}2\\1\\4\end{pmatrix}$

Equation of the plane with two direction vectors $\vec{d}_1$ and $\vec{d}_2$, containing the point a is given by

$$r = a + \lambda \vec{d}_1 + \mu \vec{d}_2 \quad \text{or} \quad r = \begin{pmatrix}2\\1\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\1\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\1\end{pmatrix} + \mu\begin{pmatrix}1\\2\\3\end{pmatrix}$$

$$\Rightarrow \quad x = 2 + \lambda + \mu \cdots\cdots\cdots(i)$$

$$y = 1 - 2\lambda + 2\mu \cdots\cdots\cdots(ii)$$

$$z = 4 + \lambda + 3\mu \cdots\cdots\cdots(iii)$$

from eq. (i), $\mu = x - 2 - \lambda$, put in eq. (ii)

$$y = 1 - 2\lambda + 2(x - 2 - \lambda) \quad \Rightarrow \quad y = 1 - 2\lambda + 2x - 4 - 2\lambda$$

$$\Rightarrow \quad y = 2x - 3 - 4\lambda \quad \Rightarrow \quad 4\lambda = 2x - 3 - y \quad \Rightarrow \quad \lambda = \frac{2x-3-y}{4} \cdots\cdots\cdots(iv)$$

$$\therefore \quad \mu = x - 2 - \frac{2x-3-y}{4}$$

$$\Rightarrow \quad \mu = \frac{4x-8-(2x-3-y)}{4} \quad \Rightarrow \quad \mu = \frac{2x-5+y}{4} \cdots\cdots\cdots(v)$$

putting the values of $\lambda$ and $\mu$ from eq. (iv) & eq. (v) into eq. (iii)

$$z = 4 + \left(\frac{2x-3-y}{4}\right) + 3\left(\frac{2x-5+y}{4}\right)$$

$$\Rightarrow \quad z = 4 + \frac{2x-3-y}{4} + \frac{6x-15+3y}{4}$$

$$\Rightarrow \quad z = \frac{16+2x-3-y+6x-15+3y}{4} \quad \Rightarrow \quad z = \frac{8x+2y-2}{4}$$

$$\Rightarrow \quad 4z = 8x + 2y - 2 \quad \Rightarrow \quad 8x + 2y - 4z = 2 \quad \text{or} \quad 4x + y - 2z = 1 \quad \text{(Ans)}$$

**8.** Let $f(x) = \dfrac{7x+4}{(2x+1)(x+1)^2}$

(i) Express $f(x)$ in partial fractions. [5]

(ii) Hence show that $\int_0^2 f(x)\,dx = 2 + \ln\dfrac{5}{3}$. [5]

*Suggested Solution:*

(i) $f(x) = \dfrac{7x+4}{(2x+1)(x+1)^2}$

as this is a proper fraction

$$\therefore \quad \frac{7x+4}{(2x+1)(x+1)^2} \equiv \frac{A}{2x+1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$\Rightarrow \quad 7x+4 \equiv A(x+1)^2 + B(2x+1)(x+1) + C(2x+1)$$

for $x = -1$

$$7(-1)+4 = A(-1+1)^2 + B(2(-1)+1)(-1+1) + C(2(-1)+1)$$

$$\Rightarrow \quad -7+4 = 0+0+C(-1) \quad \Rightarrow \quad -3 = -C \quad \Rightarrow \quad C=3$$

for $x = -\frac{1}{2}$

$$7(-\frac{1}{2})+4 = A(-\frac{1}{2}+1)^2 + B(2(-\frac{1}{2})+1)(-\frac{1}{2}+1) + C(2(-\frac{1}{2})+1)$$

$$\Rightarrow \quad -\frac{7}{2}+4 = A(\frac{1}{2})^2 + 0 + 0 \quad \Rightarrow \quad \frac{1}{2} = A(\frac{1}{4}) \quad \Rightarrow \quad A=2$$

for $x = 0$

$$7(0)+4 = A(0+1)^2 + B(2(0)+1)(0+1) + C(2(0)+1)$$

$$\Rightarrow \quad 4 = A(1)+B(1)+C(1) \quad \Rightarrow \quad 4 = A+B+C$$

putting values of $A$ and $C$

$$\Rightarrow \quad 4 = 2+B+3 \quad \Rightarrow \quad B=-1$$

$$\therefore \quad \frac{7x+4}{(2x+1)(x+1)^2} \equiv \frac{2}{2x+1} - \frac{1}{x+1} + \frac{3}{(x+1)^2} \quad \text{(Ans)}$$

(ii) $\text{L.H.S} = \int_0^2 f(x)\,dx$

$$\Rightarrow \int_0^2 \left( \frac{2}{2x+1} - \frac{1}{x+1} + \frac{3}{(x+1)^2} \right) dx$$

$$= \int_0^2 \frac{2}{2x+1}\,dx - \int_0^2 \frac{1}{x+1}\,dx + \int_0^2 \frac{3}{(x+1)^2}\,dx$$

$$= \Big[\ln|2x+1|\Big]_0^2 - \Big[\ln|x+1|\Big]_0^2 + 3\left[\frac{(x+1)^{-1}}{-1}\right]_0^2$$

$$= \Big[\ln|2(2)+1| - \ln|2(0)+1|\Big] - \Big[\ln|2+1| - \ln|0+1|\Big] - 3\left[\frac{1}{x+1}\right]_0^2$$

$$= [\ln 5 - \ln 1] - [\ln 3 - \ln 1] - 3\left[\frac{1}{2+1} - \frac{1}{0+1}\right]$$

$$= \ln 5 - \ln 3 - 3\left(\frac{1}{3} - 1\right) = \ln\left(\frac{5}{3}\right) - 3\left(-\frac{2}{3}\right) = 2 + \ln\left(\frac{5}{3}\right) = \text{R.H.S} \quad \textbf{(Shown)}$$

9. The complex number $u$ is given by

$$u = \frac{3+i}{2-i}$$

(i) Express $u$ in the form $x + iy$, where $x$ and $y$ are real. [3]

(ii) Find the modulus and argument of $u$. [2]

(iii) Sketch an Argand diagram showing the point representing the complex number $u$. Show on the same diagram the locus of the point representing the complex number $z$ such that $|z - u| = 1$. [3]

(iv) Using your diagram, calculate the least value of $|z|$ for points on this locus. [2]

*Suggested Solution:*

(i) $u = \dfrac{3+i}{2-i}$

realizing the denominator

$$u = \frac{3+i}{2-i} \times \frac{2+i}{2+i}$$

$$= \frac{6+3i+2i+i^2}{2^2 - i^2} = \frac{6+5i-1}{4-(-1)} = \frac{5+5i}{4+1} = \frac{5(1+i)}{5} = 1+i$$

$\therefore \quad u = 1 + i$ **(Ans)**

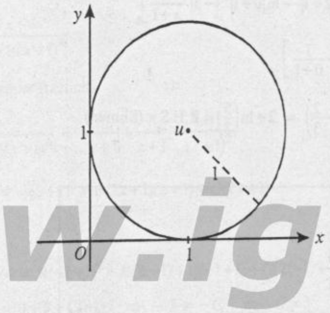
(ii) Using the result of part (i), we have

$$u = 1 + \mathrm{i}$$

$\therefore\ |u| = \sqrt{1^2 + 1^2} = \sqrt{2}$ **(Ans)**

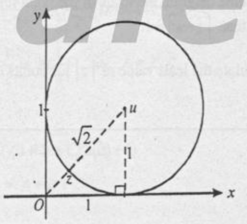
and $\arg(u) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$ radians **(Ans)**

(iii)



(iv) We have

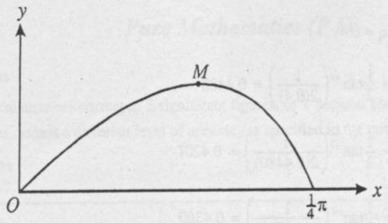
$|Ou| = \sqrt{2}$, and $|zu| = 1$



Note that $|z|$ is the distance of a variable point $z$ on the locus (circle) from the origin.

$\therefore$ Least value of $|z| = |Ou| - |zu| = \sqrt{2} - 1 = 0.414$ **(Ans)**

**10.**



The diagram shows the curve $y = x\cos 2x$ for $0 \le x \le \frac{1}{4}\pi$. The point $M$ is a maximum point.

(i) Show that the $x$-coordinate of $M$ satisfies the equation $1 = 2x\tan 2x$. [3]

(ii) The equation in part (i) can be rearranged in the form $x = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x}\right)$.

Use the iterative formula

$$x_{n+1} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_n}\right),$$

with initial value $x_1 = 0.4$, to calculate the $x$-coordinate of $M$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(iii) Use integration by parts to find the exact area of the region enclosed between the curve and the $x$-axis from 0 to $\frac{1}{4}\pi$. [5]

*Suggested Solution:*

(i) $y = x\cos 2x$

differentiating w.r.t $x$

$$\frac{d}{dx}(y) = \frac{d}{dx}(x\cos 2x)$$

$$= \cos 2x\frac{d}{dx}(x) + x\frac{d}{dx}(\cos 2x) = \cos 2x + x(-\sin 2x)(2)$$

$$= \cos 2x - 2x\sin 2x$$

for maxima or minima, $\frac{dy}{dx} = 0$

$\Rightarrow \cos 2x - 2x\sin 2x = 0$

$\Rightarrow 2x\sin 2x = \cos 2x$

$\Rightarrow \frac{2x\sin 2x}{\cos 2x} = 1 \Rightarrow 2x\tan 2x = 1$ (Shown)

(ii) $x_{n+1} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_n}\right)$,

Given that initial value $x_1 = 0.4$

$$\therefore \quad x_2 = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_1}\right) = \frac{1}{2}\tan^{-1}\left(\frac{1}{2(0.4)}\right) = 0.4480$$

$$x_3 = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_2}\right) = \frac{1}{2}\tan^{-1}\left(\frac{1}{2(0.4480)}\right) = 0.4201$$

$$x_4 = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_3}\right) = \frac{1}{2}\tan^{-1}\left(\frac{1}{2(0.4201)}\right) = 0.4360$$

$$x_5 = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_4}\right) = \frac{1}{2}\tan^{-1}\left(\frac{1}{2(0.4360)}\right) = 0.4268$$

$$x_6 = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_5}\right) = \frac{1}{2}\tan^{-1}\left(\frac{1}{2(0.4268)}\right) = 0.4321$$

$$x_7 = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_6}\right) = \frac{1}{2}\tan^{-1}\left(\frac{1}{2(0.4321)}\right) = 0.4291$$

$$x_8 = \frac{1}{2}\tan^{-1}\left(\frac{1}{2x_7}\right) = \frac{1}{2}\tan^{-1}\left(\frac{1}{2(0.4291)}\right) = 0.4308$$

$\therefore$ $x$-coordinate of $M = 0.43$ (to 2 dec.pl) **(Ans)**

(iii) Area, $A = \int_0^{\frac{\pi}{4}} x\cos 2x \, dx$

using integration by parts

$$A = \left[x(\frac{\sin 2x}{2})\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \frac{\sin 2x}{2}(1)\, dx$$

$$= \frac{1}{2}\left[x\sin 2x\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin 2x \, dx$$

$$= \frac{1}{2}\left[x\sin 2x\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\left[(-\frac{\cos 2x}{2})\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[x\sin 2x\right]_0^{\frac{\pi}{4}} + \frac{1}{4}\left[\cos 2x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{4}\sin 2(\frac{\pi}{4})\right) - (0\sin 2(0))\right] + \frac{1}{4}\left[\left(\cos 2(\frac{\pi}{4})\right) - (\cos 2(0))\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4}\sin(\frac{\pi}{2}) - 0\right] + \frac{1}{4}\left[\cos(\frac{\pi}{2}) - 1\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4}(1)\right] + \frac{1}{4}[0-1] = \frac{\pi}{8} - \frac{1}{4} = \frac{1}{8}(\pi - 2)$$

$\therefore$ required area $= \frac{1}{8}(\pi - 2)$ sq.units **(Ans)**

# June 2007 Paper 1

## *Pure Mathematics (P1)*

Answer all the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Find the value of the constant $c$ for which the line $y = 2x + c$ is a tangent to the curve $y^2 = 4x$. [4]

***Suggested Solution:***

$y = 2x + c \cdots\cdots\cdots$(i), $\qquad y^2 = 4x \cdots\cdots\cdots$(ii)

as the line is tangent to the curve, therefore solving eq. (i) and eq. (ii) simultaneously we have,

$$(2x+c)^2 = 4x$$
$$4x^2 + 4cx + c^2 = 4x$$
$$4x^2 + 4cx + c^2 - 4x = 0$$
$$4x^2 + (4c-4)x + c^2 = 0$$

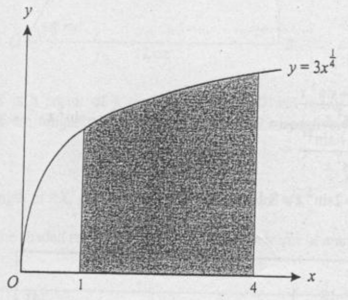
The above equation must have equal roots

$\Rightarrow$ Disc = 0

$$(4c-4)^2 - 4(4)(c^2) = 0$$
$$16c^2 - 32c + 16 - 16c^2 = 0$$
$$-32c = -16 \quad \Rightarrow \quad c = \frac{1}{2} \quad \text{(Ans)}$$

When a line is tangent to the curve then the quadratic equation emerging from their simultaneous solution has equal roots

**2.**



The diagram shows the curve $y = 3x^{\frac{1}{4}}$. The shaded region is bounded by the curve, the $x$-axis and the lines $x = 1$ and $x = 4$. Find the volume of the solid obtained when this shaded region is rotated completely about the $x$-axis, giving your answer in terms of $\pi$. [4]

*Suggested Solution:*

$$\text{Volume} = \pi\int_1^4 y^2\ \mathrm{d}x$$

$$= \pi\int_1^4 (3x^{\frac{1}{4}})^2\ \mathrm{d}x$$

$$= 9\pi\int_1^4 x^{\frac{1}{2}}\ \mathrm{d}x$$

$$= 9\pi\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4$$

$$= 9\pi\left(\frac{2}{3}\right)\left[x^{\frac{3}{2}}\right]_1^4$$

$$= 6\pi\left(4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$$= 6\pi\left((2^2)^{\frac{3}{2}} - 1\right)$$

$$= 6\pi[8-1]$$

$$= 42\pi\ \text{cube units}\quad \textbf{(Ans)}$$

Remember:

For definite integral, do not put the integration constant

3. Prove the identity $\frac{1-\tan^2 x}{1+\tan^2 x} \equiv 1-2\sin^2 x$. [4]

*Suggested Solution:*

$$\text{L.H.S} = \frac{1-\tan^2 x}{1+\tan^2 x}$$

$$= \frac{1-\frac{\sin^2 x}{\cos^2 x}}{1+\frac{\sin^2 x}{\cos^2 x}} = \frac{\frac{\cos^2 x-\sin^2 x}{\cos^2 x}}{\frac{\cos^2 x+\sin^{2x}}{\cos^2 x}} = \frac{\cos^2 x-\sin^2 x}{\cos^2 x+\sin^{2x}} = \cos^2 x-\sin^2 x$$

$$= (1-\sin^2 x)-\sin^2 x = 1-2\sin^2 x = \text{R.H.S}\quad \textbf{(Proved)}$$

4. Find the real roots of the equation $\frac{18}{x^4}+\frac{1}{x^2}=4$. [4]

**Suggested Solution:**

$$\frac{18}{x^4}+\frac{1}{x^2}=4$$

multiplying throughout by $x^4$, we have

$$18+x^2=4x^4 \quad \Rightarrow \quad 4x^4-x^2-18=0$$

let $x^2=t$

$$\Rightarrow \quad 4t^2-t-18=0$$

$$4t^2-9t+8t-18=0$$

$$t(4t-9)+2(4t-9)=0$$

$$(4t-9)(t+2)=0$$

$$t=\frac{9}{4} \quad \text{or} \quad t=-2$$

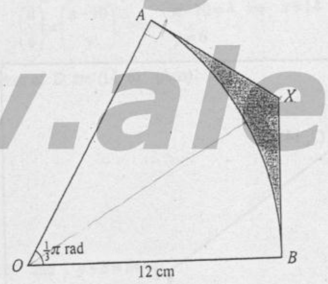
as $x^2=t$

$$\Rightarrow \quad x^2=\frac{9}{4} \quad \text{or} \quad x^2=-2 \text{ (ignored)}$$

$$x=\pm\frac{3}{2}=\pm 1.5$$

$\therefore \quad x=1.5$ or $-1.5$ **(Ans)**

5.



In the diagram, $OAB$ is a sector of a circle with centre $O$ and radius 12 cm. The lines $AX$ and $BX$ are tangents to the circle at $A$ and $B$ respectively. Angle $AOB=\frac{1}{3}\pi$ radians.

(i) Find the exact length of $AX$, giving your answer in terms of $\sqrt{3}$. [2]

(ii) Find the area of the shaded region, giving your answer in terms of $\pi$ and $\sqrt{3}$. [3]

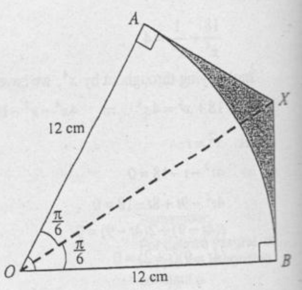
***Suggested Solution:***

(i) Given that $AX$ and $BX$ are tangents,

$\therefore$ $\Delta OAX$ and $\Delta OBX$ are right angled triangles.

In $\Delta OAX$, $\tan\frac{\pi}{6}=\frac{AX}{OA}$

$\Rightarrow AX = OA\times\tan\frac{\pi}{6} \Rightarrow AX=\frac{12}{\sqrt{3}}$ (Ans)



(ii) Area of $\Delta OAX = \frac{1}{2}\times OA\times AX = \frac{1}{2}\times 12\times\frac{12}{\sqrt{3}}=\frac{72}{\sqrt{3}}$

Area of the kite $OAXB$ = area of $\Delta OAX$ + area of $\Delta OBX$

$$=\frac{72}{\sqrt{3}}+\frac{72}{\sqrt{3}}=\frac{144}{\sqrt{3}}$$

Area of sector $\overset{\frown}{OAB}=\frac{1}{2}(12)^2\frac{\pi}{3}=24\pi$

now, area of shaded region = area of kite $OAXB$ − area of the sector $\overset{\frown}{OAB}$

$$=\frac{144}{\sqrt{3}}-24\pi$$

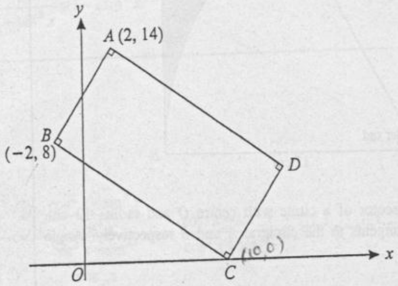
$$=\frac{144\sqrt{3}}{3}-24\pi = (48\sqrt{3}-24\pi) \text{ sq units (Ans)}$$

Remember:

- $\tan(\frac{\pi}{6})=\tan 30°=\frac{1}{\sqrt{3}}$
- $\Delta OAX$ and $\Delta OBX$ are equal in area.

  Line $OX$ bisects the angle $AOB$.
- area of sector $=\frac{1}{2}r^2\theta$

**6.**



The diagram shows a rectangle $ABCD$. The point $A$ is (2, 14), $B$ is (−2, 8) and $C$ lies on the $x$-axis. Find

(i) the equation of $BC$, [4]

(ii) the coordinates of $C$ and $D$. [3]

***Suggested Solution:***

(i) Gradient of $AB = \frac{14-8}{2+2} = \frac{6}{4} = \frac{3}{2}$

$\therefore$ gradient of $BC = -\frac{2}{3}$

Equation of $BC$ passing through the point $B(-2, 8)$ is:

$$y-8 = -\frac{2}{3}(x+2) \Rightarrow 3y-24 = -2x-4 \Rightarrow 2x+3y = 20 \quad \text{(Ans)}$$

(ii) $C$ is the point where $BC$ intersects $x$-axis..

$\therefore$ putting $y=0$ in the equation of $BC$, we have

$$2x+3(0) = 20 \Rightarrow 2x = 20 \Rightarrow x = 10$$

$\therefore$ coordinates of $C$ are $(10, 0)$ **(Ans)**

Let $D = (x, y)$

using vectors, $\overrightarrow{BA} = \overrightarrow{CD}$

$$\overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{OD} - \overrightarrow{OC}$$

$$\begin{pmatrix} 2+2 \\ 14-8 \end{pmatrix} = \begin{pmatrix} x-10 \\ y-0 \end{pmatrix}$$

$$\begin{pmatrix} 4 \\ 6 \end{pmatrix} = \begin{pmatrix} x-10 \\ y \end{pmatrix} \Rightarrow \begin{matrix} x-10 = 4 \Rightarrow x = 14 \\ y = 6 \end{matrix}$$

$\therefore$ coordinates of $D$ are $(14, 6)$ **(Ans)**

***Alternatively:***

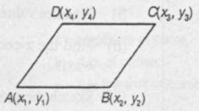
For point $D$:

$x-2 = 2+10$ and $y+8 = 14+0$

$x = 14$ $\quad$ $y = 6$

$\therefore$ $D(14, 6)$ (Ans)

For family of parallelograms, if $A$, $B$, $C$ and $D$ have the following coordinates



then $x_1 + x_3 = x_2 + x_4$

and $y_1 + y_3 = y_2 + y_4$

7. The second term of a geometric progression is 3 and the sum to infinity is 12.

(i) Find the first term of the progression. [4]

An arithmetic progression has the same first and second terms as the geometric progression.

(ii) Find the sum of the first 20 terms of the arithmetic progression. [3]

***Suggested Solution:***

(i) Given that, $T_2 = 3 \Rightarrow ar = 3 \Rightarrow r = \frac{3}{a}$

also given, $S_\infty = 12$

$$\Rightarrow \frac{a}{1-r} = 12$$

$$\frac{a}{1-\frac{3}{a}} = 12$$

$$\frac{a^2}{a-3} = 12$$

$$a^2 = 12a - 36$$

$$a^2 - 12a + 36 = 0$$

$$(a-6)(a-6) = 0$$

$$(a-6)^2 = 0$$

$$a = 6$$

$\therefore$ first term = 6 **(Ans)**

(ii) Given that, first term $a = 6$, and second term = 3

$\therefore$ common difference $= 3 - 6 = -3$

$$S_{20} = \frac{20}{2}[2(6) + (20-1)(-3)] = 10[12 + 19(-3)] = 10[-45] = -450 \quad \textbf{(Ans)}$$

The sum to infinity of a GP is:

$$S_\infty = \frac{a}{1-r}$$

Sum of $n$ terms of an A.P is:

$$S_n = \frac{n}{2}[2a + (n-1)d] \text{ and}$$

$$S_n = \frac{n}{2}[a + l]$$

where
$a$ = first term
$n$ = no. of terms.
$d$ = common difference
$l$ = the last term.

**8.** The function f is defined by $f(x) = a + b\cos 2x$, for $0 \le x \le \pi$. It is given that $f(0) = -1$ and $f\left(\frac{1}{2}\pi\right) = 7$.

(i) Find the values of $a$ and $b$. [3]

(ii) Find the $x$-coordinates of the points where the curve $y = f(x)$ intersects the $x$-axis. [3]

(iii) Sketch the graph of $y = f(x)$. [2]

***Suggested Solution:***

(i) $f(x) = a + b\cos 2x$

Given that, $f(0) = -1$

$\Rightarrow a + b\cos 2(0) = -1$

$a + b = -1 \cdots\cdots\cdots$(i)

also, $f(\frac{1}{2}\pi) = 7$

$\Rightarrow a + b\cos 2(\frac{1}{2}\pi) = 7$

$a + b\cos(\pi) = 7$

$a + b(-1) = 7$

$a - b = 7 \cdots\cdots\cdots$(ii)

solving eq. (i) and eq. (ii) simultaneously

$$\begin{array}{l} a+b=-1 \\ \underline{a-b=\ 7} \\ \underline{2a\quad =6} \end{array}$$

$\Rightarrow \quad a=\frac{6}{2}=3$ (Ans)

putting this value of $a$ in eq.(i),

$3+b=-1 \quad \Rightarrow \quad b=-4$ (Ans)

(ii) using the values of $a$ and $b$ found in part (i), the function f becomes

$f(x)=3-4\cos 2x$

For $x$-intercept, put $f(x)=0$

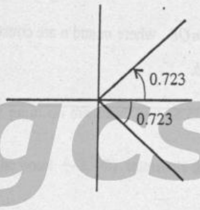
$\Rightarrow \quad 3-4\cos 2x=0$

$\cos 2x=\frac{3}{4}$

basic angle $\alpha=0.723$

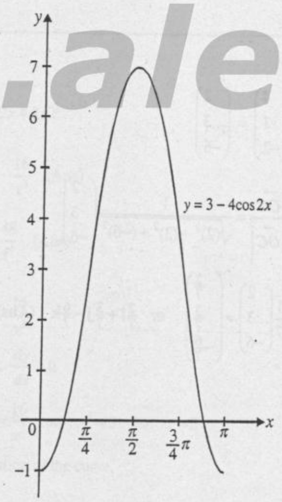
$\Rightarrow \quad 2x=0.723,\ 5.56$

$\therefore \quad x=0.36,\ 2.78$ radians (Ans)



Note that cos is +ve in (I) and (IV) quadrant.

(iii)



The period for $\sin x$ or $\cos x$ is from 0° to 360° and for $\tan x$, it is from 0° to 180°.

To sketch

$y=a\cos bx+c$ or

$y=a\sin bx+c$

where,

$a$ = amplitude of curve,

$b$ = No. of cycles,

$y=c$ is the axis of curve

1. Find the period of one cycle for $\cos x$ or $\sin x$, use $\frac{360}{b}$
2. Divide the period into four equal parts.
3. Find the corresponding values of $y$ and sketch.

**9.** Relative to an origin $O$, the position vectors of the points $A$ and $B$ are given by

$$\vec{OA} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} \text{ and } \vec{OB} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}.$$

(i) Given that $C$ is the point such that $\vec{AC} = 2\vec{AB}$, find the unit vector in the direction of $\vec{OC}$. [4]

The position vector of the point $D$ is given by $\vec{OD} = \begin{pmatrix} 1 \\ 4 \\ k \end{pmatrix}$, where $k$ is a constant, and it is given that $\vec{OD} = m\vec{OA} + n\vec{OB}$, where $m$ and $n$ are constants.

(ii) Find the values of $m$, $n$ and $k$. [4]

***Suggested Solution:***

(i) $\vec{AC} = 2\vec{AB}$

$$\Rightarrow \vec{OC} - \vec{OA} = 2(\vec{OB} - \vec{OA})$$

$$\Rightarrow \vec{OC} = 2\vec{OB} - 2\vec{OA} + \vec{OA}$$

$$\Rightarrow \vec{OC} = 2\vec{OB} - \vec{OA}$$

$$= 2\begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ -8 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$$

For any vector **v**, the unit vector $\hat{\mathbf{v}}$, is given as:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

unit vector in the direction of $\vec{OC} = \dfrac{\vec{OC}}{|\vec{OC}|} = \dfrac{1}{\sqrt{(2)^2 + (3)^2 + (-6)^2}}\begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}$

$$= \frac{1}{7}\begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} \\ \frac{3}{7} \\ -\frac{6}{7} \end{pmatrix} \text{ or } \tfrac{2}{7}\mathrm{i} + \tfrac{3}{7}\mathrm{j} - \tfrac{6}{7}\mathrm{k} \quad \textbf{(Ans)}$$

(ii) Given that, $\vec{OD} = m\vec{OA} + n\vec{OB}$

$$\begin{pmatrix} 1 \\ 4 \\ k \end{pmatrix} = m\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + n\begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 4 \\ k \end{pmatrix} = \begin{pmatrix} 4m + 3n \\ m + 2n \\ -2m - 4n \end{pmatrix}$$

$$\Rightarrow \quad 1 = 4m + 3n \cdots\cdots\cdots\text{(i)}$$

$$4 = m + 2n \cdots\cdots\cdots\text{(ii)}$$

$$k = -2m - 4n \cdots\cdots\cdots\text{(iii)}$$

from eq. (ii): $m = 4 - 2n$ ………(iv)

substitute into eq. (i): $1 = 4(4-2n)+3n$

$$1 = 16 - 8n + 3n$$
$$5n = 15$$
$$n = 3 \quad \text{(Ans)}$$

substitute $n = 3$ into eq. (iv): $m = 4 - 2(3) = -2$ (Ans)

putting the values of m and n in equation (iii), we have

$$k = -2(-2) - 4(3) = 4 - 12 = -8 \quad \text{(Ans)}$$

**10.** The equation of a curve is $y = 2x + \frac{8}{x^2}$.

(i) Obtain expressions for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. [3]

(ii) Find the coordinates of the stationary point on the curve and determine the nature of the stationary point. [3]

(iii) Show that the normal to the curve at the point $(-2, -2)$ intersects the $x$-axis at the point $(-10, 0)$. [3]

(iv) Find the area of the region enclosed by the curve, the $x$-axis and the lines $x = 1$ and $x = 2$. [3]

*Suggested Solution:*

(i) $y = 2x + \frac{8}{x^2} \Rightarrow y = 2x + 8x^{-2}$

$$\frac{dy}{dx} = 2 - 16x^{-3} = 2 - \frac{16}{x^3} \quad \text{(Ans)}$$

$$\frac{d^2y}{dx^2} = 0 + 48x^{-4} = \frac{48}{x^4} \quad \text{(Ans)}$$

Note that:

$\frac{d}{dx}x^n = nx^{n-1}$

(ii) From part (i), $\frac{dy}{dx} = 2 - \frac{16}{x^3}$

for stationary points, put $\frac{dy}{dx} = 0$

$$\Rightarrow 2 - \frac{16}{x^3} = 0 \Rightarrow 2 = \frac{16}{x^3} \Rightarrow x^3 = 8 \Rightarrow x = 2$$

putting $x = 2$ in the equation of the curve,

$$y = 2(2) + \frac{8}{(2)^2} = 4 + 2 = 6$$

$\therefore$ stationary point is $(2, 6)$ (Ans)

at $x = 2$, $\frac{d^2y}{dx^2} = \frac{48}{(2)^4} = 3 > 0$

$\therefore$ (2, 6) is a minimum point (Ans)

Nature:

If $\frac{d^2y}{dx^2} > 0$ $y$ is min.

If $\frac{d^2y}{dx^2} < 0$ $y$ is max.

(iii) gradient, $\dfrac{dy}{dx}=2-\dfrac{16}{x^3}$

at point $(-2,\ -2)$, $\dfrac{dy}{dx}=2-\dfrac{16}{(-2)^3}=2-\dfrac{16}{-8}=2+2=4$

$\therefore$ gradient of the normal $=-\dfrac{1}{4}$

equation of the normal at point $(-2,\ -2)$ with gradient $-\dfrac{1}{4}$ is:

$$y-(-2)=-\frac{1}{4}(x-(-2))$$

$$\Rightarrow \quad y+2=-\frac{1}{4}(x+2) \Rightarrow 4y+8=-x-2 \Rightarrow 4y+x+10=0$$

for $x$-intercept put $y=0$,

$$\Rightarrow 4(0)+x+10=0 \Rightarrow x=-10$$

$\therefore$ the normal cuts the $x$-axis at $(-10,\ 0)$ (Ans)

Gradient at any point $(x, y)$ on a curve is given by $\dfrac{dy}{dx}$

If $m_1$ and $m_2$ are the gradients of two perpendicular lines, then $m_1 \times m_2 = -1$

Equation of a line in point-slope form is: $y-y_1=m(x-x_1)$

(iv) $y=2x+\dfrac{8}{x^2} \Rightarrow y=2x+8x^{-2}$

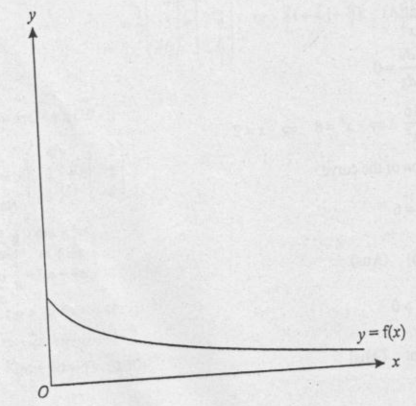
$$\text{Area}=\int_1^2 y\,dx=\int_1^2(2x+8x^{-2})\,dx$$

$$=\left[2\left(\frac{x^2}{2}\right)+8\left(\frac{x^{-1}}{-1}\right)\right]_1^2=\left[x^2-\frac{8}{x}\right]_1^2$$

$$=\left((2)^2-\frac{8}{2}\right)-\left((1)^2-\frac{8}{1}\right)$$

$$=(4-4)-(-7)=7 \text{ sq. units} \quad \text{(Ans)}$$

**11.**



The diagram shows the graph of $y = \mathrm{f}(x)$, where $\mathrm{f} : x \mapsto \dfrac{6}{2x+3}$ for $x \geq 0$.

(i) Find an expression, in terms of $x$, for $\mathrm{f}'(x)$ and explain how your answer shows that f is a decreasing function. [3]

(ii) Find an expression, in terms of $x$, for $\mathrm{f}^{-1}(x)$ and find the domain of $\mathrm{f}^{-1}$. [4]

(iii) Copy the diagram and, on your copy, sketch the graph of $y = \mathrm{f}^{-1}(x)$, making clear the relationship between the graphs. [2]

The function g is defined by $\mathrm{g} : x \mapsto \dfrac{1}{2}x$ for $x \geq 0$.

(iv) Solve the equation $\mathrm{fg}(x) = \dfrac{3}{2}$. [3]

***Suggested Solution:***

(i) $\mathrm{f}(x) = \dfrac{6}{2x+3} = 6(2x+3)^{-1}$

$$\mathrm{f}'(x) = 6\left(-(2x+3)^{-2} \times 2\right)$$

$$= -12(2x+3)^{-2} = -\frac{12}{(2x+3)^2} \quad \text{(Ans)}$$

we see that $(2x+3)^2$ is always positive for all values of $x$.

$\Rightarrow \ -\dfrac{12}{(2x+3)^2} < 0$ for all $x$.

$\therefore$ $\mathrm{f}(x)$ is a decreasing function. **(Ans)**

(ii) Let $\mathrm{f}(x) = y \ \Rightarrow \ \mathrm{f}^{-1}(y) = x$

$$\therefore \ y = \frac{6}{2x+3} \ \Rightarrow \ 2x+3 = \frac{6}{y} \ \Rightarrow \ 2x = \frac{6}{y} - 3 \ \Rightarrow \ x = \frac{1}{2}\left(\frac{6}{y} - 3\right)$$

$$\text{or } \ \mathrm{f}^{-1}(y) = \frac{1}{2}\left(\frac{6}{y} - 3\right) \ \Rightarrow \ \mathrm{f}^{-1}(x) = \frac{1}{2}\left(\frac{6}{x} - 3\right) \quad \text{(Ans)}$$

now, $\mathrm{f}(x) = \dfrac{6}{2x+3}$

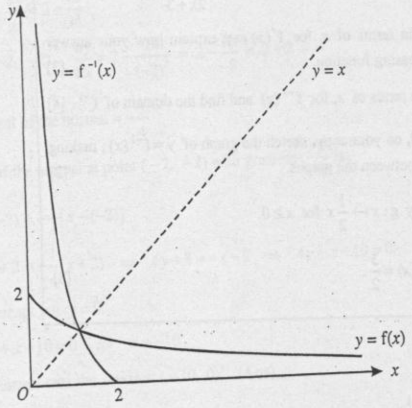
at $x = 0$, $\mathrm{f}(0) = \dfrac{6}{2(0)+3} = 2$, and from graph we see that as $x \to \infty$, $\mathrm{f}(x) \to 0$

$\therefore$ range of $\mathrm{f}(x)$ is: $0 < \mathrm{f}(x) \leq 2$

$\Rightarrow$ domain of $\mathrm{f}^{-1}(x)$: $0 < x \leq 2$ **(Ans)**

Remember that the range of f($x$) is equal to the domain of $\mathrm{f}^{-1}(x)$ and vice versa.

(iii)



$y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.

(iv)

$$fg(x) = \frac{3}{2}$$

$$f\left(\frac{1}{2}x\right) = \frac{3}{2}$$

$$\frac{6}{2\left(\frac{1}{2}x\right)+3} = \frac{3}{2}$$

$$\frac{6}{x+3} = \frac{3}{2}$$

$$12 = 3x + 9$$

$$3x = 3$$

$$x = 1 \quad \text{(Ans)}$$

The diagram shows the graph of $y = f(x)$, where $f : x \mapsto \frac{6}{2x+3}$ for $x \geq 0$.

(i) Find an expression, in terms of $x$, for $f'(x)$ and explain how your answer shows that f is a decreasing function. [3]

(ii) Find an expression, in terms of $x$, for $f^{-1}(x)$ and find the domain of $f^{-1}$. [4]

(iii) Copy the diagram and, on your copy, sketch the graph of $y = f^{-1}(x)$, making clear the relationship between the graphs. [2]

The function g is defined by $g : x \mapsto \frac{1}{2}x$ for $x \geq 0$.

(iv) Solve the equation $fg(x) = \frac{3}{2}$. [3]

***Suggested Solution:***

(i) $f(x) = \frac{6}{2x+3} = 6(2x+3)^{-1}$

$f'(x) = 6\left(-(2x+3)^{-2} \times 2\right)$

$= -12(2x+3)^{-2} = -\frac{12}{(2x+3)^2}$ (Ans)

we see that $(2x+3)^2$ is always positive for all values of $x$.

$\Rightarrow -\frac{12}{(2x+3)^2} < 0$ for all $x$.

$\therefore$ $f(x)$ is a decreasing function. **(Ans)**

(ii) Let $f(x) = y \Rightarrow f^{-1}(y) = x$

$\therefore y = \frac{6}{2x+3} \Rightarrow 2x+3 = \frac{6}{y} \Rightarrow 2x = \frac{6}{y} - 3 \Rightarrow x = \frac{1}{2}\left(\frac{6}{y} - 3\right)$

or $f^{-1}(y) = \frac{1}{2}\left(\frac{6}{y} - 3\right) \Rightarrow f^{-1}(x) = \frac{1}{2}\left(\frac{6}{x} - 3\right)$ (Ans)

now, $f(x) = \frac{6}{2x+3}$

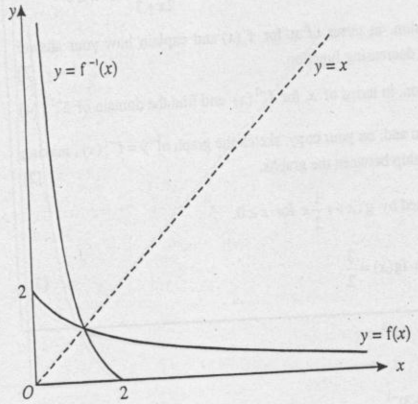
at $x = 0$, $f(0) = \frac{6}{2(0)+3} = 2$, and from graph we see that as $x \to \infty$, $f(x) \to 0$

$\therefore$ range of $f(x)$ is: $0 < f(x) \leq 2$

$\Rightarrow$ domain of $f^{-1}(x)$: $0 < x \leq 2$ (Ans)

Remember that the range of f(x) is equal to the domain of $f^{-1}(x)$ and vice versa.

(iii)



$y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.

(iv)

$$fg(x) = \frac{3}{2}$$

$$f\left(\frac{1}{2}x\right) = \frac{3}{2}$$

$$\frac{6}{2\left(\frac{1}{2}x\right)+3} = \frac{3}{2}$$

$$\frac{6}{x+3} = \frac{3}{2}$$

$$12 = 3x + 9$$

$$3x = 3$$

$$x = 1 \quad \text{(Ans)}$$

# June 2007 Paper 3

## Pure Mathematics (P3)

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Expand $(2+3x)^{-2}$ in ascending powers of $x$, up to and including the term in $x^2$, simplifying the coefficients. [4]

***Suggested Solution:***

$$(2+3x)^{-2}=\left[2\left(1+\frac{3}{2}x\right)\right]^{-2}=2^{-2}\left(1+\frac{3}{2}x\right)^{-2}=\frac{1}{4}\left(1+\frac{3}{2}x\right)^{-2}$$

Binomial Expansion:

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots$$

Using binomial expansion upto the term in $x^2$

$$\frac{1}{4}\left(1+\frac{3}{2}x\right)^{-2}=\frac{1}{4}\left[1+(-2)\left(\frac{3}{2}x\right)+\frac{(-2)(-2-1)}{2!}(\frac{3}{2}x)^2\right]$$

$$=\frac{1}{4}\left[1-3x+3(\frac{9}{4}x^2)\right]$$

$$=\frac{1}{4}\left(1-3x+\frac{27}{4}x^2\right)\quad\text{(Ans)}$$

**2.** The polynomial $x^3-2x+a$, where $a$ is a constant, is denoted by $\mathrm{p}(x)$. It is given that $(x+2)$ is a factor of $\mathrm{p}(x)$.

(i) Find the value of $a$. [2]

(ii) When $a$ has this value, find the quadratic factor of $\mathrm{p}(x)$. [2]

***Suggested Solution:***

(i) As $(x+2)$ is a factor of $\mathrm{p}(x)$

$\therefore\ \mathrm{p}(-2)=0$

$\Rightarrow\ (-2)^3-2(-2)+a=0$

$-8+4+a=0$

$a=4$ (Ans)

(ii) Using the result of part (i)

$p(x) = x^3 - 2x + 4$ and $(x+2)$ is a factor

using long division, we have

$$
\begin{array}{r|l}
 & x^2 - 2x + 2 \\
\hline
x+2 & x^3 - 2x + 4 \\
 & x^3 \quad + 2x^2 \\
 & - \qquad\quad - \\
\hline
 & -2x^2 - 2x + 4 \\
 & -2x^2 - 4x \\
 & + \qquad + \\
\hline
 & +2x + 4 \\
 & +2x + 4 \\
 & - \quad - \\
\hline
 & 0
\end{array}
$$

∴ quadratic factor of $p(x) = x^2 - 2x + 2$ (Ans)

***Alternative solution to part (ii)***

$(x+2)$ is a factor of $p(x) = x^3 - 2x + 4$.

Therefore using synthetic division,

$$
\begin{array}{r|rrr|rl}
 & x^3 & x^2 & x^1 & x^0 & \\
-2 & 1 & 0 & -2 & 4 & \\
 & & -2 & 4 & -4 & \text{adding} \\
\hline
 & 1 & -2 & 2 & 0 &
\end{array}
$$

⇒ Quadratic factor $= x^2 - 2x + 2$ (Ans)

**3.** The equation of a curve is $y = x \sin 2x$, where $x$ is in radians. Find the equation of the tangent to the curve at the point where $x = \frac{1}{4}\pi$. [4]

***Suggested Solution:***

$$y = x \sin 2x$$

gradient, $\frac{dy}{dx} = \sin 2x(1) + x(\cos 2x)(2)$

$$= \sin 2x + 2x \cos 2x$$

at $x = \frac{\pi}{4}$, $\frac{dy}{dx} = \sin 2(\frac{\pi}{4}) + 2(\frac{\pi}{4}) \cos 2(\frac{\pi}{4})$

$$= \sin(\frac{\pi}{2}) + (\frac{\pi}{2}) \cos(\frac{\pi}{2}) = 1 + \frac{\pi}{2}(0) = 1$$

∴ gradient, $m = 1$

also when $x = \frac{\pi}{4}$, $\quad y = \frac{\pi}{4}\sin 2(\frac{\pi}{4}) = \frac{\pi}{4}\sin(\frac{\pi}{2}) = \frac{\pi}{4}$

$\therefore$ equation of tangent at point $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ with gradient, m = 1 is

$$y - \frac{\pi}{4} = 1(x - \frac{\pi}{4}) \quad \Rightarrow \quad y - x = 0 \quad \textbf{(Ans)}$$

**4.** Using the substitution $u = 3^x$, or otherwise, solve, correct to 3 significant figures, the equation

$$3^x = 2 + 3^{-x}.$$

[6]

*Suggested Solution:*

$3^x = 2 + 3^{-x} \quad \Rightarrow \quad 3^x = 2 + \frac{1}{3^x}$

given that $u = 3^x$

$\Rightarrow \qquad u = 2 + \frac{1}{u}$

$u^2 = 2u + 1$

$u^2 - 2u - 1 = 0$

applying quadratic formula

$$u = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2}$$

$\therefore \quad u = \frac{2+\sqrt{8}}{2} \qquad$ or $\qquad u = \frac{2-\sqrt{8}}{2}$

$u = 2.414 \qquad\qquad u = -0.414$

$\Rightarrow \quad 3^x = 2.414 \qquad\qquad 3^x = -0.414$ (impossible)

taking log on both sides

$\log 3^x = \log 2.414$

$x \log 3 = \log 2.414$

$$x = \frac{\log 2.414}{\log 3} = 0.802 \quad \textbf{(Ans)}$$

**5.** (i) Express $\cos\theta + (\sqrt{3})\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$, giving the exact values of $R$ and $\alpha$. [3]

(ii) Hence show that $\int_0^{\frac{1}{2}\pi} \frac{1}{(\cos\theta + (\sqrt{3})\sin\theta)^2}\, d\theta = \frac{1}{\sqrt{3}}$ [4]

***Suggested Solution:***

(i) $R\cos(\theta-\alpha) \equiv \cos\theta + (\sqrt{3})\sin\theta$

$R\cos\theta\cos\alpha + R\sin\theta\sin\alpha \equiv \cos\theta + \sqrt{3}\,\sin\theta$

comparing the coefficients of $\cos\theta$ and $\sin\theta$, we have

$$R\cos\alpha = 1\cdots\cdots(\text{i}) \quad \text{and} \quad R\sin\alpha = \sqrt{3}\cdots\cdots(\text{ii})$$

eq. (ii) ÷ eq.(i) gives

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{\sqrt{3}}{1} \Rightarrow \tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$$

squaring and adding eq. (i) & eq. (ii) gives

$$(R\cos\alpha)^2 + (R\sin\alpha)^2 = 1^2 + (\sqrt{3})^2$$

$$\Rightarrow R^2(\sin^2\alpha + \cos^2) = 4 \Rightarrow R^2 = 4 \Rightarrow R = 2$$

$$\therefore \cos\theta + (\sqrt{3})\sin\theta = 2\cos(\theta - \tfrac{\pi}{3}) \quad \textbf{(Ans)}$$

(ii) $$\text{L.H.S} = \int_0^{\frac{1}{2}\pi} \frac{1}{(\cos\theta+(\sqrt{3})\sin\theta)^2}\,d\theta$$

$$= \int_0^{\frac{1}{2}\pi} \frac{1}{\left(2\cos(\theta-\frac{\pi}{3})\right)^2}\,d\theta$$

$$= \int_0^{\frac{1}{2}\pi} \frac{1}{4\cos^2(\theta-\frac{\pi}{3})}\,d\theta$$

$$= \frac{1}{4}\int_0^{\frac{1}{2}\pi} \sec^2(\theta-\tfrac{\pi}{3})\,d\theta$$

$$= \frac{1}{4}\Big[\tan(\theta-\tfrac{\pi}{3})\Big]_0^{\frac{1}{2}\pi}$$

$$= \frac{1}{4}\left(\tan(\tfrac{\pi}{2}-\tfrac{\pi}{3})\right) - \left(\tan(0-\tfrac{\pi}{3})\right)$$

$$= \frac{1}{4}\left(\tan(\tfrac{\pi}{6}) - \tan(-\tfrac{\pi}{3})\right)$$

$$= \frac{1}{4}\left(\frac{1}{\sqrt{3}} + \sqrt{3}\right)$$

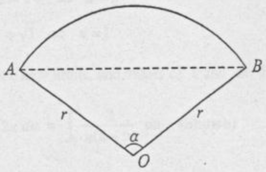
$$= \frac{1}{4}\left(\frac{1+3}{\sqrt{3}}\right)$$

$$= \frac{1}{4}\left(\frac{4}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}} \quad \textbf{(Shown)}$$

Note that:

- $\int \sec^2\theta\, d\theta = \tan\theta + K$
- $\tan(-\theta) = -\tan\theta$

**6.**



The diagram shows a sector $AOB$ of a circle with centre $O$ and radius $r$. The angle $AOB$ is $\alpha$ radians, where $0 < \alpha < \pi$. The area of triangle $AOB$ is half the area of the sector.

(i) Show that $\alpha$ satisfies the equation

$$x = 2\sin x. \qquad [2]$$

(ii) Verify by calculation that $\alpha$ lies between $\frac{1}{2}\pi$ and $\frac{2}{3}\pi$. [2]

(iii) Show that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{1}{3}(x_n + 4\sin x_n)$$

converges, then it converges to a root of the equation in part (i). [2]

(iv) Use this iterative formula, with initial value $x_1 = 1.8$, to find $\alpha$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

*Suggested Solution:*

(i) Area of $\Delta AOB = \frac{1}{2} \times r \times r \times \sin\alpha = \frac{1}{2}r^2\sin\alpha$

area of the sector $\overset{\frown}{AOB} = \frac{1}{2}r^2\alpha$

given that

the area of $\Delta AOB = \frac{1}{2}$(area of the sector $\overset{\frown}{AOB}$)

$$\Rightarrow \qquad \frac{1}{2}r^2\sin\alpha = \frac{1}{2}\left(\frac{1}{2}r^2\alpha\right)$$

$$\frac{1}{2}r^2\sin\alpha = \frac{1}{4}r^2\alpha$$

$$\sin\alpha = \frac{1}{2}\alpha$$

$$\alpha = 2\sin\alpha$$

$\therefore$ $\alpha$ satisfies the equation $x = 2\sin x$ **(Shown)**

(ii) Let $f(x) = x - 2\sin x$

Now, $f(\frac{\pi}{2}) = \frac{\pi}{2} - 2\sin(\frac{\pi}{2}) = \frac{\pi}{2} - 2 = -0.4292 \quad < 0$

$f(\frac{2}{3}\pi) = \frac{2}{3}\pi - 2\sin(\frac{2}{3}\pi) = 2.094 - 2(0.866) = 0.362 \quad > 0$

$\therefore$ the change of sign indicates that $\alpha$ lies between $\frac{1}{2}\pi$ and $\frac{2}{3}\pi$ **(Verified)**

Note:
Change the mode of your calculator to radian.

(iii) $x_{n+1} = \frac{1}{3}(x_n + 4\sin x_n)$

Removing the subscripts, we have

$$x = \frac{1}{3}(x + 4\sin x)$$

$$3x = x + 4\sin x$$

$$2x = 4\sin x$$

$$x = 2\sin x$$

which is the same equation as in part (i) **(Ans)**

(iv) $x_{n+1} = \frac{1}{3}(x_n + 4\sin x_n)$

Given that $x_1 = 1.8$

Note:
Change the mode of your calculator to radian.

$$\therefore \quad x_2 = \frac{1}{3}(x_1 + 4\sin x_1) = \frac{1}{3}(1.8 + 4\sin 1.8) = 1.8985$$

$$x_3 = \frac{1}{3}((1.8985 + 4\sin(1.8985)) = 1.8952$$

$$x_4 = \frac{1}{3}(1.8952 + 4\sin(1.8952)) = 1.8955$$

$$x_5 = \frac{1}{3}(1.8955 + 4\sin(1.8955)) = 1.8955$$

$\therefore \quad \alpha = 1.90$ (to 2 dp) **(Ans)**

**7.** Let $I = \int_1^4 \frac{1}{x(4-\sqrt{x})}\,dx.$

(i) Use the substitution $u = \sqrt{x}$ to show that $I = \int_1^2 \frac{2}{u(4-u)}\,du.$ [3]

(ii) Hence show that $I = \frac{1}{2}\ln 3.$ [6]

***Suggested Solution:***

(i) $I = \int_1^4 \frac{1}{x(4-\sqrt{x})}\,dx.$

given substitution is

$$u = \sqrt{x}$$

differentiating *w.r.t. x*

$$\frac{d}{dx}(u) = \frac{d}{dx}(\sqrt{x}) \Rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow \frac{du}{dx} = \frac{1}{2u} \Rightarrow dx = 2u\,du$$

for limits,

when $x=4 \Rightarrow u=\sqrt{4} \Rightarrow u=2$

when $x=1 \Rightarrow u=\sqrt{1} \Rightarrow u=1$

Now, substituting the new limits, and values of $x$ and $dx$, the integral becomes

$$I=\int_1^2 \frac{1}{u^2(4-u)}\times 2u\,du = \int_1^2 \frac{2}{u(4-u)}\,du \quad \textbf{(Shown)}$$

(ii) $I=\int_1^2 \frac{2}{u(4-u)}\,du$

using partial fractions

$$\frac{2}{u(4-u)} \equiv \frac{A}{u}+\frac{B}{4-u}$$

$$\Rightarrow \quad 2 \equiv A(4-u)+Bu$$

for $u=0$

$$2=A(4-0)+B(0) \quad \Rightarrow \quad 2=4A \quad \Rightarrow \quad A=\frac{1}{2}$$

for $u=4$

$$2=A(4-4)+B(4) \quad \Rightarrow \quad 2=4B \quad \Rightarrow \quad B=\frac{1}{2}$$

$$\therefore \quad \frac{2}{u(4-u)} \equiv \frac{1}{2u}+\frac{1}{2(4-u)}$$

now, $I=\int_1^2 \frac{2}{u(4-u)}\,du$

$$=\int_1^2 \left(\frac{1}{2u}+\frac{1}{2(4-u)}\right)du$$

$$=\frac{1}{2}\int_1^2 \frac{1}{u}\,du + \frac{1}{2}\int_1^2 \frac{1}{4-u}\,du$$

$$=\frac{1}{2}[\ln u]_1^2+\frac{1}{2}[-\ln(4-u)]_1^2$$

$$=\frac{1}{2}(\ln 2-\ln 1)-\frac{1}{2}(\ln(4-2)-\ln(4-1))$$

$$=\frac{1}{2}(\ln 2-0)-\frac{1}{2}(\ln 2-\ln 3)$$

$$=\frac{1}{2}\ln 2-\frac{1}{2}\ln 2+\frac{1}{2}\ln 3=\frac{1}{2}\ln 3 \quad \textbf{(Shown)}$$

5. The complex number $\frac{2}{-1+\mathrm{i}}$ is denoted by $u$.

(i) Find the modulus and argument of $u$ and $u^2$. [6]

(ii) Sketch an Argand diagram showing the points representing the complex numbers $u$ and $u^2$. Shade the region whose points represent the complex numbers $z$ which satisfy both the inequalities $|z|<2$ and $|z-u^2|<|z-u|$. [4]

*Suggested Solution:*

(i) Given that $u = \frac{2}{-1+i}$

realizing the denominator

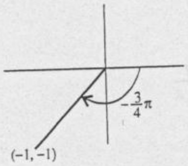
$$u = \frac{2}{-1+i} \times \frac{-1-i}{-1-i}$$

$$= \frac{2(-1-i)}{(-1)^2 - (i)^2} = \frac{-2-2i}{1+1} = \frac{2(-1-i)}{2} = -1-i$$

$\therefore \quad u = -1-i$

now, $|u| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$ (Ans)

$\arg u = \tan^{-1}\left(\frac{-1}{-1}\right) = -\frac{3}{4}\pi$ (Ans)

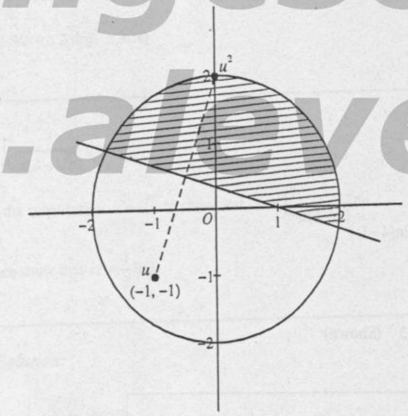


$u^2 = (-1-i)^2 = 1+2i+i^2 = 1+2i-1 = 2i$

$\therefore \quad |u^2| = \sqrt{(2)^2} = 2$ (Ans)

$\arg(u^2) = \frac{1}{2}\pi$ (Ans)

(ii)

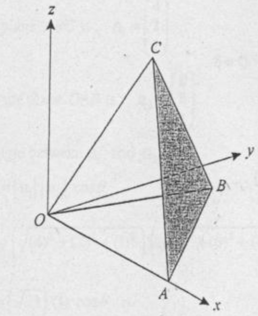


Remember:

- $i = \sqrt{-1}$
- If $z = a+ib$, then $|z| = \sqrt{a^2+b^2}$

The range of principal argument $\alpha$ is $-\pi < \alpha \leq \pi$

9.



The diagram shows a set of rectangular axes $Ox$, $Oy$ and $Oz$, and three points $A$, $B$ and $C$ with position vectors $\overrightarrow{OA} = \begin{pmatrix}2\\0\\0\end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix}1\\2\\0\end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix}1\\1\\2\end{pmatrix}$.

(i) Find the equation of the plane $ABC$, giving your answer in the form $ax + by + cz = d$.. [6]

(ii) Calculate the acute angle between the planes $ABC$ and $OAB$. [4]

***Suggested Solution:***

(i) Let us first find two direction vectors $\vec{d}_1$ and $\vec{d}_2$ parallel to the plane $ABC$

Let $\vec{d}_1 = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}1\\2\\0\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\2\\0\end{pmatrix}$

and $\vec{d}_2 = \overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix}1\\1\\2\end{pmatrix} - \begin{pmatrix}2\\0\\0\end{pmatrix} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$

Now, the normal vector $\mathbf{n}$ which is perpendicular to the plane $ABC$ is given by

$$\mathbf{n} = \overline{d_1} \times \overline{d_2} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 2 & 0\\ -1 & 1 & 2\end{vmatrix} = \mathbf{i}(4-0) - \mathbf{j}(-2-0) + \mathbf{k}(-1+2) = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$\therefore \quad \mathbf{n} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ or $\begin{pmatrix}4\\2\\1\end{pmatrix}$

equation of the plane in scalar product form is

$$\mathbf{r} \cdot \begin{pmatrix}4\\2\\1\end{pmatrix} = D$$

as point $A(2, 0, 0)$ lies on the plane

$$\Rightarrow \begin{pmatrix}2\\0\\0\end{pmatrix} \cdot \begin{pmatrix}4\\2\\1\end{pmatrix} = D \quad \Rightarrow \quad D = 8+0+0 \quad \Rightarrow \quad D = 8$$

$\therefore$ required equation of the plane is

$$\mathbf{r} \cdot \begin{pmatrix}4\\2\\1\end{pmatrix} = 8 \quad \text{or} \quad 4x+2y+z=8 \quad \text{(Ans)}$$

***Alternative Solution to part*** **(i):**

we have $\quad \overrightarrow{AB} = \begin{pmatrix}-1\\2\\0\end{pmatrix} \quad$ and $\quad \overrightarrow{AC} = \begin{pmatrix}-1\\1\\2\end{pmatrix}$

equation of the plane $ABC$ is given by

$$\mathbf{r} = \overrightarrow{OA} + \lambda \overrightarrow{AB} + \mu \overrightarrow{AC}$$

$$\Rightarrow \mathbf{r} = \begin{pmatrix}2\\0\\0\end{pmatrix} + \lambda \begin{pmatrix}-1\\2\\0\end{pmatrix} + \mu \begin{pmatrix}-1\\1\\2\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2-\lambda-\mu\\0+2\lambda+\mu\\0+0\lambda+2\mu\end{pmatrix}$$

$$\Rightarrow x = 2-\lambda-\mu \cdots\cdots\cdots \text{(i)}$$

$$y = 2\lambda+\mu \cdots\cdots\cdots \text{(ii)}$$

$$z = 2\mu \cdots\cdots\cdots \text{(iii)}$$

to eliminate $\lambda$ and $\mu$, equation (ii) can be written as

$$y = 2\lambda+\mu \quad \Rightarrow \quad \lambda = \frac{1}{2}(y-\mu)$$

from eq.(iii), $\quad z = 2\mu \quad \Rightarrow \quad \mu = \frac{z}{2}$

$$\therefore \quad \lambda = \frac{1}{2}\left(y-\frac{z}{2}\right) \quad \Rightarrow \quad \lambda = \frac{y}{2}-\frac{z}{4}$$

putting the value of $\lambda$ and $\mu$ in eq. (i)

$$x = 2-\left(\frac{y}{2}-\frac{z}{4}\right)-\left(\frac{z}{2}\right)$$

$$\Rightarrow \quad x = 2-\frac{y}{2}+\frac{z}{4}-\frac{z}{2} \quad \Rightarrow \quad 4x = 8-2y+z-2z \quad \Rightarrow \quad 4x+2y+z=8 \quad \text{(Ans)}$$

(ii) Angle between the two planes is the acute angle between their respective normals.

normal to the plane $ABC$ is: $\mathbf{n}_1 = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$

and normal to the plane $OAB$ is: $\mathbf{n}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

As $O$, $A$ and $B$, lie in the $xy$-plane, therefore any vector in the $z$ direction will act as the normal vector to the plane $OAB$.

let $\theta$ be the angle between $\mathbf{n}_1$ and $\mathbf{n}_2$

$$\therefore \quad \mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1||\mathbf{n}_2| \cos\theta$$

$$\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \left(\sqrt{(4)^2 + (2)^2 + (1)^2}\right)\left(\sqrt{(0)^2 + (0)^2 + (1)^2}\right) \cos\theta$$

$$0 + 0 + 1 = \left(\sqrt{21}\right)(1) \cos\theta$$

$$\cos\theta = \frac{1}{\sqrt{21}} \quad \Rightarrow \quad \theta = 77.39° \approx 77.4° \quad \textbf{(Ans)}$$

**10.** A model for the height, $h$ metres, of a certain type of tree at time $t$ years after being planted assumes that, while the tree is growing, the rate of increase in height is proportional to $(9 - h)^{\frac{1}{3}}$. It is given that, when $t = 0$, $h = 1$ and $\frac{\mathrm{d}h}{\mathrm{d}t} = 0.2$.

(i) Show that $h$ and $t$ satisfy the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0.1(9 - h)^{\frac{1}{3}}.$$ [2]

(ii) Solve this differential equation, and obtain an expression for $h$ in terms of $t$. [7]

(iii) Find the maximum height of the tree and the time taken to reach this height after planting. [2]

(iv) Calculate the time taken to reach half the maximum height. [1]

*Suggested Solution:*

(i) Given that

$$\frac{\mathrm{d}h}{\mathrm{d}t} \propto (9 - h)^{\frac{1}{3}}$$

$$\Rightarrow \quad \frac{\mathrm{d}h}{\mathrm{d}t} = k(9 - h)^{\frac{1}{3}}, \quad \text{where } k \text{ is the constant of variation}$$

when $h = 1$, $\frac{\mathrm{d}h}{\mathrm{d}t} = 0.2$

$$\Rightarrow \quad 0.2 = k(9 - 1)^{\frac{1}{3}} \quad \Rightarrow \quad 0.2 = k(8)^{\frac{1}{3}} \quad \Rightarrow \quad 0.2 = 2k \quad \Rightarrow \quad k = 0.1$$

$$\therefore \quad \frac{\mathrm{d}h}{\mathrm{d}t} = 0.1(9 - h)^{\frac{1}{3}} \quad \textbf{(Shown)}$$

(ii) $\frac{dh}{dt} = 0.1(9-h)^{\frac{1}{3}}$.

$\Rightarrow \quad (9-h)^{-\frac{1}{3}}\, dh = 0.1\, dt$

integrating both sides

$\int (9-h)^{-\frac{1}{3}}\, dh = 0.1 \int dt$

$\frac{(9-h)^{\frac{2}{3}}}{\left(\frac{2}{3}\right)(-1)} = 0.1t + C,$ where $C$ is the constant of integration

$-\frac{3}{2}(9-h)^{\frac{2}{3}} = 0.1t + C \cdots\cdots\cdots \text{(i)}$

when $t = 0, \ h = 1$

$\Rightarrow \quad -\frac{3}{2}(9-1)^{\frac{2}{3}} = 0.1(0) + C$

$\Rightarrow \quad -\frac{3}{2}(8)^{\frac{2}{3}} = C \quad \Rightarrow \quad -\frac{3}{2}(2^3)^{\frac{2}{3}} = C \quad \Rightarrow \quad -\frac{3}{2}(4) = C \quad \Rightarrow \quad C = -6$

$\therefore$ equation (i) becomes

$-\frac{3}{2}(9-h)^{\frac{2}{3}} = 0.1t - 6$

$\Rightarrow \quad (9-h)^{\frac{2}{3}} = -\frac{2}{3}\left(\frac{t}{10} - 6\right)$

$\Rightarrow \quad (9-h)^{\frac{2}{3}} = -\frac{t}{15} + 4$

$\Rightarrow \quad (9-h)^{\frac{2}{3}} = 4 - \frac{t}{15}$

$\Rightarrow \quad 9 - h = \left(4 - \frac{t}{15}\right)^{\frac{3}{2}}$

$h = 9 - \left(4 - \frac{t}{15}\right)^{\frac{3}{2}}$ **(Ans)**

(iii) From part (ii), height $h$ is:

$h = 9 - \left(4 - \frac{t}{15}\right)^{\frac{3}{2}}$

height of the tree is maximum when $\left(4 - \frac{t}{15}\right)^{\frac{3}{2}} = 0$

$\therefore$ maximum height $= 9$ metres, **(Ans)**

This maximum height is reached when

$\left(4 - \frac{t}{15}\right)^{\frac{3}{2}} = 0 \quad \Rightarrow \quad 4 - \frac{t}{15} = 0 \quad \Rightarrow \quad t = 60$ years. **(Ans)**

(iv) From part (ii), height $h$ is:

$$h = 9 - \left(4 - \frac{t}{15}\right)^{\frac{3}{2}}$$

given height is, $h = \frac{9}{2}$ metres

$$\therefore \quad \frac{9}{2} = 9 - \left(4 - \frac{t}{15}\right)^{\frac{3}{2}}$$

$$\Rightarrow \quad \left(4 - \frac{t}{15}\right)^{\frac{3}{2}} = 9 - \frac{9}{2}$$

$$\left(4 - \frac{t}{15}\right)^{\frac{3}{2}} = \frac{9}{2}$$

$$4 - \frac{t}{15} = \left(\frac{9}{2}\right)^{\frac{2}{3}}$$

$$4 - \frac{t}{15} = 2.726$$

$$\frac{t}{15} = 1.274$$

$$t = 19.115 \approx 19.1$$

$\therefore$ req. time = 19.1 metres (Ans)

# November 2007 Paper 1

## Pure Mathematics (P1)

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Determine the set of values of the constant $k$ for which the line $y = 4x + k$ does not intersect the curve $y = x^2$. [3]

***Suggested Solution:***

Solving the equation of the line and the curve simultaneously we have,

$$4x + k = x^2 \quad \Rightarrow \quad x^2 - 4x - k = 0$$

given that the line does not intersect the curve, therefore the above quadratic equation has imaginary roots.

i.e. Discriminent, $b^2 - 4ac < 0$

$\Rightarrow \quad (-4)^2 - 4(1)(-k) < 0$

$\Rightarrow \quad 16 + 4k < 0 \quad \Rightarrow \quad k < -4$ **Ans.**

When a line does not intersects the curve then the quadratic equation emerging from their si-multaneous solution has no real roots.

**2.** Find the area of the region enclosed by the curve $y = 2\sqrt{x}$, the $x$-axis and the lines $x = 1$ and $x = 4$. [4]

***Suggested Solution:***

Area, $A = \int_1^4 y \, dx$

$\Rightarrow \quad A = \int_1^4 2\sqrt{x} \, dx$

$$= 2\int_1^4 x^{\frac{1}{2}} \, dx$$

$$= 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^4 = \frac{4}{3}\left[x^{\frac{3}{2}}\right]_1^4 = \frac{4}{3}\left[4^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = \frac{4}{3}[8-1] = \frac{28}{3} = 9\frac{1}{3} \text{ sq. units} \quad \textbf{Ans.}$$

Remember:

For definite integral, do not put the constant of integration.

**3.** (i) Find the first three terms in the expansion of $(2 + u)^5$ in ascending powers of $u$. [3]

(ii) Use the substitution $u = x + x^2$ in your answer to part (i) to find the coefficient of $x^2$ in the expansion of $(2 + x + x^2)^5$. [2]

a line does not
ts the curve then
adratic equation
g from their si-
ous solution has
roots.

nber:
efinite integral, do
ut the constant of
ation.

***Suggested Solution:***

(i) $(2+u)^5$

$= {}^5C_0(2)^5 + {}^5C_1(2)^4(u)^1 + {}^5C_2(2)^3(u)^2$

$= 32 + 80u + 80u^2$ **Ans.**

(ii) Using the result of part (i)

$(2+u)^5 = 32 + 80u + 80u^2$

given substitution is: $u = x + x^2$

$\therefore \quad (2+x+x^2) = 32 + 80(x+x^2) + 80(x+x^2)^2$

$= 32 + 80x + 80x^2 + 80(x^2 + 2x^3 + x^4)$

$= 32 + 80x + 80x^2 + 80x^2 + 160x^3 + 80x^4$

$= 32 + 80x + 160x^2 + 160x^3 + 80x^4$

$\therefore$ coefficient of $x^2 = 160$ **Ans.**

Binomial theorem:

$(a+b)^n = {}^nC_0\,a^n + {}^nC_1\,a^{n-1}b + {}^nC_2\,a^{n-2}b^2 + \dots + {}^nC_r\,a^{n-r}b^r + \dots + {}^nC_n\,b^n$

4. The 1st term of an arithmetic progression is $a$ and the common difference is $d$, where $d \neq 0$.

(i) Write down expressions, in terms of $a$ and $d$, for the 5th term and the 15th term. [1]

The 1st term, the 5th term and the 15th term of the arithmetic progression are the first three terms of a geometric progression.

(ii) Show that $3a = 8d$. [3]

(iii) Find the common ratio of the geometric progression. [2]

***Suggested Solution:***

(i) Using $T_n = a + (n-1)d$, we have

$T_5 = a + (5-1)d \Rightarrow T_5 = a + 4d$ **Ans.**

and $T_{15} = a + (15-1)d \Rightarrow T_{15} = a + 14d$ **Ans.**

(ii) Given that the 1st, 5th and 15th terms of above A.P. are the first three terms of a G.P.

$\therefore \quad a, \quad a+4d, \quad a+14d$ are in $G.P$

common ratio, $r = \dfrac{a+4d}{a}$ or $\dfrac{a+14d}{a+4d}$

$\Rightarrow \quad \dfrac{a+4d}{a} = \dfrac{a+14d}{a+4d}$

$(a+4d)^2 = a(a+14d)$

$a^2 + 8ad + 16d^2 = a^2 + 14ad$

$16d^2 - 6ad = 0$

$2d(8d - 3a) = 0$

either $2d = 0$ or $8d - 3a = 0$

since $d$ cannot be zero

$\therefore \quad 8d - 3a = 0 \Rightarrow 3a = 8d$ **Shown.**

(iii) From part (ii), $3a = 8d \Rightarrow d = \frac{3}{8}a$

common ratio, $r = \frac{a+4d}{a}$

putting the value of $d$,

$$r = \frac{a+4\left(\frac{3}{8}a\right)}{a} = \frac{a+\frac{3}{2}a}{a} = \frac{2a+3a}{2a} = \frac{5a}{2a} = \frac{5}{2} = 2\frac{1}{2} \quad \textbf{Ans.}$$

5. (i) Show that the equation $3\sin x \tan x = 8$ can be written as

$3\cos^2 x + 8\cos x - 3 = 0$. [3]

(ii) Hence solve the equation $3\sin x \tan x = 8$ for $0^\circ \le x \le 360^\circ$. [3]

***Suggested Solution:***

(i)

$$3\sin x \tan x = 8$$

$$3\sin x\left(\frac{\sin x}{\cos x}\right) = 8$$

$$3\sin^2 x = 8\cos x$$

$$3\sin^2 x - 8\cos x = 0$$

$$3(1-\cos^2 x) - 8\cos x = 0$$

$$3 - 3\cos^2 x - 8\cos x = 0$$

$$3\cos^2 x + 8\cos x - 3 = 0 \quad \textbf{Shown.}$$

(ii) From part (ii) we see that $3\sin x \tan x = 8$ can be written as:

$$3\cos^2 x + 8\cos x - 3 = 0$$

$$3\cos^2 x + 9\cos x - \cos x - 3 = 0$$

$$3\cos x(\cos x + 3) - 1(\cos x + 3) = 0$$

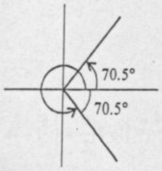
$$(\cos x + 3)(3\cos x - 1) = 0$$

$\Rightarrow \cos x = -3$ or $\cos x = \frac{1}{3}$

$\cos x = -3$ is ignored as $-1 \le \cos x \le 1$

$\therefore \cos x = \frac{1}{3}$

basic angle $\alpha = 70.5^\circ$



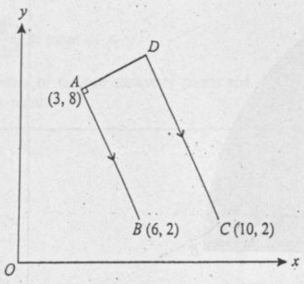
Note that:

$\cos\theta$ is positive in I and IV quadrant.

$\therefore$ in the Ist quadrant $\theta = \alpha$ and in the IV quadrant $\theta = 360 - \alpha$

$\Rightarrow x = 70.5^\circ, \ 289.5^\circ$ **Ans.**

6.



The three points $A$ (3, 8), $B$ (6, 2) and $C$ (10, 2) are shown in the diagram. The point $D$ is such that the line $DA$ is perpendicular to $AB$ and $DC$ is parallel to $AB$. Calculate the coordinates of $D$. [7]

***Suggested Solution:***

Gradient of $AB = \frac{2-8}{6-3} = -\frac{6}{3} = -2$

as $AB \parallel CD$

$\therefore$ gradient of $CD = -2$

equation of $CD$ passing through $C(10, 2)$ and gradient $-2$ is:

$$y-2=-2(x-10) \Rightarrow y-2=-2x+20 \Rightarrow y=-2x+22 \cdots\cdots\text{(i)}$$

Now, as $AB \perp AD$

$\therefore$ gradient of $AD = \frac{1}{2}$

Tangent and normal to the curve are perpendicular to each other, and for perpendicular lines (grad. of tangent) × (grad. of normal) = −1

equation of $AD$ passing through $A(3, 8)$ is:

$$y-8=\frac{1}{2}(x-3) \Rightarrow 2y-16=x-3 \Rightarrow 2y=x+13 \cdots\cdots\text{(ii)}$$

Equation of a line in point-slope form is: $y - y_1 = m(x - x_1)$

lines $CD$ and $AD$ meet at $D$

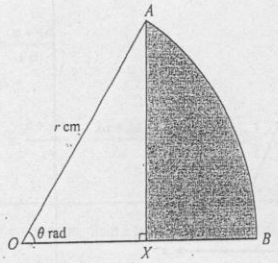
$\therefore$ solving eq. (i) and eq. (ii) simultaneously

$$2(-2x+22)=x+13$$
$$-4x+44=x+13$$
$$5x=31$$
$$x=\frac{31}{5} \quad \text{put in eq.(i)}$$

$$y=-2\left(\frac{31}{5}\right)+22 = -\frac{62}{5}+22 = \frac{48}{5}$$

$\therefore$ coordinates of $D:\left(\frac{31}{5}, \frac{48}{5}\right)$ or $(6.2, 9.6)$ Ans.

7.



In the diagram, $AB$ is an arc of a circle, centre $O$ and radius r cm, and angle $AOB = \theta$ radians. The point $X$ lies on $OB$ and $AX$ is perpendicular to $OB$.

(i) Show that the area, $A$ cm², of the shaded region $AXB$ is given by

$$A = \frac{1}{2}r^2(\theta - \sin\theta\cos\theta). \qquad [3]$$

(ii) In the case where $r = 12$ and $\theta = \frac{1}{6}\pi$, find the perimeter of the shaded region $AXB$, leaving your answer in terms of $\sqrt{3}$ and $\pi$. [4]

***Suggested Solution:***

(i) In $\Delta OAX$, $\sin\theta = \frac{AX}{OA} \Rightarrow AX = r\sin\theta$

$\cos\theta = \frac{OX}{OA} \Rightarrow OX = r\cos\theta$

$\therefore$ area of $\Delta OAX = \frac{1}{2}(OX)(AX) = \frac{1}{2}(r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2\sin\theta\cos\theta$

shaded area $AXB$ is: $A = \text{area of sector } \overset{\frown}{OAB} - \text{area of } \Delta OAX$

$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta\cos\theta$$

$$= \frac{1}{2}r^2(\theta - \sin\theta\cos\theta) \quad \textbf{Shown.}$$

Remember:

- Area of sector $= \frac{1}{2}r^2\theta$
- Length of arc $= r\theta$

where $r$ is the radius and $\theta$ is the angle in radians.

(ii) Given that, $r = 12$, $\theta = \frac{\pi}{6}$

$AX = r\sin\theta = 12\sin\frac{\pi}{6} = 12\left(\frac{1}{2}\right) = 6$ cm

$OX = r\cos\theta = 12\cos\frac{\pi}{6} = 12\left(\frac{\sqrt{3}}{2}\right) = 6\sqrt{3}$ cm

$XB = OB - OX = 12 - 6\sqrt{3}$

Arc length $\overset{\frown}{AB} = r\theta = 12\left(\frac{\pi}{6}\right) = 2\pi$ cm

$\therefore$ Perimeter of the shaded region $AXB = \overset{\frown}{AB} + XB + AX$

$$= 2\pi + 12 - 6\sqrt{3} + 6$$

$$= 2\pi + 18 - 6\sqrt{3} = 2(\pi + 9 - 3\sqrt{3}) \text{ cm} \quad \textbf{Ans.}$$

8. The equation of a curve is $y = (2x - 3)^3 - 6x$.

(i) Express $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of $x$. [3]

(ii) Find the $x$-coordinates of the two stationary points and determine the nature of each stationary point. [5]

***Suggested Solution:***

(i) $y = (2x-3)^3 - 6x$

$$\frac{dy}{dx} = 3(2x-3)^2(2) - 6$$
$$= 6\left[(2x-3)^2 - 1\right]$$
$$= 6\left[4x^2 - 12x + 9 - 1\right]$$
$$= 6\left[4x^2 - 12x + 8\right]$$
$$= 6\left[4(x^2 - 3x + 2)\right]$$
$$= 24(x^2 - 3x + 2) \quad \textbf{Ans.}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(24(x^2 - 3x + 2)\right) = 24(2x-3) \quad \textbf{Ans.}$$

(ii) $\frac{dy}{dx} = 24(x^2 - 3x + 2) = 24(x-2)(x-1)$

For stationary points, $\frac{dy}{dx} = 0$

$\Rightarrow \ 24(x-2)(x-1) = 0$

$\Rightarrow \ (x-2) = 0 \ \text{ or } \ (x-1) = 0$

$\Rightarrow \ x = 1, \ x = 2$

$\therefore$ the $x$-coordinates of the two stationary points are: $x = 1$ and $x = 2$ **Ans.**

now, $\frac{d^2y}{dx^2} = 24(2x-3)$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} = 24(2(1) - 3) = -24 < 0$$

$\therefore$ at $x = 1$, the stationary point is a maximum point. **Ans.**

$$\left.\frac{d^2y}{dx^2}\right|_{x=2} = 24(2(2) - 3) = 24 > 0$$

$\therefore$ at $x = 2$, the stationary point is a minimum point. **Ans.**

Remember:

if $\left.\frac{d^2y}{dx^2}\right|_{x = x_1} > 0$, $y$ is min

if $\left.\frac{d^2y}{dx^2}\right|_{x = x_1} < 0$, $y$ is max

9. A curve is such that $\frac{dy}{dx} = 4 - x$ and the point $P(2, 9)$ lies on the curve. The normal to the curve at $P$ meets the curve again at $Q$. Find

(i) the equation of the curve, [3]

(ii) the equation of the normal to the curve at $P$, [3]

(iii) the coordinates of $Q$. [3]

ector = $\frac{1}{2}r^2\theta$

arc = $r\theta$

he radius and
le in radians.

***Suggested Solution:***

(i) $\frac{dy}{dx} = 4 - x \Rightarrow dy = (4 - x)\,dx$

integrating both sides, we have

$$\int dy = \int (4 - x)\,dx$$

$$\Rightarrow \quad y = 4x - \frac{x^2}{2} + K$$

the curve passes through $P(2, 9)$

$$9 = 4(2) - \frac{(2)^2}{2} + K \Rightarrow 9 = 8 - 2 + k \Rightarrow K = 3$$

$\therefore$ equation of the curve is: $y = 4x - \frac{x^2}{2} + 3$ **Ans.**

When $\frac{dy}{dx}$ (or gradient) of the curve is given, always integrate both sides of the given expression to find the equation of a curve.

(ii) Gradient of tangent at $P(2, 9)$ is:

$$\frac{dy}{dx} = 4 - 2 = 2$$

$\Rightarrow$ gradient of normal at $P = -\frac{1}{2}$

$\therefore$ equation of normal passing through $P(2, 9)$ is:

$$y - 9 = -\frac{1}{2}(x - 2) \Rightarrow 2y - 18 = -x + 2 \Rightarrow 2y + x = 20$$ **Ans.**

If two lines are perpendicular than product of their gradient is equal to −1 i.e. $m_1 \times m_2 = -1$

(iii) equation of the curve: $y = 4x - \frac{x^2}{2} + 3$ ········(i)

equation of the normal: $2y + x = 20$ ········(ii)

from eq.(ii): $y = \frac{20 - x}{2}$ put in eq.(i)

$$\frac{20 - x}{2} = 4x - \frac{x^2}{2} + 3$$

multiplying throughout by 2 we have,

$$20 - x = 8x - x^2 + 6$$
$$x^2 - 9x + 14 = 0$$
$$x^2 - 7x - 2x + 14 = 0$$
$$x(x - 7) - 2(x - 7) = 0$$
$$(x - 2)(x - 7) = 0$$

$\therefore$ $x = 2$ and $x = 7$

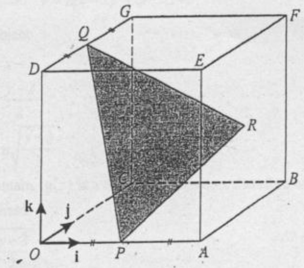
when $x = 2$, $y = \frac{20 - 2}{2} = 9$

i.e. point $P(2, 9)$, already on the curve

when $x = 7$, $y = \frac{20 - 7}{2} = \frac{13}{2} = 6.5$

$\therefore$ coordinates of $Q$: $(7, 6.5)$ **Ans.**

**10.**



The diagram shows a cube *OABCDEFG* in which the length of each side is 4 units.

The unit vectors **i**, **j** and **k** are parallel to $\overrightarrow{OA}$. $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively. The mid-points of *OA* and *DG* are *P* and *Q* respectively and *R* is the centre of the square face *ABFE*.

(i) Express each of the vectors $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ in terms of **i**, **j** and **k**. [3]

(ii) Use a scalar product to find angle *QPR*. [4]

(iii) Find the perimeter of triangle *PQR*, giving your answer correct to 1 decimal place. [3]

***Suggested Solution:***

(i) We have, $\overrightarrow{OP}=\begin{pmatrix}2\\0\\0\end{pmatrix}$, $\overrightarrow{OQ}=\begin{pmatrix}0\\2\\4\end{pmatrix}$, $\overrightarrow{OR}=\begin{pmatrix}4\\2\\2\end{pmatrix}$

$$\overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}=\begin{pmatrix}4\\2\\2\end{pmatrix}-\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}2\\2\\2\end{pmatrix}$$

$\therefore$ $\overrightarrow{PR}=2\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ **Ans.**

$$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}0\\2\\4\end{pmatrix}-\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}-2\\2\\4\end{pmatrix}$$

$\therefore$ $\overrightarrow{PQ}=-2\mathbf{i}+2\mathbf{j}+4\mathbf{k}$ **Ans.**

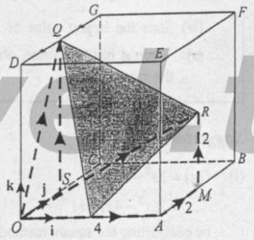
(ii) $\overrightarrow{PQ}\cdot\overrightarrow{PR}=\left|\overrightarrow{PQ}\right|\left|\overrightarrow{PR}\right|\cos Q\hat{P}R$

$$\begin{pmatrix}-2\\2\\4\end{pmatrix}\cdot\begin{pmatrix}2\\2\\2\end{pmatrix}=\left(\sqrt{(-2)^2+(2)^2+(4)^2}\right)\left(\sqrt{(2)^2+(2)^2+(2)^2}\right)\cos Q\hat{P}R$$

$$-4+4+8=\left(\sqrt{24}\right)\left(\sqrt{12}\right)\cos Q\hat{P}R$$

$$8=\left(\sqrt{288}\right)\cos Q\hat{P}R$$

$$\cos Q\hat{P}R=\frac{8}{\sqrt{288}}\quad\Rightarrow\quad Q\hat{P}R=61.874\approx 61.9\quad \textbf{Ans.}$$



Note that

$\overrightarrow{OQ}=\overrightarrow{OS}+\overrightarrow{SQ}=2\mathbf{j}+4\mathbf{k}$

$\therefore\ \overrightarrow{OQ}=\begin{pmatrix}0\\2\\4\end{pmatrix}$

similarly

$\overrightarrow{OR}=\overrightarrow{OA}+\overrightarrow{AM}+\overrightarrow{MR}$
$=4\mathbf{i}+2\mathbf{j}+2\mathbf{k}$

$\therefore\ \overrightarrow{OR}=\begin{pmatrix}4\\2\\2\end{pmatrix}$

(iii) We have,

$$\left|\overrightarrow{PR}\right| = \sqrt{(2)^2+(2)^2+(2)^2} = \sqrt{12}$$

$$\left|\overrightarrow{PQ}\right| = \sqrt{(-2)^2+(2)^2+(4)^2} = \sqrt{24}$$

$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \begin{pmatrix}4\\2\\2\end{pmatrix} - \begin{pmatrix}0\\2\\4\end{pmatrix} = \begin{pmatrix}4\\0\\-2\end{pmatrix}$$

$$\Rightarrow \left|\overrightarrow{QR}\right| = \sqrt{(4)^2+(-2)^2} = \sqrt{20}$$

$$\therefore \quad \text{perimeter of } \Delta PQR = \left|\overrightarrow{PQ}\right| + \left|\overrightarrow{PR}\right| + \left|\overrightarrow{QR}\right|$$

$$= \sqrt{24} + \sqrt{12} + \sqrt{20}$$

$$= 4.899 + 3.464 + 4.472$$

$= 12.8$ (to 3sf) units **Ans.**

**11.** The function f is defined by $f: x \mapsto 2x^2 - 8x + 11$ for $x \in \mathbb{R}$.

(i) Express f(x) in the form $a(x+b)^2 + c$, where $a$, $b$ and $c$ are constants. [3]

(ii) State the range of f. [1]

(iii) Explain why f does not have an inverse. [1]

The function g is defined by $g: x \mapsto 2x^2 - 8x + 11$ for $x \le A.$, where $A$ is a constant.

(iv) State the largest value of $A$ for which g has an inverse. [1]

(v) When $A$ has this value, obtain an expression, in terms of $x$, for $g^{-1}(x)$ and state the range of $g^{-1}$. [4]

***Suggested Solution:***

(i) $f(x) = 2x^2 - 8x + 11$

$= 2(x^2 - 4x) + 11$

by completing the square method, we have

$$f(x) = 2(x^2 - 4x + 4 - 4) + 11$$

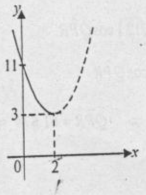
$$= 2[(x-2)^2 - 4] + 11$$

$$= 2(x-2)^2 - 8 + 11$$

$= 2(x-2)^2 + 3$ **Ans.**

(ii) From part (i), we see that the coordinates of the turning point are (2, 3).

$\therefore$ range of $f(x)$: $f(x) \ge 3$ **Ans.**



**Horizontal Line Test:**

If any horizontal line intersects at more than one point with the graph of a function then the function is not a one-one function.

(iii) For the given domain, f is not a one–one function.

(iv) The largest value of $A$ for which g(x) has an inverse is 2.

e Test:
ontal line
more than
the graph
n the func-
one-one

(v) $g(x) = 2x^2 - 8x + 11 \Rightarrow g(x) = 2(x-2)^2 + 3$

Let $g(x) = y \Rightarrow y = 2(x-2)^2 + 3$

making $x$ the subject

$$2(x-2)^2 = y - 3$$

$$(x-2)^2 = \frac{y-3}{2}$$

$$(x-2) = \pm\sqrt{\frac{y-3}{2}}$$

For the given domain, $g(x)$ is a decreasing function, therefore we keep the negative sign with the radical.

$$\therefore \quad x - 2 = -\sqrt{\frac{y-3}{2}}$$

$$\Rightarrow \quad x = 2 - \sqrt{\frac{y-3}{2}}$$

as $g(x) = y \Rightarrow g^{-1}(y) = x$

$$\therefore \quad g^{-1}(y) = 2 - \sqrt{\frac{y-3}{2}}$$

$$\Rightarrow \quad g^{-1}(x) = 2 - \sqrt{\frac{x-3}{2}}$$ **Ans.**

now, range of $g^{-1}(x)$ is the domain of $g(x)$

$\because$ domain of $g(x)$ is: $x \le 2$

$\therefore$ range of $g^{-1}(x)$ is: $g^{-1}(x) \le 2$ **Ans.**

For a one-one function $g(x)$:

Domain of $g(x)$ = range of $g^{-1}(x)$, and

Range of $g(x)$ = domain of $g^{-1}(x)$.

# November 2007 Paper 3

## *Pure Mathematics (P3)*

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Find the exact value of the constant $k$ for which $\int_1^k \frac{1}{2x-1}\,dx = 1$. [4]

***Suggested Solution:***

$$\int_1^k \frac{1}{2x-1}\,dx = 1 \Rightarrow \frac{1}{2}\int_1^k \frac{2}{2x-1}\,dx = 1 \Rightarrow \frac{1}{2}\Big[\ln|2x-1|\Big]_1^k = 1$$

$$\Rightarrow \frac{1}{2}\Big[\ln|2k-1| - \ln|2(1)-1|\Big] = 1 \Rightarrow \frac{1}{2}\Big[\ln|2k-1| - \ln 1\Big] = 1$$

$$\Rightarrow \frac{1}{2}\ln|2k-1| = 1 \Rightarrow \ln|2k-1| = 2 \Rightarrow \ln|2k-1| = 2\ln e$$

$$\Rightarrow \ln|2k-1| = \ln e^2 \Rightarrow 2k-1 = e^2 \Rightarrow 2k = 1+e^2 \Rightarrow k = \frac{1}{2}(1+e^2) \quad \textbf{Ans.}$$

Remember:
For definite integral, do not put the constant of integration.

Note that:
- $\ln e = 1$
- $\ln 1 = 0$

**2.** The polynomial $x^4 + 3x^2 + a$, where a is a constant, is denoted by p($x$). It is given that $x^2 + x + 2$ is a factor of p($x$). Find the value of a and the other quadratic factor of p($x$). [4]

***Suggested Solution:***

$p(x) = x^4 + 3x^2 + a$ and $(x^2 + x + 2)$ is a factor

using long division, we have

$$\begin{array}{r|l}
 & x^2 - x + 2 \\ \hline
x^2 + x + 2 & x^4 + 3x^2 + a \\
 & x^4 + 2x^2 \quad + x^3 \\
 & \underline{- \quad - \qquad\quad -} \\
 & -x^3 + x^2 + a \\
 & -x^3 - x^2 \quad - 2x \\
 & \underline{+ \quad + \qquad\quad +} \\
 & 2x^2 + 2x + a \\
 & +2x^2 + 2x + 4 \\
 & \underline{- \quad - \quad -} \\
 & a - 4
\end{array}$$

given that $(x^2 + x + 2)$ is a factor of p($x$)

$\therefore$ Remainder = 0

$\Rightarrow a - 4 = 0 \Rightarrow a = 4$ **Ans.**

Other quadratic factor of $p(x) = x^2 - x + 2$ **Ans.**

ember:
definite integral, do
put the constant of
gration.

that:
e = 1,
1 = 0

3. Use integration by parts to show that

$$\int_2^4 \ln x \;\mathrm{d}x = 6\ln 2 - 2.$$ [4]

***Suggested Solution:***

$$\text{L.H.S} = \int_2^4 \ln x \;\mathrm{d}x$$

using integration by parts with 1 as second factor, we have

$$\int_2^4 (\ln x)(1)\;\mathrm{d}x$$

$$= \left|(\ln x)\int 1\,\mathrm{d}x - \int\left(\left(\int 1\,\mathrm{d}x\right) \times \frac{\mathrm{d}}{\mathrm{d}x}(\ln x)\;\mathrm{d}x\right)\right|_2^4$$

$$= \left|\ln x(x) - \int x \times \frac{1}{x}\;\mathrm{d}x\right|_2^4$$

$$= \left|x\ln x - \int 1\,\mathrm{d}x\right|_2^4$$

$$= \left|x\ln x - x\right|_2^4$$

$$= (4\ln 4 - 4) - (2\ln 2 - 2)$$

$$= (4\ln 2^2 - 4) - (2\ln 2 - 2)$$

$$= 8\ln 2 - 4 - 2\ln 2 + 2 = 6\ln 2 - 2 \quad \textbf{Shown.}$$

4. The curve with equation $y = e^{-x}\sin x$ has one stationary point for which $0 \le x \le \pi$.

(i) Find the $x$-coordinate of this point. [4]

(ii) Determine whether this point is a maximum or a minimum point. [2]

***Suggested Solution:***

(i) $y = e^{-x}\sin x$

differentiating w.r.t. $x$

$$\frac{\mathrm{d}}{\mathrm{d}x}y = \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{-x}\sin x\right)$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \sin x\frac{\mathrm{d}}{\mathrm{d}x}(e^{-x}) + e^{-x}\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)$$

$$= \sin x\,(-e^{-x}) + e^{-x}\,(\cos x) = e^{-x}(\cos x - \sin x)$$

for stationary values, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$\Rightarrow \quad e^{-x}(\cos x - \sin x) = 0$

$\Rightarrow \quad e^{-x} = 0$ or $\cos x - \sin x = 0$

(not possible) $\sin x = \cos x$

$$\frac{\sin x}{\cos x} = 1 \;\Rightarrow\; \tan x = 1 \;\Rightarrow\; x = \frac{\pi}{4} \quad \textbf{Ans.}$$

(ii) $\frac{dy}{dx} = e^{-x}(\cos x - \sin x)$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(e^{-x}(\cos x - \sin x)\right)$$

$$= (\cos x - \sin x)(-e^{-x}) + e^{-x}(-\sin x - \cos x)$$

$$= e^{-x}\left(-\cos x + \sin x - \sin x - \cos x\right)$$

$$= e^{-x}\left(-2\cos x\right) = -2e^{-x}\cos x$$

at $x = \frac{\pi}{4}$,

$$\frac{d^2y}{dx^2} = -2e^{-\frac{\pi}{4}}\cos\frac{\pi}{4} = -2e^{-\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}}\right) < 0 \quad (\text{as } e^{-\frac{\pi}{4}} > 0)$$

$\therefore\ x = \frac{\pi}{4}$ is the $x$-coordinate of the maximum point. **Ans.**

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} > 0$, $y$ is min

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1} < 0$, $y$ is max

**5.** (i) Show that the equation

$$\tan(45^\circ + x) - \tan x = 2$$

can be written in the form

$$\tan^2 x + 2\tan x - 1 = 0. \qquad [3]$$

(ii) Hence solve the equation

$$\tan(45^\circ + x) - \tan x = 2,$$

giving all solutions in the interval $0^\circ \le x \le 180^\circ$. [4]

***Suggested Solution:***

(i) $\tan(45^\circ + x) - \tan x = 2$

$$\frac{\tan 45^\circ + \tan x}{1 - \tan 45^\circ \tan x} - \tan x = 2$$

$$\frac{1 + \tan x}{1 - \tan x} - \tan x = 2$$

$$\frac{1 + \tan x - \tan x(1 - \tan x)}{1 - \tan x} = 2$$

$$\frac{1 + \tan x - \tan x + \tan^2 x}{1 - \tan x} = 2$$

$$1 + \tan^2 x = 2(1 - \tan x)$$

$$1 + \tan^2 x = 2 - 2\tan x \;\Rightarrow\; \tan^2 x + 2\tan x - 1 = 0 \quad \textbf{Shown.}$$

Formula used:

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

Also note that,

$\tan 45^\circ = 1$

(ii) Using the result of part (i), we have

$$\tan^2 x + 2\tan x - 1 = 0$$

using quadratic formula

$$\tan x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

$\left.\right|_{x=x_1} > 0,\ y$ is min

$\left.\right|_{x=x_1} < 0,\ y$ is max

mula used:

$(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$

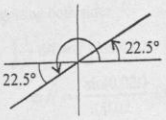
o note that,

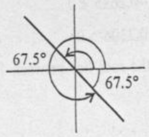
$45° = 1$

$\therefore\quad \tan x=-1+\sqrt{2}$ or $\tan x=-1-\sqrt{2}$

$\tan x=0.4142$ $\qquad$ $\tan x=-2.4142$

basic angle $\alpha=22.5°$ $\qquad$ basic angle $\alpha=67.5°$





Note that:
$\tan\theta$ is positive in 1st and 3rd quadrant and negative in 2nd and 4th quadrant.

as the given range is $0°\le x\le 180°$

$\therefore\quad x=22.5°,\ 112.5°$ **Ans.**

6. (i) By sketching a suitable pair of graphs, show that the equation
$$2-x=\ln x$$
has only one root. [2]

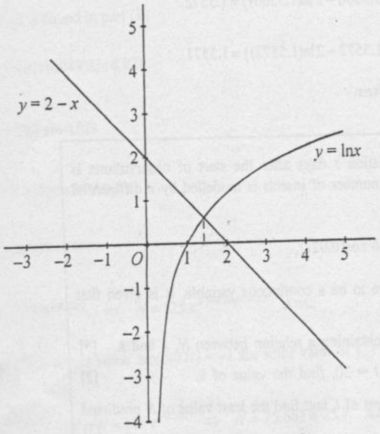
(ii) Verify by calculation that this root lies between 1.4 and 1.7. [2]

(iii) Show that this root also satisfies the equation
$$x=\frac{1}{3}(4+x-2\ln x).$$ [1]

(iv) Use the iterative formula
$$x_{n+1}=\frac{1}{3}(4+x_n-2\ln x_n),$$
with initial value $x_1=1.5$, to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

***Suggested Solution:***

(i) Sketching $y=2-x$ and $y=\ln x$ on the same diagram, we have.



$\therefore$ the two graphs have only one point of intersection.

**(ii)** $2-x=\ln x \;\Rightarrow\; \ln x+x-2=0$

Let $f(x)=\ln x+x-2$

$f(1.4)=\ln(1.4)+(1.4)-2=-0.2635<0$

$f(1.7)=\ln(1.7)+(1.7)-2=0.2306>0$

A change of sign indicates that the root lies between 1.4 and 1.7.

**(iii)** $x=\frac{1}{3}(4+x-2\ln x)$

$x=\frac{4}{3}+\frac{1}{3}x-\frac{2}{3}\ln x$

$x-\frac{1}{3}x-\frac{4}{3}=-\frac{2}{3}\ln x$

$\frac{2}{3}x-\frac{4}{3}=-\frac{2}{3}\ln x$

multiplying both sides by 3

$2x-4=-2\ln x$

$2\ln x=4-2x$

$\ln x=2-x$

Which is same equation as is given in part (i). Therefore the root also satisfies the above equation.

**(iv)** $x_{n+1}=\frac{1}{3}(4+x_n-2\ln x_n)$.

Given that initial value $x_1=1.5$

$\therefore\quad x_2=\frac{1}{3}(4+x_1-2\ln x_1)=\frac{1}{3}(4+1.5-2\ln(1.5))=1.5630$

$x_3=\frac{1}{3}(4+x_2-2\ln x_2)=\frac{1}{3}(4+1.5630-2\ln(1.5630))=1.5566$

$x_4=\frac{1}{3}(4+x_3-2\ln x_3)=\frac{1}{3}(4+1.5566-2\ln(1.5566))=1.5572$

$x_5=\frac{1}{3}(4+x_4-2\ln x_4)=\frac{1}{3}(4+1.5572-2\ln(1.5572))=1.5571$

$\Rightarrow$ root $=1.56$ (to 2 dec. places) **Ans.**

When the question did not specify the number of iterations to use, it is incorrect to use only one or two iterations of the formula. In fact, we need to continue until we get the same answer twice for the required degree of accuracy.

7. The number of insects in a population $t$ days after the start of observations is denoted by $N$. The variation in the number of insects is modelled by a differential equation of the form

$$\frac{dN}{dt}=kN\cos(0.02t),$$

where $k$ is a constant and $N$ is taken to be a continuous variable. It is given that $N = 125$ when $t = 0$.

(i) Solve the differential equation, obtaining a relation between $N$, $k$ and $t$. [5]

(ii) Given also that $N = 166$ when $t = 30$, find the value of $k$. [2]

(iii) Obtain an expression for $N$ in terms of $t$, and find the least value of $N$ predicted by this model. [3]

***Suggested Solution:***

(i) $\frac{dN}{dt} = kN\cos(0.02t) \quad \Rightarrow \quad \frac{dN}{N} = k\cos(0.02t)\ dt$

integrating both sides

$$\int \frac{1}{N}\, dN = k\int \cos(0.02t)\ dt$$

$$\ln N = k\frac{\sin(0.02t)}{0.02} + C$$

$$\ln N = 50k\sin(0.02t) + C \;\cdots\cdots\cdots\text{(i)}$$

given that $N = 125$ when $t = 0$,

$$\ln(125) = 50k\sin\left(0.02(0)t\right) + C \quad \Rightarrow \quad \ln(125) = 0 + C \quad \Rightarrow \quad C = \ln 125$$

$\therefore$ equation (i) becomes

$$\ln N = 50k\sin(0.02t) + \ln 125$$

$$\Rightarrow \quad \ln N - \ln 125 = 50k\sin(0.02t)$$

$$\Rightarrow \quad \ln\left(\frac{N}{125}\right) = 50k\sin 0.02t \quad \textbf{Ans.}$$

(ii) $\ln\left(\frac{N}{125}\right) = 50k\sin 0.02t$

given that $N = 166$ when $t = 30$, we have

$$\ln\left(\frac{166}{125}\right) = 50k\sin(0.02 \times 30)$$

$$0.2837 = 50k\sin(0.6)$$

$$0.2837 = 28.2321k$$

$$k = 0.010049 \quad \textbf{Ans.}$$

Remember to change your calculator to radian mode.

(iii) $\ln\left(\frac{N}{125}\right) = 50k\sin 0.02t$

putting the value of $k$ found in part (ii)

$$\ln\left(\frac{N}{125}\right) = 50(0.010049)\sin 0.02t$$

$$\Rightarrow \quad \ln\left(\frac{N}{125}\right) = 0.502\ \sin 0.02t$$

$$\Rightarrow \quad \ln\left(\frac{N}{125}\right) = (0.502\ \sin 0.02t)\ln e$$

$$\Rightarrow \quad \ln\left(\frac{N}{125}\right) = \ln e^{0.502\ \sin(0.02t)}$$

$$\Rightarrow \quad \frac{N}{125} = e^{0.502\ \sin(0.02t)} \quad \Rightarrow \quad N = 125\ e^{0.502\ \sin(0.02t)} \quad \textbf{Ans.}$$

now, least value of $N$ occurs when $\sin(0.02t) = -1$ for some value of $t$.
If it happens then,

$$N = 125\ e^{0.502\,(-1)} \quad \Rightarrow \quad N = 125\ e^{-0.502} \quad \Rightarrow \quad N = 75.665 \approx 75.7 \quad \textbf{Ans.}$$

8. (a) The complex number $z$ is given by $z = \frac{4-3i}{1-2i}$.

   (i) Express $z$ in the form $x + iy$, where $x$ and $y$ are real. [2]

   (ii) Find the modulus and argument of $z$. [2]

(b) Find the two square roots of the complex number $5 - 12i$, giving your answers in the form $x + iy$, where $x$ and $y$ are real. [6]

***Suggested Solution:***

**(a)** (i) $z = \frac{4-3i}{1-2i}$

realising the denominator, we have

$$z = \frac{4-3i}{1-2i} \times \frac{1+2i}{1+2i}$$

$$= \frac{4+8i-3i-6i^2}{(1)^2-(2i)^2}$$

$$= \frac{4+5i+6}{1+4} = \frac{10+5i}{5} = \frac{5(2+i)}{5} = 2+i \quad \textbf{Ans.}$$

(ii) $z = 2+i$

$\therefore$ Modulus, $|z| = \sqrt{(2)^2+(1)^2} = \sqrt{5}$ **Ans.**

$$\arg(z) = \tan^{-1}(\frac{1}{2}) = 0.464 \text{ radians} \quad \textbf{Ans.}$$

**(b)** Let $\sqrt{5-12i} = x+iy \qquad x, y \in \mathbb{R}$

$$\Rightarrow \quad 5-12i = (x+iy)^2$$

$$5-12i = x^2+2xyi+(iy)^2$$

$$5-12i = x^2+2xyi-y^2$$

$$5-12i = (x^2-y^2)+i2xy$$

equating real and imaginary parts, we have

$$x^2-y^2 = 5 \cdots\cdots\cdots(1) \quad \text{and} \quad 2xy = -12 \Rightarrow xy = -6 \cdots\cdots\cdots(2)$$

from eq.(2), $y = -\frac{6}{x}$, put in eq.(1)

$$x^2 - \left(-\frac{6}{x}\right)^2 = 5$$

$$x^2 - \frac{36}{x^2} = 5$$

$$x^4 - 36 = 5x^2$$

$$x^4 - 5x^2 - 36 = 0$$

$$x^4 - 9x^2 + 4x^2 - 36 = 0$$

$$x^2(x^2-9) + 4(x^2-9) = 0$$

$$(x^2-9)(x^2+4) = 0$$

$\therefore \quad x^2 = 9 \quad$ or $\quad x^2 = -4 \quad$ (ignored as $x$ is a real number)

$\Rightarrow \quad x = \pm 3$

when $x = 3,\ y = -\frac{6}{3} = -2$

$\therefore$ $(3-2\text{i})$ is one square root

when $x = -3,\ y = -\frac{6}{-3} = 2$

$\therefore$ $(-3+2\text{i})$ is another square root.

hence, $\sqrt{5-12\text{i}} = \pm(3-2\text{i})$ **Ans.**

*Alternative method for part (b)*

Expressing $5-12\text{i}$ in polar form, we have

$$|5-12\text{i}| = \sqrt{(5)^2+(-12)^2} = 13$$

$$\arg(5-12\text{i}) = \tan^{-1}(-\frac{12}{5}) = -1.176$$

$$\therefore \quad 5-12\text{i} = 13[\cos(-1.176)+\text{i}\sin(-1.176)]$$

$$\Rightarrow \quad \sqrt{5-12\text{i}} = \pm\sqrt{13}\,[\cos(-0.588)+\text{i}\sin(-0.588)]$$

$$= \pm\sqrt{13}[\cos(0.588)-\text{i}\sin(0.588)]$$

$$= \pm\sqrt{13}[0.83205-\text{i}(0.5547)]$$

$$= \pm(3-2\text{i}) \quad \textbf{Ans.}$$

Any complex Number $z = x+\text{i}y$ can be expressed in its polar form $z = r(\cos\theta + i\sin\theta)$, where $|z| = r$, $\arg z = \theta$ where $-\pi < \theta \le \pi$

Also when $z = r(\cos\theta + i\sin\theta)$, then $\sqrt{z} = \sqrt{r}\ (\cos\frac{\theta}{2} + \text{i}\sin\frac{\theta}{2})$

9. (i) Express $\dfrac{2-x+8x^2}{(1-x)(1+2x)(2+x)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\dfrac{2-x+8x^2}{(1-x)(1+2x)(2+x)}$ in ascending powers of $x$, up to and including the term in $x^2$. [5]

***Suggested Solution:***

(i) $$\frac{2-x+8x^2}{(1-x)(1+2x)(2+x)} \equiv \frac{A}{1-x} + \frac{B}{1+2x} + \frac{C}{2+x}$$

$$\Rightarrow \ 2-x+8x^2 \equiv A(1+2x)(2+x) + B(1-x)(2+x) + C(1-x)(1+2x)$$

for $x = 1$

$$2-1+8(1)^2 = A(1+2(1))(2+1) + B(1-1)(2+1) + C(1-1)(1+2(1))$$

$$2-1+8 = A(3)(3) \ \Rightarrow \ 9 = 9A \ \Rightarrow \ A = 1$$

for $x = -2$

$$2-(-2)+8(-2)^2 = A(1+2(-2))(2+(-2)) + B(1-(-2))(2+(-2)) + C(1-(-2))(1+2(-2))$$

$$2+2+32 = C(3)(-3) \ \Rightarrow \ 36 = -9C \ \Rightarrow \ C = -4$$

for $x = -\frac{1}{2}$

$$2-(-\frac{1}{2})+8(-\frac{1}{2})^2 = A(1+2(-\frac{1}{2}))(2+(-\frac{1}{2})) + B(1-(-\frac{1}{2}))(2+(-\frac{1}{2})) + C(1-(-\frac{1}{2}))(1+2(-\frac{1}{2}))$$

$$2+\frac{1}{2}+8\left(\frac{1}{4}\right) = B\left(1+\frac{1}{2}\right)\left(2-\frac{1}{2}\right) \ \Rightarrow \ \frac{9}{2} = B\left(\frac{3}{2}\right)\left(\frac{3}{2}\right) \ \Rightarrow \ \frac{9}{2} = B\left(\frac{9}{4}\right) \ \Rightarrow \ B = 2$$

$$\therefore \quad \frac{2-x+8x^2}{(1-x)(1+2x)(2+x)}=\frac{1}{1-x}+\frac{2}{1+2x}-\frac{4}{2+x} \quad \textbf{Ans.}$$

(ii) Using the result of part (i), we have

$$\frac{2-x+8x^2}{(1-x)(1+2x)(2+x)}=\frac{1}{1-x}+\frac{2}{1+2x}-\frac{4}{2+x}$$

$$=\frac{1}{1-x}+\frac{2}{1+2x}-\frac{4}{2\left(1+\frac{x}{2}\right)}$$

$$=\frac{1}{1-x}+\frac{2}{1+2x}-\frac{2}{1+\frac{x}{2}} = (1-x)^{-1}+2(1+2x)^{-1}-2\left(1+\frac{x}{2}\right)^{-1}$$

applying binomial expansion up to the term including $x^2$

$$=\left(1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2\right)+2\left(1+(-1)(2x)+\frac{(-1)(-2)}{2!}(2x)^2\right)$$

$$-2\left(1+(-1)\left(\frac{x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x}{2}\right)^2\right)$$

$$=\left(1+x+x^2\right)+2(1-2x+4x^2)-2\left(1-\frac{x}{2}+\frac{x^2}{4}\right)$$

$$=1+x+x^2+2-4x+8x^2-2+x-\frac{x^2}{2}$$

$$=1-2x+\frac{17}{2}x^2 \quad \textbf{Ans.}$$

Binomial Expansion:

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+...$$

**10.** The straight line $l$ has equation $\mathbf{r}=\mathbf{i}+6\mathbf{j}-3\mathbf{k}+s(\mathbf{i}-2\mathbf{j}+2\mathbf{k})$. The plane $p$ has equation $(\mathbf{r}-3\mathbf{i})\cdot(2\mathbf{i}-3\mathbf{j}+6\mathbf{k})=0$. The line $l$ intersects the plane $p$ at the point $A$.

(i) Find the position vector of $A$. [3]

(ii) Find the acute angle between $l$ and $p$. [4]

(iii) Find a vector equation for the line which lies in $p$, passes through $A$ and is perpendicular to $l$. [5]

***Suggested Solution:***

(i) Given that,

Equation of line $l$: $\mathbf{r}=\begin{pmatrix}1\\6\\-3\end{pmatrix}+s\begin{pmatrix}1\\-2\\2\end{pmatrix} \Rightarrow \mathbf{r}=\begin{pmatrix}1+s\\6-2s\\-3+2s\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1+s\\6-2s\\-3+2s\end{pmatrix} \Rightarrow \begin{matrix}x=1+s\\y=6-2s\\z=-3+2s\end{matrix} \quad \cdots\cdots\cdots(A)$$

Remember that r is the position vector of any point (x, y, z) on the line.

Equation of plane $p$: $(\mathbf{r}-3\mathbf{i})\cdot(2\mathbf{i}-3\mathbf{j}+6\mathbf{k})=0$

$$\Rightarrow \left[\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}3\\0\\0\end{pmatrix}\right]\cdot\left[\begin{pmatrix}2\\-3\\6\end{pmatrix}\right]=0 \Rightarrow \begin{pmatrix}x-3\\y\\z\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\6\end{pmatrix}=0$$

$$\Rightarrow 2x-6-3y+6z=0 \Rightarrow 2x-3y+6z=6\cdots\cdots\cdots(B)$$

putting eq. (A) into eq. (B), we have

$$2(1+s)-3(6-2s)+6(-3+2s)=6$$
$$2+2s-18+6s-18+12s=6$$
$$20s-34=6$$
$$20s=40$$
$$s=2$$

putting $s=2$ in the equation of line $l$,

$$\mathbf{r}=\begin{pmatrix}1+2\\6-2(2)\\-3+2(2)\end{pmatrix}=\begin{pmatrix}3\\2\\1\end{pmatrix}$$

$\therefore$ position vector of point $A$ is: $\overrightarrow{OA}=\begin{pmatrix}3\\2\\1\end{pmatrix}$ or $3\mathbf{i}+2\mathbf{j}+\mathbf{k}$ **Ans.**

(ii) Equation of plane $p$ in scalar product form is: $(\mathbf{r}-3\mathbf{i})\bullet(2\mathbf{i}-3\mathbf{j}+6\mathbf{k})=0$

$\therefore$ the normal **n** to the plane is: $\mathbf{n}=\begin{pmatrix}2\\-3\\6\end{pmatrix}$

let **d** be the direction vector of the line $l$, $\therefore$ $\mathbf{d}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$

now, the angle between line $l$ and plane $p$ is given by

$$\mathbf{d}\bullet\mathbf{n}=|\mathbf{d}|\;|\mathbf{n}|\sin\theta$$

$$\Rightarrow\quad \begin{pmatrix}1\\-2\\2\end{pmatrix}\bullet\begin{pmatrix}2\\-3\\6\end{pmatrix}=\left(\sqrt{(1)^2+(-2)^2+(2)^2}\right)\left(\sqrt{(2)^2+(-3)^2+(6)^2}\right)\sin\theta$$

$$\Rightarrow\quad 2+6+12=\left(\sqrt{1+4+4}\right)\left(\sqrt{4+9+36}\right)\sin\theta$$

$$\Rightarrow\quad 20=\left(\sqrt{9}\right)\left(\sqrt{49}\right)\sin\theta$$

$$\Rightarrow\quad 20=(3)(7)\sin\theta$$

$$\Rightarrow\quad \sin\theta=\frac{20}{21}\ \Rightarrow\ \theta=72.2^\circ$$

$\therefore$ acute angle $\theta=72.2^\circ$ **Ans.**

(iii) The required line lies in $p$, therefore it must be perpendicular to the normal of the plane $p$. The required line is also perpendicular to the direction vector $(\vec{d})$ of the line $l$.

$\therefore$ $\mathbf{n}=\begin{pmatrix}2\\-3\\6\end{pmatrix}$ and $\vec{d}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$

now, the direction vector $\vec{d_1}$ of the required line is given by,

$$\vec{d_1}=\mathbf{n}\times\vec{d}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&-3&6\\1&-2&2\end{vmatrix}=\mathbf{i}(-6+12)-\mathbf{j}(4-6)+\mathbf{k}(-4+3)=6\mathbf{i}+2\mathbf{j}-\mathbf{k}$$

as the line passes through point $A(3,\ 2,\ 1)$, therefore equation of the required line is:

$\mathbf{r}=3\mathbf{i}+2\mathbf{j}+\mathbf{k}+\lambda(6\mathbf{i}+2\mathbf{j}-\mathbf{k})$ **(Ans)**

# June 2008 Paper 1

## Pure Mathematics (P1)

Answer all the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. In the triangle $ABC$, $AB = 12$ cm, angle $BAC = 60°$ and angle $ACB = 45°$. Find the exact length of $BC$. [3]

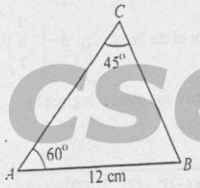
***Suggested Solution:***

Using sine rule,

$$\frac{BC}{\sin 60°}=\frac{12}{\sin 45°}$$

$$BC=\frac{12}{\sin 45°}\times \sin 60°$$

$$=\frac{12}{\frac{1}{\sqrt{2}}}\times\frac{\sqrt{3}}{2}=6\sqrt{6}\text{ cm} \quad \textbf{(Ans)}$$



2. (i) Show that the equation $2\tan^2\theta\cos\theta = 3$ can be written in the form $2\cos^2\theta + 3\cos\theta - 2 = 0$. [2]

   (ii) Hence solve the equation $2\tan^2\theta\cos\theta = 3$, for $0 \le \theta \le 360°$. [3]

***Suggested Solution:***

(i)
$$2\tan^2\theta\cos\theta = 3$$

$$2\left(\frac{\sin^2\theta}{\cos^2\theta}\right)\cos\theta = 3$$

$$2\left(\frac{\sin^2\theta}{\cos\theta}\right) = 3$$

$$2\sin^2\theta = 3\cos\theta$$

$$2(1-\cos^2\theta) = 3\cos\theta$$

$$2-2\cos^2\theta = 3\cos\theta$$

$$2\cos^2\theta + 3\cos\theta - 2 = 0 \quad \textbf{(Shown)}$$

(ii) $2\tan^2\theta\cos\theta = 3$

using the result of part (i), the above equation can be written as,

$$2\cos^2\theta + 3\cos\theta - 2 = 0$$

$$2\cos^2\theta + 4\cos\theta - \cos\theta - 2 = 0$$

$$2\cos\theta(\cos\theta + 2) - 1(\cos\theta + 2) = 0$$

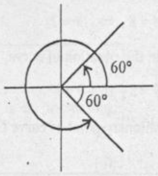
$$(\cos\theta + 2)(2\cos\theta - 1) = 0$$

$\cos\theta + 2 = 0$ or $2\cos\theta - 1 = 0$

$\Rightarrow$ $\cos\theta = -2$(invalid), or $\cos\theta = \frac{1}{2}$

basic Angle $\alpha = 60°$

$\therefore$ $\theta = 60°, 300°$ **(Ans)**



Note that:
$\cos\theta$ is positive in 1st and 4th quadrant.

> 3. (i) Find the first 3 terms in the expansion, in ascending powers of $x$, of $(2 + x^2)^5$. [3]
>
> (ii) Hence find the coefficient of $x^4$ in the expansion of $(1 + x^2)^2(2 + x^2)^5$. [3]

*Suggested Solution:*

(i) $(2 + x^2)^5$

$$= {}^5C_0(2)^5(x^2)^0 + {}^5C_1(2)^4(x^2)^1 + {}^5C_2(2)^3(x^2)^2 + \cdots\cdots$$

$$= 32 + 5(16)x^2 + 10(8)x^4 = 32 + 80x^2 + 80x^4 \quad \textbf{(Ans)}$$

Binomial theorem:

$(a+b)^n = {}^nC_0a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + .... + {}^nC_ra^{n-r}b^r + .... + {}^nC_nb^n$

(ii) $(1 + x^2)^2(2 + x^2)^5$

$$= (1 + 2x^2 + x^4)(32 + 80x^2 + 80x^4 + \cdots\cdots)$$

collecting the terms containing $x^4$ only,

$$80x^4 + 160x^4 + 32x^4 = 272x^4$$

$\therefore$ coefficient of $x^4 = 272$ **(Ans)**

> 4. The equation of a curve $C$ is $y = 2x^2 - 8x + 9$ and the equation of a line $L$ is $x + y = 3$.
>
> (i) Find the $x$-coordinates of the points of intersection of $L$ and $C$. [4]
>
> (ii) Show that one of these points is also the stationary point of $C$. [3]

*Suggested Solution:*

(i) Equation of line $L$: $x + y = 3 \Rightarrow y = 3 - x \cdots\cdots\cdots$(i)

substituting eq. (i) into equation of the curve and solving simultaneously,

$$3 - x = 2x^2 - 8x + 9 \Rightarrow 2x^2 - 7x + 6 = 0 \Rightarrow 2x^2 - 4x - 3x + 6 = 0$$

$$\Rightarrow 2x(x-2) - 3(x-2) = 0 \Rightarrow (x-2)(2x-3) = 0$$

$\Rightarrow$ $(x - 2) = 0$ or $2x - 3 = 0$

$x = 2$ or $x = \frac{3}{2}$

$\therefore$ $x$-coordinates of the points of intersection $= 2, 1\frac{1}{2}$ **(Ans)**

(ii) Equation of curve $C$: $y = 2x^2 - 8x + 9$

$$\frac{dy}{dx} = 4x - 8$$

for stationary points, $\frac{dy}{dx} = 0$

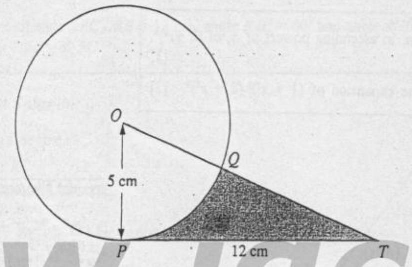
$\Rightarrow\ 4x - 8 = 0\ \Rightarrow\ 4x = 8\ \Rightarrow\ x = 2$

putting this value of $x$ in the equation of curve,

$$y = 2(2)^2 - 8(2) + 9 = 8 - 16 + 9 = 1$$

$\therefore$ (2, 1) is also the stationary point of curve $C$. **(Ans)**

**5.**



The diagram shows a circle with centre $O$ and radius 5 cm. The point $P$ lies on the circle, $PT$ is a tangent to the circle and $PT$ = 12 cm. The line $OT$ cuts the circle at the point $Q$.

(i) Find the perimeter of the shaded region. [4]

(ii) Find the area of the shaded region. [3]

***Suggested Solution:***

(i) Since $OP \perp PL$, $\therefore\ O\hat{P}T = 90°$

$\tan P\hat{O}T = \frac{12}{5}\ \Rightarrow\ P\hat{O}T = 1.176$ radians

using $S = r\theta$, arc length $\overset{\frown}{PQ} = (5)(1.176) = 5.88$ cm

Note:
Change your calculator to radian mode.

applying pythagoras theorm on $\Delta OPT$, we have,

$$(OT)^2 = (OP)^2 + (PT)^2$$
$$(OT)^2 = (5)^2 + (12)^2$$
$$(OT)^2 = 169\ \Rightarrow\ OT = 13 \text{ cm}$$

$\therefore\ QT = OT - OQ$
$= 13 - 5 = 8$

perimeter of the shaded region = arc length $\overset{\frown}{PQ} + QT + PT$
$= 5.88 + 8 + 12 = 25.88 \approx 25.9$ cm **(Ans)**

CORNER

our calculator to
le.

(ii) Area of $\Delta OPT = \frac{1}{2}(12)(5) = 30 \text{ cm}^2$

area of the sector $\widehat{OPQ} = \frac{1}{2}(5)^2(1.176) = 14.7 \text{ cm}^2$

$\therefore$ area of the shaded region = area of $\Delta OPT$ − area of the sector $\widehat{OPQ}$

$= 30 - 14.7 = 15.3 \text{ cm}^2$ **(Ans)**

When the angle is in radians, then

Area of sector $= \frac{1}{2}r^2\theta$

Arc length, $S = r\theta$.

where $r$ is the radius and $\theta$ is the angle in radians.

6. The function f is such that $f(x) = (3x + 2)^3 - 5$ for $x \geq 0$.

(i) Obtain an expression for $f'(x)$ and hence explain why f is an increasing function. [3]

(ii) Obtain an expression for $f^{-1}(x)$ and state the domain of $f^{-1}$. [4]

*Suggested Solution:*

(i) $f(x) = (3x+2)^3 - 5$

$f'(x) = \frac{d}{dx}\left((3x+2)^3 - 5\right)$

$= 3(3x+2)^2 \times 3 - 0$

$= 9(3x+2)^2$

now, $(3x+2)^2 \geq 0$ for all $x$

$\Rightarrow 9(3x+2)^2 \geq 0$ for all $x$

$\therefore$ f($x$) is an increasing function. **(Ans)**

Note that:

- If $f(x) = y$, then, $f'(x) = \frac{dy}{dx}$
- A squared quantity is always positive or $\geq 0$.
- Increasing function means 'positive gradient'.

(ii) $f(x) = (3x+2)^3 - 5$

let $y = f(x)$

$\therefore \quad y = (3x+2)^3 - 5$

$(3x+2)^3 = y + 5$

$3x + 2 = (y+5)^{\frac{1}{3}}$

$3x = (y+5)^{\frac{1}{3}} - 2$

$x = \frac{1}{3}(y+5)^{\frac{1}{3}} - \frac{2}{3}$

i.e. $f^{-1}(y) = \frac{1}{3}(y+5)^{\frac{1}{3}} - \frac{2}{3} \quad \because x = f^{-1}(y)$

$\therefore f^{-1}(x) = \frac{1}{3}(x+5)^{\frac{1}{3}} - \frac{2}{3}$ **(Ans)**

$f(x) = (3x+2)^3 - 5$ for $x \geq 0$

$\therefore$ at $x = 0$, $f(0) = (3(0)+2)^3 - 5 = 8 - 5 = 3$

$f(0) = 3$, and f($x$) is an increasing function,

$\therefore$ the range of the f($x$) is: $f(x) \geq 3$

$\Rightarrow$ domain of $f^{-1}(x)$ is: $x \geq 3$ **(Ans)**

Domain of $f^{-1}(x)$ is the range of $f(x)$ and vice versa.

7. The first term of a geometric progression is 81 and the fourth term is 24. Find

(i) the common ratio of the progression, [2]

(ii) the sum to infinity of the progression. [2]

The second and third terms of this geometric progression are the first and fourth terms respectively of an arithmetic progression.

(iii) Find the sum of the first ten terms of the arithmetic progression. [3]

***Suggested Solution:***

(i) Given that, $a = 81$, $T_4 = 24$

using $T_n = ar^{n-1}$, we have,

$T_4 = ar^{4-1}$

$$24 = 81r^3 \Rightarrow r^3 = \frac{24}{81} \Rightarrow r^3 = \frac{8}{27} \Rightarrow r = \frac{2}{3} \quad \text{(Ans)}$$

(ii) As $r = \frac{2}{3}$, $\Rightarrow |r| < 1 \Rightarrow S_\infty$ is possible

using $S_\infty = \frac{a}{1-r}$,

$$S_\infty = \frac{81}{1-\frac{2}{3}} = \frac{81}{\frac{1}{3}} = 81 \times 3 = 243 \quad \text{(Ans)}$$

(iii) 2nd term of GP is: $T_2 = ar = 81\left(\frac{2}{3}\right) = 54$

$\therefore$ first term of AP, $T_1 = 54 \Rightarrow a = 54$

3rd term of GP is: $T_3 = ar^2 = 81\left(\frac{2}{3}\right)^2 = 81\left(\frac{4}{9}\right) = 36$

$\therefore$ fourth term of AP, $T_4 = 36$

using $T_n = a + (n-1)d$, we have

$T_4 = a + (4-1)d$

$36 = 54 + 3d \Rightarrow -18 = 3d \Rightarrow d = -6$

using $S_n = \frac{n}{2}(2a + (n-1)d)$

$$S_{10} = \frac{10}{2}(2(54) + (10-1)(-6)) = 5(108 + (-54)) = 5(54) = 270 \quad \text{(Ans)}$$

8. Functions f and g are defined by

$\mathrm{f} : x \mapsto 4x - 2k$ for $x \in \mathbb{R}$, where $k$ is a constant,

$\mathrm{g} : x \mapsto \frac{9}{2-x}$ for $x \in \mathbb{R}$, $x \neq 2$.

(i) Find the values of $k$ for which the equation $\mathrm{fg}(x) = x$ has two equal roots. [4]

(ii) Determine the roots of the equation $\mathrm{fg}(x) = x$ for the values of $k$ found in part (i). [3]

*Suggested Solution:*

(i) $f(x) = 4x - 2k$, and $g(x) = \frac{9}{2-x}$

$$fg(x) = f\left(\frac{9}{2-x}\right)$$

$$= 4\left(\frac{9}{2-x}\right) - 2k = \frac{36}{2-x} - 2k$$

given that $fg(x) = x$

$$\Rightarrow \frac{36}{2-x} - 2k = x$$

$$\Rightarrow \frac{36 - 2k(2-x)}{2-x} = x \Rightarrow 36 - 4k + 2kx = 2x - x^2$$

$$\Rightarrow x^2 + 2kx - 2x + 36 - 4k = 0 \Rightarrow x^2 + (2k-2)x + (36-4k) = 0$$

given that the above equation has equal roots

$\therefore$ discriminant $= 0$

$$\Rightarrow b^2 - 4ac = 0$$

$$\Rightarrow (2k-2)^2 - 4(1)(36-4k) = 0$$

$$4k^2 - 8k + 4 - 4(36-4k) = 0$$

$$k^2 - 2k + 1 - (36-4k) = 0$$

$$k^2 - 2k + 1 - 36 + 4k = 0$$

$$k^2 + 2k - 35 = 0$$

$$k^2 + 7k - 5k - 35 = 0$$

$$k(k+7) - 5(k+7) = 0$$

$$(k+7)(k-5) = 0$$

$$\Rightarrow k = -7, \text{ or } k = 5$$

$\therefore$ $k = -7$ or $5$ **(Ans)**

(ii) From part (i), $fg(x) = x$, can be written as

$$x^2 + (2k-2)x + (36-4k) = 0$$

when $k = -7$

$$x^2 + (2(-7)-2)x + (36-4(-7)) = 0$$

$$x^2 + (-14-2)x + (36+28) = 0$$

$$x^2 - 16x + 64 = 0$$

$$x^2 - 8x - 8x + 64 = 0$$

$$x(x-8) - 8(x-8) = 0$$

$$(x-8)(x-8) = 0$$

$$(x-8)^2 = 0 \Rightarrow x = 8$$ **(Ans)**

when $k = 5$

$$x^2 + (2(5)-2)x + (36-4(5)) = 0$$

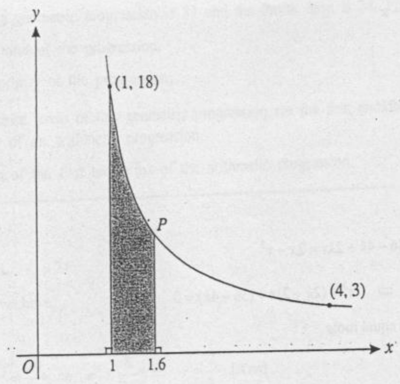
$$x^2 + 8x + 16 = 0$$

$$x^2 + 4x + 4x + 16 = 0$$

$$x(x+4) + 4(x+4) = 0$$

$$(x+4)(x+4) = 0 \Rightarrow (x+4)^2 = 0 \Rightarrow x = -4$$ **(Ans)**

9.



The diagram shows a curve for which $\frac{dy}{dx} = -\frac{k}{x^3}$, where $k$ is a constant. The curve passes through the points (1, 18) and (4, 3).

(i) Show, by integration, that the equation of the curve is $y = \frac{16}{x^2} + 2$. [4]

The point $P$ lies on the curve and has $x$-coordinate 1.6.

(ii) Find the area of the shaded region. [4]

***Suggested Solution:***

(i) $\frac{dy}{dx} = -\frac{k}{x^3} \Rightarrow dy = -kx^{-3}\,dx$

integrating both sides,

$$\int dy = -k\int x^{-3}\,dx$$

$$y = -k\left(\frac{x^{-2}}{-2}\right) + C \Rightarrow y = \frac{k}{2x^2} + C$$

The curve passes through (1, 18)

$$\therefore\ 18 = \frac{k}{2(1)^2} + C \Rightarrow 18 = \frac{k+2C}{2} \Rightarrow 36 = k + 2C \Rightarrow k = 36 - 2C \cdots\cdots\cdots \text{(i)}$$

The curve also passes through (4, 3)

$$\therefore\ 3 = \frac{k}{2(4)^2} + C \Rightarrow 3 = \frac{k}{32} + C \Rightarrow 3 = \frac{k+32C}{32} \Rightarrow 96 = k + 32C \cdots\cdots\cdots \text{(ii)}$$

substituting equation (i) into (ii)

$$96 = (36 - 2C) + 32C \Rightarrow 96 = 36 + 30C \Rightarrow 60 = 30C \Rightarrow C = 2$$

substituting the value of $C$ into eq. (i), $k = 36 - 2(2) \Rightarrow k = 32$

$\therefore$ the equation of the curve is:

$$y = \frac{32}{2x^2} + 2 \Rightarrow y = \frac{16}{x^2} + 2 \quad \textbf{(Shown)}$$

When $\frac{dy}{dx}$ (or gradient) of the curve is given, always integrate both sides of the given expression to find the equation of the curve.

(ii) Area of shaded region $= \int_{1}^{1.6} y\,dx$

$$= \int_{1}^{1.6} \left(\frac{16}{x^2}+2\right) dx$$

$$= \int_{1}^{1.6} \left(16x^{-2}+2\right) dx$$

$$= \left[16(\frac{x^{-1}}{-1})+2x\right]_{1}^{1.6}$$

$$= \left[-\frac{16}{x}+2x\right]_{1}^{1.6}$$

$$= \left(-\frac{16}{1.6}+2(1.6)\right)-\left(-\frac{16}{1}+2(1)\right)$$

$$= (-10+3.2)-(-14)$$

$$= -6.8+14 = 7.2 \text{ units}^2 \quad \textbf{(Ans)}$$

10. Relative to an origin $O$, the position vectors of points $A$ and $B$ are $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $3\mathbf{i}-2\mathbf{j}+p\mathbf{k}$ respectively.

(i) Find the value of $p$ for which $OA$ and $OB$ are perpendicular. [2]

(ii) In the case where $p = 6$, use a scalar product to find angle $AOB$, correct to the nearest degree. [3]

(iii) Express the vector $\overrightarrow{AB}$ in terms of $p$ and hence find the values of $p$ for which the length of $AB$ is 3.5 units. [4]

***Suggested Solution:***

(i) Given that, $\vec{OA}=\begin{pmatrix}2\\1\\2\end{pmatrix}$, $\vec{OB}=\begin{pmatrix}3\\-2\\p\end{pmatrix}$

since $\vec{OA}$ is perpendicular to $\vec{OB}$

$\therefore \quad \vec{OA}\cdot\vec{OB}=0$

$$\begin{pmatrix}2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\p\end{pmatrix}=0 \Rightarrow 6-2+2p=0 \Rightarrow 2p=-4 \Rightarrow p=-2 \quad \textbf{(Ans)}$$

(ii) When $p=6$, $\vec{OA}=\begin{pmatrix}2\\1\\2\end{pmatrix}$, $\vec{OB}=\begin{pmatrix}3\\-2\\6\end{pmatrix}$

applying scalar product,

$$\vec{OA}\cdot\vec{OB}=\left|\vec{OA}\right|\cdot\left|\vec{OB}\right|\cos A\hat{O}B$$

$$\begin{pmatrix}2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}3\\-2\\6\end{pmatrix}=\left(\sqrt{(2)^2+(1)^2+(2)^2}\right)\left(\sqrt{(3)^2+(-2)^2+(6)^2}\right)\cos A\hat{O}B$$

$$6-2+12=(\sqrt{9})(\sqrt{49})\cos A\hat{O}B$$

$$16=(3)(7)\cos A\hat{O}B$$

$$\cos A\hat{O}B=\frac{16}{21}$$

$$A\hat{O}B=40.367°\approx 40.4° \quad \textbf{(Ans)}$$

(iii) $\vec{AB}=\vec{OB}-\vec{OA}=\vec{OA}=\begin{pmatrix}3\\-2\\p\end{pmatrix}-\begin{pmatrix}2\\1\\2\end{pmatrix}=\begin{pmatrix}1\\-3\\p-2\end{pmatrix}$

given that, $\left|\vec{AB}\right|=3.5$

$$\sqrt{(1)^2+(-3)^2+(p-2)^2}=3.5$$

$$\sqrt{10+(p-2)^2}=3.5 \quad \text{(squaring both sides)}$$

$$10+(p-2)^2=12.25$$

$$(p-2)^2=2.25 \quad \text{(taking square root)}$$

$$p-2=\pm 1.5$$

$p=2+1.5$ or $p=2-1.5$

$p=3.5$ or $p=0.5$

$\therefore$ values of $p$ are 3.5 or 0.5 **(Ans)**

If

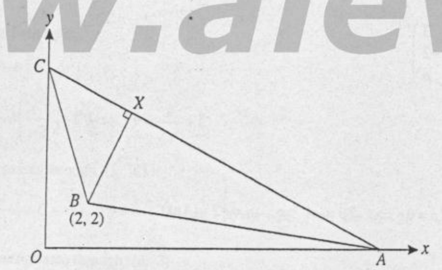
$\mathbf{a}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, then

magnitude (or length) of

**a** is:

$|\mathbf{a}|=\sqrt{(a)^2+(b)^2+(c)^2}$

**11.**



In the diagram, the points $A$ and $C$ lie on the $x$- and $y$-axes respectively and the equation of $AC$ is $2y+x=16$. The point $B$ has coordinates (2, 2). The perpendicular from $B$ to $AC$ meets $AC$ at the point $X$.

(i) Find the coordinates of $X$. [4]

The point $D$ is such that the quadrilateral $ABCD$ has $AC$ as a line of symmetry.

(ii) Find the coordinates of $D$. [2]

(iii) Find, correct to 1 decimal place, the perimeter of $ABCD$. [3]

earning CORNER

If
$\mathbf{a} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then
magnitude (or length) of
**a** is:
$|\mathbf{a}| = \sqrt{(a)^2 + (b)^2 + (c)^2}$

Learning CORNER

*Suggested Solution:*

(i) Equation of $AC$:

$$2y + x = 16 \;\Rightarrow\; 2y = -x + 16 \;\Rightarrow\; y = -\frac{1}{2}x + 8 \;\cdots\cdots\text{(i)}$$

$\therefore$ gradient of $AC = -\frac{1}{2}$

$\Rightarrow$ gradient of $BX = 2$ $(\because BX \perp AC)$

Since lines $AC$ and $BX$ are perpendicular to each other, therefore,
(grad. of $AC$) (grad. of $BX$) $= -1$

equation of $BX$:

$$y - 2 = 2(x - 2) \;\Rightarrow\; y - 2 = 2x - 4 \;\Rightarrow\; y = 2x - 2 \;\cdots\cdots\text{(ii)}$$

Equation of a straight line with gradient $m$ and passing through $(x_1, y_1)$ is:
$(y - y_1) = m(x - x_1)$.

lines $AC$ and $BX$ meet at $X$, therefore solving eq. (i) and (ii) simultaneously,

$$2x - 2 = -\frac{1}{2}x + 8$$

$$2x + \frac{1}{2}x = 10$$

$$\frac{5}{2}x = 10 \;\Rightarrow\; x = 4$$

putting $x$ in eq. (ii), $y = 2(4) - 2 = 6$

$\therefore$ coordinates of $X = (4, 6)$ **(Ans)**

(ii) Let point $D$ has coordinates $(x, y)$

As $AC$ is the line of symmetry of $ABCD$,

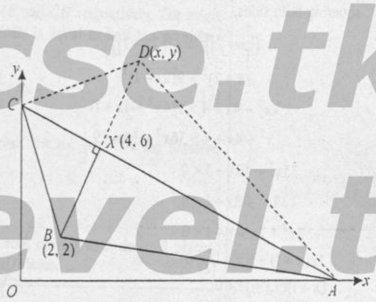
therefore $X$ is the mid-point of $BD$.

$$\Rightarrow \left(\frac{x+2}{2}, \frac{y+2}{2}\right) = (4, 6)$$

$$\Rightarrow \frac{x+2}{2} = 4 \quad\text{and}\quad \frac{y+2}{2} = 6$$

$$\Rightarrow x = 6 \quad\text{and}\quad y = 10$$

$\therefore$ coordinates of $D\,(6, 10)$ **(Ans)**



(iii) Equation of $AC$: $2y + x = 16$

for point $A$, put $y = 0$,

$\Rightarrow 2(0) + x = 16 \;\Rightarrow\; x = 16$

$\therefore$ coordinates of $A = (16, 0)$

for point $C$, put $x = 0$,

$\Rightarrow 2y + 0 = 16 \;\Rightarrow\; 2y = 16 \;\Rightarrow\; y = 8$

$\therefore$ coordinates of $C = (0, 8)$

length of $AB$: $|AB| = \sqrt{(16-2)^2 + (0-2)^2} = \sqrt{196+4} = \sqrt{200}$

length of $BC$: $|BC| = \sqrt{(0-2)^2 + (8-2)^2} = \sqrt{4+36} = \sqrt{40}$

Distance between two points,
$A(x_1, y_1)$ and $B(x_2, y_2)$ is:
$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\therefore$ perimeter of $ABCD = 2(AB + BC)$

$$= 2(\sqrt{200} + \sqrt{40})$$

$$= 2(20.4667) = 40.9334 \approx 40.9 \text{ units (3 sf)}$$ **(Ans)**

# June 2008 Paper 3

## *Pure Mathematics (P3)*

Answer all the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

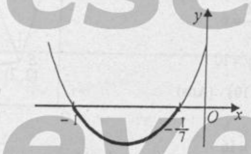
**1.** Solve the inequality $|x-2| > 3|2x+1|$ [4]

***Suggested Solution:***

$|x-2| > 3|2x+1|$

squaring both sides,

$$(|x-2|)^2 > (3|2x+1|)^2$$
$$(x-2)^2 > 9(2x+1)^2$$
$$x^2-4x+4 > 9(4x^2+4x+1)$$
$$x^2-4x+4 > 36x^2+36x+9$$
$$35x^2+40x+5<0$$
$$7x^2+8x+1<0$$
$$7x^2+7x+x+1<0$$
$$7x(x+1)+1(x+1)<0$$
$$(x+1)(7x+1)<0$$



critical values are: $x=-1$, $x=-\frac{1}{7}$

$\therefore \quad -1 < x < -\frac{1}{7}$

**2.** Solve, correct to 3 significant figures, the equation

$$e^x + e^{2x} = e^{3x}$$ [5]

***Suggested Solution:***

$e^x + e^{2x} = e^{3x}$

Let $e^x = y$

$\Rightarrow \quad y + y^2 = y^3$

$y^3 - y^2 - y = 0$

$y(y^2 - y - 1) = 0$

$\Rightarrow \quad y = 0$ or $y^2 - y - 1 = 0$

$\Rightarrow \quad y = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} \Rightarrow y = \dfrac{1 \pm \sqrt{5}}{2}$

$$y=\frac{1+\sqrt{5}}{2},\quad y=\frac{1-\sqrt{5}}{2}$$

$\therefore\quad y=0,\qquad y=1.618,\qquad y=-0.618$

as $\quad y=e^x$

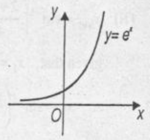
$\Rightarrow\quad e^x=0$ (impossible), $\qquad e^x=1.618,\qquad e^x=-0.618$ (impossible)

$\therefore\quad e^x=1.618$

$\ln(e^x)=\ln(1.618)$

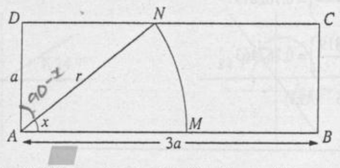
$x\ln e=\ln(1.618)$

$x=\ln(1.618)\ \Rightarrow\ x=0.481$ **(Ans)**



Exponential function is always positive.

**3.**



In the diagram, *ABCD* is a rectangle with *AB* = 3*a* and *AD* = *a*. A circular arc, with centre *A* and radius *r*, joins points *M* and *N* on *AB* and *CD* respectively. The angle *MAN* is *x* radians. The perimeter of the sector *AMN* is equal to half the perimeter of the rectangle.

(i) Show that *x* satisfies the equation $\sin x=\frac{1}{4}(2+x)$. [3]

(ii) This equation has only one root in the interval $0<x<\frac{1}{2}\pi$. Use the iterative formula $x_{n+1}=\sin^{-1}\left(\frac{2+x_n}{4}\right)$, with initial value $x_1$ = 0.8, to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

***Suggested Solution:***

(i) Using $S=r\theta$, arc length $\widehat{MN}=rx$

$\therefore$ perimeter of the sector $\widehat{AMN}=r+r+rx\ =\ 2r+rx$

consider $\Delta\, ADN$

$\angle AND=x$ radians (alternate angles)

$\therefore\ \sin x=\frac{a}{r}\ \Rightarrow\ a=r\sin x\ \cdots\cdots\cdots$(i)

given that,

perimeter of the sector $AMN=\frac{1}{2}$(perimeter of rectangle $ABCD$)

$$2r+rx=\frac{1}{2}[2(a+3a)]\ \Rightarrow\ r(2+x)=4a$$

substituting value of $a$ from eq. (i)

$$r(2+x)=4(r\sin x)$$

$$2+x=4\sin x$$

$$\sin x=\frac{2+x}{4}\quad\text{or}\quad \sin x=\frac{1}{4}(2+x)\quad \textbf{(Shown)}$$

(ii) $x_{n+1} = \sin^{-1}\left(\frac{2+x_n}{4}\right)$

given that, $x_1 = 0.8$

$\therefore \quad x_2 = \sin^{-1}\left(\frac{2+x_1}{4}\right) = \sin^{-1}\left(\frac{2+0.8}{4}\right) = 0.775397$

$x_3 = \sin^{-1}\left(\frac{2+x_2}{4}\right) = \sin^{-1}\left(\frac{2+0.775397}{4}\right) = 0.766820$

$x_4 = \sin^{-1}\left(\frac{2+x_3}{4}\right) = \sin^{-1}\left(\frac{2+0.766820}{4}\right) = 0.763847$

$x_5 = \sin^{-1}\left(\frac{2+x_4}{4}\right) = \sin^{-1}\left(\frac{2+0.763847}{4}\right) = 0.762819$

$x_6 = \sin^{-1}\left(\frac{2+x_5}{4}\right) = \sin^{-1}\left(\frac{2+0.762819}{4}\right) = 0.762463$

$\therefore$ root correct to 2 decimal places $= 0.76$ **(Ans)**

When the question did not specify the number of iterations to use, it is incorrect to use only one or two iterations of the formula. In fact, we need to continue until we get the same answer twice for the required degree of accuracy.

4. (i) Show that the equation $\tan(30° + \theta) = 2\tan(60° - \theta)$ can be written in the form $\tan^2\theta + (6\sqrt{3})\tan\theta - 5 = 0.$ [4]

(ii) Hence, or otherwise, solve the equation

$\tan(30° + \theta) = 2\tan(60° - \theta),$ for $0° \le \theta \le 180°.$ [3]

***Suggested Solution:***

(i) $\tan(30° + \theta) = 2\tan(60° - \theta)$

using compound angle formulae,

$$\frac{\tan 30° + \tan\theta}{1 - \tan 30° \tan\theta} = 2\left(\frac{\tan 60° - \tan\theta}{1 + \tan 60° \tan\theta}\right)$$

$$\frac{\frac{1}{\sqrt{3}} + \tan\theta}{1 - \frac{1}{\sqrt{3}}\tan\theta} = 2\left(\frac{\sqrt{3} - \tan\theta}{1 + \sqrt{3}\tan\theta}\right)$$

$$\frac{\frac{1+\sqrt{3}\tan\theta}{\sqrt{3}}}{\frac{\sqrt{3} - \tan\theta}{\sqrt{3}}} = \frac{2\left(\sqrt{3} - \tan\theta\right)}{1 + \sqrt{3}\tan\theta}$$

$$\frac{1+\sqrt{3}\tan\theta}{\sqrt{3} - \tan\theta} = \frac{2\left(\sqrt{3} - \tan\theta\right)}{1 + \sqrt{3}\tan\theta}$$

$$\left(1+\sqrt{3}\tan\theta\right)\left(1+\sqrt{3}\tan\theta\right) = 2\left(\sqrt{3} - \tan\theta\right)\left(\sqrt{3} - \tan\theta\right)$$

$$1 + 2\sqrt{3}\tan\theta + 3\tan^2\theta = 2\left[3 - 2\sqrt{3}\tan\theta + \tan^2\theta\right]$$

$$1 + 2\sqrt{3}\tan\theta + 3\tan^2\theta = 6 - 4\sqrt{3}\tan\theta + 2\tan^2\theta$$

$\tan^2\theta + 6\sqrt{3}\tan\theta - 5 = 0$ **(Shown)**

Remember:

$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Also note that:

$\tan 30° = \frac{1}{\sqrt{3}}$

$\tan 60° = \sqrt{3}$

(ii) $\tan(30^\circ+\theta)=2\tan(60^\circ-\theta)$

using the result of part (i), the above can be written as,

$$\tan^2\theta+6\sqrt{3}\,\tan\theta-5=0$$

using quadratic formula,

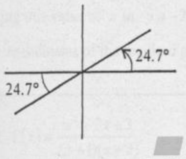
$$\tan\theta=\frac{-6\sqrt{3}\pm\sqrt{(6\sqrt{3})^2-4(1)(-5)}}{2(1)}=\frac{-6\sqrt{3}\pm\sqrt{108+20}}{2}$$

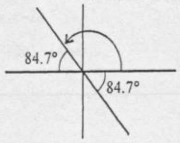
$\tan\theta=\frac{-6\sqrt{3}+\sqrt{128}}{2}$ or $\tan\theta=\frac{-6\sqrt{3}-\sqrt{128}}{2}$

$\tan\theta=0.461$ or $\tan\theta=-10.853$

basic angle $\alpha=24.8°$ or basic angle $\alpha=84.7°$

Note that:
$\tan\theta$ is positive in 1st and 3rd quadrant and negative in 2nd and 4th quadrant.





∴ for the given range, $\theta=24.7°,\ 95.3°$ **(Ans)**

5. The variable complex number $z$ is given by

$$z=2\cos\theta+\mathrm{i}(1-2\sin\theta),$$

where $\theta$ takes all values in the interval $-\pi<\theta\le\pi$.

(i) Show that $|z-\mathrm{i}|=2$, for all values of $\theta$. Hence sketch, in an Argand diagram, the locus of the point representing $z$. [3]

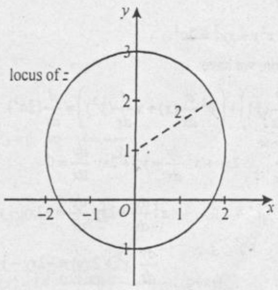
(ii) Prove that the real part of $\frac{1}{z+2-\mathrm{i}}$ is constant for $-\pi<\theta<\pi$. [4]

*Suggested Solution:*

(i) $z=2\cos\theta+\mathrm{i}(1-2\sin\theta)$

$$\begin{aligned}z-\mathrm{i}&=2\cos\theta+\mathrm{i}(1-2\sin\theta)-\mathrm{i}\\&=2\cos\theta+\mathrm{i}(1-2\sin\theta-1)\\&=2\cos\theta+\mathrm{i}(-2\sin\theta)\end{aligned}$$

$$\begin{aligned}\therefore\ |z-\mathrm{i}|&=\sqrt{(2\cos\theta)^2+(-2\sin\theta)^2}\\&=\sqrt{4\cos^2\theta+4\sin^2\theta}\\&=\sqrt{4(\cos^2\theta+\sin^2\theta)}\\&=\sqrt{4}\\&=2\quad\textbf{(Shown)}\end{aligned}$$



(ii) $z = 2\cos\theta + i(1 - 2\sin\theta)$

$$\frac{1}{z} = \frac{1}{2\cos\theta + i(1-2\sin\theta)}$$

$$\frac{1}{z+2-i} = \frac{1}{2\cos\theta + i(1-2\sin\theta) + 2 - i}$$

$$= \frac{1}{(2\cos\theta + 2) + i(1 - 2\sin\theta - 1)}$$

$$= \frac{1}{2(1+\cos\theta) + i(-2\sin\theta)}$$

$$= \frac{1}{2(1+\cos\theta - i\sin\theta)} \times \frac{2(1+\cos\theta + i\sin\theta)}{2(1+\cos\theta + i\sin\theta)}$$

$$= \frac{2(1+\cos\theta + i\sin\theta)}{4\left((1+\cos\theta)^2 - (i\sin\theta)^2\right)}$$

$$= \frac{1+\cos\theta + i\sin\theta}{2\left((1+\cos\theta)^2 - i^2\sin^2\theta\right)}$$

$$= \frac{1+\cos\theta + i\sin\theta}{2\left((1+\cos\theta)^2 + \sin^2\theta\right)}$$

$$= \frac{1+\cos\theta + i\sin\theta}{2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta)}$$

$$= \frac{1+\cos\theta + i\ \sin\theta}{2\,(2 + 2\cos\theta)}$$

$$= \frac{1+\cos\theta + i\ \sin\theta}{4\,(1+\cos\theta)}$$

$$= \frac{1+\cos\theta}{4\,(1+\cos\theta)} + \frac{i\ \sin\theta}{4\,(1+\cos\theta)} = \frac{1}{4} + i\frac{\sin\theta}{4\,(1+\cos\theta)}$$

$\therefore$ real part of $\dfrac{1}{z+2-i} = \dfrac{1}{4}$ which is a constant. **(Proved)**

> **6.** The equation of a curve is $xy(x+y) = 2a^3$, where $a$ is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the $x$-axis, and find the coordinates of this point. [8]

***Suggested Solution:***

$xy(x+y) = 2a^3 \Rightarrow x^2y + xy^2 = 2a^3$

using implicit derivation, we have

$$\left(y\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(y)\right) + \left(y^2\frac{d}{dx}(x) + x\frac{d}{dx}(y^2)\right) = \frac{d}{dx}(2a^3)$$

$$2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 0$$

$$x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$$

$$\frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2$$

$$\frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$$

as the tangent is parallel to $x$-axis, therefore $\frac{dy}{dx}=0$

$$\Rightarrow \quad \frac{-2xy-y^2}{x^2+2xy}=0 \quad \Rightarrow \quad -y(2x+y)=0$$

either $\quad -y=0 \quad$ or $\quad 2x+y=0$
$\Rightarrow \quad y=0 \quad$ or $\quad y=-2x$

putting $y=0$ in the equation of curve,

$$x(0)(x+0)=2a^3 \quad \Rightarrow \quad 2a^3=0 \quad \Rightarrow \quad a=0 \text{ which is untrue.}$$

putting $y=-2x$ in the equation of curve,

$$x(-2x)(x+(-2x))=2a^3$$

$$\Rightarrow \quad -2x^2(-x)=2a^3 \quad \Rightarrow \quad 2x^3=2a^3 \quad \Rightarrow \quad x^3=a^3 \quad \Rightarrow \quad x=a$$

putting this value of $x$ in $y=-2x$, we have $y=-2a$

$\therefore$ coordinates of the required point are: $(a, -2a)$ **(Ans)**

7. Let $\mathrm{f}(x)\equiv\frac{x^2+3x+3}{(x+1)(x+3)}$.

(i) Express f($x$) in partial fractions. [5]

(ii) Hence show that $\int_0^3 \mathrm{f}(x)\,\mathrm{d}x = 3-\frac{1}{2}\ln 2$. [4]

***Suggested Solution:***

(i) $\mathrm{f}(x)=\frac{x^2+3x+3}{(x+1)(x+3)}=\frac{x^2+3x+3}{x^2+4x+3}$

f($x$) is an improper fraction. Using long division,

$$\begin{array}{r|l} & \quad 1 \\ \hline x^2+4x+3 & x^2+3x+3 \\ & x^2+4x+3 \\ & - \quad - \quad - \\ \hline & \quad -x \end{array}$$

$$\therefore \quad \mathrm{f}(x)=1-\frac{x}{x^2+4x+3} \quad \Rightarrow \quad \mathrm{f}(x)=1-\frac{x}{(x+1)(x+3)}$$

now, $\quad \frac{x}{(x+1)(x+3)}\equiv\frac{A}{x+1}+\frac{B}{x+3}$

$$\Rightarrow \quad x\equiv A(x+3)+B(x+1)$$

for $x=-1$

$$-1=A(-1+3)+B(-1+1) \quad \Rightarrow \quad -1=A(2)+0 \quad \Rightarrow \quad A=-\frac{1}{2}$$

for $x=-3$

$$-3=A(-3+3)+B(-3+1) \quad \Rightarrow \quad -3=0-2B \quad \Rightarrow \quad B=\frac{3}{2}$$

$$\therefore \quad \mathrm{f}(x)=1-\left(-\frac{1}{2(x+1)}+\frac{3}{2(x+3)}\right) \quad \Rightarrow \quad \mathrm{f}(x)=1+\frac{1}{2(x+1)}-\frac{3}{2(x+3)} \quad \textbf{(Ans)}$$

(ii) Using the result of part (i).

$$\text{L.H.S.} = \int_2^3 f(x)\,dx$$

$$= \int_2^3 \left(1 + \frac{1}{2(x+1)} - \frac{3}{2(x+3)}\right) dx$$

$$= \left[x + \frac{1}{2}\ln|x+1| - \frac{3}{2}\ln|x+3|\right]_0^3$$

$$= \left[3 + \frac{1}{2}\ln(3+1) - \frac{3}{2}\ln(3+3)\right] - \left[0 + \frac{1}{2}\ln(0+1) - \frac{3}{2}\ln(0+3)\right]$$

$$= 3 + \frac{1}{2}\ln 4 - \frac{3}{2}\ln 6 - \frac{1}{2}\ln 1 + \frac{3}{2}\ln 3$$

$$= 3 + \frac{1}{2}\ln 2^2 - \frac{3}{2}\ln(2\times 3) + \frac{3}{2}\ln 3$$

$$= 3 + \ln 2 - \frac{3}{2}(\ln 2 + \ln 3) + \frac{3}{2}\ln 3$$

$$= 3 + \ln 2 - \frac{3}{2}\ln 2 - \frac{3}{2}\ln 3 + \frac{3}{2}\ln 3$$

$$= 3 + \ln 2 - \frac{3}{2}\ln 2$$

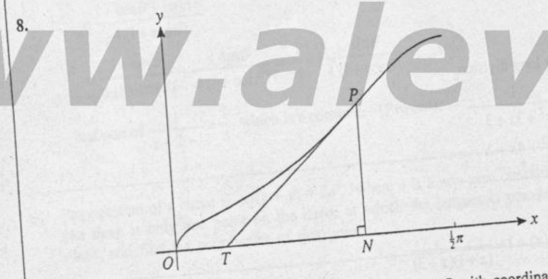
$$= 3 + (1 - \frac{3}{2})\ln 2 = 3 - \frac{1}{2}\ln 2 \quad \text{(Shown)}$$

Remember:
For definite integral, do not put the constant of integration.

Note that:

- $\ln 1 = 0$
- $\ln(a \times b) = \ln a + \ln b$
- $\ln(a^b) = b\ln a$

**8.**



In the diagram the tangent to a curve at a general point $P$ with coordinates $(x, y)$ meets the $x$-axis at $T$. The point $N$ on the $x$-axis is such that $PN$ is perpendicular to the $x$-axis. The curve is such that, for all values of $x$ in the interval $0 < x < \frac{1}{2}\pi$, the area of triangle $PTN$ is equal to $\tan x$, where $x$ is in radians.

(i) Using the fact that the gradient of the curve at $P$ is $\frac{PN}{TN}$, show that [3]

$$\frac{dy}{dx} = \frac{1}{2}y^2 \cot x.$$

(ii) Given that $y = 2$ when $x = \frac{1}{6}\pi$, solve this differential equation to find the equation of the curve, expressing $y$ in terms of $x$. [6]

***Suggested Solution:***

(i) Given that, area of $\Delta PTN = \tan x$

$\Rightarrow \quad \frac{1}{2} \times TN \times y = \tan x$ ………(i)

also given, $\frac{dy}{dx} = \frac{PN}{TN}$

$\Rightarrow \quad TN = PN \times \frac{dx}{dy} \quad \Rightarrow \quad TN = y\frac{dx}{dy}$ ………(ii)

substituting eq.(ii) into eq.(i)

$$\frac{1}{2} \times \left( y\frac{dx}{dy} \right) \times y = \tan x$$

$$\frac{1}{2} y^2 \frac{dx}{dy} = \tan x$$

$$\frac{dx}{dy} = \frac{2\tan x}{y^2}$$

$$\frac{dy}{dx} = \frac{y^2}{2\tan x}$$

$$= \frac{1}{2} y^2 \cot x \quad \textbf{(Shown)}$$

(ii) $\frac{dy}{dx} = \frac{1}{2} y^2 \cot x \quad \Rightarrow \quad y^{-2}\, dy = \frac{1}{2} \cot x \; dx$

integrating both sides

$$\int y^{-2}\, dy = \frac{1}{2} \int \cot x \, dx$$

$$\Rightarrow \quad \int y^{-2}\, dy = \frac{1}{2} \int \frac{\cos x}{\sin x} \, dx$$

$$\Rightarrow \quad \frac{y^{-1}}{-1} = \frac{1}{2} \ln|\sin x| + K \quad \Rightarrow \quad -\frac{1}{y} = \frac{1}{2} \ln|\sin x| + K \quad \text{………(A)}$$

given that $y = 2$ when $x = \frac{\pi}{6}$

$$-\frac{1}{2} = \frac{1}{2} \ln\left|\sin\frac{\pi}{6}\right| + K \quad \Rightarrow \quad -\frac{1}{2} = \frac{1}{2} \ln\frac{1}{2} + K \quad \Rightarrow \quad K = -\frac{1}{2} - \frac{1}{2} \ln\frac{1}{2}$$

$$\Rightarrow \quad K = -\frac{1}{2} - \frac{1}{2}(\ln 1 - \ln 2) \quad \Rightarrow \quad K = -\frac{1}{2} - \frac{1}{2}(-\ln 2) \quad \Rightarrow \quad K = \frac{1}{2}\ln 2 - \frac{1}{2}$$

$\therefore$ equation (A) becomes

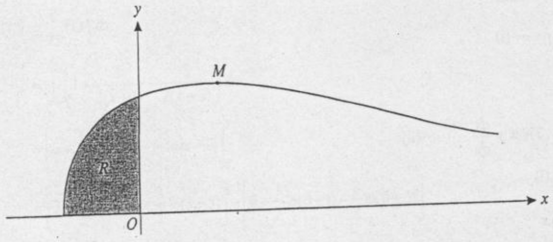
$$-\frac{1}{y} = \frac{1}{2} \ln|\sin x| + \frac{1}{2}\ln 2 - \frac{1}{2}$$

$$-\frac{2}{y} = \ln|\sin x| + \ln 2 - 1$$

$$y = -\frac{2}{\ln|\sin x| + \ln 2 - 1}$$

$$\Rightarrow \quad y = -\frac{2}{\ln|\sin x| - 0.307} \quad \text{or} \quad y = \frac{2}{0.307 - \ln|\sin x|} \quad \textbf{(Ans)}$$

9.



The diagram shows the curve $y = e^{-\frac{1}{2}x}\sqrt{(1+2x)}$ and its maximum point $M$. The shaded region between the curve and the axes is denoted by $R$.

(i) Find the $x$-coordinate of $M$. [4]

(ii) Find by integration the volume of the solid obtained when $R$ is rotated completely about the $x$-axis. Give your answer in terms of $\pi$ and e. [6]

***Suggested Solution:***

(i) $y = e^{-\frac{1}{2}x}\sqrt{(1+2x)}$

differentiating w.r.t. $x$

$$\frac{dy}{dx} = \sqrt{(1+2x)}\ \frac{d}{dx}(e^{-\frac{1}{2}x}) + e^{-\frac{1}{2}x}\ \frac{d}{dx}\sqrt{(1+2x)}$$

$$= \sqrt{(1+2x)}\left(-\frac{1}{2}e^{-\frac{1}{2}x}\right) + e^{-\frac{1}{2}x}\left(\frac{1}{2}(1+2x)^{-\frac{1}{2}} \times 2\right)$$

$$= -\frac{1}{2}e^{-\frac{1}{2}x}\sqrt{(1+2x)} + \frac{e^{-\frac{1}{2}x}}{\sqrt{(1+2x)}}$$

Remember:

$\frac{d}{dx}(uv) = v\frac{d}{dx}u + u\frac{d}{dx}v$

for maximum or minimum point, $\frac{dy}{dx} = 0$

$$\Rightarrow\ -\frac{1}{2}e^{-\frac{1}{2}x}\sqrt{(1+2x)} + \frac{e^{-\frac{1}{2}x}}{\sqrt{(1+2x)}} = 0$$

$$\frac{-e^{-\frac{1}{2}x}(1+2x) + 2e^{-\frac{1}{2}x}}{2\sqrt{(1+2x)}} = 0$$

$$-e^{-\frac{1}{2}x}(1+2x) + 2e^{-\frac{1}{2}x} = 0$$

$$e^{-\frac{1}{2}x}(-1-2x+2) = 0$$

$$e^{-\frac{1}{2}x}(1-2x) = 0$$

$\Rightarrow\ e^{-\frac{1}{2}x} = 0$ (not possible). or $(1-2x) = 0 \ \Rightarrow\ x = \frac{1}{2}$

$\therefore$ $x$-coordinate of $M$ is: $x = \frac{1}{2}$ **(Ans)**

(ii) $y = \mathrm{e}^{-\frac{1}{2}x}\sqrt{(1+2x)}$

for $x$-intercept, put $y = 0$.

$\Rightarrow \quad \mathrm{e}^{-\frac{1}{2}x}\sqrt{(1+2x)} = 0$

$\Rightarrow \quad \mathrm{e}^{-\frac{1}{2}x} = 0$ (impossible) or $\sqrt{(1+2x)} = 0 \Rightarrow 1+2x = 0 \Rightarrow x = -\frac{1}{2}$

volume of $R$: $V = \pi \int_{-\frac{1}{2}}^{0} y^2\, dx$

$$= \pi \int_{-\frac{1}{2}}^{0} \left(\mathrm{e}^{-\frac{1}{2}x}\sqrt{(1+2x)}\right)^2 dx = \pi \int_{-\frac{1}{2}}^{0} \mathrm{e}^{-x}(1+2x)\, dx$$

using integration by parts

Formula for integration by parts:

$\int uv\, dx$
$= u\int v\, dx - \int\left(\frac{d}{dx}u\right)\left(\int v\, dx\right) dx$

$$V = \pi\left[(1+2x)\int \mathrm{e}^{-x}\, dx - \int\left(\int \mathrm{e}^{-x} dx \times \frac{d}{dx}(1+2x)\right) dx\right]_{-\frac{1}{2}}^{0}$$

$$= \pi\left[(1+2x)(-\mathrm{e}^{-x}) - \int(-\mathrm{e}^{-x}) \times 2\, dx\right]_{-\frac{1}{2}}^{0}$$

$$= \pi\left[-\mathrm{e}^{-x}(1+2x) + 2\int \mathrm{e}^{-x}\, dx\right]_{-\frac{1}{2}}^{0}$$

$$= \pi\left[-\mathrm{e}^{-x}(1+2x) + 2(-\mathrm{e}^{-x})\right]_{-\frac{1}{2}}^{0}$$

$$= \pi\left[\mathrm{e}^{-x}(-1-2x-2)\right]_{-\frac{1}{2}}^{0}$$

$$= \pi\left[\mathrm{e}^{-x}(-2x-3)\right]_{-\frac{1}{2}}^{0}$$

$$= \pi\left[\left(\mathrm{e}^{-0}(-2(0)-3)\right) - \left(\mathrm{e}^{-(-\frac{1}{2})}(-2(-\frac{1}{2})-3)\right)\right]$$

$$= \pi\left(-3 - \mathrm{e}^{\frac{1}{2}}(-2)\right) = \pi\left(-3 + 2\sqrt{\mathrm{e}}\right) \text{ units}^3 \quad \textbf{(Ans)}$$

10. The points $A$ and $B$ have position vectors, relative to the origin $O$, given by

$$\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}.$$

The line $l$ has vector equation

$$\mathbf{r} = (1-2t)\mathbf{i} + (5+t)\mathbf{j} + (2-t)\mathbf{k}.$$

(i) Show that $l$ does not intersect the line passing through $A$ and $B$. [4]

(ii) The point $P$ lies on $l$ and is such that angle $PAB$ is equal to 60°. Given that the position vector of $P$ is $(1-2t)\mathbf{i} + (5+t)\mathbf{j} + (2-t)\mathbf{k}$, show that $3t^2 + 7t + 2 = 0$. Hence find the only possible position vector of $P$. [6]

*Suggested Solution:*

(i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\Rightarrow$ equation of the line $AB$ is: $\mathbf{r}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Equation of the line $l$: $\mathbf{r} = (1-2t)\mathbf{i} + (5+t)\mathbf{j} + (2-t)\mathbf{k}$

$$= (\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}) + t(-2\mathbf{i} + \mathbf{j} - \mathbf{k}) = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

from the two equations, we see that the direction vectors of line $AB$ and line $l$ i.e.

$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$ are not parallel. Therefore the two lines are not parallel.

If the two lines intersect, then

$$\mathbf{r}_1 = \mathbf{r}$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1+s \\ 2-s \\ 3 \end{pmatrix} = \begin{pmatrix} 1-2t \\ 5+t \\ 2-t \end{pmatrix}$$

equating the coefficients of i and j, we have

$1+s = 1-2t \Rightarrow s = -2t$ ………(i)

$2-s = 5+t \Rightarrow s+t = -3$ ………(ii)

substituting eq. (i) into eq. (ii) and solving simultaneously,

$-2t + t = -3 \Rightarrow -t = -3 \Rightarrow t = 3$

$\therefore\ s = -2(3) = -6$

putting these values in the $z$-components of the two lines,

$z$-component of $\mathbf{r}_1 = 3$

$z$-component of $\mathbf{r} = 2 - 3 = -1$

$\Rightarrow$ $z$-component of $\mathbf{r}_1 \neq z$-component of $\mathbf{r}$

$\therefore$ the two lines do not intersect. **(Shown)**

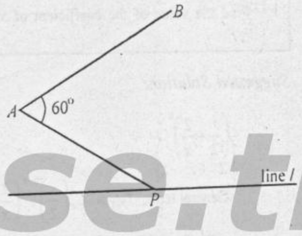
(ii) Given that, $\vec{OP} = (1-2t)\mathbf{i} + (5+t)\mathbf{j} + (2-t)\mathbf{k} = \begin{pmatrix} 1-2t \\ 5+t \\ 2-t \end{pmatrix}$

$$\vec{AP} = \vec{OP} - \vec{OA} = \begin{pmatrix} 1-2t \\ 5+t \\ 2-t \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2t \\ 3+t \\ -1-t \end{pmatrix}$$ and, from part (i), $\vec{AB} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

applying scalar product

$$\vec{AP} \cdot \vec{AB} = \left|\vec{AP}\right|\left|\vec{AB}\right| \cos P\hat{A}B$$

$$\begin{pmatrix} -2t \\ 3+t \\ -1-t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \left(\sqrt{(-2t)^2 + (3+t)^2 + (-1-t)^2}\right)\left(\sqrt{(1)^2 + (-1)^2 + (0)^2}\right)\cos 60°$$



$$-2t - 3 - t = \left(\sqrt{4t^2 + t^2 + 6t + 9 + 1 + 2t + t^2}\right)\left(\sqrt{2}\right)\left(\frac{1}{2}\right)$$

$$-3t - 3 = \left(\sqrt{6t^2 + 8t + 10}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$-3(t+1) = \left(\sqrt{6t^2 + 8t + 10}\right)\left(\frac{1}{\sqrt{2}}\right)$$

squaring both sides

$$9(t^2 + 2t + 1) = (6t^2 + 8t + 10)\left(\frac{1}{2}\right)$$

$$9t^2 + 18t + 9 = 3t^2 + 4t + 5$$

$$6t^2 + 14t + 4 = 0$$

$$3t^2 + 7t + 2 = 0 \quad \textbf{(Shown)}$$

Now, solving the above equation, we have

$$3t^2 + 7t + 2 = 0$$

$$3t^2 + 6t + t + 2 = 0$$

$$3t(t+2) + 1(t+2) = 0$$

$$(t+2)(3t+1) = 0$$

$$t + 2 = 0 \quad \text{or} \quad 3t + 1 = 0$$

$$\Rightarrow \quad t = -2 \quad \text{or} \quad t = -\frac{1}{3}$$

$t = -\frac{1}{3}$ gives $\angle PAB$ as an obtuse and is therefore rejected.

for $t = -2$,

$$\vec{OP} = \begin{pmatrix} 1-2(-2) \\ 5+(-2) \\ 2-(-2) \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$$

$\therefore$ position vector of the point $P$ is: $\begin{pmatrix} 5 \\ 3 \\ 4 \end{pmatrix}$ or $5\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ **(Ans)**

# November 2008 Paper 1

## Pure Mathematics (P1)

Answer all the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Find the value of the coefficient of $x^2$ in the expansion of $\left(\frac{x}{2}+\frac{2}{x}\right)^6$. [3]

***Suggested Solution:***

$$\left(\frac{x}{2}+\frac{2}{x}\right)^6$$

General term, $T_{r+1} = {}^6C_r\left(\frac{x}{2}\right)^{6-r}\left(\frac{2}{x}\right)^r$

collecting powers of $x$ only, we have,

$$\frac{x^{6-r}}{x^r} = x^{6-2r}$$

we need the value of $r$ for which $x^{6-2r} = x^2$

$$\Rightarrow \quad 6-2r = 2 \quad \Rightarrow \quad 2r = 4 \quad \Rightarrow \quad r = 2$$

$$\therefore \quad T_{2+1} = {}^6C_2\left(\frac{x}{2}\right)^{6-2}\left(\frac{2}{x}\right)^2$$

$$\Rightarrow \quad T_3 = 15\left(\frac{x}{2}\right)^4\left(\frac{2}{x}\right)^2 = 15\left(\frac{x^4}{16}\right)\left(\frac{4}{x^2}\right) = \frac{15}{4}x^2$$

$\therefore$ coefficient of $x^2 = \frac{15}{4} = 3\frac{3}{4}$ **(Ans).**

Expression for general te
is: $T_{r+1} = {}^nC_r(a)^{n-r}(b)^r$
You need to collect pow
of $x$ from general term an
equate them to the power o
$x$ required

**2.** Prove the identity

$$\frac{1+\sin x}{\cos x}+\frac{\cos x}{1+\sin x} \equiv \frac{2}{\cos x}.$$ [4]

***Suggested Solution:***

$$\text{L.H.S.} = \frac{1+\sin x}{\cos x}+\frac{\cos x}{1+\sin x}$$

$$= \frac{(1+\sin x)^2+\cos^2 x}{\cos x(1+\sin x)} = \frac{1+2\sin x+\sin^2 x+\cos^2 x}{\cos x(1+\sin x)} = \frac{1+2\sin x+1}{\cos x(1+\sin x)}$$

$$= \frac{2+2\sin x}{\cos x(1+\sin x)} = \frac{2(1+\sin x)}{\cos x(1+\sin x)} = \frac{2}{\cos x} = \text{R.H.S.} \quad \textbf{(Proved).}$$

Recall:
$\sin^2\theta+\cos^2\theta = 1$

sion for general term
$T_{r+1} = {}^nC_r\,(a)^{n-r}(b)^r$
eed to collect powers
from general term and
them to the power of
quired

$\theta + \cos^2\theta = 1$

3. The first term of an arithmetic progression is 6 and the fifth term is 12. The progression has $n$ terms and the sum of all the terms is 90. Find the value of $n$. [4]

***Suggested Solution:***

Given that, $a=6,\ \ T_5=12$

using $T_n = a+(n-1)d$

$$12 = 6+(5-1)d \ \Rightarrow\ 6=4d \ \Rightarrow\ d=\frac{3}{2}$$

also given that, $S_n = 90$

using $S_n = \frac{n}{2}\big(2a+(n-1)d\big)$

$$90 = \frac{n}{2}\left(2(6)+(n-1)\frac{3}{2}\right)$$

$$180 = 12n+\frac{3}{2}n(n-1)$$

$$360 = 24n+3n(n-1)$$

$$360 = 24n+3n^2-3n$$

$$3n^2+21n-360=0$$

$$n^2+7n-120=0$$

$$n^2+15n-8n-120=0$$

$$n(n+15)-8(n+15)=0$$

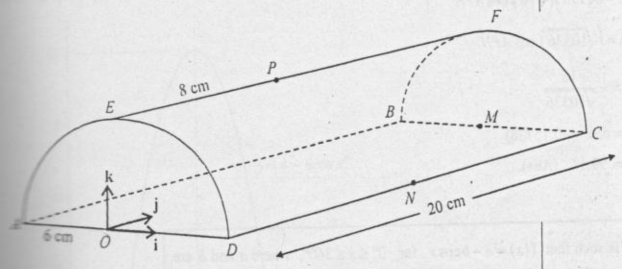
$$(n+15)(n-8)=0$$

$$n+15=0 \quad \text{or} \quad n-8=0$$

$$n=-15\text{(ignored)} \quad \text{or} \quad n=8$$

$\therefore\ n=8$ **(Ans).**

4.



The diagram shows a semicircular prism with a horizontal rectangular base $ABCD$. The vertical ends $AED$ and $BFC$ are semicircles of radius 6cm. The length of the prism is 20cm. The mid-point of $AD$ is the origin $O$, the mid-point of $BC$ is $M$ and the mid-point of $DC$ is $N$. The points $E$ and $F$ are the highest points of the semicircular ends of the prism. The point $P$ lies on $EF$ such that $EP$ = 8cm.

Unit vectors **i**, **j** and **k** are parallel to $OD$, $OM$ and $OE$ respectively.

(i) Express each of the vectors $\overrightarrow{PA}$ and $\overrightarrow{PN}$ in terms of **i**, **j** and **k**. [3]

(ii) Use a scalar product to calculate angle $APN$. [4]

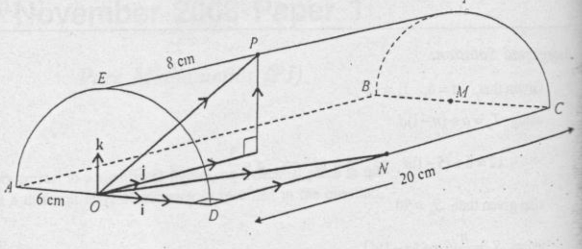
*Suggested Solution:*

(i) We have,

$$\vec{OA} = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix},$$

$$\vec{OP} = \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix},$$

$$\vec{ON} = \begin{pmatrix} 6 \\ 10 \\ 0 \end{pmatrix}$$



$$\vec{PA} = \vec{OA} - \vec{OP} = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} \Rightarrow \vec{PA} = \begin{pmatrix} -6 \\ -8 \\ -6 \end{pmatrix} \quad \textbf{(Ans).}$$

$$\vec{PN} = \vec{ON} - \vec{OP} = \begin{pmatrix} 6 \\ 10 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 8 \\ 6 \end{pmatrix} \Rightarrow \vec{PN} = \begin{pmatrix} 6 \\ 2 \\ -6 \end{pmatrix} \quad \textbf{(Ans).}$$

(ii) Applying scalar product,

$$\vec{PA} \cdot \vec{PN} = \left|\vec{PA}\right|\left|\vec{PN}\right| \cos A\hat{P}N$$

$$\begin{pmatrix} -6 \\ -8 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ -6 \end{pmatrix} = \left(\sqrt{(-6)^2 + (-8)^2 + (-6)^2}\right)\left(\sqrt{(6)^2 + (2)^2 + (-6)^2}\right) \cos A\hat{P}N$$

$$-36 - 16 + 36 = \left(\sqrt{36 + 64 + 36}\right)\left(\sqrt{36 + 4 + 36}\right) \cos A\hat{P}N$$

$$-16 = \left(\sqrt{136}\right)\left(\sqrt{76}\right) \cos A\hat{P}N$$

$$-16 = \left(\sqrt{10336}\right) \cos A\hat{P}N$$

$$\cos A\hat{P}N = -\frac{16}{\sqrt{10336}}$$

$$A\hat{P}N = \cos^{-1}(-0.15738)$$

$$= 99.1° \quad \textbf{(Ans).}$$

If

$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$\mathbf{a}.\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

and magnitude of **a** is

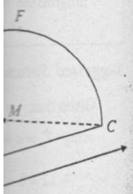
$|\mathbf{a}| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$

**5.** The function f is such that $f(x) = a - b\cos x$ for $0° \le x \le 360°$, where $a$ and $b$ are positive constants. The maximum value of f($x$) is 10 and the minimum value is −2.

(i) Find the values of $a$ and $b$. [3]

(ii) Solve the equation f($x$) = 0. [3]

(iii) Sketch the graph of $y$ = f($x$). [2]



$= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$\mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

d magnitude of **a** is

$| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$

---

***Suggested Solution:***

(i) We know that

$$-1 \le \cos x \le 1$$

multiplying the inequality by $-b$

$$(-b)(-1) \ge (-b)\cos x \ge (-b)(1)$$

$\Rightarrow \quad b \ge -b\cos x \ge -b$

$\Rightarrow \quad -b \le -b\cos x \le b$

adding $a$ throughout, we get

$$a-b \le a-b\cos x \le a+b$$

$\therefore$ minimum value $= a-b$, and maximum value $= a+b$

$\Rightarrow \quad a-b=-2 \cdots\cdots\cdots$(i)

$a+b=10 \cdots\cdots\cdots$(ii)

Note that the inequality sign changes when an inequality is multiplied by a negative number.

solving equations (i) and (ii) simultaneously,

$$\begin{array}{r} a+b=\ 10 \\ a-b=-2 \\ \hline 2a \quad =\ 8 \end{array} \quad \Rightarrow \quad a=4 \ \ \textbf{(Ans).}$$

putting the value of $a$ in eq. (ii), $4+b=10 \Rightarrow b=6$ **(Ans).**

(ii) Using the result of part (i), $f(x)$ can be written as,

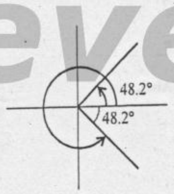
$$f(x) = 4-6\cos x$$

given that $f(x)=0$

$\Rightarrow \quad 4-6\cos x = 0 \quad \Rightarrow \quad \cos x = \frac{4}{6} \quad \Rightarrow \quad \cos x = \frac{2}{3}$

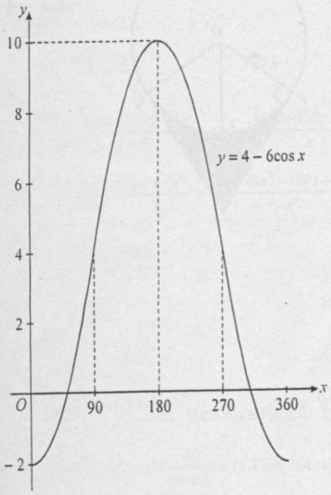
basic angle $\alpha = 48.2°$

$\therefore \quad x = 48.2°, \ 311.8°$ **(Ans).**

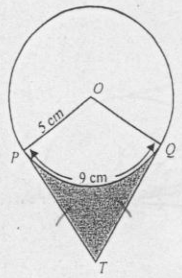


Note that $\cos\theta$ is positive in I and IV quadrant.

(iii)



6.



In the diagram, the circle has centre $O$ and radius 5cm. The points $P$ and $Q$ lie on the circle, and the arc length $PQ$ is 9cm. The tangents to the circle at $P$ and $Q$ meet at the point $T$. Calculate

(i) angle $POQ$ in radians, [2]

(ii) the length of $PT$, [3]

(iii) the area of the shaded region. [3]

***Suggested Solution:***

(i) Using $S = r\theta$ in the sector $\overset{\frown}{OPQ}$,

$$9 = (5)\theta \Rightarrow \theta = \frac{9}{5} = 1.8$$

$\therefore$ angle $POQ = 1.8$ radians **(Ans).**

Angle is in radians. Therefore change your calculator to radian mode

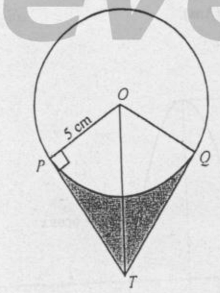
(ii) In right angled $\Delta OPT$

$$P\hat{O}T = \frac{1}{2}(1.8) = 0.9 \text{ rad.}$$

$$\tan P\hat{O}T = \frac{PT}{OP}$$

$$\tan(0.9) = \frac{PT}{5}$$

$$5\tan(0.9) = PT \Rightarrow PT = 6.3 \text{ cm} \quad \textbf{(Ans).}$$



(iii) Area of $\Delta OPT = \frac{1}{2}(6.3)(5)$

$$= 15.75 \text{ cm}^2$$

area of kite $OPTQ = 2(\text{area of } \Delta OPT)$

$$= 2(15.75) = 31.5 \text{ cm}^2$$

area of minor sector $\overset{\frown}{OPQ} = \frac{1}{2}(5)^2(1.8)$

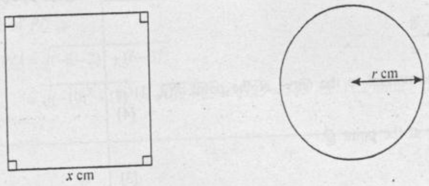
$$= 22.5 \text{ cm}^2$$

shaded area = area of the kite $OPTQ$ − area of sector $\overset{\frown}{OPQ}$

$$= 31.5 - 22.5 = 9 \text{ cm}^2 \quad \textbf{(Ans).}$$

in radians. Therefore
your calculator to
node

7.



A wire, 80cm long, is cut into two pieces. One piece is bent to form a square of side $x$ cm and the other piece is bent to form a circle of radius $r$ cm (see diagram). The total area of the square and the circle is $A$ cm$^2$.

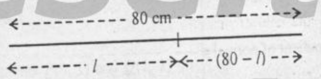
(i) Show that $A = \dfrac{(\pi+4)x^2 - 160x + 1600}{\pi}$. [4]

(ii) Given that $x$ and $r$ can vary, find the value of $x$ for which $A$ has a stationary value. [4]

***Suggested Solution:***

(i) Let the length of one piece = $l$ cm

$\therefore$ length of the other piece = $80 - l$ cm



perimeter of square = $4x$

$\Rightarrow \quad l = 4x$

perimeter of circle = $80 - l$

$\Rightarrow \quad 2\pi r = 80 - l$

eliminating $l$, we have,

$$2\pi r = 80 - 4x \;\Rightarrow\; r = \frac{80-4x}{2\pi} \;\Rightarrow\; r = \frac{40-2x}{\pi}$$

given that, total area of square and the circle is = $A$ cm$^2$

$\therefore \quad A = x^2 + \pi r^2$

substituting the value of $r$,

$$A = x^2 + \pi\left(\frac{40-2x}{\pi}\right)^2$$

$$= x^2 + \pi\left(\frac{1600-160x+4x^2}{\pi^2}\right) = x^2 + \frac{1600-160x+4x^2}{\pi}$$

$$= \frac{\pi x^2 + 1600 - 160x + 4x^2}{\pi} = \frac{(\pi+4)x^2 - 160x + 1600}{\pi} \quad \textbf{(Shown).}$$

(ii) $A = \dfrac{(\pi+4)x^2 - 160x + 1600}{\pi}$.

$$\frac{dA}{dx} = \frac{2(\pi+4)x - 160}{\pi}$$

for stationary values, $\dfrac{dA}{dx} = 0$

$$\Rightarrow \frac{2(\pi+4)x-160}{\pi} = 0 \;\Rightarrow\; 2(\pi+4)x - 160 = 0 \;\Rightarrow\; 2(\pi+4)x = 160$$

$$\Rightarrow (\pi+4)x = 80 \;\Rightarrow\; x = \frac{80}{(\pi+4)} = 11.2 \text{ cm} \quad \textbf{(Ans).}$$

8. The equation of a curve is $y = 5 - \frac{8}{x}$.

(i) Show that the equation of the normal to the curve at the point $P(2,\ 1)$ is $2y + x = 4$. [4]

This normal meets the curve again at the point $Q$.

(ii) Find the coordinates of $Q$. [3]

(iii) Find the length of $PQ$. [2]

***Suggested Solution:***

(i) $y = 5 - \frac{8}{x} \quad \Rightarrow \quad y = 5 - 8x^{-1}$

$\frac{dy}{dx} = 8x^{-2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{8}{x^2}$

gradient of tangent at $P(2,\ 1)$ is:

$\frac{dy}{dx} = \frac{8}{2^2} = 2$

$\therefore$ gradient of normal $= -\frac{1}{2}$

equation of normal at point $P$:

$y - 1 = -\frac{1}{2}(x - 2) \quad \Rightarrow \quad 2y - 2 = -x + 2 \quad \Rightarrow \quad 2y + x = 4$ **(Shown).**

Tangent and normal to the curve are perpendicular to each other, and for perpendicular lines:
(grad. of tangent) × (grad. of normal) = −1.

Equation of a straight line with gradient $m$ and passing through $(x_1,\ y_1)$ is:
$(y - y_1) = m(x - x_1)$.

(ii) Equation of curve: $y = 5 - \frac{8}{x}$ .........(i)

equation of the normal: $2y + x = 4 \quad \Rightarrow \quad y = \frac{4 - x}{2}$ .........(ii)

solving equations (i) & (ii) simultaneously, we have

$$5 - \frac{8}{x} = \frac{4 - x}{2}$$
$$\frac{5x - 8}{x} = \frac{4 - x}{2}$$
$$2(5x - 8) = x(4 - x)$$
$$10x - 16 = 4x - x^2$$
$$x^2 + 6x - 16 = 0$$
$$x^2 + 8x - 2x - 16 = 0$$
$$x(x + 8) - 2(x + 8) = 0$$
$$(x + 8)(x - 2) = 0$$

$\Rightarrow \quad x = -8, \quad$ or $\quad x = 2$

when $x = -8, \quad y = 5 - \frac{8}{-8} \quad \Rightarrow \quad y = 5 + 1 = 6$

when $x = 2, \quad y = 5 - \frac{8}{2} \quad \Rightarrow \quad y = 5 - 4 = 1$

$\Rightarrow$ the normal meets the curve at $(2,\ 1)$ and $(-8,\ 6)$

$\therefore$ coordinates of point $Q\ (-8,\ 6)$ **(Ans).**

**(iii)** From part (ii), we have $P(2,\ 1)$ and $Q(-8,\ 6)$

length of $PQ$ is:

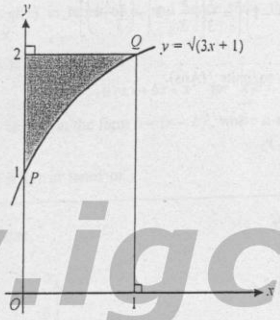
$$|PQ| = \sqrt{(-8-2)^2 + (6-1)^2}$$

$$= \sqrt{(-10)^2 + (5)^2} = \sqrt{100+25} = \sqrt{125} = 5\sqrt{5} \text{ or } 11.2 \text{ units.} \quad \textbf{(Ans).}$$

Distance formula:

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**9.**



The diagram shows the curve $y = \sqrt{(3x+1)}$ and the points $P(0,\ 1)$ and $Q(1,\ 2)$ on the curve. The shaded region is bounded by the curve, the $y$-axis and the line $y = 2$.

(i) Find the area of the shaded region. [4]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the $x$-axis. [4]

Tangents are drawn to the curve at the points $P$ and $Q$.

(iii) Find the acute angle, in degrees correct to 1 decimal place, between the two tangents. [4]

*Suggested Solution:*

(i) Area of unshaded region, under the curve $PQ$ is:

$$\int_0^1 y\,dx = \int_0^1 \sqrt{3x+1}\,dx$$

$$= \int_0^1 (3x+1)^{\frac{1}{2}}\,dx$$

$$= \left[\frac{(3x+1)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(3)}\right]_0^1 = \frac{2}{9}\left[(3x+1)^{\frac{3}{2}}\right]_0^1 = \frac{2}{9}\left((3(1)+1)^{\frac{3}{2}} - (3(0)+1)^{\frac{3}{2}}\right)$$

For definite integral, do not put the integration constant

$$= \frac{2}{9}\left((4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right) = \frac{2}{9}\left((2)^3 - 1\right) = \frac{2}{9}(8-1) = \frac{14}{9} \text{ units}^2$$

area of rectangle $= 1 \times 2 = 2$ units$^2$

$\therefore$ area of the shaded region = area of rectangle − area of under the curve $PQ$

$$= 2 - \frac{14}{9} = \frac{4}{9} = 0.444 \text{ sq.units} \quad \textbf{(Ans).}$$

*Alternative Solution to part (i):*

$$y=\sqrt{3x+1} \Rightarrow y^2=3x+1 \Rightarrow 3x=y^2-1 \Rightarrow x=\frac{1}{3}(y^2-1)$$

Shaded area, $A=\int_1^2 x\,dy = \int_1^2 \frac{1}{3}(y^2-1)\,dy = \frac{1}{3}\int_1^2 (y^2-1)\,dy$

$$=\frac{1}{3}\left[\frac{y^3}{3}-y\right]_1^2 = \frac{1}{3}\left[\left(\frac{(2)^3}{3}-2\right)-\left(\frac{(1)^3}{3}-1\right)\right] = \frac{1}{3}\left[\left(\frac{8}{3}-2\right)-\left(\frac{1}{3}-1\right)\right]$$

$$=\frac{1}{3}\left(\frac{2}{3}+\frac{2}{3}\right) = \frac{1}{3}\left(\frac{4}{3}\right) = \frac{4}{9} \text{ or } 0.444 \text{ sq. units} \quad \textbf{(Ans).}$$

(ii) Let equation of line be $y_1$ and equation of the curve be $y_2$

$\Rightarrow \quad y_1 = 2$ and $y_2=\sqrt{3x+1}$

volume of shaded region, $V=\pi\int_0^1 (y_1)^2\,dx - \pi\int_0^1 (y_2)^2\,dx$

$$=\pi\int_0^1 (2)^2\,dx - \pi\int_0^1 (\sqrt{3x+1})^2\,dx$$

$$=4\pi\int_0^1 1\,dx - \pi\int_0^1 (3x+1)\,dx$$

$$=4\pi\left[\,x\,\right]_0^1 - \pi\left[\frac{3x^2}{2}+x\right]_0^1$$

$$=4\pi[1-0]-\pi\left[\left(\frac{3(1)^2}{2}+1\right)-\left(\frac{3(0)^2}{2}+0\right)\right]$$

$$=4\pi-\pi\left[\left(\frac{3}{2}+1\right)-0\right] = 4\pi-\frac{5}{2}\pi = \frac{3}{2}\pi \text{ units}^3 \quad \textbf{(Ans).}$$

(iii) $y=\sqrt{3x+1}=(3x+1)^{\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{1}{2}(3x+1)^{-\frac{1}{2}}\times 3 = \frac{3}{2\sqrt{3x+1}}$$

gradient of tangent at $P(0,\ 1)$ is: $\frac{dy}{dx}=\frac{3}{2\sqrt{3(0)+1}}=\frac{3}{2\sqrt{1}}=\frac{3}{2}$

$\therefore \quad \tan\beta=\frac{3}{2} \quad \Rightarrow \quad \beta=56.3°$

gradient of tangent at $Q(1,\ 2)$ is: $\frac{dy}{dx}=\frac{3}{2\sqrt{3(1)+1}}=\frac{3}{2\sqrt{4}}=\frac{3}{4}$

$\therefore \quad \tan\alpha=\frac{3}{4} \quad \Rightarrow \quad \alpha=36.9°$

from the figure, we have,

$\alpha+\theta=\beta$

$\theta=\beta-\alpha$

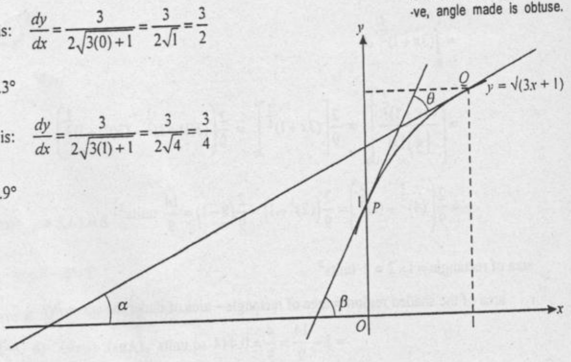
$=56.3°-36.9°$

$=19.4°$

$\therefore$ angle between the two tangents, $\theta=19.4°$ **(Ans).**

Gradient of a line is also defined as the tangent of the angle which a line makes with the +ve direction of x-axis.

If gradient is +ve, angle made is acute and if gradient is -ve, angle made is obtuse.



**10.** The function f is defined by

$$f : x \mapsto 3x - 2 \quad \text{for} \quad x \in \mathbb{R}.$$

(i) Sketch, in a single diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between the two graphs. [2]

The function g is defined by

$$g : x \mapsto 6x - x^2 \quad \text{for} \quad x \in \mathbb{R}.$$

(ii) Express $gf(x)$ in terms of $x$, and hence show that the maximum value of $gf(x)$ is 9. [5]

The function h is defined by

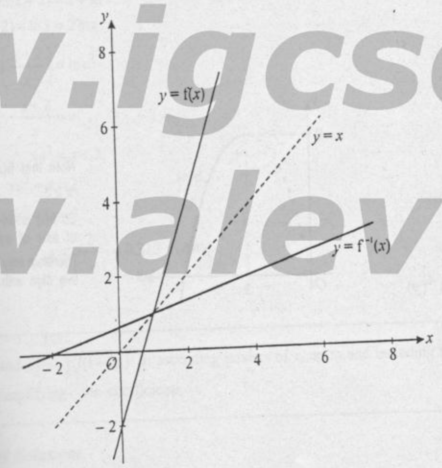
$$h : x \mapsto 6x - x^2 \quad \text{for} \quad x \geq 3.$$

(iii) Express $6x - x^2$ in the form $a - (x - b)^2$, where $a$ and $b$ are positive constants. [2]

(iv) Express $h^{-1}(x)$ in terms of $x$. [3]

***Suggested Solution:***

(i)



Graph of $y = f^{-1}(x)$ is a reflection of graph of $y = f(x)$ in the line $y = x$.

(ii) Given that, $f(x) = 3x - 2$, and $g(x) = 6x - x^2$

$$\begin{aligned} gf(x) &= 6(f(x)) - (f(x))^2 \\ &= 6(3x-2) - (3x-2)^2 \\ &= 18x - 12 - (9x^2 - 12x + 4) \\ &= 18x - 12 - 9x^2 + 12x - 4 \\ &= -9x^2 + 30x - 16 \end{aligned}$$

gf(x) represents a parabola opening downwards

$\therefore$ maximum value of $\mathrm{g\,f}(x)$ is given by,

$$y = \frac{4ac - b^2}{4a}$$

$a = -9, \quad b = 30 \quad c = -16$

$$\Rightarrow \quad y = \frac{4(-9)(-16) - (30)^2}{4(-9)} = \frac{576 - 900}{-36} = \frac{-324}{-36} = 9$$

$\therefore$ maximum value of $\mathrm{g\,f}(x) = 9$ **(Shown).**

**(iii)** $\mathrm{h}(x) = 6x - x^2$

$$= -x^2 + 6x = -(x^2 - 6x)$$

by completing the square, we have,

$$\mathrm{h}(x) = -(x^2 - 6x + (3)^2 - (3)^2)$$
$$= -((x-3)^2 - 9)$$
$$= -(x-3)^2 + 9$$
$$= 9 - (x-3)^2 \quad \textbf{(Ans).}$$

**(iv)** From part (iii), $\mathrm{h}(x)$ can be written as

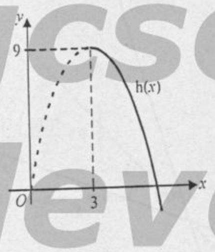
$$\mathrm{h}(x) = 9 - (x-3)^2$$

let $\quad y = \mathrm{h}(x)$

$$\therefore \quad y = 9 - (x-3)^2$$
$$(x-3)^2 = 9 - y$$
$$x - 3 = +\sqrt{9-y}$$
$$x = 3 + \sqrt{9-y}$$

i.e. $\mathrm{h}^{-1}(y) = 3 + \sqrt{9-y} \qquad \because x = \mathrm{h}^{-1}(y)$

$\therefore \quad \mathrm{h}^{-1}(x) = 3 + \sqrt{9-x}$ **(Ans)**



Note that h(x) is valid for $x \geq 3$.

So h(x) is on the right side of axis of symmetry.

Therefore only keep the positive sign with the radical.

# November 2008 Paper 3

## *Pure Mathematics (P3)*

Answer all the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Solve the equation $\ln(x+2)=2+\ln x$, giving your answer correct to 3 decimal places. [3]

***Suggested Solution:***

$$\ln(x+2)=2+\ln x$$
$$\ln(x+2)-\ln x=2\ln e$$
$$\ln\left(\frac{x+2}{x}\right)=\ln e^2$$
$$\frac{x+2}{x}=e^2$$
$$xe^2=x+2$$
$$xe^2-x=2$$
$$x(e^2-1)=2$$
$$x=\frac{2}{e^2-1}=0.313 \quad \textbf{(Ans).}$$

Note that:
- $\ln e=1$
- $\ln(\frac{a}{b})=\ln a-\ln b$

**2.** Expand $(1+x)\sqrt{(1-2x)}$ in ascending powers of $x$, up to and including the term in $x^2$, simplifying the coefficients. [4]

***Suggested Solution:***

$$(1+x)\sqrt{1-2x}$$
$$=(1+x)(1-2x)^{\frac{1}{2}}$$
$$=(1+x)\left(1+\tfrac{1}{2}(-2x)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-2x)^2+....\right)$$
$$=(1+x)\left(1-x-\tfrac{1}{8}(4x^2)+....\right)$$
$$=(1+x)\left(1-x-\tfrac{1}{2}x^2-....\right)$$
$$=1-x-\tfrac{1}{2}x^2+x-x^2-....$$
$$=1-\tfrac{3}{2}x^2 \quad \textbf{(Ans).}$$

Binomial Expansion:

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+...$$

3. The curve $y=\frac{e^x}{\cos x}$, for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, has one stationary point. Find the $x$-coordinate of this point. [5]

***Suggested Solution:***

$$y=\frac{e^x}{\cos x}$$

differentiating w.r.t. $x$

$$\frac{dy}{dx}=\frac{(\cos x)\frac{d}{dx}(e^x)-(e^x)\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$=\frac{e^x\cos x-e^x(-\sin x)}{\cos^2 x}=\frac{e^x(\cos x+\sin x)}{\cos^2 x}$$

for stationary values, put $\frac{dy}{dx}=0$

$$\Rightarrow \quad \frac{e^x(\cos x+\sin x)}{\cos^2 x}=0$$

$$e^x(\cos x+\sin x)=0$$

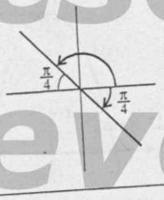
as $e^x \neq 0$

$\therefore \quad \sin x+\cos x=0 \quad \Rightarrow \quad \sin x=-\cos x \quad \Rightarrow \quad \tan x=-1$

basic angle $\alpha=\frac{\pi}{4}$

since given range is $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$,

$\therefore \quad x=-\frac{\pi}{4}$ **(Ans).**



Note that:
$\tan\theta$ is negative in 2nd and 4th quadrant.

4. The parametric equations of a curve are

$$x=a(2\theta-\sin 2\theta), \quad y=a(1-\cos 2\theta).$$

Show that $\frac{dy}{dx}=\cot\theta$. [5]

***Suggested Solution:***

$x=a\,(2\theta-\sin 2\theta) \quad , \quad y=a\,(1-\cos 2\theta)$

$\frac{dx}{d\theta}=a(2-2\cos 2\theta)=2a(1-\cos 2\theta) \quad , \quad \frac{dy}{d\theta}=a\big(0-(-\sin 2\theta\times 2)\big)=2a\sin 2\theta$

applying chain rule, $\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}$

$$\Rightarrow \quad \frac{dy}{dx}=2a\sin 2\theta\times\frac{1}{2a(1-\cos 2\theta)}$$

$$=\frac{\sin 2\theta}{1-\cos 2\theta}=\frac{2\sin\theta\cos\theta}{1-(1-2\sin^2\theta)}=\frac{2\sin\theta\cos\theta}{2\sin^2\theta}=\frac{\cos\theta}{\sin\theta}=\cot\theta \quad \textbf{(Shown).}$$

Note:

$\sin 2\theta=2\sin\theta\cos\theta$
$\cos 2\theta=1-2\sin^2\theta$
$\cos 2\theta=2\cos^2\theta-1$

lote that:
anθ is negative in 2nd
nd 4th quadrant.

Note:
$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos 2\theta = 1 - 2\sin^2\theta$
$\cos 2\theta = 2\cos^2\theta - 1$

5. The polynomial $4x^3 - 4x^2 + 3x + a$, where $a$ is a constant, is denoted by $p(x)$. It is given that $p(x)$ is divisible by $2x^2 - 3x + 3$.
   (i) Find the value of $a$. [3]
   (ii) When $a$ has this value, solve the inequality $p(x) < 0$, justifying your answer. [3]

***Suggested Solution:***

(i) $p(x) = 4x^3 - 4x^2 + 3x + a$

using long division,

$$\begin{array}{r|l} & 2x+1 \\ \cline{2-2} 2x^2-3x+3 & 4x^3-4x^2+3x+a \\ & 4x^3-6x^2+6x \\ & -\qquad+\qquad- \\ \cline{2-2} & \qquad 2x^2-3x+a \\ & \qquad 2x^2-3x+3 \\ & \qquad -\qquad+\qquad- \\ \cline{2-2} & \qquad\qquad a-3 \end{array}$$

as $(2x^2 - 3x + 3)$ is a factor of $p(x)$, therefore remainder $= 0$

$\Rightarrow\ a - 3 = 0\ \Rightarrow\ a = 3$ **(Ans).**

(ii) $p(x) < 0$

$\Rightarrow\ 4x^3 - 4x^2 + 3x + 3 < 0$

$\Rightarrow\ (2x+1)(2x^2 - 3x + 3) < 0$ (from part (i))

consider the quadratic factor,

$$\begin{aligned} & 2x^2 - 3x + 3 \\ &= 2\left(x^2 - \frac{3}{2}x\right) + 3 \\ &= 2\left[x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2\right] + 3 \\ &= 2\left[\left(x - \frac{3}{4}\right)^2 - \frac{9}{16}\right] + 3 \\ &= 2\left(x - \frac{3}{4}\right)^2 - \frac{9}{8} + 3 \\ &= 2\left(x - \frac{3}{4}\right)^2 - \frac{15}{8} > 0 \text{ for all } x \end{aligned}$$

$\therefore\ 2x + 1 < 0\ \Rightarrow\ x = -\frac{1}{2}$ **(Ans).**

6. (i) Express $5\sin x + 12\cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$, giving the value of $\alpha$ correct to 2 decimal places. [3]

(ii) Hence solve the equation

$$5\sin 2\theta + 12\cos 2\theta = 11,$$

giving all solutions in the interval $0^\circ < \theta < 180^\circ$. [5]

*Suggested Solution:*

(i) $R\sin(x+\alpha) \equiv 5\sin x + 12\cos x$

$R\sin x\cos\alpha + R\cos x\sin\alpha \equiv 5\sin x + 12\cos x$

comparing coefficients of $\sin x$ and $\cos x$, we have

$R\cos\alpha = 5 \cdots\cdots\cdots(\text{i})$ and $R\sin\alpha = 12 \cdots\cdots\cdots(\text{ii})$

squaring and adding eq.(i) and eq.(ii),

$$(R\cos\alpha)^2 + (R\sin\alpha)^2 = 5^2 + 12^2$$

$$R^2(\cos^2\alpha + \sin^2\alpha) = 169 \quad \Rightarrow \quad R^2 = 169 \quad \Rightarrow \quad R = 13$$

eq.(ii) ÷ eq.(i) gives,

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{12}{5} \quad \Rightarrow \quad \tan\alpha = \frac{12}{5} \quad \Rightarrow \quad \alpha = 67.38^\circ$$

$\therefore \quad 5\sin x + 12\cos x = 13\sin(x + 67.38^\circ)$ **(Ans).**

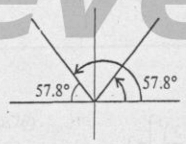
(ii) $5\sin 2\theta + 12\cos 2\theta = 11$

using the result of part (i), and replacing $x$ by $2\theta$, we have,

$$13\sin(2\theta + 67.38^\circ) = 11$$

$$\Rightarrow \quad \sin(2\theta + 67.38^\circ) = \frac{11}{13}$$

basic angle $\alpha = 57.8^\circ$



$$\therefore \quad 2\theta + 67.38^\circ = 122.2^\circ, \quad 57.8^\circ + 360$$

$$2\theta = 122.2^\circ - 67.38^\circ, \quad 417.8^\circ - 67.38^\circ$$

$$\theta = 27.4^\circ, \quad 175.2^\circ$$ **(Ans).**

7. Two planes have equations $2x - y - 3z = 7$ and $x + 2y + 2z = 0$.

(i) Find the acute angle between the planes. [4]

(ii) Find a vector equation for their line of intersection. [6]

*Suggested Solution:*

(i) We have,

$$2x-y-3z=7 \Rightarrow \mathbf{r}\cdot\begin{pmatrix}2\\-1\\-3\end{pmatrix}=7, \quad \therefore \text{ normal, } \vec{n}_1=\begin{pmatrix}2\\-1\\-3\end{pmatrix}$$

$$x+2y+2z=0 \Rightarrow \mathbf{r}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=0, \quad \therefore \text{ normal, } \vec{n}_2=\begin{pmatrix}1\\2\\2\end{pmatrix}$$

the angle between two planes is the angle between their respective normals

$$\therefore \quad \vec{n}_1\cdot\vec{n}_2=\left|\vec{n}_1\right|\cdot\left|\vec{n}_2\right|\cos\theta$$

$$\begin{pmatrix}2\\-1\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=\left(\sqrt{(2)^2+(-1)^2+(-3)^2}\right)\left(\sqrt{(1)^2+(2)^2+(2)^2}\right)\cos\theta$$

$$2-2-6=\left(\sqrt{4+1+9}\right)\left(\sqrt{1+4+4}\right)\cos\theta$$

$$-6=3\sqrt{14}\cos\theta$$

$$\cos\theta=-\frac{2}{\sqrt{14}} \Rightarrow \theta=122.3°$$

$\therefore$ acute angle $=180-122.3=57.7°$ **(Ans).**

(ii) $2x-y-3z=7$ ………(i)
$x+2y+2z=0$ ………(ii)

eliminating two variables one by one from the equations of planes,

eq.(i): $2x-y-3z=7$
eq.(ii) × 2: $2x+4y+4z=0$
$\underline{- \quad - \quad -}$
$-5y-7z=7 \Rightarrow z=\frac{-5y-7}{7}$ ………(iii)

eq.(i) × 2: $4x-2y-6z=14$
eq.(ii): $\underline{x+2y+2z=0}$
$5x-4z=14 \Rightarrow z=\frac{5x-14}{4}$ ………(iv)

from equations (iii) and (iv), we have,

$$z=\frac{-5y-7}{7}=\frac{5x-14}{4}$$

$$\frac{z-0}{1}=\frac{-5(y+\frac{7}{5})}{7}=\frac{5(x-\frac{14}{5})}{4}$$

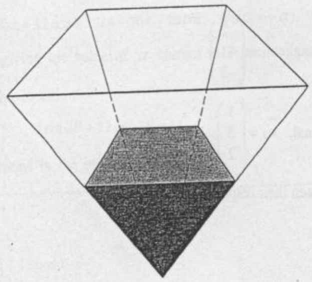
$$\frac{z-0}{1}=\frac{y+\frac{7}{5}}{-\frac{7}{5}}=\frac{x-\frac{14}{5}}{\frac{4}{5}}$$

$\therefore$ the line of intersection of the two planes is:

$$r=\left(\frac{14}{5}\mathrm{i}-\frac{7}{5}\mathrm{j}+0\mathrm{k}\right)+\lambda\left(\frac{4}{5}\mathrm{i}-\frac{7}{5}\mathrm{j}+\mathrm{k}\right)$$

or $r=\left(\frac{14}{5}\mathrm{i}-\frac{7}{5}\mathrm{j}\right)+\lambda(4\mathrm{i}-7\mathrm{j}+5\mathrm{k})$ **(Ans).**

**8.**



An underground storage tank is being filled with liquid as shown in the diagram. Initially the tank is empty. At time $t$ hours after filling begins, the volume of liquid is $V$ m³ and the depth of liquid is $h$ m. It is given that $V = \frac{4}{3}h^3$.

The liquid is poured in at a rate of 20 m³ per hour, but owing to leakage, liquid is lost at a rate proportional to $h^2$. when $h = 1$, $\frac{dh}{dt} = 4.95$.

(i) Show that $h$ satisfies the differential equation [4]

$$\frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}.$$

(ii) Verify that $\frac{20h^2}{100-h^2} \equiv -20 + \frac{2000}{(10-h)(10+h)}$. [1]

(iii) Hence solve the differential equation in part (i), obtaining an expression for $t$ in terms of $h$. [5]

***Suggested Solution:***

(i) Given that, $\frac{dV}{dt} = 20$ ………(i) and $\frac{dV}{dt} \propto -h^2 \Rightarrow \frac{dV}{dt} = -kh^2$ ………(ii)

combining equations (i) and (ii),

$$\frac{dV}{dt} = 20 - kh^2$$

also given, $V = \frac{4}{3}h^3$

differentiating w.r.t $h$, $\frac{dV}{dh} = \frac{4}{3}(3h^2) = 4h^2$

now, $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$$= (\frac{1}{4h^2}) \times (20 - kh^2) = \frac{5}{h^2} - \frac{k}{4} \text{ ………(iii)}$$

given that, when $h = 1$, $\frac{dh}{dt} = 4.95$

$\Rightarrow 4.95 = \frac{5}{1^2} - \frac{k}{4} \Rightarrow 4.95 = 5 - \frac{k}{4} \Rightarrow \frac{k}{4} = 0.05 \Rightarrow k = 0.2$

$\therefore$ equation (iii) becomes,

$$\frac{dh}{dt} = \frac{5}{h^2} - \frac{0.2}{4} \Rightarrow \frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20} \text{ (Shown).}$$

(ii) R.H.S. $= -20 + \dfrac{2000}{(10+h)(10-h)}$

$$= -20 + \frac{2000}{100-h^2}$$

$$= \frac{-20(100-h^2)+2000}{100-h^2}$$

$$= \frac{-2000+20h^2+2000}{(100-h^2)} = \frac{20h^2}{100-h^2} = \text{L.H.S.} \quad \textbf{(Verified).}$$

(iii) From part (i),

$$\frac{dh}{dt} = \frac{5}{h^2} - \frac{1}{20}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{100-h^2}{20h^2} \quad \Rightarrow \quad \frac{20h^2}{100-h^2}\,dh = dt$$

using the result of part (ii),

$$\left(-20 + \frac{2000}{(10-h)(10+h)}\right) dh = dt$$

integrating both sides

$$\int\left(-20 + \frac{2000}{(10-h)(10+h)}\right) dh = \int dt$$

$$\int\left(-20 + 2000\left(\frac{1}{(10^2-h^2)}\right)\right) dh = \int dt$$

$$-20h + 2000\left(\frac{1}{2(10)}\ln\left(\frac{10+h}{10-h}\right)\right) = t + C$$

$$-20h + 100\ln\left(\frac{10+h}{10-h}\right) = t + C$$

Note that:

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + K$$

when $t = 0$, $h = 0$

$$\Rightarrow \quad -20(0) + 100\ln\left(\frac{10+0}{10-0}\right) = 0 + C \quad \Rightarrow \quad 100\ln(1) = C \quad \Rightarrow \quad C = 0$$

$$\therefore \quad -20h + 100\ln\left(\frac{10+h}{10-h}\right) = t \quad \textbf{(Ans).}$$

8. The constant $a$ is such that $\int_0^a x e^{\frac{1}{2}x}\,dx = 6$.

(i) Show that $a$ satisfies the equation

$$x = 2 + e^{-\frac{1}{2}x}.$$ [5]

(ii) By sketching a suitable pair of graphs, show that this equation has only one root. [2]

(iii) Verify by calculation that this root lies between 2 and 2.5. [2]

(iv) Use an iterative formula based on the equation in part (i) to calculate the value of $a$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

*Suggested Solution:*

(i) $\int_0^a xe^{\frac{1}{2}x}\,dx = 6$

using integration by parts

$$\Rightarrow \left[x\int e^{\frac{1}{2}x}dx - \int\left(\frac{d}{dx}(x)\int e^{\frac{1}{2}x}\,dx\right)dx\right]_0^a = 6$$

$$\left[2xe^{\frac{1}{2}x} - 2\int e^{\frac{1}{2}x}\,dx\right]_0^a = 6$$

$$\left[2xe^{\frac{1}{2}x} - 4e^{\frac{1}{2}x}\right]_0^a = 6$$

$$\left(2ae^{\frac{1}{2}a} - 4e^{\frac{1}{2}a}\right) - \left(2(0)e^{\frac{1}{2}(0)} - 4e^{\frac{1}{2}(0)}\right) = 6$$

$$\left(2ae^{\frac{1}{2}a} - 4e^{\frac{1}{2}a}\right) + 4 = 6$$

$$2ae^{\frac{1}{2}a} - 4e^{\frac{1}{2}a} = 2$$

$$ae^{\frac{1}{2}a} - 2e^{\frac{1}{2}a} = 1$$

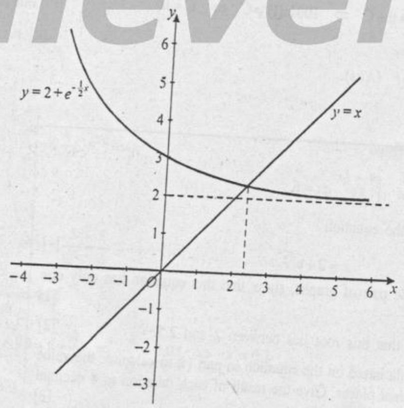
$$e^{\frac{1}{2}a}(a-2) = 1$$

$$a - 2 = e^{-\frac{1}{2}a} \quad \Rightarrow \quad a = 2 + e^{-\frac{1}{2}a}$$

$\therefore$ $a$ satisfies the equation $x = 2 + e^{-\frac{1}{2}x}$ **(Shown).**

Remember:
For definite integral, do not put the constant of integration.

(ii) Sketching $y = x$ and $y = 2 + e^{-\frac{1}{2}x}$, we have,



since the two graphs meet at only one point,

$\therefore$ equation $x = 2 + e^{-\frac{1}{2}x}$ has only one root. **(Shown).**

(iii) $x = 2 + e^{-\frac{1}{2}x} \Rightarrow x - 2 - e^{-\frac{1}{2}x} = 0$

Let $f(x) = x - 2 - e^{-\frac{1}{2}x}$

$$f(2) = 2 - 2 - e^{-\frac{1}{2}(2)} = -e^{-1} = -\frac{1}{e} < 0$$

$$f(2.5) = 2.5 - 2 - e^{-\frac{1}{2}(2.5)} = 0.5 - e^{-1.25} = 0.5 - 0.2865 = 0.213 > 0$$

change of sign indicates that there is a root between 2 and 2.5 **(Verified).**

(iv) Iterative formula: $x_{n+1} = 2 + e^{-\frac{1}{2}x_n}$

using initial value, $x_1 = 2$

$$x_2 = 2 + e^{-\frac{1}{2}x_1} = 2 + e^{-\frac{1}{2}(2)} = 2.3679$$

$$x_3 = 2 + e^{-\frac{1}{2}x_2} = 2 + e^{-\frac{1}{2}(2.3679)} = 2.3061$$

$$x_4 = 2 + e^{-\frac{1}{2}x_3} = 2 + e^{-\frac{1}{2}(2.3061)} = 2.3157$$

$$x_5 = 2 + e^{-\frac{1}{2}x_4} = 2 + e^{-\frac{1}{2}(2.3157)} = 2.3142$$

$$x_6 = 2 + e^{-\frac{1}{2}x_5} = 2 + e^{-\frac{1}{2}(2.3142)} = 2.3144$$

$\therefore$ $a = 2.31$ (2 dp) **(Ans).**

Note:
Keep your calculator on radian mode.

10. The complex number $w$ is given by $w = -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}$.

(i) Find the modulus and argument of $w$. [2]

(ii) The complex number $z$ has modulus $R$ and argument $\theta$, where $-\frac{1}{3}\pi < \theta < \frac{1}{3}\pi$.

State the modulus and argument of $wz$ and the modulus and argument of $\frac{z}{w}$. [4]

(iii) Hence explain why, in an Argand diagram, the points representing $z$, $wz$ and $\frac{z}{w}$ are the vertices of an equilateral triangle. [2]

(iv) In an Argand diagram, the vertices of an equilateral triangle lie on a circle with centre at the origin. One of the vertices represents the complex number 4 + 2i. Find the complex numbers represented by the other two vertices. Give your answers in the form $x + \mathrm{i}y$, where $x$ and $y$ are real and exact. [4]

*Suggested Solution:*

(i) $\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$

modulus. $|\omega|=\sqrt{\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{3}{4}}=1$ **(Ans).**

$\arg(\omega)=\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right)=\tan^{-1}(-\sqrt{3}),$

basic angle $=\frac{1}{3}\pi$ radians

$\therefore$ required $\arg(\omega)=\pi-\frac{1}{3}\pi=\frac{2}{3}\pi$ radians **(Ans).**

Note that:

If $z=x+iy$, then

- $|z|=\sqrt{x^2+y^2}$
- $\arg(z)=\tan^{-1}\left(\frac{y}{x}\right)$

The range of principal argument $\alpha$ is $-\pi<\alpha\le\pi$

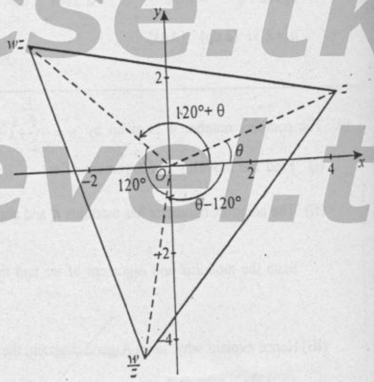
(ii) Modulus of $(wz)$: $|wz|=|w|\times|z|=1\times R=R$ **(Ans).**

$\arg(wz)$: $\arg(w)+\arg(z)=\frac{2}{3}\pi+\theta$ **(Ans).**

Modulus of $\left(\frac{z}{w}\right)$: $\left|\frac{z}{w}\right|=\frac{|z|}{|w|}=\frac{R}{1}=R$ **(Ans).**

$\arg\left(\frac{z}{w}\right)=\arg(z)-\arg(w)=\theta-\frac{2}{3}\pi$ **(Ans).**

(iii) All the three complex numbers represent points in the Argand diagram with modulus $R$. Also the angle between any two terminal positions is $\frac{2\pi}{3}=120°$. Therefore the three complex numbers represent the vertices of an equilateral triangle.



(iv) Using the results of part (ii) and part (iii), we see that the three vertices of the equilateral triangle are $z$, $wz$, and $\frac{z}{w}$.

$\Rightarrow$ $z=4+2i$

$wz=\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(4+2i)$

$=-2-i+2\sqrt{3}i+\sqrt{3}\,i^2$

$=-2-i+2\sqrt{3}i-\sqrt{3}$

$=(-2-\sqrt{3})+i(2\sqrt{3}-1)$ **(Ans).**

then

$+y^2$

$an^{-1}\left(\frac{y}{x}\right)$

f principal

is $-\pi<\alpha\leq\pi$

$$\frac{z}{w}=\frac{4+2i}{-\frac{1}{2}+i\frac{\sqrt{3}}{2}}$$

$$=\frac{4+2i}{-\frac{1}{2}+i\frac{\sqrt{3}}{2}}\times\frac{-\frac{1}{2}-i\frac{\sqrt{3}}{2}}{-\frac{1}{2}-i\frac{\sqrt{3}}{2}}$$

$$=\frac{-2-i2\sqrt{3}-i-\sqrt{3}\,i^2}{\left(-\frac{1}{2}\right)^2-\left(i\frac{\sqrt{3}}{2}\right)^2}$$

$$=\frac{-2-2\sqrt{3}\,i-i+\sqrt{3}}{\left(\frac{1}{4}\right)-\left(-\frac{3}{4}\right)}$$

$$=(-2+\sqrt{3})+i(-2\sqrt{3}-1)\quad\textbf{(Ans).}$$

# June 2009 Paper 1

## Pure Mathematics (P1)

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Prove the identity $\frac{\sin x}{1-\sin x}-\frac{\sin x}{1+\sin x}=2\tan^2 x$. [3]

***Suggested Solution:***

$$\text{L.H.S}=\frac{\sin x}{1-\sin x}-\frac{\sin x}{1+\sin x}$$

$$=\frac{\sin x(1+\sin x)-\sin x(1-\sin x)}{(1-\sin x)(1+\sin x)}$$

$$=\frac{\sin x+\sin^2 x-\sin x+\sin^2 x}{1-\sin^2 x}=\frac{2\sin^2 x}{\cos^2 x}=2\tan^2 x=\text{R.H.S}\quad \textbf{(Shown).}$$

Remember:

- $\sin^2 x+\cos^2 x=1$
- $1-\sin^2 x=\cos^2 x$

**2.** Find the set of values of $k$ for which the line $y=kx-4$ intersects the curve $y=x^2-2x$ at two distinct points. [4]

***Suggested Solution:***

Let $y=kx-4$............(i) and $y=x^2-2x$............(ii)

substituting equation (i) into equation (ii),

$$kx-4=x^2-2x$$

$$x^2-2x-kx+4=0 \quad\Rightarrow\quad x^2-x(2+k)+4=0$$

the line intersects the curve at two distinct points.

$\therefore$ Discriminant, $b^2-4ac>0$

$$\Rightarrow\quad (-(2+k))^2-4(1)(4)>0$$

$$4+4k+k^2-16>0$$

$$k^2+4k-12>0$$

$$k^2+6k-2k-12>0$$

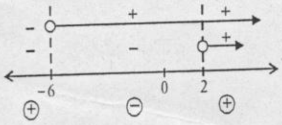
$$k(k+6)-2(k+6)>0$$

$$(k+6)(k-2)>0$$

critical values are $k=-6$, and $k=2$

using line method, the set of values of $k$ are:

$k<-6$, $k>2$ **(Ans).**



$x + \cos^2 x = 1$
$\sin^2 x = \cos^2 x$

3. (i) Find the first 3 terms in the expansion of $(2 + 3x)^5$ in ascending powers of $x$. [3]

 (ii) Hence find the value of the constant $a$ for which there is no term in $x^2$ in the expansion of $(1 + ax)(2 + 3x)^5$. [2]

***Suggested Solution:***

(i) Using Binomial expansion,

$$(2+3x)^5 = {}^5C_0\,(2)^5(3x)^0 + {}^5C_1\,(2)^4(3x)^1 + {}^5C_2(2)^3(3x)^2 + \cdots\cdots$$
$$= 32 + (5)(16)(3x) + (10)(8)(9x^2)$$
$$= 32 + 240x + 720x^2 \quad \textbf{(Ans).}$$

Binomial theorem:

$(a+b)^n = {}^nC_0\,a^n + {}^nC_1\,a^{n-1}b + {}^nC_2\,a^{n-2}b^2 + .... + {}^nC_r\,a^{n-r}b^r + .... + {}^nC_n\,b^n$

(ii) $(1+ax)(2+3x)^5$

Using the result of part (i)

$= (1+ax)(32+240x+720x^2)$

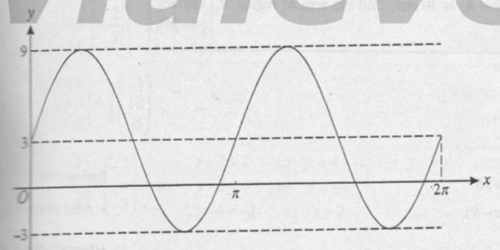
Terms containing $x^2$ are,

$= 720x^2 + 240ax^2 = x^2(720+240a)$

given that there is no term in $x^2$, therefore coefficient of $x^2 = 0$

$$\Rightarrow \quad 720 + 240a = 0 \quad \Rightarrow \quad a = -\frac{720}{240} \quad \Rightarrow \quad a = -3 \quad \textbf{(Ans).}$$

4.



The diagram shows the graph of $y = a\sin(bx) + c$ for $0 \le x \le 2\pi$.

(i) Find the values of $a$, $b$ and $c$. [3]

(ii) Find the smallest value of $x$ in the interval $0 \le x \le 2\pi$ for which $y = 0$. [3]

***Suggested Solution:***

(i) $y = a\sin(bx) + c$

$a$ = amplitude of the curve

$\therefore$ from the graph, $a = 9 - 3 = 6$ **(Ans).**

As there are two cycles in the normal period of sin.

$\therefore$ $b = 2$ **(Ans).**

$c$ indicates vertical shift in the graph from horizontal axis.

$\therefore$ $c = 3$ units **(Ans).**

*a:* Indicates the amplitude of the curve which is the maximum or minimum range of the curve from the mean position.

*b:* Indicates the periodicity of the curve. We see that in the given figure there are two cycles in a normal period of sine.

*c:* Indicates the vertical shift in the graph from the horizontal axis, which in this case is 3 units.

(ii) Using the result of part (i), we have

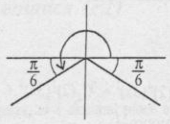
$$y = 6\sin(2x) + 3 \qquad \text{for } 0 \le x \le 2\pi$$

at $y = 0$, $6\sin 2x + 3 = 0 \Rightarrow 6\sin 2x = -3 \Rightarrow \sin 2x = -\frac{1}{2}$

basic angle, $\alpha = \frac{\pi}{6}$.

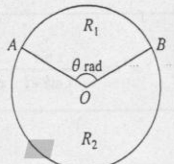
$\therefore$ for smallest value, $2x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$

$\Rightarrow x = \frac{7\pi}{12}$ **(Ans).**



Note that:
sin$\theta$ is negative in III and IV quadrants.

**5.**



The diagram shows a circle with centre $O$. The circle is divided into two regions, $R_1$ and $R_2$, by the radii $OA$ and $OB$, where angle $AOB = \theta$ radians. The perimeter of the region $R_1$ is equal to the length of the major arc $AB$.

(i) Show that $\theta = \pi - 1$. [3]

(ii) Given that the area of region $R_1$ is 30 cm², find the area of region $R_2$, correct to 3 significant figures. [4]

***Suggested Solution:***

(i) Let $r$ be the radius of the circle.

using $s = r\theta$,

arc length of region $R_1$ is : $\overset{\frown}{AB} = r\theta$

arc length of region $R_2$ is : $r(2\pi - \theta)$

given that,

perimeter of region $R_1$ = arc length of $R_2$

$$\Rightarrow r + r + r\theta = r(2\pi - \theta)$$
$$2r + r\theta = r(2\pi - \theta)$$
$$r(2 + \theta) = r(2\pi - \theta)$$
$$2 + \theta = 2\pi - \theta$$
$$2\theta = 2\pi - 2 \Rightarrow \theta = \pi - 1 \quad \textbf{(Shown).}$$

Remember:

- Area of sector = $\frac{1}{2}r^2\theta$
- Length of arc = $r\theta$

where $r$ is the radius and $\theta$ is the angle in radians.

(ii) Area of $R_1 = 30 \Rightarrow \frac{1}{2}r^2\theta = 30 \Rightarrow r^2 = \frac{60}{\theta} \Rightarrow r^2 = \frac{60}{\pi - 1}$

area of $R_2 = \frac{1}{2}r^2(2\pi - \theta)$

putting the value of $r^2$ and $\theta$,

$$\text{area of } R_2 = \frac{1}{2}\left(\frac{60}{\pi - 1}\right)(2\pi - (\pi - 1))$$
$$= \left(\frac{30}{\pi - 1}\right)(\pi + 1) = \frac{30(\pi + 1)}{(\pi - 1)} = 58.0 \text{ cm}^2 \quad \textbf{(Ans).}$$

hat:
negative in III and
drants.

ber:
of sector $= \frac{1}{2}r^2\theta$
h of arc $= r\theta$
r is the radius and
angle in radians.

6. Relative to an origin $O$, the position vectors of the points $A$ and $B$ are given by

$$\overrightarrow{OA} = 2\mathbf{i} - 8\mathbf{j} + 4\mathbf{k} \quad \text{and} \quad \overrightarrow{OB} = 7\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

(i) Find the value of $\overrightarrow{OA}.\overrightarrow{OB}$ and hence state whether angle $AOB$ is acute, obtuse or a right angle. [3]

(ii) The point $X$ is such that $\overrightarrow{AX} = \frac{2}{5}\overrightarrow{AB}$. Find the unit vector in the direction of $OX$. [4]

***Suggested Solution:***

(i) $\overrightarrow{OA}\cdot\overrightarrow{OB} = \begin{pmatrix}2\\-8\\4\end{pmatrix}\cdot\begin{pmatrix}7\\2\\-1\end{pmatrix} = 14 - 16 - 4 = -6$ (i.e. negative)

$\therefore$ angle $AOB$ is obtuse. **(Ans).**

If
$\mathbf{a} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$
$\mathbf{b} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then
$\mathbf{a}\cdot\mathbf{b} = ax + by + cz$
and magnitude of $\mathbf{a}$ is
$|\mathbf{a}| = \sqrt{(a)^2 + (b)^2 + (c)^2}$

(ii) Let $\overrightarrow{OX} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

given that, $\overrightarrow{AX} = \frac{2}{5}\overrightarrow{AB}$

$$\Rightarrow \quad \overrightarrow{OX} - \overrightarrow{OA} = \frac{2}{5}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} - \begin{pmatrix}2\\-8\\4\end{pmatrix} = \frac{2}{5}\left[\begin{pmatrix}7\\2\\-1\end{pmatrix} - \begin{pmatrix}2\\-8\\4\end{pmatrix}\right]$$

$$\Rightarrow \quad \begin{pmatrix}x-2\\y+8\\z-4\end{pmatrix} = \frac{2}{5}\begin{pmatrix}5\\10\\-5\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}x-2\\y+8\\z-4\end{pmatrix} = \begin{pmatrix}2\\4\\-2\end{pmatrix} \quad \Rightarrow \quad \begin{aligned} x-2 &= 2 &\Rightarrow\ x &= 4\\ y+8 &= 4 &\Rightarrow\ y &= -4\\ z-4 &= -2 &\Rightarrow\ z &= 2\end{aligned}$$

$\therefore \quad \overrightarrow{OX} = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

unit vector in the direction of $OX$

$$= \frac{\overrightarrow{OX}}{\left|\overrightarrow{OX}\right|} = \frac{4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}}{\sqrt{(4)^2 + (-4)^2 + (2)^2}}$$

$$= \frac{4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}}{\sqrt{16 + 16 + 4}}$$

$$= \frac{4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}}{6}$$

$$= \frac{2\mathbf{i} - 2\mathbf{j} + \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k} \quad \textbf{(Ans).}$$

For any vector $\mathbf{v}$, the unit vector $\hat{\mathbf{v}}$, is given as:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

7. (a) Find the sum to infinity of the geometric progression with first three terms 0.5, $0.5^3$ and $0.5^5$. [3]

(b) The first two terms in an arithmetic progression are 5 and 9. The last term in the progression is the only term which is greater than 200. Find the sum of all the terms in the progression. [4]

***Suggested Solution:***

**(a)** 0.5. $0.5^3$. $0.5^5$,............ are in G.P

$\Rightarrow \quad a = 0.5. \quad r = 0.5^2$

as $|r| < 1$. therefore using $S_\infty = \dfrac{a}{1-r}$.

$$S_\infty = \frac{0.5}{1-(0.5)^2}$$

$$= \frac{0.5}{1-0.25} = \frac{0.5}{0.75} = \frac{50}{75} = \frac{2}{3} \quad \textbf{(Ans).}$$

**(b)** We have. $a = 5. \quad d = 9-5 = 4$

Let $T_n = 200$.

using $T_n = a + (n-1)d$

$5 + (n-1)4 = 200$

$\Rightarrow \quad 4(n-1) = 195 \quad \Rightarrow \quad n-1 = 48.75 \quad \Rightarrow \quad n = 49.75$

given that the last term is greater than 200.

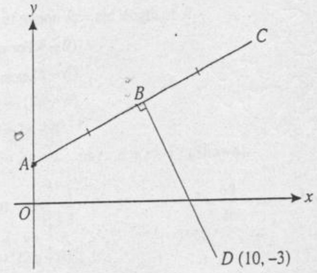
$\Rightarrow \quad n = 50$

now using $S_n = \dfrac{n}{2}[2a + (n-1)d]$

$$S_{50} = \frac{50}{2}[2(5) + (50-1)4]$$

$$= 25[10 + (49)4)] = (25)(206) = 5150 \quad \textbf{(Ans).}$$

8.



The diagram shows points $A$, $B$ and $C$ lying on the line $2y = x + 4$. The point $A$ lies on the $y$-axis and $AB = BC$. The line from $D(10, -3)$ to $B$ is perpendicular to $AC$. Calculate the coordinates of $B$ and $C$. [7]

***Suggested Solution:***

Equation of line $ABC$ : $2y = x + 4$

For $y$-intercept, put $x = 0$.

$\Rightarrow \quad 2y = 0 + 4 \quad \Rightarrow \quad y = 2$

$\therefore$ coordinates of $A(0, 2)$

$2y = x + 4 \quad \Rightarrow \quad y = \frac{1}{2}x + 2 \cdots\cdots\cdots(\text{i})$

$\therefore$ gradient of line $ABC = \frac{1}{2}$

$\Rightarrow$ gradient of line $BD = -2$

$\therefore$ equation of line $BD$ passing through $D(10, -3)$ is,

$y + 3 = -2(x - 10) \quad \Rightarrow \quad y + 3 = -2x + 20 \quad \Rightarrow \quad y = -2x + 17 \cdots\cdots\cdots(\text{ii})$

When two lines are perpendicular to each other then the product of their gradients is equal to −1
i.e. $m_1 \times m_2 = -1$

Equation of a line in point-slope form is:
$y - y_1 = m(x - x_1)$

solving equation (i) and equation (ii) simultaneously for point $B$.

$\frac{1}{2}x + 2 = -2x + 17 \quad \Rightarrow \quad x + 4 = -4x + 34 \quad \Rightarrow \quad 5x = 30 \quad \Rightarrow \quad x = 6$

putting $x = 6$, in equation (i) : $\quad y = \frac{1}{2}(6) + 2 \quad \Rightarrow \quad y = 5$

$\therefore$ coordinates of $B$ are $(6, 5)$ **(Ans).**

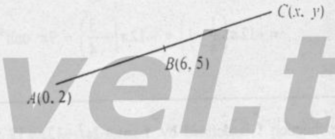
Let the coordinates of $C$ be $(x, y)$

coordinates of midpoint of $AC$ = coordinates of point $B$

$\Rightarrow \left(\frac{x+0}{2}, \frac{y+2}{2}\right) = (6, 5)$

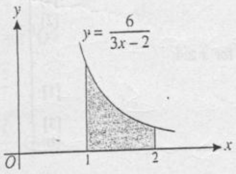
$\Rightarrow \frac{x+0}{2} = 6, \qquad \frac{y+2}{2} = 5$

$\Rightarrow \qquad x = 12 \qquad\qquad y = 10 - 2 = 8$



$\therefore$ coordinates of $C$ are $(12, 8)$ **(Ans).**

9.



The diagram shows part of the curve $y = \frac{6}{3x-2}$.

(i) Find the gradient of the curve at the point where $x = 2$. [3]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the $x$-axis, giving your answer in terms of $\pi$. [5]

*Suggested Solution:*

(i) $y = \frac{6}{3x-2} = 6(3x-2)^{-1}$

gradient, $\frac{dy}{dx} = 6\left(-(3x-2)^{-2}(3)\right) = \frac{-18}{(3x-2)^2}$

at $x = 2$, $\frac{dy}{dx} = -\frac{18}{(3(2)-2)^2} = -\frac{18}{16} = -\frac{9}{8}$ **(Ans).**

(ii) Volume, $V = \pi\int_1^2 y^2\,dx$

$$\Rightarrow \quad V = \pi\int_1^2\left(\frac{6}{3x-2}\right)^2 dx$$

$$= 36\,\pi\int_1^2 (3x-2)^{-2}\,dx$$

$$= 36\,\pi\left[\frac{(3x-2)^{-1}}{(-1)(3)}\right]_1^2$$

$$= \frac{36\pi}{-3}\left[\frac{1}{3x-2}\right]_1^2$$

$$= -12\pi\left(\frac{1}{3(2)-2} - \frac{1}{3(1)-2}\right)$$

$$= -12\pi\left(\frac{1}{4}-1\right) = -12\pi\left(-\frac{3}{4}\right) = 9\pi \text{ unit}^3 \quad \textbf{(Ans).}$$

**10.** The function f is defined by $f : x \mapsto 2x^2 - 12x + 13$ for $0 \le x \le A$, where $A$ is a constant.

(i) Express f($x$) in the form $a(x+b)^2 + c$, where $a$, $b$ and $c$ are constants. [3]

(ii) State the value of $A$ for which the graph of $y = f(x)$ has a line of symmetry. [1]

(iii) When $A$ has this value, find the range of f. [2]

The function g is defined by $g : x \mapsto 2x^2 - 12x + 13$ for $x \ge 4$.

(iv) Explain why g has an inverse. [1]

(v) Obtain an expression, in terms of $x$, for $g^{-1}(x)$. [3]

*Suggested Solution:*

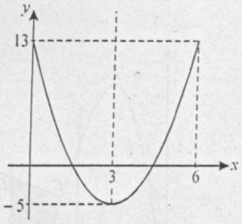
(i) $f(x) = 2x^2 - 12x + 13$ for $0 \le x \le A$

$= 2(x^2 - 6x) + 13$

$= 2\,(x^2 - 6x + 9 - 9) + 13$

$= 2\left((x-3)^2 - 9\right) + 13$

$= 2(x-3)^2 - 18 + 13 = 2(x-3)^2 - 5$ **(Ans).**



**(ii)** From part (i), $f(x) = 2(x-3)^2 - 5$

$\Rightarrow$ coordinates of turning points are $(3, -5)$

$\therefore$ $y = f(x)$ has a line of symmetry at $x = 3$.

the function is defined for $0 \le x \le A$, and because it has a line of symmetry at $x = 3$, therefore $A$ then must be equal to 6.

$\Rightarrow$ $A = 6$ **(Ans).**

**(iii)** $f : x \mapsto 2x^2 - 12x + 13$ for $0 \le x \le 6$

$\therefore$ range of $f(x)$ is: $-5 \le f(x) \le 13$ **(Ans).**

**(iv)** $g(x) = 2x^2 - 12x + 13$

given domain of $g(x)$ is: $x \ge 4$

As the graph of $g(x)$ is to the right side of the minimum point, therefore $g(x)$ is a one-one function and has an inverse.

**(v)** $g(x) = 2x^2 - 12x + 13$

using the result of part (i).

$$g(x) = 2(x-3)^2 - 5$$

Let $g(x) = y$

$$\Rightarrow \quad y = 2(x-3)^2 - 5$$

$$y + 5 = 2(x-3)^2$$

$$\frac{y+5}{2} = (x-3)^2$$

$$x - 3 = \pm\sqrt{\frac{y+5}{2}}$$

we keep the positive sign as the graph of $g(x)$ is to the right of axis of symmetry

$$\therefore \quad x - 3 = \sqrt{\frac{y+5}{2}}$$

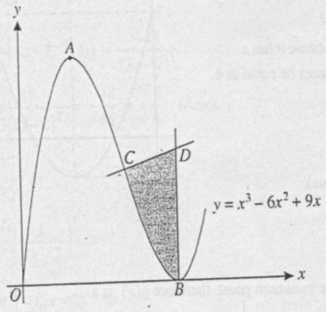
$$\Rightarrow \quad x = 3 + \sqrt{\frac{y+5}{2}}$$

as $g(x) = y \Rightarrow x = g^{-1}(y)$

$$\therefore \quad g^{-1}(y) = 3 + \sqrt{\frac{y+5}{2}}$$

$$\Rightarrow \quad g^{-1}(x) = 3 + \sqrt{\frac{x+5}{2}} \quad \textbf{(Ans).}$$

11.



The diagram shows the curve $y = x^3 - 6x^2 + 9x$ for $x \geq 0$. The curve has a maximum point at $A$ and a minimum point on the $x$-axis at $B$. The normal to the curve at $C$ (2, 2) meets the normal to the curve at $B$ at the point $D$.

(i) Find the coordinates of $A$ and $B$. [3]

(ii) Find the equation of the normal to the curve at $C$. [3]

(iii) Find the area of the shaded region. [5]

*Suggested Solution:*

(i) $y = x^3 - 6x^2 + 9x$

$\frac{dy}{dx} = 3x^2 - 12x + 9$

for stationary values. $\frac{dy}{dx} = 0$

$\Rightarrow \quad 3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$\therefore \quad x = 1.$ and $x = 3$

when $x = 1.$ $y = (1)^3 - 6(1)^2 + 9(1) = 1 - 6 + 9 = 4$

$\therefore$ coordinates of $A(1.\ 4)$ **(Ans).**

when $x = 3.$ $y = (3)^3 - 6(3)^2 + 9(3) = 27 - 54 + 27 = 0$

$\therefore$ coordinates of $B(3.\ 0)$ **(Ans).**

(ii) Gradient of the tangent at $C(2.\ 2)$ is,

$\frac{dy}{dx} = 3(2)^2 - 12(2) + 9 = 12 - 24 + 9 = -3$

$\Rightarrow$ gradient of normal at $C = \frac{1}{3}$

$\therefore$ equation of normal passing through $C(2,\ 2)$ is.

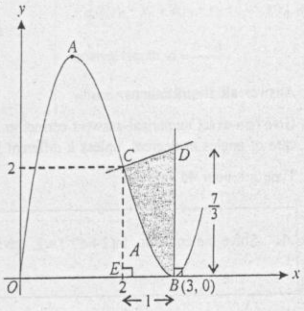
$y - 2 = \frac{1}{3}(x-2) \quad \Rightarrow \quad 3y - 6 = x - 2 \quad \Rightarrow \quad 3y - x = 4$ **(Ans).**

(iii) From part (i), coordinates of point $B$ are (3, 0).

From part (ii), equation of $CD$ is: $3y - x = 4$

$\therefore$ for point $D$, put $x = 3$,

$$\Rightarrow \quad 3y - 3 = 4 \quad \Rightarrow \quad y = \frac{7}{3}$$



from figure,

$$\text{area of trapezium } BDCE = \frac{1}{2}(1)\left(2 + \frac{7}{3}\right)$$

$$= \frac{1}{2}\left(\frac{13}{3}\right) = \frac{13}{6} = 2\frac{1}{6} \text{ units}^2$$

Area $A$ under the curve is,

$$A = \int_2^3 y \, dx$$

$$= \int_2^3 (x^3 - 6x^2 + 9x) \, dx$$

$$= \left[\frac{1}{4}x^4 - \frac{6}{3}x^3 + \frac{9}{2}x^2\right]_2^3$$

$$= \left(\frac{1}{4}(3)^4 - \frac{6}{3}(3)^3 + \frac{9}{2}(3)^2\right) - \left(\frac{1}{4}(2)^4 - \frac{6}{3}(2)^3 + \frac{9}{2}(2)^2\right)$$

$$= \left(\frac{81}{4} - 54 + \frac{81}{2}\right) - (4 - 16 + 18)$$

$$= \frac{27}{4} - 6 = \frac{3}{4}$$

Area of Shaded Region = area of trapezium $BDCE$ – area $A$ under the curve

$$= \frac{13}{6} - \frac{3}{4}$$

$$= \frac{17}{12} = 1\frac{5}{12} \text{ unit}^2 \quad \textbf{(Ans).}$$

# June 2009 Paper 3

## *Pure Mathematics (P3)*

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. Solve the equation $\ln(2+e^{-x})=2$, giving your answer correct to 2 decimal places. [4]

***Suggested Solution:***

$$\ln(2+e^{-x})=2$$
$$2+e^{-x}=e^2$$
$$e^{-x}=e^2-2$$
$$-x=\ln(e^2-2)$$
$$x=-\ln(e^2-2)$$
$$x=-\ln(5.389056)$$
$$x=-1.68 \quad \textbf{(Ans).}$$

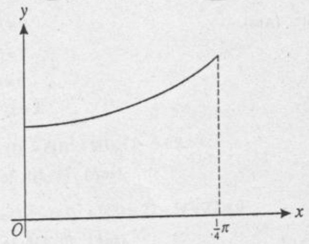
Remember:

If $\ln x=u \Rightarrow x=e^u$

Conversely

If $x=e^u \Rightarrow u=\ln x$

**2.**



The diagram shows the curve $y=\sqrt{(1+2\tan^2 x)}$ for $0 \le x \le \frac{1}{4}\pi$.

(i) Use the trapezium rule with three intervals to estimate the value of

$$\int_0^{\frac{1}{4}\pi} \sqrt{(1+2\tan^2 x)}\,dx,$$

giving your answer correct to 2 decimal places. [3]

**(ii)** The estimate found in part (i) is denoted by $E$. Explain, without further calculation, whether another estimate found using the trapezium rule with six intervals would be greater than $E$ or less than $E$. [1]

ber:

$= u \Rightarrow x = e^u$

rsely

$e^u \Rightarrow u = \ln x$

***Suggested Solution:***

(i) Interval length, $d = \dfrac{\frac{\pi}{4} - 0}{3} = \dfrac{\pi}{12}$

$x_0 = 0, \quad y_0 = \sqrt{1 + 2\tan^2(0)} = \sqrt{1+0} = 1$

$x_1 = \dfrac{\pi}{12}, \quad y_1 = \sqrt{1 + 2\tan^2(\frac{\pi}{12})} = \sqrt{1.14359} = 1.06939$

$x_2 = \dfrac{\pi}{6}, \quad y_2 = \sqrt{1 + 2\tan^2(\frac{\pi}{6})} = \sqrt{1.666667} = 1.29099$

$x_3 = \dfrac{\pi}{4}, \quad y_3 = \sqrt{1 + 2\tan^2(\frac{\pi}{4})} = \sqrt{3} = 1.73205$

applying trapezium rule,

$$\int_0^{\frac{1}{4}\pi} \sqrt{(1 + 2\tan^2 x)}\ dx \approx \frac{1}{2}(\frac{\pi}{12})[1 + 1.73205 + 2(1.06939 + 1.29099)]$$

$$\approx \frac{1}{2}(\frac{\pi}{12})(7.452819) \approx 0.98 \quad \textbf{(Ans).}$$

Trapezium rule:

$\int_a^b y\ dx$

$\approx \frac{1}{2} d[y_0 + y_n + 2(y_1 + y_2 + ....... + y_{n-1})]$

Interval length, $d = \dfrac{b-a}{n}$

where $n$ = no. of intervals

(ii) As the estimation $E$ found in part (i) is an overestimation and increase in number of intervals increases the degree of accuracy, therefore, the new estimation will be less than $E$.

3\. (i) Prove the identity $\operatorname{cosec}2\theta + \cot 2\theta \equiv \cot\theta$. [3]

(ii) Hence solve the equation $\operatorname{cosec}2\theta + \cot 2\theta = 2$. for $0° \le \theta \le 360°$. [2]

***Suggested Solution:***

(i) $\text{L.H.S} = \operatorname{cosec}2\theta + \cot 2\theta$

$= \dfrac{1}{\sin 2\theta} + \dfrac{\cos 2\theta}{\sin 2\theta}$

$= \dfrac{1 + \cos 2\theta}{\sin 2\theta}$

$= \dfrac{2\cos^2\theta}{2\sin\theta\cos\theta}$

$= \dfrac{\cos\theta}{\sin\theta} = \cot\theta = \text{R.H.S}$ **(Shown).**

Note that:

- $1 + \cos 2\theta = 2\cos^2\theta$
- $2\sin\theta\cos\theta = \sin 2\theta$

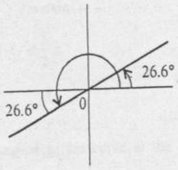
(ii) $\operatorname{cosec}2\theta + \cot 2\theta = 2$

using the result of part (i), the above equation can be written as,

$\cot\theta = 2 \Rightarrow \tan\theta = \dfrac{1}{2}$

basic angle $\alpha = 26.6°$

$\therefore \theta = 26.6°, 206.6°$ **(Ans).**



4. The equation $x^3 - 2x - 2 = 0$ has one real root.

(i) Show by calculation that this root lies between $x = 1$ and $x = 2$. [2]

(ii) Prove that, if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2 - 2}$$

converges, then it converges to this root. [2]

(iii) Use this iterative formula to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

***Suggested Solution:***

(i) Let $f(x) = x^3 - 2x - 2$

$f(1) = (1)^3 - 2(1) - 2 = -3 < 0$

$f(2) = (2)^3 - 2(2) - 2 = 2 > 0$

A change of sign indicates that the root lies between $x = 1$ and $x = 2$ **(Shown).**

(ii) $x_{n+1} = \dfrac{2x_n^3 + 2}{3x_n^2 - 2}$

Removing the subscripts, we have,

$$x = \frac{2x^3 + 2}{3x^2 - 2}$$

$x(3x^2 - 2) = 2x^3 + 2$

$3x^3 - 2x = 2x^3 + 2$

$x^3 - 2x - 2 = 0$

which is the given equation. **(Shown).**

(iii) $x_{n+1} = \dfrac{2x_n^3 + 2}{3x_n^2 - 2}$

taking $x_1 = \dfrac{1+2}{2} = 1.5$,

$$x_2 = \frac{2x_1^3 + 2}{3x_1^2 - 2} = \frac{2(1.5)^3 + 2}{3(1.5)^2 - 2} = 1.8421$$

$$x_3 = \frac{2x_2^3 + 2}{3x_2^2 - 2} = \frac{2(1.8421)^3 + 2}{3(1.8421)^2 - 2} = 1.7728$$

$$x_4 = \frac{2x_3^3 + 2}{3x_3^2 - 2} = \frac{2(1.7728)^3 + 2}{3(1.7728)^2 - 2} = 1.7693$$

$$x_5 = \frac{2x_4^3 + 2}{3x_4^2 - 2} = \frac{2(1.7693)^3 + 2}{3(1.7693)^2 - 2} = 1.7693$$

$\therefore$ root $\alpha = 1.77$ (2dp) **(Ans).**

5. When $(1+2x)(1+ax)^{\frac{2}{3}}$, where $a$ is a constant, is expanded in ascending powers of $x$, the coefficient of the term in $x$ is zero.

(i) Find the value of $a$. [3]

(ii) When $a$ has this value, find the term in $x^3$ in the expansion of $(1+2x)(1+ax)^{\frac{2}{3}}$, simplifying the coefficient. [4]

***Suggested Solution:***

(i) $(1+2x)(1+ax)^{\frac{2}{3}}$

Using binomial expansion,

$$=(1+2x)\left(1+\frac{2}{3}(ax)+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)}{2!}(ax)^2+\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}(ax)^3\ldots\ldots\right)$$

$$=(1+2x)\left(1+\frac{2}{3}ax-\frac{1}{9}a^2x^2+\frac{4}{81}a^3x^3\ldots\ldots\right)$$

terms containing $x$ only are

$$\frac{2}{3}ax+2x=\left(\frac{2}{3}a+2\right)x$$

given that coefficient of $x$ is 0,

$\Rightarrow \frac{2}{3}a+2=0 \Rightarrow a=-3$ **(Ans).**

(ii) Using the expansion in part (i), the terms containing $x^3$ only are,

$$\frac{4}{81}a^3x^3-\frac{2}{9}a^2x^3$$

$$=\left(\frac{4}{81}a^3-\frac{2}{9}a^2\right)x^3$$

putting $a=-3$ found in part (i),

the term in $x^3$ $=\left(\frac{4}{81}(-3)^3-\frac{2}{9}(-3)^2\right)x^3=\left(-\frac{4}{3}-2\right)x^3=-\frac{10}{3}x^3$ **(Ans).**

6. The parametric equations of a curve are

$$x=a\cos^3 t,\quad y=a\sin^3 t,$$

where $a$ is a positive constant and $0<t<\frac{1}{2}\pi$.

(i) Express $\frac{dy}{dx}$ in terms of $t$. [3]

(ii) Show that the equation of the tangent to the curve at the point with parameter $t$ is

$$x\sin t+y\cos t=a\sin t\cos t.$$ [3]

(iii) Hence show that, if this tangent meets the $x$-axis at $X$ and the $y$-axis at $Y$, then the length of $XY$ is always equal to $a$. [2]

***Suggested Solution:***

(i) $x = a\cos^3 t$, $\qquad y = a\sin^3 t$

$$\frac{dx}{dt} = 3a\cos^2 t(-\sin t), \qquad \frac{dy}{dt} = 3a\sin^2 t(\cos t)$$

$$= -3a\sin t\cos^2 t. \qquad = 3a\sin^2 t\cos t$$

applying chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 3a\sin^2 t\cos t \times \frac{1}{-3a\sin t\cos^2 t} \Rightarrow \frac{dy}{dx} = -\tan t$$ **(Ans).**

**(ii)** From part (i), gradient of tangent, $\frac{dy}{dx} = -\tan t$

given point is: $(a\cos^3 t,\ a\sin^3 t)$

$\therefore$ equation of tangent is,

$$y - a\sin^3 t = -\tan t(x - a\cos^3 t)$$

$$y - a\sin^3 t = -\frac{\sin t}{\cos t}(x - a\cos^3 t)$$

$$y\cos t - a\sin^3 t\cos t = -x\sin t + a\sin t\cos^3 t$$

$$x\sin t + y\cos t = a\sin t\cos^3 t + a\sin^3 t\cos t$$

$$x\sin t + y\cos t = a\sin t\cos t(\cos^2 t + \sin^2 t)$$

$$x\sin t + y\cos t = a\sin t\cos t$$ **(Shown).**

**(iii)** Equation of the tangent is: $x\sin t + y\cos t = a\sin t\cos t$

for $x$-intercept put $y = 0$,

$\Rightarrow x\sin t + 0 = a\sin t\cos t \Rightarrow x\sin t = a\sin t\cos t \Rightarrow x = a\cos t$

For $y$-intercept put $x = 0$,

$\Rightarrow 0 + y\cos t = a\sin t\cos t \Rightarrow y = a\sin t$

$\therefore$ coordinates of $X$ are $(a\cos t,\ 0)$ and
coordinates of $Y$ are $(0,\ a\sin t)$

for length of $XY$,

$$|XY| = \sqrt{(a\cos t)^2 + (a\sin t)^2}$$

$$= \sqrt{a^2\cos^2 t + a^2\sin^2 t} = \sqrt{a^2(\cos^2 t + \sin^2 t)} = \sqrt{a^2} = a$$ **(Shown).**

7. (i) Solve the equation $z^2 + (2\sqrt{3})\mathrm{i}z - 4 = 0$, giving your answers in the form $x + \mathrm{i}y$, where $x$ and $y$ are real. [3]
   (ii) Sketch an Argand diagram showing the points representing the roots. [1]
   (iii) Find the modulus and argument of each root. [3]
   (iv) Show that the origin and the points representing the roots are the vertices of an equilateral triangle. [1]

*Suggested Solution:*

(i) $z^2+\left(2\sqrt{3}\right)iz-4=0$

using quadratic formula

$$z=\frac{(-2\sqrt{3}i)\pm\sqrt{(2\sqrt{3}i)^2-4(1)(-4)}}{2(1)}$$

$$=\frac{-2\sqrt{3}i\ \pm\sqrt{-12+16}}{2}$$

$$=\frac{-2\sqrt{3}i\ \pm 2}{2}$$

$$=\frac{2(-\sqrt{3}i\ \pm 1)}{2}=-\sqrt{3}i\ \pm 1$$

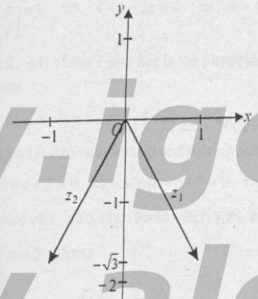
$\therefore\quad z=1-i\sqrt{3}$ or $-1-i\sqrt{3}$ **(Ans).**

Remember:

$i=\sqrt{-1}$

$\therefore\quad i^2=-1$

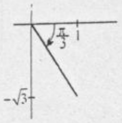
(ii)



(iii) Let $z_1=1-i\sqrt{3}$ and $z_2=-1-i\sqrt{3}$

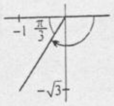
$|z_1|=\sqrt{(1)^2+(-\sqrt{3})^2}=\sqrt{1+3}=2$ **(Ans).**

$\arg(z_1)=\tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)=-\tan^{-1}\sqrt{3}=-\frac{\pi}{3}$ **(Ans).**



$|z_2|=\sqrt{(-1)^2+(-\sqrt{3})^2}=\sqrt{1+3}=2$ **(Ans).**

$\arg(z_2)=\tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right)=\tan^{-1}\sqrt{3},$



basic angle $=\frac{\pi}{3}$

$\therefore$ required argument $=-\frac{2}{3}\pi$ **(Ans).**

Note that:

If $z=x+iy$, then

- $|z|=\sqrt{x^2+y^2}$
- $\arg(z)=\tan^{-1}\left(\frac{y}{x}\right)$

The range of principal argument $\alpha$ is $-\pi<\alpha\le\pi$

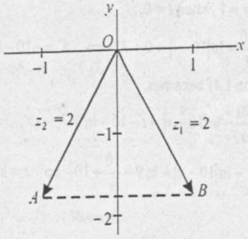
(iv) From the Argand diagram,

$|OB|=|z_1|=2$

$|OA|=|z_2|=2$

$|AB|=2$

$\therefore$ OAB is an equilateral triangle. **(Ans).**



8. (i) Express $\dfrac{100}{x^2(10-x)}$ in partial fractions. [4]

(ii) Given that $x = 1$ when $t = 0$, solve the differential equation

$$\frac{dx}{dt} = \frac{1}{100}x^2(10-x),$$

obtaining an expression for $t$ in terms of $x$. [6]

***Suggested Solution:***

(i) Let $\dfrac{100}{x^2(10-x)} \equiv \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{10-x}$

$\Rightarrow \quad 100 \equiv Ax(10-x) + B(10-x) + Cx^2$

for $x = 0$

$100 = 0 + B(10-0) + C(0)^2 \quad \Rightarrow \quad 100 = 10B \quad \Rightarrow \quad B = 10$

for $x = 10$

$100 = A(10)(10-10) + B(10-10) + C(10)^2 \quad \Rightarrow \quad 100 = 100C \quad \Rightarrow \quad C = 1$

for $x = 1$

$100 = A(1)(10-1) + B(10-1) + C(1)^2$

$\Rightarrow \quad 100 = 9A + 9B + C \quad \Rightarrow \quad 100 = 9A + 9(10) + 1 \quad \Rightarrow \quad 9 = 9A \quad \Rightarrow \quad A = 1$

$\therefore \quad \dfrac{100}{x^2(10-x)} \equiv \dfrac{1}{x} + \dfrac{10}{x^2} + \dfrac{1}{10-x}$ **(Ans).**

(ii) $\dfrac{dx}{dt} = \dfrac{1}{100}x^2(10-x) \quad \Rightarrow \quad \dfrac{100}{x^2(10-x)}\,dx = dt$

using the result of part (i),

$\left(\dfrac{1}{x} + \dfrac{10}{x^2} + \dfrac{1}{10-x}\right) dx = dt$

Integrating both sides

$$\int\left(\frac{1}{x} + \frac{10}{x^2} + \frac{1}{10-x}\right) dx = \int dt$$

$\Rightarrow \quad \ln x + \dfrac{10x^{-1}}{-1} + \dfrac{\ln|10-x|}{-1} = t + K$

$\Rightarrow \quad \ln x - \dfrac{10}{x} - \ln|10-x| = t + K \quad \cdots\cdots\cdots (A)$

given that $x = 1$ when $t = 0$,

$\Rightarrow \quad \ln 1 - \dfrac{10}{1} - \ln|10-1| = 0 + K \quad \Rightarrow \quad K = -10 - \ln 9$

$\therefore$ equation $(A)$ becomes.

$\ln x - \dfrac{10}{x} - \ln|10-x| = t - 10 - \ln 9$

$\Rightarrow \quad t = \ln x - \ln|10-x| + \ln 9 - \dfrac{10}{x} + 10 \quad \Rightarrow \quad t = \ln\left|\dfrac{9x}{10-x}\right| + 10 - \dfrac{10}{x}$ **(Ans).**

Note that:

- $\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$
- $\ln(a \times b) = \ln a + \ln b$
- $\ln(a^b) = b\ln a$

9. The line $l$ has equation $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k} + t(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$. It is given that $l$ lies in the plane with equation $2x + by + cz = 1$, where $b$ and $c$ are constants.

(i) Find the values of $b$ and $c$. [6]

(ii) The point $P$ has position vector $2\mathbf{j} + 4\mathbf{k}$. Show that the perpendicular distance from $P$ to $l$ is $\sqrt{5}$. [5]

***Suggested Solution:***

(i) Equation of line $l$: $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k} + t(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

Equation of plane: $2x + by + cz = 1 \Rightarrow \mathbf{r}\cdot(2\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 1$

Line $l$ lies in the plane, therefore angle between the normal of plane and the direction vector of line $l$ is 90°.

$$\Rightarrow \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ b \\ c \end{pmatrix} = 0 \Rightarrow 4 - b - 2c = 0 \Rightarrow b = 4 - 2c \cdots\cdots\cdots(1)$$

The point (4, 2, −1) of line $l$ also lies in the plane, therefore it will satisfy the plane's equation

$$2(4) + b(2) + c(-1) = 1 \Rightarrow 8 + 2b - c = 1 \Rightarrow 2b - c + 7 = 0 \cdots\cdots\cdots(2)$$

putting equation (1) into equation (2) and solving simultaneously.

$$2(4 - 2c) - c + 7 = 0 \Rightarrow 8 - 4c - c + 7 = 0 \Rightarrow 5c = 15 \Rightarrow c = 3$$

putting the value of $c$ in eq. (1): $b = 4 - 2(3) \Rightarrow b = -2$

$\therefore$ $b = -2$, $c = 3$ **(Ans).**

(ii) Equation of line $l$: $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

Let $Q$ be the general point on line $l$. $\therefore$ $\vec{OQ} = \begin{pmatrix} 4+2t \\ 2-t \\ -1-2t \end{pmatrix}$

position vector of point $P$ is: $\vec{OP} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{pmatrix} 4+2t \\ 2-t \\ -1-2t \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4+2t \\ -t \\ -5-2t \end{pmatrix}$$

Let $\vec{d}$ be the direction vector of line $l$.

As $\vec{PQ}$ is perpendicular to line $l$, $\therefore$ $\vec{PQ} \cdot \vec{d} = 0$

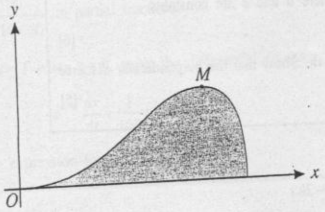
$$\Rightarrow \begin{pmatrix} 4+2t \\ -t \\ -5-2t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0 \Rightarrow 8 + 4t + t + 10 + 4t = 0 \Rightarrow 18 + 9t = 0 \Rightarrow t = -2$$

$$\therefore \vec{PQ} = \begin{pmatrix} 4+2(-2) \\ -(-2) \\ -5-2(-2) \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

Hence, the distance from $P$ to the line $l$ is: $\left|\vec{PQ}\right| = \sqrt{(2)^2 + (-1)^2} = \sqrt{5}$ **(Shown).**

lna − lnb
) = lna + lnb
= blna

**10.**



The diagram shows the curve $y = x^2\sqrt{(1-x^2)}$ for $x \geq 0$ and its maximum point $M$.

(i) Find the exact value of the $x$-coordinate of $M$. [4]

(ii) Show, by means of the substitution $x = \sin\theta$, that the area $A$ of the shaded region between the curve and the $x$-axis is given by

$$A = \frac{1}{4}\int_0^{\frac{1}{2}\pi} \sin^2 2\theta \ d\theta.$$ [3]

(iii) Hence obtain the exact value of $A$. [4]

***Suggested Solution:***

(i) $y = x^2\sqrt{1-x^2}$ for $x \geq 0$

differentiating w.r.t $x$.

$$\frac{dy}{dx} = \sqrt{1-x^2}\,\frac{d}{dx}(x^2) + x^2\,\frac{dy}{dx}\left(\sqrt{1-x^2}\right)$$

$$= 2x\sqrt{1-x^2} + x^2\left(\frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times (-2x)\right)$$

$$= 2x\sqrt{1-x^2} - \frac{x^3}{\sqrt{1-x^2}} = \frac{2x(1-x^2)-x^3}{\sqrt{1-x^2}} = \frac{2x-3x^3}{\sqrt{1-x^2}}$$

Remember:

$$\frac{d}{dx}(uv) = v\frac{d}{dx}u + u\frac{d}{dx}v$$

for maximum value, $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{2x-3x^3}{\sqrt{1-x^2}} = 0 \Rightarrow 2x - 3x^3 = 0 \Rightarrow x(2-3x^2) = 0$$

either $x = 0$ (rejected). or $2 - 3x^2 = 0 \Rightarrow x^2 = \frac{3}{2} \Rightarrow x = \pm\sqrt{\frac{3}{2}}$

$\therefore$ according to the given figure, $x$-coordinate of $M$ is: $x = \sqrt{\frac{2}{3}}$ **(Ans).**

(ii) $y = x^2\sqrt{(1-x^2)}$

for $x$-intercept put $y = 0$

$\Rightarrow x^2\sqrt{(1-x^2)} = 0$

$\Rightarrow$ either $x = 0$ (rejected), or $1 - x^2 = 0 \Rightarrow x = 1$

mber:

$$(uv) = v\frac{d}{dx}u + u\frac{d}{dx}v$$

Area of the shaded region. $A = \int_0^1 y \, dx = \int_0^1 x^2\sqrt{(1-x^2)} \, dx$

given substitution is: $x = \sin\theta \Rightarrow \frac{dx}{d\theta} = \cos\theta \Rightarrow dx = \cos\theta \, d\theta$

for limits: when $x = 0$, $\sin\theta = 0 \Rightarrow \theta = 0$

when $x = 1$, $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

substituting the new limits and the values of $x$ and $dx$,

$$\text{Area } A = \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{1-\sin^2\theta} \; \cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{\cos^2\theta} \; \cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta \; d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} 4\sin^2\theta\cos^2\theta \; d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^2 \; d\theta = \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta \; d\theta \quad \textbf{(Shown).}$$

(iii) Area, $A = \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta \; d\theta$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \frac{1-\cos4\theta}{2} \; d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{2}} \left(1-\cos4\theta\right) \; d\theta$$

$$= \frac{1}{8}\left[\theta - \frac{\sin4\theta}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{8}\left[\left(\frac{\pi}{2} - \frac{\sin4(\frac{\pi}{2})}{4}\right) - \left(0 - \frac{\sin4(0)}{4}\right)\right]$$

$$= \frac{1}{8}\left[\left(\frac{\pi}{2} - \frac{\sin2\pi}{4}\right) - 0\right]$$

$$= \frac{1}{8}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{16} \text{ sq. units} \quad \textbf{(Ans).}$$

Remember:

$\cos2\theta = 1 - 2\sin^2\theta$

$\Rightarrow \sin^2\theta = \frac{1-\cos2\theta}{2}$

$\Rightarrow \sin^2 2\theta = \frac{1-\cos4\theta}{2}$

# November 2009 Paper 1

## *Pure Mathematics (P1)*

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** The equation of a curve is such that $\frac{dy}{dx} = \frac{3}{\sqrt{x}} - x$. Given that the curve passes through the point (4, 6), find the equation of the curve. [4]

***Suggested Solution:***

$$\frac{dy}{dx} = \frac{3}{\sqrt{x}} - x \quad \Rightarrow \quad dy = \left(3x^{-\frac{1}{2}} - x\right) dx$$

integrating both sides

$$\int dy = \int \left(3x^{-\frac{1}{2}} - x\right) dx$$

$$\Rightarrow \quad y = 3\left(\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right) - \frac{x^2}{2} + K = 6\sqrt{x} - \frac{1}{2}x^2 + K$$

as the curve passes through (4. 6).

$$\therefore \quad 6 = 6\sqrt{4} - \frac{1}{2}(4)^2 + K \quad \Rightarrow \quad 6 = 12 - 8 + K \quad \Rightarrow \quad K = 2$$

$\therefore$ the equation of the curve is,

$$y = 6\sqrt{x} - \frac{1}{2}x^2 + 2 \quad \textbf{(Ans).}$$

When $\frac{dy}{dx}$ (or gradient) of the curve is given, always integrate both sides of the given expression to find the equation of the curve.

**2.** (i) Find, in terms of the non-zero constant $k$, the first 4 terms in the expansion of $(k + x)^8$ in ascending powers of $x$. [3]

(ii) Given that the coefficients of $x^2$ and $x^3$ in this expansion are equal, find the value of $k$. [2]

***Suggested Solution:***

(i) $(k + x)^8 = {}^8C_0\, k^8 x^0 + {}^8C_1\, k^7 x^1 + {}^8C_2\, k^6 x^2 + {}^8C_3\, k^5 x^3$

$= k^8 + 8k^7x + 28k^6x^2 + 56k^5x^3$ **(Ans).**

Binomial theorem:

$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}b + {}^nC_2\, a^{n-2}b^2 + .... + {}^nC_r\, a^{n-r}b^r + .... + {}^nC_n\, b^n$

CORNER

(or gradient) of
given, always
th sides of the
ession to find
n of the curve.

rem:
$a^n + {}^nC_1 a^{n-1}b +$
$\ldots + {}^nC_r a^{n-r}b^r +$

(ii) coefficient of $x^2$ = coefficient of $x^3$

$\Rightarrow \quad 28k^6 = 56k^5$

$28k^6 - 56k^5 = 0$

$28k^5(k-2) = 0$

$\Rightarrow \quad 28k^5 = 0 \quad \text{or} \quad (k-2) = 0$

$\Rightarrow \quad k = 0$ (ignored) or $k = 2$

$\therefore \quad k = 2$ **(Ans).**

3. A progression has a second term of 96 and a fourth term of 54. Find the first term of the progression in each of the following cases:

(i) the progression is arithmetic, [3]

(ii) the progression is geometric with a positive common ratio. [3]

***Suggested Solution:***

(i) Given that, $T_2 = 96$

$\Rightarrow \; a + (2-1)d = 96 \;\Rightarrow\; a + d = 96 \cdots\cdots\cdots(i)$

also given that, $T_4 = 54$

$\Rightarrow \; a + (4-1)d = 54 \;\Rightarrow\; a + 3d = 54 \cdots\cdots\cdots(ii)$

solving equations (i) & (ii) simultaneously,

3 × eq (i): $\quad 3a + 3d = 288$

1 × eq (ii): $\quad a + 3d = 54$

$$\underline{\;-\quad-\quad\;\;-\;}$$

$$2a + 0 = 234 \quad\Rightarrow\quad a = 117 \quad \textbf{(Ans).}$$

(ii) Using $T_n = ar^{n-1}$

$T_2 = 96$

$\Rightarrow \; ar^{2-1} = 96 \;\Rightarrow\; ar = 96 \cdots\cdots\cdots(i)$

$T_4 = 54$

$\Rightarrow \; ar^{4-1} = 54 \;\Rightarrow\; ar^3 = 54 \cdots\cdots\cdots(ii)$

dividing equation (ii) by (i),

$$\frac{ar^3}{ar} = \frac{54}{96} \;\Rightarrow\; r^2 = \frac{9}{16} \;\Rightarrow\; r = \frac{3}{4} \text{ (using positive sign only)}$$

putting value of $r$ in equation (i),

$$a\left(\frac{3}{4}\right) = 96 \;\Rightarrow\; a = 128 \quad \textbf{(Ans).}$$

4. The function f is defined by $f : x \mapsto 5 - 3\sin 2x$ for $0 \le x \le \pi$.

(i) Find the range of f. [2]

(ii) Sketch the graph of $y = f(x)$. [3]

(iii) State, with a reason, whether f has an inverse. [1]

*Suggested Solution:*

(i) $f : x \mapsto 5 - 3\sin 2x$

we know that, $-1 \le \sin x \le 1$

$\Rightarrow \quad -1 \le \sin 2x \le 1$

multiplying by $-3$,

$3 \ge -3\sin 2x \ge -3$

$\Rightarrow \quad -3 \le -3\sin 2x \le 3$

adding 5,

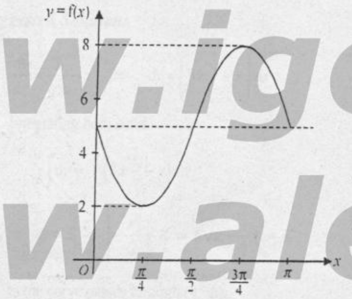
$5 - 3 \le 5 - 3\sin 2x \le 3 + 5$

$\Rightarrow \quad 2 \le 5 - 3\sin 2x \le 8$

$\therefore$ range of f is: $2 \le f(x) \le 8$ **(Ans).**

**(ii)**

| $x$ | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $\frac{3\pi}{4}$ | $\pi$ |
|---|---|---|---|---|---|
| $y = f(x)$ | 5 | 2 | 5 | 8 | 5 |



For better sketching of the graph of the type $f(x) = a\sin bx + c$, take the interval $\frac{90°}{b}$ or $\left(\frac{\pi}{2} \div b\right)$ according to the range given in degrees or in radians.

**(iii)** f(x) does not have an inverse because it is not a (1-1) function.

> **5.** (i) Prove the identity $(\sin x + \cos x)(1 - \sin x\cos x) \equiv \sin^3 x + \cos^3 x$. [3]
>
> (ii) Solve the equation $(\sin x + \cos x)(1 - \sin x\cos x) = 9\sin^3 x$ for $0° \le x \le 360°$. [3]

*Suggested Solution:*

(i) $\text{R.H.S.} = \sin^3 x + \cos^3 x$

$= (\sin x)^3 + (\cos x)^3$

$= (\sin x + \cos x)(\sin^2 x + \cos^2 x - \sin x\cos x)$

$= (\sin x + \cos x)(1 - \sin x\cos x) = \text{L.H.S.}$ **(Proved).**

Remember:

$(a)^3 + (b)^3$

$= (a + b)(a^2 + b^2 - ab)$

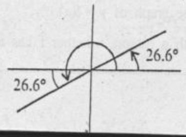
(ii) $(\sin x + \cos x)(1 - \sin x\cos x) = 9\sin^3 x$

using the result in part (i), we have,

$\sin^3 x + \cos^3 x = 9\sin^3 x$

$\cos^3 x = 8\sin^3 x \Rightarrow \frac{\sin^3 x}{\cos^3 x} = \frac{1}{8} \Rightarrow \tan^3 x = \frac{1}{8} \Rightarrow \tan x = \frac{1}{2}$

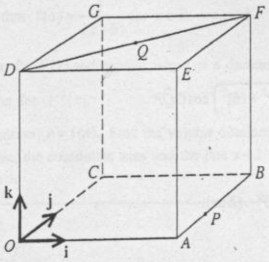
Note that:
$\tan\theta$ is positive in 1st and 3rd quadrant.



basic angle, $\alpha = 26.6°$

$\therefore x = 26.6°, 206.6°$ (Ans)

6.



In the diagram, *OABCDEFG* is a cube in which each side has length 6. Unit vectors **i**, **j** and **k** are parallel to $\vec{OA}$, $\vec{OC}$. and $\vec{OD}$ respectively. The point *P* is such that $\vec{AP} = \frac{1}{3}\vec{AB}$ and the point *Q* is the mid-point of *DF*.

(i) Express each of the vectors $\vec{OQ}$ and $\vec{PQ}$ in terms of **i**, **j** and **k**. [3]

(ii) Find the angle *OQP*. [4]

***Suggested Solution:***

(i) We have.

$$\vec{OD} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}, \qquad \vec{OF} = \vec{OA} + \vec{AB} + \vec{BF} = 6\mathbf{i} + 6\mathbf{j} + 6\mathbf{k} = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$$

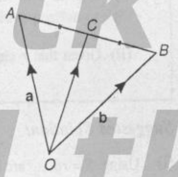
$$\therefore \vec{OQ} = \frac{\vec{OD} + \vec{OF}}{2} = \begin{pmatrix} \frac{0+6}{2} \\ \frac{0+6}{2} \\ \frac{6+6}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} \quad \text{(Ans).}$$

$$\vec{OP} = \vec{OA} + \vec{AP}$$

$$= \vec{OA} + \frac{1}{3}\vec{AB} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}$$

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{pmatrix} 3 \\ 3 \\ 6 \end{pmatrix} - \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 6 \end{pmatrix}$$

$$\therefore \vec{PQ} = \begin{pmatrix} -3 \\ 1 \\ 6 \end{pmatrix} \text{ or } -3\mathbf{i} + \mathbf{j} + 6\mathbf{k} \quad \text{(Ans).}$$



If

position vector of $A = \mathbf{a}$

position vector of $B = \mathbf{b}$

then the position vector of *C* is:

$$\vec{OC} = \frac{\mathbf{a} + \mathbf{b}}{2}$$

where *C* is the midpoint of straight line *AB*.

(ii) Applying scalar product,

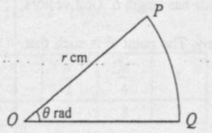
$$\vec{OQ}\cdot\vec{PQ}=\left|\vec{OQ}\right|\left|\vec{PQ}\right|\cos O\hat{Q}P$$

$$\begin{pmatrix}3\\3\\6\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\6\end{pmatrix}=\left(\sqrt{(3)^2+(3)^2+(6)^2}\right)\left(\sqrt{(-3)^2+(1)^2+(6)^2}\right)\cos O\hat{Q}P$$

$$-9+3+36=\left(\sqrt{54}\right)\left(\sqrt{46}\right)\cos O\hat{Q}P$$

$$\cos O\hat{Q}P=\frac{30}{\sqrt{2484}}\quad\Rightarrow\quad O\hat{Q}P=52.99°\approx 53°\quad \textbf{(Ans).}$$

7.



A piece of wire of length 50 cm is bent to form the perimeter of a sector $POQ$ of a circle. The radius of the circle is $r$ cm and the angle $POQ$ is $\theta$ radians (see diagram).

(i) Express $\theta$ in terms of $r$ and show that the area, $A$ cm², of the sector is given by

$$A=25r-r^2. \qquad [4]$$

(ii) Given that $r$ can vary, find the stationary value of $A$ and determine its nature. [4]

***Suggested Solution:***

(i) Using $S=r\theta$, arc length, $\overset{\frown}{PQ}=r\theta$

perimeter of the sector $POQ=50$ cm

$\Rightarrow\ r+r+r\theta=50$

$$2r+r\theta=50\ \Rightarrow\ \theta=\frac{50-2r}{r}\ \Rightarrow\ \theta=\frac{50}{r}-2$$

Area of the sector, $A=\frac{1}{2}r^2\theta$

$$\Rightarrow\ A=\frac{1}{2}r^2\left(\frac{50}{r}-2\right)\ \Rightarrow\ A=25r-r^2\quad \textbf{(Shown).}$$

(ii) $A=25r-r^2$

differentiating w.r.t. $r$

$$\frac{dA}{dr}=25-2r$$

for stationary values, $\frac{dA}{dr}=0$

$\Rightarrow\ 25-2r=0\ \Rightarrow\ r=\frac{25}{2}=12.5$ cm.

putting $r=12.5$ in the equation of area,

$\Rightarrow\ A=25(12.5)-(12.5)^2=156.25$

$\therefore$ stationary value of $A=156.25$ cm² **(Ans).**

$\frac{d^2A}{dr^2}=-2<0\quad\therefore$ stationary value of $A$ is a maximum **(Ans).**

Remember:

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1}>0$, $y$ is min.

if $\left.\frac{d^2y}{dx^2}\right|_{x=x_1}<0$, $y$ is max.

8. The function f is such that $f(x)=\frac{3}{2x+5}$ for $x\in\mathbb{R}$, $x\neq -2.5$.

(i) Obtain an expression for $f'(x)$ and explain why f is a decreasing function. [3]

(ii) Obtain an expression for $f^{-1}(x)$. [2]

(iii) A curve has the equation $y=f(x)$. Find the volume obtained when the region bounded by the curve, the coordinate axes and the line $x=2$ is rotated through 360° about the $x$-axis. [4]

***Suggested Solution:***

(i) $f(x)=\frac{3}{2x+5}=3(2x+5)^{-1}$

Differentiating w.r.t. $x$

$\Rightarrow f'(x)=-3(2x+5)^{-2}(2)=-\frac{6}{(2x+5)^2}$

Since $(2x+5)^2$ is always positive.

$\frac{-6}{(2x+5)^2}$ is always negative $\Rightarrow f'(x)=\frac{-6}{(2x+5)^2}$ is always negative

$\therefore$ f(x) is a decreasing function **(Ans).**

(ii) $f(x)=\frac{3}{2x+5}$

Let $f(x)=y$

$\Rightarrow y=\frac{3}{2x+5} \Rightarrow 2x+5=\frac{3}{y} \Rightarrow 2x=\frac{3}{y}-5 \Rightarrow x=\frac{1}{2}\left(\frac{3}{y}-5\right)$

as $f(x)=y \Rightarrow x=f^{-1}(y)$

$\Rightarrow f^{-1}(y)=\frac{1}{2}\left(\frac{3}{y}-5\right)$

$\therefore f^{-1}(x)=\frac{1}{2}\left(\frac{3}{x}-5\right)$ for $x\neq 0$ **(Ans).**

(iii) $f(x)=\frac{3}{2x+5}=3(2x+5)^{-1}$

For $y$-intercept put $x=0$. $\Rightarrow y=\frac{3}{2(0)+5}=0.6$

$\therefore$ the curve meets the $y$-axis at (0, 0.6)

$\therefore$ Volume, $V=\pi\int_0^2 y^2\,dx$

$\Rightarrow V=\pi\int_0^2\left(3(2x+5)^{-1}\right)^2\,dx$

$=9\pi\int_0^2(2x+5)^{-2}\,dx$

Note that:

- If $f(x)=y$, then, $f'(x)=\frac{dy}{dx}$
- A function $f(x)$ is increasing if $f'(x)>0$
- A function $f(x)$ is decreasing if $f'(x)<0$
- A squared quantity is always positive or $\geq 0$.

is min.

is max.

$$= 9\pi\left[\frac{(2x+5)^{-1}}{-1(2)}\right]_0^2$$

$$= \frac{9\pi}{-2}\left[\frac{1}{2x+5}\right]_0^2$$

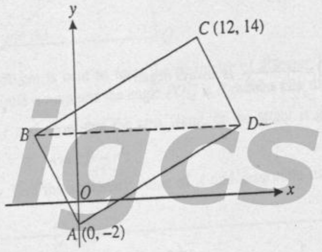
$$= -\frac{9\pi}{2}\left(\frac{1}{2(2)+5}\right)-\left(\frac{1}{2(0)+5}\right)$$

$$= -\frac{9\pi}{2}\left(\frac{1}{9}-\frac{1}{5}\right)$$

$$= -\frac{9\pi}{2}\left(-\frac{4}{45}\right)$$

$= \frac{2}{5}\pi$ units$^3$ **(Ans).**

**9.**



The diagram shows a rectangle $ABCD$. The point $A$ is $(0, -2)$ and $C$ is $(12, 14)$. The diagonal $BD$ is parallel to the $x$-axis.

(i) Explain why the $y$-coordinate of $D$ is 6. [1]

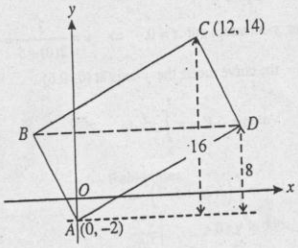
The $x$-coordinate of $D$ is $h$.

(ii) Express the gradients of $AD$ and $CD$ in terms of $h$. [3]

(iii) Calculate the $x$-coordinates of $D$ and $B$. [4]

(iv) Calculate the area of the rectangle $ABCD$. [3]

***Suggested Solution:***



(i) $BD$ is parallel to $x$-axis.

$\Rightarrow$ vertical distance between $A$ and $C = 16$ units

$\therefore$ vertical distance between $A$ and $D = 8$ units.

$\therefore$ $y$-coordinate of $D = 6$ **(Ans).**

(ii) $A(0, -2)$ and $D(h, 6)$

$\therefore$ gradient of $AD = \frac{6+2}{h-0} = \frac{8}{h}$ **(Ans).**

$C(12, 14)$ and $D(h, 6)$

$\therefore$ gradient of $CD = \frac{6-14}{h-12} = \frac{-8}{h-12} = \frac{8}{12-h}$ **(Ans).**

(iii) $AD$ is ⊥ to $CD$

∴ (gradient of $AD$)(gradient of $CD$) $= -1$

$$\Rightarrow \quad \left(\frac{8}{h}\right)\left(\frac{8}{12-h}\right) = -1$$

$$64 = -h(12-h)$$

$$64 = -12h + h^2$$

$$h^2 - 12h - 64 = 0$$

$$(h-16)(h+4) = 0$$

$\Rightarrow$ $h = -4$ and $h = 16$.

∴ $x$-coordinate of $B$ is $-4$. and $x$-coordinates of $D$ is 16. **(Ans).**

(iv) We have, $A(0, -2)$, $B(-4, 6)$. $D(16, 6)$.

applying distance formula.

$$|AB| = \sqrt{(-4-0)^2 + (6+2)^2} = \sqrt{16+64} = \sqrt{80}$$

$$|AD| = \sqrt{(16-0)^2 + (6+2)^2} = \sqrt{256+64} = \sqrt{320}$$

∴ area of rectangle $ABCD = |AB| \times |AD|$

$$= \sqrt{80} \times \sqrt{320} = 160 \text{ unit}^2 \quad \textbf{(Ans).}$$

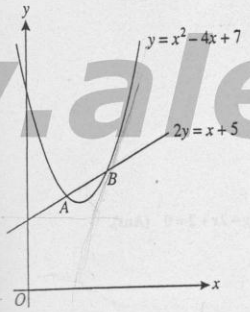
Distance formula:

Distance between two points,

$A(x_1, y_1)$ and $B(x_2, y_2)$ is:

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

8.



(i) The diagram shows the line $2y = x + 5$ and the curve $y = x^2 - 4x + 7$, which intersect at the points $A$ and $B$. Find

(a) the $x$-coordinates of $A$ and $B$, [3]

(b) the equation of the tangent to the curve at $B$, [3]

(c) the acute angle, in degrees correct to 1 decimal place, between this tangent and the line $2y = x + 5$. [3]

(ii) Determine the set of values of $k$ for which the line $2y = x + k$ does not intersect the curve $y = x^2 - 4x + 7$. [4]

12, 14)

**Suggested Solution:**

(i) (a) Equation of line is: $2y = x + 5 \Rightarrow y = \frac{x+5}{2}$ ………(i)

Equation of curve: $y = x^2 - 4x + 7$ ………(ii)

substituting equation (i) into equation (ii) and solving simultaneously,

$$\frac{x+5}{2} = x^2 - 4x + 7$$
$$x + 5 = 2x^2 - 8x + 14$$
$$2x^2 - 9x + 9 = 0$$
$$2x^2 - 6x - 3x + 9 = 0$$
$$2x(x-3) - 3(x-3) = 0$$
$$(x-3)(2x-3) = 0$$
$$\Rightarrow x = 3 \text{ or } x = \frac{3}{2}$$

$\therefore$ $x$-coordinate of $A$ is: $x = \frac{3}{2}$ **(Ans).**

$x$-coordinate of $B$ is: $x = 3$ **(Ans).**

(b) Equation of line $AB$: $2y = x + 5$

$x$-coordinate of $B$ is 3. For $y$-coordinate, put $x = 3$ in the equation of line $AB$.

$\Rightarrow$ $2y = 3 + 5 \Rightarrow y = 4$

$\therefore$ coordinates of $B(3, 4)$

Equation of curve is: $y = x^2 - 4x + 7$

gradient of curve, $\frac{dy}{dx} = 2x - 4$

at $x = 3$, $\frac{dy}{dx} = 2(3) - 4 = 2$

$\therefore$ equation of tangent to the curve at $B$ is,

$y - 4 = 2(x - 3) \Rightarrow y - 4 = 2x - 6 \Rightarrow y - 2x + 2 = 0$ **(Ans).**

Gradient of a line is also defined as the tangent of the angle which a line makes with the positive direction of the x-axis.

If the gradient is +ve, angle made is acute and if the gradient is –ve, angle made is obtuse.

(c) Let this tangent makes an angle $\alpha$ with the $x$-axis.

as gradient of the tangent = 2

$\Rightarrow$ $\tan\alpha = 2 \Rightarrow \alpha = 63.4°$

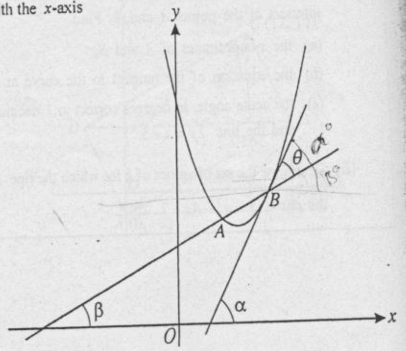
Let $\beta$ be the angle which the line $2y = x + 5$ makes with the $x$-axis

gradient of the line is: $2\frac{dy}{dx} = 1 + 0 \Rightarrow \frac{dy}{dx} = \frac{1}{2}$

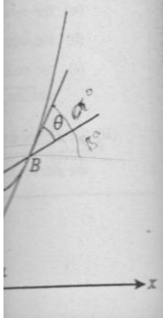
$\Rightarrow$ $\tan\beta = \frac{1}{2} \Rightarrow \beta = 26.6°$

acute angle $\theta$ between the two lines is,

$\theta = \alpha - \beta = 63.4° - 26.6° = 36.8°$ **(Ans).**



adient of a line is also
fined as the tangent of
e angle which a line
kes with the positive
ection of the x-axis.
the gradient is +ve,
gle made is acute and
he gradient is –ve, angle
ade is obtuse.



(ii) Equation of line: $2y = x + k \Rightarrow y = \frac{x+k}{2}$

Equation of the curve: $y = x^2 - 4x + 7$

solving the two equations simultaneously,

$$\frac{x+k}{2} = x^2 - 4x + 7$$

$$2x^2 - 8x + 14 = x + k$$

$$2x^2 - 9x + (14 - k) = 0$$

the line does not intersect the curve, therefore the above equation has imaginary roots,

$\therefore$ Discriminant. $b^2 - 4ac < 0$

$$\Rightarrow (-9)^2 - 4(2)(14 - k) < 0$$

$$81 - 8(14 - k) < 0$$

$$81 - 112 + 8k < 0$$

$$-31 + 8k < 0$$

$$8k < 31$$

$$k < \frac{31}{8} \quad \text{or} \quad k < 3\tfrac{7}{8}$$ **(Ans).**

# November 2009 Paper 3

## Pure Mathematics (P3)

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** Solve the equation

$$\ln(5-x)=\ln 5-\ln x,$$

giving your answers correct to 3 significant figures. [4]

***Suggested Solution:***

$$\ln(5-x)=\ln 5-\ln x$$

$$\ln(5-x)=\ln\left(\frac{5}{x}\right)$$

$$5-x=\frac{5}{x}$$

$$5x-x^2=5$$

$$x^2-5x+5=0$$

applying quadratic formula,

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(5)}}{2(1)}=\frac{5\pm\sqrt{25-20}}{2}=\frac{5\pm\sqrt{5}}{2}$$

$$\Rightarrow\quad x=\frac{5+\sqrt{5}}{2}\quad\text{or}\quad x=\frac{5-\sqrt{5}}{2}$$

$$\Rightarrow\quad x=3.62\ \text{or}\ x=1.38\quad\textbf{(Ans).}$$

Note that:

- $\ln\left(\frac{a}{b}\right)=\ln a-\ln b$
- $\ln(a\times b)=\ln a+\ln b$
- $\ln(a^b)=b\ln a$

**2.** The equation $x^3-8x-13=0$ has one real root.

(i) Find the two consecutive integers between which this root lies. [2]

(ii) Use the iterative formula

$$x_{n+1}=(8x_n+13)^{\frac{1}{3}}$$

to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

*Suggested Solution:*

(i) Let $f(x) = x^3 - 8x - 13$

by trial and error.

$f(0) = -13 < 0$

$f(1) = 1 - 8 - 13 = -20 < 0$

$f(2) = 8 - 16 - 13 = -21 < 0$

$f(3) = 27 - 24 - 13 = -10 < 0$

$f(4) = 64 - 32 - 13 = 19 > 0$

The change of sign indicates that the root lies between $x = 3$ and $x = 4$ **(Ans).**

(ii) $x_{n+1} = (8x_n + 13)^{\frac{1}{3}}$

Let the initial value be any intermediate value of the interval $3 < x < 4$.

$\therefore$ taking $x_1 = \frac{3+4}{2} = 3.5$

$x_2 = (8x_1 + 13)^{\frac{1}{3}} = (8(3.5) + 13)^{\frac{1}{3}} = 3.4482$

$x_3 = (8x_2 + 13)^{\frac{1}{3}} = (8(3.4482) + 13)^{\frac{1}{3}} = 3.4366$

$x_4 = (8x_3 + 13)^{\frac{1}{3}} = (8(3.4366) + 13)^{\frac{1}{3}} = 3.4339$

$x_5 = (8x_4 + 13)^{\frac{1}{3}} = (8(3.4339) + 13)^{\frac{1}{3}} = 3.4333$

$x_6 = (8x_5 + 13)^{\frac{1}{3}} = (8(3.4333) + 13)^{\frac{1}{3}} = 3.4332$

$x_7 = (8x_6 + 13)^{\frac{1}{3}} = (8(3.4332) + 13)^{\frac{1}{3}} = 3.4332$

$\therefore$ root, $\alpha = 3.43$ (to 2 dp) **(Ans).**

3. The equation of a curve is $x^3 - x^2y - y^3 = 3$.

(i) Find $\frac{dy}{dx}$ in terms of $x$ and $y$. [4]

(ii) Find the equation of the tangent to the curve at the point (2, 1), giving your answer in the form $ax + by + c = 0$. [2]

*Suggested Solution:*

(i) $x^3 - x^2y - y^3 = 3$

differentiating w.r.t. $x$

$$\frac{d}{dx}(x^3 - x^2y - y^3) = \frac{d}{dx}(3)$$

$$\Rightarrow 3x^2 - \left(y(2x) + x^2\frac{dy}{dx}\right) - 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 - 2xy - x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 - 2xy = (x^2 + 3y^2)\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 + 3y^2}$$ **(Ans).**

Remember:

$$\frac{d}{dx}(uv) = v\frac{d}{dx}u + u\frac{d}{dx}v$$

(ii) Using the result of part (i), gradient of the tangent at point (2, 1) is,

$$\frac{dy}{dx}=\frac{3(2)^2-2(2)(1)}{(2)^2+3(1)^2}=\frac{12-4}{4+3}=\frac{8}{7}$$

∴ equation of the tangent at (2, 1) is,

$$y-1=\frac{8}{7}(x-2) \;\Rightarrow\; 7y-7=8x-16 \;\Rightarrow\; 8x-7y-9=0$$ **(Ans).**

4. The angles $\alpha$ and $\beta$ lie in the interval $0° < x < 180°$, and are such that

$$\tan\alpha = 2\tan\beta \quad \text{and} \quad \tan(\alpha+\beta)=3.$$

Find the possible values of $\alpha$ and $\beta$. [6]

***Suggested Solution:***

Let $\tan\alpha = 2\tan\beta$ ............(i) and $\tan(\alpha+\beta)=3$ ............(ii)

Consider equation (ii)

$$\tan(\alpha+\beta)=3 \;\Rightarrow\; \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\ \tan\beta}=3$$

Remember:

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\ \tan B}$$

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\ \tan B}$$

substituting the value of $\tan\alpha$ from equation (i),

$$\frac{2\tan\beta+\tan\beta}{1-2\tan^2\beta}=3$$

$$3\tan\beta = 3-6\tan^2\beta$$

$$6\tan^2\beta+3\tan\beta-3=0$$

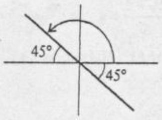
$$2\tan^2\beta+\tan\beta-1=0$$

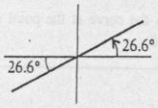
$$(\tan\beta+1)(2\tan\beta-1)=0$$

$\Rightarrow$ $\tan\beta+1=0$ or $2\tan\beta-1=0$

$\tan\beta=-1$ or $\tan\beta=0.5$

basic angle = 45° basic angle = 26.6°





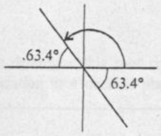
Note that:
$\tan\theta$ is positive in 1st and 3rd quadrant and negative in 2nd and 4th quadrant.

∴ $\beta = 180° - 45° = 135°$ ∴ $\beta = 26.6°$

now, when $\tan\beta=-1$,

$\tan\alpha = 2(-1)=-2$

basic angle = 63.4°

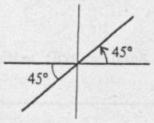


∴ $\alpha = 116.6°$

when $\tan\beta = 0.5$,

$\tan\alpha = 2(0.5)=1$

basic angle = 45°



∴ $\alpha = 45°$

$\Rightarrow$ $\alpha = 45°, 116.6°$ and $\beta = 26.6°, 135°$ **(Ans).**

5. The polynomial $2x^3 + ax^2 + bx - 4$, where $a$ and $b$ are constants, is denoted by $p(x)$. The result of differentiating $p(x)$ with respect to $x$ is denoted by $p'(x)$. It is given that $(x+2)$ is a factor of $p(x)$ and of $p'(x)$.

(i) Find the values of $a$ and $b$. [5]

(ii) When $a$ and $b$ have these values, factorise $p(x)$ completely. [3]

*Suggested Solution:*

(i) $p(x) = 2x^3 + ax^2 + bx - 4$

given that $(x+2)$ is a factor of $p(x)$

$\therefore \quad p(-2) = 0$

$\Rightarrow \quad 2(-2)^3 + a(-2)^2 + b(-2) - 4 = 0$

$-16 + 4a - 2b - 4 = 0$

$\Rightarrow \quad 4a - 2b = 20 \quad \Rightarrow \quad 2a - b = 10$ ............(i)

$$p'(x) = \frac{d}{dx}(2x^3 + ax^2 + bx - 4) = 6x^2 + 2ax + b$$

given that $(x+2)$ is a factor of $p'(x)$

$\therefore \quad p'(-2) = 0$

$\Rightarrow \quad 6(-2)^2 + 2a(-2) + b = 0$

$\Rightarrow \quad 24 - 4a + b = 0 \quad \Rightarrow \quad -4a + b = -24$ ............(ii)

solving equations (i) and (ii) simultaneously,

$$\begin{array}{r} 2a - b = \ \ 10 \\ \underline{-4a + b = -24} \\ -2a \quad = -14 \quad \Rightarrow \quad a = 7 \end{array}$$

putting the value of $a$ in eq. (i),

$2(7) - b = 10 \quad \Rightarrow \quad b = 4$

$\therefore \quad a = 7$ and $b = 4$ **(Ans).**

(ii) Substituting the values of $a$ and $b$, we have,

$p(x) = 2x^3 + 7x^2 + 4x - 4$

using synthetic division, with $(x+2)$ as factor,

$$\begin{array}{r|rrr|rl} & x^3 & x^2 & x^1 & x^0 & \\ -2 & 2 & 7 & 4 & -4 & \\ & & -4 & -6 & 4 & \text{adding} \\ \hline & 2 & 3 & -2 & 0 & \end{array}$$

$\Rightarrow$ quadratic factor of $p(x) = 2x^2 + 3x - 2$

$= 2x^2 + 4x - x - 2$

$= 2x(x+2) - 1(x+2)$

$= (x+2)(2x-1)$

$\therefore \quad p(x) = (x+2)(x+2)(2x-1) = (x+2)^2(2x-1)$ **(Ans).**

**6.** (i) Use the substitution $x = 2\tan\theta$ to show that

$$\int_0^2 \frac{8}{(4+x^2)^2}\,dx = \int_0^{\frac{1}{4}\pi} \cos^2\theta \ d\theta. \qquad [4]$$

(ii) Hence find the exact value of

$$\int_0^2 \frac{8}{(4+x^2)^2}\,dx. \qquad [4]$$

***Suggested Solution:***

(i) $\int_0^2 \frac{8}{(4+x^2)^2}\,dx$

given substitution is: $x = 2\tan\theta$

differentiating w.r.t. $\theta$

$$\Rightarrow \frac{d}{d\theta}(x) = \frac{d}{d\theta}(2\tan\theta) \Rightarrow dx = 2\sec^2\theta \ d\theta$$

$$\text{new limits: } \begin{cases} \text{when } x = 0, & 2\tan\theta = 0 \Rightarrow \tan\theta = 0 \Rightarrow \theta = 0 \\ \text{when } x = 2, & 2\tan\theta = 2 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4} \end{cases}$$

$\therefore$ the above integral is transformed to,

$$\int_0^{\frac{\pi}{4}} \frac{8}{\left(4+(2\tan\theta)^2\right)^2}(2\sec^2\theta)\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{16\sec^2\theta}{(4+4\tan^2\theta)^2}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{16\sec^2\theta}{\left(4(1+\tan^2\theta)\right)^2}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{16\sec^2\theta}{16(\sec^2\theta)^2}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2\theta \ d\theta \quad \textbf{(Shown).}$$

Remember:

For definite integral, do not put the constant of integration.

(ii) Using the result of part (i), we have,

$$\int_0^2 \frac{8}{(4+x^2)^2}\,dx = \int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left[2\theta + \sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left[\left(2(\frac{\pi}{4}) + \sin 2(\frac{\pi}{4})\right) - (2(0) + \sin 2(0))\right]$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - 0\right] = \frac{1}{4}\left(\frac{\pi}{2}+1\right) = \frac{1}{8}(\pi+2) \quad \textbf{(Ans).}$$

7. The complex numbers $-2+\mathrm{i}$ and $3+\mathrm{i}$ are denoted by $u$ and $v$ respectively.

(i) Find, in the form $x + \mathrm{i}y$, the complex numbers

(a) $u + v$, [1]

(b) $\frac{u}{v}$, showing all your working. [3]

(ii) State the argument of $\frac{u}{v}$. [1]

In an Argand diagram with origin $O$, the points $A$, $B$ and $C$ represent the complex numbers $u$, $v$ and $u + v$ respectively.

(iii) Prove that angle $AOB = \frac{3}{4}\pi$. [2]

(iv) State fully the geometrical relationship between the line segments $OA$ and $BC$. [2]

***Suggested Solution:***

(i) (a) $u+v=(-2+\mathrm{i})+(3+\mathrm{i})=1+2i$ **(Ans).**

(b) $\frac{u}{v}=\frac{-2+\mathrm{i}}{3+\mathrm{i}}$

realising the denominator,

$$\frac{u}{v}=\frac{-2+\mathrm{i}}{3+\mathrm{i}}\times\frac{3-\mathrm{i}}{3-\mathrm{i}}$$

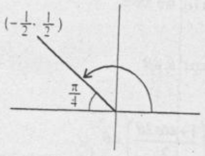
$$=\frac{-6+2\mathrm{i}+3\mathrm{i}-\mathrm{i}^2}{3^2-\mathrm{i}^2} \qquad (\because\ \mathrm{i}^2=-1)$$

$$=\frac{-6+5\mathrm{i}-(-1)}{9-(-1)}=\frac{-5+5\mathrm{i}}{10}=-\frac{1}{2}+\frac{1}{2}\mathrm{i} \quad \textbf{(Ans).}$$

**(ii)** $\arg\left(\frac{u}{v}\right) = \tan^{-1}\left(\frac{\frac{1}{2}}{-\frac{1}{2}}\right) = \tan^{-1}(-1)$

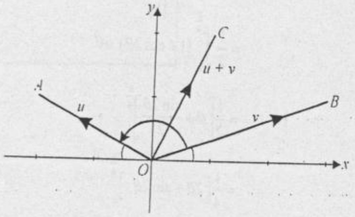
basic angle. $\alpha = \frac{\pi}{4}$ radians.

$\therefore$ required $\arg\left(\frac{u}{v}\right) = \pi - \frac{\pi}{4} = \frac{3}{4}\pi$ **(Ans).**



**(iii)** $A\hat{O}B = \arg(u) - \arg(v)$

$= \arg\left(\frac{u}{v}\right) = \frac{3}{4}\pi$ **(Proved).**



**(iv)** $OA = BC = u$

$\therefore$ $OA$ is parallel to $BC$. **(Ans).**

also. $|OA| = |u| = \sqrt{(-2)^2 + (1)^2} = \sqrt{5}$

$|BC| = \sqrt{(1-3)^2 + (2-1)^2} = \sqrt{4+1} = \sqrt{5}$

$\Rightarrow$ $|OA| = |BC|$ **(Ans).**

8. (i) Express $\frac{1+x}{(1-x)(2+x^2)}$ in partial fractions. [5]

(ii) Hence obtain the expansion of $\frac{1+x}{(1-x)(2+x^2)}$ in ascending powers of $x$, up to and including the term in $x^2$. [5]

***Suggested Solution:***

(i) $\frac{1+x}{(1-x)(2+x^2)} \equiv \frac{A}{1-x} + \frac{Bx+C}{2+x^2}$

$\Rightarrow$ $1 + x \equiv A(x^2+2) + (Bx+C)(1-x)$

for $x = 1$.

$2 = A(1+2) + (B(1)+C)(1-1) \Rightarrow 3A = 2 \Rightarrow A = \frac{2}{3}$

for $x = 0$.

$1 + 0 = A(0+2) + (B(0)+C)(1-0)$

$\Rightarrow 1 = 2A + C \Rightarrow 1 = 2(\frac{2}{3}) + C \Rightarrow C = 1 - \frac{4}{3} = -\frac{1}{3}$

for $x = -1$.

$1 + (-1) = A((-1)^2+2) + (B(-1)+C)(1-(-1))$

$\Rightarrow 0 = 3A - 2B + 2C \Rightarrow 3(\frac{2}{3}) - 2B + 2(-\frac{1}{3}) = 0 \Rightarrow 2B = \frac{4}{3} \Rightarrow B = \frac{2}{3}$

$$\therefore \quad \frac{1+x}{(1-x)(2+x^2)} = \frac{2}{3(1-x)} + \frac{\frac{2}{3}x - \frac{1}{3}}{2+x^2}$$

$$= \frac{2}{3(1-x)} + \frac{2x-1}{3(2+x^2)} \quad \textbf{(Ans).}$$

(ii) $$\frac{1+x}{(1-x)(2+x^2)}=\frac{2}{3(1-x)}+\frac{2x-1}{3(2+x^2)}$$

$$=\frac{2}{3(1-x)}+\frac{2x-1}{6\left(1+\frac{x^2}{2}\right)}=\frac{2}{3}(1-x)^{-1}+\frac{1}{6}(2x-1)\left(1+\frac{x^2}{2}\right)^{-1}$$

applying binomial theorem up to the term in $x^2$

$$=\frac{2}{3}\left[1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2\right]+\frac{1}{6}(2x-1)\left[1+(-1)(\frac{x^2}{2})\right]$$

$$=\frac{2}{3}(1+x+x^2)+\frac{1}{6}\left[(2x-1)\left(1-\frac{x^2}{2}\right)\right]$$

$$=\frac{2}{3}+\frac{2}{3}x+\frac{2}{3}x^2+\frac{1}{6}\left(2x-1+\frac{x^2}{2}\right)$$

$$=\frac{2}{3}+\frac{2}{3}x+\frac{2}{3}x^2+\frac{1}{3}x-\frac{1}{6}+\frac{1}{12}x^2$$

$$=\frac{1}{2}+x+\frac{3}{4}x^2 \quad \textbf{(Ans).}$$

Binomial Expansion:

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+.......+x^n$$

9. The temperature of a quantity of liquid at time $t$ is $\theta$. The liquid is cooling in an atmosphere whose temperature is constant and equal to $A$. The rate of decrease of $\theta$ is proportional to the temperature difference $(\theta - A)$. Thus $\theta$ and $t$ satisfy the differential equation

$$\frac{d\theta}{dt}=-k(\theta-A),$$

where $k$ is a positive constant.

(i) Find, in any form, the solution of this differential equation, given that $\theta = 4A$ when $t = 0$. [5]

(ii) Given also that $\theta = 3A$ when $t = 1$, show that $k=\ln\frac{3}{2}$. [1]

(iii) Find $\theta$ in terms of $A$ when $t = 2$, expressing your answer in its simplest form. [3]

***Suggested Solution:***

(i) $$\frac{d\theta}{dt}=-k(\theta-A) \Rightarrow \frac{1}{\theta-A}\,d\theta=-k\,dt$$

integrating both sides

$$\int\frac{1}{\theta-A}\,d\theta=-k\int dt \Rightarrow \ln|\theta-A|=-kt+C\cdots\cdots\cdots(i)$$

given that, $\theta = 4A$ when $t = 0$.

$\Rightarrow \ln|4A-A|=-k(0)+C \Rightarrow C=\ln 3A$

$\therefore$ equation (i) becomes,

$\ln|\theta-A|=-kt+\ln 3A$ **(Ans).**

(ii) From part (i): $\ln|\theta - A| = -kt + \ln 3A$ .

given that $\theta = 3A$ when $t = 1$

$\Rightarrow \ln|3A - A| = -k(1) + \ln 3A$

$\ln|2A| = -k + \ln 3A$

$k = \ln 3A - \ln 2A \Rightarrow k = \ln\dfrac{3A}{2A} \Rightarrow k = \ln\dfrac{3}{2}$ **(Shown).**

(iii) We have. $\ln|\theta - A| = -kt + \ln 3A$

putting $k = \ln\frac{3}{2}$. from part (ii).

$\ln|\theta - A| = -t\ \ln\dfrac{3}{2} + \ln 3A$

$t\ln\dfrac{3}{2} = \ln 3A - \ln|\theta - A|$

$\ln\left(\dfrac{3}{2}\right)^t = \ln\left|\dfrac{3A}{\theta - A}\right| \Rightarrow \left(\dfrac{3}{2}\right)^t = \dfrac{3A}{\theta - A}$

when $t = 2$

$\left(\dfrac{3}{2}\right)^2 = \dfrac{3A}{\theta - A}$

$\dfrac{9}{4} = \dfrac{3A}{\theta - A} \Rightarrow 9\theta - 9A = 12A \Rightarrow 9\theta = 21A \Rightarrow \theta = \dfrac{7}{3}A$ **(Ans).**

Note that:

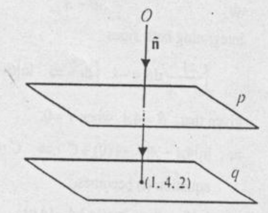
- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
- $\ln(a \times b) = \ln a + \ln b$
- $\ln(a^b) = b\ln a$

10. The plane $p$ has equation $2x - 3y + 6z = 16$. The plane $q$ is parallel to $p$ and contains the point with position vector $\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.
    (i) Find the equation of $q$, giving your answer in the form $ax + by + cz = d$. [2]
    (ii) Calculate the perpendicular distance between $p$ and $q$. [3]
    (iii) The line $l$ is parallel to the plane $p$ and also parallel to the plane with equation $x - 2y + 2z = 5$.
    Given that $l$ passes through the origin, find a vector equation for $l$. [5]

***Suggested Solution:***

(i) Equation of plane $p$: $2x - 3y + 6z = 16 \Rightarrow \mathbf{r}\cdot\begin{pmatrix}2\\-3\\6\end{pmatrix} = 16$

plane $q$ is parallel to plane $p$. therefore they have same normal vectors.



$\Rightarrow$ equation of plane $q$: $\mathbf{r}\cdot\begin{pmatrix}2\\-3\\6\end{pmatrix} = D$

also. plane $q$ passes through the point $\begin{pmatrix}1\\4\\2\end{pmatrix}$

$\Rightarrow \begin{pmatrix}1\\4\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\6\end{pmatrix} = D \Rightarrow D = 2 - 12 + 12 \Rightarrow D = 2$

$\therefore$ equation of plane $q$ is: $\mathbf{r}\cdot\begin{pmatrix}2\\-3\\6\end{pmatrix} = 2$ or $2x - 3y + 6z = 2$ **(Ans).**

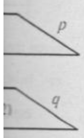
ning CORNER

e that:

$\left(\frac{a}{b}\right) = \ln a - \ln b$

$(a \times b) = \ln a + \ln b$

$(a^b) = b \ln a$



**(ii)** Perpendicular distance between two planes = perpendicular distance of a fixed point of a plane from the other plane.

Equation of plane $p$: $2x-3y+6z-16=0$

Plane $q$ contains the point (1, 4. 2))

$\therefore$ distance $D$ of point (1. 4, 2) from the plane $p$ is,

$$D=\left|\frac{2(1)-3(4)+6(2)-16}{\sqrt{2^2+(-3)^2+(6)^2}}\right|$$

$$=\left|\frac{2-12+12-16}{\sqrt{4+9+36}}\right|=\left|\frac{-14}{\sqrt{49}}\right|=\frac{14}{7}=2 \text{ units} \quad \textbf{(Ans).}$$

***Alternative Solution:***

We can find the distance of both planes from the origin by transforming their normal vectors into unit vectors.

Distance of plane $p$ from origin is : $\dfrac{16}{\sqrt{(2)^2+(-3)^2+(6)^2}}=\dfrac{16}{\sqrt{49}}=\dfrac{16}{7}$

Distance of plane $q$ from origin is : $\dfrac{2}{\sqrt{(2)^2+(-3)^2+(6)^2}}=\dfrac{2}{\sqrt{49}}=\dfrac{2}{7}$

$\therefore$ distance between the two parallel planes $=\dfrac{16}{7}-\dfrac{2}{7}$

$$=\frac{14}{7}=2 \text{ units} \quad \textbf{(Ans).}$$

**(iii)** Line $l$ is parallel to plane $p$ and the plane $x-2y+2z=5$. Therefore direction vector of the line $l$ will be perpendicular to the normal vectors of both planes.

Let the normal vectors of the two planes be $\mathbf{n}_1$ and $\mathbf{n}_2$.

$\therefore$ the direction vector $\vec{d}$ of line $l$ is,

$$\vec{d}=\mathbf{n}_1\times\mathbf{n}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -3 & 6\\ 1 & -2 & 2\end{vmatrix}=\mathbf{i}(-6+12)-\mathbf{j}(4-6)+\mathbf{k}(-4+3)=6\mathbf{i}+2\mathbf{j}-\mathbf{k}$$

given that the line passes through the origin.

$\therefore$ equation of line is.

$$\mathbf{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}6\\2\\-1\end{pmatrix}$$

$$\Rightarrow \quad \mathbf{r}=+\lambda\begin{pmatrix}6\\2\\-1\end{pmatrix} \quad \text{or} \quad \mathbf{r}=\lambda(6\mathbf{i}+2\mathbf{j}-\mathbf{k}) \quad \textbf{(Ans).}$$

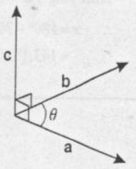
Learning CORNER

Perpendicular distance $D$ from a point $P(x_1, y_1, z_1)$ to the plane $Ax+By+Cz+D=0$, is

$$D=\left|\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right|$$

**Cross Product**

The Cross product ($\mathbf{a}\times\mathbf{b}$) helps us to calculate a new vector **c** that is perpendicular to two existing non-parallel vectors **a** and **b**.



If

$\mathbf{a}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, and

$\mathbf{b}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then

vector **c** is:

$$\mathbf{c}=\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a & b & c\\ x & y & z\end{vmatrix}$$

$$=\mathbf{i}(bz-yc)-\mathbf{j}(az-cx)+\mathbf{k}(ay-bx)$$

# June 2010 Paper 1

## Pure Mathematics (P1)

Answer **all** the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

**1.** (i) Show that the equation

$$3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$$

can be written in the form $\tan x = -\frac{3}{4}$. [2]

(ii) Solve the equation $3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$, for $0° \le x \le 360°$. [2]

***Suggested Solution:***

(i) $3(2\sin x - \cos x) = 2(\sin x - 3\cos x)$

$6\sin x - 3\cos x = 2\sin x - 6\cos x$

$4\sin x = -3\cos x$

$\frac{\sin x}{\cos x} = -\frac{3}{4}$

$\tan x = -\frac{3}{4}$ **(Shown).**

(ii) From part (i), $\tan x = -\frac{3}{4}$

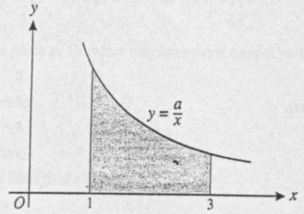
basic Angle $\alpha = 36.9°$

$\therefore \quad x = 180° - 36.9° \ . \ 360° - 36.9°$

$= 143.1°, \ 323.1°$ **(Ans).**

Note that:
$\tan\theta$ is negative in II and IV quadrants.

**2.**



The diagram shows part of the curve $y = \frac{a}{x}$, where $a$ is a positive constant. Given that the volume obtained when the shaded region is rotated through 360° about the $x$-axis is $24\pi$, find the value of $a$. [4]

at
negative in II and
trants.

*Suggested Solution:*

Volume $= \pi \int_1^3 y^2 \, dx = \pi \int_1^3 \left(\frac{a}{x}\right)^2 dx = \pi a^2 \int_1^3 x^{-2} \, dx$

given volume $= 24\pi$

$\Rightarrow \quad 24\pi = \pi a^2 \int_1^3 x^{-2} \, dx$

$\Rightarrow \quad 24 = a^2 \int_1^3 x^{-2} \, dx$

$\Rightarrow \quad 24 = a^2 \left[-\frac{1}{x}\right]_1^3$

$\Rightarrow \quad 24 = a^2 \left[\left(-\frac{1}{3}\right) - \left(-\frac{1}{1}\right)\right]$

$\Rightarrow \quad 24 = a^2 \left(-\frac{1}{3} + 1\right) \Rightarrow 24 = a^2 (\frac{2}{3}) \Rightarrow a^2 = 36 \Rightarrow a = \pm 6$

given that $a$ is positive. $\therefore \quad a = 6$ **(Ans).**

3. The functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto 4x - 2x^2,$$
$$g : x \mapsto 5x + 3.$$

(i) Find the range of f. [2]

(ii) Find the value of the constant $k$ for which the equation $gf(x) = k$ has equal roots. [3]

*Suggested Solution:*

(i) $f(x) = 4x - 2x^2$, for $x \in \mathbb{R}$

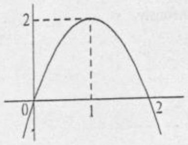
$\Rightarrow \quad f(x) = 2x(2 - x)$

Critical values are $x = 0$ and $x = 2$ and the parabolic curve open downwards.

$\therefore$ maximum value of $f(x)$ is at $x = 1$, which is

$f(1) = 4(1) - 2(1)^2 \Rightarrow f(1) = 2$

$\therefore$ range of $f(x)$ is: $f(x) \leq 2$ **(Ans).**



*Alternative Solution:*

Using completing the square method,

$f(x) = 4x - 2x^2$

$= -2(x^2 - 2x) = -2(x^2 - 2x + 1 - 1) = -2\left((x-1)^2 - 1\right) = -2(x-1)^2 + 2$

$\Rightarrow$ max. point has coordinates $(1, 2)$

$\therefore$ range of $f(x)$ is: $f(x) \leq 2$ **(Ans).**

(ii) $gf(x) = g(4x - 2x^2)$

$= 5(4x - 2x^2) + 3 = 20x - 10x^2 + 3$

given that $gf(x) = k$, has equal roots,

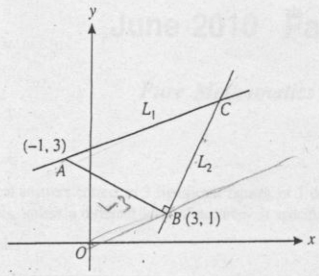
$\Rightarrow \quad 20x - 10x^2 + 3 = k \Rightarrow 10x^2 - 20x + (k - 3) = 0$

Discriminant, $b^2 - 4ac = 0$

$\Rightarrow \quad (-20)^2 - 4(10)(k - 3) = 0$

$\Rightarrow \quad 400 - 40(k - 3) = 0 \Rightarrow 40(k - 3) = 400 \Rightarrow k - 3 = 10 \Rightarrow k = 13$ **(Ans).**

4.



In the diagram, $A$ is the point $(-1, 3)$ and $B$ is the point $(3, 1)$. The line $L_1$ passes through $A$ and is parallel to $OB$. The line $L_2$ passes through $B$ and is perpendicular to $AB$. The lines $L_1$ and $L_2$ meet at $C$. Find the coordinates of $C$. [6]

***Suggested Solution:***

Gradient of $L_1$ = Gradient of $OB = \frac{1}{3}$

equation of $L_1$ passing through $(-1, 3)$ is.

$$y-3=\frac{1}{3}(x+1) \;\Rightarrow\; 3y-9=x+1 \;\Rightarrow\; 3y=x+10 \text{ ............(i)}$$

Gradient of $AB=\frac{1-3}{3+1}=-\frac{2}{4}=-\frac{1}{2}$

$AB \perp BC$

$\Rightarrow$ gradient of $BC = 2$

equation of $L_2$ passing thorugh (3, 1) with gradient 2 is:

$$y-1=2(x-3) \;\Rightarrow\; y-1=2x-6 \;\Rightarrow\; y=2x-5 \text{ ............(ii)}$$

Solving (i) and (ii) simultaneously,

$$3(2x-5)=x+10$$
$$6x-15=x+10$$
$$5x=25$$
$$x=5$$

substitute $x = 5$ in eq. (ii). $y=2(5)-5 \;\Rightarrow\; y=5$

$\therefore$ coordinates of $C$ are (5, 5) **(Ans).**

When two lines are perpendicular to each other then the product of their gradients is equal to −1

i.e. $m_1 \times m_2 = -1$

Equation of a line in point-slope form is:
$y - y_1 = m(x - x_1)$

5. Relative to an origin $O$, the position vectors of the points $A$ and $B$ are given by

$$\vec{OA}=\begin{pmatrix}-2\\3\\1\end{pmatrix} \text{ and } \vec{OB}=\begin{pmatrix}4\\1\\p\end{pmatrix}.$$

(i) Find the value of $p$ for which $\vec{OA}$ is perpendicular to $\vec{OB}$. [2]

(ii) Find the values of $p$ for which the magnitude of $\vec{AB}$ is 7. [4]

two lines are per-
cular to each other
the product of their
ents is equal to −1
$m_1 \times m_2 = -1$

tion of a line in point-
form is:
$y_1 = m(x - x_1)$

---

*Suggested Solution:*

(i) If $\vec{OA} \perp \vec{OB}$ then,

$$\vec{OA} \cdot \vec{OB} = 0$$

$$\Rightarrow \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \\ p \end{pmatrix} = 0 \Rightarrow -8+3+p=0 \Rightarrow p=5 \quad \textbf{(Ans).}$$

If a ⊥ b then a·b = 0

(ii) $\vec{AB} = \vec{OB} - \vec{OA}$

$$= \begin{pmatrix} 4 \\ 1 \\ p \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4+2 \\ 1-3 \\ p-1 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \\ p-1 \end{pmatrix}$$

given that, $\left|\vec{AB}\right| = 7$

$$\Rightarrow \sqrt{(6)^2 + (-2)^2 + (p-1)^2} = 7$$

$$\Rightarrow \sqrt{40 + (p-1)^2} = 7 \Rightarrow 40 + (p-1)^2 = 49 \Rightarrow (p-1)^2 = 9 \Rightarrow p-1 = \pm 3$$

either $p-1=+3$, or $p-1=-3$

$\Rightarrow \quad p=4$. or $\quad p=-2$ **(Ans).**

If
$\mathbf{a} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$
$\mathbf{b} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then
$\mathbf{a} \cdot \mathbf{b} = ax + by + cz$
and magnitude of a is
$|\mathbf{a}| = \sqrt{(a)^2 + (b)^2 + (c)^2}$

6. (i) Find the first 3 terms in the expansion of $(1+ax)^5$ in ascending powers of $x$. [2]

(ii) Given that there is no term in $x$ in the expansion of $(1-2x)(1+ax)^5$, find the value of the constant $a$. [2]

(iii) For this value of $a$, find the coefficient of $x^2$ in the expansion of $(1-2x)(1+ax)^5$. [3]

*Suggested Solution:*

(i) $(1+ax)^5 = {}^5C_0\,(ax)^0 + {}^5C_1\,(ax)^1 + {}^5C_2\,(ax)^2 + \cdots\cdots$

$$= 1 + 5ax + 10a^2x^2 \quad \textbf{(Ans).}$$

Binomial theorem:
$(a+b)^n = {}^nC_0\,a^n + {}^nC_1\,a^{n-1}b +$
${}^nC_2\,a^{n-2}b^2 + \ldots. + {}^nC_r\,a^{n-r}b^r +$
$\ldots. + {}^nC_n\,b^n$

(ii) $(1-2x)(1+ax)^5$

$= (1-2x)(1+5ax+10a^2x^2+\cdots)$

collecting the terms containing $x$ only,

$$5ax - 2x = (5a-2)x$$

as there is no term in $x$. $\Rightarrow 5a-2=0 \Rightarrow a = \frac{2}{5}$ **(Ans).**

(iii) $(1-2x)(1+ax)^5 = (1-2x)(1+5ax+10a^2x^2+\cdots)$

Collecting the terms containing $x^2$ only, we have,

$$10a^2x^2 - 10ax^2 = x^2(10a^2 - 10a)$$

$\therefore$ coefficient of $x^2 = 10a^2 - 10a$

substituting $a = \frac{2}{5}$, from part (ii),

coefficient of $x^2 = 10\left(\frac{2}{5}\right)^2 - 10\left(\frac{2}{5}\right) = 10\left(\frac{4}{25}\right) - 4 = \frac{8}{5} - 4 = -\frac{12}{5} = -2.4$ **(Ans).**

7. (a) Find the sum of all the multiples of 5 between 100 and 300 inclusive. [3]

(b) A geometric progression has a common ratio of $-\frac{2}{3}$ and the sum of the first 3 terms is 35. Find

(i) the first term of the progression, [3]

(ii) the sum to infinity. [2]

***Suggested Solution:***

**(a)** Multiples of 5 between 100 and 300 inclusive are.

$$100+105+110+115+120..........+300$$

$\therefore\ a=100.\ \ d=5.\ \ T_n=300$

using. $T_n=a+(n-1)d$.

$$300=100+(n-1)5 \Rightarrow 200=5(n-1) \Rightarrow n-1=40 \Rightarrow n=41$$

using. $S_n=\frac{n}{2}(a+l)$.

$$S_{41}=\frac{41}{2}(100+300) = \frac{41}{2}(400) = 8200 \quad \textbf{(Ans).}$$

**(b)** (i) Given that. $r=-\frac{2}{3}$. $S_3=35$. $n=3$

using. $S_n=\frac{a(1-r^n)}{1-r}$.

$$35=\frac{a\left(1-\left(-\frac{2}{3}\right)^3\right)}{1-\left(-\frac{2}{3}\right)} \Rightarrow 35=\frac{a\left(1-\left(-\frac{8}{27}\right)\right)}{1+\frac{2}{3}}$$

$$\Rightarrow 35=\frac{a\left(1+\frac{8}{27}\right)}{\frac{5}{3}} \Rightarrow 35\left(\frac{5}{3}\right)=a\left(\frac{35}{27}\right) \Rightarrow a=45 \quad \textbf{(Ans).}$$

(ii) $S_x=\frac{a}{1-r}$

$$=\frac{45}{1-\left(-\frac{2}{3}\right)}=\frac{45}{1+\frac{2}{3}}=\frac{45}{\frac{5}{3}} = 45\times\frac{3}{5} = 27 \quad \textbf{(Ans).}$$

8. A solid rectangular block has a square base of side $x$ cm. The height of the block is $h$ cm and the total surface area of the block is 96 cm².

(i) Express $h$ in terms of $x$ and show that the volume, $V$ cm³, of the block is given by

$$V=24x-\frac{1}{2}x^3. \quad [3]$$

Given that $x$ can vary,

(ii) find the stationary value of $V$, [3]

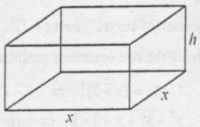
(iii) determine whether this stationary value is a maximum or a minimum. [2]

*Suggested Solution:*

(i) Total surface area $= 2(x \times x) + 2(x \times h) + 2(x \times h)$

$\Rightarrow \quad 2(x \times x) + 2(x \times h) + 2(x \times h) = 96$

$\Rightarrow \quad 2x^2 + 4xh = 96 \;\Rightarrow\; h = \dfrac{96 - 2x^2}{4x} \;\Rightarrow\; h = \dfrac{48 - x^2}{2x}$



Volume. $V$ = base area × $h$

$$= x^2\left(\frac{48 - x^2}{2x}\right) = \frac{48x^2 - x^4}{2x} = \frac{48x^2}{2x} - \frac{x^4}{2x} = 24x - \frac{1}{2}x^3 \quad \textbf{(Shown).}$$

(ii) $V = 24x - \dfrac{1}{2}x^3$

Differentiating w.r.t. $x$

$$\frac{dV}{dx} = 24 - \frac{1}{2}(3x^2)$$

for stationary values. $\dfrac{dV}{dx} = 0$

$\Rightarrow \quad 24 - \dfrac{3}{2}x^2 = 0 \;\Rightarrow\; \dfrac{3}{2}x^2 = 24 \;\Rightarrow\; x^2 = \dfrac{48}{3} \;\Rightarrow\; x^2 = 16 \;\Rightarrow\; x = 4$

$\therefore$ stationary value of $V = 24(4) - \dfrac{1}{2}(4)^3 = 96 - 32 = 64$ **(Ans).**

(iii) $\dfrac{dV}{dx} = 24 - \dfrac{3}{2}x^2$

Differentiating w.r.t. $x$

$$\frac{d^2V}{dx^2} = 0 - \frac{3}{2}(2x) = -3x$$

at $x = 4$, $\dfrac{d^2V}{dx^2} = -3(4) = -12 < 0$

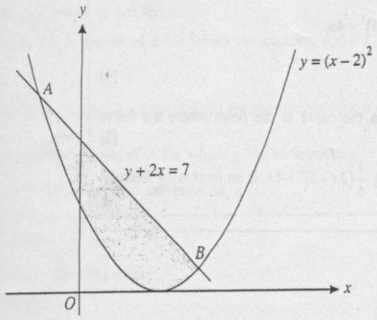
$\therefore$ the stationary value is a maximum. **(Ans).**

Remember:

if $\left.\dfrac{d^2y}{dx^2}\right|_{x = x_1} > 0$, $y$ is min.

if $\left.\dfrac{d^2y}{dx^2}\right|_{x = x_1} < 0$, $y$ is max.

9.



The diagram shows the curve $y = (x-2)^2$ and the line $y + 2x = 7$, which intersect at points $A$ and $B$. Find the area of the shaded region. [8]

***Suggested Solution:***

Equation of curve: $y=(x-2)^2$. equation of line: $y+2x=7 \Rightarrow y=7-2x$

solving the two equations simultaneously.

$$7-2x=(x-2)^2 \Rightarrow 7-2x=x^2-4x+4 \Rightarrow x^2-2x-3=0$$

$$\Rightarrow x^2-3x+x-3=0 \Rightarrow x(x-3)+1(x-3)=0 \Rightarrow (x-3)(x+1)=0$$

$$\Rightarrow x=-1,\ 3$$

$\therefore$ lower limit $=-1$, upper limit $=3$

Area of shaded region. $A=\int_{-1}^{3}\left((7-2x)-(x-2)^2\right)\,dx$

$$=\int_{-1}^{3}\left((7-2x)-(x^2-4x+4)\right)\,dx$$

$$=\int_{-1}^{3}(7-2x-x^2+4x-4)\,dx$$

$$=\int_{-1}^{3}(3+2x-x^2)\,dx$$

$$=\left(3x+\frac{2x^2}{2}-\frac{x^3}{3}\right)_{-1}^{3}$$

$$=\left(3x+x^2-\frac{x^3}{3}\right)_{-1}^{3}$$

$$=\left(3(3)+(3)^2-\frac{(3)^3}{3}\right)-\left(3(-1)+(-1)^2-\frac{(-1)^3}{3}\right)$$

$$=(9+9-9)-\left(-3+1+\frac{1}{3}\right)$$

$$=9-\left(-\frac{5}{3}\right)$$

$$=\frac{32}{3}=10\tfrac{2}{3}\text{ unit}^2 \quad \textbf{(Ans).}$$

**10.** The equation of a curve is $y=\frac{1}{6}(2x-3)^3-4x$.

(i) Find $\frac{dy}{dx}$. [3]

(ii) Find the equation of the tangent to the curve at the point where the curve intersects the $y$-axis. [3]

(iii) Find the set of values of $x$ for which $\frac{1}{6}(2x-3)^3-4x$ is an increasing function of $x$. [3]

***Suggested Solution:***

(i) $y=\frac{1}{6}(2x-3)^3-4x$

$$\frac{dy}{dx}=\frac{1}{6}(3)(2x-3)^2\times\frac{d}{dx}(2x-3)-4=\frac{1}{2}(2x-3)^2\times 2-4=(2x-3)^2-4 \quad \textbf{(Ans).}$$

(ii) $y = \frac{1}{6}(2x-3)^3 - 4x$

The curve intersects the $y$-axis at $x = 0$.

$\Rightarrow \quad y = \frac{1}{6}(2(0)-3)^3 - 4(0) = \frac{1}{6}(-3)^3 = -\frac{9}{2}$

$\therefore$ the curve meets the $y$-axis at point $(0.\ -\frac{9}{2})$

from part (i). $\frac{dy}{dx} = (2x-3)^2 - 4$

gradient of the tangent at $x = 0$ is,

$\frac{dy}{dx} = (2(0)-3)^2 - 4 = (-3)^2 - 4 = 5$

$\therefore$ equation of the tangent is.

$y - (-\frac{9}{2}) = 5(x-0) \Rightarrow y + \frac{9}{2} = 5x \Rightarrow 2y - 10x + 9 = 0$ **(Ans).**

(iii) $\frac{dy}{dx} = (2x-3)^2 - 4$

For increasing function. $\frac{dy}{dx} > 0$

$\Rightarrow (2x-3)^2 - 4 > 0$

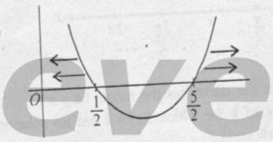
$\Rightarrow (2x-3)^2 - (2)^2 > 0$

$\Rightarrow ((2x-3)+2)((2x-3)-2) > 0$

$\Rightarrow (2x-1)(2x-5) > 0$

critical values are $x = \frac{1}{2}.\ x = 2\frac{1}{2}$

$\therefore x < \frac{1}{2}.\ x > 2\frac{1}{2}$ **(Ans).**



Note that:

- f($x$) is increasing if f′($x$) > 0,

where $f'(x) = \frac{dy}{dx}$

11. The function $f : x \mapsto 4 - 3\sin x$ is defined for the domain $0 \le x \le 2\pi$.

(i) Solve the equation $f(x) = 2$. [3]

(ii) Sketch the graph of $y = f(x)$. [2]

(iii) Find the set of values of $k$ for which the equation $f(x) = k$ has no solution. [2]

The function $g : x \mapsto 4 - 3\sin x$ is defined for the domain $\frac{1}{2}\pi \le x \le A$.

(iv) State the largest value of $A$ for which g has an inverse. [1]

(v) For this value of $A$, find the value of $g^{-1}(3)$. [2]

***Suggested Solution:***

(i) $f(x) = 4 - 3\sin x$ for $0 \le x \le 2\pi$

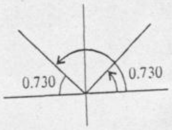
given that $f(x) = 2$

$\Rightarrow 4 - 3\sin x = 2$

$\Rightarrow 3\sin x = 2 \Rightarrow \sin x = \frac{2}{3}$

basic angle $\alpha = 0.730$ radian

$\therefore x = 0.730$ radian. 2.41 radians **(Ans).**

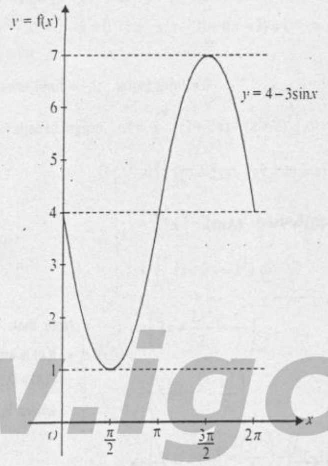


Note:

Change your calculator to radian mode.

(ii)

| $x$ | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| $y = f(x)$ | 4 | 1 | 4 | 7 | 4 |



(iii) From graph, we see that the curve has the range $1 \le y \le 7$.
Hence, the set of values of $k$ for which $f(x) = k$ has no solution are:
$k < 1$ or $k > 7$. **(Ans).**

(iv) $g(x) = 4 - 3\sin x$. for $\frac{1}{2}\pi \le x \le A$

From graph, the largest value of $A$ in the given domain for which g has an inverse is $= \frac{3\pi}{2}$ **(Ans).**

(v) $g(x) = 4 - 3\sin x$

Let $g^{-1}(3) = x$

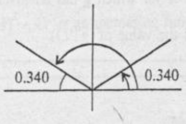
$\Rightarrow \quad g(x) = 3 \Rightarrow 4 - 3\sin x = 3 \Rightarrow 3\sin x = 1 \Rightarrow \sin x = \frac{1}{3}$

basic angle $\alpha = 0.340$ radians.

given range is $\frac{1}{2}\pi \le x \le \frac{3}{2}\pi$.

$\Rightarrow \quad x = \pi - 0.340 = 2.80$ radians.

$\therefore \quad g^{-1}(3) = 2.80$ **(Ans).**



# June 2010 Paper 3

## *Pure Mathematics (P3)*

Answer all the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. Solve the equation

$$\frac{2^x+1}{2^x-1}=5,$$

giving your answer correct to 3 significant figures. [4]

*Suggested Solution:*

$$\frac{2^x+1}{2^x-1}=5$$

Let $2^x = y$

$$\Rightarrow \quad \frac{y+1}{y-1}=5$$

$$y+1=5(y-1)$$

$$4y=6$$

$$y=1.5$$

$$\Rightarrow \quad 2^x=1.5$$

$$\log(2^x)=\log(1.5)$$

$$x\log(2)=\log(1.5)$$

$$x=\frac{\log 1.5}{\log 2}=0.585 \quad \textbf{(Ans).}$$

Note that:

- $\log\left(\frac{a}{b}\right)=\log a-\log b$
- $\log(a\times b)=\log a+\log b$
- $\log(a^b)=b\log a$

2. Show that $\int_0^{\pi} x^2 \sin x \, dx = \pi^2 - 4.$ [5]

*Suggested Solution:*

$$\int_0^{\pi} x^2 \sin x \, dx$$

using integration by parts,

$$=\left[x^2\int \sin x-\int\left(\int \sin x \, dx\right)\times\frac{d}{dx}(x^2)\, dx\right]_0^{\pi}$$

$$=\left[x^2(-\cos x)-\int(-\cos x)(2x)\, dx\right]_0^{\pi}$$

$$=\left[-x^2\cos x+2\int x\cos x \, dx\right]_0^{\pi}$$

Formula for integration by parts:

$$\int uv \, dx$$

$$=u\int v\, dx-\int\left(\frac{d}{dx}u\right)\left(\int v\, dx\right)dx$$

again using integration by parts.

$$= \left[-x^2 \cos x + 2\left(x(\sin x) - \int \sin x(1)\, dx\right)\right]_0^{\pi}$$

$$= \left[-x^2 \cos x + 2(x \sin x + \cos x)\right]_0^{\pi}$$

$$= \left[-x^2 \cos x + 2x \sin x + 2\cos x)\right]_0^{\pi}$$

$$= \left[2x \sin x + (2 - x^2)\cos x\right]_0^{\pi}$$

$$= \left(2\pi \sin \pi + (2 - \pi^2)\cos \pi\right) - \left(2(0)\sin(0) + (2 - 0^2)\cos(0)\right)$$

$$= \left(2\pi(0) + (2 - \pi^2)(-1)\right) - \left(0 + 2(1)\right)$$

$$= (-2 + \pi^2) - 2$$

$= \pi^2 - 4$ **(Shown).**

3. It is given that $\cos a = \frac{3}{5}$. where $0° < a < 90°$. Showing your working and without using a calculator to evaluate $a$,

   (i) find the exact value of $\sin(a - 30°)$. [3]

   (ii) find the exact value of $\tan 2a$, and hence find the exact value of $\tan 3a$. [4]

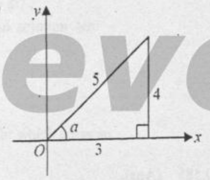
***Suggested Solution:***

Given that $\cos a = \frac{3}{5}$, $0° < a < 90°$.

(i) $\sin(a - 30°) = \sin a \cos 30° - \cos a \sin 30°$

$$= \left(\frac{4}{5} \times \frac{\sqrt{3}}{2}\right) - \left(\frac{3}{5} \times \frac{1}{2}\right)$$

$= \frac{4\sqrt{3}}{10} - \frac{3}{10} = \frac{4\sqrt{3} - 3}{10}$ **(Ans).**



(ii) From figure, $\tan a = \frac{4}{3}$

using $\tan 2a = \frac{2\tan a}{1 - \tan^2 a}$.

$$\tan 2a = \frac{2\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2} = \frac{\frac{8}{3}}{1 - \frac{16}{9}} = \frac{\frac{8}{3}}{-\frac{7}{9}} = \frac{8}{3} \times -\frac{9}{7} = -\frac{24}{7}$$

$\therefore$ $\tan 2a = -\frac{24}{7}$ **(Ans).**

Now, $\tan 3a = \tan(2a + a)$

$$= \frac{\tan 2a + \tan a}{1 - \tan 2a \tan a}$$

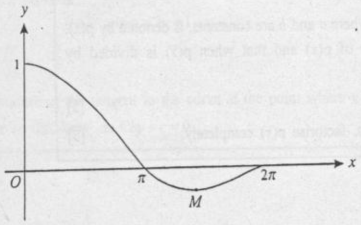
$$= \frac{-\frac{24}{7} + \frac{4}{3}}{1 - \left(-\frac{24}{7}\right)\left(\frac{4}{3}\right)}$$

$= \frac{\frac{-72+28}{21}}{1 + \frac{32}{7}} = \frac{-\frac{44}{21}}{\frac{39}{7}} = -\frac{44}{21} \times \frac{7}{39} = -\frac{44}{117}$ **(Ans).**

Remember:

- $\sin(A \pm B)$
  $= \sin A \cos B \pm \cos A \sin B$
- $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
- $\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$

4.



The diagram shows the curve $y=\frac{\sin x}{x}$ for $0 \le x \le 2\pi$, and its minimum point $M$.

(i) Show that the $x$-coordinate of $M$ satisfies the equation

$$x = \tan x.$$ [4]

(ii) The iterative formula

$$x_{n+1} = \tan^{-1}(x_n) + \pi$$

can be used to determine the $x$-coordinate of $M$. Use this formula to determine the $x$-coordinate of $M$ correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

***Suggested Solution:***

(i) $y=\frac{\sin x}{x}, \quad 0 \le x \le 2\pi$

differentiating w.r.t. $x$

$$\frac{d}{dx}y = \frac{d}{dx}\left(\frac{\sin x}{x}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{x\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(x)}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

Remember:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}u - u\frac{d}{dx}v}{v^2}$$

for maxima and minima, $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{x\cos x - \sin x}{x^2} = 0$$

$$x\cos x - \sin x = 0$$

$$x\cos x = \sin x$$

$$x = \tan x \quad \textbf{(Shown).}$$

(ii) $x_{n+1} = \tan^{-1}(x_n) + \pi$

taking $x_1 = \pi$

$$x_2 = \tan^{-1}(x_1) + \pi = \tan^{-1}(\pi) + \pi = 4.4042$$

$$x_3 = \tan^{-1}(x_2) + \pi = \tan^{-1}(4.4042) + \pi = 4.4891$$

$$x_4 = \tan^{-1}(x_3) + \pi = \tan^{-1}(4.4891) + \pi = 4.4932$$

$$x_5 = \tan^{-1}(x_4) + \pi = \tan^{-1}(4.4932) + \pi = 4.4933$$

Keep your calculator on radian mode.

$\therefore$ $x$-coordinate of $M = 4.49$ (2dp) **(Ans).**

5. The polynomial $2x^3+5x^2+ax+b$, where $a$ and $b$ are constants, is denoted by $p(x)$. It is given that $(2x+1)$ is a factor of $p(x)$ and that when $p(x)$ is divided by $(x+2)$ the remainder is 9.

   (i) Find the values of $a$ and $b$. [5]

   (ii) When $a$ and $b$ have these values, factorise $p(x)$ completely. [3]

***Suggested Solution:***

(i) $p(x)=2x^3+5x^2+ax+b$

Given that $(2x+1)$ is a factor of $p(x)$.

$$\therefore \quad p\left(-\frac{1}{2}\right)=0$$

$$\Rightarrow \quad 2\left(-\frac{1}{2}\right)^3+5\left(-\frac{1}{2}\right)^2+a\left(-\frac{1}{2}\right)+b=0$$

$$\Rightarrow \quad -\frac{1}{4}+\frac{5}{4}-\frac{a}{2}+b=0 \;\Rightarrow\; 1-\frac{a}{2}+b=0 \;\Rightarrow\; b=\frac{a}{2}-1 \;\cdots\cdots\cdots(1)$$

given that, when $p(x)$ is divided by $(x+2)$, remainder is 9,

$$\therefore \quad p(-2)=9$$

$$\Rightarrow \quad 2(-2)^3+5(-2)^2+a(-2)+b=9$$

$$\Rightarrow \quad -16+20-2a+b=9 \;\Rightarrow\; 4-2a+b=9 \;\Rightarrow\; b=2a+5 \;\cdots\cdots\cdots(2)$$

solving equation (1) and equation (2) simultaneously.

$$2a+5=\frac{a}{2}-1$$

$$2a-\frac{a}{2}=-1-5 \;\Rightarrow\; \frac{3a}{2}=-6 \;\Rightarrow\; a=-4$$

substituting the value of $a$ in eq. (2).

$$b=2(-4)+5 \;\Rightarrow\; b=-3$$

$$\therefore \quad a=-4,\; b=-3 \quad \textbf{(Ans).}$$

(ii) $p(x)=2x^3+5x^2-4x-3$

using long division.

$$\begin{array}{r r}
 & x^2+2x-3 \\
2x+1 \big) & \overline{2x^3+5x^2-4x-3} \\
 & 2x^3+x^2 \quad\quad\quad \\
 & \underline{-\quad\; -\quad\quad\quad\quad} \\
 & 4x^2-4x-3 \\
 & 4x^2+2x \quad\; \\
 & \underline{-\quad\;\; -\quad\;\;} \\
 & -6x-3 \\
 & -6x-3 \\
 & \underline{+\quad\; +} \\
 & 0 \quad
\end{array}$$

$$\begin{aligned}
\therefore \quad p(x) &= (2x+1)(x^2+2x-3) \\
&= (2x+1)(x^2+3x-x-3) \\
&= (2x+1)\big(x(x+3)-1(x+3)\big) \\
&= (2x+1)(x+3)(x-1) \quad \textbf{(Ans).}
\end{aligned}$$

6. The equation of a curve is

$$x \ln y = 2x + 1.$$

(i) Show that $\frac{dy}{dx} = -\frac{y}{x^2}$. [4]

(ii) Find the equation of the tangent to the curve at the point where $y = 1$, giving your answer in the form $ax + by + c = 0$. [4]

***Suggested Solution:***

(i) $x \ln y = 2x + 1 \;\Rightarrow\; \ln y = 2 + \frac{1}{x}$

differentiating w.r.t. $x$.

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(2) + \frac{d}{dx}(x^{-1})$$

$$\frac{1}{y}(\frac{dy}{dx}) = 0 + (-x^{-2}) \;\Rightarrow\; \frac{1}{y}(\frac{dy}{dx}) = -\frac{1}{x^2} \;\Rightarrow\; \frac{dy}{dx} = \frac{y}{x^2}$$ **(Shown).**

(ii) $x \ln y = 2x + 1$

When $y = 1$,

$$x \ln(1) = 2x + 1 \;\Rightarrow\; 0 = 2x + 1 \;\Rightarrow\; x = -\frac{1}{2}$$

$\therefore$ coordinates of point are $(-\frac{1}{2}, 1)$

gradient of tangent at point $(-\frac{1}{2}, 1)$ is

$$\frac{dy}{dx} = -\frac{1}{(-\frac{1}{2})^2} = -\frac{1}{\frac{1}{4}} = -4$$

$\therefore$ equation of tangent is:

$$y - 1 = -4(x + \frac{1}{2})$$

$\Rightarrow$ $y - 1 = -4x - 2$

$\Rightarrow$ $4x + y + 1 = 0$ **(Ans).**

7. The variables $x$ and $t$ are related by the differential equation

$$e^{2t}\frac{dx}{dt} = \cos^2 x,$$

where $t \geq 0$. When $t = 0$, $x = 0$.

(i) Solve the differential equation, obtaining an expression for $x$ in terms of $t$. [6]

(ii) State what happens to the value of $x$ when $t$ becomes very large. [1]

(iii) Explain why $x$ increases as $t$ increases. [1]

***Suggested Solution:***

(i) $e^{2t}\dfrac{dx}{dt} = \cos^2 x$

Separating the variables, we have,

$$\frac{1}{\cos^2 x}\,dx = \frac{1}{e^{2t}}\,dt \quad \Rightarrow \quad \sec^2 x\,dx = e^{-2t}\,dt$$

integrating both sides

$$\int \sec^2 x\,dx = \int e^{-2t}\,dt$$

$$\Rightarrow \qquad \tan x = \frac{e^{-2t}}{-2} + K$$

when $t = 0,\ x = 0$

$$\Rightarrow \quad \tan(0) = -\frac{1}{2}e^{-2(0)} + K \quad \Rightarrow \quad 0 = -\frac{1}{2} + K \quad \Rightarrow \quad K = \frac{1}{2}$$

$$\therefore \quad \tan x = -\frac{1}{2}e^{-2t} + \frac{1}{2}$$

$$\Rightarrow \quad \tan x = \frac{1}{2}(1 - e^{-2t})$$

$$\Rightarrow \qquad x = \tan^{-1}\left(\frac{1}{2}(1 - e^{-2t})\right) \quad \textbf{(Ans).}$$

(ii) As $t \to \infty$

$$e^{-2t} = \frac{1}{e^{2t}} \to 0$$

$$\Rightarrow \quad x \to \tan^{-1}(\frac{1}{2})$$

$\therefore$ when $t$ becomes very large, $x$ approaches to $\tan^{-1}(\frac{1}{2})$ **(Ans).**

(iii) $e^{2t}\dfrac{dx}{dt} = \cos^2 x$

$\Rightarrow \quad \dfrac{dx}{dt} = e^{-2t}\cos^2 x > 0,$ for all values of $t$

$\Rightarrow \quad x$ is an increasing function.

$\therefore \quad x$ increases as $t$ increases.

**8.** The variable complex number $z$ is given by

$$z = 1 + \cos 2\theta + \mathrm{i}\sin 2\theta,$$

where $\theta$ takes all values in the interval $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$.

(i) Show that the modulus of $z$ is $2\cos\theta$ and the argument of $z$ is $\theta$. [6]

(ii) Prove that the real part of $\dfrac{1}{z}$ is constant. [3]

*Suggested Solution:*

(i) $z = 1 + \cos 2\theta + \mathrm{i}\sin 2\theta$

$$|z| = \sqrt{(1+\cos 2\theta)^2 + (\sin 2\theta)^2}$$
$$= \sqrt{1 + 2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}$$
$$= \sqrt{1 + 2\cos 2\theta + 1}$$
$$= \sqrt{2 + 2\cos 2\theta} = \sqrt{2(1+\cos 2\theta)} = \sqrt{2(2\cos^2\theta)} = \sqrt{4\cos^2\theta} = 2\cos\theta \quad \textbf{(Shown).}$$

$$\arg(z) = \tan^{-1}\left(\frac{\sin 2\theta}{1+\cos 2\theta}\right)$$
$$= \tan^{-1}\left(\frac{2\sin\theta\cos\theta}{2\cos^2\theta}\right) = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta \quad \textbf{(Shown).}$$

(ii) $z = 1 + \cos 2\theta + \mathrm{i}\sin 2\theta$

$$= 2\cos^2\theta + \mathrm{i}(2\sin\theta\cos\theta)$$
$$= 2\cos\theta(\cos\theta + \mathrm{i}\sin\theta)$$

$$\therefore \quad \frac{1}{z} = \frac{1}{2\cos\theta(\cos\theta + \mathrm{i}\sin\theta)} \times \frac{(\cos\theta - \mathrm{i}\sin\theta)}{(\cos\theta - \mathrm{i}\sin\theta)}$$
$$= \frac{\cos\theta - \mathrm{i}\sin\theta}{2\cos\theta(\cos^2\theta - \mathrm{i}^2\sin^2\theta)}$$
$$= \frac{\cos\theta - \mathrm{i}\sin\theta}{2\cos\theta(\cos^2\theta + \sin^2\theta)}$$
$$= \frac{\cos\theta - \mathrm{i}\sin\theta}{2\cos\theta} = \frac{\cos\theta}{2\cos\theta} - \mathrm{i}\frac{\sin\theta}{2\cos\theta} = \frac{1}{2} - \mathrm{i}\frac{\sin\theta}{2\cos\theta}$$

$\therefore$ real part of $\frac{1}{z} = \frac{1}{2}$ which is a constant. **(Proved).**

Note that:

If $z = x + \mathrm{i}y$, then

- $|z| = \sqrt{x^2 + y^2}$
- $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

The range of principal argument $\alpha$ is $-\pi < \alpha \leq \pi$

Recall:

- $\sin^2\theta + \cos^2\theta = 1$
- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta$
- $2\cos^2\theta = 1 + \cos 2\theta$

9. The plane $p$ has equation $3x + 2y + 4z = 13$. A second plane $q$ is perpendicular to $p$ and has equation $ax + y + z = 4$, where $a$ is a constant.

(i) Find the value of $a$. [3]

(ii) The line with equation $\mathbf{r} = \mathbf{j} - \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ meets the plane $p$ at the point $A$ and the plane $q$ at the point $B$. Find the length of $AB$. [6]

*Suggested Solution:*

(i) Equation of $p$: $3x + 2y + 4z = 13 \Rightarrow \mathbf{r}\cdot\begin{pmatrix}3\\2\\4\end{pmatrix} = 13$. $\quad\therefore$ normal $\mathbf{n}_1$ to the plane $p$ is $\mathbf{n}_1 = \begin{pmatrix}3\\2\\4\end{pmatrix}$

Equation of $q$: $ax + y + z = 4 \Rightarrow \mathbf{r}\cdot\begin{pmatrix}a\\1\\1\end{pmatrix} = 4$. $\quad\therefore$ normal $\mathbf{n}_2$ to the plane $q$ is $\mathbf{n}_2 = \begin{pmatrix}a\\1\\1\end{pmatrix}$

as the two planes $p$ and $q$ are perpendicular,

$\therefore \quad \mathbf{n}_1 \cdot \mathbf{n}_2 = 0$

$$\Rightarrow \begin{pmatrix}3\\2\\4\end{pmatrix}\cdot\begin{pmatrix}a\\1\\1\end{pmatrix} = 0 \Rightarrow 3a + 2 + 4 = 0 \Rightarrow 3a = -6 \Rightarrow a = -2 \quad \textbf{(Ans).}$$

(ii) Equation of line is: $\mathbf{r}=\begin{pmatrix}0\\1\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\2\end{pmatrix} \Rightarrow \mathbf{r}=\begin{pmatrix}\lambda\\1+2\lambda\\-1+2\lambda\end{pmatrix}$ ............(1)

Equation of plane $p$ is: $\mathbf{r}\cdot\begin{pmatrix}3\\2\\4\end{pmatrix}=13$ ............(2)

solving equations (1) and (2) simultaneously,

$$\begin{pmatrix}\lambda\\1+2\lambda\\-1+2\lambda\end{pmatrix}\cdot\begin{pmatrix}3\\2\\4\end{pmatrix}=13$$

$$\Rightarrow \ 3\lambda+2(1+2\lambda)+4(-1+2\lambda)=13$$

$$\Rightarrow \ 3\lambda+2+4\lambda-4+8\lambda=13 \ \Rightarrow \ 15\lambda-2=13 \ \Rightarrow \ 15\lambda=15 \ \Rightarrow \ \lambda=1$$

substituting $\lambda=1$ in equation (1).

$$\mathbf{r}=\begin{pmatrix}1\\1+2(1)\\-1+2(1)\end{pmatrix}=\begin{pmatrix}1\\3\\1\end{pmatrix}. \qquad \therefore \ \text{point } A \text{ is } (1, 3, 1)$$

Equation of plane $q$ is: $\mathbf{r}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}=4$ ............(3)

solving equations (1) and (3) simultaneously.

$$\begin{pmatrix}\lambda\\1+2\lambda\\-1+2\lambda\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}=4$$

$$\Rightarrow \ -2\lambda+1+2\lambda-1+2\lambda=4 \ \Rightarrow \ 2\lambda=4 \ \Rightarrow \ \lambda=2$$

substituting $\lambda=2$ in equation (1).

$$\mathbf{r}=\begin{pmatrix}2\\1+2(2)\\-1+2(2)\end{pmatrix}=\begin{pmatrix}2\\5\\3\end{pmatrix}. \qquad \therefore \ \text{point } B \text{ is } (2, 5, 3)$$

$\therefore$ using distance formula

$$|AB|=\sqrt{(2-1)^2+(5-3)^2+(3-1)^2}$$

$$=\sqrt{1+4+4}=\sqrt{9}=3 \text{ units} \quad \textbf{(Ans).}$$

**10.** (i) Find the values of the constants $A$, $B$, $C$ and $D$ such that

$$\frac{2x^3-1}{x^2(2x-1)}\equiv A+\frac{B}{x}+\frac{C}{x^2}+\frac{D}{2x-1}.$$ [5]

(ii) Hence show that

$$\int_1^2 \frac{2x^3-1}{x^2(2x-1)}\,dx=\frac{3}{2}+\frac{1}{2}\ln\left(\frac{16}{27}\right).$$ [5]

*Suggested Solution:*

(i) $\frac{2x^3-1}{x^2(2x-1)}=\frac{2x^3-1}{2x^3-x^2}$ is an improper fraction

∴ using long division.

$$\begin{array}{r|l} & 1 \\ 2x^3-x^2 & 2x^3-1 \\ & \underline{\overset{-}{}2x^3 \overset{+}{-x^2}} \\ & x^2-1 \end{array}$$

$$\therefore \frac{2x^3-1}{x^2(2x-1)}\equiv 1+\frac{x^2-1}{x^2(2x-1)}$$

now, $\frac{x^2-1}{x^2(2x-1)}\equiv\frac{B}{x}+\frac{C}{x^2}+\frac{D}{2x-1}$

$\Rightarrow x^2-1\equiv Bx(2x-1)+C(2x-1)+Dx^2$

For $x=0$,

$-1=B(0)(0-1)+C(0-1)+D(0)^2 \;\Rightarrow\; -1=-C \;\Rightarrow\; C=1$

For $x=\frac{1}{2}$,

$\frac{1}{4}-1=B\left(\frac{1}{2}\right)(1-1)+C(1-1)+D\left(\frac{1}{2}\right)^2 \;\Rightarrow\; -\frac{3}{4}=\frac{1}{4}D \;\Rightarrow\; D=-3$

For $x=1$,

$1-1=B(1)(2-1)+C(2-1)+D(1)^2 \;\Rightarrow\; 0=B+C+D$

substitute $C=1$, $D=-3$,

$0=B+1+(-3) \;\Rightarrow\; B-2=0 \;\Rightarrow\; B=2$

$\Rightarrow \frac{2x^3-1}{x^2(2x-1)}\equiv 1+\frac{2}{x}+\frac{1}{x^2}-\frac{3}{2x-1}$

∴ $A=1$, $B=2$, $C=1$, $D=-3$ **(Ans).**

(ii) From part (i),

$$\int_1^2\frac{2x^3-1}{x^2(2x-1)}\,dx\equiv\int_1^2\left(1+\frac{2}{x}+\frac{1}{x^2}-\frac{3}{2x-1}\right)dx$$

$$=\int_1^2\left(1+\frac{2}{x}+x^{-2}-\frac{3}{2x-1}\right)dx$$

$$=\left|x+2\ln x+\frac{x^{-1}}{-1}-\frac{3}{2}\ln(2x-1)\right|_1^2$$

$$=\left|x-\frac{1}{x}+\frac{4}{2}\ln x-\frac{3}{2}\ln(2x-1)\right|_1^2=\left|x-\frac{1}{x}+\frac{1}{2}(4\ln x-3\ln(2x-1))\right|_1^2$$

$$=\left|x-\frac{1}{x}+\frac{1}{2}\left(\ln x^4-\ln(2x-1)^3\right)\right|_1^2=\left|x-\frac{1}{x}+\frac{1}{2}\ln\frac{x^4}{(2x-1)^3}\right|_1^2$$

$$=\left(2-\frac{1}{2}+\frac{1}{2}\ln\frac{2^4}{(2(2)-1)^3}\right)-\left(1-\frac{1}{1}+\frac{1}{2}\ln\frac{1^4}{(2(1)-1)^3}\right)$$

$$=\left(\frac{3}{2}+\frac{1}{2}\ln(\frac{16}{27})\right)-\left(0+\frac{1}{2}\ln(1)\right)=\frac{3}{2}+\frac{1}{2}\ln(\frac{16}{27})\quad\textbf{(Shown).}$$

# November 2010 Paper 1

## *Pure Mathematics (P1)*

Answer all the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. (i) Find the first 3 terms in the expansion, in ascending powers of $x$, of $(1-2x^2)^8$. [2]

   (ii) Find the coefficient of $x^4$ in the expansion of $(2-x^2)(1-2x^2)^8$. [2]

***Suggested Solution:***

(i) $(1-2x^2)^8$

$= {}^8C_0(-2x^2)^0 + {}^8C_1(-2x^2)^1 + {}^8C_2(-2x^2)^2 + \cdots\cdots$

$= 1-16x^2+112x^4+\cdots\cdots$ **(Ans).**

Binomial theorem:

$(1+x)^n = {}^nC_0\,x^0 + {}^nC_1\,x^1 + {}^nC_2\,x^2 + \ldots\ldots + {}^nC_r\,x^r + \ldots\ldots + {}^nC_n\,x^n$

(ii) $(2-x^2)(1-2x^2)^8$

using the result of part (i),

$(2-x^2)(1-16x^2+112x^4)$

collecting the terms containing $x^4$ only.

$2(112x^4)-x^2(-16x^2)$

$= 224x^4+16x^4 = 240x^4$

$\therefore$ coefficient of $x^4 = 240$ **(Ans).**

2. Prove the identity

$$\tan^2 x - \sin^2 x \equiv \tan^2 x \sin^2 x. \qquad [4]$$

***Suggested Solution:***

$$\text{L.H.S} = \tan^2 x - \sin^2 x$$

$$= \frac{\sin^2 x}{\cos^2 x} - \frac{\sin^2 x}{1}$$

$$= \frac{\sin^2 x - \sin^2 x\cos^2 x}{\cos^2 x}$$

$$= \frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x}$$

$$= \frac{\sin^2 x(\sin^2 x)}{\cos^2 x}$$

$$= \sin^2 x\left(\frac{\sin^2 x}{\cos^2 x}\right) = \sin^2 x\tan^2 x = \text{R.H.S}$$ **(Shown).**

Recall:

$1-\sin^2\theta \equiv \cos^2\theta$

$1-\cos^2\theta \equiv \sin^2\theta$

l theorem:

$= {}^nC_0 x^0 + {}^nC_1 x^1 +$

$\ldots + {}^nC_r x^r +$

$x^n$

$= \cos^2\theta$

$\theta = \sin^2\theta$

3. The length, $x$ metres, of a Green Anaconda snake which is $t$ years old is given approximately by the formula

$$x = 0.7\sqrt{(2t-1)}.$$

where $1 \le t \le 10$. Using this formula, find

(i) $\frac{dx}{dt}$. [2]

(ii) the rate of growth of a Green Anaconda snake which is 5 years old. [2]

***Suggested Solution:***

(i) $x = 0.7\sqrt{(2t-1)}$

differentiating w.r.t. $t$

$$\frac{dx}{dt} = 0.7\left(\frac{1}{2}(2t-1)^{-\frac{1}{2}} \times 2\right) = \frac{0.7}{\sqrt{(2t-1)}} \quad \textbf{(Ans).}$$

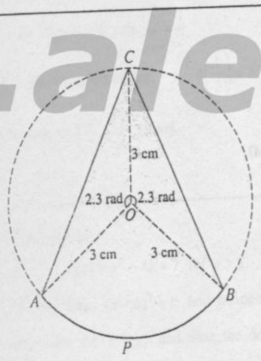
(ii) when $t = 5$ years

$$\frac{dx}{dt} = \frac{0.7}{\sqrt{(2(5)-1)}}$$

$$= \frac{0.7}{\sqrt{9}} = \frac{0.7}{3} = 0.233$$

$\therefore$ rate of growth = 0.233 m/year **(Ans).**

4.



The diagram shows points $A$, $C$, $B$, $P$ on the circumference of a circle with centre $O$ and radius 3 cm.

Angle $AOC$ = angle $BOC$ = 2.3 radians.

(i) Find angle $AOB$ in radians, correct to 4 significant figures. [1]

(ii) Find the area of the shaded region $ACBP$, correct to 3 significant figures. [4]

***Suggested Solution:***

(i) $A\hat{O}B = 2\pi - (2.3 + 2.3) = 1.683$ radians **(Ans).**

(ii) Area of $\Delta OAC = \frac{1}{2}(3)(3)\sin(2.3) = 3.3557 \text{ cm}^2$

$\Rightarrow$ Area of $\Delta OBC = 3.3557 \text{ cm}^2$

Area of sector $AOB = \frac{1}{2}(3)^2(1.683) = 7.5735 \text{ cm}^2$

$\therefore$ Area of the shaded region $= 3.3557 + 3.3557 + 7.5735$

$= 14.2849 \approx 14.3 \text{ cm}^2$ **(Ans).**

Learning CORNER

Note:

- Area of sector $= \frac{1}{2}r^2\theta$
- Length of arc $= r\theta$

where $r$ is the radius and $\theta$ is the angle in radians.

5. (a) The first and second terms of an arithmetic progression are 161 and 154 respectively. The sum of the first $m$ terms is zero. Find the value of $m$. [3]

(b) A geometric progression, in which all the terms are positive, has common ratio $r$. The sum of the first $n$ terms is less than 90% of the sum to infinity. Show that $r^n > 0.1$. [3]

*Suggested Solution:*

(a) First term, $a = 161$, second term $= 154$.

$\therefore$ common difference, $d = 154 - 161 = -7$

given that, $S_m = 0$

$\Rightarrow \frac{m}{2}\left(2a + (m-1)d\right) = 0$

$\Rightarrow \frac{m}{2}\left(2(161) + (m-1)(-7)\right) = 0$

$\frac{m}{2}(322 - 7m + 7) = 0$

$\frac{m}{2}(329 - 7m) = 0$

either $m = 0$ (ignored) or $329 - 7m = 0$

$m = 47$ **(Ans).**

(b) We know that, $S_n = \frac{a(1-r^n)}{1-r}$ and $S_\infty = \frac{a}{1-r}$

According to given condition,

$S_n < 90\%$ of $S_\infty$

$\Rightarrow \frac{a(1-r^n)}{(1-r)} < \frac{90}{100}\left(\frac{a}{1-r}\right)$

$\Rightarrow 1 - r^n < 0.9 \Rightarrow -r^n < -0.1 \Rightarrow r^n > 0.1$ **(Shown).**

6. A curve has equation $y = kx^2 + 1$ and a line has equation $y = kx$, where $k$ is a non-zero constant.

(i) Find the set of values of $k$ for which the curve and the line have no common points. [3]

(ii) State the value of $k$ for which the line is a tangent to the curve and, for this case, find the coordinates of the point where the line touches the curve. [4]

ea of sector $= \frac{1}{2}r^2\theta$

ngth of arc $= r\theta$

e $r$ is the radius and
e angle in radians.

***Suggested Solution:***

(i) Equation of curve : $y = kx^2 + 1$, equation of line : $y = kx$

solving the two equations simultaneously,

$kx^2 + 1 = kx \Rightarrow kx^2 - kx + 1 = 0$ ·········(1)

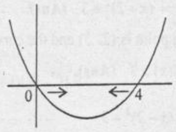
as there are no common points,

$\therefore$ discriminant, $b^2 - 4ac < 0$

$\Rightarrow (-k)^2 - 4(k)(1) < 0$

$k^2 - 4k < 0$

$k(k-4) < 0$



using sketch method, the set of values of $k$ are: $0 < k < 4$ **(Ans).**

(ii) Using equation (1), from part (i),we have

$kx^2 - kx + 1 = 0$

as the line is a tangent to the curve, the above equation must have equal roots

$\Rightarrow$ discriminant $= 0$

$\Rightarrow (-k)^2 - 4(k)(1) = 0$

$k^2 - 4k = 0$

$k(k-4) = 0$

either $k = 0$ or $k = 4$

given that $k$ is a non-zero constant, $\therefore k = 4$ **(Ans).**

substituting $k = 4$ into equation (1),

$4x^2 - 4x + 1 = 0$

$\Rightarrow (2x-1)^2 = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

substituting $x = \frac{1}{2}$ in the equation of line, $y = 4\left(\frac{1}{2}\right) = 2$

$\therefore$ point of intersection is $\left(\frac{1}{2}, 2\right)$ **(Ans).**

7. The function f is defined by

$$f(x) = x^2 - 4x + 7 \text{ for } x > 2.$$

(i) Express f($x$) in the form $(x-a)^2 + b$ and hence state the range of f. [3]

(ii) Obtain an expression for $f^{-1}(x)$ and state the domain of $f^{-1}$. [3]

The function g is defined by

$$g(x) = x - 2 \text{ for } x > 2.$$

The function h is such that f = hg and the domain of h is $x > 0$.

(iii) Obtain an expression for h($x$). [1]

*Suggested Solution:*

(i) $f(x) = x^2 - 4x + 7$

$= x^2 - 2(x)(2) + (2)^2 - (2)^2 + 7$

$= (x-2)^2 - 4 + 7 = (x-2)^2 + 3$ **(Ans).**

we see that the turning point is (2, 3) and the curve open upwards.

$\therefore$ range of f is: $f(x) \geq 3$ **(Ans).**

(ii) From part (i), $f(x) = (x-2)^2 + 3$

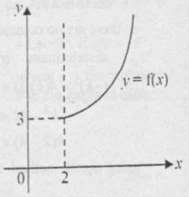
let $y = f(x)$, $\Rightarrow f^{-1}(y) = x$

$\therefore \quad y = (x-2)^2 + 3 \Rightarrow (x-2)^2 = y - 3 \Rightarrow x - 2 = \pm\sqrt{y-3}$

The graph of f(x) is to the right of the axis of symmetry, therefore using positive sign with the radical, we have

$x - 2 = \sqrt{y-3} \Rightarrow x = 2 + \sqrt{y-3}$

or $f^{-1}(y) = 2 + \sqrt{y-3} \Rightarrow f^{-1}(x) = 2 + \sqrt{x-3}$ **(Ans).**



Note that f($x$) is valid for $x > 2$.

So f($x$) is on the right side of the axis of symmetry. Therefore only keep the positive sign with the radical.

Now, $f(x) = (x-2)^2 + 3$

range of f is: $f(x) \geq 3$

$\Rightarrow$ domain of $f^{-1}(x)$ is: $x \geq 3$ **(Ans).**

Domain of $f^{-1}(x)$ is the range of f($x$) and vice versa.

(iii) $f(x) = (x-2)^2 + 3$

Given that, f = hg

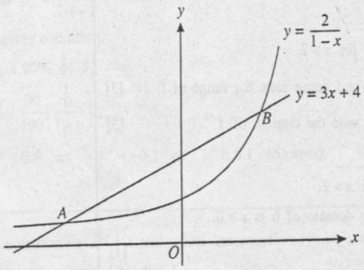
$\Rightarrow \quad hg(x) = (x-2)^2 + 3$

$\Rightarrow \quad h(g(x)) = (x-2)^2 + 3$

$\Rightarrow \quad h(g(x)) = (g(x))^2 + 3$ (since $g(x) = x - 2$)

$\Rightarrow \quad h(x) = (x)^2 + 3$ **(Ans).**

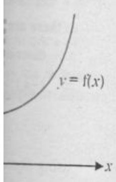
**8.**



The diagram shows part of the curve $y = \frac{2}{1-x}$ and the line $y = 3x + 4$. The curve and the line meet at points $A$ and $B$.

(i) Find the coordinates of $A$ and $B$. [4]

(ii) Find the length of the line $AB$ and the coordinates of the mid-point of $AB$. [3]



t f(x) is valid for

on the right side
is of symmetry.
only keep the
ign with the radi-

$f^{-1}(x)$ is the
(x) and vice

---

*Suggested Solution:*

(i) Equation of curve: $y = \dfrac{2}{1-x}$. equation of the line: $y = 3x + 4$

solving the two equations simultaneously.

$$3x + 4 = \frac{2}{1-x}$$

$$\Rightarrow (3x+4)(1-x) = 2 \Rightarrow -3x^2 - x + 4 = 2 \Rightarrow 3x^2 + x - 2 = 0$$

$$\Rightarrow 3x^2 + 3x - 2x - 2 = 0 \Rightarrow 3x(x+1) - 2(x+1) = 0 \Rightarrow (x+1)(3x-2) = 0$$

$$\Rightarrow x = -1 \text{ or } x = \frac{2}{3}$$

when $x = -1$, $y = 3(-1) + 4 = 1$.

∴ coordinates of $A$ are $(-1, 1)$ **(Ans).**

when $x = \dfrac{2}{3}$, $y = 3\left(\dfrac{2}{3}\right) + 4 = 6$.

∴ coordinates of $B$ are $\left(\dfrac{2}{3}, 6\right)$ **(Ans)**

(ii) $$AB = \sqrt{\left(\frac{2}{3} - (-1)\right)^2 + (6-1)^2}$$

$$= \sqrt{\left(\frac{5}{3}\right)^2 + (5)^2} = \sqrt{\frac{25}{9} + 25} = \sqrt{\frac{250}{9}} = \frac{5\sqrt{10}}{3}$$

Distance formula:

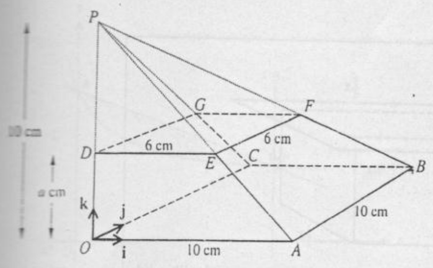
Distance between two points, $A(x_1, y_1)$ and $B(x_2, y_2)$ is:

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

∴ length of line $AB = \dfrac{5\sqrt{10}}{3}$ or 5.27 units. (3 sf) (Ans).

Mid point of $AB = \left(\dfrac{-1+\frac{2}{3}}{2}, \dfrac{1+6}{2}\right) = \left(-\dfrac{1}{6}, \dfrac{7}{2}\right)$ (Ans).

9.



The diagram shows a pyramid $OABCP$ in which the horizontal base $OABC$ is a square of side 10 cm and the vertex $P$ is 10 cm vertically above $O$. The points $D$, $E$, $F$, $G$ lie on $OP$, $AP$, $BP$, $CP$ respectively and $DEFG$ is a horizontal square of side 6 cm. The height of $DEFG$ above the base is $a$ cm. Unit vectors **i**, **j** and **k** are parallel to $OA$, $OC$ and $OD$ respectively.

(i) Show that $a = 4$. [2]

(ii) Express the vector $\overrightarrow{BG}$ in terms of **i**, **j** and **k**. [2]

(iii) Use a scalar product to find angle $GBA$. [4]

***Suggested Solution:***

(i) $\Delta PDE$ is similar to $\Delta POA$

$$\therefore \quad \frac{PD}{PO} = \frac{DE}{OA}$$

$$\Rightarrow \quad \frac{10-a}{10} = \frac{6}{10} \quad \Rightarrow \quad 10-a=6 \quad \Rightarrow \quad a=4 \quad \textbf{(Shown).}$$

(ii) We have, $\vec{OB} = \begin{pmatrix} 10 \\ 10 \\ 0 \end{pmatrix}$, $\quad \vec{OG} = \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix}$

$$\therefore \quad \vec{BG} = \vec{OG} - \vec{OB} = \begin{pmatrix} 0 \\ 6 \\ 4 \end{pmatrix} - \begin{pmatrix} 10 \\ 10 \\ 0 \end{pmatrix} = \begin{pmatrix} -10 \\ -4 \\ 4 \end{pmatrix} \qquad \therefore \quad \vec{BG} = -10\mathbf{i} - 4\mathbf{j} + 4\mathbf{k} \quad \textbf{(Ans).}$$

(iii) Position vector of $A$ is: $\vec{OA} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \quad \vec{BA} = \vec{OA} - \vec{OB} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 10 \\ 10 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -10 \\ 0 \end{pmatrix}$$

applying scalar product.

$$\vec{BG} \cdot \vec{BA} = \left|\vec{BG}\right| \left|\vec{BA}\right| \cos G\hat{B}A$$

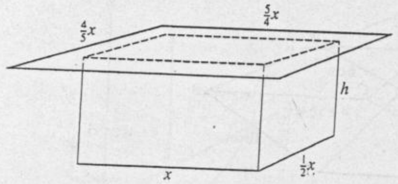
$$\begin{pmatrix} -10 \\ -4 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -10 \\ 0 \end{pmatrix} = \left(\sqrt{(-10)^2 + (-4)^2 + (4)^2}\right)\left(\sqrt{(-10)^2}\right) \cos G\hat{B}A$$

$$0 + 40 + 0 = \left(\sqrt{132}\right)\left(\sqrt{100}\right) \cos G\hat{B}A$$

$$40 = 10\sqrt{132} \cos G\hat{B}A$$

$$\cos G\hat{B}A = \frac{40}{10\sqrt{132}} \quad \Rightarrow \quad G\hat{B}A = 69.6° \quad \textbf{(Ans).}$$

**10.**



The diagram shows an open rectangular tank of height $h$ metres covered with a lid. The base of the tank has sides of length $x$ metres and $\frac{1}{2}x$ metres and the lid is a rectangle with sides of length $\frac{5}{4}x$ metres and $\frac{4}{5}x$ metres. When full the tank holds $4\ \text{m}^3$ of water. The material from which the tank is made is of negligible thickness. The external surface area of the tank together with the area of the top of the lid is $A\ \text{m}^2$.

(i) Express $h$ in terms of $x$ and hence show that $A = \frac{3}{2}x^2 + \frac{24}{x}$. [5]

(ii) Given that $x$ can vary, find the value of $x$ for which $A$ is a minimum, showing clearly that $A$ is a minimum and not a maximum. [5]

**Suggested Solution:**

(i) Volume of tank $= (x)(\frac{1}{2}x)(h)$ m$^3$

given volume of tank $= 4$ m$^3$

$\Rightarrow (x)(\frac{1}{2}x)(h) = 4 \Rightarrow hx^2 = 8 \Rightarrow h = \frac{8}{x^2}$ **(Ans).**

External surface area. $A = (x)(\frac{1}{2}x) + +2(x)(h) + 2(\frac{1}{2}x)(h) + (\frac{4}{5}x)(\frac{5}{4}x)$

$$= \frac{1}{2}x^2 + 2xh + xh + x^2 = \frac{3}{2}x^2 + 3xh$$

substituting the value of $h$

$A = \frac{3}{2}x^2 + 3x(\frac{8}{x^2}) = \frac{3}{2}x^2 + \frac{24}{x}$ **(Shown).**

(ii) $A = \frac{3}{2}x^2 + \frac{24}{x}$

Differentiating w.r.t. $x$

$\frac{dA}{dx} = \frac{3}{2}(2x) + 24(-x^{-2}) = 3x - \frac{24}{x^2}$

for maxima and minima. $\frac{dA}{dx} = 0$

$\Rightarrow 3x - \frac{24}{x^2} = 0 \Rightarrow 3x = \frac{24}{x^2} \Rightarrow x^3 = 8 \Rightarrow x = 2$

then, $\frac{d^2A}{dx^2} = 3 + \frac{48}{x^3}$

at $x = 2$, $\frac{d^2A}{dx^2} = 3 + \frac{48}{(2)^3} = 3 + 6 = 9 > 0$

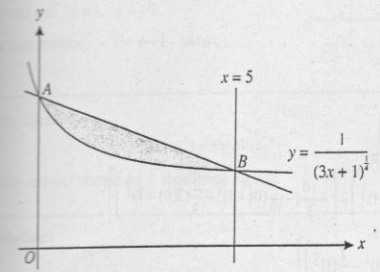
$\Rightarrow$ $A$ is a minimum at $x = 2$ **(Ans).**

Remember:

if $\left.\frac{d^2y}{dx^2}\right|_{x = x_1} > 0$, $y$ is min.

if $\left.\frac{d^2y}{dx^2}\right|_{x = x_1} < 0$, $y$ is max.

2.



The diagram shows part of the curve $y = \frac{1}{(3x+1)^{\frac{1}{4}}}$. The curve cuts the $y$-axis at $A$ and the line $x = 5$ at $B$.

(i) Show that the equation of the line $AB$ is $y = -\frac{1}{10}x + 1$. [4]

(ii) Find the volume obtained when the shaded region is rotated through 360° about the $x$-axis. [9]

*Suggested Solution:*

(i) Equation of the curve is: $y = \dfrac{1}{(3x+1)^{\frac{1}{4}}}$.

at $x = 0$. $y = \dfrac{1}{(3(0)+1)^{\frac{1}{4}}} = 1$

$\therefore$ coordinates of point $A$ are $(0.\ 1)$

at $x = 5$. $y = \dfrac{1}{(3(5)+1)^{\frac{1}{4}}} = \dfrac{1}{(16)^{\frac{1}{4}}} = \dfrac{1}{2}$

$\therefore$ coordinates of point $B$ are $(5.\ \frac{1}{2})$

gradient of $AB = \dfrac{\frac{1}{2}-1}{5-0} = \dfrac{-\frac{1}{2}}{5} = -\dfrac{1}{10}$

equation of line $AB$ passing through $A(0.\ 1)$ is.

$$y-1 = -\frac{1}{10}(x-0) \Rightarrow y = -\frac{1}{10}x+1 \quad \textbf{(Shown).}$$

(ii) Let $y_1 = \dfrac{1}{(3x+1)^{\frac{1}{4}}}$ and $y_2 = -\dfrac{1}{10}x+1$

volume of the shaded region, $V = \pi\displaystyle\int_0^5 \left((y_2)^2-(y_1)^2\right)\,dx$

$$\Rightarrow V = \pi\int_0^5\left((-\tfrac{1}{10}x+1)^2-\left((3x+1)^{-\frac{1}{4}}\right)^2\right)dx$$

$$= \pi\int_0^5\left((-\tfrac{1}{10}x+1)^2-(3x+1)^{-\frac{1}{2}}\right)dx$$

$$= \pi\left(\frac{(-\frac{1}{10}x+1)^3}{(3)(-\frac{1}{10})}-\frac{(3x+1)^{\frac{1}{2}}}{(\frac{1}{2})(3)}\right)_0^5$$

$$= \pi\left(-\frac{10}{3}(-\tfrac{1}{10}x+1)^3-\frac{2}{3}(3x+1)^{\frac{1}{2}}\right)_0^5$$

$$= \pi\left[\left(-\frac{10}{3}(-\tfrac{1}{10}(5)+1)^3-\frac{2}{3}(3(5)+1)^{\frac{1}{2}}\right)-\left(-\frac{10}{3}(-\tfrac{1}{10}(0)+1)^3-\frac{2}{3}(3(0)+1)^{\frac{1}{2}}\right)\right]$$

$$= \pi\left[\left(-\frac{10}{3}(\tfrac{1}{2})^3-\frac{2}{3}(16)^{\frac{1}{2}}\right)-\left(-\frac{10}{3}(1)^3-\frac{2}{3}(1)^{\frac{1}{2}}\right)\right]$$

$$= \pi\left[\left(-\frac{10}{3}(\tfrac{1}{8})-\frac{2}{3}(4)\right)-\left(-\frac{10}{3}-\frac{2}{3}\right)\right].$$

$$= \pi\left(-\frac{37}{12}+4\right) = \frac{11}{12}\pi \text{ units}^3 \quad \textbf{(Ans).}$$

# November 2010 Paper 3

## Pure Mathematics (P3)

Answer all the questions

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

Time : 1 hour 45 minutes

1. Solve the inequality $2|x-3|>|3x+1|$. [4]

***Suggested Solution:***

$2|x-3|>|3x+1|$

Squaring both sides

$(2|x-3|)^2>(|3x+1|)^2$

$4(x-3)^2>(3x+1)^2$

$4(x^2-6x+9)>9x^2+6x+1$

$4x^2-24x+36>9x^2+6x+1$

$5x^2+30x-35<0$

$x^2+6x-7<0$

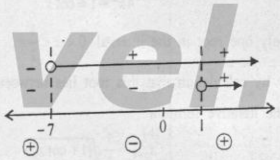
$x^2+7x-x-7<0$

$x(x+7)-1(x+7)<0$

$(x+7)(x-1)<0$

critical values are: $x=1.$ $x=-7$

$\therefore$ using line method, $-7<x<1$ **(Ans).**



2. Solve the equation

$$\ln(1+x^2)=1+2\ln x,$$

giving your answer correct to 3 significant figures. [4]

***Suggested Solution:***

$\ln(1+x^2)=1+2\ln x$

$\Rightarrow \ln(1+x^2)-2\ln x=1 \Rightarrow \ln(1+x^2)-\ln x^2=1 \Rightarrow \ln\left(\frac{1+x^2}{x^2}\right)=1$

$\Rightarrow \frac{1+x^2}{x^2}=e \Rightarrow 1+x^2=ex^2 \Rightarrow ex^2-x^2=1 \Rightarrow x^2(e-1)=1$

$\Rightarrow x^2=\frac{1}{(e-1)} \Rightarrow x=\pm\frac{1}{\sqrt{e-1}}$

$\Rightarrow x=0.763$ or $x=-0.763$ (rejected)

$\therefore x=0.763$ **(Ans).**

Recall:

- $\ln(\frac{a}{b})=\ln a-\ln b$
- $\ln(a\times b)=\ln a+\ln b$
- $\ln(a^b)=b\ln a$
- $\ln e=1$
- $\ln x=1 \Rightarrow x=e^1$
- Remember that log of a negative number is not defined.

3. Solve the equation

$$\cos(\theta + 60°) = 2\sin\theta.$$

giving all solutions in the interval $0° \le \theta \le 360°$. [5]

*Suggested Solution:*

$\cos(\theta + 60°) = 2\sin\theta$

$\Rightarrow \cos\theta\cos 60° - \sin\theta\sin 60° = 2\sin\theta$

$\Rightarrow \cos\theta\left(\frac{1}{2}\right) - \sin\theta\left(\frac{\sqrt{3}}{2}\right) = 2\sin\theta$

$\Rightarrow \cos\theta - \sqrt{3}\sin\theta = 4\sin\theta$

$\Rightarrow \cos\theta = 4\sin\theta + \sqrt{3}\sin\theta$

$\Rightarrow \cos\theta = (4+\sqrt{3})\sin\theta$

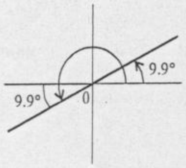
$\Rightarrow \frac{\sin\theta}{\cos\theta} = \frac{1}{(4+\sqrt{3})}$

$\Rightarrow \tan\theta = \frac{1}{4+\sqrt{3}}$

$\Rightarrow \tan\theta = 0.174457$

basic angle $\alpha = 9.9°$

$\therefore \quad \theta = 9.9°.\ \ 189.9°$ **(Ans).**



Recall:

$\cos(A+B)$
$= \cos A\cos B - \sin A\sin B$

$\cos(A-B)$
$= \cos A\cos B + \sin A\sin B$

Also remember that:

- $\cos 60° = \frac{1}{2}$
- $\sin 60° = \frac{\sqrt{3}}{2}$

$\tan\theta$ is positive in 1st and 3rd quadrant and negative in 2nd and 4th quadrant.

4. (i) By sketching suitable graphs, show that the equation

$$4x^2 - 1 = \cot x$$

has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(ii) Verify by calculation that this root lies between 0.6 and 1. [2]

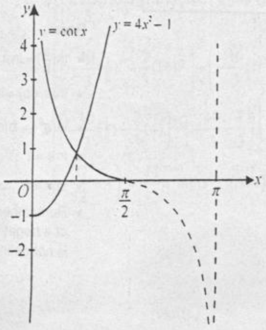
(iii) Use the iterative formula

$$x_{n+1} = \frac{1}{2}\sqrt{(1+\cot x_n)}$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

*Suggested Solution:*

(i)



The two graphs intersect only at one point in the interval $0 < x < \frac{\pi}{2}$, therefore only one real root lies in this interval.

ecall:

os(A + B)
= cos A cos B − sin A sin B

os(A − B)
cos A cos B + sin A sin B

so remember that:

cos 60° = $\frac{1}{2}$

sin 60° = $\frac{\sqrt{3}}{2}$

θ is positive in 1st and
quadrant and negative
d and 4th quadrant.

rval $0 < x < \frac{\pi}{2}$.

---

(ii) Let $f(x) = 4x^2 - 1 - \cot x$

$f(0.6) = 4(0.6)^2 - 1 - \cot(0.6) = 0.44 - 1.4617 = -1.022 < 0$

$f(1) = 4(1)^2 - 1 - \cot(1) = 3 - 0.6421 = 2.338 > 0$

change of sign indicates that the root lies between 0.6 and 1. **(Verified).**

Remember to keep the mode of calculator at radian.

(iii) $x_{n+1} = \frac{1}{2}\sqrt{(1 + \cot x_n)}$

using $x_1 = \frac{0.6 + 1}{2} = 0.8$

$x_2 = \frac{1}{2}\sqrt{(1 + \cot x_1)} = \frac{1}{2}\sqrt{1 + \cot(0.8)} = 0.7020$

$x_3 = \frac{1}{2}\sqrt{(1 + \cot x_2)} = \frac{1}{2}\sqrt{1 + \cot(0.7020)} = 0.7387$

$x_4 = \frac{1}{2}\sqrt{(1 + \cot x_3)} = \frac{1}{2}\sqrt{1 + \cot(0.7387)} = 0.7242$

$x_5 = \frac{1}{2}\sqrt{(1 + \cot x_4)} = \frac{1}{2}\sqrt{1 + \cot(0.7242)} = 0.7298$

$x_6 = \frac{1}{2}\sqrt{(1 + \cot x_5)} = \frac{1}{2}\sqrt{1 + \cot(0.7298)} = 0.7276$

$x_7 = \frac{1}{2}\sqrt{(1 + \cot x_6)} = \frac{1}{2}\sqrt{1 + \cot(0.7276)} = 0.7285$

root correct to 2 decimal places = 0.73 **(Ans).**

7. Let $I = \int_0^1 \frac{x^2}{\sqrt{(4 - x^2)}}\, dx$.

(i) Using the substitution $x = 2\sin\theta$, show that

$$I = \int_0^{\frac{1}{6}\pi} 4\sin^2\theta \, d\theta. \qquad [3]$$

(ii) Hence find the exact value of $I$. [4]

*Suggested Solution:*

(i) $I = \int_0^1 \frac{x^2}{\sqrt{(4 - x^2)}}\, dx$

given substitution is: $x = 2\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta \Rightarrow dx = 2\cos\theta \, d\theta$

new limits $\begin{cases} \text{when } x = 0, \ 2\sin\theta = 0 \Rightarrow \theta = 0 \\ \text{when } x = 1, \ 2\sin\theta = 1 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \end{cases}$

substituting the new limits and the values of $x$ and $dx$, we have

$$I = \int_0^{\frac{\pi}{6}} \frac{(2\sin\theta)^2}{\sqrt{4-(2\sin\theta)^2}} \times 2\cos\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{8\sin^2\theta\cos\theta}{\sqrt{4-4\sin^2\theta}} \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{8\sin^2\theta\cos\theta}{\sqrt{4(1-\sin^2\theta)}} \, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{8\sin^2\theta\cos\theta}{\sqrt{4\cos^2\theta}} \, d\theta = \int_0^{\frac{\pi}{6}} \frac{8\sin^2\theta\cos\theta}{2\cos\theta} \, d\theta = \int_0^{\frac{\pi}{6}} 4\sin^2\theta \, d\theta \quad \textbf{(Shown).}$$

(ii) $I = \int_0^{\frac{\pi}{6}} 4\sin^2\theta \, d\theta$

$$= 4\int_0^{\frac{\pi}{6}} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= 2\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$$

$$= 2\left[\left(\frac{\pi}{6} - \frac{1}{2}\sin 2(\frac{\pi}{6})\right) - \left(0 - \frac{1}{2}\sin 2(0)\right)\right]$$

$$= 2\left[\left(\frac{\pi}{6} - \frac{1}{2}\sin(\frac{\pi}{3})\right) - 0\right] = 2\left(\frac{\pi}{6} - \frac{1}{2}(\frac{\sqrt{3}}{2})\right) = 2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) = \frac{\pi}{3} - \frac{\sqrt{3}}{2} \quad \textbf{(Ans).}$$

Note that:

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

6. The complex number $z$ is given by

$$z = (\sqrt{3}) + i.$$

(i) Find the modulus and argument of $z$. [2]

(ii) The complex conjugate of $z$ is denoted by $z^*$. Showing your working, express in the form $x + iy$, where $x$ and $y$ are real,

(a) $2z + z^*$.

(b) $\frac{iz^*}{z}$. [4]

(iii) On a sketch of an Argand diagram with origin $O$, show the points $A$ and $B$ representing the complex numbers $z$ and $iz^*$ respectively. Prove that angle $AOB = \frac{1}{6}\pi$. [3]

***Suggested Solution:***

(i) $z = (\sqrt{3}) + i$

modulus, $|z| = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = 2$ **(Ans).**

$\arg(z) = \tan^{-1}(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$ **(Ans).**

(ii) (a) $2z + z^* = 2(\sqrt{3} + i) + (\sqrt{3} - i)$

$= 2\sqrt{3} + 2i + \sqrt{3} - i = 3\sqrt{3} + i$ **(Ans).**

Note that:

If $z = x + iy$, then

- $|z| = \sqrt{x^2 + y^2}$
- $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

The range of principal argument $\alpha$ is $-\pi < \alpha \le \pi$

- If $z = x + iy$, then $z^* = x - iy$
- $i^2 = -1$

Note that:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Note that:

If $z = x + iy$, then

- $|z| = \sqrt{x^2+y^2}$
- $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$

The range of principal argument $\alpha$ is $-\pi < \alpha \le \pi$

- If $z = x + iy$, then $z^* = x - iy$
- $i^2 = -1$

(ii) $\frac{iz^*}{z} = \frac{i(\sqrt{3}-i)}{\sqrt{3}+i} = \frac{i\sqrt{3}-i^2}{\sqrt{3}+i} = \frac{1+i\sqrt{3}}{\sqrt{3}+i}$

Realising the denominator,

$$= \frac{1+i\sqrt{3}}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}$$

$$= \frac{\sqrt{3}-i+3i-i^2\sqrt{3}}{(\sqrt{3})^2-(i)^2}$$

$$= \frac{\sqrt{3}+2i+\sqrt{3}}{3+1}$$

$$= \frac{2\sqrt{3}+2i}{4} = \frac{\sqrt{3}+i}{2} = \frac{\sqrt{3}}{2} + i\,\frac{1}{2} \quad \textbf{(Ans).}$$

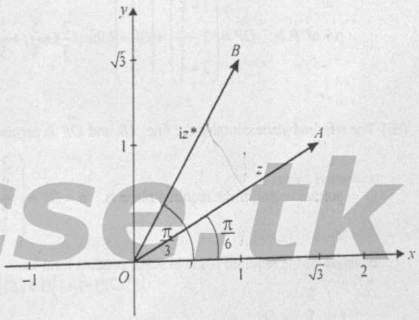
(iii) From parts (i) and (ii), we have,

$z = \sqrt{3} + i$, $\quad \arg(z) = \frac{\pi}{6}$, $\quad iz^* = 1 + i\sqrt{3}$

$$\arg(iz^*) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$$\angle AOB = \frac{\pi}{3} - \frac{\pi}{6}$$

$$= \frac{2\pi-\pi}{6} = \frac{1}{6}\pi \quad \textbf{(Shown).}$$



12. With respect to the origin $O$, the points $A$ and $B$ have position vectors given by $\vec{OA} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\vec{OB} = 3\mathbf{i} + 4\mathbf{j}$. The point $P$ lies on the line $AB$ and $OP$ is perpendicular to $AB$.

(i) Find a vector equation for the line $AB$. [1]

(ii) Find the position vector of $P$. [4]

(iii) Find the equation of the plane which contains $AB$ and which is perpendicular to the plane $OAB$, giving your answer in the form $ax + by + cz = d$. [4]

***Suggested Solution:***

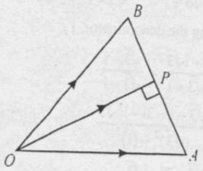
(i) Given that, $\vec{OA} = \begin{pmatrix}1\\2\\2\end{pmatrix}$, $\vec{OB} = \begin{pmatrix}3\\4\\0\end{pmatrix}$

$$\therefore \vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix}3\\4\\0\end{pmatrix} - \begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}2\\2\\-2\end{pmatrix}, \text{ or } \begin{pmatrix}1\\1\\-1\end{pmatrix}$$

Vector equation of $AB$ is: $\mathbf{r} = \vec{OA} + \lambda\,\vec{AB}$

$$= \begin{pmatrix}1\\2\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix} \quad \text{or} \quad \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} + \mathbf{j} - \mathbf{k}) \quad \textbf{(Ans).}$$

**(ii)** Taking point $P$ as a general point on $\overrightarrow{AB}$. we have.



$$\overrightarrow{OP}=\begin{pmatrix}1+\lambda\\2+\lambda\\2-\lambda\end{pmatrix},\quad \vec{d}=\begin{pmatrix}1\\1\\-1\end{pmatrix}$$

since $\overrightarrow{OP}\perp\overrightarrow{AB}$

$$\therefore\quad \overrightarrow{OP}\cdot\overrightarrow{AB}=0$$

$$\Rightarrow\quad \begin{pmatrix}1+\lambda\\2+\lambda\\2-\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=0 \;\Rightarrow\; 1+\lambda+2+\lambda-2+\lambda=0 \;\Rightarrow\; 3\lambda+1=0 \;\Rightarrow\; \lambda=-\frac{1}{3}$$

$$\therefore\quad \text{p.v of } P \text{ is, } \overrightarrow{OP}=\begin{pmatrix}1-\frac{1}{3}\\2-\frac{1}{3}\\2+\frac{1}{3}\end{pmatrix}=\begin{pmatrix}\frac{2}{3}\\\frac{5}{3}\\\frac{7}{3}\end{pmatrix} \quad\text{or}\quad \frac{2}{3}\mathrm{i}+\frac{5}{3}\mathrm{j}+\frac{7}{3}\mathrm{k} \quad \textbf{(Ans).}$$

**(iii)** The required plane contains the line $AB$. and $\overrightarrow{OP}$ is perpendicular to $AB$.

$$\therefore\quad \text{normal vector of the required plane is, } \mathbf{n}=\overrightarrow{OP}=\begin{pmatrix}\frac{2}{3}\\\frac{5}{3}\\\frac{7}{3}\end{pmatrix} \quad\text{or}\quad \mathbf{n}=\begin{pmatrix}2\\5\\7\end{pmatrix}$$

equation of the required plane in scalar product form is,

$$\mathbf{r}\cdot\begin{pmatrix}2\\5\\7\end{pmatrix}=D$$

this plane contains the point $A(1, 2, 2)$

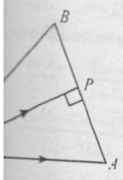
$$\therefore\quad \begin{pmatrix}1\\2\\2\end{pmatrix}\cdot\begin{pmatrix}2\\5\\7\end{pmatrix}=D \;\Rightarrow\; 2+10+14=D \;\Rightarrow\; D=26$$

equation of the required plane is: $\mathbf{r}\cdot\begin{pmatrix}2\\5\\7\end{pmatrix}=26 \;\Rightarrow\; 2x+5y+7z=26$ **(Ans).**

**8.** Let $f(x)=\dfrac{3x}{(1+x)(1+2x^2)}$.

**(i)** Express $f(x)$ in partial fractions. [5]

**(ii)** Hence obtain the expansion of $f(x)$ in ascending powers of $x$, up to and including the term in $x^3$. [5]



Suggested Solution:

(i) Let $\frac{3x}{(1+x)(1+2x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+2x^2}$

$\Rightarrow 3x = A(1+2x^2) + (Bx+C)(1+x)$

Set $x = -1$

$3(-1) = A(1+2(-1)^2) + (B(-1)+C)(1+(-1))$

$\Rightarrow -3 = A(3) + (-B+C)(0) \Rightarrow -3 = 3A \Rightarrow A = -1$

Set $x = 0$

$3(0) = A(1+2(0)^2) + (B(0)+C)(1+(0))$

$\Rightarrow 0 = A + C \Rightarrow C = -A \Rightarrow C = -(-1) = 1$

Set $x = 1$

$3(1) = A(1+2(1)^2) + (B(1)+C)(1+(1))$

$\Rightarrow 3 = A(3) + (B+C)(2) \Rightarrow 3 = 3A + 2B + 2C \Rightarrow 3 = 3(-1) + 2B + 2(1)$

$\Rightarrow 4 = 2B \Rightarrow B = 2$

$\therefore f(x) = \frac{-1}{1+x} + \frac{2x+1}{1+2x^2}$ **(Ans).**

Binomial Expansion:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

(ii) Using the result of part (i),

$f(x) = \frac{-1}{1+x} + \frac{2x+1}{1+2x^2} = -1(1+x)^{-1} + (2x+1)(1+2x^2)^{-1}$

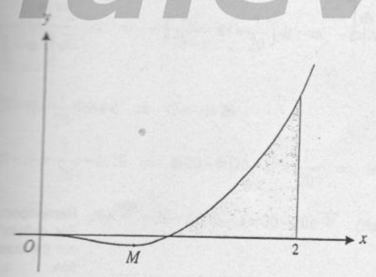
applying binomial theorem up to the term in $x^3$,

$= -\left(1 + (-1)x + \frac{(-1)(-2)}{2!}(x)^2 + \frac{(-1)(-2)(-3)}{3!}(x)^3\right) + (2x+1)\left(1 + (-1)(2x^2)\right)$

$= -(1 - x + x^2 - x^3) + (2x+1)(1-2x^2)$

$= -1 + x - x^2 + x^3 + 2x - 4x^3 + 1 - 2x^2$

$= 3x - 3x^2 - 3x^3$ **(Ans).**



The diagram shows the curve $y = x^3 \ln x$ and its minimum point $M$.

(i) Find the exact coordinates of $M$. [5]

(ii) Find the exact area of the shaded region bounded by the curve, the $x$-axis and the line $x = 2$. [5]

***Suggested Solution:***

(i) $y = x^3 \ln x$

Differentiating w.r.t. $x$

$$\frac{dy}{dx} = \ln x \frac{d}{dx}(x^3) + x^3 \frac{d}{dx}(\ln x)$$

$$= 3x^2 \ln x + x^3(\frac{1}{x}) = 3x^2 \ln x + x^2$$

Remember:

$\frac{d}{dx}(uv) = v\frac{d}{dx}u + u\frac{d}{dx}v$

for minimum point, $\frac{dy}{dx} = 0$

$\Rightarrow \ 3x^2 \ln x + x^2 = 0 \ \Rightarrow \ x^2(3\ln x + 1) = 0$

$\Rightarrow \ x^2 = 0 \quad$ or $\quad 3\ln x + 1 = 0$

$\Rightarrow \ x = 0$ (rej) or $\quad \ln x = -\frac{1}{3} \ \Rightarrow \ x = e^{-\frac{1}{3}}$

substituting this value of $x$ into the equation of curve.

$$y = (e^{-\frac{1}{3}})^3 \ln(e^{-\frac{1}{3}}) = e^{-1}(-\frac{1}{3}\ln e) = e^{-1}(-\frac{1}{3}) = -\frac{1}{3e}$$

$\therefore \ M$ is $\left(e^{-\frac{1}{3}}, -\frac{1}{3e}\right)$ **(Ans).**

(ii) Equation of curve: $y = x^3 \ln x$

at $y = 0$, $\ x^3 \ln x = 0$

$\Rightarrow \ x^3 = 0 \quad$ or $\quad \ln x = 0$

$\Rightarrow \ x = 0 \quad$ or $\quad x = e^0 \ \Rightarrow \ x = 1$

area of the shaded region between $x = 1$ and $x = 2$ is.

$$A = \int_1^2 y \, dx \ \Rightarrow \ A = \int_1^2 x^3 \ln x \, dx$$

using integration by parts,

$$A = \left| \ln x \int x^3 \, dx - \int\left(\frac{d}{dx}(\ln x)\right)\left(\int x^3 \, dx\right) dx \right|_1^2$$

$$= \left| \ln x(\frac{x^4}{4}) - \int(\frac{1}{x})(\frac{x^4}{4}) \, dx \right|_1^2$$

$$= \left| \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 \, dx \right|_1^2$$

$$= \left| \frac{1}{4}x^4 \ln x - \frac{1}{4}(\frac{x^4}{4}) \right|_1^2$$

$$= \left| \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 \right|_1^2$$

$$= \left(\frac{1}{4}(2)^4 \ln(2) - \frac{1}{16}(2)^4\right) - \left(\frac{1}{4}(1)^4 \ln(1) - \frac{1}{16}(1)^4\right)$$

$$= (4\ln 2 - 1) - (-\frac{1}{16}) = 4\ln 2 - 1 + \frac{1}{16} = (4\ln 2 - \frac{15}{16}) \text{ units}^2 \ \textbf{(Ans).}$$

Formula for integration by parts:

$\int uv \, dx$

$= u\int v \, dx - \int\left(\frac{d}{dx}u\right)\left(\int v \, dx\right) dx$

Remember:
For definite integral, do not put the constant of integration.

Remember:

$\frac{d}{dx}(uv) = v\frac{d}{dx}u + u\frac{d}{dx}v$

...ula for integration by
...s:

...dx

...dx $- \int\left(\frac{d}{dx}u\right)\left(\int v\ dx\right) dx$

...nber:
...finite integral, do not
... constant of integra-

---

(8) A certain substance is formed in a chemical reaction. The mass of substance formed $t$ seconds after the start of the reaction is $x$ grams. At any time the rate of formation of the substance is proportional to $(20 - x)$. When $t = 0$, $x = 0$ and $\frac{dx}{dt} = 1$.

(i) Show that $x$ and $t$ satisfy the differential equation

$$\frac{dx}{dt} = 0.05(20 - x). \qquad [2]$$

(ii) Find, in any form, the solution of this differential equation. [5]

(iii) Find $x$ when $t = 10$, giving your answer correct to 1 decimal place. [2]

(iv) State what happens to the value of $x$ as $t$ becomes very large. [1]

**Suggested Solution:**

(i) According to given information,

$$\frac{dx}{dt} \propto (20-x) \Rightarrow \frac{dx}{dt} = k(20-x) \text{ ............(1)}$$

given that, $t = 0$, $\frac{dx}{dt} = 1$, we have,

$$1 = k(20-0) \Rightarrow 20k = 1 \Rightarrow k = \frac{1}{20}$$

∴ equation (1) becomes,

$$\frac{dx}{dt} = \frac{1}{20}(20-x) \Rightarrow \frac{dx}{dt} = 0.05(20-x) \text{ (Shown).}$$

(ii) $\frac{dx}{dt} = \frac{1}{20}(20-x) \Rightarrow \frac{1}{20-x}\ dx = \frac{1}{20}\ dt$

integrating both sides

$$\int \frac{1}{20-x}\ dx = \frac{1}{20}\int dt \Rightarrow -\int \frac{-1}{20-x}\ dx = \frac{1}{20}\int dt \Rightarrow -\ln|20-x| = \frac{1}{20}t + C$$

when $t = 0$, $x = 0$

$$\Rightarrow -\ln|20-0| = \frac{1}{20}(0)t + C \Rightarrow C = -\ln 20$$

$$\therefore -\ln|20-x| = \frac{1}{20}t - \ln 20 \Rightarrow \ln 20 - \ln|20-x| = \frac{1}{20}t \Rightarrow \ln\left|\frac{20}{20-x}\right| = \frac{1}{20}t,$$

$$\Rightarrow \frac{20}{20-x} = e^{\frac{1}{20}t} \Rightarrow 20e^{-\frac{1}{20}t} = 20 - x \Rightarrow x = 20 - 20e^{-\frac{1}{20}t} \text{ (Ans).}$$

(iii) When $t = 10$

$$x = 20 - 20e^{-\frac{1}{20}(10)} = 20 - 20e^{-\frac{1}{2}} = 20 - 12.131 = 7.869 \approx 7.9 \text{ (1 dp) (Ans).}$$

(iv) $x = 20 - 20e^{-\frac{1}{20}t}$

As $t \to \infty$, $e^{-\frac{1}{20}t} \to 0$ and $x \to 20$

∴ $x$ approaches to 20 as $t$ becomes very large. **(Ans).**