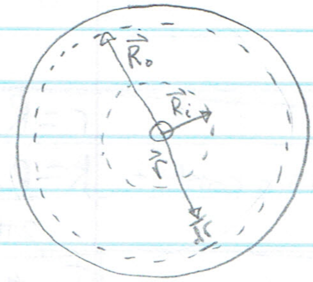
How long is the groove of a record?

Suppose a record plays for 25.23 minutes at $33\frac{1}{3}$ rpm.

$25.23\text{ min} \times 33.3333 \frac{\text{rev}}{\text{min}} =$ 841 revolutions



R_o = outer radius of vinyl

R_i = inner radius of vinyl

N = # of revolutions completed by the record

k = a constant

θ = angle in radians

Assume the groove is a spiral described by a polar equation of the form $r = R_o - \underbrace{k\theta}_{dr}$, where k is a constant and θ is measured in radians

1a. Express k in terms of R_o, R_i, and N.

N is complete revolutions; so they only care about when $\theta = 2\pi$

$r = R_i$ when $\overbrace{k\theta}^{dr} = R_o - R_i$ $\therefore k = \dfrac{R_o - R_i}{\theta N} = \dfrac{R_o - R_i}{2\pi\text{ rad} \cdot 841\text{ rev}}$

$R_i = R_o - k\theta$

N will be # of revolutions (841 total)

Finish conditions

N is max revolutions given $t \cdot$ rpm

$r = R_o$ when $\theta = 0$ $r = R_o = R_o - k(0)$

Start conditions

N is minimum

$-k\theta = r - R_o$ $k\theta = R_o - r$

N is min when $r = R_o$ $N = 0$; when $r = R_i$ N is max

average Δr per revolution $= \dfrac{R_o - R_i}{N}$

$N = \dfrac{R_o - r}{\Delta r} = \dfrac{dr}{\Delta r} = \dfrac{k\theta}{\Delta r}$

$\dfrac{k\theta}{N} = \Delta r = \dfrac{R_o - r}{N}$

Length of arc
P. 222
Ch6 review
Applications of integration

$s=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ ← cartesian

Polar → $s=\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

b. Show that the length L of the record groove is given by

$$L=\frac{1}{k}\int_{R_i}^{R_o}\sqrt{k^2+u^2}\,du$$

Coordinate Transform
$(x,y)=(r\cos\theta, r\sin\theta)$
$=((R_o-k\theta)\cos\theta, (R_o-k\theta)\sin\theta)$
to cartesian from polar
to graph a polar plot on (x,y)

$r=R_o-k\theta$ Polar coordinate

k is a constant $k=\frac{R_o-R_i}{2\pi\,\text{rad}\cdot N}$ $\therefore r=R_o-\left(\frac{R_o-R_i}{2\pi\,\text{rad}\cdot N}\right)\theta$

In the swan lake side 2 example: $R_o=146$ mm
$R_i=62$ mm
$N\approx 841$ revolutions

By length of arc:

$s=\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$

u substitution $u=r$ and $du=dr$

$=\int_0^{2\pi N}\sqrt{(R_o-k\theta)^2+(-k)^2}\,d\theta$, $u=r$

$r=R_o-k\theta$ Solve for θ and $d\theta$ from $\frac{d\theta}{dr}$

$\therefore \theta=\frac{R_o-r}{k}=\frac{R_o}{k}-\frac{r}{k}$

$=\int_0^{2\pi N}\sqrt{(R_o-k\theta)^2+k^2}\,d\theta$

$\frac{d\theta}{dr}=-\frac{1}{k}$ $\therefore d\theta=\frac{-1}{k}dr$

$=\int_0^{2\pi N}\sqrt{R_o^2-2R_ok\theta+k^2\theta^2+k^2}\,d\theta$

$\theta_2=2\pi N$ $r_2=R_i$

$\theta_1=0$ $r_1=R_o$

solve for new limits r_1, r_2 in terms of r not θ, using $r=R_o-k\theta$

$$=-\frac{1}{k}\int_{R_o}^{R_i}\sqrt{k^2+r^2}\,dr$$

$$=\frac{1}{k}\int_{R_i}^{R_o}\sqrt{k^2+r^2}\,dr$$

Flipped the limits so the value changes sign ex online: $\sqrt{9-x} = 3\left(1-\frac{x}{9}\right)^{\frac{1}{2}} = 3\left(1+\left(\frac{-x}{9}\right)\right)^{\frac{1}{2}}$

to rewrite in the correct form: $(1+x)^k$

1 c. Use a binomial series to establish the approximation

$$\sqrt{k^2+u^2} \approx u\left[1+\frac{1}{2}\left(\frac{k}{u}\right)^2\right]$$

We need to rewrite the term a little to put it in the form required.

$$\sqrt{k^2+u^2} = \sqrt{u^2+k^2} = u\left(1+\frac{k^2}{u^2}\right)^{\frac{1}{2}}$$

Here the exponent in the form (k) is $\frac{1}{2}$

Here the term in the form (x) is $\frac{k^2}{u^2}$

Form k is any $\mathbb{R}$

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$$

$$= 1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3 + \cdots$$

The first four terms in the binomial series is then,

$$\sqrt{k^2+u^2} = u\left(1+\frac{k^2}{u^2}\right)^{\frac{1}{2}}$$

$$= u\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\left(\frac{k^2}{u^2}\right)^n$$

$$= u\left[\binom{\frac{1}{2}}{0}\left(\frac{k^2}{u^2}\right)^0+\binom{\frac{1}{2}}{1}\left(\frac{k^2}{u^2}\right)^1+\binom{\frac{1}{2}}{2}\left(\frac{k^2}{u^2}\right)^2+\binom{\frac{1}{2}}{3}\left(\frac{k^2}{u^2}\right)^3+\cdots\right]$$

$$= u\left[1(1)+\frac{1}{2}\left(\frac{k^2}{u^2}\right)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}\left(\frac{k^2}{u^2}\right)^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}\left(\frac{k^2}{u^2}\right)^3+\cdots\right]$$

$$= u\left[1+\frac{1}{2}\left(\frac{k^2}{u^2}\right)-\frac{1}{8}\left(\frac{k^2}{u^2}\right)^2+\frac{1}{16}\left(\frac{k^2}{u^2}\right)^3+\cdots\right]$$

$$\approx u\left[1+\frac{1}{2}\left(\frac{k}{u}\right)^2\right]$$

QED

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1 d. In part b, use the approximation in part c to show that the Length L of the groove is given by the approximation

$$L \approx \pi N(R_i+R_o)+\frac{R_o-R_i}{4\pi N}\ln\frac{R_o}{R_i}$$

diff of 2 squares

$$= \frac{(R_o+R_i)(R_o-R_i)}{\frac{R_o-R_i}{\pi N}}+\frac{R_o-R_i}{4\pi N}\ln\frac{R_o}{R_i}$$

By substitution of c into b:

$$L = \frac{1}{k}\int_{R_i}^{R_o}\sqrt{k^2+u^2}\,du$$

$$= \frac{1}{k}\int_{R_i}^{R_o}u\left[1+\frac{1}{2}\left(\frac{k}{u}\right)^2\right]du$$

$$= \frac{1}{k}\int_{R_i}^{R_o}u+\frac{1}{2}\frac{k^2}{u}\,du$$

$$= \frac{1}{k}\int_{R_i}^{R_o}u+\frac{k^2}{2}\frac{1}{u}\,du$$

$$= \frac{1}{k}\left[\frac{u^2}{2}+\frac{k^2}{2}\ln u\right]\Bigg|_{R_i}^{R_o}$$

$$= \frac{1}{k}\left[\frac{R_o^2}{2}+\frac{k^2}{2}\ln R_o-\left(\frac{R_i^2}{2}+\frac{k^2}{2}\ln R_i\right)\right]$$

$$= \frac{1}{k}\left[\frac{R_o^2}{2}-\frac{R_i^2}{2}+\frac{k^2}{2}\ln R_o-\frac{k^2}{2}\ln R_i\right]$$

$$= \frac{1}{k}\left[\frac{R_o^2-R_i^2}{2}+\frac{k^2}{2}\ln\frac{R_o}{R_i}\right]$$

$$= \frac{R_o^2-R_i^2}{2k}+\frac{k}{2}\ln\frac{R_o}{R_i}$$

another clean approximation

$$L \approx \pi N(R_o+R_i)+\frac{R_o-R_i}{4\pi N}\ln\frac{R_o}{R_i}$$

1e. Use the result in part d to approximate the groove length if $R_o = 6$ in, $R_i = 2.5$ in

20 min @ $33\frac{1}{3}$ rpm
$N = 666.67$ revolutions

$$\boxed{L \approx \pi N(R_o + R_i) + \frac{R_o - R_i}{4\pi N} \ln \frac{R_o}{R_i}}$$

$$\approx \pi(666.67)(6\text{ in} + 2.5\text{ in}) + \frac{6\text{ in} - 2.5\text{ in}}{4\pi \cdot 666.67} \ln \frac{6}{2.5}$$

$$= 17802.44738\text{ in} + 3.65753 \times 10^{-4}\text{ in}$$

≈ 17802.44775 inches
≈ 1483.5 ft
≈ 0.281 miles

if Swan Lake
$R_o = 146$ mm
$R_i = 62$ mm
$N = t \cdot \text{rpm} = 25.23\text{ min} \times 33\frac{1}{3}\text{ rpm}$
$= 841$ revolutions

$$L \approx \pi(841)(146\text{mm} + 62\text{mm}) + \frac{(146-62)}{4\pi(841)} \ln \frac{146}{62}$$

$L \approx$ 549552.5197 mm + .006807 mm
≈ 549552.5265 mm
≈ 549.5525 m $\approx .3415$ miles
≈ 1802.995 ft

MATLAB says 1802.993831 ft for my loop estimator

Yes!

1f. Use an appropriate trig? substitution to evaluate the integral in part b using $R_o = 6$ in, $R_i = 2.5$ in from part e. Compare this to what is approximated in part e from integrating the binomial series approximation terms.

$$L = \frac{1}{k} \int_{R_i}^{R_o} \sqrt{k^2 + u^2}\, du$$

For $\sqrt{a^2 + x^2}$ p. 319
set $x = a \tan\theta$

$$k = \frac{R_o - R_i}{2\pi\text{rad} \cdot N} = \frac{6\text{in} - 2.5\text{in}}{2\pi \cdot 666.67} = 8.3556 \times 10^{-4}$$

$$= \frac{1}{k} \int_{2.5}^{6} \sqrt{k^2 + u^2}\, du$$

$\therefore u = k \tan x$
$\therefore \frac{du}{dx} = k \cdot \sec^2 x \quad \therefore du = k \sec^2 x\, dx$

Change u and du to r and dr

Wolfram will do more du's

$$L = \frac{1}{K}\int_{R_i}^{R_o} \sqrt{K^2 + r^2}\, dr$$

$r = K\tan(x)$ trig substitution

$dr = K\sec^2(x)\, dx$ solve for dr

trig identity $1+\tan^2(x) = \sec^2(x)$

$$L = \frac{1}{K}\int_{x_i}^{x_o} \underbrace{\sqrt{K^2 + K^2\tan^2(x)}}\, K\sec^2(x)\, dx = \frac{1}{K}\int_{x_i}^{x_o} K^2\sec^3(x)\, dx$$

$$\sqrt{K^2\sec^2(x)} = K\sec(x)$$

$\therefore x = \tan^{-1}\left(\frac{r}{K}\right)$

$\therefore x_o = \tan^{-1}(R_o/K)$

$\therefore x_i = \tan^{-1}(R_i/K)$

factor out constants

$$= \frac{1}{k}\cdot k^2\int_{x_i}^{x_o} \sec^3(x)\, dx$$

Wolfram says use reduction formula

$$\int \sec^m(x)\, dx = \frac{\sin(x)\sec^{m-1}(x)}{m-1} + \frac{m-2}{m-1}\int \sec^{-2+m}(x)\, dx$$

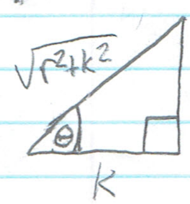
where $m=3$

but blue packet of integrals has #71

$$\int \sec^3 x\, dx = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

$\sec(x) = \frac{1}{\cos(x)} = \frac{\sqrt{r^2+k^2}}{k}$

used tan to make triangle



$$= \frac{1}{k}\cdot k^2\left[\frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x)|\Big|_{x_i}^{x_o}\right]$$

Substitute $x = \tan^{-1}\left(\frac{r}{k}\right)$

$$= \frac{1}{k}\cdot k^2\left[\frac{1}{2}\sec\left(\tan^{-1}\left(\frac{r}{k}\right)\right)\frac{r}{k} + \frac{1}{2}\ln\left|\sec\left(\tan^{-1}\left(\frac{r}{k}\right)\right) + \frac{r}{k}\right|\Big|_{R_i}^{R_o}\right]$$

$$= \frac{1}{k}\cdot k^2\left[\frac{1}{2}\frac{r\sqrt{r^2+k^2}}{k^2} + \frac{1}{2}\ln\left|\frac{\sqrt{r^2+k^2}}{k} + \frac{r}{k}\right|\Big|_{R_i}^{R_o}\right]$$

$$= k\left[\frac{1}{2}\left(\frac{r\sqrt{r^2+k^2}}{k^2} + \ln\left|\frac{\sqrt{r^2+k^2}}{k} + r/k\right|\right)\Big|_{R_i}^{R_o}\right]$$

Evaluate the limits now

$$= k\left[\frac{1}{2}\left(\frac{R_o\sqrt{R_o^2+k^2}}{k^2} + \ln\left|\frac{\sqrt{R_o^2+k^2}}{k} + \frac{R_o}{k}\right| - \left(\frac{R_i\sqrt{R_i^2+k^2}}{k^2} + \ln\left|\frac{\sqrt{R_i^2+k^2}}{k} + \frac{R_i}{k}\right|\right)\right)\right]$$

Combine like terms and the natural logs

$$= \frac{1}{2}k\left[\frac{R_o\sqrt{R_o^2+k^2} - R_i\sqrt{R_i^2+k^2}}{k^2} + \ln\left|\frac{\frac{\sqrt{R_o^2+k^2}+R_o}{k}}{\frac{\sqrt{R_i^2+k^2}+R_i}{k}}\right|\right]$$

$$L = \frac{1}{2}k\left[\frac{R_o\sqrt{R_o^2+k^2} - R_i\sqrt{R_i^2+k^2}}{k^2} + \ln\left|\frac{\sqrt{R_o^2+k^2}+R_o}{\sqrt{R_i^2+k^2}+R_i}\right|\right]$$

for $k = (6-2.5)/(2\pi\cdot 666.67)$ $L = 17802.44775$ in

$R_o = 6$ in

$R_i = 2.5$ in

compared to part e $L = 17802.44775$ in — same!

For $k = (146mm - 62mm)/(2\pi\cdot 841)$

$R_o = 146mm$ $L = 549.5525265$ meters

$R_i = 62mm$ same!

$$k = \frac{R_o - R_i}{2\pi N}$$