

The calculus of analytic functors

Lecture 1

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Innolec lectures - Masaryk University

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Plan

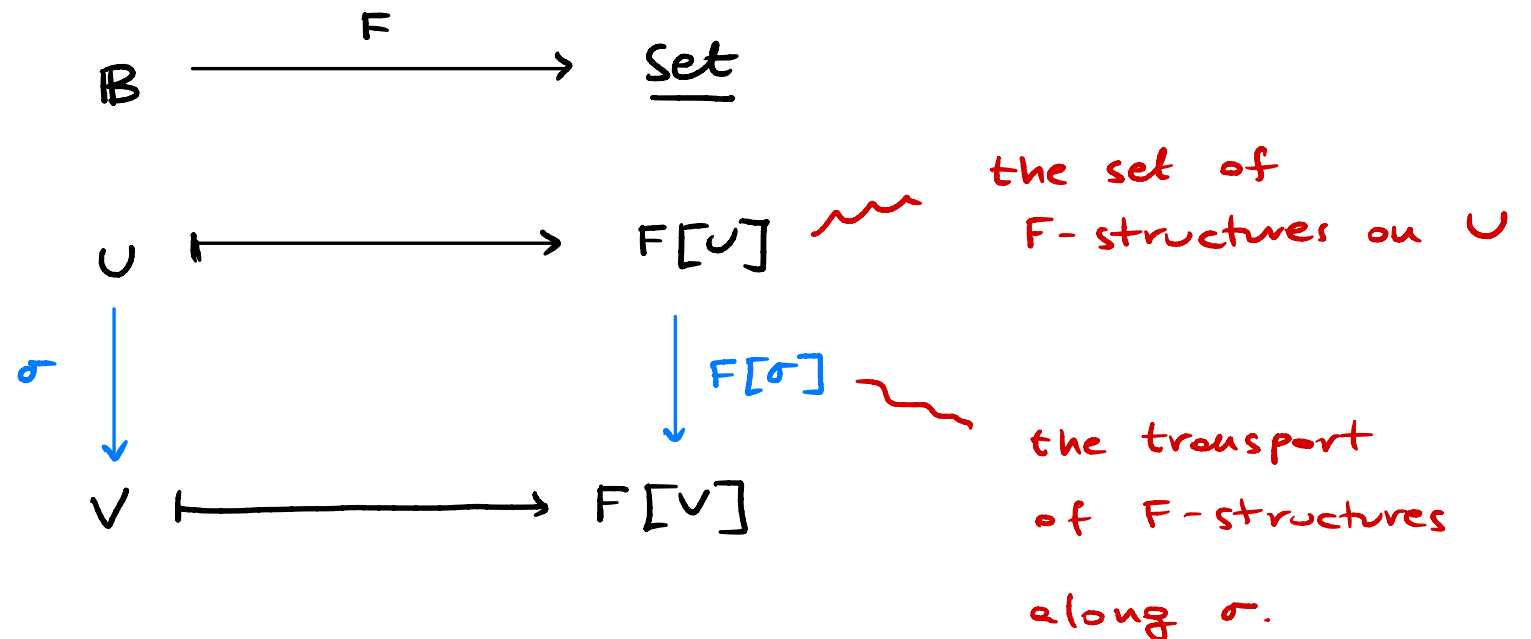
1. Joyal's species of structures and analytic functors
2. The bicategory of analytic functors
3. Connection with Linear Logic
4. Differentiation

1) Species of structures

$\mathbb{B}$ = category of

- finite sets $U, V, W, \dots$
- bijections $\sigma: U \rightarrow V, \tau: V \rightarrow W, \dots$

Definition A species of structures is a functor



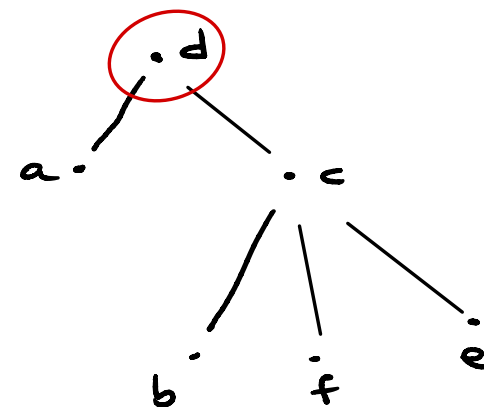
Example

$$\mathbb{B} \xrightarrow{\mathcal{A}} \underline{\text{Set}}$$

$$U \longmapsto \mathcal{A}[U] =_{\text{def}}$$

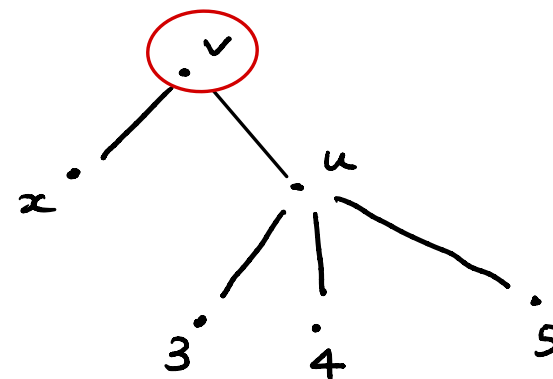
set of rooted
trees labelled by U

$$U = \{a, b, c, d, e, f\} \longmapsto \mathcal{A}[U] \ni$$



$\mathcal{A}[\sigma]$

$$V = \{x, 3, u, v, 5, 4\} \longmapsto \mathcal{A}[V] \ni$$



Examples

$$\mathbb{B} \longrightarrow \underline{\text{Set}}$$

$$(i) \quad U \longmapsto X[U] = \begin{cases} \{U\} & \text{if } |U| = 1 \\ \emptyset & \text{otw} \end{cases}$$

$$(ii) \quad U \longmapsto E[U] = \{U\}$$

$$(iii) \quad U \longmapsto S[U] = \begin{array}{c} \text{set of permutations} \\ \text{on } U \end{array}$$

$$(iv) \quad U \longmapsto \text{Der}[U] = \begin{array}{c} \text{set of derangements} \\ \text{on } U. \end{array}$$

General problem Let $F: \mathbb{B} \rightarrow \underline{\text{Set}}$ s.t. $F[U]$ finite $\forall U$

$\Rightarrow$ Find $f_n = |F[n]|$, for $[n] = \{1, \dots, n\} \in \mathbb{B}$

Generating series Consider $f(x) = \sum_{n \in \mathbb{N}} f_n \frac{x^n}{n!}$

Examples

$$(i) \quad X(x) = \sum_{n \in \mathbb{N}} |X[n]| \frac{x^n}{n!} = x$$

$$(ii) \quad E(x) = \sum_{n \in \mathbb{N}} |E[n]| \frac{x^n}{n!} = \sum_{n \in \mathbb{N}} \frac{x^n}{n!} = e^x$$

$$(iii) \quad S(x) = \sum_{n \in \mathbb{N}} \cancel{|S[n]|} \frac{\cancel{x^n}}{\cancel{n!}} = \sum_{n \in \mathbb{N}} x^n = \frac{1}{1-x}$$

Joyal '82 Analogue of calculus of exponential power series for species of structures

$$F + G, \quad F \cdot G, \quad G \circ F, \quad F', \dots$$

$\Rightarrow$ bijective proofs!

Une théorie combinatoire des séries formelles, Adv. Math 1982

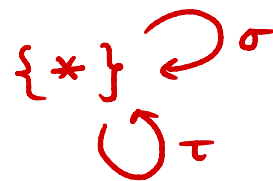
Joyal '86 Introduced analytic functors $F: \underline{\text{Set}} \rightarrow \underline{\text{Set}}$
as functorial analogues of exponential power series.

Foncteurs analytiques et espèces de structures, LNM 1234, 1986

2) Symmetric sequences and analytic functors

- S_n = n -th symmetric group, regarded as a one-object

category



- $\mathbb{P} = \bigsqcup_{n \in \mathbb{N}} S_n$ category, ie.

- $\text{Ob}(\mathbb{P}) = \mathbb{N}$

- $\mathbb{P}[n, m] = \begin{cases} S_n & \text{if } n = m \\ \emptyset & \text{otw.} \end{cases}$

Definition A symmetric sequence is a functor

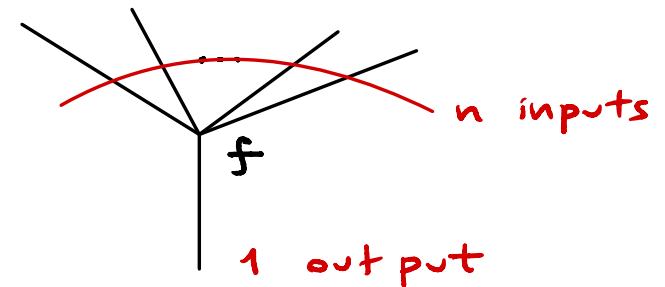
$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{F} & \underline{\text{Set}} \\ n & \longmapsto & F[n] \end{array}$$

Remark

$\mathbb{P} \longleftrightarrow \mathbb{B}$ equivalence $\Rightarrow$ sym sequences $\simeq$ species of structures

Idea

$f \in F[n] \Leftrightarrow$ an n -ary operation $\Leftrightarrow$



Let $F: \mathbb{P} \longrightarrow \underline{\text{Set}}$ be a symmetric sequence.

We define the analytic functor associated to F as

$$\underline{\text{Set}} \xrightarrow{F} \underline{\text{Set}}$$

$$X \longmapsto \sum_{n \in \mathbb{N}} F[n] \times X^n / S_n$$

action of S_n
on $F[n] \times X^n$

Note $f(x) = \sum_{n \in \mathbb{N}} f_n \frac{x^n}{n!}$

$$(f, (x_1, \dots, x_n)) \cdot \sigma = (f \cdot \sigma, (x_{\sigma(1)}, \dots, x_{\sigma(n)}))$$

Theorem (Joyal '86)

Analytic functors are closed

under the following operations

- Sum : $(F + G)(X) = F(X) + G(X)$
- Product : $(F \cdot G)(X) = F(X) \times G(X)$
- Multiplication by scalar : $C F(X) = C \times F(X)$
- Composition : $(G \circ F)(X) = G(F(X))$

Furthermore, the identity functor $1_{\text{Set}} : \underline{\text{Set}} \longrightarrow \underline{\text{Set}}$

is analytic. It is the analytic functor associated to

$$\mathbb{P} \longleftrightarrow \mathbb{B} \xrightarrow{X} \text{Set}$$

We use some category theory to develop this calculus.

Key ingredients

- (i) Presheaf construction
- (ii) Day convolution
- (iii) Universal property of IP .

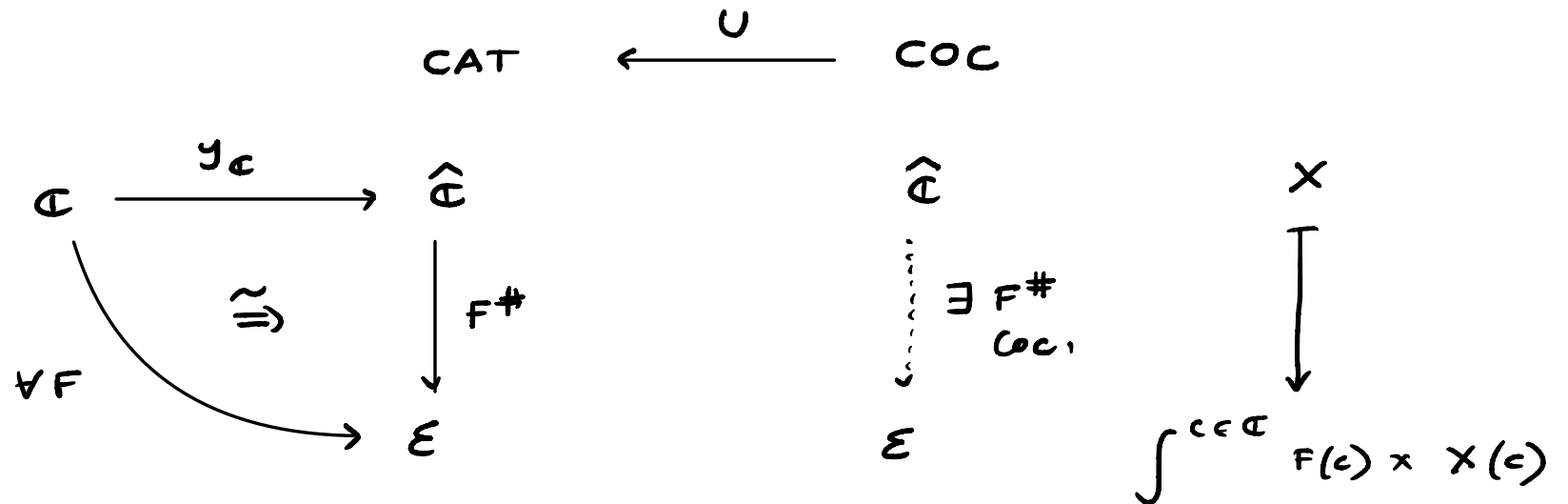
G.B. Im & G.M. Kelly, A universal property of the convolution monoidal structure, JPAA 1986

G.M. Kelly, On the operads of J.P. May, 1972.

(i) Presheaf construction

- $\mathcal{C}$ small category.
- $\hat{\mathcal{C}} = [\mathcal{C}^{op}, \mathbf{Set}]$ = category of presheaves over $\mathcal{C}$
- $y_{\mathcal{C}} : \mathcal{C} \longrightarrow \hat{\mathcal{C}}$ Yoneda embedding

Universal property $\hat{\mathcal{C}}$ is the **free cocompletion** of $\mathcal{C}$, i.e.



(ii) Day convolution

$(\mathbb{C}, \otimes, \mathbb{I})$ symmetric monoidal small category

$\Rightarrow \hat{\mathbb{C}}$ admits a symmetric monoidal structure :

$$\begin{array}{ccc}
 \mathbb{C} \times \mathbb{C} & \xrightarrow{y_{\mathbb{C}} \times y_{\mathbb{C}}} & \hat{\mathbb{C}} \times \hat{\mathbb{C}} \\
 \searrow \otimes & & \downarrow \otimes \\
 & \mathbb{C} & \hat{\mathbb{A}} \\
 & \searrow y_{\mathbb{C}} & \\
 & & \hat{\mathbb{A}}
 \end{array}
 \quad \cong$$

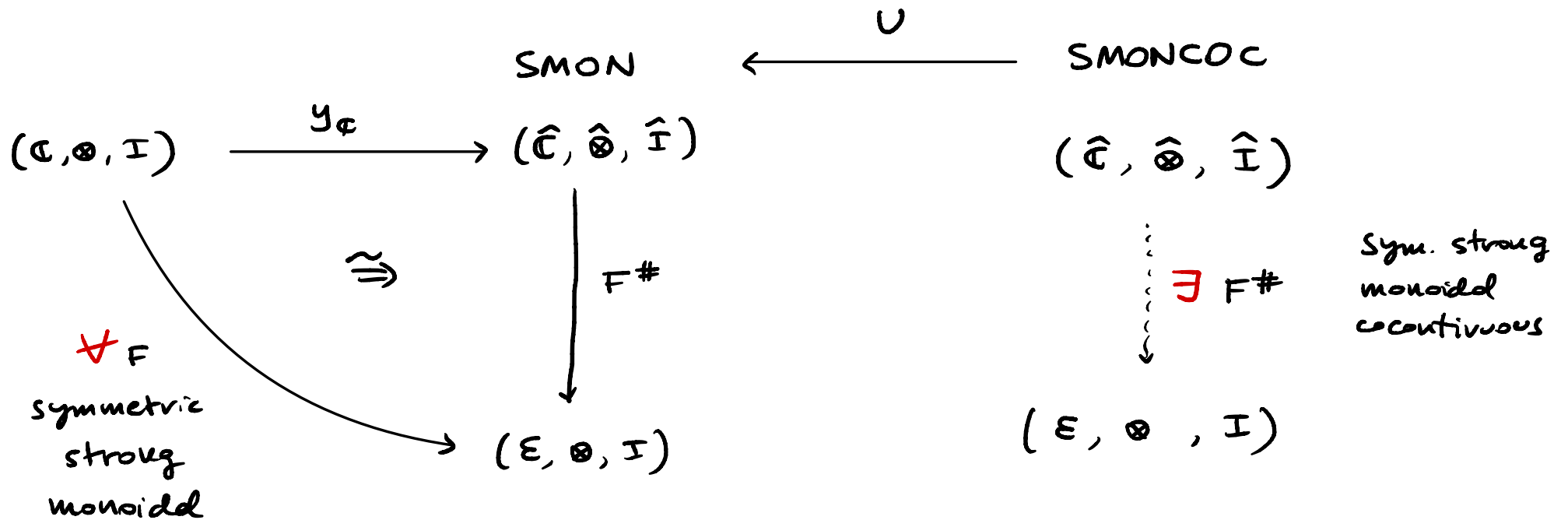
For $x_1, x_2 \in \hat{\mathbb{C}}$

$$\underbrace{(x_1 \hat{\otimes} x_2)}_{\text{Day convolution}}(c) = \int^{c_1, c_2 \in \mathbb{C}} x_1(c_1) \times x_2(c_2) \times \mathbb{C}(c, c_1 \otimes c_2)$$

Unit $\hat{\mathbb{I}} = y_{\mathbb{C}}(\mathbb{I})$.

Universal property

$(\hat{\mathbb{C}}, \hat{\otimes}, \hat{I})$ is the free symmetric monoidally cocomplete category on $(\mathbb{C}, \otimes, I)$



(iii) Universal property of $\mathbb{P}$

- $\mathbb{P}$ has symmetric strict monoidal structure

$$\begin{array}{ccc} \mathbb{P} \times \mathbb{P} & \xrightarrow{\oplus} & \mathbb{P} \\ (n, m) & \longmapsto & n+m \end{array} \quad \text{with unit } 0$$

- $\mathbb{P}$ is the free symmetric strict monoidal category on $\underline{1}$

$$\begin{array}{ccc} & \text{Cat} & \xleftarrow{\quad \cup \quad} \text{SMonCat} \\ & \downarrow & \\ \underline{1} & \xrightarrow{\quad 1 \quad} \mathbb{P} & \downarrow \text{!F} \\ & \downarrow & (\mathbb{P}, \oplus, 0) \\ \text{!X} & \searrow & (\mathcal{E}, \otimes, \mathbb{I}) \end{array}$$

$$\begin{array}{c} n \\ \downarrow \\ \underbrace{X \otimes \dots \otimes X}_{n \text{ times}} = X^{\otimes n} \end{array}$$

Remark Given $F[-] : \mathbb{P} \rightarrow \underline{\text{Set}}$, we can view the analytic functor

$$F(-) : \underline{\text{Set}} \rightarrow \underline{\text{Set}}$$

as a left Kan extension:

$$\begin{array}{ccc} \mathbb{P} & \xrightarrow{\text{incl}} & \underline{\text{Set}} \\ & \searrow F[-] & \downarrow F(-) \\ & & \underline{\text{Set}} \end{array} \Rightarrow$$

$$\begin{array}{c} X \\ \downarrow \\ F(X) = \int^{n \in \mathbb{P}} F[n] \times X^n \\ \downarrow \\ = \sum_{n \in \mathbb{N}} F[n] \times X^n / S_n \end{array}$$

Note: ' $\Rightarrow$ ' not invertible

Back to calculus : $(\mathbb{P}, \oplus, 0)$ symmetric monoidal

$\Rightarrow [\mathbb{P}, \text{set}]$ symmetric monoidal by Day convolution.

$$(F_1 \hat{\oplus} F_2)[n] = \int^{n_1, n_2 \in \mathbb{P}} F_1[n_1] \times F_2[n_2] \times \mathbb{P}[n_1 + n_2, n]$$

We get :

$$\begin{aligned} (F_1 \hat{\oplus} F_2)(X) &= \int^n (F_1 \hat{\oplus} F_2)[n] \times X^n \\ &\cong \int^n \int^{n_1, n_2} F_1[n_1] \times F_2[n_2] \times \mathbb{P}[n_1 + n_2, n] \times X^n \\ &\cong \int^{n_1, n_2} F_1[n_1] \times F_2[n_2] \times \underbrace{X^{n_1} \times X^{n_2}}_{X^{n_1 + n_2}} \\ &\cong F_1(X) \times F_2(X). \end{aligned}$$

Substitution Let $F: IP \rightarrow \underline{\text{Set}}$, $G: IP \rightarrow \underline{\text{Set}}$.

We want $G \circ F: IP \rightarrow \underline{\text{Set}}$ such that

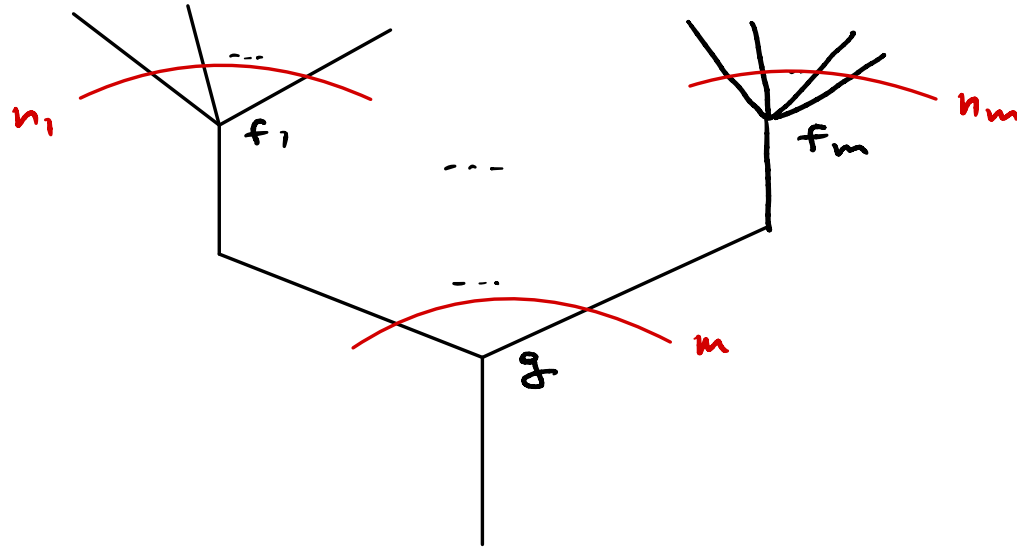
$$(G \circ F)(x) \cong G(F(x))$$

for all $x \in \underline{\text{Set}}$

$\Rightarrow$ We define

$$(G \circ F)[n] = \int^m \int^{n_1, \dots, n_m} F[n_1] \times \dots \times F[n_m] \times \\ G[m] \times IP[n_1 + \dots + n_m, n]$$

Idea For $n \in \mathbb{P}$, an element of $(G \circ F)[n]$ is



$$(G \circ F)[n] = \int^m \int^{n_1, \dots, n_m} F[n_1] \times \dots \times F[n_m] \times \\ G[m] \times \mathbb{P}[n, t \dots + n_m, n]$$

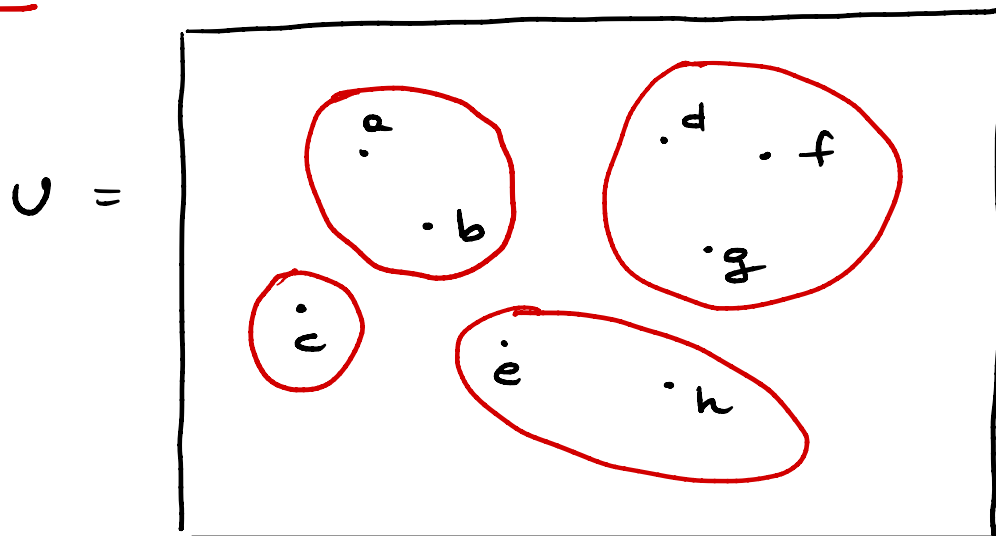
Substitution for species of structures

For $F, G: \mathcal{B} \rightarrow \underline{\text{Set}}$

s.t. $F[\emptyset] = \emptyset$, we have

$$(G \circ F)[U] = \sum_{\substack{\pi \text{ partition} \\ \text{of } U}} G[\pi] \times \prod_{V \in \pi} F[V]$$

Idea



- G -structure on $\{\{a, b\}, \{c\}, \{d, e, f, g\}, \{h\}\}$

- F -structures on $\{a, b\}, \dots$

Example (Joyal) $\text{End} \cong S \circ A$

"an endomorphism is a permutation of rooted trees"

Theorem (Kelly)

(i) The substitution operation determines a monoidal structure on $[P, \text{Set}]$, with unit $I : P \rightarrow \text{Set}$ s.t.

$$I(x) \cong x, \text{ for all } x \in \underline{\text{Set}}.$$

(ii) The monoids wrt substitution monoidal structure, i.e.

$$M : P \rightarrow \text{Set} \quad \text{with}$$

$$M \circ M \Rightarrow M, \quad I \Rightarrow M$$

are exactly symmetric operads.

Proof of (i) (Carboni)

Fix $F : \mathbb{P} \longrightarrow \underline{\text{Set}}$. So have

$$1 \xrightarrow{F} [\mathbb{P}, \text{Set}]$$

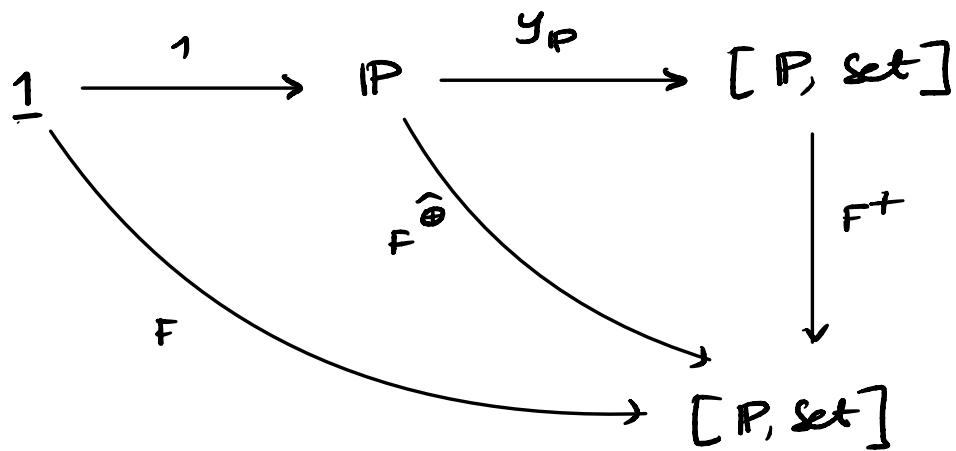
by univ prop of $\mathbb{P}$

$$\mathbb{P} \xrightarrow{F^{\hat{\Theta}}} ([\mathbb{P}, \text{Set}], \hat{\Theta}, \hat{O}) \quad \text{sym. strong monoidal}$$

by univ. prop of presheaves

$$[\mathbb{P}, \text{Set}] \xrightarrow{F^+} [\mathbb{P}, \text{Set}]$$

sym. strong monoidal coc.



$$\ni G \xrightarrow{\quad} \ni F^+(G) \cong G \circ F$$

$$F^{\hat{\Theta}}[m] = \underbrace{F \hat{\Theta} \dots \hat{\Theta} F}_{m \text{ times}}, \text{ so}$$

$$\underbrace{(F \hat{\Theta} \dots \hat{\Theta} F)}_{m \text{ times}}[n] = \int^{n_1, \dots, n_m} F[n_1] \times \dots \times F[n_m] \times P[n_1 + \dots + n_m, n]$$

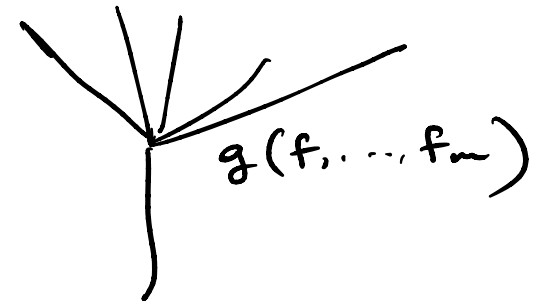
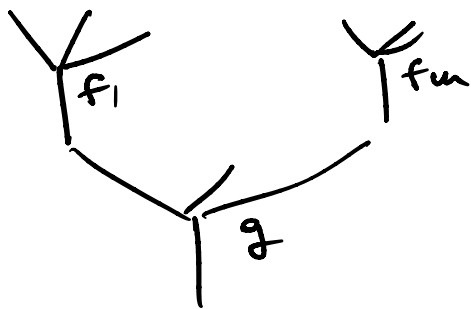
$$F^+(G) = \int^m F^{\hat{\Theta}}[m] \times G[m] \cong (G \circ F)$$

Associativity and unitarity of substitution now follow

$$\text{from } (F^+ G)^+ \cong F^+ G^+ \quad \text{and} \quad I^+ \cong \text{Id}_{\text{set}}.$$

Proof of (ii) : see Kelly's paper.

In an operad $M \circ M \xrightarrow{\mu} M$



Sym operads = sym. multicategories with one object.

Summary

Three equivalent notions

- species of structures
- symmetric sequences
- analytic functors

These support a rich calculus, including an operation of substitution $\Rightarrow$ analytic functors are closed under composition.