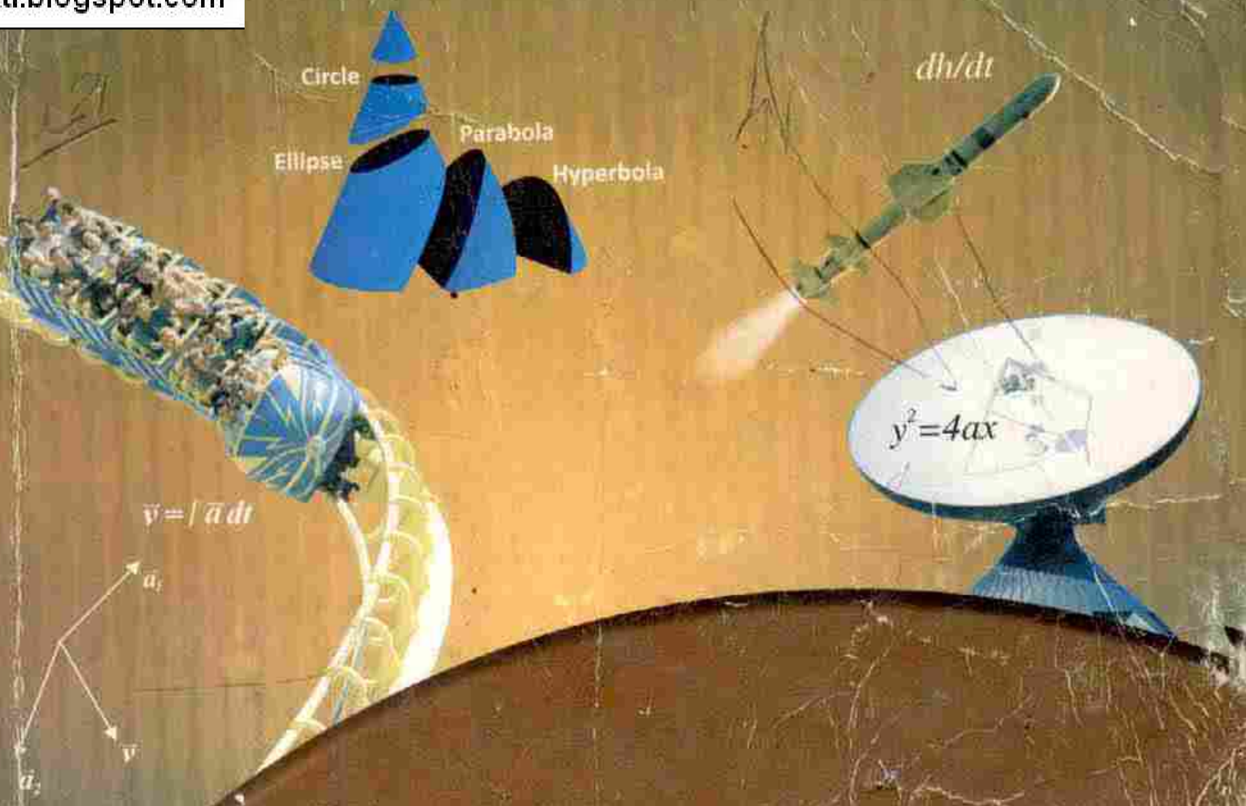


College MATHEMATICS INTERMEDIATE PART-II

COLLEGE Mathematics

CALCULUS & ANALYTIC GEOMETRY

12

Farukh Mahmood

Tahir Kamran Ranjha

Pearl Educational
College Publishing Co.

PEARL Educational

PREFACE

"If you have knowledge,let the others candle at it"

This book is specially designed by keeping in mind the demand of securing top class marks as well as the difficulties of an average student in understanding of Text Book. A significant feauture of this book is

- All important definitions .
- Formulas in the begening of every exercise.
- Complete and comprehensive notes of every chapter.
- Easy approach towards every solution.
- The questions are supported with comprehensive diagrams.
- Each and every important question is highlighted.
- This book is a complete replacement of text book, students need not bother about text book when they have it.

Each chapter is provided with the important questions at its end.This book is a tremendous equalizer with its main focus to save students from any kind of perplexity and preparing them for the examenitions of all the boards of punjab and Federal. Underlying all the aspects, this book will prove to be a great asset, not just for students but for all knowledge seekers.

A special care has been made to avoid mistakes of every kind; therefore this note book has been read repeatedly so that before printing, all sorts of mistakes or shortcomings can be overcome. In this regard, I am highly indebted to Prof. Rafique Bloach, Prof. Nadeem Iqbal, Prof. Farooq Khan, Prof. M. Farooq, Prof. Babar Zaheer, Dr. Zafar Iqbal, Prof. Tanveer Iqbal , Prof. Shafiq-ur-Rehman for exhaustive proof reading and giving their very valuable directions to keep the book according to the level of the students.

I am highly obliged to my worthy principle **Prof. Shaukat Ali** for his motivation and encouragment to write this book.

It is hoped that this book will serve the purpose well for which it has been compiled. I am a staunch believer of the fact that the students will certainly find a great boosting difference by comparing it with the other books.

Dedication

This book is dedicated to the sacred one Almighty who bestowed knowledge upon me, and endowed me with honour and esteem, and rendered me capacity and ability to toil and labour, no doubt I was ignorant and nameless.

To the Professors

This book will also be beneficial to our worthy teachers as this will make a speedy and quick overview to the lecture.

Moreover this will be helpful in pointing out and highlighting all the important definitions and questions.

The questions at the end of the chapter are of M.C Qs, short and long questions type. Studying the Concepts reviews the content of the chapter and requires that students write out their answers. "Testing your skills" of the questions.

All the convincing comments and patronizingly forwarded Suggestions will be thankfully entertained for making this Book more effective.

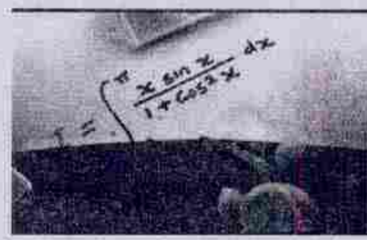
Farukh Mahmood

Tahir Kamran Ranjha

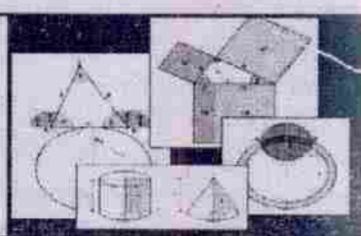
Contents

Unit 1
1
FUNCTIONS AND LIMITS

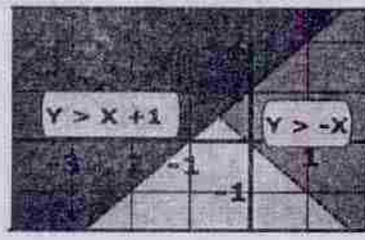
Unit 2
54
DIFFERENTIATION



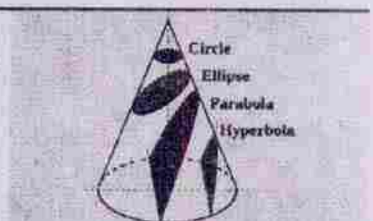
Unit 3
139
INTEGRATION



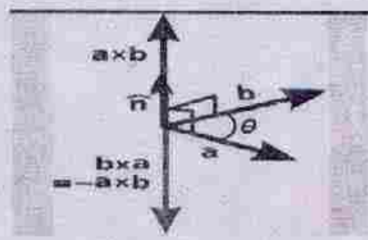
Unit 4
251
INTRODUCTION TO ANALYTIC GEOMETRY



Unit 5
326
LINEAR INEQUALITIES AND LINEAR PROGRAMMING



Unit 6
368
CONIC SECTION



Unit 7
482
VECTORS

Functions and Limits

1

Definitions:

1. Function:

A function is a rule that assigns to each element x in X a unique element y in Y.

Example: $A = x^2$ (A is a function of x)

2. Domain:

In a function f: x → y the set X is called the domain of f.

3. Range:

In a function f: x → y the set of corresponding elements y in Y is called the range of f.

4. Independent and dependent Variables

In y = f (x) , the variable x is called independent variable and y is called dependent variable of f.

5. Real valued Function:

If variables used in function are real numbers then function is called real valued function.

6. Algebraic Functions:

Functions which are defined by algebraic expressions.

7. Polynomial Function:

A function of the form $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1x + a_0$ where $a_n, a_{n-1}, a_{n-2} \ldots\ldots, a_1, a_0$ are real numbers and exponent are non–negative integers is called polynomial function.

8. Linear Function:

Sargodha 2011

If degree of polynomial function is one then it is called linear function.

Example: $f(x) = 3x + 4$

9. Identity Function: Sargodha 2011

For any set X, a function $f: X \to X$ of the form $I(x) = x$ is called identity function

Example : $f(x) = x$

10. Constant Function:

A function $C : X \to Y$ defined by $C(x) = a$ is called constant functions.

Example : $c(x) = 2$

11. Rational Function:

A function of the form $\frac{P(x)}{Q(x)}$ where P (x) and Q (x) are polynomial functions and $Q(x) \neq 0$ is called rational function.

12. Exponential Function:

A Function in which the variable appears as exponent is called exponential function.

Example: $y = e^x$, $y = 2^x$

13. Logarithmic Function:

If $x = a^y$ then $y = \log_a x$ $(a>0\ a \neq 1)$ is called logarithmic function. If a = 10 then $y = \log_{10} x$ is called common log. If a = e then $y = \log_e x = \ln x$ is called natural log.

14. Explicit Function: Sargodha 2008

If y is easily expressed in term of independent variable x then y is called explicit function.

Example: $y = x^2 + 2x - 1$

15. Implicit Function: Sargodha 2008

If y is not easily expressed in term of independent variable x then y is called implicit function.

Example: $x^2 + xy + y^2 = -1$

Important Formulas

1. Sin hx = $\dfrac{e^{x}-e^{-x}}{2}$

2. Cos hx = $\dfrac{e^{x}+e^{-x}}{2}$

3. Tan hx = $\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

4. Even Function f (–x) = f(x)

5. Odd Function f (–x) = –f(x)

6. Perimeter of Square = 4x

7. Area of Square = x^2

8. Area of Circle = πr^2

9. Circumference of Circle = $2\pi r$

10. Volume of cube = $V=x^3$

11. fog (x) = f (g(x))

12. $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=na^{n-1}$

13. $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{n}\right)^{n}=e$

14. $\lim\limits_{x\to+0}(1+x)^{1/n}=e$

15. $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log_e a=\ln a$

16. A function f is continuous at c if satisfy three conditions

i. f (c) is defined

ii. $\lim\limits_{x\to c} f(x)$ *exists*

iii. $\lim\limits_{x\to c} f(x)=f(c)$

17. Discontinuous if one or more above three conditions are not satisfied.

18. Limit Exist if L.H. Limit = R.H. Limit

Exercise 1.1

Q.1 Given that (a) $f(x) = x^2 - x$ (b) $f(x) = \sqrt{x+4}$,

Find (i) f(−2), (ii) f(0), (iii) f(x−1), (iv), $f(x^2+4)$

1. (a) $f(x) = x^2 - x$ —— I

(i) **Find f (−2)**

Put x = −2 in I

$f(-2) = (-2)^2 - (-2)$

$= 4 + 2 = 6$

(ii) **Find f (0)**

put x = 0 in I

$f(0) = (0)^2 - (0)$

$= 0 - 0 = 0$

(iii) **Find f(x−1)**

replace x by x −1 in I

$f(x-1) = (x-1)^2 - (x-1) = x^2 - 2x + 1 - x + 1$

$= x^2 - 3x + 2$

(iv) **Find $f(x^2+4)$** **Sargodha 2009**

replace x by (x^4+4) in I

$f(x^2+4) = (x^2+4)^2 - (x^2+4)$

$= x^4 + 8x^2 + 16 - x^2 - 4$

$= x^4 + 7x^2 + 12$

(b) $f(x) = \sqrt{x+4}$ ______ II

(i) **Find f (−2)**

put x = −2 in II

$f(-2) = \sqrt{-2+4} = \sqrt{2}$

(ii) **Find f (0)**

put x = 0 in II

$f(0) = \sqrt{0+4} = \sqrt{4} = 2$

(iii) **Find f (x – 1)**

replace x by x – 1 in II

$$f(x-1) = \sqrt{x-1+4}$$

$$= \sqrt{x+3}$$

(iv) **Find $f(x^2+4)$** **(Sargodha 2009)**

replace x by x^2+4

$$f(x^2+4) = \sqrt{x^2+4+4}$$

$$= \sqrt{x^2+8}$$

Q.2 **Find $\frac{f(a+h)-f(a)}{h}$ and simplify where (i) f(x) = 6x–9, (ii) f(x) = sin x ,**

(iii) $f(x)= x^3+2x^2-1$, (iv) f(x) = Cosx

2. **(i)** **f (x) = 6x – 9**

$$\frac{f(a+h)-f(a)}{h} = \frac{(6(a+h)-9)-(6a-9)}{h}$$

$$= \frac{6a+6h-9-6a+9}{h} = \frac{6h}{h} = 6$$

(ii) **f (x) = Sin x**

$$\frac{f(a+h)-f(a)}{h} = \frac{Sin(a+h)-Sin\,a}{h}$$

$$= \frac{2\cos\left(\frac{a+h+a}{2}\right)Sin\left(\frac{a+h-a}{2}\right)}{h}$$

$$= \frac{2}{h}\cos\left(\frac{2a+h}{2}\right)Sin\frac{h}{2}$$

$$= \frac{2}{h}Cos\left(a+\frac{h}{2}\right)Sin\frac{h}{2}$$

(iii) **$f(x) = x^3 + 2x^2 - 1$** **(Sargodha 2009)**

$f(a+h) = (a+h)^3 + 2(a+h)^2 - 1$

$= a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 2h^2 + 4ah - 1$

$f(a) = a^3 + 2a^2 - 1$

$$\frac{f(a+h)-f(a)}{h}$$

$$= \frac{a^3 + h^3 + 3a^2h + 3ah^2 + 2a^2 + 2h^2 + 4ah - 1 - a^3 - 2a^2 + 1}{h}$$

$$= \frac{h(h^2 + 3a^2 + 3ah + 2h + 4a)}{h}$$

$= h^2 + 3a^2 + 3ah + 2h + 4a$

(iv) $f(x) = \text{Cos}\, x$ **(Sargodha 2010)**

$$\frac{f(a+h)-f(a)}{h} = \frac{\text{Cos}(a+h)-\text{Cos}\,a}{h}$$

$$= \frac{1}{h}\left(-2\,\text{Sin}\frac{a+h+a}{2}\,\text{Sin}\frac{a+h-a}{2}\right)$$

$$= \frac{-2}{h}\text{Sin}\frac{2a+h}{2}\,\text{Sin}\frac{h}{2}$$

$$= \frac{-2}{h}\text{Sin}\left(a+\frac{h}{2}\right)\text{Sin}\frac{h}{2}$$

Q3. Express the following (a) The perimeter P of square as a function of its area A.

(b) The area A of a circle as a function of its circumference C.

(c) The volume V of a cube as a function of the area A of its base.

(a) Let each side of square be x then **(Lahore 2010)**

Perimeter = P = 4x ——— I

And Area = A = x. x = x^2 ——— II

From II $A = x^2 \Rightarrow \sqrt{x^2} = \sqrt{A} \Rightarrow x = \sqrt{A}$

Put in I $P = 4\sqrt{A}$

(b) Let r be the radius of circle then

Area = A = πr^2 ——— I

Circumference = C = $2\pi r$ ——— II

From II C = $2\pi r \Rightarrow r = \dfrac{c}{2\pi}$

Put in I

$$A = \pi\left(\frac{c}{2\pi}\right)^2 = \pi\left(\frac{c^2}{4\pi^2}\right) \Rightarrow A = \frac{c^2}{4\pi}$$

(c) Let each side of cube be x then

Volume = V = x. x. x = x^3

Area of Base = A = x. x. = x^2

From II $A = x^2 \Rightarrow x = \sqrt{A}$

Put in I $V = (\sqrt{A})^3 \Rightarrow V = \left(A^{\frac{1}{2}}\right)^3 \Rightarrow V = A^{\frac{3}{2}}$

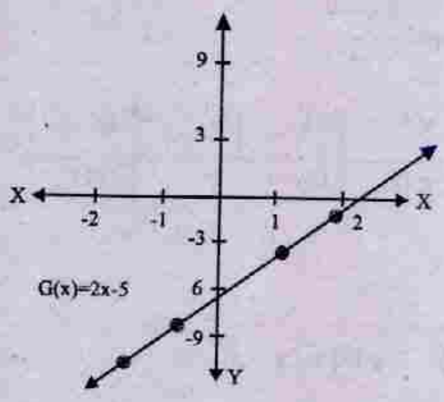
Q4. Find domain and range and sketch the graph.

4. (i) g(x) = 2x – 5

Domain = Set of real numbers

Range = Set of real numbers

For Graph



X	–2	–1	0	1	2
g (x)	–9	–7	–5	–3	–1

(ii) $g(x) = \sqrt{x^2 - 4}$

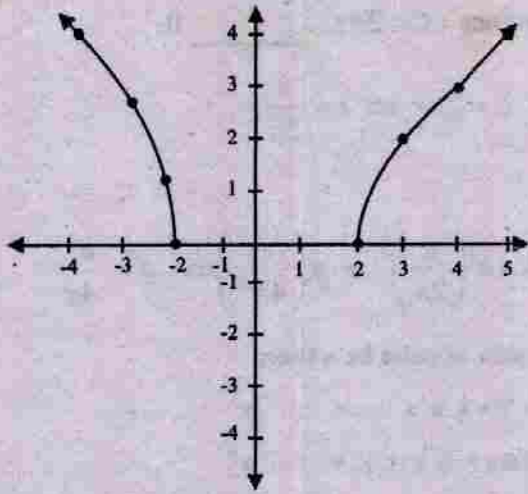
Domain = R – (–2, 2)

Range = [0, ∞]

We cannot put (–2, 2) because g (x) become imaginary in this interval.

Graph:

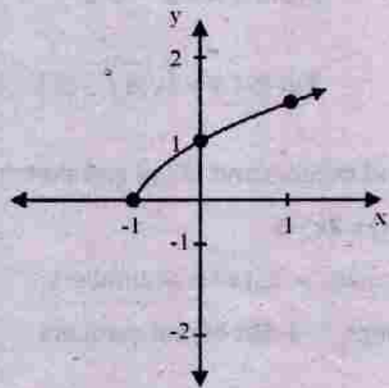
g (x)	3.46	2.23	0	0	2.23	3.46	4.58
X	–4	–3	–2	2	3	4	5



(iii) $g(x) = \sqrt{x+1}$

Domain = [−1, ∞]

Range = [0 , ∞]



Graph

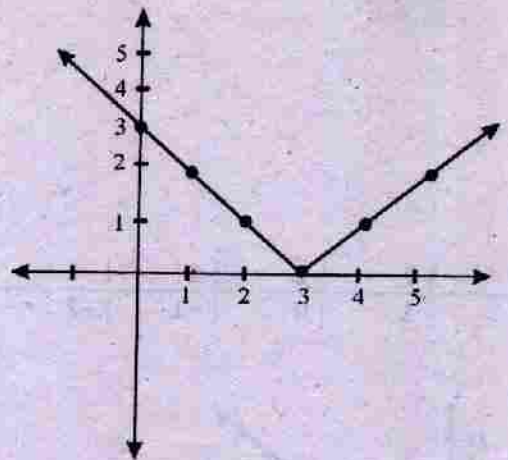
X	−1	0	1
g (x)	0	1	1.41

(iv) $g(x) = |x - 3|$

Domain = Set of real number

Range = [0 , ∞]

x	0	1	2	3	4	5
y	3	2	1	0	1	2

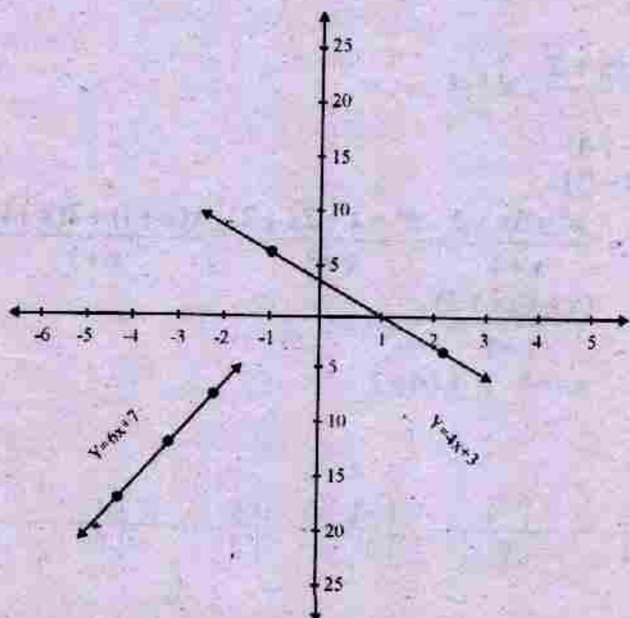


(v) $g(x) = \begin{cases} 6x+7, & x \leq -2 \\ 4-3x, & -2 < x \end{cases}$

Domain = $(-\infty, -5) \cup (10, +\infty)$

Range = $(-\infty, \infty)$

x	–5	–4	–3	–2	–1	0	1	2
y	–23	–17	–11	–5	7	4	1	–2

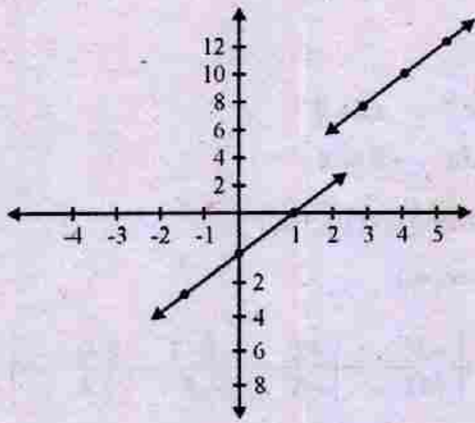


(vi) $g(x)=\begin{cases} x-1\ , & x<3 \\ 2x+1\ , & 3\ \text{£}\ x \end{cases}$

Domain = $(-\infty,\infty)$

Range = $(-\infty,2)\cup[7,x)$

x	5	4	3	2	1	0	−1
g(x)	11	9	7	1	0	−1	−2



(vii) $g(x)=\dfrac{x^2+3x+2}{x+1}\ , x\ ^{\prime}\ -1$

Domain = R − {−1}

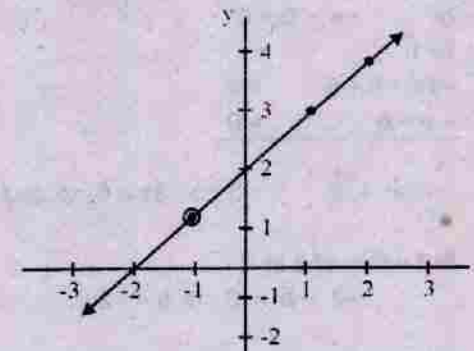
Range = R − {1}

Because $g(x) = \dfrac{x^2+3x+2}{x+1}=\dfrac{x^2+x+2x+2}{x+1}=\dfrac{x(x+1)+2(x+1)}{x+1}$

$g(x) = \dfrac{\cancel{(x+1)}(x+2)}{\cancel{x+1}}=x+2$

at $x=-1$, $g(x)=1$

x	−3	−2	−1	0	1	2
g(x)	−1	0	1	2	3	4



(viii) $g(x) = \dfrac{x^2 - 16}{x - 4}, x \neq 4$

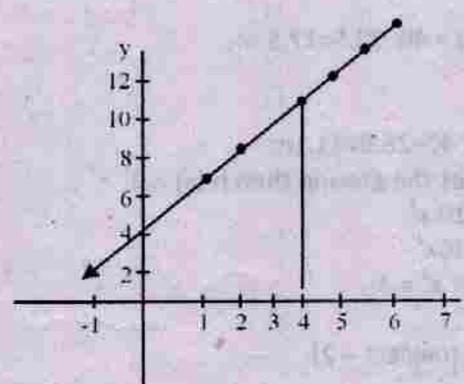
$g(x) = \dfrac{(x-4)(x+4)}{x-4} = x + 4$

When x = 4 then g (x) = 8

Domain = R – {4}

Range = R – {8}

x	−1	0	2	3	4	5
g (x)	3	4	6	7	8	9



Q.5 **If f(2)=−3 and f(−1)=0 Find the value of a and b**

Given $f(x) = x^3 - ax^2 + bx + 1$

if f (2) = −3 and f (−1) = 0

Now $f(2) = (2)^3 - a(2)^2 + b(2) + 1$

$-3 = 8 - 4a + 2b + 1$

or $-4a + 2b + 12 = 0$

÷ by 2 $-2a + b + 6 = 0$

Also $f(-1) = (-1)^3 - a(-1)^2 + b(-1) + 1$

$0 = -1 - a - b + 1$

or $\quad -a - b = 0$

I + II

$$\begin{aligned} -2a + b + 6 &= 0 \\ -a - b &= 0 \end{aligned}$$

$-3a + 6 = 0 \Rightarrow 3a = 6 \Rightarrow a = 2$

Put value of a in II

$-2 - b = 0 \Rightarrow b = -2$

Q.6 A Stone fall from a height of 60m on the ground , the height h after x second is approximately given by $h(x) = 40 - 10x^2$.

(i) What is the height of the stone when. (a) x = 1sec (b) x=1.5 sec (c) x=1.7 sec

(ii) When does the stone strike the ground?

(i) $h(x) = 40 - 10x^2$

(a) **x = 1 sec**

$h(1) = 40 - 10(1)^2$

$= 40 - 10 = 30\text{m}$

(b) **x = 1. 5 sec**

$h(1.5) = 40 - 10(1.5)^2$

$= 40 - 10(2.25) = 40 - 22.5 = 17.5\text{ m}$

(c) **x=1.7 sec**

$h(1.7) = 40 - 10(1.7)^2$

$= 40 - 10(2.89) = 40 - 28.9 = 11.1\text{m}$

(ii). When does stone strikes the ground then h (x) = 0

So
$$\begin{aligned} h(x) &= 40 - 10x^2 \\ 0 &= 40 - 10x^2 \\ 10x^2 &= 40 \Rightarrow x^2 = 4 \\ x &= \pm 2 \\ x &= 2 \text{ sec (neglect } -2) \end{aligned}$$

Q7. Show that the parametric equations :

(i) $x = at^2$, $y = 2at$ represent the equation of parabola $y^2 = 4ax$

Sol $\quad x = at^2 \rightarrow \text{I} \quad , \quad y = 2at \rightarrow \text{II}$

From II $\quad t = \dfrac{y}{2a}$

I become $\quad x = a\left(\dfrac{y}{2a}\right)^2 = \dfrac{y^2}{4a} \Rightarrow y^2 = 4ax$

$= a\left[\dfrac{y^2}{4a^2}\right] = \dfrac{y^2}{4a}$

(ii) x=acos θ , y = b sin θ represent the equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$x = a\cos\theta \rightarrow \text{I}$, $y = b\sin\theta \rightarrow \text{II}$

$\frac{x}{a} = \cos\theta$, $\frac{y}{b} = \sin\theta$

$\frac{x^2}{a^2} = \cos^2\theta$, $\frac{y^2}{b^2} = \sin^2\theta$

Adding

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta = 1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(iii) x=a sec θ , y =b tan θ represent the equation of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$x = a\sec\theta$, $y = b\tan\theta$

$\frac{x}{a} = \sec\theta$, $\frac{y}{b} = \tan\theta$

$\frac{x^2}{a^2} = \sec^2\theta \rightarrow \text{I}$, $\frac{y^2}{b^2} = \tan^2\theta \rightarrow \text{II}$

I – II

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2\theta - \tan^2\theta$$

$$= 1 + \tan^2\theta - \tan^2\theta \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

M.S.Q

Note:
$\sec^2\theta = 1+\tan^2\theta$
$\sin hx = \frac{e^x - e^{-x}}{2}$
$\cos hx = \frac{e^x + e^{-x}}{2}$
$\tan hx = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Q8. **Prove the identities:**

(i) **Sin h2x = 2Sin hx Cos hx**

R.H.S = 2 Sin hx Cos hx

$$= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = \frac{e^{2x} - e^{-2x}}{2} = \sin h2x = L.H.S$$

(ii) Sec h^2x = $1 - \tan h^2 x$

R.H.S = $1 - \tan h^2 x$

$$= 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2$$

$$= 1 - \frac{\left(e^x - e^{-x}\right)^2}{\left(e^x + e^{-x}\right)^2}$$

M.S.Q

Note:
$e^x . e^{-x} = e^{x-x} = e^0 = 1$

$$= \frac{(e^{2x}+e^{-2x}+2e^{x}.e^{-x})-(e^{2x}+e^{-2x}-2e^{x}.e^{-x})}{(e^{x}+e^{-x})^2}$$

$$= \frac{e^{2x}+e^{-2x}+2-e^{2x}-e^{-2x}+2}{(e^{x}+e^{-x})^2} = \frac{4}{(e^{x}+e^{-x})^2} = \left(\frac{2}{e^{x}+e^{-x}}\right)^2$$

$$= \frac{1}{\left(\frac{e^{x}+e^{-x}}{2}\right)^2} = \frac{1}{\cosh^2 x} = \sec h^2 x = L.H.S$$

(iii) **Cosec h^2x = Cot $h^2x - 1$**

R.H.S = Cot h^2 $x - 1$

$$= \frac{(e^{x}+e^{-x})^2}{(e^{x}-e^{-x})^2} - 1 = \frac{(e^{x}+e^{-x})^2}{(e^{x}-e^{-x})^2} - 1$$

$$= \frac{(e^{x}+e^{-x})^2-(e^{x}-e^{-x})^2}{(e^{x}-e^{-x})^2}$$

$$= \frac{(e^{2x}+e^{-2x}+2e^{x}.e^{-x})-(e^{2x}+e^{-2x}-2e^{x}.e^{-x})}{(e^{x}-e^{-x})^2}$$

$$= \frac{e^{2x}+e^{-2x}+2.1-e^{2x}-e^{-2x}+2.1}{(e^{x}-e^{-x})^2}$$

$$= \frac{4}{(e^{x}-e^{-x})^2} = \left(\frac{2}{e^{x}-e^{-x}}\right)^2$$

$$= \frac{1}{\left(\frac{e^{x}-e^{-x}}{2}\right)^2} = \frac{1}{\sinh^2 x} = Cosec\, h^2 x = L.H.S$$

Q9. Determine whether the given function f is even or odd.

(i) **f(x) = $x^3 + x$**

Replace x by –x

$f(-x) = (-x)^3 + (-x)$

$= -x^3 - x$

$= -(x^3 + x) = -f(x)$

f (x) is odd function

> **Note:**
> Even $f(-x) = f(x)$
> Odd $f(-x) = -f(x)$

(ii) $f(x) = (x+2)^2$

Replace x by −x

$F(-x) = (-x+2)^2 \neq f(x)$

Neither Even nor Odd

(iii) $f(x) = x\sqrt{x^2+5}$

Replace x by −x

$f(-x) = -x\sqrt{(-x)^2+5}$

$= -(x\sqrt{x^2+5}) = -f(x)$

f (x) is odd function

(iv) $f(x) = \dfrac{x-1}{x+1}$

Replace x by −x

$f(-x) = \dfrac{-x-1}{-x+1} = \dfrac{-(x+1)}{1-x} \neq f(x)$

Neither Even nor Odd

(v) $f(x) = x^{\frac{2}{3}} + 6$

Replace x by −x

$f(-x) = (-x)^{\frac{2}{3}} + 6$

$= [(-x)^2]^{\frac{1}{3}} + 6$

$= (x^2)^{1/3} + 6 = x^{2/3} + 6 = f(x)$

f(x) is Even function

(vi) $f(x) = \dfrac{x^3-x}{x^2+1}$ **(Sargodha 2007)**

Replace x by −x

$f(-x) = \dfrac{(-x)^3-(-x)}{(-x)^2+1} = \dfrac{-x^3+x}{x^2+1}$

$= -\left(\dfrac{x^3-x}{x^2+1}\right) = -f(x)$

f (x) is odd function

Exercise 1.2

Q.1 **The real valued functions f and g are defined below. Find (a) fog(x) (b)gof(x) (c) fof(x) (d) gog(x)**

(i) $f(x) = 2x + 1,\ g(x) = \dfrac{3}{x-1},\ x \neq 1$

(a) $fog(x) = f(g(x)) = f\left(\dfrac{3}{x-1}\right)$ **(Sargodha 2012)**

$$= 2\left(\frac{3}{x-1}\right) + 1 = \frac{6}{x-1} + 1$$

$$= \frac{6+x-1}{x-1} = \frac{5+x}{x-1}$$

(b) $gof(x) = g(f(x)) = g(2x+1)$

$$= \frac{3}{2x+1-1} = \frac{3}{2x}$$

(c) $fof(x) = f(f(x)) = f(2x+1)$

$$= 2(2x+1) + 1 = 4x + 2 + 1$$

$$= 4x + 3$$

(d) $gog(x) = g(g(x)) = g\left(\dfrac{3}{x-1}\right)$

$$= \frac{3}{\frac{3}{x-1} - 1} = \frac{3}{\frac{3-x+1}{x-1}} = \frac{3}{\frac{4-x}{x-1}} = \frac{3(x-1)}{4-x}$$

(ii) $f(x) = \sqrt{x+1},\ g(x) = \dfrac{1}{x^2}$

(a) $fog(x) = f(g(x)) = f\left(\dfrac{1}{x^2}\right)$

$$= \sqrt{\frac{1}{x^2} + 1} = \sqrt{\frac{1+x^2}{x^2}} = \frac{\sqrt{1+x^2}}{x}$$

(b) $gof(x) = g(f(x))$

$$= g\left(\sqrt{x+1}\right) = \frac{1}{\left(\sqrt{x+1}\right)^2} = \frac{1}{x+1}$$

(c) $fof(x) = f(f(x)) = f\left(\sqrt{x+1}\right) = \sqrt{\sqrt{x+1} + 1}$

(d) $gog(x) = g(g(x)) = g\left(\dfrac{1}{x^2}\right)$

$$= \frac{1}{\left(\frac{1}{x^2}\right)^2} = \frac{1}{\frac{1}{x^4}} = x^4$$

(iii) $f(x) = \frac{1}{\sqrt{x-1}}, x \neq 1, g(x) = (x^2+1)^2$

(a) $fog(x) = f(g(x)) = f((x^2+1)^2)$

$$= \frac{1}{\sqrt{(x^2+1)^2 - 1}} = \frac{1}{\sqrt{x^4+2x^2+1-1}} = \frac{1}{\sqrt{x^4+2x^2}}$$

(b) $gof(x) = g(f(x))$

$$= g\left(\frac{1}{\sqrt{x-1}}\right) = \left[\left(\frac{1}{\sqrt{x-1}}\right)^2 + 1\right]^2$$

$$= \left(\frac{1}{x-1} + 1\right)^2 = \left(\frac{1+x-1}{x-1}\right)^2 = \left(\frac{x}{x-1}\right)^2$$

(c) $fof(x) = f(f(x)) = f\left(\frac{1}{\sqrt{x-1}}\right)$

$$= \frac{1}{\sqrt{\frac{1}{\sqrt{x-1}} - 1}} = \frac{1}{\frac{\sqrt{1-\sqrt{x-1}}}{\sqrt{\sqrt{x-1}}}} = \frac{(x-1)^{1/4}}{\sqrt{1-\sqrt{x-1}}}$$

(d) $gog(x) = g(g(x)) = g((x^2+1)^2)$

$$= [((x^2+1)^2)^2+1]^2 = ((x^2+1)^4+1)^2$$

(iv) $f(x) = 3x^4 - 2x^2, g(x) = \frac{2}{\sqrt{x}}, x \neq 0$

(a) $fog(x) = f(g(x)) = f\left(\frac{2}{\sqrt{x}}\right)$

$$= 3\left(\frac{2}{\sqrt{x}}\right)^4 - 2\left(\frac{2}{\sqrt{x}}\right)^2 = 3\left(\frac{16}{x^2}\right) - 2\left(\frac{4}{x}\right)$$

$$= \frac{48}{x^2} - \frac{8}{x} = \frac{48-8x}{x^2} = \frac{8(6-x)}{x^2}$$

(b) $gof(x) = g(f(x)) = g(3x^4 - 2x^2)$

$$= \frac{2}{\sqrt{3x^4-2x^2}} = \frac{2}{\sqrt{x^2(3x^2-2)}} = \frac{2}{x\sqrt{3x^2-2}}$$

(c) fof (x) $= f(f(x))$
$= f(3x^4 - 2x^2)$
$= 3(3x^4-2x^2)^4 - 2(3x^4-2x^2)^2$

(d) $gog(x) = g(g(x)) = g\left(\frac{2}{\sqrt{x}}\right)$

$$= \frac{2}{\sqrt{\frac{2}{\sqrt{x}}}} = \frac{2}{\frac{\sqrt{2}}{\sqrt{\sqrt{x}}}} = \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}} \times (x)^{1/4} = \sqrt{2}\, x^{\frac{1}{4}}$$

2. For the real valued function f defined below, find (a) $f^{-1}(x)$,(b) $f^{-1}(-1)$ and verify $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(i) $f(x) = -2x + 8$ Sargodha 2008

Let $f(x) = y \Rightarrow x = f^{-1}(y)$ ------------- I

Then $y = -2x + 8 \Rightarrow y - 8 = -2x$

$= -y + 8 = 2x \Rightarrow x = \frac{8-y}{2}$

Replace y by x

$$f^{-1}(x) = \frac{8-x}{2}$$

put $x = -1$

$$f^{-1}(-1) = \frac{8-(-1)}{2} = \frac{8+1}{2} = \frac{9}{2}$$

verification $f(f^{-1}(x)) = f\left(\frac{8-x}{2}\right)$

$$= -2\left(\frac{8-x}{2}\right) + 8 = -8 + x + 8 = x$$

= Again $f^{-1}(f(x)) = f^{-1}(-2x+8)$

$$= \frac{8-(-2x+8)}{2} = \frac{8+2x-8}{2} = \frac{2x}{2} = x$$

Hence $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(ii) $f(x) = 3x^3 + 7$

Take $f(x) = y \Rightarrow x = f^{-1}(y)$) ________ I

Then $y = 3x^3 + 7 \Rightarrow y - 7 = 3x^3$

$$\frac{y-7}{3} = x^3 \Rightarrow x = \left(\frac{y-7}{3}\right)^{\frac{1}{3}} \text{ ________ II}$$

Compare I and II

$$f^{-1}(y) = \left(\frac{y-7}{3}\right)^{1/3}$$

Replace y by x

$$f^{-1}(x) = \left(\frac{x-7}{3}\right)^{1/3}$$

put $x=-1$, $f^{-1}(-1) = \left(\frac{-1-7}{3}\right)^{\frac{1}{3}} = \left(\frac{-8}{3}\right)^{1/3}$

Verification:

$$f(f^{-1}(x)) = f\left(\left(\frac{x-7}{3}\right)^{\frac{1}{3}}\right) = 3\left(\frac{x-7}{3}\right)^{3\times\frac{1}{3}} + 7$$

$$= \not{3}\left(\frac{x-7}{\not{3}}\right) + 7 = x - \not{7} + \not{7} = x$$

And $f^{-1}(f(x)) = f^{-1}(3x^3 + 7)$

$$= \left(\frac{3x^3 + \not{7} - \not{7}}{3}\right)^{1/3} = \left(\frac{\not{3}x^3}{\not{3}}\right)^{1/3} = x^{3\times\frac{1}{3}} = x$$

Hence $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(iii). $f(x) = (-x+9)^3$ **(Lahore 2010)**

Let $f(x) = y \Rightarrow x = f^{-1}(y)$ ______ I

Then $y = (-x+9)^3 \Rightarrow y^{1/3} = -x+9$

$\Rightarrow x = 9 - y^{1/3}$ ______ II

Compare I and II

$f^{-1}(y) = 9 - y^{1/3} \Rightarrow f^{-1}(x) = 9 - x^{1/3}$ (Replace y by x)

Put $x=-1$, $f^{-1}(-1) = 9 - (-1)^{1/3}$

Verification:

$f(f^{-1}(x)) = f(9-x^{1/3})$

$= (-((9-x^{1/3})+9)^3$

$= (-\not{9} + x^{1/3} + \not{9})^3 = x\, x^{3\times\frac{1}{3}} = x$

Also $f^{-1}(f(x)) = f^{-1}((-x+9)^3)$

$= 9-[(-x+9)^3]^{1/3} = 9 - (-x+9)$

$= \not{9} + x - \not{9} = x$

Hence $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

(iv) $f(x) = \dfrac{2x+1}{x-1}$ (Sargodha 2011)

Let $f(x) = y \Rightarrow x = f^{-1}(y)$ ________ I

Then $y = \dfrac{2x+1}{x-1} \Rightarrow y(x-1) = 2x+1$

$yx - y = 2x + 1 \Rightarrow yx - 2x = y + 1$

$x(y-2) = y+1 \quad \Rightarrow x = \dfrac{y+1}{y-2}$ ________ II

Compare I and II

$f^{-1}(y) = \dfrac{y+1}{y-2}$ Replace y by x

$f^{-1}(x) = \dfrac{x+1}{x-2}$

Put $x = -1 \quad \Rightarrow f^{-1}(-1) = \dfrac{-1+1}{-1-2} = \dfrac{0}{-3} = 0$

Verification;

$$f(f^{-1}(x)) = f\left(\frac{x+1}{x-2}\right)$$

$$= \frac{2\left(\frac{x+1}{x-2}\right)+1}{\frac{x+1}{x-2}-1} = \frac{\frac{2x+2+x-2}{x-2}}{\frac{x+1-x+2}{x-2}}$$

$$= \frac{3x}{x-2} \times \frac{x-2}{3} = x$$

Now $f^{-1}(f(x)) = f^{-1}\left(\dfrac{2x+1}{x-1}\right)$

$$= \frac{\left(\frac{2x+1}{x-1}\right)+1}{\frac{2x+1}{x-1}-2} = \frac{\frac{2x+1+x-1}{x-1}}{\frac{2x+1-2x+2}{x-1}}$$

$$= \frac{3x}{x-1} \times \frac{x-1}{3} = x$$

Hence $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

3. without finding the inverse, state the domain and range of f^{-1}

(i) $f(x) = \sqrt{x+2}$

Domain of f(x) = $[-2, \infty)$
Range of f(x) = $[0, \infty)$
Domain of $f^{-1}(x) = [0, \infty)$
Range of $f^{-1}(x) = [-2, \infty)$

(ii) $f(x) = \dfrac{x-1}{x-4},\ x \neq 4$

Domain of f(x) = $R - \{4\}$
Range Of f(x) = $R - \{1\}$
Domain of $f^{-1}(x) = R - \{1\}$
Range of $f^{-1}(x) = R - \{4\}$

(iii) $f(x) = \dfrac{1}{x+3},\ x \neq -3$

Domain of f(x) = $R - \{-3\}$
Range of f(x) = $R - \{0\}$
Domain of $f^{-1}(x) = R - \{0\}$
Range of $f^{-1}(x) = R - \{-3\}$

(iv) $f(x) = (x-5)^2$

Domain of f(x) = $[5, \infty]$
Range of f(x) = $[0, \infty]$
Domain of $f^{-1}(x) = [0, \infty]$
Range of $f^{-1}(x) = [5, \infty]$

Theorem : Prove that $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x-a} = na^{n-1}\ ,\ n \in Z$

Proof : case I (when n is +ve)

We know that $x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots\ldots + a^{n-1})$

Then $$\lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a} \frac{(x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots\ldots + a^{n-1})}{(x-a)}$$

$$= \lim_{x\to a} (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots\ldots + a^{n-1})$$

$$= a^{n-1} + a.a^{n-2} + a^2.a^{n-3} + \ldots\ldots + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \ldots\ldots + a^{n-1} (\text{n time}) = na^{n-1}$$

Thus $$\lim_{x\to a} \frac{x^n - a^n}{x-a} = na^{n-1} \longrightarrow I$$

Case II Take n –ve So n = –m

Then $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=\lim\limits_{x\to a}\frac{x^{-m}-a^{-m}}{x-a}=\lim\limits_{x\to a}\frac{1}{(x-a)}\left[\frac{1}{x^m}-\frac{1}{a^m}\right]$

$$=\lim_{x\to a}\frac{1}{(x-a)}\left[\frac{a^m-x^m}{x^m\,a^m}\right]=\lim_{x\to a}\frac{-1(x^m-a^m)}{x^m\,a^m(x-a)}$$

$$=\lim\frac{-1}{x^m.a^m}\times\lim\frac{x^m-a^m}{x-a}$$

$$=\frac{-1}{a^m.a^m}\times ma^{m-1}(use\,I)=\frac{-ma^{m-1}}{a^{2m}}$$

$$=-ma^{m-1-2m}=-ma^{-m-1}$$

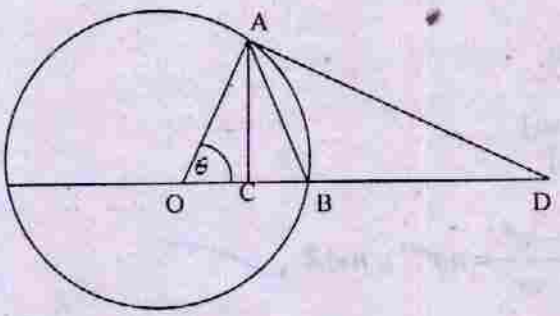
$\Rightarrow \lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$ (replace $-m$ by n) $\longrightarrow II$

By I and II we can calculate that

$$=\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

Theorem : **Prove that $\lim\limits_{\theta\to 0}\frac{Sin\theta}{\theta}=1$**

Proof : Draw a unit circle (radius I) in which



Area of triangle OAB $\angle$ *Area of sector OAB* $\angle$ Area of triangle OAD $\rightarrow I$

Now Area of triangle OAB $=\frac{1}{2}(base)(perpendicular)$

$=\frac{1}{2}|OB||AC|$ where $\left|\frac{AC}{OA}\right|=Sin\theta$

$=\frac{1}{2}(1)(\sin\theta)$ $|AC|=|OA|Sin\theta$

$=\frac{1}{2}Sin\theta\rightarrow II$ $|AC|=(1)Sin\theta$

Radius = $|OA| = |OB| = 1$

Area of sector OAB $= \frac{1}{2} r^2\theta = \frac{1}{2}(1)^2(\theta) = \frac{1}{2}\theta \longrightarrow III$

Area of triangle OAD = $\frac{1}{2}$ (Base) (perpendicular)

$= \frac{1}{2}|OA||AD| \qquad where \left|\frac{AD}{OA}\right| = Tan\theta$

$= \frac{1}{2}(1)(\tan\theta) \longrightarrow IV \qquad |AD| = (1)Tan\theta$

Put II, III, IV in I $\qquad |AD| = Tan\theta$

$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$

or $\quad \sin\theta < \theta < \tan\theta$ ('x' by 2)

or $\quad \frac{Sin\theta}{Sin\theta} < \frac{\theta}{Sin\theta} < \frac{Sin\theta}{Cos\theta} \times \frac{1}{Sin\theta} (\div by\ Sin\theta)$

or $\quad 1 < \frac{\theta}{Sin\theta} < \frac{1}{Cos\theta}$

Take reciprocal and limit $\theta \to 0$

$= \lim_{\theta\to 0}(1) > \lim_{\theta\to 0}\frac{Sin\theta}{\theta} > \lim_{\theta\to 0}\frac{Cos\theta}{1}$

$= 1 > \lim_{\theta\to 0}\frac{Sin\theta}{\theta} > 1 \Rightarrow \lim_{\theta\to 0}\frac{Sin\theta}{\theta} = 1$ (Applying sandwich theorem)

Theorem: Prove that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ **(Sargodha 2007)**

Proof : We know by Binomial Series that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots$$

So $$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n^3}\right) + \ldots\ldots$$

Therefore $$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1 + 1 + \frac{n^2\left(1-\frac{1}{n}\right)}{2}\frac{1}{n^2} + \frac{n^3\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3.2.1}\frac{1}{n^3} + \ldots\ldots\right)$$

$$= \lim_{n\to\infty}\left(2+\frac{1}{2}\left(1-\frac{1}{n}\right)+\frac{1}{6}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\ldots\ldots\right)$$

When n $\to \infty$ then $\frac{1}{n},\frac{2}{n},\ldots\ldots \to 0$ So

$$= \left(2+\frac{1}{2}(1-0)+\frac{1}{6}(1-0)(1-0)+\ldots\ldots\right)$$

$$= 2+0.5+0.16+\ldots\ldots$$

Hence $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

Theorem : Prove that $\lim_{x\to 0}\frac{a^x-1}{x} = \log_e a$ **(Sargodha 2010)**

Proof: Put $a^x - 1 = y \Rightarrow a^x = 1+y \Rightarrow x = \log_a(1+y)$

Also when $x \to 0$ Then $y \to 0$

So $\lim_{x\to 0}\frac{a^x-1}{x} = \lim_{y\to 0}\frac{y}{\log_a(1+y)} = \lim_{y\to 0}\frac{1}{\frac{1}{y}\log_a(1+y)}$

$= \lim_{y\to 0}\frac{1}{\log_a(1+y)^{1/y}} = \frac{1}{\log_a e} = \log_e a = \ell na$

> ***Note:***
> $a^x = 1+y$
> $\log a^x = \log(1+y)$
> $x \log a = \log(1+y)$
> $x = \frac{\log_{10}(1+y)}{\log_{10} a}$
> $x = \log_{10} a \times \log_{10}(1+y)$
> $x = \log_a(1+y)$

> ***Note:***
> *use* $\lim_{y\to\infty}(1+y)^{1/y} = e$

Exercise 1.3

Q1. Evaluate each limit by using theorems of limits:

1.(i) $\lim\limits_{x\to 3}(2x+4)$

$$= \lim_{x\to 3} 2x + \lim_{x\to 3} 4$$
$$= 2\lim_{x\to 3}(x) + (4)$$
$$= 2(3) + 4 = 6 + 4 = 10$$

(ii) $\lim\limits_{x\to 1}(3x^2-2x+4)$

$$= 3\lim_{x\to 1} x^2 - 2\lim_{x\to 1}(x) + \lim_{x\to 1} 4$$
$$= 3(1)^2 - 2(1) + 4 = 3 - 2 + 4 = 5$$

(iii) $\lim\limits_{x\to 3}\sqrt{x^2+x+4}$

$$= \sqrt{(3)^2+3+4} = \sqrt{9+7} = \sqrt{16} = 4$$

(iv) $\lim\limits_{x\to 2} \times\sqrt{x^2-4}$

$$= \lim_{x\to 2} x > \lim_{x\to 2}\sqrt{x^2-4}$$
$$= 2\times\sqrt{(2)^2-4} = 2\times\sqrt{4-4} = 2\times 0 = 0$$

(v) $\lim\limits_{x\to 2}\left(\sqrt{x^3+1}-\sqrt{x^2+5}\right)$

$$= \lim_{x\to 2}\sqrt{x^3+1} - \lim_{x\to 2}\sqrt{x^2+5}$$
$$= \sqrt{(2)^3+1} - \sqrt{(2)^2+5}$$
$$= \sqrt{9} - \sqrt{9} = 0$$

(vi) $\lim\limits_{x\to 2}\dfrac{2x^3+5x}{3x-2}$

$$= \frac{\lim\limits_{x\to -2}(2x^3+5x)}{\lim\limits_{x\to -2}(3x-2)} = \frac{2(-2)^{+3}+5(-2)}{3(-2)-2}$$
$$= \frac{-16-10}{-6-2} = \frac{-26}{-8} = \frac{13}{4}$$

Q2. Evaluate each limit by using algebraic techniques.

(i) $\lim\limits_{x\to -1}\dfrac{x^3-x}{x+1}$

$$= \lim_{x\to -1} \frac{x(x^2-1)}{x+1}$$

$$= \lim_{x\to -1} \frac{x(x-1)(x+1)}{(x+1)} = \lim_{x\to -1} x(x-1)$$

$$= -1(-1-1) = -1(-2) = 2$$

(ii) $\lim\limits_{x\to 1} \dfrac{4x^3+4x}{x^2+x}$

$$= \frac{\lim\limits_{x\to 1}(4x^3+4x)}{\lim\limits_{x\to 1}(x^2+x)} = \frac{4(1)^3+4(1)}{(1)^2+1} = \frac{8}{2} = 4$$

(iii) $\lim\limits_{x\to 2} \dfrac{x^3-8}{x^2+x-6}$

$$= \lim_{x\to 2} \frac{x^3-(2)^3}{x^2+3x-2x-6}$$

$$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x(x+3)-2(x+3)}$$

$$= \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x+3)}$$

$$= \lim_{x\to 2} \frac{x^2+2x+4}{x+3}$$

$$= \frac{(2)^2+2(2)+4}{2+3} = \frac{4+4+4}{5} = \frac{12}{5}$$

(iv) $\lim\limits_{x\to 1} \dfrac{x^3-3x^2+3x-1}{x^3-x}$

$$= \lim_{x\to 1} \frac{x^3-1-3x^2+3x}{x(x^2-1)}$$

$$= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)-3x(x-1)}{x(x-1)(x+1)}$$

$$= \lim_{x\to 1} \frac{(x-1)(x^2+x+1-3x)}{x(x-1)(x+1)}$$

$$= \lim_{x\to 1} \frac{x^2+1-2x}{x(x+1)}$$

$$= \frac{1+1-2}{1(1+1)} = \frac{2-2}{2} \frac{0}{2} = 0$$

(v) $\lim\limits_{x\to -1} \dfrac{x^3+x^2}{x^2-1}$

$$= \lim_{x\to -1} \frac{x^2(x+1)}{(x-1)(x+1)}$$

$$= \lim_{x\to -1} \frac{x^2}{x-1} = \frac{(-1)^2}{-1-1} = \frac{1}{-2}$$

(vi) $\lim\limits_{x\to 4} \dfrac{2x^2-32}{x^3-4x^2}$ **Sargodha 2012**

$$= \lim_{x\to 4} \frac{2(x^2-16)}{x^2(x-4)}$$

$$= \lim_{x\to 4} \frac{2(x-4)(x+4)}{x^2(x-4)}$$

$$= \lim_{x\to 4} \frac{2(x+4)}{x^2} = \frac{2(4+4)}{(4)^2} = \frac{16}{16} = 1$$

(vii) $\lim\limits_{x\to 2} \dfrac{\sqrt{x}-\sqrt{2}}{x-2}$ **Sargodha 2011**

$$= \lim_{x\to 2} \frac{\sqrt{x}-\sqrt{2}}{x-2} \times \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}}$$

$$= \lim_{x\to 2} \frac{\left(\sqrt{x}\right)^2-\left(\sqrt{2}\right)^2}{(x-2)(\sqrt{x}+\sqrt{2})}$$

$$= \lim_{x\to 2} \frac{(x-2)}{(x-2)(\sqrt{x}+\sqrt{2})}$$

$$= \lim_{x\to 2} \frac{1}{\sqrt{x}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

(viii) $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ **(Faisalabad 2011)**

$$= \lim_{h\to 0} \frac{\sqrt{x-h}-\sqrt{x}}{h} \times \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x+0}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

(ix) $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x^m - a^m}$

$$= \frac{\lim\limits_{x\to a} \dfrac{x^n - a^n}{x-a}}{\lim\limits_{x\to a} \dfrac{x^m - a^m}{x-a}}$$ (÷ Num , and dino by (x –a))

$$= \frac{na^{n-1}}{ma^{m-1}} (Use\,Theorem)$$

$$= \frac{n}{m} a^{n-1-m+1}$$

$$= \frac{n}{m} a^{n-m}$$

Q3. Evaluate the following limits.

(i) $\lim\limits_{x\to a} \dfrac{Sin7x}{x}$

$$= \lim_{x\to 0} \frac{Sin7x}{x} \quad (\text{"x" \& "÷" } by\, 7)$$

$$= 7\left(\lim_{x\to 0} \frac{Sin7x}{7x}\right) = 7(1) = 7$$

Note:
$$x^0 = x \times 1^0 = x \times \frac{\pi}{180}$$
$$= \frac{x\pi}{180}$$

(ii) $\lim\limits_{x\to 0} \dfrac{Sinx^\theta}{x} = \lim\limits_{x\to 0} \dfrac{Sin\dfrac{x\pi}{180}}{x}$

$$= \lim_{x\to 0} \frac{\dfrac{Sin\pi x}{180}}{\dfrac{x\pi}{180}} \times \frac{\pi}{180} \quad (\text{"×" \& "÷" } by \frac{\pi}{180})$$

$$= \frac{\pi}{180} \lim_{x\to 0} \frac{Sin\dfrac{\pi x}{180}}{\dfrac{\pi x}{180}} = \frac{\pi}{180}(1) = \frac{\pi}{180}$$

(iii) $\lim_{x\to 0}\frac{1-Cos\theta}{Sin\theta}$

$1-Cos2\theta = 2Sin^2\theta$

$Sin2\theta = 2Sin\theta \cdot Cos\theta$

$$=\lim_{\theta\to 0}\frac{2Sin^2\left(\frac{\theta}{2}\right)}{2Sin\left(\frac{\theta}{2}\right)Cos\left(\frac{\theta}{2}\right)}$$

$$=\frac{\lim_{\theta\to 0}Sin\left(\frac{\theta}{2}\right)}{\lim_{\theta\to 0}Cos\left(\frac{\theta}{2}\right)}=\frac{Sin0}{Cos0}=\frac{0}{1}=0$$

(iv) $\lim_{x\to\pi}\frac{Sinx}{\pi-x}$

Put $\pi-x=\theta \Rightarrow x=\pi-\theta$

When $x\to\pi$ then $\theta\to 0$

$$=\lim_{\theta\to 0}\frac{Sin(\pi-\theta)}{\theta}=\lim_{\theta\to 0}\frac{Sin\pi.Cos\theta-Cos\pi.Sin\theta}{\theta}$$

$Sin(\alpha-\beta)$

$$=\lim_{\theta\to 0}\frac{0.Cos\theta-(-1)Sin\theta}{\theta}=\lim_{\theta\to 0}\frac{Sin\theta}{\theta}=1$$

$\frac{0.Cos\theta+1.Sin\theta}{\theta}=\frac{Cos\theta+Sin\theta}{\theta}=\frac{1}{\theta}=1$

(v) $\lim_{x\to 0}\frac{Sin\,ax}{Sin\,bx}$ Sargodha 2008

$$=\frac{\lim_{x\to 0}\frac{Sin\,ax}{ax}\times ax}{\lim_{x\to 0}\frac{Sin\,bx}{bx}\times bx}$$ ("×"&"÷" *Numerator by ax and Dinumerator by bx*)

$$=\frac{a.\lim_{x\to 0}\frac{Sin\,ax}{ax}}{b.\lim_{x\to 0}\frac{Sin\,bx}{bx}}=\frac{a(1)}{b(1)}=\frac{a}{b}$$

(vi) $\lim_{x\to 0}\frac{x}{Tan\,x}=\lim_{x\to 0}\frac{x}{\frac{Sin\,x}{Cos\,x}}$

$$=\lim_{x\to 0}x\times\frac{Cos\,x}{Sin\,x}$$

$$=\lim_{x\to 0}\frac{x}{Sin\,x}\times\lim_{x\to 0}Cos\,x$$

$$= \frac{1}{\lim\limits_{x\to 0}\frac{Sin\,x}{x}} \times \lim_{x\to 0} Cos\,x$$

$$= \frac{1}{1} \times 1 = 1$$

(vii) $\lim\limits_{x\to 0}\dfrac{1-Cos\,2x}{x^2}$

$$= \lim_{x\to 0}\frac{2Sin^2 x}{x^2} = 2\left(\lim_{x\to 0}\frac{Sinx}{x}\right)^2$$

$$= 2(1)^2 = 2$$

(viii) $\lim\limits_{x\to 0}\dfrac{1-Cos\,x}{Sin^2 x}$ **Sargodha 2008**

$$= \lim_{x\to 0}\frac{1-Cos\,x}{1-Cos^2 x}$$

$$= \lim_{x\to 0}\frac{1-Cos\,x}{(1-Cos\,x)(1+Cos\,x)}$$

$$= \lim_{x\to 0}\frac{1}{1+Cos\,x} = \frac{1}{1+Cos\,0} = \frac{1}{1+1} = \frac{1}{2}$$

(ix) $\lim\limits_{\theta\to 0}\dfrac{Sin^2\theta}{\theta}$ **Sargodha 2009**

$$= \lim_{\theta\to 0}\frac{Sin^2\theta}{\theta^2}\times\theta \qquad (\text{"}\times\text{" \& "}\div\text{" by } \theta)$$

$$= \left(\lim_{\theta\to 0}\frac{Sin\theta}{\theta}\right)^2 \lim_{\theta\to 0}\theta$$

$$= (1)^2 \times 0 = 0$$

(x) $\lim\limits_{x\to 0}\dfrac{Sec\,x - Cos\,x}{x}$ **Sargodha 2008**

$$= \lim_{x\to 0}\frac{\frac{1}{\cos x} - \cos x}{x}$$

$$= \lim_{x\to 0}\left(\frac{1-\cos^2 x}{\cos x}\right)\left(\frac{1}{x}\right)$$

$$= \lim_{x\to 0}\frac{Sin^2 x}{x}\times\frac{1}{Cos\,x}$$

$$= \lim_{x\to 0} \frac{Sin^2 x}{x^2} \times x \times \frac{1}{Cos\, x}$$

$$= \left(\lim_{x\to 0} \frac{Sin\, x}{x}\right)^2 \times \lim_{x\to 0} x \times \lim_{x\to 0} \frac{1}{Cos\, x}$$

$$= (1)^2 \times 0 \times \frac{1}{1} = 0$$

(xi) $\lim_{\theta\to 0} \frac{1 - Cos\, p\theta}{1 - Cos\, q\theta}$ **Sargodha 2009**

$$= \lim_{\theta\to 0} \frac{2 Sin^2 \frac{p\theta}{2}}{2 Sin^2 \frac{q\theta}{2}}$$

$$= \left(\lim_{\theta\to 0} \frac{Sin \frac{p\theta}{2}}{Sin \frac{q\theta}{2}}\right)^2$$

$$= \left(\frac{\lim_{\theta\to 0} \frac{Sin \frac{p\theta}{2}}{\frac{p\theta}{2}} \times \frac{p\theta}{2}}{\lim_{\theta\to 0} \frac{Sin \frac{q\theta}{2}}{\frac{q\theta}{2}} \times \frac{q\theta}{2}}\right)^2 = \left(\frac{1\times p}{1\times q}\right)^2 = \frac{p^2}{q^2}$$

(xii) $\lim_{\theta\to 0} \frac{Tan\,\theta - Sin\,\theta}{Sin^3\theta}$ **Sargodha 2008**

$$= \lim_{\theta\to 0} \left(\frac{Sin\,\theta}{Cos\,\theta} - Sin\,\theta\right) \frac{1}{Sin^3\theta}$$

$$= \lim_{\theta\to 0} \left(\frac{Sin\theta - Sin\theta Cos\theta}{Cos\theta}\right) \frac{1}{Sin^3\theta}$$

$$= \lim_{\theta\to 0} \frac{Sin\theta(1 - Cos\theta)}{Cos\theta} \frac{1}{Sin\theta\, Sin^2\theta}$$

$$= \lim_{\theta\to 0} \frac{1 - Cos\theta}{Cos\theta(1 - Cos^2\theta)}$$

$$= \lim_{\theta \to 0} \frac{(1-Cos\theta)}{Cos\theta(1-Cos\theta)(1+Cos\theta)}$$

$$= \lim_{\theta \to 0} \frac{1}{Cos\theta(1+Cos\theta)}$$

$$= \frac{1}{1(1+1)} = \frac{1}{2}$$

Q4. **Express each limit in terms of e:**

(i) $\lim\limits_{n \to \infty}\left(1+\frac{1}{n}\right)^{2n}$ **Sargodha 2011**

$$= \left[\lim_{n \to \infty}\left(1+\frac{1}{n}\right)^{n}\right]^{2} = e^{2}$$

(ii) $\lim\limits_{n \to \infty}\left(1+\frac{1}{n}\right)^{n/2}$

$$= \left[\lim_{n \to \infty}\left(1+\frac{1}{n}\right)^{n}\right]^{1/2} = e^{1/2}$$

(iii) $\lim\limits_{n \to \infty}\left(1-\frac{1}{n}\right)^{n} = \lim\limits_{n \to \infty}\left[1+\left(-\frac{1}{n}\right)\right]^{n}$ **Sargodha 2009**

$$= \left[\lim_{n \to \infty}\left(1+\left(\frac{-1}{n}\right)\right)^{-n}\right]^{-1} = e^{-1} = \frac{1}{e}$$

(iv) $\lim\limits_{n \to \infty}\left(1+\frac{1}{3n}\right)^{n}$

$$= \left[\lim_{n \to \infty}\left(1+\frac{1}{3n}\right)^{3n}\right]^{1/3} = e^{1/3}$$

(v) $\lim\limits_{n \to \infty}\left(1+\frac{4}{n}\right)^{n}$

$$= \lim_{n \to \infty}\left(1+\frac{4}{n}\right)^{4n/4}$$

$$= \left[\lim_{n \to \infty}\left(1+\frac{4}{n}\right)^{n/4}\right]^{4} = e^{4}$$

(vi) $\lim_{x\to 0}(1+3x)^{2/x}$

$= \lim_{x\to 0}(1+3x)^{3\times 2/3x}$

$= \left[\lim_{x\to 0}(1+3x)^{1/3x}\right]^6 = e^6$

(vii) $\lim_{x\to 0}(1+2x^2)^{1/x^2}$

$= \lim_{x\to 0}(1+2x^2)^{2/2x^2}$

$= \left[\lim_{x\to 0}\left(1+2x^2\right)^{1/2x^2}\right]^2 = e^2$

(viii) $\lim_{h\to 0}(1-2h)^{1/h}$

$= \lim_{h\to 0}(1+(-2h))^{1/h}$

$= \lim_{h\to 0}(1+(-2h))^{-2/-2h}$

$= \left[\lim_{h\to 0}(1+(-2h))^{1/-2h}\right]^{-2} = e^{-2} = \frac{1}{e^2}$

(ix) $\lim_{x\to 0}\left(\frac{x}{1+x}\right)^x = \lim_{x\to 0}\left(\frac{1+x}{x}\right)^{-x}$ **(Gujrawala 2010)**

$= \lim_{x\to 0}\left(\frac{1}{x}+\frac{x}{x}\right)^{-x}$

$= \lim_{x\to 0}\left(\frac{1}{x}+1\right)^{-x} = \left[\lim_{x\to 0}\left(1+\frac{1}{x}\right)^x\right]^{-1} = e^{-1} = \frac{1}{e}$

(x) $\lim_{x\to 0}\frac{e^{1/x}-1}{e^{1/x}+1}, x<0,$ $\Rightarrow$ *Put* $x=-y$ *when* $x\to 0$ *then* $y\to 0$

$= \lim_{y\to 0}\frac{e^{1/-y}-1}{e^{1/-y}+1} = \lim_{y\to 0}\frac{\frac{1}{e^{1/y}}-1}{\frac{1}{e^{1/y}+1}}$

$= \frac{\frac{1}{\infty}-1}{\frac{1}{\infty}+1} = \frac{0-1}{0+1} = -1$

(xi) $\lim\limits_{x\to 0}\dfrac{e^{1/x}-1}{e^{1/x}+1}$, $x>0$

$$= \lim_{x\to 0}\frac{e^{1/x}\left(1-\frac{1}{e^{1/x}}\right)}{e^{1/x}\left(1+\frac{1}{e^{1/x}}\right)} = \frac{1-\frac{1}{\infty}}{1+\frac{1}{\infty}} = \frac{1-0}{1+0} = 1$$

Important Example 2 : Sargodha (2007 & 2009)

Example : Discuss continuity $f(x) = \dfrac{x^2-1}{x-1}$ at $x=1$

Answer: Here f(1) is not defined so discontinuous at x = 1

Further $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1}\dfrac{x^2-1}{x-1} = \lim\limits_{x\to 1}\dfrac{(x-1)(x+1)}{(x-1)}$

$= \lim\limits_{x\to 1}(x+1) = 1 + 1 = 2$

f(x) is continuous at any number except x = 1

Exercise 1.4

Q1. Determining the left hand limit and right hand limit and then find limits of the following function at x = c.

(i) $f(x) = 2x^2 + x - 5$, c = 1

L.H.L $= \lim_{x\to 1^-} f(x) = \lim_{x\to 1} (2x^2+x-5)$

$= 2(1)^2 + 1 - 5 = 2+1-5 = -2$

R.H.L $= \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x^2+x-5)$

$= 2(1)^2+1-5 = 2+1-5 = -2$

L.H.L = R.H.L = −2 so $\lim_{x\to 1} f(x) = -2$

(ii) $f(x) = \dfrac{x^2-9}{x-3}, c = -3$ Sargodha 2011

L.H.L $= \lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \dfrac{x^2-9}{x-3}$

$= \dfrac{(-3)^2-9}{-3-3} = \dfrac{9-9}{-6} = 0$

R.H.L $= \lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \dfrac{x^2-9}{x-3}$

$= \dfrac{(-3)^2-9}{-3-3} = \dfrac{9-9}{-6} = 0$

L.H.L = R.H.L = 0 so $\lim_{x\to 1} f(x) = 0$

(iii) $f(x) = |x-5|$, c = 5

L.H.L $= \lim_{x\to -5^-} f(x) = \lim_{x\to -5^-} -x-5) = -(5-5) = 0$

R.H.L $= \lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} (x-5) = 5-5 = 0$

L.H.L = R.H.L = 0 so $\lim_{x\to 5} f(x) = 0$

Q2. Discuss the continuity of f(x) at x = c

(i) $f(x) = \begin{cases} 2x+5 & \text{if } x \le 2 \\ 4x+1 & \text{if } x > 2 \end{cases}$ $c = 2$ **(Federal 2008, Lahore 2008)**

L.H.L $= \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x+5)$

$= 2(2) + 5 = 4 + 5 = 9$

R.H.L $= \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (4x+1) = 4(2) + 1 = 9$

f(2) $= 2(2) + 5 = 4+5 = 9$

L.H.L = R.H.L = f(2) so f(x) is Continuous at x = 2

(ii) $f(x) = \begin{cases} 3x-1 & \text{if } x<1 \\ 4 & \text{if } x=1 \\ 2x & \text{if } x>1 \end{cases}$ $c = 1$ **Sargodha 2008**

L.H.L $= \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(3x-1) = 3(1)-1 = 3-1 = 2$

R.H.L $= \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(2x) = 2(1) = 2$

$f(1) = 4$

$f(1) \neq$ L.H.L = R. H. L so f(x) is Discontinuous at x = 1

3. $f(x) = \begin{cases} 3x & \text{if } x \le -2 \\ x^2-1 & \text{if } -2<x<2 \\ 3 & \text{if } x \ge 2 \end{cases}$ **(Faisalabad 2011)**

Case I for x = 2

L.H.L $= \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (x^2-1) = (2)^2-1 = 4-1 = 3$

R.H.L $= \lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} 3 = 3$

$f(2) = 3$

L.H.L = R.H.L = f(2)

Continuous at c = 2

Case II For $x = -2$

L.H.L = $\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2^-} (3x) = 3(-2) = -6$

R.H.L $= \lim\limits_{x\to -2^+} f(x) = \lim\limits_{x\to -2^+} (x^2-1)$

$= (-2)^2-1 = 4-1 = 3$

$f(-2) = 3(-2) = -6$

L.H.L= $f(-2) \neq$ R.H.L

F(x) is Discontinuous at $x = -2$

4. $f(x) = \begin{cases} x+2, & x \le -1 \\ c+2, & x > -1 \end{cases}$ **find c so that $\lim\limits_{x\to -1}$ f(x) exist** **(Sargodha 2008,09,11)**

L.H.L $= \lim\limits_{x\to -1^-} f(x) = \lim\limits_{x\to -1^-} (x+2) = -1+2 = 1$

R.H.L $= \lim\limits_{x\to -1^+} f(x) = \lim\limits_{x\to -1^+} (c+2) = c+2$

Given limit exist mean

L.H.L = R.H.L

$1 = c+2 \Rightarrow c = -1$

Note:
Limit exist mean
L.H.L = R.H.L

Q5. Find the values m and n, so that given function f is continuous.

(i) $f(x) = \begin{cases} mx & \text{if } x<3 \\ n & \text{if } x=3 \\ -2x+9 & \text{if } x>3 \end{cases}$ **(Sargodha 2011,12)**

L.H.L $= \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (mx) = 3m$

R.H.L $= \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (-2x+9)$

$= -2(3)+9 = -6+9 = 3$

f(3) = n Given f(x) Continuous

So L.H.L = R.H.L = f(3) = 3m = 3 = n

$\Rightarrow 3m = 3$ and $n = 3$ or $m = 1$ and $n = 3$

(ii) $f(x) = \begin{cases} mx & \text{if } x<4 \\ x^2 & \text{if } x\geq 4 \end{cases}$ **(Sargodha 2009)**

L.H.L $= \lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (mx) = 4m$

R.H.L $= \lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} x^2 = (4)^2 = 16$

$f(4) = (4)^2 = 16$

Given f(x) continuous mean

L.H.L = R.H.L = f(4) = 4m = 16 = 16 $\Rightarrow$ m = 4

6. $f(x) = \begin{cases} \dfrac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}, & x\neq 2 \\ k & x=2 \end{cases}$ **Find value of K so that f is continuous at x =2**

(Sargodha 2007,11, Lhr 2010, Fsd 2010, Gujarawala 2012)

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} \times \frac{\sqrt{2x+5}+\sqrt{x+7}}{\sqrt{2x+5}+\sqrt{x+7}}$$

$$\lim_{x\to 2} \frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2} = \lim_{x\to 2} \frac{\left(\sqrt{2x+5}\right)^2-\left(\sqrt{x+7}\right)^2}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$

$$= \lim_{x\to 2} \frac{2x+5-x-7}{(x-2)(\sqrt{2x+5}+\sqrt{x+7})}$$

$$= \lim_{x\to 2} \frac{(x-2)}{(x-2)\left(\sqrt{2x+5}+\sqrt{x+7}\right)}$$

$$= \frac{1}{\sqrt{2(2)+5}+\sqrt{2+7}} = \frac{1}{\sqrt{9}+\sqrt{9}} = \frac{1}{3+3} = \frac{1}{6}$$

Given f(x) is continuous so $\lim_{x\to 2} f(x) = f(2) \Rightarrow \frac{1}{6} = k$ or $k = \frac{1}{6}$

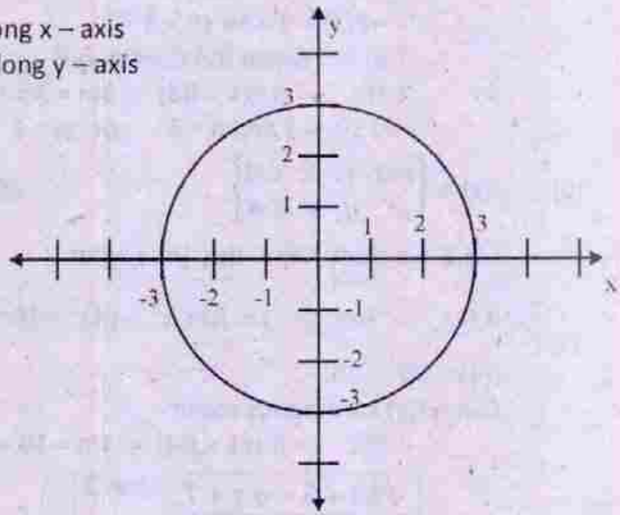
Exercise 1.5

1. Draw the graphs of the following equations.

(i) $x^2+y^2=9 \Rightarrow y=\pm\sqrt{9-x^2}$

X	−3	−2	−1	0	1	2	3
Y	0	± 2.2	± 2.8	± 3	± 2.8	± 2.2	0

Scale : 2 small square = 1 unit along x − axis
2 small square = 1 unit along y − axis



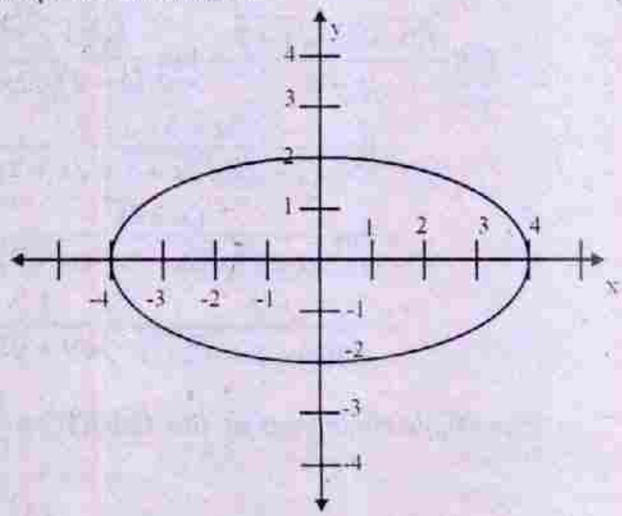
(ii) $\dfrac{x^2}{16}+\dfrac{y^2}{4}=1 \Rightarrow \dfrac{x^2}{(4)^2}+\dfrac{y^2}{(2)^2}=1$

Here when $y=0 \Rightarrow x=\pm 4$

$x=0 \Rightarrow y=\pm 2$

and the graph is symmetric with respect to both axes.

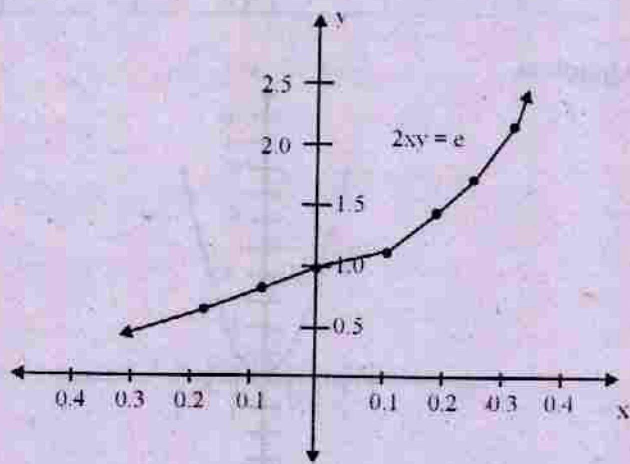
The graph is



(iii) $y = e^{2x}$

X	–0.3	–0.2	–0.1	0	0.1	0.2	0.3	0.4
Y	0.5	0.7	0.8	1	1.2	1.5	1.8	2.2

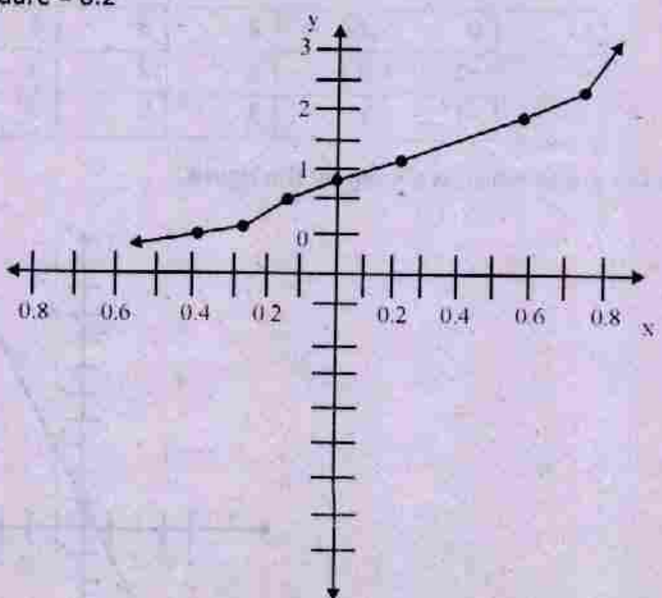
Scale 1 small square = 0.1



(iv) $y = 3^x$

X	–0.6	–0.4	–0.2	0	0.2	0.4	0.6	0.8	1
Y	0.5	0.6	0.8	1	1.2	1.6	1.9	2.4	3

Scale 1 small square = 0.2



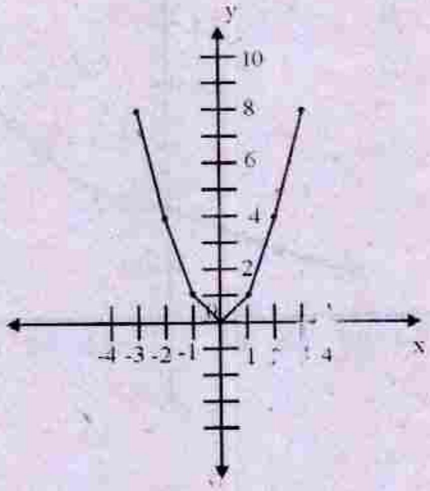
2. **Graph the curves that has the parametric equations given below**

(i) $x = t, y = t^2, -3 \le t \le 3$

Since $t \in [-3, 3]$, we have the table values as

T	−3	−2	−1	0	1	2	3
X	−3	−2	−1	0	1	2	3
Y	9	4	1	0	1	4	9

Now we plot a graph as

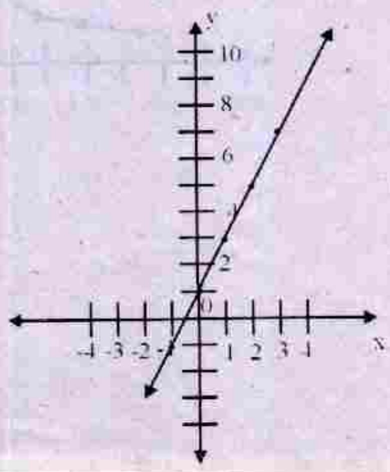


(ii) $x = t - 1, y = 2t - 1, -1 < t < 5$

Since $t \in [-1, 5]$ The corresponding values of x and y are given by the table

t	0	1	2	3	4
x	−1	0	1	2	3
y	−1	1	3	5	7

Now plot the graph which is shown by the figure.

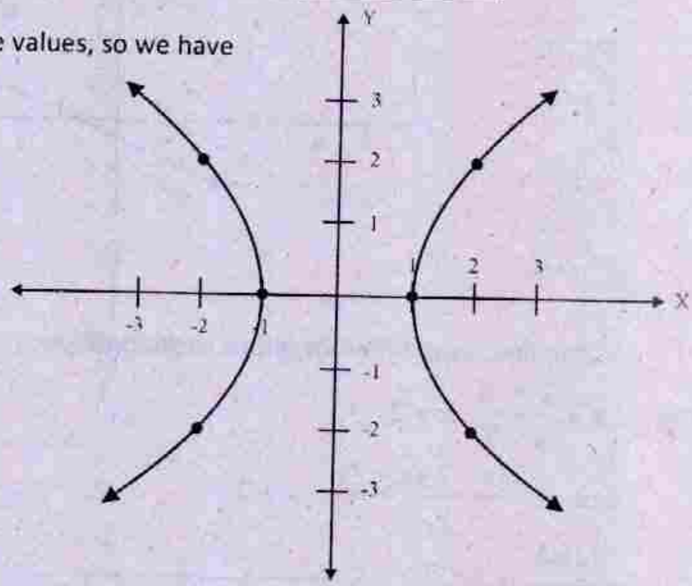


(iii) $x = \sec\theta$, $y = \tan\theta$

The corresponding values of x and y for real number θ are

θ	−180°	−120°	−60°	0°	60°	120°	180°
$X=\sec\theta$	−1	−2	2	1	2	−2	−1
$Y=\tan\theta$	0	1.73	−1.73	0	1.73	−1.73	0

Plot the graph by using table values, so we have



3. **Draw the graphs of the functions defined below and find whether they are continuous**

(i) $y = \begin{cases} x-1 \text{ if } x<3 \\ 2x+1 \text{ if } x\geq 3 \end{cases}$

Here $y = x-1$ when $x < 3$

and $y = 2x+1$ when $x \leq 3$, we have the table values as

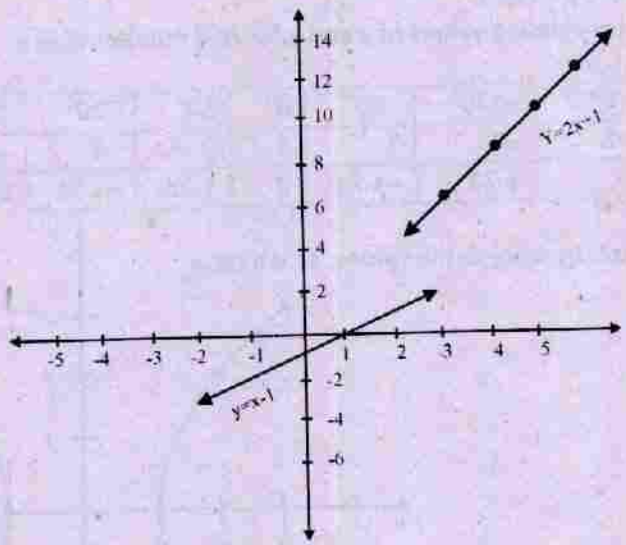
$y = x - 1$

x	−3	−2	−1	0	1	2
y	−4	−3	−2	−1	0	1

$Y = 2x + 1$

x	−2	−1	0	1	2	3
y	−3	−1	1	3	5	7

The graphs for both shown by the figure.



both lines have different slopes so discontinuo' :

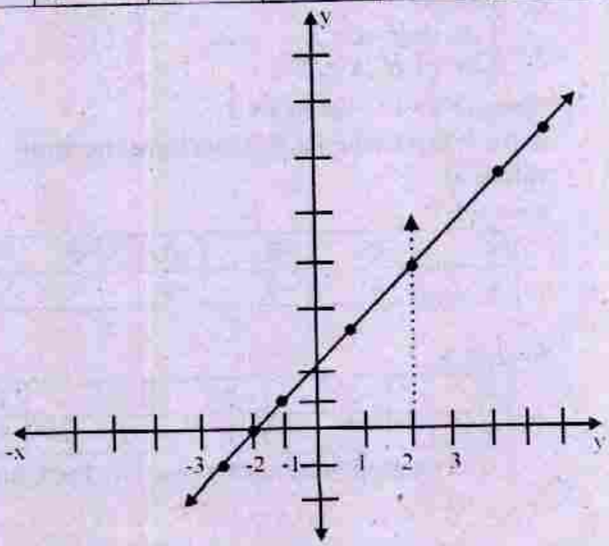
(ii). $Y = \dfrac{x^2-4}{x-2}, x \neq 2$

$\Rightarrow y = \dfrac{(x-2)(x+2)}{x-2} = x+2$

Table

x	−3	−2	−1	0	1	2	3	4
y	−1	0	1	2	3	4	5	6

The graph is



The graph shown by the figure is broken at x=2 hence the function is discontinuous at x = 2.

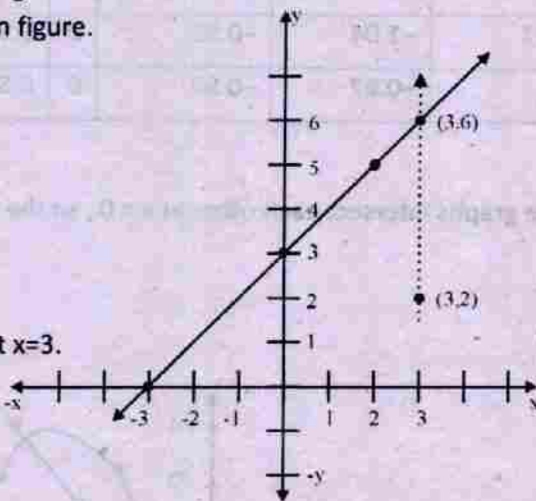
(iii). $y = \begin{cases} x+3 & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$

Table values are

x	−3	−2	−1	0	1	2	3
y	0	1	2	3	y4	5	6

and y = 2 for x = 3

The graph is as shown in figure.



Here at x = 3, y=2 and y=6.
so domain is respected.
Hence discontinuous function at x=3.

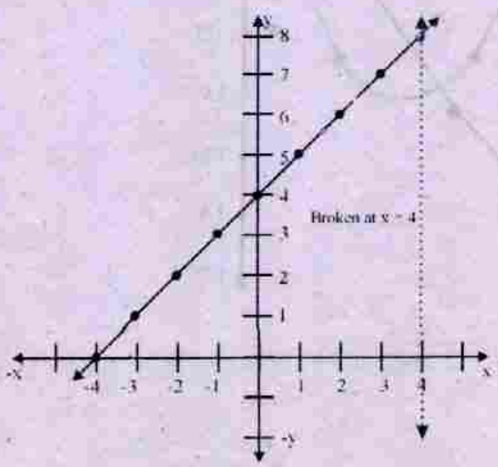
(iv). $\dfrac{x^2-16}{x-4}, x \neq 4$

$$f(x)=\frac{x^2-16}{x-4}=x+4$$

For different values of x , we have the corresponding values of y, these are given in the following table.

X	−3	−2	−1	0	1	2	3	4
Y	1	2	3	4	5	6	7	$\frac{0}{0}$

Its graph is



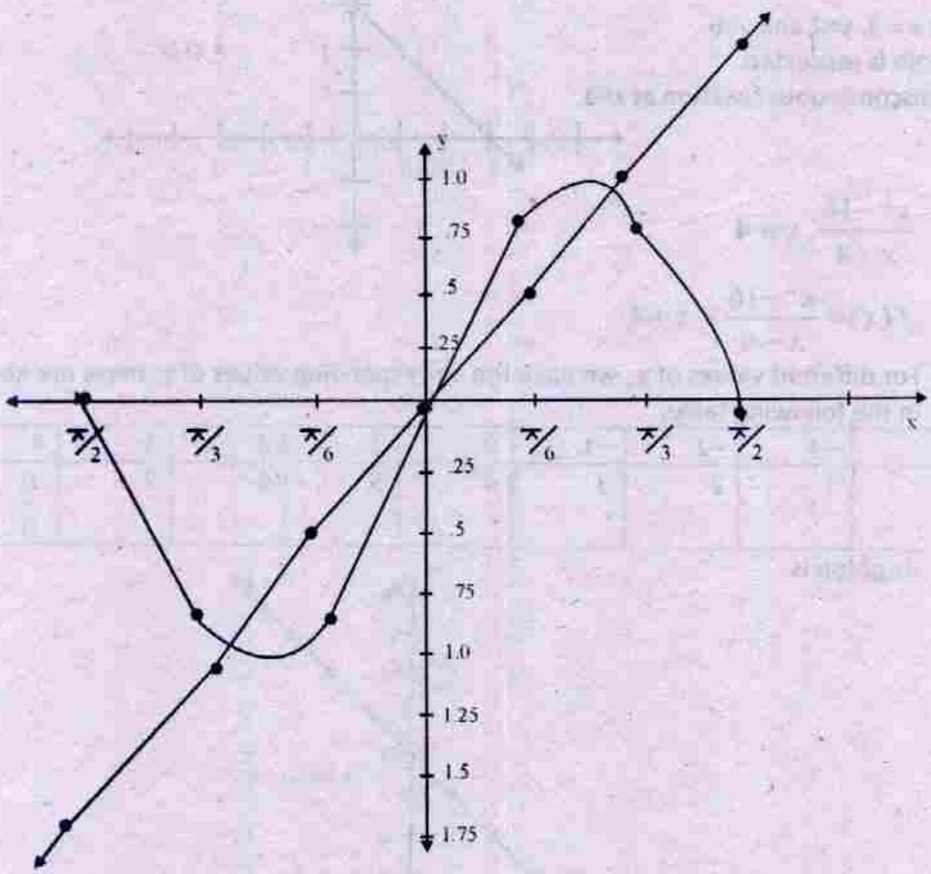
4. Find the graphical solution of the following equations.

(i) x = sin 2x

take $y_1 = x$ and $y_2 = \sin 2x$

x	$\frac{-\pi}{2}=1.57$	$\frac{-\pi}{3}=1.04$	$\frac{-\pi}{6}=-0.52$	0	$\frac{\pi}{6}=0.52$	$\frac{\pi}{3}=1.04$	$\frac{\pi}{2}=1.57$
y_1	−1.57	−1.04	−0.52	0	0.52	1.04	1.57
y_2	0	−0.87	−0.87	0	0.87	0.87	0

Both the graphs intersect each other at x = 0 , so the graphical solution is x = 0.



(ii) $\frac{x}{2} = \cos x$

For $y_1 = \frac{x}{2}$ take $\pi = 3.1416$

For $y_2 = \text{Cos}x$ take $\pi = 180^\circ$

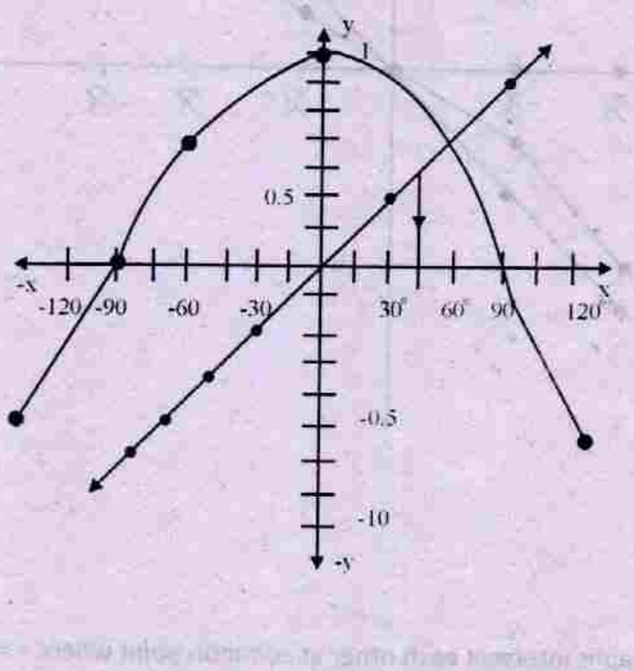
Table for $y_1 = \frac{x}{2}$

x	$\frac{-\pi}{3} = -1.04$	$\frac{-\pi}{6} = -.52$	0	$\frac{\pi}{6} = .52$	$\frac{\pi}{3} = 1.04$
y_1	–0.52	–0.26	0	0.26	0.52

Table for $y_2 = \cos x$

X	–90°	–60°	–30°	0	30°	60°	90°
y_2	0	0.5	0.87	1	0.87	0.5	0

Now we plot the graphs for $y_1 = \frac{x}{2}$ and $y_2 = \cos x$



Both the graph meet at x = 42° approximately.

(iii) **2x = Tan x**

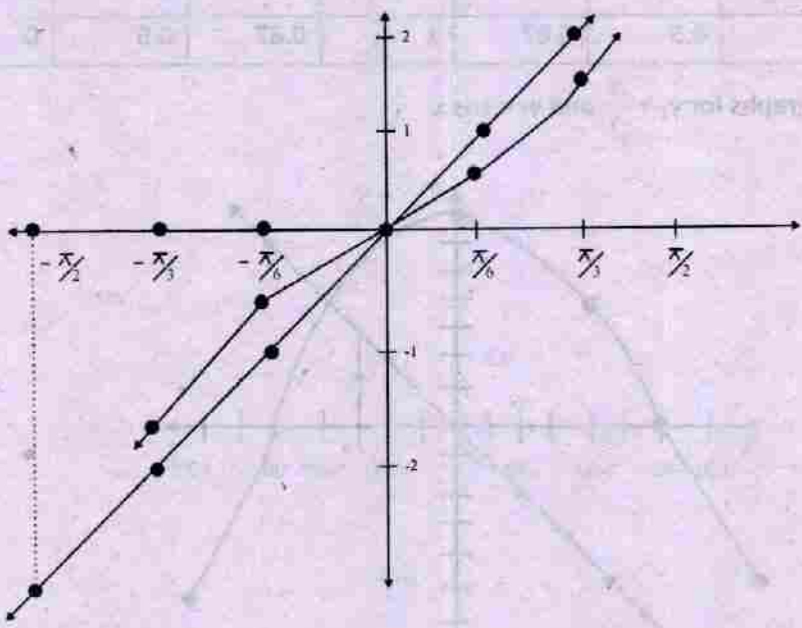
Let $y_1 = 2x$, $y_2 = \tan x$

Table values for both y_1 and y_2 are

x	$\frac{-\pi}{3} = -1.04$	$\frac{-\pi}{6} = -.52$	0	$\frac{\pi}{6} = -.52$	$\frac{\pi}{3} = 1.04$
y_1	−2.08	−1.04	0	1.04	2.08

x	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
y_2	∞	−1.73	−0.577	0	0.577	1.73	∞

Now we plot the graph as



Both the graphs intersect each other at common point where x = 0 , so the graphical solution is x = 0.

TEST YOUR SKILLS Marks 100

OBJECTIVE

Q.No.1 Given below are a few possible answers to each statement of which one is correct, identify the correct one. **(20)**

1. **Functions play a central role in the study of**
   (a) Mechanics (b) Trigonometry
   (c) Calculus (d) Both (b) & (c)
2. **A correspondence that assigns to each element x in X a unique element y in Y is called** ·
   (a) Vector (b) Determinant
   (c) Function (d) Matrix
3. **A function from X to Y is written as**
   (a) $f: Y \to X$ (b) $f: Y \to Y$
   (c) $f: X \to X$ (d) $f: X \to Y$
4. **If $f: X \to Y$ then the set Y is called**
   (a) Domain of f (b) Range of f
   (c) Both (a) & (b) (d) None of these
5. **If $f: X \to Y$ then the set of corresponding elements y in Y is called the**
   (a) Domain of f (b) Range of f
   (c) Both (a) & (b) (d) None of these
6. **If a variable y depends upon a variable x in such a way that each value of x determines exactly one value of y, then we say that**
   (a) x is a function of y (b) y is a function y
   (c) y is a function of x (d) x is a function x
7. **If $y = f(x)$ then $f(x)$ is called**
   (a) The value of f at x (b) Image of x under f
   (c) Both (a) & (b) (d) None of these
8. **If $f(x) = x^3 - 2x^2 + 4x - 1$, then $f(1 + x) =$**
   (a) $x^3 + x^2 + 3x + 2$ (b) $x^3 - x^2 + 3x + 2$

(c) $x^3 + x^2 - 3x + 2$ (d) $x^3 + x^2 + 3x - 2$

9. If $f(x) = x^2$ then the domain of f is

(a) $\{-1, 1\}$ (b) Set of integers

(c) Set of all real numbers (d) Set of natural numbers

10. If $f(x) = \sqrt{x^2 - 9}$ then range of f is

(a) Set of integers (b) Set of natural numbers

(c) Set of all real numbers (d) $[0, +\infty)$

11. A function in which the variable appears as exponent is call an

(a) Identity Function (b) Exponential Function

(c) Inverse Function (d) Constant function

12. The equations of the type $x = f(t)$ and $y = g(t)$ are called the

(a) Implicit functions (b) Parametric equations

(c) Reciprocals equations (d) None of these

13. If $f(x) = \frac{3x}{x^2 + 1}$, then it is

(a) Odd function (b) Even function

(c) Neither function (d) None of these

14. The area A of a circle as a function of its circumference C, is

(a) $A = \frac{1}{2\pi}C$ (b) $A = \frac{1}{4\pi}C$

(c) $A = \frac{1}{4\pi}C^2$ (d) $A = \frac{1}{\pi}C^2$

15. The volume V of a cube as a function of the area A of its base is

(a) $V = (A)^{2/3}$ (b) $V = A^{3/2}$

(c) $V = A^{1/3}$ (d) $V = A^{3/4}$

16. The function $f(x) = (x + 2)^2$ is

(a) Odd function (b) Even function

(c) Neither function (d) None of these

17. $\lim_{x \to a} \frac{x^n - a^n}{x - a} =$

(a) a^{n-1} (b) na^{n-1}

(c) na^{n+1} (d) $\frac{1}{n}a$

18. $\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} =$

(a) $\frac{1}{\sqrt{x}}$ (b) $\frac{1}{2\sqrt{x}}$

(c) $2\sqrt{x}$ (d) $\frac{2}{\sqrt{x}}$

19. **The graph of $f(x) = a^x$ lies in**

(a) 1st quadrant (b) 2nd quadrant

(c) 1st & 2nd quadrant (d) 4th quadrant

20. Graph of $y = \begin{cases} x & \text{when } 0 \le x \le 1 \\ x - 1 & \text{when } 1 < x \le 2 \end{cases}$ is

(a) Circle (b) Parabola

(c) Broken straight line (d) Hyperbola

SECTION I

SUBJECTIVE

Attempt any 25 (twenty five) short question from Question 2,3 and 4, at least 12 short questions from question 2, 12 short question from question 3 and 13 short question from question 4. All questio.. carry equal marks. (25x2=50)

Q.No. 2

i. Find domain & range of the function defined by $f(x) = x^2$

ii. Find domain & range of the function defined by $f(x) = \frac{x}{x^2 - 4}$

iii. Find domain & range of the function defined by $f(x) = \sqrt{x^2 - 9}$

iv. Find domain & range of the function defined by $g(x) = \sqrt{x+1}$

v. Find domain & range of the function defined by $g(x) = |x-3|$

vi. Find domain & range of the function defined by $g(x) = \frac{x^2-16}{x-4}\ x \neq 4$

vii. Express the perimeter *P* of a square as a function of its area *A*

viii. Express the area *A* of a circle as a function of its circumference '*C*'

ix. Express the volume *V* of a cube as a function of the area A of its base

x. Prove the identities $\text{Cos}^2 hx - \text{Sin}^2 hx = 1$

xi. Prove the identities $\text{Cos}^2 hx + \text{Sin}^2 hx = \text{Cos}\,h\,2x$

xii. Prove the identities $\text{Sec}^2 hx = 1 - \text{Tan}\,h^2 x$

Q.No. 3

i. Prove the identities $\text{Cosec}^2 hx = \text{Cot}^2 hx - 1$

ii. Determine whether the given function '*f*' is even or odd $f(x) = x^{2/3} + 6$

iii. Determine whether the given function '*f*' is even or odd $f(x) = \frac{x^3 - x}{x^2+1}$

iv. Determine whether the given function '*f*' is even or odd $f(x) = \frac{x-1}{x+1}$

v. Determine whether the given function '*f*' is even or odd $f(x) = \text{Sin}\,x + \text{Cos}\,x$

vi. Without finding the inverse, state the domain & range of f^{-1} where

$$f(x) = 2 + \sqrt{x+2}$$

vii. Without finding the inverse, state the domain & range of f^{-1} where

$$f(x) = \frac{x-1}{x-4},\ x \neq 4$$

viii. Without finding the inverse, state the domain & range of f^{-1} where

$$f(x) = (x-5)^2\ x \geq 5$$

ix. Let $f(x) = 3x^3 + 7$ find $f^{-1}(x)$. Show $f[f^{-1}(x)] = x$

x. Let $f(x) = 3x^3 + 7$ find $f^{-1}(x)$. Show $f^{-1}(f(x)) = x$

xi. Let $f(x)=\dfrac{2x+1}{x-1}$ find $f^{-1}(x)$. Show $f^{-1}[f(x)]=x$

xii. Let $f(x)=\dfrac{2x+1}{x-1}$ find $f^{-1}(x)$. Show $f^{-1}(f(x))=x$

Q.No. 4

i. Define limit of a function & what is the criteria for existence of limit of a function?

ii. Evaluate $\underset{x\to-\infty}{\text{Lt}}\dfrac{2-3x}{\sqrt{3+4x^2}}$, x < 0

iii. Evaluate $\underset{x\to\infty}{\text{Lt}}\dfrac{2-3x}{\sqrt{3+4x^2}}$, x > 0

iv. State Sandwitch Theorem.

v. Evaluate $\underset{n\to\infty}{\text{Lt}}\left(1+\dfrac{3}{n}\right)^{2n}$

vi. Evaluate $\underset{\theta\to0}{\text{Lt}}\dfrac{\text{Sin}\,7\theta}{\theta}$

vii. Evaluate $\underset{x\to0}{\text{Lt}}\dfrac{\text{Sin}\,ax}{\text{Sin}\,bx}$

viii. Evaluate $\underset{x\to\pi}{\text{Lt}}\dfrac{\text{Sin}\,x}{\pi-x}$

ix. Evaluate $\underset{\theta\to0}{\text{Lt}}\dfrac{\text{Sin}^2\theta}{\theta}$

x. Evaluate $\underset{h\to0}{\text{Lt}}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

xi. Evaluate $\underset{n\to\infty}{\text{Lt}}\left(1-\dfrac{1}{n}\right)^{n}$

xii. Evaluate $\underset{h\to0}{\text{Lt}}(1-2h)^{1/h}$

xiii. Evaluate $\underset{x\to\infty}{\text{Lt}}\left(\dfrac{x}{1+x}\right)^{x}$

SECTION II

Attempt any 3 (three) questions. **(3x10=30)**

Q.No.5

(a) Show that parametric equation $x=a\cos\theta$, $y=b\sin\theta$ represent the equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(b) Prove that $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$, where n is integer and a > 0.

Q.No.6

(a) Evaluate $\lim_{\theta\to 0}\frac{\tan\theta-\sin\theta}{\sin^3\theta}$

(b) Prove that $\lim_{x\to 0}\frac{\sqrt{x+a}-\sqrt{a}}{x}=\frac{1}{2\sqrt{a}}$

Q.No.7

(a) If $f(x)=\begin{cases}\frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}, & x\neq 2\\ k, & x=2\end{cases}$ find value of k so that f is continuous at x=2

(b) Evaluate $\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x}$

Q.No.8

(a) Prove that $\lim_{x\to+\infty}\left(1+\frac{1}{n}\right)^n=e$

(b) If $f(x)=\begin{cases}x+2, & x\leq -1\\ c+2, & x>-1\end{cases}$ find "C" so that $\lim_{x\to -1}$ f (x) exists

Q.No.9

(a) Without finding the inverse, state the domain and range of f^{-1}
$f(x)=(x-5)^2$, $x\geq 5$

(b) Express the limit in terms of e; $\lim_{x\to\infty}\left(\frac{x}{1+x}\right)^x$

Previous Board Questions

1. If $f(x) = 2x^3 - 5x^2 + 6x + 3$, then find $f\left(\frac{1}{2x}\right)$, $x \neq 0$. (Lhr – 2006)
2. Without finding the inverse, state the domain and range of $f^{-1}(x)$ when $f(x) = (x - 5)^2$, $x \geq 5$. (Grw – 2007)
3. Without finding the inverse, state the domain and range of $f^{-1}(x)$ when $f(x) = \sqrt{x+2}$. (Fsd – 2009)
4. Find $f^{-1}(x)$ for $f(x) = \frac{1}{x+3}$. (Lhr – 2007)
5. If $f(x) = 2x + 1$, then show that $f^{-1}(f(x)) = x$. (Mtn, Lhr – 2009)
6. If $f(x) = -2x + 8$, then that $f^{-1}(x)$. (Lhr – 2009)
7. Define a polynomial function of degree n. (Grw – 2005)
8. Evaluate $\lim_{x \to 0} \frac{1 - \cos x}{\sin \theta}$ (Lhr – 2009)
9. Evaluate $\lim_{x \to 0} \frac{x}{\tan x}$ (Lhr – 2008)
10. Evaluate $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$ (Grw – 2007)
11. Express the perimeter P of square as function of its area A. (Lahore – 2010)
12. Define continuous function at a number c. (Lahore – 2010)
13. If $f(x) = (-x + 9)^3$, find $f^{-1}(x)$. (Lahore – 2010)
14. Define the limit of a function. (Lahore – 2010)
15. Define the inverse of a function. (Gujranwala– 2010)
16. Express $\lim_{x \to \infty} \left(\frac{x}{1+x}\right)^x$ in terms of e. (Gujranwala– 2010)

****

DIFFERENTIATION

2

Definitions:

1. Average Rate of change: (Sargodha 2011, Fsd 2010)

Let f be a real valued function the (difference quotient) $\frac{f(x_1)-f(x)}{x_1-x}$ is called average rate of change.

2. Derivative: (Sargodha 2009)

Instantaneous rate of change of one variable with respect to other variable is called derivative or if limit of $\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$ exist then it is called derivative denoted by $\frac{dy}{dx}$.

3. Maclaurin Seriens:

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + ...$ *is called* maclaurin series.

4. Taylor Series:

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + ...$ is called taylor Series

5. Increasing: (Fsd 2010)

f is increasing on the interval (a,b) if $f(x_2) > f(x_1)$ wherever $x_2 > x_1$

6. Decreasing: (Sargodha 2010)

f is decreasing on the interval (a,b) if $f(x_2) < f(x_1)$ where ever $x_2 > x_1$

7. Stationary Point: (Gujranwala 2010)

Any point where f is neither increasing nor decreasing.

8. Critical value or Critical Point:

If $c \in f$ and $f'(c) = 0$ or $f'(c)$ does not exist then c is called critical value or critical point.

9. Relative Maxima:

f has relative maxima at c if $f''(c) < 0$

10. Relative Minima:

f has relative minima at c if $f''(c) > 0$

11. Point of Inflection:

The function f is increasing before x = 0 and also after x = 0 such point is called point of inflection.

Important Formulas

1. $\frac{d}{dx}(c)=0$
2. $\frac{d}{dx}(x)=1$
3. $\frac{d}{dx}(cx)=c.1=c$
4. $\frac{d}{dx}(x^n)=nx^{n-1}$
5. $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$
6. $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
7. $\frac{d}{dx}(\ln x)=\frac{1}{x}$
8. $\frac{d}{dx}(e^x)=e^x$
9. $\frac{d}{dx}(a^x)=a^x\ln a$
10. $\frac{d}{dx}(Sin x)=Cos x$
11. $\frac{d}{dx}(Cos x)=-Sin x$
12. $\frac{d}{dx}(Tan x)=Sec^2 x$
13. $\frac{d}{dx}(Cot x)=-Cosec^2 x$
14. $\frac{d}{dx}(Sec x)=Sec x Tan x$
15. $\frac{d}{dx}(Cosec x)=-Cosec x Cot x$
16. $\frac{d}{dx}(Sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$
17. $\frac{d}{dx}(Cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}$
18. $\frac{d}{dx}(Tan^{-1}x)=\frac{1}{1+x^2}$
19. $\frac{d}{dx}(Cot^{-1}x)=\frac{-1}{1+x^2}$
20. $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}}$
21. $\frac{d}{dx}(Cosec^{-1}x)=\frac{-1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}(Sinhx)=Coshx$
23. $\frac{d}{dx}(Coshx)=Sinhx$
24. $\frac{d}{dx}(Tanhx)=Sech^2x$
25. $\frac{d}{dx}(Tanh^{-1}x)=\frac{1}{1-x^2}$
26. $\frac{d}{dx}(Cothx)=-Cosech^2x$
27. $\frac{d}{dx}(Sechx)=SechxTanhx$
28. $\frac{d}{dx}(Cosechx)=-CosechxCothx$
29. $\frac{d}{dx}(Sinh^{-1}x)=\frac{1}{\sqrt{1+x^2}}$

30. $\frac{d}{dx}(Cosh^{-1}x)=\frac{1}{\sqrt{x^2-1}}$

31. $\frac{d}{dx}(Coth^{-1}x)=\frac{1}{1-x^2}$

32. $\frac{d}{dx}(Sech^{-1}x)=\frac{-1}{x\sqrt{1-x^2}}$

33. $\frac{d}{dx}(Cosech^{-1}x)=\frac{-1}{x\sqrt{1+x^2}}$

Exercise 2.1

Q1. Find by definition, the derivatives w.r.t 'x' of the following function defined as.

(i) $2x^2+1$ (Sargodha 2012)

Let $y = 2x^2+1$

Then $y+\delta y = 2(x+\delta x)^2+1$

$\delta y = 2(x^2+2x\delta x+\delta x^2)+1-y$

$\delta y = 2x^2+4x\delta x+2\delta x^2+1-(2x^2+1)$

$\delta y = 2x^2+4x\delta x+2\delta x^2+1-2x^2-1$

$= 4x\delta x+2\delta x^2$

$= \delta x(4x+2\delta x)$

Divide both side by δx and take limit $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\delta x(4x+2\delta x)}{\delta x}$$

$$= \frac{dy}{dx} = 4x$$

(ii) $2-\sqrt{x}$

Let $y = 2-\sqrt{x}$

Then $y+\delta y = 2-\sqrt{x+\delta x}$

$\delta y = 2-\sqrt{x+\delta x}-y$

$= 2-\sqrt{x+\delta x}-2+\sqrt{x}$

$= \sqrt{x}-\sqrt{x+\delta x}$

$$= (\sqrt{x}-\sqrt{x+\delta x})\times\frac{\sqrt{x}+\sqrt{x+\delta x}}{\sqrt{x}+\sqrt{x+\delta x}}$$

$$= \frac{(\sqrt{x})^2-(\sqrt{x+\delta x})^2}{\sqrt{x}+\sqrt{x+\delta x}} = \frac{x-x-\delta x}{\sqrt{x}+\sqrt{x+\delta x}}$$

= Divide by δx and take limit $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{-\delta x}{\delta x\left(\sqrt{x}+\sqrt{x+\delta x}\right)}$$

$$= \lim_{\delta x\to 0}\frac{-1}{\left(\sqrt{x}+\sqrt{x+\delta x}\right)}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{x}+\sqrt{x+0}} = \frac{-1}{\sqrt{x}+\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{x}}$$

(iii) $\frac{1}{\sqrt{x}}$

Let y $= \frac{1}{\sqrt{x}}$

Then $y + \delta y = \frac{1}{\sqrt{x+\delta x}}$

$$\delta y = \frac{1}{\sqrt{x+\delta x}} - y = \frac{1}{\sqrt{x+\delta x}} - \frac{1}{\sqrt{x}}$$

$$= \frac{\sqrt{x}-\sqrt{x+\delta x}}{(x+\delta x)(\sqrt{x})}$$

$$= \frac{\sqrt{x}-\sqrt{x+\delta x}}{\left(\sqrt{x+\delta x}\right)\sqrt{x}} \times \frac{\sqrt{x}+\sqrt{x+\delta x}}{\sqrt{x}+\sqrt{x+\delta x}}$$

$$= \frac{\left(\sqrt{x}\right)^2 - \left(\sqrt{x+\delta x}\right)^2}{\sqrt{x+\delta x}\sqrt{x}\left(\sqrt{x}+\sqrt{x+\delta x}\right)} = \frac{x - x - \delta x}{\sqrt{x+\delta x}\sqrt{x}\left(\sqrt{x}+\sqrt{x+\delta x}\right)}$$

= Divide by δx and take limit $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\delta x}\frac{-\delta x}{\sqrt{x+\delta x}\sqrt{x}\left(\sqrt{x}+\sqrt{x+\delta x}\right)}$$

$$\frac{dx}{dy} = \lim_{\delta x\to 0}\frac{-1}{\sqrt{x+\delta x}\sqrt{x}\left(\sqrt{x}+\sqrt{x+\delta x}\right)}$$

$$= \frac{-1}{\sqrt{x+0}\sqrt{x}\left(\sqrt{x}+\sqrt{x+0}\right)}$$

$$\frac{dy}{dx} = \frac{-1}{x(2\sqrt{x})} = -\frac{1}{2x^{3/2}}$$

(iv) $\frac{1}{x^3}$

Let $y = \frac{1}{x^3}$

Then $y + \delta y = \frac{1}{(x+\delta x)^3}$

$$\delta y = \frac{1}{(x+\delta x)^3} - y = \frac{1}{(x+\delta x)^3} - \frac{1}{x^3}$$

$$= \frac{x^3 - (x+\delta x)^3}{(x+\delta x)^3 . x^3}$$

$$= \frac{x^3 - (x^3 + 3x^2\delta x + 3x\delta x^2 + \delta x^3)}{(x+\delta x)^3 . x^3}$$

$$= \frac{\delta x\left(-3x^2 - 3x\delta x - \delta x^2\right)}{(x+\delta x)^3 . x^3}$$

Divide by δ x and take limit δ x→0

$$\lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \lim_{\delta x\to 0} \frac{\delta x\left(-3x^2 - 3x\delta x - \delta x^2\right)}{\delta x(x+\delta x)^3 . x^3}$$

$$\frac{dy}{dx} = \lim_{\delta x\to 0} \frac{-3x^2 - 0 - 0}{(x+0)^3 . x^3} = \frac{-3x^2}{x^6}$$

$$\frac{dy}{dx} = \frac{-3}{x^4}$$

(v) $\dfrac{1}{x-a}$

$$y = \frac{1}{x-a}$$

Then $y + \delta y = \dfrac{1}{x+\delta x - a}$

$$\delta y = \frac{1}{x+\delta x - a} - y = \frac{1}{x+\delta x - a} - \frac{1}{x-a}$$

$$= \frac{(x-a) - (x+\delta x - a)}{(x+\delta x - a)(x-a)}$$

$$= \frac{x - a - x - \delta x + a}{(x+\delta x - a)(x-a)}$$

$$= \frac{-\delta x}{(x+\delta x - a)(x-a)}$$

Take limit δ x →0 and ÷ by δ x

$$\lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \lim_{\delta x\to 0} \frac{-\delta x}{(x+\delta x - a)(x-a)}$$

$$= \lim_{\delta x\to 0} \frac{-1}{(x+\delta x - a)(x-a)}$$

$$\frac{dy}{dx} = \frac{-1}{(x+0-a)(x-a)} = \frac{-1}{(x-a)^2}$$

(vi) $x(x-3)$

Let $y = x(x-3) = x^2-3x$

Then $y+\delta y = (x+\delta x)^2-3(x+\delta x)$

$\delta y = x^2+2x\delta x+\delta x^2-3x-3\delta x-y$

$= x^2+2x\delta x+\delta x^2-3x-3\delta x-x^2+3x$

$= \delta x(2x+\delta x-3)$

Divide by δx and take limit $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\delta x(2x+\delta x-3)}{\delta x}$$

$$\frac{dy}{dx} = 2x+0-3 = 2x-3$$

(vii) $\frac{2}{x^4} = 2x^{-4}$

Let $y = 2x^{-4}$

Then $y+\delta y = 2(x+\delta x)^{-4}$

$\delta y = 2(x+\delta x)^{-4}-y$

$\delta y = 2(x+\delta x)^{-4}-2x^{-4}$

$= 2(x+\delta x)^{-4}-2x^{-4}$

$$= 2x^{-4}\left(1+\frac{\delta x}{x}\right)^{-4}-2x^{-4}$$

$$= 2x^{-4}\left[\left(1+\frac{\delta x}{x}\right)^{-4}-1\right]$$

$$= 2x^{-4}\left(1+(-4)\frac{\delta x}{x}+\frac{-4(-4-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\ldots\ldots-1\right)$$

$$= 2x^{-4}\left(\frac{\delta x}{x}\right)\left[-4+\frac{(-4)(-5)}{2!}\frac{\delta x}{x}+\ldots\ldots\right]$$

Divide by δx, Take limit $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{2x^{-4-1}\delta x\left(-4+\frac{(-4)(-5)}{2!}\frac{\delta x}{x}+\ldots\ldots\right)}{\delta x}$$

$$= 2x^{-5}(-4+0) = -8x^{-5} = \frac{-8}{x^5}$$

(viii) $(x+4)^{1/3}$

Let $y = (x+4)^{1/3}$

Then $y+\delta y = (x+\delta x+4)^{1/3}$

$\delta y = (x+4+\delta x)^{1/3}-y$

$$= ((x+4)+\delta x)^{1/3}-(x+4)^{1/3}$$

$$= (x+4)^{1/3}\left(1+\frac{\delta x}{x+4}\right)^{1/3}-(x+4)^{1/3}$$

$$= (x+4)^{1/3}\left[\left(1+\frac{\delta x}{x+4}\right)^{1/3}-1\right]$$

$$= (x+4)^{1/3}\left[1+\frac{1}{3}\left(\frac{\delta x}{x+4}\right)+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{\delta x}{x+4}\right)^2+\ldots\ldots-1\right]$$

$$= (x+4)^{1/3}\left[\frac{1}{3}\left(\frac{\delta x}{x+4}\right)+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2}\left(\frac{\delta x}{x+4}\right)^2+\ldots\ldots\right]$$

Take limit $\delta x\to 0$ *and* $\div$ by δx

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{(x+4)^{1/3}\left(\frac{\delta x}{x+4}\right)}{\delta x}\left[\frac{1}{3}+\frac{\frac{1}{3}\left(\frac{-2}{3}\right)}{2}\left(\frac{\delta x}{x+4}\right)+\ldots\ldots\right]$$

$$\frac{dy}{dx}=\lim_{\delta x\to 0}(x+4)^{1/3-1}\left[\frac{1}{3}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}\left(\frac{\delta x}{x+4}\right)+\ldots\ldots\right]$$

$$\frac{dy}{dx}=(x+4)^{-2/3}\left(\frac{1}{3}+0\right)$$

$$f'(x)=\frac{1}{3}(x+4)^{-2/3}$$

(ix) $x^{3/2}$

Let $y=x^{3/2}$ then $y+\delta y=(x+\delta x)^{3/2}$

$\delta y=(x+\delta x)^{3/2}-y=(x+\delta x)^{3/2}-x^{3/2}$

$$=x^{3/2}\left(1+\frac{\delta x}{x}\right)^{3/2}-x^{3/2}$$

$$=x^{3/2}\left[\left(1+\frac{\delta x}{x}\right)^{3/2}-1\right]$$

$$= x^{3/2}\left[1+\frac{3}{2}\frac{\delta x}{x}+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2}\left(\frac{\delta x}{x}\right)^2+\ldots\ldots-1\right]$$

÷by δ x and lim δ x $\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{x^{3/2}.\frac{\delta x}{x}}{\delta x}\left[\frac{3}{2}+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2}.\frac{\delta x}{x}+\ldots\ldots\right]$$

$$\frac{dy}{dx}=x^{3/2-1}\left(\frac{3}{2}+0\right)=\frac{3}{2}x^{1/2}$$

(x) $x^{5/2}$

Let y $= x^{5/2} \Rightarrow y + \delta y = (x+\delta x)^{5/2}$

δ y $= (x+\delta x)^{5/2} - y \Rightarrow \delta y = (x+\delta x)^{5/2} - x^{5/2}$

$$= x^{5/2}\left[\left(1+\frac{\delta x}{x}\right)^{5/2}-1\right]$$

$$= x^{5/2}\left[1+\frac{5}{2}\frac{\delta x}{x}+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\ldots\ldots-1\right]$$

$$= x^{5/2}\frac{\delta x}{x}\left[\frac{5}{2}+\frac{\frac{5}{2}\left(\frac{5}{2}-1\right)}{2!}\frac{\delta x}{x}+\ldots\ldots\right]$$

÷ by δ x and take lim δ x $\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{x^{5/2-1}.\delta x}{\delta x}\left[\frac{5}{2}+\frac{\frac{5}{2}\left(\frac{3}{2}\right)}{2!}\frac{\delta x}{x}+\ldots\ldots\right]$$

$$\frac{dy}{dx} = x^{3/2}\left(\frac{5}{2}+0\right)=\frac{5}{2}x^{3/2}$$

(xi) x^m

Let $y = x^m$

Then $y + \delta y = (x+\delta x)^m$

$\delta y = (x+\delta x)^m - y = (x+\delta x)^m - x^m$

$\delta y = x^m + m_{c_1} x^{m-1}(\delta x)^1 + mc_2 x^{m-2}(\delta x)^2 + \ldots {}^m c_m x^0 \delta x^m - x^m$

$$\delta y = mx^{m-1}\delta x + \frac{m(m-1)}{2}x^{m-2}(\delta x)^2 + \ldots + (\delta x)^m$$

$$\delta y = mx^{m-1}\,\delta x + \left[\frac{m(m-1)}{2}x^{m-2}.\delta x + \ldots(\delta x)^{m-1}\right]$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\delta x\left[mx^{m-1} + \frac{m(m-1)}{2}x^{m-2}\,\delta x + \ldots + (\delta x)^{m-1}\right]}{\delta x}$$

$$\frac{dy}{dx} = mx^{m-1} + 0$$

$$\Rightarrow \frac{dy}{dx} = mx^{m-1}$$

(xii) $\frac{1}{x^m} = x^{-m}$

Let $y = x^{-m} \Rightarrow y + \delta y = (x+\delta x)^{-m}$

$$\delta y = x^{-m}\left[1+\frac{\delta x}{x}\right]^{-m} - x^{-m}$$

$$\delta y = x^{-m}\left[1+(-m)\frac{\delta x}{x} + \frac{(-m)(-m-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \ldots\ldots -1\right]$$

$$\delta y = x^{-m}.\frac{\delta x}{x}\left[-m + \frac{(-m)(-m-1)}{2!}\frac{\delta x}{x} + \ldots\ldots\right]$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{x^{-m-1}.\delta x}{\delta x}\left(-m + \frac{(-m)(-m-1)}{2!}\frac{\delta x}{x} + \ldots\ldots\right)$$

$$\frac{dy}{dx} = x^{-m-1}(-m+0) \Rightarrow \frac{dy}{dx} = -mx^{-m-1}$$

(xiii) x^{40} **Let $y = x^{40}$**

Then $y + \delta y = (x+\delta x)^{40} \Rightarrow \delta y = (x+\delta x)^{40} - y$

$\delta y = (x+\delta x)^{40} - x^{40}$

$\delta y = {}^{40}c_0\, x^{40}(\delta x)^0 + {}^{40}c_1\, x^{39}(\delta x)^1 + {}^{40}c_2\, x^{38}(\delta x)^2 + \ldots + {}^{40}c_{40}\, x^0(\delta x)^{40} - x^{40}$

$$\delta y = x^{40} + 40x^{39}\delta x + \frac{40(39)}{2}x^{38}\delta x^2 + \ldots + (\delta x)^{40} - x^{40}$$

$$\delta y = \delta x(40x^{39} + 20(39)x^{38}\delta x^2 + \ldots + (\delta x)^{39})$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\delta x\left(40x^{39} + 20(39)x^{38}.\delta x + \ldots + (\delta x)^{39}\right)}{\delta x}$$

$$\frac{dy}{dx} = 40x^{39} + 0 \Rightarrow \quad \frac{dy}{dx} = 40x^{39}$$

(xiv) x^{-100}

Let $y = x^{-100}$ then $y + \delta y = (x+\delta x)^{-100}$

$$= (x+\delta x)^{-100} - x^{-100}$$

$$= x^{-100}\left[\left(1+\frac{\delta x}{x}\right)^{-100} - 1\right]$$

$$= x^{-100}\left[1 + (-100)\frac{\delta x}{x} + \frac{(-100-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \ldots - 1\right]$$

$$= \lim_{\delta x\to 0}\frac{x^{-100}.\left(\frac{\delta x}{x}\right)\left[-100 + \frac{(-100-1)}{2!}\frac{\delta x}{x} + \ldots\right]}{\delta x}$$

$$\frac{dy}{dx} = x^{-100-1}(-100+0) = -100x^{-101}$$

2. Find $\frac{dy}{dx}$ from first principle if *(i)* $\sqrt{x+2}$ *(ii)* $\frac{1}{\sqrt{x+a}}$

(i) $\left(\sqrt{x+2}\right)$

Let $y = (x+2)^{1/2}$

Then $y + \delta y = (x + \delta x + 2)^{1/2}$

$$\delta y = (x+2+\delta x)^{1/2} - y$$

$$\delta y = ((x+2) + \delta x)^{1/2} - (x+2)^{1/2}$$

$$= ((x+2) + \delta x)^{1/2} - (x+2)^{1/2}$$

$$= (x+2)^{1/2}\left[\left(1+\frac{\delta x}{x+2}\right)^{1/2} - 1\right]$$

$$= (x+2)^{1/2}\left[1 + \frac{1}{2}\left(\frac{\delta x}{x+2}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x+2}\right)^2 + \ldots - 1\right]$$

$$= (x+2)^{1/2} \cdot \frac{\delta x}{x+2}\left[\frac{1}{2}+\frac{\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x+2}\right)+\ldots\right]$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{(x+2)^{1/2-1}.\quad \delta x}{\delta x}\left[\frac{1}{2}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x+2}\right)+\ldots\right]$$

$$= \frac{dy}{dx}=(x+2)^{1/2}\left(\frac{1}{2}+0\right)$$

$$= \frac{dy}{dx}=\frac{1}{2}(x+2)^{-1/2}$$

(ii) $y = \dfrac{1}{\sqrt{x+a}} = (x+a)^{-1/2}$ **(Lahore 2010)**

$$y+\delta y = (x+\delta x+a)^{-1/2}$$

$$\delta y = (x+a+\delta x)^{-1/2}-y = (x+a+\delta x)^{-1/2}-(x+a)^{-1/2}$$

$$\delta y = (x+a)^{-1/2}\left[\left(1+\frac{\delta x}{x+a}\right)^{-1/2}-1\right]$$

$$\frac{\delta y}{\delta x}=\frac{\left[1+\left(-\frac{1}{2}\right)\frac{\delta x}{x+a}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x+a}\right)^2+\ldots\ldots-1\right]}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{(x+a)^{-1/2}.\ \frac{\delta x}{(x+a)}\left[-\frac{1}{2}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\frac{\delta x}{x+a}+\ldots\right]}{\delta x}$$

$$\frac{dy}{dx}=(x+a)^{-3/2}\left(-\frac{1}{2}+0\right) \quad\Rightarrow\quad \frac{dy}{dx}=-\frac{1}{2}(x+a)^{-3/2}$$

Exercise 2.2

1. Find from first principles the derivatives of the following expressions w.r.t their respective independent variable.

(i) Let y = $(ax + b)^3$ **(Sargodha 2012, Faisalabad 2010)**

Then $y+\delta y = \left[a\left(x+\delta x\right)+b\right]^3 = \left(ax+b+a\delta x\right)^3$

$$\Rightarrow \delta y = (ax+b+a\delta x)^3 - y = (ax+b+a\delta x)^3 - (ax+b)^3$$

$$\Rightarrow \delta y = (ax+b)^3\left[1+\frac{a\delta x}{ax+b}\right]^3 - (ax+b)^3$$

$$= (ax+b)^3\left[\left(1+\frac{a\delta x}{ax+b}\right)^3 - 1\right]$$

$$= (ax+b)^3\left[1+\frac{3a\delta x}{ax+b}+\frac{3(3-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)^2+\ldots\ldots-1\right]$$

$$= (ax+b)^3\frac{a\delta x}{ax+b}\left[3+\frac{3(2)}{2!}\frac{a\delta x}{(ax+b)}+\ldots\ldots\right]$$

Divide by δ x and take lim δ v $\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{(ax+b)^{3-1}.a\delta x\left[3+\frac{3(2)}{2!}\left(\frac{a\delta x}{ax+b}\right)+\ldots\ldots\right]}{\delta x}$$

$$\frac{dx}{dy} = (ax+b)^2 a(3+0) = 3a(ax+b)^2$$

(ii) Let y = $(2x+3)^5$

Then y + δ y = $[2(x+\delta x)+3]^5 = (2x+2\delta x+3)^5 = (2x+3+2\delta x)^5$

$\delta y = (2x+3+2\delta x)^5 - y = (2x+3+2\delta x)^5 - (2x+3)^5$

$$\delta y = (2x+3)^5\left[\left(1+\frac{2\delta x}{2x+3}\right)^5\right] - (2x+3)^5$$

$$\delta y = (2x+3)^5\left[\left(1+\frac{2\delta x}{2x+3}\right)^5 - 1\right]$$

$$= (2x+3)^5\left[1+5\left(\frac{2\delta x}{2x+3}\right)+\frac{5(5-1)}{2!}\left(\frac{2\delta x}{2x+3}\right)^2+\ldots\ldots-1\right]$$

$$= (2x+3)^5\left[5\left(\frac{2\delta x}{2x+3}\right)+\frac{5(4)}{2!}\left(\frac{2\delta x}{2x+3}\right)+\ldots\ldots\right]$$

$$\delta y = (2x+3)^{5-1}\, 2\delta x\left(5+\frac{5(4)}{2!}\frac{2\delta x}{(2x+3)}+........\right)$$

Divide by δ x and take lim δ x$\to$0

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{(2x+3)^4.2\delta x\left[5+\frac{5(4)}{2!}\left(\frac{2\delta x}{2x+3}\right)+.....\right]}{\delta x}$$

$$\frac{dy}{dx}=(2x+3)^4 2(5+0)=5\times 2(2x+3)^4=10(2x+3)^4$$

(iii) **Let** $y = (3t+2)^{-2}$

Then $y+\delta y = [3(t+\delta t)+2]^{-2} = (3t+3\delta t+2)^{-2}$

$y+\delta y = (3t+2+3\delta t)^{-2} \quad\Rightarrow\quad \delta y = (3t+2+3\delta t)^{-2} - y$

$$\delta y = (3t+2+3\delta t)^{-2}-(3t+2)^{-2} = (3t+2)^{-2}\left[\left(1+\frac{3\delta t}{3t+2}\right)^{-2}-1\right]$$

$$=(3t+2)^{-2}\left[1+\frac{(-2)3\delta t}{(3t+2)}+\frac{(-2)(-2-1)}{2!}\left(\frac{3\delta t}{3t+2}\right)^2+.....-1\right]$$

$$=(3t+2)^{-2}.\left[\frac{(-2)3\delta t}{(3t+2)}+\frac{(-2)(-3)}{2!}\left(\frac{3\delta t}{3t+2}\right)+.......\right]$$

$$=(3t+2)^{-2-1}.(3\delta t)\left[-2+\frac{(-2)(-3)}{2!}\left(\frac{3\delta t}{3t+2}\right)+.......\right]$$

Divide by δt and take limit $\delta t \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{(3t+2)^{-3}3\delta t\left[-2+\frac{(-2)(-3)}{2!}\left(\frac{3\delta x}{3t+2}\right)+.....\right]}{\delta x}$$

$$\frac{dy}{dt}= (et+2)^{-3}.\,3(-2+0) = -6(3t+2)^{-3}$$

(iv) **Let** $y = \frac{1}{(ax+b)^{+5}} = (ax+b)^{-5}$ **(Sargodha 2010)**

Then $y+\delta y = (a(x+\delta x)+b)^{-5} = (ax+a\delta x+b)^{-5}$

$$\delta y = (ax+b+a\delta x)^{-5} - (ax+b)^{-5}$$

$$=(ax+b)^{-5}\left[\left(1+\frac{a\delta x}{ax+b}\right)^{-5}-1\right]$$

$$= (ax+b)^{-5}\left[1+(-5)\left(\frac{a\delta x}{ax+b}\right)+\frac{(-5)(4)}{2!}\left(\frac{a\delta x}{ax+b}\right)^2+.....-1\right]$$

$$= (ax+b)^{-5}\left(\frac{a\delta x}{ax+b}\right)\left[-5+\frac{(-5)(4)}{2!}\left(\frac{a\delta x}{ax+b}\right)+........\right]$$

Dividing both sides by δ x and take limit δ x $\rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{(ax+b)^{-5-1}a\delta x\left[-5+\frac{(-5)(4)}{2!}\left(\frac{a\delta x}{ax+b}\right)+.....\right]}{\delta x}$$

$$\frac{dy}{dx}=(ax+b)^{-6}a(-5+0)=-5a(ax+b)^{-6}$$

(v) Let $y=\frac{1}{(az-b)^7}=(az-b)^{-7}$ **(Sargodha 2010, Lahore 2010)**

Then $y+\delta y = (a(z+\delta z)-b)^{-7} = (az+a\delta z-b)^{-7}$

$$= (az-b+a\delta z)^{-7} = (az-b)^{-7}\left(1+\frac{a\delta z}{az-b}\right)^{-7}$$

$$\delta y = (az-b)^{-7}\left(1+\frac{a\delta z}{az-b}\right)^{-7}-(az-b)^{-7}$$

$$\delta y = (az-b)^{-7}\left[1+(-7)\left(\frac{a\delta z}{az-b}\right)+\frac{(-7)(-7-1)}{2!}\left(\frac{a\delta z}{az-b}\right)^2+......-1\right]$$

$$\delta y = (az-b)^{-7}\frac{a\delta z}{(az-b)}\left[-7+\frac{(-7)(-7-1)}{2!}\left(\frac{a\delta z}{az-b}\right)+......\right]$$

Divide by δ z and take lim δ z $\rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{(az-b)^{-7-1}a\delta z\left[-7+\frac{(-7)(-8)}{2!}\left(\frac{a\delta z}{az-b}\right)+.....\right]}{\delta z}$$

$$\frac{dy}{dz}=(az-b)^{-8}(a)(-7+0)=(az-b)^{-8}(-7a)=-7a(az-b)^{-8}$$

Exercise 2.3

Q. Differentiate w.r.t 'x'

1. Let $y = x^4 + 2x^3 + x^2$

Then $\frac{dy}{dx} = \frac{d}{dx}x^4 + 2\frac{d}{dx}x^3 + \frac{d}{dx}x^2 = 4x^3 + 2.3x^2 + 2x$

$$\frac{dy}{dx} = 4x^3 + 6x^2 + 2x.$$

2. Let $y = x^{-3} + 2x^{-3/2} + 3$ then

$$\frac{dy}{dx} = \frac{d}{dx}x^{-3} + 2\frac{d}{dx}x^{-3/2} + \frac{d}{dx}3 = -3x^{-4} + 2\left(-\frac{3}{2}\right)x^{-3/2-1} + 0$$

$$\frac{dy}{dx} = -3x^{-4} - 3x^{-5/2} = \frac{-3}{x^4} - \frac{3}{x^{5/2}} = -3\left(\frac{1}{x^4} + \frac{1}{x^{5/2}}\right)$$

3. Let $y = \frac{a+x}{a-x}$ *then* $\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{a+x}{a-x}\right)$ (Sgd 2008,11 Fsd 2010, Lhr 2010)

$$\frac{dy}{dx} = \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2} = \frac{1}{(a-x)^2}\left[(a-x)(0+1) - (a+x)(0-1)\right]$$

$$\frac{dy}{dx} = \frac{1}{(a-x)^2}\left[a - x + a + x\right] = \frac{2a}{(a-x)^2}$$

4. Let $y = \frac{2x-3}{2x+1}$ *then* $\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{2x-3}{2x+1}\right)$

$$\frac{dy}{dx} = \frac{(2x+1)\frac{d}{dx}(2x-3) - (2x-3)\frac{d}{dx}(2x+1)}{(2x+1)^2} = \frac{(2x+1)(2-0) - (2x-3)(2+0)}{(2x+1)^2}$$

$$\frac{dy}{dx} = \frac{4x+2-4x+6}{(2x+1)^2} = \frac{2(4)}{(2x+1)^2} = \frac{8}{(2x+1)^2}$$

5. Let $y = (x-5)(3-x)$ then $\Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x-5)(3-x)$

$$\frac{dy}{dx} = (x-5)\frac{d}{dx}(3-x) + (3-x)\frac{d}{dx}(x-5) = (x-5)(0-1) + (3-x)(1-0)$$

$$\frac{dy}{dx} = -x + 5 + 3 - x = 8 - 2x$$

6. Let $y = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 = \left(\sqrt{x}\right)^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2\sqrt{x}\frac{1}{\sqrt{x}}$ (Gujranwala 2010)

$y = x + \frac{1}{x} - 2 \Rightarrow y = x + x^{-1} - 2 \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}x^{-1} - \frac{d}{dx}(2)$

$$\frac{dy}{dx} = 1 + (-1)x^{-2} - 0 = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$$

7. Let $y = \frac{\left(1+\sqrt{x}\right)\left(x - x^{3/2}\right)}{\sqrt{x}}$

$$y = \frac{x\left(1+\sqrt{x}\right)\left(1-x^{1/2}\right)}{\sqrt{x}} = \frac{\sqrt{x} \times \sqrt{x}\left(1+\sqrt{x}\right) + \left(1-\sqrt{x}\right)}{\sqrt{x}}$$

$$y = \sqrt{x}\left((1)^2 - \left(\sqrt{x}\right)^2\right) = \sqrt{x}(1-x) = (x)^{1/2} - x^{1/2+1}$$

$y = x^{1/2} - x^{3/2} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}x^{1/2} - \frac{d}{dx}x^{3/2}$

$$\frac{dy}{dx} = \frac{1}{2}x^{1/2-1} - \frac{3}{2}x^{3/2-1} = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2}$$

$$\frac{dy}{dx} = \frac{1-3x}{2\sqrt{x}}$$

8. Let $y = \frac{\left(x^2+1\right)^2}{x^2-1}$ *then*

$$\frac{dy}{dx} = \frac{\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right) - \left(x^2+1\right)^2}{\left(x^2-1\right)^2} \times \frac{d}{dx}\left(x^2-1\right)$$

$$= \frac{2x\left(x^2+1\right)\left[2\left(x^2-1\right) - \left(x^2+1\right)\right]}{\left(x^2+1\right)^2}$$

$$\frac{dy}{dx} = \frac{2x\left(x^2+1\right)\left(2x^2-2-x^2-1\right)}{\left(x^2-1\right)^2} = \frac{2x(x^2+1)(x^2-3)}{(x^2-1)^2}$$

9. Let $y = \frac{x^2+1}{x^2-3}$ *then*

$$\frac{dy}{dx} = \frac{\left(x^2-3\right)\frac{d}{dx}\left(x^2+1\right) - \left(x^2+1\right)\frac{d}{dx}\left(x^2-3\right)}{\left(x^2-3\right)^2}$$

$$= \frac{(x^2-3)(2x)-(x^2+1)(2x)}{(x^2-3)^2} = \frac{2x(x^2-3-x^2-1)}{(x^2-3)^2}$$

$$\frac{dy}{dx} = \frac{2x(-4)}{(x^2-3)^2} = \frac{-8x}{(x^2-3)^2}$$

10. **Let** $y = \frac{\sqrt{1+x}}{\sqrt{1-x}} = \frac{(1+x)^{1/2}}{(1-x)^{1/2}}$

$$\frac{dy}{dx} = \frac{(1-x)^{1/2}\frac{d}{dx}(1+x)^{1/2}-(1+x)^{1/2}\frac{d}{dx}(1-x)^{1/2}}{\left[(1-x)^{1/2}\right]^2}$$

$$\frac{dy}{dx} = \frac{(1-x)^{1/2}\frac{1}{2}(1+x)^{-1/2}(1)-(1+x)^{1/2}\frac{1}{2}(1-x)^{-1/2}(-1)}{(1-x)}$$

$$\frac{dy}{dx} = \frac{1}{(1-x)}\left[\frac{\sqrt{1-x}}{2\sqrt{1+x}}+\frac{\sqrt{1+x}}{2\sqrt{1-x}}\right] = \frac{1}{(1-x)}\left[\frac{1-x+1+x}{2\sqrt{1+x}\sqrt{1-x}}\right]$$

$$\frac{dy}{dx} = \frac{1}{(1-x)}\left[\frac{2}{2\sqrt{1+x}(1-x)^{1/2}}\right] = \frac{1}{\sqrt{1+x}(1-x)^{3/2}}$$

11. **Let** $y = \frac{2x-1}{\sqrt{x^2+1}}$

$$\frac{dy}{dx} = \frac{(x^2+1)^{1/2}\frac{d}{dx}(2x-1)-(2x-1)\frac{d}{dx}(x^2+1)^{1/2}}{\left[(x^2+1)^{1/2}\right]^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{(x^2+1)}(2)-(2x-1)\frac{1}{2}(x^2+1)^{-1/2}(2x)}{(x^2+1)}$$

$$\frac{dy}{dx} = \left[2\sqrt{(x^2+1)}-\frac{x(2x-1)}{\sqrt{x^2+1}}\right]\frac{1}{(x^2+1)}$$

$$= \frac{1}{(x^2+1)}\left[\frac{2(x^2+1)-(2x^2-x)}{\sqrt{x^2+1}}\right]\frac{2x^2+2-2x^2+x}{(x^2+1)(x^2+1)^{1/2}}$$

$$\frac{dy}{dx} = \frac{x+2}{(x^2+1)^{3/2}}$$

12. Let $y = \sqrt{\dfrac{a-x}{a+x}} = \dfrac{(a-x)^{1/2}}{(a+x)^{1/2}}$

$$\frac{dy}{dx} = \frac{(a+x)^{1/2}\frac{d}{dx}(a-x)^{1/2} - (a-x)^{1/2}\frac{d}{dx}(a+x)^{1/2}}{\left[(a+x)^{1/2}\right]^2}$$

$$\frac{dy}{dx} = \frac{(a+x)^{1/2}\frac{1}{2}(a-x)^{-1/2}(-1) - (a-x)^{1/2}\frac{1}{2}(a+x)^{-1/2}(+1)}{(a+x)}$$

$$\frac{dy}{dx} = \left[\sqrt{a+x}\frac{-1}{2\sqrt{a-x}} - \frac{\sqrt{a-x}}{2\sqrt{a+x}}\right]\frac{1}{(a+x)}$$

$$= \frac{1}{(a+x)}\left[\frac{-(a+x)-(a-x)}{2\sqrt{a-x}\sqrt{a+x}}\right] = \frac{-a-x-a+x}{2\sqrt{(a-x)}(a+x)^{3/2}}$$

$$\frac{dy}{dx} = \frac{-2a}{2\sqrt{a-x}(a+x)^{3/2}} = \frac{-a}{\sqrt{(a-x)}(a+x)^{3/2}}$$

13. Let $y = \dfrac{\sqrt{x^2+1}}{\sqrt{x^2-1}} = \dfrac{(x^2+1)^{1/2}}{(x^2-1)^{1/2}}$ **(Sargodha 2011)**

Then $$\frac{dy}{dx} = \frac{(x^2-1)^{1/2}\frac{d}{dx}(x^2+1)^{1/2} - (x^2+1)^{1/2}\frac{d}{dx}(x^2-1)^{1/2}}{\left[(x^2+1)^{1/2}\right]^2}$$

$$\frac{dy}{dx} = \frac{1}{(x^2-1)}\left[\sqrt{x^2-1}\,\frac{1}{2}(x^2+1)^{-1/2}(2x) - \sqrt{x^2+1}\,\frac{1}{2}(x^2-1)^{-1/2}(2x)\right]$$

$$\frac{dy}{dx} = \frac{1}{(x^2-1)}\left[\frac{x\sqrt{x^2-1}}{\sqrt{x^2+1}} - \frac{x\sqrt{x^2+1}}{\sqrt{x^2-1}}\right] = \frac{1}{(x^2-1)}\left[\frac{x(x^2-1)-x(x^2+1)}{\sqrt{x^2+1}\sqrt{x^2-1}}\right]$$

$$= \frac{x}{(x^2-1)}\left[\frac{x^2-1-x^2-1}{\sqrt{x^2+1}(x^2-1)^{1/2}}\right] = \frac{-2x}{\sqrt{x^2+1}(x^2-1)^{3/2}}$$

14. Let $y = \dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

$$y = \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} \times \frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}} = \frac{\left(\sqrt{1+x}-\sqrt{1-x}\right)^2}{\left(\sqrt{1+x}\right)^2 - \left(\sqrt{1-x}\right)^2}$$

$$y = \frac{\left(\sqrt{1+x}\right)^2+\left(\sqrt{1-x}\right)^2-2\sqrt{1+x}\sqrt{1-x}}{(1+x)-(1-x)} = \frac{1+x+1-x-2\sqrt{(1+x)(1-x)}}{1+x-1+x}$$

$$y = \frac{2-2\sqrt{1-x^2}}{2x} = \frac{2\left(1-\sqrt{1-x^2}\right)}{2x} = \frac{1-\sqrt{1-x^2}}{x}$$

Then $$\frac{dy}{dx} = \frac{x\frac{d}{dx}\left(1-\sqrt{1-x^2}\right)-\left(1-\sqrt{1-x^2}\right)\frac{d}{dx}(x)}{x^2}$$

$$\frac{dy}{dx} = \frac{1}{x^2}\left[x\left\{0-\frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)\right\}-\left(1-\sqrt{1-x^2}\right).1\right]$$

$$= \frac{1}{x^2}\left[\frac{x^2}{\sqrt{1-x^2}}-1+\sqrt{1-x^2}\right] = \frac{1}{x^2}\left[\frac{x^2-\sqrt{1-x^2}+1-x^2}{\sqrt{1-x^2}}\right]$$

$$= \frac{1-\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}$$

15. Let $$y = \frac{x\sqrt{a+x}}{\sqrt{a-x}}$$

Then $$\frac{dy}{dx} = x\frac{d}{dx}\frac{\sqrt{a+x}}{\sqrt{a-x}}+\frac{\sqrt{a+x}}{\sqrt{a-x}}\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x\left[\frac{\sqrt{a-x}\frac{d}{dx}(a+x)^{1/2}-\sqrt{a+x}\frac{d}{dx}(a-x)^{1/2}}{\left(\sqrt{a-x}\right)^2}\right]+\frac{\sqrt{a+x}}{\sqrt{a-x}}.1$$

$$\frac{dy}{dx} = \frac{x}{(a-x)}\left[\sqrt{a-x}\frac{1}{2}(a+x)^{-1/2}(1)-\sqrt{a+x}\frac{1}{2}(a-x)^{-1/2}(-1)\right]+\frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$\frac{dy}{dx} = \frac{x}{(a-x)}\left[\frac{\sqrt{a-x}}{2\sqrt{a+x}}+\frac{\sqrt{a+x}}{2\sqrt{a-x}}\right]+\frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$= \frac{x}{(a-x)}\left[\frac{a-x+a+x}{2\sqrt{a+x}\sqrt{a-x}}\right]+\frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$= \frac{x}{(a-x)}\left[\frac{2a}{2\sqrt{a+x}\sqrt{a-x}}\right]+\frac{\sqrt{a+x}}{\sqrt{a-x}}$$

$$= \frac{ax}{(a-x)\sqrt{a+x}\sqrt{a-x}}+\frac{\sqrt{a+x}}{\sqrt{a-x}} = \frac{ax+(a-x)(a+x)}{(a-x)\sqrt{a+x}\sqrt{a-x}}$$

$$= \frac{ax + a^2 - x^2}{\sqrt{a+x}(a-x)^{3/2}}$$

16. $y = \sqrt{x} - \frac{1}{\sqrt{x}} = x^{1/2} - x^{-1/2}$ **(Sargodha 2012, Lahore 2010)**

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \left(-\frac{1}{2}\right)x^{-3/2}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{2x^{3/2}}$$

$$\text{L.H.S} = 2x\frac{dy}{dx} + y = 2x\left(\frac{1}{2\sqrt{x}} + \frac{1}{2x^{3/2}}\right) + \sqrt{x} - \frac{1}{\sqrt{x}}$$

$$= \sqrt{x} + \frac{1}{\sqrt{x}} + \sqrt{x} - \frac{1}{\sqrt{x}}$$

$= 2\sqrt{x}$ Hence Proved

17. $y = x^4+2x^2+2$ **(Sargodha 2011)**

$$\frac{dy}{dx} = 4x^3+2.2x+0 = 4x^3+4x$$

$$\frac{dy}{dx} = 4x(x^2+1)$$

From I $y = x^4+2x^2+2$

or $y-1 = x^4+2x^2+2-1$

or $y-1 = x^4+2x^2+1$

or $y-1 = (x^2+1)^2$

or $\sqrt{y-1} = (x^2+1)$

Put value of x^2+1 1 in II then

$\frac{dy}{dx} = 4x\sqrt{y-1}$ Hence proved

$y = u^2$ $\quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$x = 2u$

Exercise 2.4

1. Find $\frac{dy}{dx}$ by making suitable substitutions in the following functions defined as:

(i) $y = \sqrt{\frac{1-x}{1+x}}$ Put $\frac{1-x}{1+x} = u$

Then $y = \sqrt{u} = (u)^{1/2}$ $\quad \Rightarrow \frac{dy}{du} = \frac{1}{2}(u)^{-1/2} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{\frac{1-x}{1+x}}}$

So $\frac{dy}{du} = \frac{1}{2}\sqrt{\frac{1+x}{1-x}}$ $\quad$ and $\quad u = \frac{1-x}{1+x}$

$$\frac{du}{dx} = \frac{(1+x)\frac{d}{dx}(1-x) - (1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{dy}{du} = \frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2} \cdot \frac{-1-x-1+x}{(1+x)^2} = \frac{-2}{(1+x)^2}$$

$$= \frac{dy}{du} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2}\sqrt{\frac{1+x}{1-x}} \cdot \frac{(-2}{(1+x)^2} = \frac{-1}{\sqrt{1-x}(1+x)^{2-1/2}}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x}(1+x)^{3/2}}$$

(ii) $y = \sqrt{x+\sqrt{x}}$ $\quad$ ***Put*** $\quad u = x + \sqrt{x}$ $\quad$ **(Sargodha 2010)**

then $y = \sqrt{u}$ $\quad \Rightarrow \quad \frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} = \frac{1}{2\sqrt{x+\sqrt{x}}}$

$u = x + \sqrt{x}$ $\quad \Rightarrow \quad \frac{du}{dx} = 1 + \frac{1}{2}(x)^{-1/2} = 1 + \frac{1}{2\sqrt{x}}$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{x+\sqrt{x}}}\left(1 + \frac{1}{2\sqrt{x}}\right)$$

(iii) $y = x\sqrt{\frac{a+x}{a-x}} = \sqrt{\frac{x^2(a+x)}{a-x}} = \left(\frac{ax^2+x^3}{a-x}\right)^{1/2}$ $\quad$ **(Faisalabad 2010)**

Take $\frac{ax^2+x^3}{a-x} = u$ $\quad$ *then*

$$y = (u)^{1/2} \Rightarrow \frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}} = \frac{1}{2}.\frac{1}{\sqrt{\frac{ax^2+x^3}{a-x}}}$$

$$\frac{dy}{du} = \frac{\sqrt{a-x}}{2\sqrt{ax^2+x^3}} \qquad Now \quad u = \frac{ax^2+x^3}{a-x}$$

$$\frac{du}{dx} = \frac{(a-x)\frac{d}{dx}(ax^2+x^3) - (ax^2+x^3)\frac{d}{dx}(a-x)}{(a-x)^2}$$

$$= \frac{(a-x)(2ax+3x^2)-(ax^2+x^3)(-1)}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{2a^2x+3ax^2-2ax^2-3x^3+ax^2+x^3}{(a-x)^2}$$

$$\frac{du}{dx} = \frac{-2x^3+2ax^2+2a^2x}{(a-x)^2} = \frac{x.2(-x^2+ax+a^2)}{(a-x)^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = \frac{\sqrt{a-x}}{2\sqrt{ax^2+x^3}}\times\frac{2x(a^2+ax-x^2)}{(a-x)^2}$$

$$= \frac{a^2+ax-x^2}{\sqrt{a+x}(a-x)^{3/2}}$$

(iv) **$y = (3x^2-2x+7)^6$ put $u = 3x^2-2x+7$ (Sargodha 2012)**

$$\frac{du}{dx} = 6x-2$$

$$\frac{dy}{du} = 6u^5 = 6(3x^2-2x+7)^5$$

Now $\frac{dy}{du} = \frac{dy}{du}.\frac{du}{dx} = 6(3x^2-2x+7)^5\times(6x-2)$

$= 6(6x-2)(3x^2-2x+7)^5$

(v) $y = \sqrt{\frac{a^2+x^2}{a^2-x^2}}$ *put* $u = \frac{a^2+x^2}{a^2-x^2}$

$$y = \sqrt{u} = (u)^{1/2} \quad , \frac{du}{dx} = \frac{(a^2-x^2)\frac{d}{dx}(a^2+x^2)-(a^2+x^2)\frac{d}{dx}(a^2-x^2)}{(a^2-x^2)^2}$$

$$\frac{dy}{dx}=\frac{1}{2}u^{-1/2}=\frac{1}{2\sqrt{u}}, \quad =\frac{(a^2-x^2)(2x)-(a^2+x^2)(-2x)}{(a^2-x^2)^2}$$

$$=\frac{1}{2\sqrt{\frac{a^2+x^2}{a^2-x^2}}}, =\frac{2x(a^2-x^2+a^2+x^2)}{(a^2-x^2)^2}$$

$$=\frac{1}{2}\cdot\sqrt{\frac{a^2-x^2}{a^2+x^2}}, =\frac{2x.2a^2}{(a^2-x^2)^2}=\frac{4a^2x}{(a^2-x^2)^2}$$

Now $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

$$\frac{dy}{dx}=\frac{1}{2}.\sqrt{\frac{a^2-x^2}{a^2+x^2}}\frac{^2 4a^2x}{(a^2-x^2)^2}=\frac{2a^2x}{\sqrt{a^2+x^2}(a^2-x^2)^{2-1/2}}$$

$$\frac{dy}{dx}=\frac{2a^2x}{\sqrt{a^2+x^2(a^2-x^2)^{3/2}}}$$

2. **Find $\frac{dy}{dx}$ if:**

(i) **3x+4y + 7 = 0**

Take derivative both sides

$$3(1)+4\frac{dy}{dx}+0=0 \Rightarrow 4.\frac{dy}{dx}=-3 \Rightarrow \frac{dy}{dx}=\frac{-3}{4}$$

(ii) **$xy + y^2 = 2$** **(Sargodha 2008,11)**

Take derivative both sides

$$x\frac{dy}{dx}+y.1+2y\frac{dy}{dx}=0 \Rightarrow (x+2y)\frac{dy}{dx}+y=0$$

$$(x+2y)\frac{dy}{dx}=-y \Rightarrow \frac{dy}{dx}=\frac{-y}{x+2y}$$

(iii) **$x^2-4xy-5y=0$** **(Sargodha 2011, Gujranwala 2010)**

Take derivative both sides

$$2x-4\left(x\frac{dy}{dx}+y.1\right)-5\frac{dy}{dx}=0 \Rightarrow 2x-4x\frac{dy}{dx}-4y-5\frac{dy}{dx}=0$$

$$2x-4y=4x\frac{dy}{dx}+5\frac{dy}{dx} \Rightarrow 2x-4y=(4x+5)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{2x-4y}{4x+5}$$

(iv) $4x^2+2hxy+by^2+2gx+2fy+c=0$

Taking derivative both sides

$$4.2x+2h\left(x\frac{dy}{dx}+y.1\right)+b.2y\frac{dy}{dx}+2g.1+2f\frac{dy}{dx}+0=0$$

$$8x+2hx\frac{dy}{dx}+2hy+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

$$(2hx+2by+2f)\frac{dy}{dx}=-8x-2hy-2g$$

$$2(hx+by+f)\frac{dy}{dx}=-2(4x+hy+y)$$

$$\frac{dy}{dx}=\frac{-(4x+hy+g)}{(hx+by+f)}$$

(v) $x\sqrt{1+y}+y\sqrt{1+x}=0$ **(Sargodha 2011)**

Taking derivatives

$$x\frac{1}{2}(1+y)^{-1/2}\frac{dy}{dx}+\sqrt{1+y}.1+y\frac{1}{2}(1+x)^{-1/2}.1+\sqrt{1+x}\frac{dy}{dx}=0$$

$$+\frac{x}{2\sqrt{1+y}}\frac{dy}{dx}+\sqrt{1+y}+\frac{y}{2\sqrt{1+x}}+\sqrt{1+x}\frac{dy}{dx}=0$$

$$\left(\sqrt{1+x}+\frac{x}{2\sqrt{1+y}}\right)\frac{dy}{dx}=-\sqrt{1+y}-\frac{y}{2\sqrt{1+x}}=-\left(\sqrt{1+y}+\frac{y}{2\sqrt{1+x}}\right)$$

$$\frac{dy}{dx}=\frac{-\left(\sqrt{1+y}+\frac{y}{2\sqrt{1+x}}\right)}{\left(\sqrt{1+x}+\frac{x}{2\sqrt{1+y}}\right)}=\frac{\frac{-(2\sqrt{1+y}\sqrt{1+x}+y)}{2\sqrt{1+x}}}{\frac{(2\sqrt{1+x}\sqrt{1+y}+x)}{2\sqrt{1+y}}}$$

$$\frac{dy}{dx}=\frac{-\left(2\sqrt{1+y}\sqrt{1+x}+y\right)}{\sqrt{1+x}}\times\frac{\sqrt{1+y}}{\left(2\sqrt{1+x}\sqrt{1+y}+x\right)}$$

$$\frac{dy}{dx}=\frac{-\sqrt{1+y}\left(y+2\sqrt{1+y}\sqrt{1+x}\right)}{\sqrt{1+x}\left(x+2\sqrt{1+x}\sqrt{1+y}\right)}$$

(vi) $y(x^2-1)=x\sqrt{x^2+4}$

Taking derivative both sides

$$y\,2x+(x^2-1)\frac{dy}{dx}=x\frac{1}{2}(x^2+4)^{-1/2}2x+\sqrt{x^2+4}.1$$

$$2xy + (x^2-1)\frac{dy}{dx} = \frac{x^2}{\sqrt{x^2+4}} + \sqrt{x^2+4} = \frac{x^2+x^2+4}{\sqrt{x^2+4}}$$ form –1

$$(x^2-1)\frac{dy}{dx} =$$

$$\frac{2x^2+4}{\sqrt{x^2+4}} - 2xy = \frac{2x^2+4}{\sqrt{x^2+4}} - \frac{2x^2\sqrt{x^2+4}}{(x^2-1)}$$

$$y(x^2-1) = x\sqrt{x^2+4}$$

$$(x^2-1)\frac{dy}{dx} = \frac{(2x^2+4)(x^2-1) - 2x^2(x^2+4)}{(x^2-1)\sqrt{x^2+4}}$$

Multiplying by 2x

$$\frac{dy}{dx} = \frac{2x^4 - 2x^2 + 4x^2 - 4 - 2x^4 - 8x^2}{(x^2-1)^2\sqrt{x^2+4}}$$

$$2xy(x^2-1)2x^2\sqrt{x^2+4}$$

$$= \frac{-6x^2-4}{(x^2-1)^2\sqrt{x^2+4}} = \frac{-2(3x^2+2)}{(x^2-1)^2\sqrt{x^2+4}}$$

$$\Rightarrow 2xy = \frac{2x^2\sqrt{x^2+1}}{x^2-1}$$

3. **Find $\frac{dy}{dx}$ of the following parametric functions:** **Gujranwala 2012**

(i) $x = \theta + \frac{1}{\theta}$, $y = \theta + 1$

$x = \theta + \theta^{-1}$, $y = \theta + 1$

$\frac{dx}{d\theta} = 1 + (-1)\,\theta^{-2}$, $\frac{dy}{d\theta} = 1 + 0$

$\frac{dx}{d\theta} = 1 - \frac{1}{\theta^2} = \frac{\theta^2-1}{\theta^2}$, $\frac{dy}{d\theta} = 1$

$$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = 1 \times \frac{\theta^2}{\theta^2-1} = \frac{\theta^2}{\theta^2-1}$$

(ii) $x = \frac{a(1-t^2)}{1+t^2}$, $y = \frac{2bt}{1+t^2}$

$$\frac{dx}{dt} = a\left[\frac{(1+t^2)\frac{d}{dt}(1-t^2) - (1-t^2)\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right] , \frac{dy}{dt} = 2b\left[\frac{(1+t^2)\frac{d}{dt}(t) - t\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right]$$

$$\frac{dx}{dt} = a\left(\frac{(1+t^2)(-2t) - (1-t^2)(2t)}{(1+t^2)^2}\right) , \frac{dy}{dt} = \frac{2b[(1+t^2)1 - t(2t)]}{(1+t^2)^2}$$

$$\frac{dx}{dt}=\frac{2at(-2)}{\left(1+t^2\right)^2} \quad , \quad \frac{dy}{dt}=\frac{2b\left(1-t^2\right)}{\left(1+t^2\right)^2}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{2b\left(1-t^2\right)}{(1+t^2)^2}\times\frac{\left(1+t^2\right)^2}{-4at}=-\frac{b\left(1-t^2\right)}{2at}$$

4. **Prove that y $\frac{dy}{dx}+x=0$ if $x=\frac{1-t^2}{1+t^2}$, $y=\frac{2t}{1+t^2}$** **(Sargodha 2009)**

x = $\frac{1-t^2}{1+t^2}$, $y=\frac{2t}{1+t^2}$

$$\frac{dx}{dt}=\frac{\left(1+t^2\right)\frac{d}{dt}\left(1-t^2\right)-\left(1-t^2\right)\frac{d}{dt}\left(1+t^2\right)}{\left(1+t2\right)^2}$$

$$\frac{dy}{dt}=\frac{\left(1+t^2\right)\frac{d}{dt}(2t)-(2t)\frac{d}{dt}\left(1+t^2\right)}{\left(1+t^2\right)^2}$$

$$\frac{dx}{dt}=\frac{\left(1+t^2\right)(-2t)-\left(1-t^2\right)(2t)}{\left(1+t^2\right)^2} \quad , \quad \frac{dy}{dt}=\frac{\left(1+t^2\right)2-2t.2t}{\left(1+t^2\right)^2}$$

$$=\frac{2t\left(-1-t^2-1+t^2\right)}{\left(1+t^2\right)^2} = \frac{-4t}{\left(1+t^2\right)^2} \quad , \quad \frac{dy}{dt}=\frac{2+2t^2-4t^2}{\left(1+t^2\right)^2}=\frac{2-2t^2}{\left(1+t^2\right)^2}$$

$$\frac{dy}{dt}=\frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}\times\frac{\left(1+t^2\right)^2}{-4t}=\frac{1-t^2}{-2t}$$

Now $\Rightarrow y\frac{dy}{dx}+x=\frac{2t}{\left(1+t^2\right)}\cdot\frac{\left(1-t^2\right)}{(-2t)}+\left(\frac{1-t^2}{1+t^2}\right)$

$$=-\left(\frac{1-t^2}{1+t^2}\right)+\left(\frac{1-t^2}{1+t^2}\right)=0$$

Hence $\Rightarrow y\frac{dy}{dx}+x=0$

5. Differentiate:

(i) $x^2 - \frac{1}{x^2}$ **w.r.t** x^4

Let $y = x^2 - \frac{1}{x^2} = x^2 - x^{-2}$ & $u = x^4$

$$\frac{dy}{dx} = (2x - (-2)x^{-3} = 2x + \frac{2}{x^3} \qquad \frac{du}{dx} = 4x^3$$

$$= \frac{2x^4+2}{x^3} = \frac{2(x^4+1)}{x^3} \qquad \frac{du}{dx} = \frac{1}{4x^3}$$

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du} = \frac{2(x^4+1)}{x^3} \times \frac{1}{4x^3} = \frac{x^4+1}{2x^6}$$

(ii) $(1+x^2)^n$ **w.r.t.** x^2

Let $y = (1+x^2)^n$ & $u = x^2$

$$\frac{dy}{dx} = n(1+x^2)^{n-1}(2x) \quad \& \quad \frac{du}{dx} = 2x$$

$$\frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du} = n(2x)(1+x^2)^{n-1} \times \frac{1}{2x} = n(1+x^2)^{n-1}$$

(iii) $\frac{x^2+1}{x^2-1}$ ***w.r.t*** $\frac{x-1}{x+1}$

Let $y = \frac{x^2+1}{x^2-1}$ & $u = \frac{x-1}{x+1}$

$$\frac{dy}{dx} = \frac{(x^2-1)2x - (x^2+1)2x}{(x^2-1)^2} \quad \& \quad \frac{du}{dx} = \frac{(x+1)-(x-1).1}{(x+1)^2}$$

$$= \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2} \quad \& \quad \frac{du}{dx} = \frac{x+1-x+1}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{2x(-2)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2} \quad \& \quad \frac{du}{dx} = \frac{2}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dx}\cdot\frac{dx}{du} = \frac{-4x}{(x^2-1)^2} \times \frac{(x+1)^2}{2} = \frac{-2x(x+1)^2}{[(x-1)(x+1)]^2}$$

$$= \frac{-2x(x+1)^2}{(x-1)^2(x+1)^2} = \frac{-2x}{(x-1)^2}$$

(iv) $\dfrac{ax+b}{cx+d}$ w.r.t $\dfrac{ax^2+b}{ax^2+d}$ and $u=\dfrac{ax^2+b}{ax^2+d}$

$$\frac{dy}{dx}=\frac{(cx+d)\frac{d}{dx}(ax+b)-(ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2} \quad \& \quad \frac{du}{dx}=\frac{(ax^2+d)\frac{d}{dx}(ax^2+b)-(ax^2+b)\frac{d}{dx}(ax^2+d)}{(ax^2+d)^2}$$

$$\frac{dy}{dx}=\frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2} \quad \& \quad \frac{du}{dx}=\frac{(ax^2+d)(2ax)-(ax^2+b)(2ax)}{(ax^2+d)^2}$$

$$\frac{dy}{dx}=\frac{acx+ad-acx-bc}{(cx+d)^2} \quad \& \quad \frac{du}{dx}=\frac{2ax(ax^2+d-ax^2-b)}{(ax^2+d)^2}$$

$$\frac{dy}{dx}=\frac{ad-bc}{(cx+d)^2} \quad \& \quad \frac{du}{dx}=\frac{2ax(d-b)}{(ax^2+d)^2}$$

$$\frac{dy}{du}=\frac{dy}{dx}\cdot\frac{dx}{du} \quad =\frac{ab-bc}{(cx+d)^2}\times\frac{(ax^2+d)^2}{2ax(d-b)}$$

(v) Let $y=\dfrac{x^2+1}{x^2-1}$ & $u=x^3 \Rightarrow \dfrac{du}{dx}=3x^2$

$$\frac{dy}{dx}=\frac{(x^2-1)2x-(x^2+1)2x}{(x^2-1)^2}=\frac{2x(x^2-1-x^2-1)}{(x^2-1)^2}=\frac{-4x}{(x^2-1)^2}$$

$$\frac{dy}{du}=\frac{dy}{dx}\times\frac{dx}{du}=\frac{-4x}{(x^2-1)^2}\times\frac{1}{3x^2}=\frac{-4}{3x(x^2-1)^2}=\frac{-4}{3x(x^2-1)^2}$$

Exercise 2.5

1. **Differentiate the following trigonometric functions from the first principle.**

(i) **Differentiate from the first principal Sin 2x**

Let $y = \text{Sin } 2x$

Then $y + \delta y = \text{Sin}2(x + \delta x) = \text{Sin}(2x + 2\delta x)$

$\delta y = \text{Sin}(2x + 2\delta x) - \text{Sin } 2x$

Use $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$$\delta y = 2\cos\left(\frac{2x + 2\delta x + 2x}{2}\right)\sin\left(\frac{2x + 2\delta x - 2x}{2}\right)$$

$$\delta y = 2\cos\left(\frac{4x + 2\delta x}{2}\right)\sin\left(\frac{2\delta x}{2}\right)$$

Divide both sides by δx & take limit $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{2\cos(2x + \delta x)\sin\delta x}{\delta x}$$

$$\frac{dy}{dx} = 2\lim_{\delta x\to 0}\cos(2x + \delta x).\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}$$

$$= 2\cos(2x + 0).1 = 2\cos 2x$$

(ii) **Tan3x**

Let $y = \text{Tan } 3x$

Then $y + \delta y = \text{Tan}(3x + 3\delta x) = \text{Tan}(3x + 3\delta x)$

$\delta y = \text{Tan}(3x + 3\delta x) - \text{Tan } 3x$

$$= \frac{\sin(3x + 3\delta x)}{\cos(3x + 3\delta x)} - \frac{\sin 3x}{\cos 3x}$$

$$= \frac{\sin(3x + 3\delta x)\cos 3x - \cos(3x + 3\delta x)\sin 3x}{\cos(3x + 3\delta x)\cos 3x}$$

Use $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \frac{\sin(3x + 3\delta x - 3x)}{\cos(3x + 3\delta x)\cos 3x}$$

Divide by δx & take limit $\delta x \to 0$ both sides

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{\sin 3\delta x}{\delta x} \times \lim_{\delta x\to 0}\frac{1}{\delta x}\frac{\sin 3\delta x}{\cos(3x + 3\delta x)\cos 3x}$$

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\sin 3\delta x}{\delta x} \times \lim_{\delta x\to 0}\frac{1}{\cos(3x + 3\delta x)\cos 3x}$$

$$= 3\lim_{\delta x\to 0}\frac{\sin 3\delta x}{3\delta x} \times \lim_{\delta x\to 0}\frac{1}{\cos(3x + 3\delta x)\cos 3x}$$

$$= 3(1)\times\frac{1}{Cos(3x+0)Cos3x} = 3\frac{1}{Cos3x.Cos3x} = 3\frac{1}{Cos^2 3x}$$

$$\frac{dy}{dx} = 3Sec^2 3x$$

(iii) **Sin2x+Cos2x** **(Faisalabad 2010)**

Let $y = Sin2x+Cos2x$

Then $y+\delta y = Sin2(x+\delta x)+Cos2(x+\delta x)$

$= Sin(2x+2\delta x)+Cos(2x+2\delta x)$

$\delta y = Sin(2x+2\delta x)+Cos(2x+2\delta x)-(Sin2x+Cos2x)$

$= Sin(2x+2\delta x)-Sin2x+Cos(2x+2\delta x)-Cos2x$

$$\delta y = 2Cos\frac{(2x+2\delta x+2x)}{2}Sin\frac{2x+2\delta x-2x}{2}+$$

$$(-2)Sin\left(\frac{2x+2\delta x+2x}{2}\right)Sin\left(\frac{2x+2\delta x-2x}{2}\right)$$

$$= 2Cos\left(\frac{4x+2\delta x}{2}\right)Sin\left(\frac{2\delta x}{2}\right)-2Sin\left(\frac{4x+2\delta x}{2}\right)Sin\left(\frac{2\delta x}{2}\right)$$

$\delta y = 2Cos(2x+\delta x)\,Sin\,\delta x - 2Sin(2x+\delta x)Sin\,\delta x$

$= [2Cos(2x+\delta x)-2Sin(2x+\delta x)]\,Sin\,\delta x$

Divide both sides by δx & Take limit $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{dx} = \lim_{\delta x\to 0}\frac{[2Cos(2x+\delta x)-2Sin(2x+\delta x)]}{\delta x}Sin\delta x$$

$$\frac{dy}{dx} = \lim_{\delta x\to 0}2Cos(2x+\delta x)-2Sin(2x+\delta x)]\times\lim_{\delta x\to 0}\frac{Sin\delta x}{\delta x}$$

$$\frac{dy}{dx} = [2Cos(2x+0)-2Sin(2x+0)]\times 1$$

$= 2Cos2x-2Sin2x$

(iv) **Cosx^2**

Let $y = Cos\,x^2$

Then $y+\delta y = Cos(x+\delta x)^2 \Rightarrow \delta y = Cos\,(x+\delta x)^2 - x^2$

Use $Cos\alpha - Cos\beta = -2Sin\frac{\alpha+\beta}{2}Sin\frac{\alpha-\beta}{2}$

$$\delta y = -2Sin\left(\frac{(x+\delta x)^2+x^2}{2}\right)Sin\frac{(x+\delta x)^2-x^2}{2}$$

$$= -2Sin\left(\frac{x^2+2x\delta x+\delta x^2+x^2}{2}\right)Sin\left(\frac{x^2+2x\delta x+\delta x^2-x^2}{2}\right)$$

$$= -2\text{Sin}\left(\frac{2x^2+2x\delta x+\delta x^2}{2}\right)Sin\left(\frac{2x+\delta x}{2}\right)\delta x$$

$$= -2\text{Sin}\left(x^2+x\delta x+\frac{\delta x^2}{x}\right)Sin\left(\frac{2x+\delta x}{2}\right)\delta x$$

Divide both side by δ x& take limit δ x $\rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{-2Sin\left(x2+x\delta x+\frac{\delta x2}{2}\right)Sin\left(x+\frac{\delta x}{2}\right)\delta x}{\delta x}$$

Multiplying and dividing by $\left(x+\frac{\delta x}{2}\right)$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}-2Sin\left(x^2+x\delta x+\frac{\delta x^2}{2}\right)\times\lim_{\delta x\to 0}\frac{Sin\left(x+\frac{\delta x}{2}\right)}{\left(x+\frac{\delta x}{2}\right)\delta x}\times\lim_{\delta x\to 0}\left(x+\frac{\delta x}{2}\right)$$

$= -2\text{Sin}x^2 \times (1) \times (x+0)$

$= -2\text{Sin}x^2 \times x = -2x\ \text{Sin}\ x^2$

(v) **Tan^2 x**

Let $y = \text{Tan}^2 x$ then $y + \delta y = \text{Tan}^2(x+\delta x)$

Use $a^2-b^2 = (a + b)(a - b)$

$\delta y = \text{Tan}^2(x+\delta x) - \text{Tan}^2x = [\text{Tan}(x+\delta x)+\text{Tan}x]\ [\text{Tan}(x+\delta x)-\text{Tan}\ x]$

$$\delta y = \left[\frac{Sin(x+\delta x)}{\cos(x+\delta x)}+\frac{Sinx}{Cosx}\right]\left[\frac{Sin(x+\delta x)}{Cos(x+\delta x)}-\frac{Sinx}{Cosx}\right]$$

$$\delta y = \left[\frac{Sin(x+\delta x)Cosx+Cos(x+\delta x)Sinx}{Cos(x+\delta x)Cosx}\right]\left[\frac{Sin(x+\delta x)Cosx-Cos(x+\delta x)Sinx}{Cos(x+\delta x)Cosx}\right]$$

$$\delta y = \frac{Sin(x+\delta x+x)}{Cos(x+\delta x)Cosx}\times\frac{Sin(x+\delta x-x)}{Cos(x+\delta x)Cosx}=\frac{Sin(2x+\delta x)Sin\delta x}{Cos^2(x+\delta x)Cos^2x}$$

Divide by δx and take limit $\delta x \rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{Sin(2x+\delta x)Sin\delta x}{Cos2(x+\delta x)Cos^2x\delta x}=\frac{Sin(2x+\delta x)Sin\delta x}{Cos^2(x+\delta x)Cos^2x\delta x}$$

$$\frac{dy}{dx}=\lim_{\delta x\to 0}\frac{Sin(2x+\delta x)}{Cos^2(x+\delta x)Cos^2x}\times\lim_{\delta x\to 0}\frac{Sin\delta x}{\delta x}$$

$$\frac{dy}{dx}=\frac{Sin(2x+0)}{Cos^2(x+0)Cos^2x}\times 1=\frac{Sin2x}{Cos^2x\times Cos^2x}$$

$$\frac{dy}{dx}=\frac{2SinxCosx}{Cos^2x}\times\frac{1}{Cos^2}=2\frac{Sinx}{Cosx}\times\frac{1}{Cos^2x}$$

$$\frac{dy}{dx}=2Tanx\,Sec^2x$$

(vi) $\sqrt{Tan\,x}$ **(Sargodha 2010)**

Let $y=\sqrt{Tanx} \Rightarrow y+\delta y=\sqrt{Tan(x+\delta x)}$

$$\delta y=\left(\sqrt{Tan(x+\delta x)}-\sqrt{Tanx}\right)\times\frac{\left(\sqrt{Tan(x+\delta x)}+\sqrt{Tanx}\right)}{\left(\sqrt{Tan(x+\delta x)}+\sqrt{Tanx}\right)}$$

$$\delta y=\frac{\left(\sqrt{Tan(x+\delta x)}\right)^2-\left(\sqrt{Taxx}\right)^2}{\sqrt{Tan(x+\delta x)}+\sqrt{Tanx}}=\frac{Tan(x+\delta x)-Tanx}{\sqrt{Tan(x+\delta x)}+\sqrt{Tanx}}$$

$$\delta y=\frac{\frac{Sin(x+\delta x)}{Cos(x+\delta x)}-\frac{Sinx}{Cosx}}{\sqrt{Tan(x+\delta x)}+\sqrt{Tanx}}=\frac{\frac{Sin(x+\delta x)Cosx-Cos(x+\delta x)Sinx}{Cos(x+\delta x)+Cosx}}{\sqrt{Tan(x+\delta x)}+\sqrt{Tanx}}$$

$$=\frac{Sin(x+\delta x-x)}{\sqrt{(Tan(x+\delta x)}+\sqrt{Tanx})(Cos(x+\delta x).Cosx)}$$

Divide both sides by δ x & take limit δ x$\rightarrow$0

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left[\frac{Sin\delta x}{\delta x}\times\lim_{\delta x\to 0}\frac{1}{Cos(x+\delta x).Cosx}\times\lim_{\delta x\to 0}\frac{1}{\sqrt{Tan(x+\delta x)}+\sqrt{Tanx}}\right]$$

$$\frac{dy}{dx}=(1)\times\frac{1}{Cosx.Cosx}\times\frac{1}{\sqrt{Tan\,x}+\sqrt{Tan\,x}}=\frac{1}{Cos^2x.2\sqrt{Tan\,x}}$$

$$=\frac{1}{2\sqrt{Tan\,x}}Sec^2x$$

(vii) Cos $\sqrt{x}$ **(Sargodha 2009)**

Find Simple derivative

Let y = Cos $\sqrt{x}$

Then $y+\delta y=Cos\sqrt{x+\delta x} \Rightarrow \delta y=Cos\sqrt{x+\delta x}-Cos\sqrt{x}$

$$\delta y=-2\,Sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)Sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)$$

Divide both sides by δx & take limit $\delta x\rightarrow 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=-2\lim_{\delta x\to 0}\left[\frac{Sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)Sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}\right]$$

Use $4\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)=\delta x\ then$

$$\frac{dy}{dx}=-2\lim_{\delta x\to 0}\left[\frac{Sin\left(\frac{\sqrt{x+\delta x+\sqrt{x}}}{2}\right)Sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{4\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}\right]$$

$$=-\frac{1}{2}\lim_{\delta x\to 0}\frac{Sin\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)}{\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}}\times\lim_{\delta x\to 0}\frac{Sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

$$=-\frac{1}{2}\frac{Sin\left(\frac{\sqrt{x+0}+\sqrt{x}}{2}\right)}{\frac{\sqrt{x+0}+\sqrt{x}}{2}}\times 1$$

$$=-\frac{1}{2}\frac{Sin\frac{2\sqrt{x}}{2}}{\frac{2\sqrt{x}}{2}}=-\frac{1}{2\sqrt{x}}Sin\sqrt{x}$$

2. • Differentiate the following w.r.t the variable involved.

(i) Let $y = x^2Sec4x$

Then $\frac{dy}{dx}=x^2\frac{d}{dx}Sec4x+Sec4x\frac{d}{dx}x^2$

$=x^2\left(Sec4xTan4x\frac{d}{dx}(4x)\right)+(Sec4x.2x)$

$=x^2Sec4xTan4x(4)+2xSec4x$

$=2x\ Sec4x\ (1+2x\ Tan4x)$

(ii) Let $y = Tan^3\theta Sec^2\theta$

$$\frac{dy}{d\theta} = Tan^3\theta\frac{d}{d\theta}Sec^2\theta + Sec^2\theta\frac{d}{d\theta}Tan^3\theta$$

$$= Tan^3\theta\left(2Sec\theta\frac{d}{d\theta}Sec\theta\right) + Sec^2\theta\left(3Tan^2\theta\frac{d}{d\theta}Tan\theta\right)$$

$= Tan^3\theta \cdot 2Sec\theta \cdot Sec\theta\ Tan\theta + Sec^2\theta\ .3Tan^2\theta\ .Sec^2\theta$

$= 2Tan^4\theta\ Sec^2\theta + 3tan^2\theta\ Sec^4\theta$

$= Tan^2\theta Sec^2\theta\ (2Tan^2\theta + 3Sec^2\theta)$

(iii) Let $y = (Sin2\theta - Cos3\theta)^2$

$$\frac{dy}{d\theta} = 2(Sin2\theta - Cos3\theta)^1\frac{d}{d\theta}(Sin2\theta - Cos3\theta)$$

$$= 2(Sin2\theta - Cos3\theta)\left[Cos2\theta\frac{d}{d\theta}2\theta - (-Sin3\theta)\frac{d}{d\theta}3\theta\right]$$

$= 2(Sin2\theta - Cos3\theta)[Cos2\theta . 2 + Sin3\theta]$

$= 2(Sin2\theta - Cos3\theta)(2Cos2\theta + 3Sin3\theta)$

(iv) Let $y = Cos\sqrt{x} + \sqrt{Sin\,x} = Cos(x)^{1/2} + (Sin\,x)^{1/2}$

$$\frac{dy}{dx} = -Sin(x)^{1/2}\frac{d}{dx}(x)^{1/2} + \frac{1}{2}(Sin\,x)^{-1/2}\frac{d}{dx}Sin\,x$$

$$= -Sin\sqrt{x}\frac{1}{2}x^{-1/2} + \frac{1}{2}(Sinx)^{-1/2}\frac{d}{dx}Sinx$$

$$= -Sin\sqrt{x}\frac{1}{2}x^{-1/2} + \frac{1}{2}\frac{1}{(Sinx)^{1/2}}Cosx$$

$$= \frac{1}{2}\left[\frac{-Sin\sqrt{x}}{\sqrt{x}} + \frac{Cos\,x}{\sqrt{Sin\,x}}\right]$$

3. Find $\frac{dy}{dx}$ if

(i) $y = x\,Cos\,y$ **(Sargodha 2009)**

Taking derivative w.r.t.x both sides

$$\frac{dy}{dx} = x\frac{d}{dx}Cos\,y + Cos\,y\frac{d}{dx}(x)$$

$$= x\left(-Sin\,y\frac{dy}{dx}\right) + Cos\,y.1 = -x\,Sin\,y\frac{dy}{dx} + Cos\,y$$

$$\frac{dy}{dx} + x\,Sin\,y\frac{dy}{dx} = Cos\,y$$

$(1+x\,\text{Sin}\,y)\dfrac{dy}{dx}=\text{Cos}\,y \Rightarrow \dfrac{dy}{dx}=\dfrac{\text{Cos}\,y}{1+x\,\text{Sin}\,y}$

(ii) **x = y Sin y** **(Sargodha 2011)**

Taking derivative w.r.t.x both sides

$$1 = y\frac{d}{dx}\text{Sin}\,y+\text{Sin}y\frac{d}{dx}y$$

$$1 = y\,\text{Cos}\,y\frac{dy}{dx}+\text{Sin}y\frac{dy}{dx}$$

$$1 = (y\,\text{Cos}y+\text{Sin}y)\frac{dy}{dx} \Rightarrow \frac{1}{y\text{Cos}\,y+\text{Sin}y}=\frac{dy}{dx}$$

4. **Find the derirative w.r.t. x.**

(i) **Let** $y = \text{Cos}\sqrt{\dfrac{1+x}{1+2x}}$ **(Sargodha 2008)**

Then $\dfrac{dy}{dx}=-\text{Sin}\sqrt{\dfrac{1+x}{1+2x}}\dfrac{d}{dx}\sqrt{\dfrac{1+x}{1+2x}}$

$$=-\text{Sin}\sqrt{\frac{1+x}{1+2x}}\frac{d}{dx}\frac{(1+x)^{1/2}}{(1+2x)^{1/2}}$$

$$=-\text{Sin}\sqrt{\frac{1+x}{1+2x}}\left[\frac{(1+2x)^{1/2}\frac{d}{dx}(1+x)^{1/2}-(1+x)^{1/2}\frac{d}{dx}(1+2x)^{1/2}}{\left[(1+2x)^{1/2}\right]^2}\right]$$

$$=-\text{Sin}\sqrt{\frac{1+x}{1+2x}}\left[\frac{(1+2x)^{1/2}\frac{1}{2}(1+x)^{-1/2}(0+1)-(1+x)^{1/2}\frac{1}{2}(1+2x)^{-1/2}(0+2)}{(1+2x)}\right]$$

$$=-\text{Sin}\sqrt{\frac{1+x}{1+2x}}\frac{1}{(1+2x)}\left[\frac{\sqrt{1+2x}}{2\sqrt{1+x}}-\frac{2\sqrt{1+x}}{2\sqrt{1+2x}}\right]$$

$$=-\text{Sin}\sqrt{\frac{1+x}{1+2x}}\frac{1}{(1+2x)}\left[\frac{(1+2x)-2(1+x)}{2\sqrt{1+x}\sqrt{1+2x}}\right]$$

$$=-\text{Sin}\sqrt{\frac{1+x}{1+2x}}\frac{1}{(1+2x)}\left[\frac{1+2x-2-2x}{2\sqrt{1+x}\sqrt{1+2x}}\right]$$

$$= -\text{Sin}\sqrt{\frac{1+x}{1+2x}}\frac{1}{(1+2x)}\left[\frac{-1}{2\sqrt{1+x}\sqrt{1+2x}}\right]$$

$$= +\,Sin\sqrt{\frac{1+x}{1+2x}}\left(\frac{1}{2\sqrt{1+x}\,(1+2x)^{3/2}}\right)$$

(ii). **Let** $y = \text{Sin}\sqrt{\frac{1+2x}{1+x}}$

$$\frac{dy}{dx} = Cos\sqrt{\frac{1+2x}{1+x}}\frac{d}{dx}\sqrt{\frac{1+2x}{1+x}} = Cos\sqrt{\frac{1+2x}{1+x}}\frac{d}{dx}\frac{(1+x)^{1/2}}{(1+x)^{1/2}}$$

$$= \text{Cos}\sqrt{\frac{1+2x}{1+x}}\left[\frac{(1+x)^{1/2}\frac{d}{dx}(1+2x)^{1/2} - (1+2x)^{1/2}\frac{d}{dx}(1+x)^{1/2}}{\left[(1+x)^{1/2}\right]^2}\right]$$

$$= \text{Cos}\sqrt{\frac{1+2x}{1+x}}\frac{1}{(1+x)}\left[\frac{2\sqrt{1+x}}{2\sqrt{1+2x}} - \frac{\sqrt{1+2x}}{2\sqrt{1+x}}\right]$$

$$= \text{Cos}\sqrt{\frac{1+2x}{1+x}}\frac{1}{1+x}\left[\frac{2(1+x)-(1+2x)}{2\sqrt{1+2x}\sqrt{1+x}}\right]$$

$$= \text{Cos}\sqrt{\frac{1+2x}{1+x}}\left[\frac{2+2x-1-2x}{2\sqrt{1+2x}(1+x)^{3/2}}\right]$$

$$= \text{Cos}\sqrt{\frac{1+2x}{1+x}}\left(\frac{1}{2\sqrt{1+2x}(1+x)^{3/2}}\right)$$

5. **Differentiate**

(i) **Sin x w.r.t Cot x** **(Sargodha 2008,11, Gujranwala 2012)**

Let $y = \text{Sin } x$ & $u = \text{Cot } x$

$$\frac{dy}{dx} = Cos\,x \quad \& \quad \frac{du}{dx} = -Co\sec^2 x$$

$$\frac{dy}{du} = \frac{dy}{dx}\times\frac{dx}{du} = Cosx\times\frac{-1}{Co\sec^2 x} = -Cosx\,Sin^2x$$

(ii) **Sin^2x w.r.t Cos^4x**

Let $y = \text{Sin}^2x$ and $u = \text{Cos}^4x$

$$\frac{dy}{dx} = 2\text{Sin}^1x\,\frac{d}{dx}(Sinx) = 2Sinx\,Cosx$$

$$\frac{du}{dx} = 4\,Cos^3x\frac{d}{dx}Cosx = 4Cos^3x(-Sinx) = -4Cos^3xSinx$$

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du} = 2Sinx\,Cosx \times \frac{1}{-4Cos^3x\,Sinx}$$

$$\frac{dy}{du} = \frac{1}{-2Cos^2x} = -\frac{1}{2}Sec^2x$$

6. If tany (1+tanx)=1–tanx . Show that $\frac{dy}{dx} = -1$ (Gujranwala 2010)

if tany (1+Tanx) = 1 – Tanx $\Rightarrow$ $Tany = \frac{1-Tanx}{1+Tanx}$

Tany = $\frac{1-Tanx}{1+1.Tanx}$ Put $1 = Tan\frac{\pi}{4}$

Tany = $\frac{Tan\frac{\pi}{4} - Tanx}{1+Tan\frac{\pi}{4}Tanx} = Tan\left(\frac{\pi}{4} - x\right)$

Use
Tan $(\alpha - \beta) = \frac{Tan\alpha - Tan\beta}{1+Tan\alpha\,Tan\beta}$

y = $\frac{\pi}{4} - x \Rightarrow \frac{dy}{dx} = -1$

Example Differentiate Cos^4x w.r.t.x

y = Cos^4x

$\frac{dy}{dx} = 4Cos^3x\left(\frac{d}{dx}Cos\,x\right)$

= $4Cos^3x\,(-Sinx)$

= $-4Cos^3x\,Sinx$

7. If y = $\sqrt{Tanx + \sqrt{Tanx + \sqrt{Tanx + ... + \infty}}}$ prove that (2y–1) $\frac{dy}{dx}$ = sec^2x

Let y = $\sqrt{Tanx + \sqrt{Tanx + \sqrt{Tanx + ... + \infty}}}$

We can observe that

y = $\sqrt{Tanx + y}$

Squaring both sides

y^2 = Tanx + y

Taking derivatives both sides

$$2y\frac{dy}{dx} = Sec^2x + \frac{dy}{dx}$$

$$\Rightarrow \quad 2y\frac{dy}{dx} - \frac{dy}{dx} = Sec^2x$$

$$\Rightarrow \quad (2y-1)\frac{dy}{dx} = Sec^2x \quad \textit{Hence proved.}$$

8. **if** $x = aCos^3\theta$, $y = bSin^3\theta$ **show that a** $\frac{dy}{dx}$ **+btan** θ **=0**

$x = aCos^3\theta$, $y = bSin^3\theta$

$$\frac{dx}{d\theta}=3aCos^2\theta\frac{d}{d\theta}(Cos\theta) \quad , \quad \frac{dy}{d\theta}=3bSin^2\theta\frac{d}{d\theta}Sin\theta$$

$$\frac{dx}{d\theta}=3aCos^2\theta(-Sin\theta) \quad , \quad \frac{dy}{d\theta}=3bSin^2\theta Cos\theta$$

$$=-3aCos^2\theta\,Sin\theta$$

Now $\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = 3bSin^2\theta\,Cos\theta \times \frac{-1}{3aCos^2\theta\,Sin\theta}$

$$\frac{dy}{dx} = \frac{-b\,Sin\theta}{a\,Cos\theta} \Rightarrow \frac{dy}{dx}=-\frac{b}{a}Tan\theta$$

$$\Rightarrow a\frac{dy}{dx}=-bTan\theta \Rightarrow a\frac{dy}{dx}+bTan\theta=\theta$$

9. **Find** $\frac{dy}{dx}$ **if** $x = a\,(Cost + Sint)$, $y = a\,(Sint - tCost)$ **(Sargodha 2009)**

$x = a\,(Cost + Sint)$, $y = a\,(Sint - tCost)$

$$\frac{dx}{dt}=a(-Sint+Cost) \quad , \quad \frac{dy}{dt}=a[Cost-(t(-Sint)+Cost.1)]$$

$$= a\,(Cost - Sint) = a[Cost + t\,Sin\,t - Cost]$$

$$= at\,Sin\,t$$

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx} = at\,Sint\times\frac{1}{a(Cost-Sint)}=\frac{tSint}{Cost-Sint}$$

10. **Differentiate w.r.t. x** 12

(i) **Let** $y = Cos^{-1}\frac{x}{a}$

Then $\frac{dy}{dx}=\frac{-1}{\sqrt{1-\frac{x^2}{a^2}}}\frac{d}{dx}\left(\frac{x}{a}\right)$

$$=\frac{-1}{\sqrt{\frac{a^2-x^2}{a^2}}}\left(\frac{1}{a}\right)=\frac{-1}{\sqrt{\frac{a^2-x^2}{a^2}}}\left(\frac{1}{a}\right)=\frac{-a}{\sqrt{a^2-x^2}}\left(\frac{1}{a}\right)$$

$$\frac{dy}{dx}=\frac{-1}{\sqrt{a^2-x^2}}$$

Formulas:

$$\frac{d}{dx}Sin^{-1}x=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}Cos^{-1}x=\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}Tan^{-1}x=\frac{1}{1+x^2}$$

(ii) Let $y = Cot^{-1}\frac{x}{a}$ (Sargodha 2008)

Then

$$\frac{dy}{dx} = \frac{-1}{1+\frac{x^2}{a^2}}\frac{d}{dx}\left(\frac{x}{a}\right) = \frac{-1}{\frac{a^2+x^2}{a^2}}\left(\frac{1}{a}\right)$$

$$\frac{dy}{dx} = \frac{-1}{1+\frac{x^2}{a^2}}\frac{d}{dx}\left(\frac{x}{a}\right) = \frac{-1}{\frac{a^2+x^2}{a^2}}\left(\frac{1}{a}\right)$$

$$= \frac{-a^2}{a^2+x^2}\left(\frac{1}{a}\right) = \frac{-a}{a^2+x^2}$$

$$\frac{d}{dx}Cot^{-1}x = \frac{-1}{1+x^2}$$
$$\frac{d}{dx}Sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}Cose^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$$

(iii) Let $y = \frac{1}{a}Sin^{-1}\frac{a}{x}$ (Sargodha 2011,12 Lahore 2010, Gujranwala 2010)

Then $$\frac{dy}{dx} = \frac{1}{a}\frac{1}{\sqrt{1-\frac{a^2}{x^2}}}\frac{d}{dx}\left(\frac{a}{x}\right) = \frac{1}{a}\frac{1}{\sqrt{\frac{x^2-a^2}{x^2}}}\frac{d}{dx}ax^{-1}$$

$$\frac{dy}{dx} = \frac{1}{a}\frac{1}{\sqrt{\frac{x^2-a^2}{x^2}}}a(-1)x^{-2} = \frac{1}{a}\frac{x}{\sqrt{x^2-a^2}}.\frac{-a}{x^2} = \frac{-1}{x\sqrt{x^2-a^2}}$$

(iv) Let $y = Sin^{-1}\sqrt{1-x^2}$

Then $$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\frac{d}{dx}\sqrt{1-x^2} = \frac{1}{\sqrt{1-(1-x^2)}}\frac{d}{dx}(1-x^2)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-1+x^2}}\frac{1}{2}(1-x^2)^{-1/2}(0-2x) = \frac{1}{\sqrt{x^2}}\frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= \frac{1}{x}.\frac{-x}{\sqrt{1-x^2}} = \frac{-1}{\sqrt{1-x^2}}$$

(v) Let $y = Sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$ *Then*

$$\frac{dy}{dx} = \frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2-1}}\frac{d}{dx}\frac{x^2+1}{x^2-1}$$

$$=\frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\frac{\left(x^2+1\right)^2-\left(x^2-1\right)^2}{\left(x^2-1\right)^2}}}\left[\frac{\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right)-\left(x^2+1\right)\frac{d}{dx}\left(x^2-1\right)}{\left(x^2-1\right)^2}\right]$$

$$=\frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\frac{x^4+2x^2+1-x^4+2x^2-1}{\left(x^2-1\right)^2}}}\left[\frac{2x\left(x^2-1-x^2-1\right)}{\left(x^2-1\right)^2}\right]$$

$$=\frac{\left(x^2-1\right)^2}{\left(x^2+1\right)\sqrt{4x^2}}\left[\frac{-4x}{\left(x^2-1\right)^2}\right]=\frac{-4x}{\left(x^2+1\right)(2x)}$$

$$=\frac{-2}{\left(x^2+1\right)}$$

(vi) **Let** $y=\mathrm{Cot}^{-1}\left(\frac{2x}{1-x^2}\right)$

Then $$\frac{dy}{dx}=\frac{-1}{1+\left(\frac{2x}{1-x^2}\right)^2}\frac{d}{dx}\left(\frac{2x}{1-x^2}\right)$$

$$=\frac{-1}{1+\frac{4x^2}{\left(1-x^2\right)^2}}\left[\frac{\left(1-x^2\right)\frac{d}{dx}(2x)-2x\frac{d}{dx}\left(1-x^2\right)}{\left(1-x^2\right)^2}\right]$$

$$=\frac{-1}{\frac{\left(1-x^2\right)^2+4x^2}{\left(1-x^2\right)^2}}\left[\frac{\left(1-x^2\right)2-2x(0-2x)}{\left(1-x^2\right)^2}\right]$$

$$=\frac{-\left(1-x^2\right)^2}{1-2x^2+x^4+4x^2}\left(\frac{2-2x^2+4x^2}{\left(1-x^2\right)^2}\right)$$

$$\frac{dy}{dx}=\frac{-1}{1+2x^2+x^4}\left(2+2x^2\right)=\frac{-2\left(1+x^2\right)}{\left(1+x^2\right)^2}=\frac{-2}{1+x^2}$$

(vii) Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Then $$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}}\frac{d}{dx}\frac{1-x^2}{1+x^2}$$

$$= \frac{-1}{\sqrt{1-\frac{(1-x^2)^2}{(1+x^2)^2}}}\left[\frac{(1+x^2)\frac{d}{dx}(1-x^2)-(1-x^2)\frac{d}{dx}(1+x^2)}{(1+x^2)^2}\right]$$

$$= \frac{-1}{\sqrt{\frac{(1+x^2)^2-(1-x^2)^2}{(1+x^2)^2}}}\left[\frac{(1+x^2)(-2x)-(1-x^2)2x}{(1+x^2)^2}\right]$$

$$= \frac{-1}{\sqrt{\frac{1+2x^2+x^4-(1-2x^2+x^4)}{(1+x^2)^2}}}\frac{2x(-1-x^2-1+x^2)}{(1+x^2)^2}$$

$$= \frac{(1+x^2)}{\sqrt{1+2x^2+x^4-(1-2x^2+x^4)}}\frac{2x(-2)}{(1+x^2)^2} = \frac{-4x}{\sqrt{4x^2}(1+x^2)} = \frac{-4x}{2x(1+x^2)}$$

$$= \frac{dy}{dx} = \frac{-2}{(1+x^2)}$$

11. Show that $\frac{dy}{dx} = \frac{y}{x}$ if $\frac{y}{x} = Tan^{-1}\frac{x}{y}$ (Sargodha 2010,12)

$$\frac{y}{x} = Tan^{-1}\frac{x}{y} \quad \Rightarrow \quad y = x\,Tan^{-1}\frac{x}{y}$$

Taking derivative both sides.

$$\frac{dy}{dx} = x\frac{d}{dx}Tan^{-1}\frac{x}{y} + Tan^{-1}\frac{x}{y}\frac{d}{dx}x$$

$$= x\left[\frac{1}{1+\frac{x^2}{y^2}}\frac{d}{dx}\frac{x}{y}\right] + Tan^{-1}\frac{x}{y}\frac{d}{dx}x$$

$$= x\left[\frac{1}{\frac{y^2+x^2}{y^2}}\right]\left(\frac{y.1 - x\frac{dy}{dx}}{y^2}\right) + Tan^{-1}\frac{x}{y}$$

$$= \frac{xy^2}{x^2+y^2}\left(\frac{y - x\frac{dy}{dx}}{y^2}\right) + Tan^{-1}\frac{x}{y}$$

We know given $Tan^{-1}\frac{x}{y} = \frac{y}{x}$

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2} - \frac{x^2}{x^2+y^2}\frac{dy}{dx} + \frac{y}{x}$$

Or $$\frac{dy}{dx} + \frac{x^2}{x^2+y^2}\frac{dy}{dx} = \frac{xy}{x^2+y^2} + \frac{y}{x}$$

$$\left(1 + \frac{x^2}{x^2+y^2}\right)\frac{dy}{dx} = y\left(\frac{x}{x^2+y^2} + \frac{1}{x}\right)$$

$$\left(\frac{x^2+y^2+x^2}{x^2+y^2}\right)\frac{dy}{dx} = y\left(\frac{x^2+x^2+y^2}{x(x^2+y^2)}\right)$$

$$\left(\frac{2x^2+y^2}{x^2+y^2}\right)\frac{dy}{dx} = y\left(\frac{2x^2+y^2}{x(x^2+y^2)}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\left(\frac{2x^2+y^2}{x^2+y^2}\right) \times \left(\frac{x^2+y^2}{2x^2+y^2}\right)$$

$$\frac{dy}{dx} = \frac{y}{x} \text{ hence proved}$$

12. If $y = \tan(p \tan^{-1}x)$ show that $(1+x^2)\, y_1 - p(1+y^2)=0$ (Sargodha 2007)

Let $y = \tan(p \tan^{-1}x)$

$\tan^{-1}y = p\tan^{-1}x$

Take derivative both sides

$$\frac{1}{1+y^2}\, y_1 = p.\frac{1}{1+x^2}$$

$\Rightarrow \quad (1+x^2)\, y_1 = p(1+y^2)$

$\Rightarrow \quad (1+x^2)\, y_1 - p(1+y^2) = 0$

Example $y = \log_{10}(ax^2+bx+c)$ **(Sargodha 2009, 10 Lahore 2010)**

$$\frac{dy}{dx} = \frac{1}{(ax^2+bx+c)\,\ell n 10}\frac{d}{dx}\left(ax^2+bx+c\right)$$

$$= \frac{2ax+b}{\left(ax^2+bx+c\right) ln10}$$

Example $y = a^{\sqrt{x}}$

$$\frac{dy}{dx} = a^{\sqrt{x}}(\ln a)\frac{d}{dx}(x)^{1/2} = a^{\sqrt{x}}\ln a.\frac{1}{2}x^{-1/2}$$

$$= a^{\sqrt{x}}\ln a.\frac{1}{2\sqrt{x}}$$

Exercise 2.6

1. Find $f'(x)$ if

$2^x;\ 3^x;\ a^x\ \ e^x$

i) explanation
ii) long

(i) $f(x) = e^{\sqrt{x}-1}$

Then $f'(x) = e^{\sqrt{x}-1}\dfrac{d}{dx}\left(\sqrt{x}-1\right)$

$$= e^{\sqrt{x}-1}\left[\frac{1}{2}x^{-1/2}-0\right]$$

$$= e^{\sqrt{x}-1}\cdot\frac{1}{2\sqrt{x}}$$

(ii) $f(x) = x^3e^{1/x}$

Then $f'(x) = x^3\dfrac{d}{dx}e^{1/x} + e^{1/x}\dfrac{d}{dx}x^3$

$$f'(x) = x^3e^{1/x}\frac{d}{dx}\frac{1}{x} + e^{1/x}.3x^2$$

$$= x^3e^{1/x}(-1)x^{-2} + 3x^2e^{1/x}$$

$$= e^{1/x}\left(-\frac{x^3}{x^2} + 3x^2\right)$$

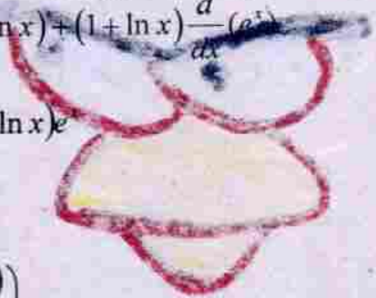
$$= e^{1/x}(-x+3x^2)$$

$$= e^{1/x}.x(-1+3x)$$

$$\Rightarrow \quad = xe^{1/x}(3x-1)$$

(iii) $f(x) = e^x(1+\ln x)$ **(Sargodha 2012)**

Then $f'(x) = e^x\dfrac{d}{dx}(1+\ln x) + (1+\ln x)\dfrac{d}{dx}(e^x)$



$$= e^x\left(0+\frac{1}{x}\right) + (1+\ln x)e^x$$

$$= e^x\left[\frac{1}{x} + (1+\ln x)\right]$$

$$= e^x\left(\frac{1+x(1+\ln x)}{x}\right)$$

(iv) $f(x) = \dfrac{e^x}{e^{-x}+1}$

Then $f'(x) = \dfrac{\left(e^{-x}+1\right)\dfrac{d}{dx}e^x - e^x\dfrac{d}{dx}\left(e^{-x}+1\right)}{\left(e^{-x}+1\right)^2}$

$$= \frac{(e^{-x}+1)e^{x} - e^{x}(e^{-x}(-1)+0)}{(e^{-x}+1)^2}$$

$$= \frac{(e^{-x}+1)e^{x} + e^{x}.e^{-x}}{(e^{-x}+1)^2}$$

$$= \frac{\frac{1}{e^{x}}\times e^{x} + e^{x} + e^{x}.\frac{1}{e^{x}}}{(e^{-x}+1)^2}$$

$$= \frac{1+e^{x}+1}{(e^{-x}+1)^2} = \frac{e^{x}+2}{(e^{-x}+1)^2}$$

(v) **f(x) = ln(e^x+e^{-x})**

Then $f'(x) = \frac{1}{(e^{x}+e^{-x})}\frac{d}{dx}(e^{x}+e^{-x})$

$$= \frac{1}{(e^{x}+e^{-x})}(e^{x}+e^{-x}(-1))$$

$$= \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}} = \frac{e^{x} - \frac{1}{e^{x}}}{e^{x}+\frac{1}{e^{x}}} = \frac{\frac{e^{2x}-1}{e^{x}}}{\frac{e^{2x}+1}{e^{x}}} = \frac{e^{2x}-1}{e^{2x}+1}$$

(vi) **f(x) =** $\frac{e^{ax}-e^{-ax}}{e^{ax}+e^{-ax}}$

$$f'(x) = \frac{(e^{ax}+e^{-ax})\frac{d}{dx}(e^{ax}-e^{-ax}) - (e^{ax}-e^{-ax})\frac{d}{dx}(e^{ax}+e^{-ax})}{(e^{ax}+e^{-ax})^2}$$

$$= \frac{(e^{ax}+e^{-ax})(e^{ax}.a - e^{-ax}(-a)) - (e^{ax}-e^{-ax})a(e^{ax}-e^{-ax})}{(e^{ax}+e^{-ax})^2}$$

$$= \frac{a\left[(e^{ax}+e^{-ax})^2 - (e^{ax}-e^{-ax})^2\right]}{(e^{ax}+e^{-ax})^2}$$

$$= \frac{a\left[e^{2ax}+e^{-2ax}+2e^{ax}.e^{-ax} - e^{2ax} - e^{-2ax} + 2e^{ax}.e^{-ax}\right]}{(e^{ax}+e^{-ax})^2}$$

$$= \frac{a\left[e^{2ax}+e^{-2ax}+2e^{ax}.e^{-ax} - e^{2ax}(-e^{-2ax} + 2e^{ax}.e^{-ax}\right]}{(e^{ax}+e^{-ax})^2}$$

$e^{ax}.e^{-ax}$, e^{ax-ax}, $e^{0} = 1$

$$= \frac{a\{e^{2ax}+e^{-2ax}+2e^{ax}.e^{-ax} - e^{2ax} + e^{-2ax+2a}}{(e^{ax}+e^{-ax})^2}$$

$$= \frac{a\left[2e^{ax}.\frac{1}{e^{ax}} + 2e^{ax}.\frac{1}{e^{ax}}\right]}{\left(e^{ax} + e^{-ax}\right)^2}$$

$$= \frac{a\left[2e^{ax}.\frac{1}{e^{ax}} + 2e^{ax}.\frac{1}{e^{ax}}\right]}{\left(e^{ax} + e^{-ax}\right)^2}$$

$$= \frac{a(2+2)}{(e^{ax} + e^{-ax})^2} = \frac{4a}{\left(e^{ax} + e^{-ax}\right)^2}$$

(vii) **f(x)** $= \sqrt{\ln\left(e^{2x} + e^{-2x}\right)}$

Then $f'(x) = \frac{1}{2}\left(\ln\left(e^{2x} + e^{-2x}\right)\right)^{-1/2} . \frac{d}{dx}\ln\left(e^{2x} + e^{-2x}\right)$

$$= \frac{1}{2\left[\ln\left(e^{2x} + e^{-2x}\right)\right]^{+1/2}} \frac{1}{\left(e^{2x} + e^{-2x}\right)} \frac{d}{dx}\left(e^{2x} + e^{-2x}\right)$$

$$= \frac{1}{2\sqrt{\ln\left(e^{2x} + e^{-2x}\right)}} \frac{1}{\left(e^{2x} + e^{-2x}\right)}\left(e^{2x}.2 + e^{-2x}(-2)\right)$$

$$= \frac{1}{2\sqrt{\ln\left(e^{2x} + e^{-2x}\right)}} \frac{1}{\left(e^{2x} + e^{-2x}\right)} 2\left(e^{2x} - e^{-2x}\right)$$

$$= \frac{e^{2x} - e^{-2x}}{\left(e^{2x} + e^{-2x}\right)\sqrt{\ln\left(e^{2x} + e^{-2x}\right)}}$$

(viii) **f (x) = ln** $\left(\sqrt{e^{2x} + e^{-2x}}\right)$

f (x) = ln $\left(e^{2x} + e^{-2x}\right)^{1/2}$

$$= \frac{1}{2}\ln\left(e^{2x} + e^{-2x}\right)$$

Taking derivative both sides

$$f'(x) = \frac{1}{2}\frac{1}{e^{2x} + e^{-2x}}\frac{d}{dx}\left(e^{2x} + e^{-2x}\right)$$

$$= \frac{1}{2\left(e^{2x} + e^{-2x}\right)}\left(e^{2x}(+2) + e^{-2x}(-2)\right)$$

$$= \frac{2\left(e^{2x} - e^{-2x}\right)}{2\left(e^{2x} + e^{-2x}\right)} = Tan\,h2x$$

2. Find $\frac{dy}{dx}$ if

(i) $y = x^2 \ln \sqrt{x} = x^2 \ln (x)^{1/2}$

$= x^2 . \frac{1}{2}\ln x = \frac{1}{2}x^2 \ln x$

Differentiating both sides w.r.t.x

$$\frac{dy}{dx} = \frac{1}{2}\left(x^2 \frac{d}{dx}\ln x + \ln x \frac{d}{dx}x^2\right)$$

$$= \frac{1}{2}\left(x^2 . \frac{1}{x} + \ln x . 2x\right)$$

$$= \frac{1}{2}(x + 2x\ln x)$$

$$= \frac{x}{2}(1 + 2\ln x) = x\left(\frac{1}{2} + \ln x\right)$$

(ii) $y = x\sqrt{\ln x} = x(\ln x)^{1/2}$ **(Sargodha 2009)**

Taking derivative both sides w.r.t.x

$$\frac{dy}{dx} = x\frac{d}{dx}(\ln x)^{+1/2} + (\ln x)^{1/2}\frac{d}{dx}x$$

$$= x.\frac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x) + (\ln x)^{1/2}.1$$

$$= \frac{x}{2(\ln x)^{1/2}}.\frac{1}{x} + \sqrt{\ln x}$$

$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x} = \frac{1 + 2\ln x}{2\sqrt{\ln x}}$$

(iii) $y = \frac{x}{\ln x}$ **(Sargodha 2011)**

Taking derivative both sides w.r.t.x

$$\frac{dy}{dx} = \frac{\ln x \frac{d}{dx}(x) - x\frac{d}{dx}(\ln x)}{(\ln x)^2} = \frac{(\ln x).1 - x.\frac{1}{x}}{(\ln x)^2}$$

$$\frac{dy}{dx} = \frac{\ln x - 1}{(\ln x)^2}$$

Log a + Log b

(iv) $y = x^2 \ln\left(\frac{1}{x}\right) = x^2 \ln x^{-1} = x^2(-1)\ln x = -x^2 \ln x$ (Sgd 2009, Lhr 2010)

$$\frac{dy}{dx} = -\left(x^2 \frac{d}{dx}\ln x + \ln x \frac{d}{dx}x^2\right) = -\left(x^2.\frac{1}{x} + \ln x.2x\right)$$

$= -x\,(1+2\ln x)$

(v) $y = \ln\sqrt{\frac{x^2-1}{x^2+1}} = \ln\left(\frac{x^2-1}{x^2+1}\right)^{1/2} = \frac{1}{2}\ln\left(\frac{x^2-1}{x^2+1}\right)$

Taking derivative both sides

$$\frac{dy}{dx} = \frac{1}{2}.\frac{1}{\left(\frac{x^2-1}{x^2+1}\right)}\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right)$$

$$= \frac{1}{2}\left(\frac{x^2+1}{x^2-1}\right)\left[\frac{(x^2+1)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}\right]$$

$$= \frac{(x^2+1)}{2(x^2-1)}\left[\frac{(x^2+1)2x - (x^2-1)2x}{(x^2+1)^2}\right] = \frac{2x(x^2+1-x^2+1)}{2(x^2-1)(x^2+1)}$$

$$= \frac{2x}{x^4-1}$$

(vi) $y = \ln(x + \sqrt{x^2+1})$

Taking derivative w.r.t x both sides

$$\frac{dy}{dx} = \frac{1}{(x+\sqrt{x^2+1})}\frac{d}{dx}\left(x+\sqrt{x^2+1}\right) = \frac{1}{(x+\sqrt{x^2+1})}\left[1+\frac{1}{2}(x^2+1)^{-1/2}.2x\right]$$

$$= \frac{1}{(x+\sqrt{x^2+1})}\left[1+\frac{x}{\sqrt{x^2+1}}\right] = \frac{1}{(x+\sqrt{x^2+1})}\left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right]$$

$$= \frac{1}{\sqrt{x^2+1}}$$

(vii) $y = \ln(9-x^2)$ **(Sargodha 2008)**

Taking derivative

$$\frac{dy}{dx} = \frac{1}{9-x^2}\frac{d}{dx}(-x^2) = \frac{1}{9-x^2}(-2x) = \frac{-2x}{9-x^2}$$

(viii) $y = e^{-2x}\text{Sin}2x$ **(Sargodha 2011)**

Log(a/b) = Log a - Log b

Taking derivative both sides

Log(a) = bLog a

$$\frac{dy}{dx} = e^{-2x}\frac{d}{dx}Sin2x + Sin2x\frac{d}{dx}e^{-2x}$$

$$= e^{-2x}\text{Cos}2x\frac{d}{dx}(2x) + Sin2xe^{-2x}\frac{d}{dx}(-2x)$$

$$= e^{-2x}(\text{Cos}2x - \text{Sin}2x)2 = 2e^{-2x}(\text{Cos}2x - \text{Sin}2x)$$

(ix) $y = e^{-x}(x^3+2x^2+1)$

Taking derivative both sides

$$\frac{dy}{dx} = e^{-x}\frac{d}{dx}\left(x^3+2x^2+1\right) + \left(x^3+2x^2+1\right)\frac{d}{dx}e^{-x}$$

$$= e^{-x}(3x^2+2.2x+0)+(x^3+2x^2+1)(-e^{-x})$$

$$= e^{-x}[3x^2+4x-x^3-2x^2-1]$$

$$= e^{-x}(-x^3+x^2+4x-1)$$

(x) $y = xe^{\text{Sin}x}$ **(Sargodha 2011)**

$$\frac{dy}{dx} = x\frac{d}{dx}e^{Sinx} + e^{Sinx}\frac{d}{dx}(x)$$

$$= xe^{\text{Sin}x}\frac{d}{dx}(Sinx) + e^{Sinx}.1$$

$$= e^{\text{Sin}x}(x\text{Cos}x + 1)$$

(xi) $y = 5e^{3x-4}$

Taking derivative w.r.t.x

$$\frac{dy}{dx} = 5e^{3x-4}\frac{d}{dx}(3x-4) = 5e^{3x-4}(3) = 15e^{3x-4}$$

(xii) $y = (x+1)^x$ **(Sargodha 2009)**

Take ln on both sides

$\ln y = \ln (x+l)^x \Rightarrow \ln y = x\ln (x+1)$

Taking derivative

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}\ln(x+1) + \ln(x+1)\frac{d}{dx}(x)$$

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{(x+1)}\frac{d}{dx}(x+1) + \ln(x+1).1$$

$$\frac{1}{y}\frac{dy}{dx} = \left(\frac{x}{(x+1)} + \ln(x+1)\right)$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{x}{(x+1)} + \ln(x+1)\right)$$

$$\frac{dy}{dx} = (x+1)^x\left[\frac{x}{(x+1)} + \ln(x+1)\right]$$

(xiii) $y = (\ln x)^{\ln x}$ **(Sargodha 2009, Faisalabad 2010)**

Taking ln both sides

$\ln y = \ln(\ln x)^{\ln x}$

$\ln y = \ln x \ln (\ln . x)$

$$\frac{1}{y}\frac{dy}{dx} = \ln x\frac{d}{dy}\ln(\ln x) + \ln(\ln x)\frac{d}{dx}(\ln x)$$

$$\frac{1}{dy}\frac{dy}{dx} = \ln x\left[\frac{1}{\ln x}\frac{d}{dx}(\ln x) + \ln(\ln x)\frac{1}{x}\right]$$

$$= \ln x\left[\left[\frac{1}{\ln x}\left(\frac{1}{x}\right)\right] + \left[\frac{1}{x}\ln(\ln x)\right]\right]$$

$$\frac{1}{y}\frac{dx}{dy} = \frac{1}{x} + \frac{1}{x}\ln(\ln x) = \frac{1}{x}(1+\ln(\ln x))$$

$$\Rightarrow \frac{dy}{dx} = y\frac{1}{x}(1+\ln(\ln x)) = (\ln x)^{\ln x}\frac{1}{x}(1+\ln(\ln x))$$

(xiv) $y = \dfrac{\sqrt{x^2-1}(x+1)}{(x^3+1)^{3/2}} = \dfrac{(x-1)^{1/2}(x+1)^{1/2}(x+1)}{\left[(x+1)(x^2-x+1)\right]^{3/2}}$

$y = \dfrac{\sqrt{x-1}(x+1)^{3/2}}{(x+1)^{3/2}(x^2-x+1)^{3/2}} = \dfrac{\sqrt{x-1}}{(x^2-x+1)^{3/2}} = \dfrac{(x-1)^{1/2}}{(x^2-x+1)^{3/2}}$

$\ln y = \ln \dfrac{(x-1)^{1/2}}{(x^2-x+1)^{3/2}} = \ln(x-1)^{1/2} - \ln(x^2-x+1)^{3/2}$

$= \dfrac{1}{2}\ln(x-1) - \dfrac{3}{2}\ln(x^2-x+1)$

Taking derivatives

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{1}{(x-1)}\dfrac{d}{dx}(x-1) - \dfrac{3}{2}\dfrac{1}{(x^2-x+1)}\dfrac{d}{dx}(x^2-x+1)$

$= \dfrac{1}{2}\cdot\dfrac{1}{(x-1)}(1-0) - \dfrac{3}{2}\dfrac{1}{(x^2-x+1)}(2x-1+0)$

$= \dfrac{1}{2(x-1)} - \dfrac{3(2x-1)}{2(x^2-x+1)} = \dfrac{(x^2-x+1)-3(2x-1)(x-1)}{2(x-1)(x^2-x+1)}$

$= \dfrac{(x^2-x+1)-3(2x^2-2x-x+1)}{2(x-1)(x^2-x+1)}$

$= \dfrac{x^2-x+1-6x^2+6x+3x-3}{2(x-1)(x^2-x+1)} = \dfrac{-5x^2+8x-2}{2(x-1)(x^2-x+1)}$

$\dfrac{dy}{dx} = y\left(\dfrac{-5x^2+8x-2}{2(x-1)(x^2-x+1)}\right) = \dfrac{(x-1)^{1/2}}{(x^2-x+1)^{3/2}}\left[\dfrac{-5x^2+8x-2}{2(x-1)(x^2-x+1)}\right]$

$= \dfrac{-5x^2+8x-2}{2(x-1)^{1-1/2}(x^2-x+1)^{3/2+1}} = \dfrac{-5x^2+8x-2}{2\sqrt{x-1}(x^2-x+1)^{5/2}}$

3. Find $\dfrac{dy}{dx}$ if

(i) $y = \text{Cos h}2x$

Taking derivative w.r.t.x

$\dfrac{dy}{dx} = \text{Sinh}2x\dfrac{d}{dx}(2x) = \text{Si h}2x\,(2) = 2\text{Sinh}2x$

(ii) y = Sinh3x

$$\frac{dy}{dx} = \text{Cos h}3x\frac{d}{dx}(3x) = \text{Cos h}3x(3) = 3\text{Cosh}3x$$

(iii) y = Tanh^{-1} (Sinx) $\Rightarrow$ $\frac{dy}{dx} = \frac{1}{1-Sin^2x}\frac{d}{dx}(Sinx)$

$$\frac{dy}{dx} = \frac{Cosx}{Cos^2x} \Rightarrow \frac{dy}{dx} = \frac{1}{Cosx} \Rightarrow \frac{dy}{dx} = Secx$$

(iv) y = Sin h^{-1}(x^3) $\frac{dy}{dx} = \frac{1}{\sqrt{1+(x^3)^2}}\frac{d}{dx}(x)^3$ **(Sargodha 2011)**

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^6}}(3x^2) = \frac{3x^2}{\sqrt{1+x^6}}$$

(v) y = ln (Tan hx)

$$\frac{dy}{dx} = \frac{1}{Tanhx}\frac{dy}{dx}Tan\,hx = \frac{1}{Tan\,hx}Sec^2hx$$

$$\frac{dy}{dx} = \frac{1}{\frac{Sinhx}{Coshx}} \times \frac{1}{Cos^2hx} = \frac{Coshx}{Sinhx} \times \frac{1}{Cos^2hx} = \frac{1}{SinhxCoshx}$$

'×' & $\div$ by 2

$$= \frac{2}{2Sinh \times Coshx} = \frac{2}{Sinh2x} = 2Co\sec h2x$$

(vi) Sinh$^{-1}\left(\frac{x}{2}\right)$ $\Rightarrow$ $\frac{dy}{dx} = \frac{1}{\sqrt{1+\left(\frac{x}{2}\right)^2}}\frac{d}{dx}\left(\frac{x}{2}\right)$ **(Sargodha 2008)**

$$\frac{dy}{dx} = \frac{\frac{1}{2}}{\sqrt{1+\frac{x^2}{4}}} = \frac{\frac{1}{2}}{\frac{\sqrt{x^2+4}}{2}} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \times \frac{2}{\sqrt{x^2+4}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2+4}}$$

Exercise 2.7

1. Find y_2 if

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

$y_1 = 2(5x^4) - 3(4x^3) + 4(3x^2) + 1 - 0$

$y_1 = 10x^4 - 12x^3 + 12x^2 + 1$

$y_2 = 10(4x^3) - 12(3x^2) + 12(2x) = 40x^3 - 36x^2 + 24x$

(ii) $y = (2x+5)^{3/2} \Rightarrow y_1 = \frac{3}{2}(2x+5)^{3/2-1}\frac{d}{dx}(2x+5)$ **(Sargodha 2009, Fsd 2010)**

$y_1 = \frac{3}{2}(2x+5)^{1/2}(2) = 3(2x+5)^{1/2}$

$y_2 = 3\left(\frac{1}{2}\right)(2x+5)^{1/2-1}\frac{d}{dx}(2x+5) = \frac{3}{2}(2x+5)^{-1/2}(2)$

$y_2 = 3(2x+5)^{-1/2} = \frac{3}{(2x+5)^{1/2}} = \frac{3}{\sqrt{2x+5}}$

(iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}} \Rightarrow y_1 = \frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}}$

$y_2 = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)x^{-3/2} + \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)x^{-5/2}$

$y_2 = -\frac{1}{4x^{3/2}} + \frac{3}{4x^{5/2}}$

2. Find y_2 if

(i) $y = x^2 e^{-x}$ **(Sargodha 2011, Gujranwala 2010)**

$y_1 = x^2 \frac{d}{dx}(e^{-x}) + e^{-x}\frac{d}{dx}(x^2)$

$= x^2(-e^{-x}) + e^{-x}(2x) = e^{-x}(-x^2+2x)$

$y^2 = e^{-x}\frac{d}{dx}(-x^2+2x) + (-x^2+2x)\frac{d}{dx}(e^{-x})$

$= e^{-x}(-2x+2) + (-x^2+2x)(-e^{-x})$

$= e^{-x}(-2x+2+x^2-2x) = e^{-x}(x^2-4x+2)$

Example (Sgd 2009–11)

$y_1 = e^{ax}$ $\quad y_2 = ?$

$y_1 = a.e^{ax}.$ $\quad a = a^2e^{ax}$

(ii) $y = \ln\left(\dfrac{2x+3}{3x+2}\right) = \ln(2x+3) - \ln(3x+2)$ (Sargodha 2011,12)

$$y_1 = \frac{1}{2x+3}\frac{d}{dx}(2x+3) - \frac{1}{(3x+2)}\frac{d}{dx}(3x+2)$$

$$= (2x+3)^{-1}.2-(3x+2)^{-1}.3 = 2(2x+3)^{-1} - 3(3x+2)^{-1}$$

$$y_2 = 2(-1)(2x+3)^{-2}\frac{d}{dx}(2x+3) - 3(-1)(3x+2)^{-2}\frac{d}{dx}(3x+2)$$

$$y_2 = -2(2x+3)^{-2}.2 + 3(3x+2)^{-2}.3$$

$$= -4(2x+3)^{-2} + 9(3x+2)^{-2} = \frac{-4}{(2x+3)^2} + \frac{9}{(3x+2)^2}$$

$$= \frac{-4(3x+2)^2 + 9(2x+3)^2}{(2x+3)^2(3x+2)^2}$$

$$= \frac{-4(9x^2+12x+4) + 9(4x^2+12x+9)}{(2x+3)^2(3x+2)^2}$$

$$= \frac{-36x^2 - 48x - 16 + 36x^2 + 108x + 81}{(2x+3)^2(3x+2)^2}$$

$$= \frac{60x+65}{(2x+3)^2(3x+2)^2}$$

3. find y_2 if

(i) $x^2+y^2=a^2$ ———————— I (Sargodha 2007,08)

Take derivative both sides

$$2x+2yy_1 = 0 \quad \Rightarrow \quad y_1 = \frac{-2x}{2y} = -\frac{x}{y}$$

$$y_2 = \frac{y(-1)-(-x)y_1}{y^2} = \frac{-y+xy_1}{y^2} = \frac{1}{y^2}\left[-y + x\left(-\frac{x}{y}\right)\right]$$

$$y_2 = \frac{1}{y^2}\left[-y - \frac{x^2}{y}\right] = \frac{1}{y^2}\left(\frac{-y^2-x^2}{y}\right) = \frac{-(x^2+y^2)}{y^3} = \frac{-a^2}{y^3} \quad \text{Use} - \text{I}$$

(ii) $x^3 - y^3 = a^3$ ——————— I

Taking derivative

$$3x^2 - 3y^2 y_1 = 0 \quad \Rightarrow \quad y_1 = \frac{3x^2}{3y^2} = \frac{x^2}{y^2}$$

$$y_2 = \frac{y^2.2x - x^2.2yy_1}{y^4} = \frac{1}{y^4}\left(2xy^2 - 2x^2yy_1\right)$$

$$y_2 = \frac{1}{y^4}\left(2xy^2 - 2x^2y.\frac{x^2}{y^2}\right) \rightarrow \text{put value of } y_1$$

$$y_2 = \frac{1}{y^4}\left(2xy^2 - \frac{2x^4y}{y^2}\right) = \frac{1}{y^4}\left(\frac{2xy^4 - 2x^4y}{y^2}\right)$$

$$= \frac{2xy\left(y^3 - x^3\right)}{y^6} = \frac{2x\left(y^3 - x^3\right)}{y^5} = \frac{-2x\left(x^3 - y^3\right)}{y^5} = \frac{-2xa^3}{y^5} \quad \text{use I}$$

(iii). $X = a\cos\theta$, $y = a\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta \quad , \quad \frac{dy}{d\theta} = a\cos\theta$$

$$y_1 = \frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = a\cos\theta \times \frac{-1}{a\sin\theta} = -\cot\theta$$

$$y_2 = -(-\operatorname{Cosec}^2\theta)\frac{d\theta}{dx} = \operatorname{Cosec}^2\theta . \frac{1}{-a\sin\theta}$$

$$= \frac{1}{\sin^2\theta}.\frac{1}{-a\sin\theta} = \frac{1}{-a\sin^3\theta}$$

(iv). $x = at^2$, $y = bt^4$

$$\frac{dx}{dt} = 2at \quad , \quad \frac{dy}{dt} = 4bt^3$$

$$\frac{dy}{dx} = y_1 = \frac{dy}{dt}.\frac{dt}{dx} = 4bt^3 \times \frac{1}{2at} = \frac{2bt^2}{a}$$

$$y_2 = \frac{2b}{a}.2t.\frac{dt}{dx} = \frac{4bt}{a}.\frac{1}{2at} = \frac{{}^2 4b}{2a^2} = \frac{2b}{a^2}$$

(v). $x^2+y^2+2gx+2fy+c = 0$

Taking derivative both sides.

$$2x + 2yy_1 + 2g + 2fy_1 + 0 = 0 \quad \Rightarrow \quad (2y + 2f)\, y_1 = -2x - 2g$$

$$y_1 = \frac{-2x - 2g}{2y + 2f} = \frac{-2(x+g)}{2(y+f)} = -\left(\frac{x+g}{y+f}\right) \text{———— } II$$

$$y_2 = \frac{(y+f)(-1)(+(x+g))y_1}{(y+f)^2} = \frac{(y+f)(-1)-(-(x+g))y_1}{(y+f)^2}$$

$$= \frac{-y-f+(x+g)y_1}{(y+f)^2} = \frac{1}{(y+f)^2}\left[-y-f+(x+g)\left[\frac{-(x+g)}{y+f}\right]\right]$$

$$= \frac{1}{(y+f)^2}\left[-(y+f)-\frac{(x+g)^2}{y+f}\right] = \frac{1}{(y+f)^2}\left[\frac{-(y+f)^2-(x+g)^2}{(y+f)}\right]$$

$$= \frac{-(y^2+f^2+2fy+x^2+g^2+2gx)}{(y+f)^3}$$

$$= \frac{-(x^2+y^2+2gx+2fy+g^2+f^2)}{(y+f)^3}$$

= From I $x^2+y^2+2gx+2fy=-c$

Use in III

$$y_2 = \frac{-(-c+g^2+f^2)}{(y+f)^3} = \frac{c-g^2-f^2}{(y+f)^3}$$

4. find y_4 if

(i). $y = Sin3x$ **(Sargodha 2008, Lahore 2010)**

$$y_1 = Cos3x\,\frac{d}{dx}(3x) = Cos3x.3 = 3Cos3x$$

$y_2 = 3(-Sin\,3x.3) = -9Sin3x$

$y_3 = -9Cos3x.3) = -27Cos3x$

$\Rightarrow$ $y_4 = -27(-Sin3x).3 = 81Sin3x$

(ii). $y = Cos^3x$

We know that

Note:
$Cos3x = 4Cos^3x - 3cosx$

$$Cos3x\,\frac{1}{4}(3x) = Cos3x. + 3Cosx$$

So $y = \frac{1}{4}(Cos3x + 3Cosx)$

$$y_1 = \frac{1}{4}(-Sin3x.3 + 3(-Sinx)) = -\frac{3}{4}(Sin3x + Sinx)$$

$$y_2 = \frac{-3}{4}(Cos3x.3 + Cosx) = \frac{-3}{4}(3Cos3x + Cosx)$$

$$y_3 = \frac{-3}{4}(3(-Sin3x.3) - Sinx) = \frac{3}{4}(9Sin3x + Sinx)$$

$$y_4 = \frac{3}{4}(9Cos3x.3 + Cosx) = \frac{3}{4}(27Cos3x + Cosx)$$

From I $Cos3x = 4Cos^3x - 3Cosx$ so

$$y_4 = \frac{3}{4}\left(27\left(4Cos^3x - 3Cosx\right) + Cosx\right)$$

$$= \frac{3\times 27\times 4}{4}Cos^3x - \frac{3\times 27\times 3}{4}Cosx + \frac{3}{4}Cosx$$

$$= 81Cos^3x - \frac{243}{4}Cosx + \frac{3}{4}Cosx$$

$$= 81\,Cos^3x + \left(\frac{3-243}{4}\right)Cosx = 81Cos^3x - \frac{240}{4}Cosx$$

$$y_4 = 81\,Cos^3x - 60Cos\,x$$

Example

$y = Cos(ax + b)$

$y_1 = -\,Sin\,(ax + b)\,.a = -\,a\,Sin\,(ax + b)$

$y_2 = -a\,Cos\,(ax + b)\,.a = -a^2\,Cos\,(ax + b)$

$y_3 = -a^2\,(-Sin\,(ax + b)\,.\,a) = a^3\,Sin\,(ax + b)$

$y_4 = a^3\,.\,Cos\,(ax + b)\,.\,a = a^4\,Cos\,(ax + b)$

(iii). $y = \ln(x^2 - 9) = \ln(x-3)(x+3)$ **(Sargodha 2009)**

$= \ln(x-3) + \ln(x+3)$

$$y_1 = \frac{1}{(x-3)}.1 + \frac{1}{(x+3)}.1 = (x-3)^{-1} + (x+3)^{-1}$$

$y_2 = (-1)\,(x-3)^{-2} + (-1)\,(x+3)^{-2}$

$y_3 = (-1)\,(-2)\,(x-3)^{-3} + (-1)(-2)\,(x+3)^{-4}$

$$y_4 = \frac{-6}{(x-3)^4} + \frac{(-6)}{(x+3)^4} = -6\left[\frac{1}{(x-3)^4} + \frac{1}{(x+3)^4}\right]$$

5. **if $x = Sin\,\theta$, $y = Sinm\,\theta$ show that $(1-x^2)y_2 - xy_1 + m^2y = 0$**

$x = Sin\,\theta$, $y = Sinm\,\theta$

$y = Sinm\,\theta$, $x = Sin\,\theta \Rightarrow \theta = Sin^{-1}x$

So $y = Sin\,(mSin^{-1}x)$

$$y_1 = Cos\,(mSin^{-1}x).\,\frac{d}{dx}(mSin^{-1}x) = Cos\left(mSin^{-1}x\right)\frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_1 = mCos\left(mSin^{-1}x\right)$$

Again taking derivative

$$\sqrt{1-x^2}\,y_2 + y_1\frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x) = m\left(-Sin\left(mSin^{-1}x\right)\right)\frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_2 - \frac{xy_1}{\sqrt{1-x^2}} = \frac{-m^2 Sin\left(mSin^{-1}x\right)}{\sqrt{1-x^2}}$$

'X' by $\sqrt{1-x^2}$ *bothsides*

$(1-x^2)\,y_2 - xy_1 = -m^2$ Sin $(mSin^{-1}x)$ Use I

$(1-x^2)\,y_2 - xy_1 = -m^2y$

$\Rightarrow (1-x^2)\,y_2 - xy_1 + m^2y = 0$

Hence proved

6. **if $y = e^x$ Sinx show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$** **(Sargodha 2011, Lhr 2010)**

$y = e^x$ Sinx

$$\frac{dy}{dx} = e^x Cosx + Sinx e^x \qquad \_\_\_\_\_\_\_\_ I$$

$$\frac{d^2y}{dx^2} = e^x(-Sinx) + Cosx e^x + Cosx e^x + Sinx e^x$$

$= -e^x$ Sinx + e^x Cosx + e^x Cosx + e^x Sinx

$$\frac{d^2y}{dx^2} = 2e^x Cosx$$

Now L.H.S = $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 2e^x Cosx - 2\left(e^x Cosx + e^x Sinx\right) + 2e^x Sinx$

= 0 = R.H.S

7. **if $y = e^{ax}$ Sinbx show that $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0$**

$y = e^{ax}$ Sinbx

$\frac{dy}{dx} = e^{ax}$ Cosbx . b + Sinbx e^{ax} . a = e^{ax} [aSinbx + bCos bx]

$$\frac{d^2y}{dx^2} = e^{ax}\left[aCosbx.b + b\left(-Sinbx\right).b\right] + e^{ax}.a\left(aSinbx + bCosbx\right)$$

= e^{ax} (abCosbx – b^2Sinbx + a^2 Sinbx + abCosbx)

= e^{ax} (2abCosbx – b^2 Sinbx + a^2Sinbx)

Now $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2+b^2\right)y = e^{ax}\left(2abCosbx - b^2 Sinbx + a^2 Sinbx\right)$

–2a [e^{ax} (aSinbx + bCosbx)] + (a^2+b^2) e^{ax} Sinbx

= e^{ax} 2ab Cosbx – e^{ax} b^2Sinbx + e^{ax} a^2Sinbx – $2a^2$ e^{ax} Sinbx – 2abe^{ax} Cosbx

= 0 hence proved

8. **if $y = (Cos^{-1} x)^2$,Prove that $(1-x^2)y_2 - xy_1 - 2 = 0$**

$y = (Cos^{-1} x)^2$

$$y_1 = 2(Cos^{-1}x)\frac{d}{dx}Cos^{-1}x = 2\,Cos^{-1}x\frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\,y_1 = -2\,Cos^{-1}x$$

Again taking derivative

$$\sqrt{1-x^2}\,y_2 + y_1\frac{1}{2}(1-x^2)^{-1/2}(-2x) = -2\frac{-1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,y_2 - \frac{xy_1}{\sqrt{1-x_2}} = \frac{2}{\sqrt{1-x^2}}$$

'X' both sides by $\sqrt{1-x^2}$

$(1-x^2)\,y_2 - xy_1 = 2 \Rightarrow (1-x^2)\,y - xy_1 - 2 = 0$

Hence proved

9. **If y = a Cos (lnx) + bSin(lnx) Prove that**

y = a Cos (lnx) + bSin(lnx) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

$$\frac{dy}{dx} = a(-Sin\ln x).\frac{1}{x} + b\,Cos(\ln x).\frac{1}{x}$$

= 'x' both sides by x

$$x\frac{dy}{dx} = -a\,Sin\ln x + b\,Cos\ln x$$

Again taking derivative

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}.1 = -a\,Cos(\ln x).\frac{1}{x} - b\,Sin\ln x.\frac{1}{x}$$

'x' both side by x

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx}. = -a\,Cos\ln x - b\,Sin\ln x$$

$= -(a\,Cos\ln x + b\,Sin\ln x) = -y$

$$\Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$ Hence proved

Maclaurin Series **(Sargodha 2010, Lahore 2010)**

Theorem : $f(x) = f(0) + x\,f(0) + \dfrac{x^2}{2!}f''(0) + \dfrac{x^3}{3!}f'''(0) + \ldots\ldots + \dfrac{x^n}{n!}f^{(n)}(0) + \ldots\ldots$

Proof: we know that

put x = 0

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots\ldots + a_nx^n + \ldots\ldots \Rightarrow f(0) = a_0$

put x = 0

$f'(x) = a_1 + 2a_2x + a_1x + 3a_3x^2 + 4a_4x^3 + \ldots\ldots + na^nx^{n-1} + \ldots\ldots \Rightarrow f'(0) = a_1$

$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \ldots\ldots + n(n-1)a^nx^{n-2} + \ldots\ldots \Rightarrow f''(0) = 2a_2$

$$\Rightarrow \frac{f''(0)}{2!} = a_2$$

$f'''(x) = 6a_3 + 24a_4x + \ldots\ldots + n(n-1)(n-2)a^nx^{n-3} + \ldots\ldots \Rightarrow f'''(0) = 6a_3$

Put value of a_1, a_2, a_3 in I $\Rightarrow \dfrac{f'''(0)}{6} = a_3$

$f(x) = f(0) + xf'(0) + \dfrac{f''(0)}{2!}x^2 + \dfrac{f'''(0)}{3!}x^3 + \ldots\ldots\ldots$ or $\dfrac{-f'''(0)}{3!} = a_3$

Example : **Prove by Maclaurin series** **(Sargodha 2009,10)**

$\text{Sin}^{-1}x = x + \dfrac{1}{2}.\dfrac{x^3}{3} + \dfrac{1.3}{2.4}.\dfrac{x^5}{5} + \ldots\ldots\ldots$

$f(x) = \text{Sin}^{-1}x \qquad \Rightarrow f(0) = \text{Sin}^{-1}(0) = 0$

$f'(x) = \dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} \qquad \Rightarrow f'(0) = \dfrac{1}{\sqrt{1-0}} = 1$

$f''(x) = \dfrac{-1}{2}(1-x^2)^{-3/2}(-2x) = x(1-x^2)^{-3/2} \qquad \Rightarrow f''(0) = 0$

$f'''(x) = x\left(\dfrac{-3}{2}\right)(1-x^2)^{-5/2}(-2x) + (1-x^2)^{-3/2}.1 \quad f'''(0) = 1$

Maclaurim series is

$f(x) = f(0) + x\,f'(0) + \dfrac{x^2}{2}f''(0) + \dfrac{x^3}{3}f'''(0) + \ldots\ldots\ldots$

$\text{Sin}^{-1}x = 0 + x(1) + \dfrac{x^2}{2}(0) + \dfrac{x^3}{3.2.1}(1) + \ldots\ldots\ldots\ldots$

$\text{Sin}^{-1}x = x + \dfrac{1}{2}.\dfrac{x^3}{3} + \dfrac{1.3}{2.4}.\dfrac{x^5}{5} + \ldots\ldots\ldots\ldots$

Hence proved..

$f(x) = f(0) + xf'(0) + \frac{x^2}{2}$ $\quad f(x) = f(0) + xf'(0) + \frac{x^2}{2!}$

Exercise 2.8

Maclaurin Series:

$$f(x) = f(0) + f'(0)x + f''\frac{(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f'(0)}{4!}x^4 . + \ldots\ldots$$

1. Apply the Maclaurin series expansion to prove that

(i). $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$

$f(x) = \ln(x+1) \Rightarrow f(0) = \ln(0+1) = \ln(1) = 0$

$$f'(x) = \frac{1}{(1+x)}(1) = (1+x)^{-1} \quad \Rightarrow f'(0) = (1+0)^{-1} = 1$$

$$f''(x) = (-1)(1+x)^{-2}.1 \Rightarrow f''(0) = (-1)(1+0)^{-2} = (-1)(1) = -1$$

$$f'''(x) = (-1)(-2)(1+x)^{-3} \quad \Rightarrow f'''(0) = (-1)(-2)(1+0)^{-3} = 2$$

$$f^{iv}(x) = (-1)(-2)(-3)(1+x)^{-4} \Rightarrow f^{iv}(0) == (-1)(-2)(-3)(1+0)^{-4} = -6$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \frac{f^{iv}(0)}{4!}x^4 + \ldots\ldots$$

$$\ln(1+x) = 0 + 1 \times x + \frac{-1}{2!}x^2 + \frac{2}{3.2.1}x^3 + \frac{6}{4.3.2.1}x^4 + \ldots\ldots$$

$$\ln(1+x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots$$

Hence proved

(ii). $\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \ldots\ldots$

$f(x) = \cos x \Rightarrow f(0) = \cos 0 = 1$

Then $f'(x) = -\sin x \Rightarrow f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = 0$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \frac{f'''(0)}{3!}x^3 + \ldots\ldots\ldots$$

$$f(x) = f(0) + f'(0)x + f''\frac{(0)}{\lfloor 2}x^2 + f'''\frac{(0)}{\lfloor 3}x^3 + f^{iv}\frac{(0)}{\lfloor 4}x^4 + \ldots\ldots\ldots$$

$$\cos x = 1 + 0 \times x + \frac{-1}{\lfloor 2}x^2 + \frac{0 \times x^3}{\lfloor 3}x^3 + \frac{1 \times x^4}{\lfloor 4} + \ldots\ldots$$

$= 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \ldots\ldots$ Hence proved

(iii). $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots\ldots$

$f(x) = \sqrt{1+x} = (1+x)^{1/2} \quad \Rightarrow \quad f(0) = (1+0)^{1/2} = 1$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}.1 \quad \Rightarrow \quad f'(0) = \frac{1}{2}(1+0)^{-1/2} = +\frac{1}{2}(1) = +\frac{1}{2}$

$f''(x) = \frac{-1}{4}(1+x)^{-3/2} \quad \Rightarrow \quad f''(0) = \frac{-1}{4}(1+0)^{-3/2} = \frac{-1}{4}$

$f'''(x) = \frac{3}{8}\left(-\frac{5}{2}\right)(1+x)^{-7/2} \quad \Rightarrow \quad f'''(0) = \frac{3}{8}(1+0)^{-5/2} = \frac{3}{8}$

$f^{iv}(x) = \frac{3}{8}\left(-\frac{5}{2}\right)(1+x)^{-7/2} \quad \Rightarrow \quad f^{iv}(0) = \frac{-15}{16}(1+0)^{-7/2} = \frac{-15}{16}$

Now $f(x) = f(0) + f'(0)x + f''\frac{(0)}{\underline{|2}}x^2 + f'''\frac{(0)}{\underline{|3}}x^3 + f^{iv}\frac{(0)}{\underline{|4}}x^4. + \ldots\ldots$

$\sqrt{1+x} = 1 + \frac{1}{2}x + -\frac{1}{4} \times \frac{1}{\underline{|2}}x^2 + \frac{3}{8}.\frac{1}{\underline{|3}}x^3 + \left(\frac{-15}{16}\right)\frac{1}{\underline{|4}}x^4 + \ldots\ldots$

$= 1 + \frac{x}{2} - \frac{1}{4}.\frac{1}{2}x^2 + \frac{3}{8}.\frac{1}{6}.x^3 - \frac{15^5}{16}.\frac{1}{24_8}x^4 + \ldots\ldots$

$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \ldots\ldots$

Hence proved

(iv). $e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots$

$f(x) = e^x \Rightarrow \quad f(0) = e^0 = 1 \quad , \quad f''(x) = e^x \quad \Rightarrow \quad f''(0) = e^0 = 1$

$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1 \quad , \quad f'''(x) = e^x \quad \Rightarrow \quad f'''(0) = e^o = 1$

$f(x) = f(0) + f'(0).x + f''\frac{(0)}{\underline{|2}}x^2 + \frac{f'''(0)}{\underline{|3}}x^3 + \ldots\ldots$

$e^x = 1 + 1. + \frac{1.x^2}{\underline{|2}} + \frac{1.x^3}{\underline{|3}} + \ldots\ldots$

$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \ldots\ldots$

(v). $e^{2x} = 1 + 2x + \frac{x^2}{\lfloor 2} + \frac{8x^3}{\lfloor 3} + \ldots\ldots\ldots$ **(Sargodha 2011,12)**

$f(x) = e^{2x}$ $\quad\quad f(0) = e^0 = 1$

$f'(x) = 2e^{2x}$ $\quad\quad f'(0) = 2e^0 = 2(1) = 2$

$f''(x) = 2.e^{2x}.2 = 4e^{2x}$ $\quad\quad f''(0) = 4e^0 = 4(1) = 4$

$f'''(x) = 4e^{2x}.2 = 8e^{2x}$ $\quad\quad f'''(0) = 8e^0 = 8$

Maclaurin 's series is

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \ldots\ldots\ldots$$

$$e^{2x} = 1 + x\,(2) + \frac{x^2}{\lfloor 2}.4 + \frac{x^3}{\lfloor 3}(8) + \ldots\ldots$$

$$e^{2x} = 1 + 2x + \frac{4x^2}{\lfloor 2} + \frac{8x^3}{\lfloor 3} + \ldots\ldots\ldots$$

Hence Proved

2. **Show that** $\cos(x+h) = \cos x - h\sin x - \frac{h^2}{\lfloor 2}\cos x + \frac{h^3}{\lfloor 3}\sin x + \ldots\ldots$ ***and evaluate*** $\cos 61^o$

Cos (x+ h) = Cosx – hSinx $- \frac{h^2}{\lfloor 2}\cos x + \frac{h^3}{\lfloor 3}\sin x +$ **(Lahore 2010)**

f(x + h) = Cos (x + h) . then f (x) = Cosx

$f'(x) = -\sin x \quad \Rightarrow \quad f''(x) = -\cos x \Rightarrow \quad f'''(x) = -(-\sin x) = \sin x$

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{\lfloor 2}h^2 + \frac{f'''(x)}{\lfloor 3}h^3 + \ldots\ldots\ldots$$

$$\cos(x+h) = \cos x + (-\sin x)\,h + \frac{(-\cos x)}{\lfloor 2}h^2 + \frac{\sin x}{\lfloor 3}h^3 + \ldots\ldots\ldots\ldots$$

$$\cos(x+h) = \cos x - h\sin x - \frac{h^2}{\lfloor 2}\cos x + \frac{h^2}{\lfloor 3}\sin x + \ldots\ldots\ldots$$, Hence proved

Now take $x = 60^\circ$ & $h = 1^\circ = 0.01745$

$$\cos(60^\circ + 1^\circ) = \cos 60^\circ - 1^\circ \sin 60^\circ - \frac{(1^\circ)^2}{\lfloor 2}\cos 60^\circ + \frac{(1^\circ)^3}{\lfloor 3}\sin 60^\circ + \ldots\ldots$$

$$\cos 61^\circ = 0.5 - (0.0174)(.866) - \frac{(.0174)^2}{2}(0.5) + \frac{(0.0174)^3}{3.2.1}(0.866)$$

$\cos 61^\circ = 0.5 - 0.015116 - 0.000076$

$\cos 61^\circ = 0.4848$

3. **Show that** $2^{x+h} = 2x\left[1+(\ln 2)h+(\ln 2)^2\frac{h^2}{\lfloor 2}+(\ln 2)^3\frac{h^3}{\lfloor 3}+\ldots\ldots\right]$

$$2^{x+h} = 2x\left[1+(\ln 2)h+(\ln 2)^2\frac{h^2}{\lfloor 2}+(\ln 2)^3\frac{h^3}{\lfloor 3}+\ldots\ldots\right]$$

$f(x+h) = 2^{x+h} \Rightarrow f(x) = 2^x \Rightarrow f'(x) = 2^x(\ln 2)$

$f''(x) = (\ln 2)2^x.(\ln 2) = (\ln 2)^2.2^x \Rightarrow f'''(x) = (\ln 2)^2 2^x(\ln 2) = (\ln 2)^3 2^x$

Now

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\lfloor 2}f''(x) + \frac{h^3}{\lfloor 3}(\ln 2)^3 2^x + \ldots\ldots$$

$$2^{x+h} = 2^x\left\{1+(\ln 2)h+\frac{(\ln 2)^2 h^2}{\lfloor 2}+\frac{(\ln 2)^3 h^3}{\lfloor 3}+\ldots\ldots\right\}$$

Hence proved

Example (2.9)

Examine $f(x) = 1 + x^3$ for extreme values

$f(x)\ 1 + x^3\ \text{I} \Rightarrow f'(x) = 3x^2 \Rightarrow f''(x) = 6x$ III

Put $f'(x) = 0 \Rightarrow 3x^2 = 0 \Rightarrow x = 0$

Put in II $x = 0$ $f''(0) = 6(0) = 0$ Given information

Now put $x = 0 - \in$

$x = 0 + \in$ in I

$f'(0-\in) = 3(0-\in)^2 = 3\in^2 > 0$

$f'(0-\in) = 3(0+\in)2 = 3\in^2 > 0$

first derivative does not change sign at x = 0

put x = 0 in I $\Rightarrow f(0) = 1 + (0)^3 = 1$

So (0 , 1) is point of inflection

f'(x) > 0 Increase   [1,5] Close
f'(x) < 0 Decrease   ]1,5[Half
f'(x) = 0   [1,5[   ]1,5]

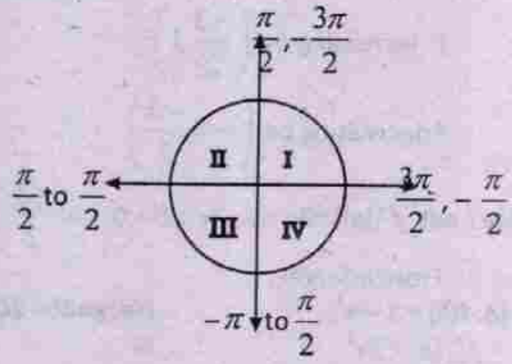
Exercise 2.9

Increasing function: **(Sargodha 2010, Faisalabad 2010)**

If $f'(x)$ *is* positive in given internal (a , b)

Decreasing function:

If f' (x) is negative in given internal (a, b)



1. (i). **f(x) = Sinx : $x \in (-\pi \; to \; \pi)$**

f' (x) = Cosx

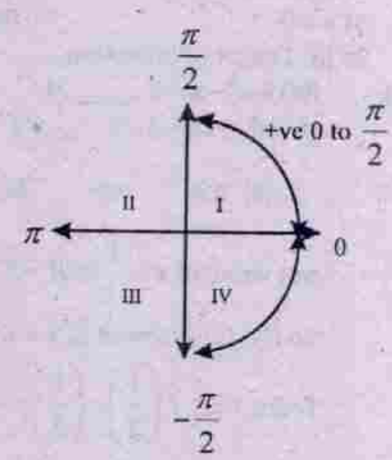
Cos x is +ve in I & IV

Cos x is –ve in II & III quadrant

So increasing on $\left(\frac{-\pi}{2} to \frac{\pi}{2}\right)$

Decreasing on $\left(-\pi \; to \frac{-\pi}{2}\right)$ *and* $\left(\frac{\pi}{2} to \; \pi\right)$

(ii) **f(x) = Cos x , $\left(\frac{-\pi}{2} to \frac{\pi}{2}\right)$** **(Sargodha 2011)**



$f'(x) = -Sinx$

Sinx is –ve in IV

Sinx is +ve in I

But –Sinx is +ve in IV

–Sinx is –ve in I

Increasing on $\left(\frac{-\pi}{2} to \; 0\right)$ & *decreasing on* $\left(0 to \frac{\pi}{2}\right)$

(iii) **f = (x) = 4 –x²** : **$x \in (-2 , 2)$**

at –2 : f' (x) = –2x

–2x = –2 (–2) = 4 : f' (x) is +ve in (–2 to 0) so increasing on –2 to 0)

at –1 f' x) is –ve in (0 , to 2) so decreasing on (0 , 2)

02 (–1) = 2

at 1 –2 (1) = –2

at 2 –2 (2) = –4

(iv) **f (x) = x² + 3x + 2**

f' (x) = 2x + 3

So f' (x) is +ve in $\left(-\frac{3}{2} to 1\right)$

$f'(x)$ is –ve in $\left(-4\,to\,\frac{-3}{2}\right)$

f increasing on $\left(\frac{-3}{2},1\right)$

f decreasing on $\left(-4,\frac{-3}{2}\right)$

Note : put $f'(x) = 0 \Rightarrow 2x + 3 = 0$ $x = \frac{-3}{2}$

From interval

2. (i). $f(x) = 1 - x^3$ **(Sargodha 2008, Lahore 2010)**

$f'(x) = -3x^2$ I & $f''(x) = -6x$ II

Take $f'(x) = 0 \Rightarrow$ $f''(0) = -6(0) = 0$

Gives no information so

Put $x = 0 - \in \,\&\, 0 + \in$ in I

$f'(0 - \in) = -3(0 + \in)^2 = -3\in^2 < 0$

First derivative does not change sign at x = 0

At x = 0 , f(0) = 1 – 0 = 1

So (0, 1) is pt of inflection.

(ii). $f(x) = x^2 - x - 2$ _____ I **(Lahore 2010)**

$f'(x) = 2x - 1$ & $f''(x) = 2$ _____ II

put $f'(x) = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

put value of $x = \frac{1}{2}\,in\,II$ $f''\left(\frac{1}{2}\right) = 2 = +ve$

So f(x) is minimum at x = ½ now for min value

From I $\Rightarrow f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 2 = \frac{1}{4} - \frac{1}{2} - 2 = \frac{1-2-8}{4} = \frac{-9}{4}$

f (x) min at $x = \frac{1}{2}\,\&\,f\left(\frac{1}{2}\right) = \frac{-9}{4}$

(iii). $f(x) = 5x^2 - 6x + 2$ **I**

$f'(x) = 10x - 6$ & $f'(x) = 10$ **II**

Put $f'(x) = 0 \Rightarrow 10x - 6 = 0 \Rightarrow x = \frac{6}{10} = \frac{3}{5}$

Put value of $x = \frac{3}{5}\,in\;\;II$

$f''\left(\frac{3}{5}\right)=10=+ve$

f (x) is min and for min value $f\left(\frac{3}{5}\right)=5\left(\frac{3}{5}\right)^2-6\left(\frac{3}{5}\right)+2$

$f\left(\frac{3}{5}\right)=50\left(\frac{9}{25}\right)-\frac{18}{5}+2=\frac{9}{5}-\frac{18}{5}+2=\frac{9-18+10}{5}=\frac{1}{5}$

So f(x) is minimum at $x=\frac{3}{5}$ & $f\left(\frac{3}{5}\right)=\frac{1}{5}$

(iv). $f(x) = 3x^2$

$f'(x) = 6x$ & $f''(x) = 6$ _____II

put $f'(x) = 0$ & $\Rightarrow$ $f''(0) = 6$ = +ve

Put x = 0 in II $f''(0) = 3(0)^2 = 0$

So f (x) is min at x = 0 & f (0) = 0

(v). $f(x) = 3x^2 - 4x + 5$_____I

$f'(x) = 6x - 4$ & $f''(x) = 6$ _____II

put x $f'(x) = 0 \Rightarrow 6x - 4 = 0 \Rightarrow x = \frac{4}{6}=\frac{2}{3}$

put $x=\frac{2}{3}$ *in II then* $f''\left(\frac{2}{3}\right)=6=+ve$

f(x) is min , for min value I $\Rightarrow$ $f\left(\frac{2}{3}\right)=3\left(\frac{2}{3}\right)^2-4\left(\frac{2}{3}\right)+5$

$f\left(\frac{2}{3}\right)=3\left(\frac{4}{9^3}\right)-\frac{8}{3}+5=\frac{4}{3}-\frac{8}{3}+5=\frac{4-8+15}{3}=\frac{11}{3}$

f(x) is min at $x=\frac{2}{3}$ *and* $f\left(\frac{2}{3}\right)=\frac{11}{3}$

(vi). $f(x) = 2x^3 - 2x^2 - 36x + 3$

$f'(x) = 6x^2 - 4x - 36$

$f''(x) = 12x - 4$

Put $f'(x) = 0 \Rightarrow 6x^2 - 4x - 36 = 0$ ÷ by 62

$3x^2 - 2x - 18 = 0$

$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(3)(-18)}}{2(3)}=\frac{2\pm\sqrt{4+12\times18}}{6}$

$x=\frac{2\pm\sqrt{4(1+3\times18)}}{6}=\frac{2\pm2\sqrt{1+54}}{6}=\frac{2(1\pm\sqrt{55})}{6^3}$

$$x = \frac{1 \pm \sqrt{55}}{3} \Rightarrow x = \frac{1\sqrt{55}}{3} \ \& \ x = \frac{1-\sqrt{55}}{3}$$

when x = $\frac{1+\sqrt{55}}{3}$ *then* $f''\left(\frac{1+\sqrt{55}}{3}\right) = 12^4\left(\frac{1+\sqrt{55}}{3}\right) - 4$

$$f''\left(\frac{1+\sqrt{55}}{3}\right) = 4 + 4\sqrt{55} - 4 = 4\sqrt{55} = +ve$$

So f (x) is min at x = $\frac{1+\sqrt{55}}{3}$ now for min value

I $\Rightarrow$ $$f\left(\frac{1+\sqrt{55}}{3}\right) = 2\left(\frac{1+\sqrt{55}}{3}\right)^3 - 2\left(\frac{1+\sqrt{55}}{3}\right)^2 - 36^{12}\left(\frac{1+\sqrt{55}}{3}\right) + 3$$

$$= 2\left(\frac{1+\sqrt{55}+3(55)+\left(\sqrt{55}\right)^3}{27}\right) - 2\left(\frac{1+55+2\sqrt{55}}{9}\right) - 12 - 12\sqrt{55} + 3$$

$$= \frac{2}{27}(1 + 3\sqrt{55} + 165 + 55\sqrt{55}) - \frac{2}{9}\left(56 + 2\sqrt{55}\right) - 9 - 12\sqrt{55}$$

$$= \frac{2}{27}\left(166 + 58\sqrt{55}\right) - \frac{2}{9}\left(56 + 2\sqrt{55}\right) - 9 - 12\sqrt{55}$$

$$= \frac{332}{27} + \frac{116\sqrt{55}}{27} - \frac{112}{9} - \frac{4\sqrt{55}}{9} - 9 - 12\sqrt{55}$$

$$= \frac{332}{27} - \frac{112}{9} - 9 + \left(\frac{116}{27} + \frac{4}{9} + 12\right)\sqrt{55}$$

$$= \left(\frac{332 - 336 - 243}{27}\right) + \left(\frac{116 - 12 - 324}{27}\right)\sqrt{55}$$

$$= \frac{-247}{27} - \frac{220}{27}\sqrt{55} = -\frac{1}{27}\left(247 + 220\sqrt{55}\right)$$

Now for x = $\frac{1-\sqrt{55}}{3}$

$$f''\left(\frac{1-\sqrt{55}}{3}\right) = 12^4\left(\frac{1-\sqrt{55}}{3}\right) - 4 = 4 - 4\sqrt{55} - 4$$

$$= -4\sqrt{55} = -ve$$

So f is max at x = $\frac{1-\sqrt{55}}{3}$

For min value same above process we get

$$f\left(\frac{1-\sqrt{55}}{3}\right)=\frac{1}{27}\left(-247+220\sqrt{55}\right)$$

(vii) $f(x) = x^4 - 4x^2$ ________ I

$f'(x)=4x^3-8x$

$f''(x)=12x^2-8x$ ________ II

put $f'(x) = 0 \Rightarrow 4x^3 - 8x = 0 \Rightarrow 4x(x^2-2) = 0$

$4x^2 = 0$ or $x^2 - 2 = 0 \Rightarrow x =$ or $x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

When x = 0 then $f''(0)=12(0)-84(0)=-8=-ve$

fn is Max at x = 0, I $\Rightarrow f(0) = (0)^4 - 4(0) = 0$

When x = $\sqrt{2}$ then $f''(\sqrt{2})=12(\sqrt{2})^2-8=12(2)-8=24-8=16=+ve$

f(x) is Min at x = $\sqrt{2}$ now for Min value

I $\Rightarrow f(\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 = 4 - 4(2) = 4 - 8 = -4$

When x = $-\sqrt{2}$ then $f''(-\sqrt{2}) = 12(-\sqrt{2})^2 - 8 = 12(2) - 8 = 24 - 8 = 16 = +ve$

f(x) is Min at x = $-\sqrt{2}$

I $\Rightarrow f(-\sqrt{2}) = (\sqrt{2})^4 - 4(-\sqrt{2})^2 = 4 - 4(2) = 4 - 8 = -4$

f(x0 is Max at x = 0, & f(0) = 0

f(x) is Min at x = $\sqrt{2}$ & $f(-\sqrt{2}) = -4$

(viii) $f(x) = (x-2)^2(x-1)$

$$f'(x)=(x-2)^2\frac{d}{dx}(x-1)+(x-1)\frac{d}{dx}(x-2)^2$$

$= (x-2)^2 . 1 + (x-1) .2 (x-2)^1 1$

$= (x-2) [x-2+2(x-1)]$

$= (x-2)(x-2+2x-2)$

$= (x-2)(3x-4)$

$f''(x)=(x-2)(3)+(3x-4).1$

$= 3x - 6 + 3x - 4 = 6x - 10$

= Put $f'(x)=0 \Rightarrow f'(x)=0 \Rightarrow (x-2)(3x-4)=0$

$x - 2 = 0$ or $x - 4 = 0$

$x = 2$ or $x = \frac{4}{3}$

When x = 2 then $f''(2) = 6(2) - 10 = 12 - 10 = 2 = +ve$

f(x) is Min for Min value I $\Rightarrow f(2) = (2-2)^2(2-1) = 0 \times 1 = 0$

When x = $\frac{4}{3}$ then $f''\left(\frac{4}{3}\right)= {}^26\left(\frac{4}{3}\right)-10=8-10-2=-ve$

f(x) is Max , I ⇒ $f\left(\frac{4}{3}\right)=\left(\frac{4}{3}-2\right)^2\left(\frac{4}{3}-1\right)$

$=\left(\frac{4-6}{3}\right)^2\left(\frac{4-3}{3}\right)=\left(\frac{-2}{3}\right)^2\left(\frac{1}{3}\right)=\frac{4}{9}$

f(x) is Min at x = 2 & f(2) = 0

f(x) is Max at x = $\frac{4}{3}$ & $\left(\frac{4}{3}\right)=\frac{4}{9}$ I

(ix). **f(x) = 5 + 3x − x³** **(Sargodha 2012)**

$f'(x) = 3 - 3x^2$

$f''(x) = -6x$

Put $f'(x) = 0$ ⇒ $3 - 3x^2 = 0$ ⇒ $3(1-x^2)=0$ ⇒ $1-x^2=0$

$x^2 = 1$ ⇒ $x=\pm1$

When x = 1 then $f''(1)=-6(+1)=-6=-ve$

f(x) is Max : $f(1) = 5 + 3(1) - (1)^3 = 5 + 3 - 1 = 7$

When x = −1 then $f''(-1)=-6(-1)=6=+ve$ $f(x)$ *is Min*

So $f(-1) = 5 + 3(-1) - (-1)^3 = 5 - 3 - (-1) = 5 - 3 + 1 = 3$

f(x) is Max at x = 1 & f(1) = 7

f(x) is Min at x = −1 & f(−1) = 3

3. **f(x) = Sinx + Cosx** I

$f'(x) = Cosx - Sinx$

$f''(x) = -Sinx - Cosx = -(Sinx + Cosx)$

Put $f'(x) = 0$ ⇒ Cos x − Sinx = 0 ÷ *by Cos x*

1 − tanx = 0

⇒ Tanx = 1 ⇒ $x = Tan^{-1}(1) = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots\ldots$

just $\frac{\pi}{4}$ & $\frac{5\pi}{4}$ *lies in* $[0, 2\pi]$ *ignor other values*

Whn x = $\frac{\pi}{4}$ then $f''\left(\frac{\pi}{4}\right)=-\left(Sin\frac{\pi}{4}+Cos\frac{\pi}{4}\right)=-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)=-ve$

f (x) is Max at x $\frac{\pi}{4}$ Now Far Max value

I ⇒ $f\left(\frac{\pi}{4}\right)=Sin\frac{\pi}{4}+Cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{\sqrt{2}\times\sqrt{2}}{\sqrt{2}}=\sqrt{2}$

When x = $\frac{5\pi}{4}$ *Then* $f''\left(\frac{5\pi}{4}\right)=-\left(Sin\frac{5\pi}{4}+Cos\frac{5\pi}{4}\right)$

$$= -\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = -\left(\frac{-2}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = +ve$$

f(x) is Min at x = $\frac{5\pi}{4}$ Now for min value.

$$\text{I} \Rightarrow f\left(\frac{5\pi}{4}\right) = Sin\frac{5\pi}{4} + Cos\frac{5\pi}{4}$$

$$= -\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = -\left(\frac{-2}{\sqrt{2}}\right) = \frac{2}{\sqrt{2}} = +ve$$

f(x) is Min at x = $\frac{5\pi}{4}$ Now for minimum value.

$$\text{I} \Rightarrow \quad f\left(\frac{5\pi}{4}\right) = Sin\frac{5\pi}{4} + Cos\frac{5\pi}{4}$$

$$f\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = \frac{-\sqrt{2}\times\sqrt{2}}{\sqrt{2}} = -\sqrt{2}$$

f(x0 is Max at = $\frac{\pi}{4}, f\left(\frac{\pi}{4}\right) = \sqrt{2}$

f(x) is Min at x = $\frac{5\pi}{4}, f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$

4. y = $\frac{\ln x}{x}$ **(Sargodha 2011)**

$$\frac{dy}{dx} = \frac{x.\frac{1}{x} - \ln x.1}{x^2} = \frac{1 - \ln x}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x).2x}{x^4} = \frac{-x - (1 - \ln x)2x}{x^4}$$

$$= \frac{-x(1 + 2(1 - \ln x))}{x^4} = \frac{-(1 + 2(1 - \ln x))}{x^3}$$

put $\frac{dy}{dx} = 0 \Rightarrow \quad \frac{1 - \ln x}{x^2} = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow \ln x = 1$

$\Rightarrow \quad x = e^1 = e$

When x = e then $\frac{d^2y}{dx^2} = \frac{-(1 + 2(1 - 1))}{e^3} = \frac{-1 + 0}{e^3} = \frac{-1}{e^3}$

$$\frac{d^2y}{dx^2} = -ve$$

Hence f(x) is Max at x= e

5. $f(x) = x^x$ **(Sargodha 2008)**

Taking ln both sides

$\ln f(x) = \ln x^x = \ln x$

$$\frac{1}{f(x)} f'(x) = x.\frac{1}{x} + \ln x.1 = 1 + \ln x$$

$f'(x) = f(x)(1+\ln x) = x^x(1+\ln x)$

$$f''(x) = x^x \frac{d}{dx}(1+\ln x) + (1+\ln x)\frac{d}{dx}x^x$$

$$= x^x\left(\frac{1}{x}\right) + (1+\ln x).x^x(1+\ln x)$$

$$= x^x\left[\frac{1}{x} + (1+\ln x)^2\right]$$

Put $f'(x) = 0 \Rightarrow x^x(1+\ln x) = 0$

$\Rightarrow 1+\ln x = 0 \Rightarrow \ln x = -1$ & $x^x \neq 0$

$\Rightarrow x = e^{-1} = \frac{1}{e}$

At $x = \frac{1}{e}$ then $\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{1/e}\left[e + (1-1)^2\right] = \left(\frac{1}{e}\right)^{1/e}(e+0) = +ve$

f(x) is Min at $x = \frac{1}{e}$

Hence proved:

Exercise 2.10

1. (i). Suppose two numbers are x & y and x + y = 30 **(Sargodha 2012, Lhr 2010)**

$\Rightarrow \quad y = 30 - x$ *then* product $= f(x) = xy = x(30 - x) = 30x - x^2$

$f'(x) = 30 - 2x \Rightarrow \quad f''(x) = -2$

When $f'(x) = 0 \Rightarrow 30 - 2x = 0 \quad \Rightarrow \quad 2x = 30 \quad \Rightarrow \quad x = 15$

At x = 15 , $f''(15) = -2 = -ve$ *So* $f(x)$ *is Max at* $x = 15$

Now first number = x = 15

Second number = 30 – x = 30 – 15 = 15

2. Suppose two numbers are x and y also x + y = 20 $\Rightarrow \quad y = 20 - x$,

then According to given condition

$f(x) = x^2 + y^2 = x^2 + (20 - x)^2 = x^2 + 400 - 40x + x^2$

$f(x) = 2x^2 - 40x + 400$

$f'(x) = 2.2x - 40 = 4x - 40$

$f''(x) = 4 \quad , \quad put\ f'(x) = 0 \Rightarrow 4x - 40 = 0$

$\Rightarrow 4x = 40 \Rightarrow \quad x = 10$

When x = 10 then $f''(10) = 4 = +ve$

So function is Min now

First number = x = 10

Second number = 20 – x = 20– 10 = 10

3. Supose two numbers are x and y also x + y = 12 $\Rightarrow \; y = 12 - x$ **then according to the given condition.**

$f(x) = yx^2 = (12 - x)\ x^2 = 12x^2 - x^3$

$f'(x) = 24x - 3x^2 \quad , \quad f'(x) = 24 - 6x$

put $f'(x) = 0 \quad \Rightarrow \quad 24x - 3x^2 = 0 \Rightarrow 3x(8 - x) = 0$

$3x = 0$ or $8 - x = 0 \Rightarrow \quad x = 0$ or $x = 8$

x = 0 is not possible so

x = 8 , at x = 8 , $f''(8) = 24 - 6(8) = 24 - 48 = -24 = -ve$

So function is Max at x = 8

First number = x = 8

Second number = 12 – x = 12 – 8 = 4

4. Supose length of one side is x secod side length is given 6 cm then third side is =16–x–6 = 10 – x so sides are x, 6, 10 – x Suppose f(x0 represent square of Area then.

$f(x) = S(S - a)(S - b)(S - c) \quad$ I $\quad a = x, b = 6, c = 10 - x$

$S = \dfrac{a+b+c}{2} = \dfrac{x + 6 + 10 - x}{2} = \dfrac{16}{2} = 8$

I become

$f(x) = 8(8 - x)(8 - 6)(8 - 10 + x)$

$= 8(8 - x)(2)(x - 2)$

$= 16(8 - x)(x - 2) = 16(8x - 16 - x^2 + 2x)$

$= 16(-x^2 + 10x - 16)$

$f'(x) = 16(-2x + 10)$

$f''(x) = 16(-2) = -32$

put $f'(x) = 0$ $\Rightarrow 16(-2x+10)=0 \Rightarrow -2x+10=0$

$2x = 10 \Rightarrow x = 5$

When x = 5 then $f''(5) = -32 = -ve$ So function is Max now sides are second side = 6 and third side = 10 – 5 = 5

5. **Suppose length = x and width = y**

Then perimeter = 120 cm $\Rightarrow 2x + 2y = 120cm$

$\Rightarrow x + y = 60$ c, I

Area = A = xy II

From I y = 60 – x Put in II

$A = x(60-x) = 60x = x^2$

$\frac{dA}{dn} = 60 - 2x$ & $\frac{d^2A}{dx^2} = -2$

Put $\frac{dA}{dx} = 0 \Rightarrow 60 - 2x = 0 \Rightarrow 2x = 60 \Rightarrow x = 30$

Area of rectangle is Max then dimensions are

Length = x = 30 cm

Wedth = y = 60 – 30 = 30cm

6. **Suppose length = x & width = y**

Then Area = xy = 36 $\Rightarrow y = \frac{36}{x}$

And P = Perimeter = 2x + 2y $\Rightarrow P = 2x + 2\left(\frac{36}{x}\right)$

$P = 2x + 72x^{-1} \Rightarrow \frac{dp}{dx} = 2 + 72(-1)x^{-2}$

$\frac{d^2P}{dx} = 0 + 72(-1)(-2)x^{-3} = \frac{144}{x^3}$

put $\frac{dp}{dx} = 0 \Rightarrow 2 + 72(-1)x^{-2} = 0 \Rightarrow 2 - \frac{72}{x^2} = 0$

$\Rightarrow \frac{72}{x^2} = 2 \Rightarrow 72 = 2x^2 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$

x = 6 (–Ignore)

When x = 6 then $\frac{d^2p}{dx^2} = \frac{144}{(6)^3} = +ve$

So perimeter is Min then sides are

Length = x = 6 & width = y = $\frac{36}{x}=\frac{36}{6}=6$

7. **Let length = x , width = x, Height = h**

Then volume = $x^2h \Rightarrow 4 = x^2h \Rightarrow h = \frac{4}{x^2}$

And $S = x^2 + 4xh = x^2 + 4x\left(\frac{4}{x^2}\right) = x^2 + 16x^{-1}$

$$\frac{dS}{dx} = 2x + 16(-1)x^{-2} \ \& \ \frac{d^2S}{dx^2} = 2 + 16(-1)(-2)(x^{-3})$$

$$= 2 + \frac{32}{x^3}$$

Put $\frac{dS}{dx} = 0 \Rightarrow 2x - \frac{16}{x^2} = 0 \Rightarrow \frac{2x^3-16}{x^2} = 0$

$\Rightarrow 2x^3 - 16 = 0 \Rightarrow 2x^3 = 16 \Rightarrow x^3 = 8 \Rightarrow x^3 = 2^3 \Rightarrow x = 2$

When x = 2 then $\frac{d^2S}{dx^2} = 2 + \frac{32}{2^3} = +ve$

So S is Min so least material required

When Length = x = 2

h = Height = $\frac{4}{x^2} = \frac{4}{(2)^2} = \frac{4}{4} = 1$

8. **Suppose length = x & Width = y**

Then perimeter = $2x+2y = 80 \Rightarrow x + y = 40 \Rightarrow y = 40 - x$

$A = xy \Rightarrow A = x(40-x) = 40x - x^2$

$$\frac{dA}{dx} = 40 - 2x \quad \& \quad \frac{d^2A}{dx^2} = -2 \quad Put\ \frac{dA}{dx} = 0$$

$\Rightarrow 40 - 2x = 0 \Rightarrow 2x = 40 \Rightarrow x = 20$

When x = 20 then $\frac{d^2A}{dx^2} = -2 = -ve$

So at x = 20 Area is Max then

Length = x = 20 & width = y = 40 – 20 = 20 meters

9. **Suppose quantity = q & Depth = h**

Then $q = x^2h \Rightarrow h = \frac{q}{x^2}$ *and*

Surface area = $S = x^2 + 4xh = x^2 + 4x\left(\frac{q}{x^2}\right)$

$S = x^2 + 4qx^{-1}$, $\frac{dS}{dx} = 2x + 4q(-1)x^{-2}$

$$\frac{d^2S}{dx^2}=2+4q(-1)(-2)x^{-3}=2+\frac{8q}{x^3}$$

Put $\frac{dS}{dx}=0 \Rightarrow 2x-\frac{4q}{x^2}=0 \Rightarrow \frac{2x^3-4q}{x^2}=0$

$\Rightarrow 2x^3-4q=0 \Rightarrow 2x^3=4q \Rightarrow x^3=2q \Rightarrow q=\frac{x^3}{2}$ *I*

$x=(2q)^{1/3}$ when $x=(2q)^{1/3}$ then

$\frac{d^2S}{dx^2}=2+4q\ \frac{8q}{(2q)}=+ve\ So$

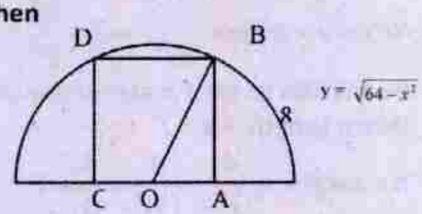
S in Min or least expense required

At $x=(2q)^{1/3}$ then

Depth = h = $\frac{q}{x^2}=\frac{\frac{x^3}{2}}{x^2}$ *from I* $=\frac{x^3}{2}\times\frac{1}{x^2}=\frac{x}{2}$ *Unit*

10. **Suppose OA = x and O is center and AB = y then**

Suppose OA = x & O is center & AB = y then



By pathgoras theorem

$(OA)^2+(AB)^2=(OB)^2$

$\Rightarrow x^2+y^2=(8)^2 \Rightarrow y^2=64-x^2$

Then length of rectangle = AC = 2x

Width of rectangle = $\sqrt{64-x^2}$

Area = $2x\times\sqrt{64-x^2}$ $\Rightarrow (Area)^2=4x^2(64-x^2)$

Take $(Area)^2=S$ then $S=4x^2(64-x^2)$

$$\frac{dS}{dx}=8x(64-x^2)+4x^2(-2x)=512x-8x^3-8x^3$$

$=512x-16x^3$, $\frac{d^2}{dx^2}=512-64x^2$

put $\frac{dS}{dx}=0 \Rightarrow 512x-16x^3=0 \Rightarrow 16x(32-x^2)=0$

$16x=0$ or $32-x^2=0$

$x=0$ or $x^2=32 \Rightarrow x=\sqrt{32}$

$x=\sqrt{2\times2\times2\times2\times2}=2\times2\sqrt{2}=4\sqrt{2}$

$$\frac{d^2S}{dx^2} = 512 - 64\left(4\sqrt{2}\right)^2$$

When $x = 4\sqrt{2}$ then $= 512 - 64(16\times 2) = 512 - 2048$

$= -1536 < 0 = -ve$

So Area is Max Dimensions are

Length = $2x = 2(4\sqrt{2}) = 8\sqrt{2}$ cm

Width = $y = \sqrt{64 - x^2} = \sqrt{64 - \left(4\sqrt{2}\right)^2} = \sqrt{64 - 16\times 2}$

$= \sqrt{32} = \sqrt{2\times 2\times 2\times 2\times 2} = 4\sqrt{2}cm$

11. Suppose (x, y) is required pt

then distance = $d = \sqrt{(x-3)^2 + (y-(-1))^2}$ given is (3, –1)

$\Rightarrow d^2 = (x-3)^2 + (y+1)^2$

Curve is given $y = x^2 - 1 \Rightarrow y + 1 = x^2$ use this value

$d^2 = (x-3)^2 + (x^2)^2 = x^2 - 6x + 9 + x^4$

Suppose $d^2 = l$ then

$l = x^4 + x^2 - 6x + 9$

$\frac{dl}{dx} = 4x^3 + 2x - 6$ & $\frac{d^2l}{dx^2} = 12x^2 + 2$

put $\frac{dl}{dx} = 0 \Rightarrow 4x^3 + 2x - 6 = 0$

1	4	0	2	-6
		4	4	6
	4	4	6	0

By synthetic division

$\Rightarrow (x-1)(4x^2 + 4x + 6) = 0$

$x-1 = 0$ or $4x^2 + 4x + 6 = 0$ $\qquad x = \frac{-2 \pm \sqrt{4-24}}{2(2)}$

$x = 1$ or $2x^2 + 2x + 3 = 0$ $\qquad = \frac{-2 \pm \sqrt{-20}}{4}$

Complex roots ignore.

When, $x = 1$ then $\frac{d^2l}{dx^2} = 12(1)^2 + 2 = 14 == +ve$

So distance is closest now

When $x = 1$, then $y = x^2 - 1 = (1)^2 - 1 = 1 - 1 = 0$

So (1, 0) is required pt on curve $y = x^2 - 1$ and closest to (3, –1)

12. Find pt on the curve $y = x^2 + 1$ closest to (18,1)

Suppose (x, y) is required point then

$$d = \sqrt{(x-18)^2 + (y-1)^2} \Rightarrow d^2 = (x-18)^2 + (y-1)^2$$

take $d^2 = l$ then $l = x^2 - 36x + 324 + (x^2)^2$ use I $y = x^2 + 1 \Rightarrow y - 1 = x^2$

$l = x^4 + x^2 - 36x + 324$

$$\frac{dl}{dx} = 4x^3 + 2x - 36 \quad \& \quad \frac{d^2l}{dx^2} = 12x^2 + 2 \qquad II$$

Put $\frac{dl}{dx} = 0 \Rightarrow \quad 4x3 + 2x - 36 = 0 \quad or\ 2x^3 + x - 18 = 0$

	2	0	1	-18
2		4	8	18
	2	4	9	0

$2x^3 + x - 18 = 0$

or $(x-2)(2x^2+4x+9) = 0$

$x-2 = 0$ or $2x^2 + 4x + 9 = 0$

$$x = 2 = \text{ or } x = \frac{-4 \pm \sqrt{(4)^2 - 4(2)(9)}}{2(2)} = \frac{-4 \pm \sqrt{16-72}}{4}$$

$$x = \frac{-4 \pm \sqrt{-56}}{4} \qquad \textit{ignore due to Imaginary}$$

put x = 2 in II $\frac{d^2l}{dx^2} = 12(2)^2 + 2 = 48 + 2 = 50 > 0$

function is Min or point is closest at x = 2

$y = x^2 + 1 = (2)^2 + 1 = 5$

So (2, 5) s required closest point

TEST YOUR SKILLS — Marks 100

OBJECTIVE

Q.No.1 Given below are a few possible answers to each statement of which one is correct, identify the correct one. (20)

1. **The difference quotient $\frac{f(x_1)-f(x)}{x_1-x}$ represents the**
   - (a) Instantaneous rate of change
   - (b) Average rate of change
   - (c) Both (a) & (b)
   - (d) None of these
2. $\lim\limits_{x_1 \to x} \frac{f(x_1)-f(x)}{x_1-x}$**, provided the limit exists, it is called**
   - (a) Instantaneous rate of change
   - (b) Average rate of change
   - (c) Both (a) & (b)
   - (d) None of these
3. $\lim\limits_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x}$**, provided the limit exists, is called**
   - (a) Derivative
   - (b) Differential coefficient
   - (c) Both (a) & (b)
   - (d) None of these
4. **The process of finding f' is called**
   - (a) Differentiation
   - (b) Integration
   - (c) Both (a) & (b)
   - (d) None of these
5. **The Notation $f'(x)$ is used for derivative by**
   - (a) Cauchy
   - (b) Newton
   - (c) Leibniz
   - (d) None of these
6. **The Notation $\frac{df}{dx}$ is used for derivative by**
   - (a) Leibniz
   - (b) Newton
   - (c) Lagrange
   - (d) Cauchy
7. **The Notation $Df(x)$ is used for derivative by**
   - (a) Leibniz
   - (b) Newton
   - (c) Lagrange
   - (d) Cauchy
8. $f(x) = x^n$, $n \neq 1$ **then** $f'(x) =$
   - (a) x^{n-1}
   - (b) nx^{n+1}
   - (c) nx^{n-1}
   - (d) $(n+1)x^n$
9. **If** $f(x) = c^3$ **then** $f'(x) =$
   - (a) $3c^2$
   - (b) c^2
   - (c) $\frac{3}{c}$
   - (d) 0
10. **If** $f(x) = \frac{1}{(ax+b)^n}$ **then** $f'(x) =$
    - (a) $na(ax+b)^{n+1}$
    - (b) $-na(ax+b)^{n-1}$
    - (c) $-na(ax+b)^{-(n+1)}$
    - (d) None of these

11. If $y = \sqrt{x} - \frac{1}{\sqrt{x}}$, then $2x\frac{dy}{dx} + y =$
   (a) $\sqrt{x}$ (b) $2\sqrt{x}$
   (c) 0 (d) None of these

12. If $y = \text{Sin}^{-1}x$, $-1 < x < 1$ then $\frac{dy}{dx} =$
   (a) $\text{Cos}^{-1}x$ (b) $\frac{1}{\sqrt{1-x^2}}$
   (c) $\frac{1}{\sqrt{1+x^2}}$ (d) $\frac{1}{\text{Sin}^{-1}x}$

13. If $y = \text{Cot}^{-1}\frac{x}{a}$ then $\frac{dy}{dx} =$
   (a) $\frac{-a}{a^2+x^2}$ (b) $\frac{1}{a^2+x^2}$
   (c) $\frac{-1}{a^2+x^2}$ (d) $\frac{a}{a^2+x^2}$

14. If $y = e^{ax}$ then $\frac{dy}{dx} =$
   (a) $\frac{1}{e^x}$ (b) ae^{ax}
   (c) e^{ax} (d) $\frac{1}{a}e^{ax}$

15. $y = \text{Cosec}\,h\,x$ then $\frac{dy}{dx} =$
   (a) $-\text{Coth}\,x\,\text{Cosec}\,hx$ (b) $-\text{Tan}\,h\,x\,\text{Sech}\,x$
   (c) $\text{Coth}\,x\,\text{Cosec}\,hx$ (d) $\text{Cot}\,h\,x\,\text{Sinh}\,x$

16. If $y = \text{Sech}^{-1}x$, then $\frac{dy}{dx} =$
   (a) $\frac{1}{x\sqrt{1-x^2}}$ (b) $\frac{1}{x\sqrt{1+x^2}}$
   (c) $\frac{-1}{x\sqrt{1-x^2}}$ (d) $\frac{-1}{x\sqrt{1+x^2}}$

17. $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ... + a_nx^n + ...$ is called a
   (a) Binomial series (b) Power series
   (c) Maclaurin series (d) Tailor series

18. $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ... + \frac{f^n(0)}{n!}x^n + ...$ this expansion of $f(x)$ is called

(a) Binomial series (b) Power series

(c) MacLaurin's series (d) Tailor series

19. If $f(x) = \text{Cos } x$ then by MacLaurin Series $f(x) =$

(a) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$ (b) $1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + ...$

(c) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$ (d) None of these

20. For stationary point for a function f we have $f'(x) =$

(a) 0 (b) +ve

(c) −ve (d) ∞

SECTION I
SUBJECTIVE

Attempt any 25 (twenty five) short question from Question 2,3 and 4, at least 12 short questions from question 2, 12 short question from question 3 and 13 short question from question 4. All question carry equal marks. ***(25x2=50)***

Q.No. 2

i. Define derivative of a function $y = f(x)$

ii. Find the derivative w.r.t. independent variables $y = \frac{1}{(az-b)^2}$

iii. Find the derivative w.r.t. independent variables $y = \frac{1}{(ax+b)^n}$

iv. Find the derivative of $y = (x^2 + 5)(x^3 + 7)$ w.r.t. 'x'.

v. If $y = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$ find $\frac{dy}{dx}$.

vi. If $y = \sqrt{x} - \frac{1}{\sqrt{x}}$ show $2x\frac{dy}{dx} + y = 2\sqrt{x}$

vii. If $y = \frac{x^2+1}{x^2-3}$ find $\frac{dy}{dx}$

viii. Find $\frac{dy}{dx}$ if $y = 2at, x = at^2$

ix. Find $\frac{dy}{dx}$ if $x = \theta + \frac{1}{\theta}, y = \theta + 1$

x. Find $\frac{dy}{dx}$ if $x^2 + y^2 = 4$

xi. Find $\frac{dy}{dx}$ if $y^2 - xy - x^2 + 4 = 0$

xii. $y = \sqrt{x + \sqrt{x}}$ find $\frac{dy}{dx} = ?$

Q.No. 3

i. If $y = \text{Cos}\sqrt{x} + \sqrt{\text{Sin}\,x}$ find $\frac{dy}{dx}$

ii. $\frac{dy}{dx} \equiv ?$ If $y = x\,\text{Cos}\,y$

iii. $\frac{dy}{dx} \equiv ?$ If $x = y\,\text{Sin}\,y$

iv. Differentiate Sin x w.r.t. Cot x

v. $\frac{dy}{dx} \equiv ?$ if $y = \text{Cos}^{-1}\frac{x}{a}$

vi. $\frac{dy}{dx} \equiv ?$ if $y = \text{Cot}^{-1}\frac{x}{a}$

vii. $\frac{dy}{dx} \equiv ?$ if $y = \frac{1}{a}\text{Sin}^{-1}\frac{a}{x}$

viii. $\frac{dy}{dx} \equiv ?$ If $y = a^{\sqrt{x}}$

ix. $\frac{dy}{dx} \equiv ?$ If $y = x^3 e^{\frac{1}{x}}$

x. $\frac{dy}{dx} \equiv ?$ If $y = \ln\frac{1}{x} + \frac{1}{\ln x}$

xi. $\frac{dy}{dx} \equiv ?$ If $y = \ln\sqrt{x} + \sqrt{\ln x}$

xii. $\frac{dy}{dx} \equiv ?$ If $y = \ln(x + \sqrt{x^2 + 1})$

Q.No. 4

i. $\frac{dy}{dx} \equiv ?$ If $y = \text{Cos}^{-1}h(\text{Sec}\,x)$

ii. $\frac{dy}{dx} \equiv ?$ If $y = \ln(\text{Tan}\,hx)$

iii. $\frac{dy}{dx} \equiv ?$ If $y = \text{Tan}\,h^{-1}(\text{Sin}\,x)$ $\frac{\pi}{2} < x < \frac{\pi}{2}$

iv. $\frac{dy}{dx} \equiv ?$ If $y = \text{Sin}\,h^{-1}\frac{x}{2}$

v. State MacLaurin series.

vi. State Taylor's series.

vii. State what is the geometrical meaning of derivative.

viii. Define increasing function.

ix. Define decreasing function.

x. What is point of maximum and minimum?

xi. What are extreme values?

xii. Define point of inflexion or stationary point.

xiii. Determine the interval in which $f(x) = 4 - x^2$ is increasing or decreasing in the interval $(-2, 2)$.

SECTION II

Attempt any 3 (three) questions. (3x10=30)

Q.No.5

(a) Find the derivative of the functions $f(x)=x^2$ by definition

(b) Find $\frac{dy}{dx}$ if $x^2 - 4xy - 5y = 0$

Q.No.6

(a) Find $\frac{dy}{dx}$ of $f(x)= \frac{1}{\sqrt{x+a}}$ by first principle

(b) Differentiate $(1+x^2)^n$ w.r.t. x^2

Q.No.7

(a) Find $\frac{dy}{dx}$ if $y = \frac{(\sqrt{x}+1)(x^{3/2}-1)}{x^{1/2}-1}$ $(x \neq 1)$

(b) Show that $\frac{dy}{dx} = \frac{y}{x}$ if $\frac{y}{x} = \text{Tan}^{-1}\frac{x}{y}$

Q.No.8

(a) Find $\frac{dy}{dx}$ if $y^3 - 2xy^2 + x^2y + 3x = 0$

(b) Find $\frac{dy}{dx}$ if $y = \log_{10}(ax^2 + bx + c)$

Q.No.9

(a) If $y = e^x \sin x$, show that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

(b) Show that $y = x^x$ has a minimum value at $x = \frac{1}{e}$

Previous Board Questions

1. Show that $\frac{d}{dx}\left(\frac{1}{g(x)}\right)=\frac{-g'(x)}{(g(x))^2}$. (Mtn – 2009)
2. Find $\frac{dy}{dx}$ if $y=\sqrt{x+2}$. (Fsd – 2009)
3. Find $\frac{dy}{dx}$ if $y = x.\sqrt{\ln x}$. (Mtn – 2009)
4. Differentiate $\sqrt{x+\sqrt{x}}$ w.t.r. to x. (Grw – 2007)
5. Differentiate $\frac{x^2+1}{x^2-3}$ with respect to x. (Lhr – 2009)
6. Differentiate $x^2 + 2x^3 + x^2$ with respect to x. (Lhr – 2005)
7. Differentiate ln (x2 + 2x) w.r.t x. (Fsd – 2009)
8. Differentiate $\frac{1}{\sqrt{a^2-x^2}}$ with respect to x. (Grw – 2005)
9. Find y_2, if $y = (2x+5)^{3/2}$. (MirPur – 2005)
10. What is power series?
11. Differentiate sin x with respect to cox x. (Lhr – 2005)
12. What is increasing function?
13. What is stationary point? (Lhr – 2005)
14. State Maclaurin's series expansion? (Mtn – 2009)
15. Find $\frac{dy}{dx}$, if $y^2 + x^2 - 4x + 5$ (Lahore – 2010)
16. Differentiate $\frac{1}{a}\sin^{-1}\frac{a}{x}$ w.r.t 'x'. (Lahore – 2010)
17. Differentiate $\cos\sqrt{x}+\sqrt{\sin x}$ w.r.t 'x'. (Lahore – 2010)
18. State Maclaurin Series Expansion. (Lahore – 2010)
19. Find the derivative of the function $f(x) = x^2$ by definition. (Gujranwala – 2010)
20. Differentiate $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2$ with respect to 'x'. (Gujranwala – 2010)

INTERGRATION

3

Definitions

1. **Integration or Antiderivative:**
   Inverse process of differentiation is called integration.
2. $df = f'(x)\,dx$, $f'(x)$ is called **differential co-efficient**.
3. **Fundamental Theorem of Calculus:**

   If f is continuous on $[a, b]$ and $\varphi'(x) = f(x)$ then $\int_a^b f(x)\,dx = \varphi(b) - \varphi(a)$

4. **Differential Equations:**
   An equation containing at least one derivation of a dependent variable w.r.t. an independent variable. e.g. $y\dfrac{dy}{dx} + 2x = 0$
5. The order of a differential equation is the order of the highest derivative in the equation.
6. **Initial Conditions:**
   The arbitrary constants involving in the solution of differential equation can be determined by the given condition. Such conditions are called initial value conditions.

Important Formulae

Derivative	Integration
1. $\dfrac{d}{dx}(c) = 0$	1. $\int 0\,dx = c$
2. $\dfrac{d}{dx}(x) = 1$	2. $\int 1\,dx = x$
3. $\dfrac{d}{dx}(x^n) = nx^{n-1}$	3. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$
4. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$	4. $\int \dfrac{1}{x}\,dx = \ln\lvert x\rvert + c$
5. $\dfrac{d}{dx}(a^x) = a^x.\ln a$	5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + c$

6. $\frac{d}{dx}(e^x) = e^x$

7. $\frac{d}{dx}(\text{Sin}x) = \text{Cos}x$

8. $\frac{d}{dx}(\text{Cos}x) = -\text{Sin}x$

9. $\frac{d}{dx}(\text{Tan}x) = \text{Sec}^2x$

10. $\frac{d}{dx}(\text{Cot}x) = -\text{Cosec}^2x$

11. $\frac{d}{dx}(\text{Sec}x) = \text{Sec}x\ \text{Tan}x$

12. $\frac{d}{dx}(\text{Cosec}x) = -\text{Cosec}x\ \text{Cot}x$

13. $\frac{d}{dx}(\text{Tan}^{-1}x) = \frac{1}{1+x^2}$

14. $\frac{d}{dx}(\text{Sin}n\,x) = \text{Cos}nx\ .\ n$

15. $\frac{d}{dx}(e^{nx}) = e^{nx}\ .\ n$

Integration By Parts:

$I = \int UV.dx$

UV = Product of two functions i.e,

$f(x) = U.$

$g(x) = V.$

It is easy to remenber if we say that.

$I = (1^{st} fn)\ (Integral\ of\ 2^{nd} fn) -$
$\int (Integral\ of\ 2^{nd}\ fn) \times (derivative\ of\ 1^{st}\ fn)\,dx$

Note:

Log x, Inverse trigonometric always 1^{st} function. If the product does not include any one of these two then power of x i.e, x^n always 1^{st} *fn*.

6. $\int e^x\,dx = e^x + c$

7. $\int \text{Sin}x\,dx = -\text{Cos}x$

8. $\int \text{Cos}x\,dx = \text{Sin}x$

9. $\int \text{Sec}^2dx = \text{Tan}x$

10. $\int \text{Cosec}^2x\,dx = -\text{Cot}x$

11. $\int \text{Sec}x\ \text{Tan}x\,dx = \text{Sec}x$

12. $\int \text{Cosec}x\ \text{Cot}x\,dx = -\text{Cosec}x$

13. $\int \frac{1}{1+x^2}dx = \text{Tan}^{-1}x$

14. $\int \text{Sin}nx\,dx = \frac{-\text{Cos}\,nx}{n}$

15. $\int e^{nx}\,dn = \frac{e^{nx}}{n}$

16. $\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\text{Tan}^{-1}\frac{x}{a}$

17. $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}$

18. $\int \frac{dx}{\sqrt{x^2-b^2}} = \frac{1}{a}\sin^{-1}\frac{x}{a}$

19. $\int \frac{1}{\sqrt{a^2-x^2}} = ln(x+\sqrt{a^2+x^2})+c$

20. $\int (f(x))^n.f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1}$

21. $\int \frac{f'(x)\,dx}{f(x)} = ln|f(x)|$

Important Formulae

	Derivation	Integration
1.	$\frac{d}{dx}(c)=0$	$\int 0\,dx=c$
2.	$\frac{d}{dx}(x)=1$	$\int 1\,dx=x$
3.	$\frac{d}{dx}(x^n)=nx^{n-1}$	$\int x^n\,dx=\frac{x^n+1}{n+1}+c$
4.	$\frac{d}{dx}(\ln x)=\frac{1}{x}$	$\int \frac{1}{x}\,dx=ln\lvert x\rvert+c$
5.	$\frac{d}{dx}(a^x)=a^x\,ln\,a$	$\int a^x\,dx=\frac{a^x}{ln\,a}+c$
6.	$\frac{d}{dx}(e^x)=e^x$	$\int e^n\,dx=e^x+c$
7.	$\frac{d}{dx}(Sinx)=Cosx$	$\int Sinx\,dx=-Cosx$
8.	$\frac{d}{dx}(Cosx)=Sinx$	$\int Cosx\,dx=Sinx$
9.	$\frac{d}{dx}(Tanx)=Sec^2x$	$\int Sec^2 dx=Tanx$
10.	$\frac{d}{dx}(Cotx)=-Co\sec^2 x$	$\int Co\sec^2 x\,dx=-Cotx$
11.	$\frac{d}{dx}(Secx)=Secx\;Tanx$	$\int Secx\;Tanx\,dx=Secx$
12.	$\frac{d}{dx}(Co\sec x)=-Co\sec x\;Cotx$	$\int Co\sec x\;Cotx\,dx=-Co\sec x$
13.	$\frac{d}{dx}(Tan^{-1}x)=\frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx=Tan^{-1}x$
14.	$\frac{d}{dx}(Sinn\,x)=Cosnx\,.\,n$	$\int Sinnx\,dx=\frac{-Cos\;nx}{n}$
15.	$\frac{d}{dx}(e^{nx})=e^{nx}\,.\,n$	$\int e^{nx}\,dn=\frac{e^{nx}}{n}$
16.		$\int \frac{1}{a^2+x^2}\,dx=\frac{1}{a}Tan^{-1}\frac{x}{a}$

Exercise 3.1

1. i. $y = x^2 - 1$ $\quad x = 3, \delta x = dx = 3.02 - 3 = 0.02$

$y = (3)^2 - 1 = 9 - 1 = 8$

Now

Then $y + \delta y = (x + \delta x)^2 - 1 \quad \Rightarrow \quad \delta y = (x + \delta x)^2 - 1 - y$

$\delta y = (3 + 0.02)^2 - 1 - 8 \quad \Rightarrow \quad \delta y = (3.02)^2 - 9 = 9.1204 - 9$

$\boxed{\delta y = 0.1204}$ Again $y = x^2 - 1$

Taking deferential

$dy = 2xdx = 2(2)\ (0.02) \quad \Rightarrow \quad \boxed{dy = 0.12}$

ii. $y = x^2 + 2x$, $\quad x = 2, \delta x = dx = 1.8 - 2 = -0.2$

$y + \delta y = (x + \delta x)^2 + 2(x + \delta x) \qquad y = x^2 + 2x$

$\delta y = (x + \delta x)^2 + 2(x + \delta x) - y \qquad y = (2)^2 + 2(2)$

$\delta y = (2 - 0.2)^2 + 2(2.02) - 8 \qquad = 4 + 4 = 8$

$\delta y = (1.8)^2 + 2(1.8) - 8 \quad \Rightarrow \delta y = 3.24 + 3.6 - 8$

$\Rightarrow \boxed{\delta y = 1.16}$ Again $y = x^2 + 2x$

Taking differential

$dy = (2x + 2)dx$

$dy = (2(2) + 2)\ (-0.2) \quad \Rightarrow \quad dy = 6\,(-0.2) \quad \Rightarrow \quad \boxed{dy = -1.2}$

iii. $y = \sqrt{x}$ $\quad x = 4, \delta x = dx = 4.41 - 4 = 0.41$ **(Sargodha 2008)**

$y = \sqrt{4} = 2$ *Now* $y + \delta y = \sqrt{x + \delta x}$

$\delta y = \sqrt{x + \delta y} - y = \sqrt{4 + 0.41} - 2 = \sqrt{4.41} - 2 = 2.1 - 2$

$\boxed{\delta y = 0.1}$ *Again* $y = \sqrt{x} \quad \Rightarrow \quad dy \frac{1}{2} x^{-1/2} dx$

$dy = \frac{1}{2\sqrt{x}} dx \quad \Rightarrow \quad dy = \frac{1}{2\sqrt{4}}(0.41) = \frac{0.41}{2 \times 2} \quad \Rightarrow \quad \boxed{dy = 0.1025}$

2. i. $xy = +x = 4$ **(Sargodha 2011)**

Taking differential

$d(xy + x) = d(4) \quad \Rightarrow \quad xdy + ydx + dx = 0$

$xdy + (y + 1)dx = 0 \quad \Rightarrow \quad xdy = -(y + 1)dx$

$\boxed{\frac{dy}{dx}=\frac{-(y+1)}{x}}$ *Taking reciprocal* $\boxed{\frac{dx}{dy}=\frac{-x}{y+1}}$

ii. $x^2+2y^2=16$ ***Taking differential*** $2xdx+2.2ydy=0$ **(Sargodha 2011)**

$2xdx+4ydy=0 \quad \div \text{ by } 2 \quad \Rightarrow \quad xdx+2y\ dy=0$

$xdx=-2y\ dy \quad \Rightarrow \quad \boxed{\frac{-x}{2y}=\frac{dy}{dx}}$ & $\boxed{\frac{dx}{dy}=\frac{-2y}{x}}$

iii. $x^4+y^2=xy^2$ ***Taking differential***

$4x^3dx+2ydy=x.2ydy+y^2dx$

$4x^3dx+2ydy=2xy\ dy+y^2dx$

$2ydy-2xy\ dy=y^2dx-4x^3dx$

$(2y-2xy)dy=(y^2-4x^3)dx \Rightarrow \boxed{\frac{dy}{dx}=\frac{y^2-4x^3}{2y-2xy}}$

$2y(1-x)dy \qquad 2y(1-x) \qquad 2y(1-x)$

Taking reciprocal $\boxed{\frac{dx}{dy}=\frac{2y-2xy}{y^2-4x^2}}$

iv. $xy-\ln x=c$ ***Taking differential***

$xdy+ydx-\frac{1}{x}dx=0 \quad \Rightarrow \quad xdy+\left(y-\frac{1}{x}\right)dx=0$

$xdy=-\left(y-\frac{1}{x}\right)dx \quad \Rightarrow \quad xdy=-\left(\frac{xy-1}{x}\right)dx$

$xdy=\left(\frac{1-xy}{x}\right)dx \quad \Rightarrow \quad \boxed{\frac{dy}{dx}\left(\frac{1-xy}{x^2}\right)}$

Taking reciprocal $\boxed{\frac{dx}{dy}=\frac{x^2}{1-xy}}$

3. i. $\sqrt[4]{17}$ *Let* $f(x)=\sqrt[4]{x}=(x)^{1/4}$

Then $f(x+\delta x)\sqrt[4]{x+\delta x}$ *and* $f'(x)=\frac{1}{4}x^{-3/4}$

Also $dy=f'(x)dx$ *Now Take*

Using $f(x+\delta x)=f(x)+dy \qquad x=16$ & $dx=1$

$f(16+1)=f(16)+f'(16)dx \qquad dx=\delta x$

$f(17)=(16)^{1/4}+\frac{1}{4(16)^{3/4}}(1)$

$\Rightarrow \quad f(17)=2^{4\times1/4}+\frac{1}{4(2)^{4\times3/4}} \quad \Rightarrow \quad \sqrt[4]{17}=2+\frac{1}{4\times8}=2+0.03125$

So $\sqrt[4]{17}=2.03125$

ii. $(31)^{1/5}$ *Let* $f(x)=(x)^{1/5}$ *then* $f(x+\delta x)=(x+\delta x)^{1/5}$ *and* $f'(x)=\frac{1}{5}x^{-4/5}$ *also*

$dy=f'(x)dx$

Take $x=32$ *and* $dx=31-32=-1 \qquad dx=\delta x$

Using $f(x+\delta x)=f(x)+dy$

$f(32-1)=f(32)+f'(32)(-1)$

$f(31)=(32)^{1/5}+\frac{1}{5(32)^{4/5}}(-1)=2^{5\times 1/5}-\frac{1}{5(2)^{5\times 4/5}}$

$f(31)=2-\frac{1}{5(2)^4}=2-\frac{1}{80}=2-0.0125$

$\Rightarrow (31)^{1/5}=1.9875$

iii. $Cos 29^\circ$, Let $f(x)=Cosx$ then $f(x+\delta x)=Cos(x+\delta x)$ and $f'(x)=-Sin$ also

$dy=f'(x)\,dx$

Take $x=30^\circ$ and $dx=\delta x=29^\circ=-1^\circ=-1\times\frac{\pi}{180}=-0.0175$

Using $f(x+\delta x)=f(x)+dy$

$f(30^\circ-1^\circ)=f(30^\circ)+f'(30^\circ)dx$

$f(29^\circ)=Cos 30^\circ+(-Sin30^\circ)(-0.0175)$

$Cos 29^\circ=0.866+(0.5)(0.0175)$

$=0.866+0.00875=0.8747$

iv $Sin61^\circ$ Let $f(x)=Sinx \Rightarrow f'(x)=Cosx$

$f(x+\delta x)=Sin(x+\delta x)$

$dy=f'(x)dx$

Take $x=60^\circ$ & $dx=\delta x=61^\circ-60^\circ=1^\circ=1\times\frac{\pi}{180}=0.0175$

Using $f(x+\delta x)=f(x)dy$

$f(61^\circ+1)=f(60^\circ)+f'(60^\circ)(0.0175)$

$f(61^\circ)=Sin60^\circ+(Cos60^\circ)(0.0175)$

$Sin61^\circ=0.866+(0.5)(0.0175)$

$Sin61^\circ=0.866+0.00875=0.87475$

4. *Suppose l is length of edge*

$Volume=v=l^3$

$dv=3l^2dl$ $\quad l=5,\ dl=5.02-5=0.02$

$=3(5)^2(0.02)$ $\quad =0.02$

$=1.5(Cubic unit)$

5. $Area=A=\pi r^2$

Diameter $=44 \Rightarrow r=radius=\frac{44}{2}=22$

$dr=\frac{44.4-44}{2}=\frac{0.4}{2}=0.2$

$A=\pi r^2$

$dA=\pi.\ 2rdr$

$dA=(3.1416)(2)(22)(0.02)$

$=27.64$ Sq Unit

Example of 3.2

Example: $\int \frac{Sin x + Cos^3 x}{Cos^2 x \; Sin x} dx$ **(Sargodha 2008)**

$$= \int \frac{Sinx}{Cos^2 x \; Sinx} dx + \int \frac{Cos^3 x}{Cos^2 x Sinx} dx$$

$$= \int \frac{1}{Cos^2 x} dx + \int \frac{Cosx}{Sinx} dx$$

$$= \int Sec^2 x \; dx + \int \frac{Cosx}{Sinx} dx$$

$$= Tanx + ln|Sinx| + c$$

Un Seen: $\int \left(\frac{1}{3x-1} + \frac{2}{(x-1)^2} \right) dx$

$$= \int \frac{1}{3x-1} dx + \int 2(x-1)^{-2} dx$$

$$= \frac{1}{3} \int \frac{3}{3x-1} dx + 2 \int (x-1)^{-2} dx$$

$$= \frac{1}{3} ln\, |3x-1| + 2 \frac{(x-1)^{-2+1}}{-2+1} + c$$

$$= \frac{1}{3} ln\, |3x-1| - 2 \frac{1}{(x-1)} + c$$

Exercise 3.2

1. **i** $\int (3x^2 - 2x + 1) dx = 3 \int x^2 dx - 2 \int x dx + \int 1 dx$

$$= 3 \frac{x^3}{3} - 2 \cdot \frac{x^2}{2} + x + c + = x^2 + x + c$$

ii $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) dx$

$$= \int (x)^{1/2} dx + \int (x)^{-1/2} dx$$

$$= \frac{x^{1/2+1}}{\frac{1}{2}+1} + \frac{x^{-1/2+1}}{\frac{-1}{2}+1} + c$$

$$= \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{1/2}}{\frac{1}{2}} + c$$

$$= \frac{2}{3} x^{3/2} + 2\sqrt{x} + c$$

iii. $\int x\left(\sqrt{x}+1\right) dx = \int x\left(x^{1/2}+1\right) dx$ **(Sargodha 2009)**

$$= \int \left(x^{1/2+1} + 1 + x\right) dx = \int x^{3/2} dx + \int x\, dx$$

$$= \frac{x^{3/2+1}}{\frac{3}{2}+1} + \frac{x^{1+1}}{1+1} + c = \frac{x^{5/2}}{\frac{5}{2}} + \frac{x^2}{2} + c$$

$$= \frac{2}{5} x^{5/2} + \frac{x^2}{2} + c$$

iv. $\int (2x+3)^{1/2} dx = \frac{1}{2}\int (2x+3)^{1/2} \cdot 2\,dx$

$$= \frac{1}{2}\frac{(2x+3)^{1/2+1}}{\frac{1}{2}+1} + c = \frac{1}{2}\frac{(2x+3)^{3/2}}{\frac{3}{2}} + c$$

$$= \frac{1}{2}\cdot\frac{2}{3}(2x+3)^{3/2} + c = \frac{1}{3}(2x+3)^{3/2} + c$$

v. $\int \left(\sqrt{x}+1\right)^2 dx = \int \left(x + 2\sqrt{x} + 1\right) dx$

$$= \int x dx + 2\int x^{1/2} dx + \int 1 dx = \frac{x^2}{2} + 2.\frac{x^{1/2+1}}{\frac{1}{2}+1} + x + c$$

$$= \frac{x^2}{2} + \frac{2x^{3/2}}{\frac{3}{2}} + x + c = \frac{x^2}{2} + 2.\frac{2}{3}x^{3/2} + x + c$$

$$= \frac{x^2}{2} + \frac{4}{3}x^{3/2} + x + c$$

vi. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx = \int \left(x + \frac{1}{x} - 2\sqrt{x}\frac{1}{\sqrt{x}}\right) dx$

$$\int x dx + \int \frac{1}{x} dx - 2\int 1 dx = \frac{x^2}{2} + ln|x| - 2x + c$$

vii. $\int \frac{3x+2}{\sqrt{x}}\,dx = \int \left(3x^{1-1/2} + 2x^{-1/2}\right)dx$ (Sargodha 2010)

$$= 3\int x^{1/2}dx + 2\int x^{-1/2}dx = 3\ .\ \frac{x^{1/2+1}}{\frac{1}{2}+1} + 2\frac{x^{-1/2+1}}{\frac{-1}{2}+1} + c$$

$$= \frac{3x^{3/2}}{\frac{3}{2}} + 2\ .\ \frac{x^{1/2}}{\frac{1}{2}} + c = \not{3}\ .\ \frac{2}{\not{3}}x^{3/2} + 2\ .2\ x^{1/2} + c$$

$$= 2x^{3/2} + 4x^{1/2} + c$$

viii. $\int \frac{\sqrt{y}(y+1)}{y}\,dy = \int \frac{y^{1/2}(y+1)}{y}\,dy = \int \frac{y^{1/2+1} + y^{1/2}}{y}\,dy$

$$= \int \frac{y^{3/2} + y^{1/2}}{y}\,dy = \int\left(\frac{y^{3/2}}{y} + \frac{y^{1/2}}{y}\right)dy = \int \left(y^{3/2-1} + y^{1/2-1}\right)dy$$

$$= \int y^{1/2}\ dy + \int \hat{y}^{1/2}\ dy = \frac{y^{1/2+1}}{\frac{1}{2}+1} + \frac{y^{-1/2+1}}{\frac{-1}{2}+1} + c$$

$$= \frac{y^{3/2}}{\frac{3}{2}} + \frac{y^{1/2}}{\frac{1}{2}} + c = \frac{2}{3}y^{3/2} + 2y^{1/2} + c$$

ix. $\int \frac{\left(\sqrt{\theta}-1\right)^2}{\sqrt{\theta}}\,d\theta = \int \left(\frac{\theta - 2\sqrt{\theta} + 1}{\sqrt{\theta}}\right)d\theta$

$$= \int\left(\frac{\theta}{(\theta)^{1/2}} - \frac{2(\theta)^{1/2}}{(\theta)^{1/2}} + \frac{1}{(\theta)^{1/2}}\right) = \int \left(\theta^{1-1/2} - 2 + \theta^{-1/2}\right)d\theta$$

$$= \int \theta^{1/2}\ d\theta - 2\int 1d\theta + \int \theta^{-1/2}\ d\theta$$

$$= \int\frac{\theta^{1/2+1}}{\frac{1}{2}+1} - 2\theta + \frac{\theta^{-1/2+1}}{\frac{-1}{2}+1} + c$$

$$= \frac{\theta^{3/2}}{\frac{3}{2}} - 2\theta + \frac{\theta^{1/2}}{\frac{1}{2}} + c = \frac{2}{3}\theta^{3/2} - 2\theta + 2\theta^{1/2} + c$$

x. $\int \frac{\left(1-\sqrt{x}\right)^2}{\sqrt{x}}\,dx = \int \left(\frac{1 - 2\sqrt{x} + x}{\sqrt{x}}\right)dx$

$$=\int\left(\frac{1}{(x)^{1/2}}-\frac{2\sqrt{x}}{\sqrt{x}}+\frac{x}{(x)^{1/2}}\right)dx=\int x^{-1/2}\,dx-2\int 1dx+\int x^{1/2}\,dx$$

$$\frac{x^{-1/2+1}}{\frac{-1}{2}+1}-2x+\frac{x^{1/2+1}}{\frac{1}{2}+1}+c=\frac{x^{1/2}}{\frac{1}{2}}-2x+\frac{x^{3/2}}{\frac{3}{2}}+c=2x^{1/2}-2x+\frac{2}{3}x^{3/2}+c$$

xi. $\int\left(\frac{e^{2x}+e^{x}}{e^{x}}\right)dx=\int\left(\frac{e^{2x}}{e^{x}}+\frac{e^{x}}{e^{x}}\right)dx$ **(Sargodha 2008)**

$$\int\left(\frac{e^{x}.e^{x}}{e^{x}}+\frac{e^{x}}{e^{x}}\right)dx=\int e^{x}\,dx+\int 1dx=e^{x}+x+c$$

2. i. $\int\frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$

$$=\int\left(\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\times\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}\right)dx$$

$$=\int\frac{\left(\sqrt{x+a}-\sqrt{x+b}\right)}{\left(\sqrt{x+a}\right)^2-\left(\sqrt{x+b}\right)^2}dx=\int\frac{\sqrt{x+a}-\sqrt{x+b}}{x+a-(x+b)}dx$$

$$=\int\frac{\sqrt{x+a}-\sqrt{x+b}}{x+a-x-b}dx=\frac{1}{(a-b)}\int\left(\sqrt{x+a}-\sqrt{x+b}\right)dx$$

$$=\frac{1}{(a-b)}\left[\int(x+a)^{1/2}\,dx-\int(x+b)^{1/2}\,dx\right]$$

$$=\frac{1}{(a-b)}\left[\frac{(x+a)^{1/2+1}}{\frac{1}{2}+1}-\frac{(x+b)^{1/2+1}}{\frac{1}{2}+1}\right]+c$$

$$=\frac{1}{(a-b)}\left[\frac{(x+a)^{3/2}}{\frac{3}{2}}-\frac{(x+b)^{3/2}}{\frac{3}{2}}\right]+c$$

$$=\frac{1}{(a-b)}\left[\frac{2}{3}(x+a)^{3/2}-\frac{2}{3}(x+b)^{3/2}\right]+c$$

$$=\frac{2}{3(a-b)}\left[(x+a)^{3/2}-(x+b)^{3/2}\right]+c$$

ii. $\int\frac{1-x^2}{1+x^2}\,dx$

$$=\int\left(-1+\frac{2}{1+x^2}\right)dx$$

$$=\int\frac{2}{1+x^2}\,dx-\int 1dx$$

$$= 2\int \frac{1}{1+x^2}\,dx - \int 1dx = 2\,Tan^{-1}x - x + c$$

$\frac{1}{1+x^2} = Tan^{-1}x$

iii. $$\int\frac{dx}{\sqrt{x+a}+\sqrt{x}} = \int\left(\frac{1}{\sqrt{x+a}+\sqrt{x}}\times\frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}\right)dx$$

$$= \int \frac{\sqrt{x+a}-\sqrt{x}}{\left(\sqrt{x+a}\right)^2-\left(\sqrt{x}\right)^2}\,dx = \int\frac{\sqrt{x+a}-\sqrt{x}}{x+a-x}$$

$$= \frac{1}{a}\left[\int (x+a)^{1/2}\,dx - \int (x)^{1/2}\,dx\right]$$

$$= \frac{1}{a}\left[\frac{(x+a)^{1/2+1}}{\frac{1}{2}+1} - \frac{x^{1/2+1}}{\frac{1}{2}+1}\right] + c = \frac{1}{a}\left[\frac{(x+a)^{3/2}}{\frac{3}{2}} - \frac{x^{3/2}}{\frac{3}{2}}\right] + c$$

$$= \frac{1}{a}\left[\frac{2}{3}(x+a)^{3/2} - \frac{2}{3}x^{3/2}\right] + c = \frac{2}{3a}\left[(x+a)^{3/2} - x^{3/2}\right] + c$$

iv. $$\int(a-2x)^{3/2}\,dx = -\frac{1}{2}\int (a-2x)^{3/2}(-2)\,dx$$

$$= -\frac{1}{2}\frac{(a-2x)^{3/2+1}}{\frac{3}{2}+1} + c = -\frac{1}{2}\frac{(a-2x)^{5/2}}{\frac{5}{2}} + c$$

$$= -\frac{1}{2}\cdot\frac{2}{5}(a-2x)^{5/2} + c = -\frac{1}{5}(a-2x)^{5/2} + c$$

v. $$\int\frac{(1+e^x)^3}{e^x}\,dx = \int\left(\frac{1+3e^x+3e^{2x}+e^{3x}}{e^x}\right)dx$$

$1+3e^x+3e^{2x}+c$

Note $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$$= \int\left(\frac{1}{e^x}+\frac{3e^x}{e^x}+\frac{3e^{2x}}{e^x}+\frac{e^{3x}}{e^x}\right)dx$$

$$= \int\left(e^{-x}+3+3e^x+e^{2x}\right)dx$$

$$= \int e^{-x}dx + 3\int 1dx + 3\int e^x dx + \int e^{2x}dx$$

$$= \frac{e^{-x}}{-1} + 3x + 3e^x + \frac{e^{2x}}{2} + c$$

$$= -e^{-x} + 3x + 3e^x + \frac{1}{2}e^{2x} + c$$

vi. $$\int Sin\,(a+b)x\,dx = \frac{1}{(a+b)}\int Sin\,(a+b)\,x\,(a+b)\,dx$$

$$= \frac{1}{(a+b)}\left[-Cos\,(a+b)x\right] + c = \frac{-Cos\,(a+b)x}{a+b}$$

vii. $\int\sqrt{1-Cos2x}\,dx$ **Note** $1-Cos2x = 2Sin^2x$

$= \int \sqrt{2Sin^2x}\,dx = \int \sqrt{2}\,Sinx\,dx = \sqrt{2}\int Sinx\,dx$

$\sqrt{2}(-Cosx)+c = -\sqrt{2}\,Cosx + c$

viii. $\int lnx \times \frac{1}{x}dx = \int (lnx)\frac{1}{x}dx = \frac{(lnx)^2}{2} + c$

ix. $\int Sin^2x\,dx = \int \frac{1-Cos2x}{2}dx = \frac{1}{2}\left[\int 1dx - \int Cos2x\,dx\right]$ **(Sargodha 2010)**

$= \frac{1}{2}\left[x - \frac{Sin2x}{2}\right] + c = \frac{1}{2}x - \frac{Sin2x}{4} + c$

x. $\int \frac{1}{1+Cosx}dx = \int \frac{1}{2Cos^2\frac{x}{2}}dx = \frac{1}{2}\int \frac{1}{Cos^2\frac{x}{2}}$ $\quad 1+Cosx = 2Cos^2\frac{x}{2}$ (sgd 2009)

$= \frac{1}{2}\int Sec^2\frac{x}{2}dx = \frac{1}{2}\frac{Tan\frac{x}{2}}{\frac{1}{2}} + c = Tan\frac{x}{2} + c$ $\quad 1 - Cosx = 2Sin^2\frac{x}{2}$

xi. $\int \frac{ax+b}{ax^2+2bx+x}dx = \frac{1}{2}\int \frac{2ax+2b}{ax^2+2bx+c}dx$

$= \frac{1}{2}ln\left|ax^2 + 2bx + c\right| + c'$

xii. $\int Cos3x\,Sin2x\,dx = \frac{1}{2}\int 2Cos3x\,Sin2x\,dx$

Use 2Cosα Sinβ = Sin (α + β) – Sin (α – β)

$= \frac{1}{2}\int \left(Sin(3x+2x) - Sin(3x-2x)\right)dx$

$= \frac{1}{2}\int (Sin5x - Sinx)dx = \frac{1}{2}\left[\int Sin5x\,dx - \int Sinx\,dx\right]$

$= \frac{1}{2}\left[\frac{-Cos5x}{5} - (-Cosx)\right] + c = \frac{1}{2}\left[\frac{-Cos5x}{5} + Cosx\right] + c$

xiii. $\int \frac{Cos2x-1}{1+Cos^2x}dx = \int \frac{1-2Sin^2x-1}{2Cos^2x}dx = \int \frac{-2Sin^2x}{2Cos^2x}dx$ (Sgd 2011)

$= -\int Tan^2x\,dx = -\int (Sec^2x - 1)dx$

$= -\int Sec^2x\,dx + \int 1dx = -Tanx + x + c$

xiv. $\int Tan^2x\,dx = \int (Sec^2x - 1)dx = \int Sec^2x\,dx - \int 1dx$

$= Tanx - x + c$

Example: 3.3

$$\int \frac{Cot\sqrt{x}}{\sqrt{x}}dx$$

Put $\sqrt{x} = u \Rightarrow \frac{1}{2}x^{-1/2}dx = du$ or $\frac{1}{\sqrt{x}}dx = 2du$

$$\int \frac{Cot\sqrt{x}}{\sqrt{x}}dx = \int Cotu\,(2du)$$

$$= 2\int Cotudu = 2\int \frac{Cosu}{Sinu}du$$

$$= 2ln\left|Sinu\right| + 3$$

$$= 2ln\left|Sin\sqrt{x}\right| + c$$

Example:

$\int Secx\, dx$ **(Sargodha 2008,11)**

Multiply and divide by (Secx + Tanx)

$$= \int \frac{Sec\,x\,(Secx + Tanx)}{(Secx + Tanx)}dx$$

$$= \int \frac{(Sec^2x + Secx\,Tanx)}{Secx + Tanx}dx$$

Put $Secx + Tanx = Z$

$$(SecxTanx + Sec^2x)dx = dZ$$

$$= \int \frac{dZ}{Z} = ln|z| + c$$

$$= ln|Secx + Tanx| + c$$

Example:

$\int Cosecx\, dx$ **(Sargodha 2007)**

Multiply and divide by (COsecx – Cotx)

$$= \int \frac{Cosecx(Cosecx - Cotx)}{Cosecx + Cotx}dx$$

$$= \int \frac{Cosec^2x + Cotx\,Cosecx}{Cosecx - Cotx}dx$$

Put $Cosecx - Cotx = z$

$$(-Cosecx\,Cotx + Cosec^2x)dx = dz$$

$$= \int \frac{1}{z}\, dz = ln|z| + c$$

$$= ln|Cosecx - Cotx| + c$$

Example:

$$\int \sqrt{1 + Sinx}\; dx \qquad \text{(Sargodha 2007)}$$

$$= \int \sqrt{1 + Sinx} \times \sqrt{\frac{1 - Sinx}{1 - Sinx}}\, dx$$

$$= \int \frac{\sqrt{1 - Sin^2 x}}{1 - Sinx}\, dx = \int \sqrt{\frac{Cos2x}{1 - Sinx}}\, dx$$

$$= \int \frac{Cosx}{\sqrt{1 - Sinx}}\, dx \qquad Put \quad Sinx = t \;;\; Cosxdx = dt$$

$$= \int \frac{dt}{\sqrt{1-t}} = \int (1-t)^{-1/2} dt$$

$$= -\frac{(1-t)^{-1/2+1}}{\frac{-1}{2}+1} + c = \frac{(1-t)^{1/2}}{\frac{-1}{2}} + c = -2\sqrt{1-Sinx} + c$$

Exercise 3.3

1. $$\int \frac{-2x}{\sqrt{4 - x^2}} \qquad \text{(Sargodha 2011,12)}$$

Put $4 - x^2 = t \Rightarrow -2xdx = dt$

$$= \int \frac{dt}{\sqrt{t}} = \int (t)^{-1/2} dt = \frac{t^{-1/2+1}}{\frac{-1}{2}+1} + c$$

$$= \frac{t^{1/2}}{\frac{1}{2}} + c = 2\sqrt{t} + c = 2\sqrt{4 - x^2} + c$$

2. $$\int \frac{dx}{\sqrt{x^2 + 4x + 13}} = \int \frac{dx}{\sqrt{x^2 + 4x + 4 + 9}}$$

Note $\int \frac{1}{a^2 + x^2}\, dx + \frac{1}{a} Tan^{-1}\frac{x}{a}$

$$\int \frac{1}{(x+2)^2 + (3)^2}\, dx$$

$$= \frac{1}{3} Tan^{-1}\frac{x+2}{3} + c$$

3. $\int \frac{x^2}{\sqrt{4+x^2}}dx$ **(Sargodha 2010)**

$$= \int\left(1 - \frac{4}{x^2+4}\right)dx$$

$$\begin{array}{r} x^2+4\,\overline{)\,x^2\quad}\;1 \\ -x^2 \pm 4 \\ \hline -4 \end{array}$$

$$= \int 1\,dx - 4\int \frac{1}{x^2+(2)^2}dx$$

$$= x - 4 \cdot \frac{1}{2} Tan^{-1}\frac{x}{2} + c$$

$$= x - 2\,Tan^{-1}\frac{x}{2} + c$$

4. $\int \frac{x^2}{x\,ln}dx = \int \frac{1}{ln}\cdot\frac{1}{x}dx$ **(Sargodha 2007,10)**

Put $lnx = t \quad \Rightarrow \quad \frac{1}{x}dx = dt$

$$= \int \frac{1}{t}dt = ln|t| + c$$

$$= ln\,|ln\,x| + c$$

5. $\int \frac{e^2}{e^x+3}dx$ **(Sargodha 2008)**

Put $e^x + 3 = t \quad \Rightarrow \quad e^x dx = dt$

$$= \int \frac{dt}{t} = ln|t| + c$$

$$= ln|e^x+3| + c$$

6. $\int \frac{x+b}{\left(x^2+2bx+c\right)^{1/2}}dx$

'×' & '+' by 2

$$= \frac{1}{2}\int \frac{2x+2b}{\left(x^2+2bx+c\right)^{1/2}}dx$$

$$= \frac{1}{2}\int \left(x^2+2bx+c\right)^{-1/2}(2x+2b)\,dx$$

$$=\frac{1}{2}\frac{\left(x^2+2bx+c\right)^{-1/2+1}}{\frac{-1}{2}+1}+c$$

$$=\frac{1}{2}\frac{\left(x^2+2bx+c\right)^{1/2}}{\frac{1}{2}+1}+c$$

$$=\sqrt{x^2 2bx+c+c}$$

7. $\int\frac{Sec^2x}{\sqrt{Tanx}}\,dx$

$$=\int\left(Tanx\right)^{-1/2}Sec^2x\,dx$$

$$=\frac{(Tanx)^{-1/2+1}}{\frac{-1}{2}+1}+c=\frac{(Tanx)^{1/2}}{\frac{1}{2}}+c=2\sqrt{Tanx}+c$$

8. **a.** $\int\frac{dx}{\sqrt{x^2-a}}$ ***Put*** $x=a\,Sec\theta\ dx=aSec\theta\,Tan\theta d\,\theta$

$$=\int\frac{aSec\theta\,Tan\theta d\theta}{\sqrt{a^2Sec^2\theta-a^2}}=\int\frac{aSec\theta\,Tan\theta d\theta}{\sqrt{a^2\left(Sec^2\theta-1\right)}}$$

$$=\int\frac{aSec\theta\,Tan\theta d\theta}{\sqrt{a^2Tan^2\theta}}=\int\frac{aSec\theta\,Tan\theta}{a\,Tan\theta}$$

$$=\int Sec\theta d\theta=ln\left|Sec^2\theta+Tan\theta\right|+c$$

$$=ln\left|Sec\theta+\sqrt{Sec^2\theta-1}\right|+c\qquad Because:\ \ Tan0=\sqrt{Sec^2 0-1}$$

$$=ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^a}-1}\right|+c$$

$$=ln\left|\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a}}\right|+c$$

$$=ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+c\qquad Note\ \ -ln\ a+c=c_1$$

$$=ln\left(x+\sqrt{x^2-a^2}\right)-ln\,a+c$$

$$=ln\left(x+\sqrt{x^2-a^2}\right)+c_1$$

b. $\int\sqrt{a^2-x^2}\,dx$ Put $x = a\,Sin\theta \Rightarrow dx = aCos\theta d\theta$ **(Sargodha 2010)**

$= \int \sqrt{a^2 - a^2 Sin^2\theta}\; Cos\theta d\theta = \sqrt{a^2\left(1-Sin^2 a\right)}\, aCos\theta d\theta$

$= \int \sqrt{a^2Cos^2\theta}.\; a\,Cos\theta d\theta = \int aCos\theta.\; aCos\theta d\theta$

$= \int a^2Cos^2\theta d\theta = a^2\int\left(\frac{1+Cos2\theta}{2}\right)d\theta$

$= \frac{a^2}{2}\left[\int 1d\theta + \int Cos2\theta d\theta\right]$

$= \frac{a^2}{2}\left[0+\frac{Sin2\theta}{2}\right]+c = \frac{a^2}{2}\left[\theta+\frac{2Sin\theta\,Cos\theta}{2}\right]+c$

$= \frac{a^2}{2}\left[\theta + Sin\theta\sqrt{1-Sin^2\theta}\right]+c$

$= \frac{a^2}{2}\left[Sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right]+c$

$= \frac{a^2}{2}\left[Sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{\frac{a^2-x^2}{a}}\right]+c$

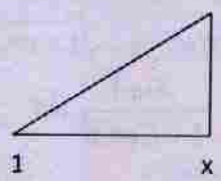
$= \frac{a^2}{2}\left[Sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{\frac{a^2-x^2}{a}}\right]+c$

$= \frac{a^2}{2}Sin^{-1}\frac{x}{a}+\frac{x\sqrt{a^2-x^2}}{2}+c$

9. $\int\frac{dx}{\left(1+x^2\right)^{3/2}}$ **Put** $x = Tan\theta$ $dx = Sec^2\theta\, d\theta$

$= \int\frac{Sec^2\theta d\theta}{\left(1+Tan^2\theta^{3/2}\right)} = \int\frac{Sec^2\theta d\theta}{\left(Sec^2\theta\right)^{3/2}} = \int\frac{Sec^2\theta d\theta}{Sec^3\theta}$

$= \int\frac{1}{Sec\theta}\,d\theta = \int Cos\theta d\theta = Sin\theta + c$ I



Note $Sin\theta = ?$

$Tan\theta = \frac{x}{1}$

By Pythagoras

$$So\ Sin\theta = \frac{x}{\sqrt{1+x^2}}$$

$$hyp = \sqrt{1+x^2}$$

$$= \frac{x}{\sqrt{1+x^2}} + c$$

10. $$\int \frac{1}{(1+x^2)\,Tan^{-1/2}}dx = \int \frac{1}{Tan^{-1}x}\cdot\frac{1}{(1+x^2)}dx$$

Put $Tan^{-1}x = t \quad \Rightarrow \quad \frac{1}{(1+x^2)}dx = dt$

$$= \int \frac{1}{t}dt = ln|t| + c = ln|Tan^{-1}x| + c$$

11. $$\sqrt{\frac{1+x}{1-x}}dx = \int \sqrt{\frac{1+x}{1-x}} \times \sqrt{\frac{1+x}{1+x}}\,dx$$

Put $x = Sin\theta \qquad dx = Cos\theta d\theta$

$$= \int \frac{\left(\sqrt{1+x}\right)^2}{\sqrt{1-x^2}}dx \int \frac{1+x}{\sqrt{1-x^2}}dx$$

$$= \int \frac{(1+Sin\theta)}{\sqrt{1-Sin^2\theta}}Cos\theta d\theta = \int \frac{1+Sin\theta}{\sqrt{Cos^2}}Cos\theta d\theta$$

$$= \int \frac{(1+Sin\theta)\,Cos}{Cos\theta}d\theta = \int 1d\theta + \int Sin\theta d\theta$$

$$= \int \frac{(1+Sin\theta)\,Cos}{Cos\theta}d\theta = \int 1d\theta + \int Sin\theta d\theta$$

$$= \theta - Cos\theta + c = \theta - \sqrt{1-Sin^2\theta} + c$$

$$= Sin^{-1/2}x - \sqrt{1-x^2} + c$$

12. $$\int \frac{Sin\theta}{1+Cos^2\theta}d\theta$$

Put $Cos\theta = t \qquad -Sin\theta d\theta = dt$

$$= a\int \frac{-Sin\theta d\theta}{1+Cos^2\theta} = 1\int \frac{dt}{1+t^2} = -Tan^{-1}t + c$$

$$= -Tan^{-1}(Cos\theta) + c$$

13. $\int \frac{ax}{\sqrt{a^2 - x^4}}$

Put $x^2 = aSin\theta \qquad 2xdx = aCos\theta d\theta$

$= \int \frac{xdx}{\sqrt{a^2 - x^4}} = \frac{a}{2}\int \frac{2xdx}{\sqrt{a^2 - x^4}} = \frac{a}{2}\int \frac{aCos\theta d\theta}{\sqrt{a^2 - a^2 Sin^2\theta}}$

$= \frac{a^2}{2}\int \frac{Cos\theta d\theta}{\sqrt{a^2\left(1 - Sin^2\theta\right)}} = \frac{a^2}{2}\int \frac{Cos\theta d\theta}{\sqrt{a^2 Cos^2\theta}} = \frac{a^2}{2}\int \frac{Cos\theta d\theta}{\sqrt{aCos\theta}}$

$= \frac{a}{2}\int 1 d\theta = \frac{a}{2}\theta + c = \frac{a}{2} Sin^{-1}\frac{x^2}{a} + c$

14. $\int \frac{dx}{\sqrt{7 - 6x - x^2}} = \int \frac{dx}{\sqrt{7 - 6x - x^2 9 + 9}} = \int \frac{dx}{\sqrt{7 + 9 - \left(x^2 + 6x + 9\right)}}$

$= \int \frac{dx}{\sqrt{16 - (x+3)^2}} = \int \frac{dx}{\sqrt{(4)^2 - (x+3)^2}} = Sin^{-1}\frac{(x+3)}{4} + c$

15. $\int \frac{Cosx}{Sinx\ ln\ Sinx} dx = \int \frac{1}{lnSinx} \cdot \frac{Cosx}{Sinx} dx$ **(Sargodha 2010)**

Put $ln\ Sinx = t \quad \Rightarrow \quad \frac{1}{Sinx} \times Cosx\ dx = dt$

or $\frac{Cos}{Sinx} dx = dt$

$= \int \frac{1}{t} dt = ln|t| + c$

$= ln|ln\ Sinx| + c$

16. $\int Cosx \frac{ln\ Sinx}{Sinx} dx = \int (ln\ Sinx)\frac{Cosx}{Sinx} dx$

Put $ln\ Sinx = t \quad \Rightarrow \quad \frac{Cosx}{Sinx} dx = dt$

$= \int t\ dt = \frac{t^2}{2} + c = \frac{(ln\ Sinx)^2}{2} + c$

17. $\int \frac{xdx}{x^2 + 2x + 4} = \frac{1}{2}\int \frac{2x\ dx}{x^2 + 2x + 4} = \frac{1}{2}\int \frac{2x + 2 - 2}{x^2 + 2x + 4} dx$ **(Sargodha 2011)**

$= \frac{1}{2}\left[\int \frac{2x + 2}{x^2 + 2x + 4} dx - \int \frac{2}{x^2 + 2x + 4}\right] + c$

$$= \frac{1}{2}\left[ln\left|x^2+2x+4\right| - 2\int \frac{1}{x^2+2x+1+3}dx \right] + c$$

$$= \frac{1}{2} ln\left|x^2+2x+4\right| - \frac{1}{2} \times 2 \int \frac{1}{(x+1)^2+\left(\sqrt{3}\right)^2}dx + c$$

$$= \frac{1}{2} ln\left|x^2+2x+4\right| - \frac{1}{2} \times 2 \int \frac{1}{(x+1)^2+\left(\sqrt{3}\right)^2}dx + c$$

$$= \frac{1}{2} ln\left|x^2+2x+4\right| - \frac{1}{\sqrt{3}} Tan^{-1}\frac{(x+1)}{\sqrt{3}} + c$$

18. $\int \frac{x}{x^4+2x^2+5}dx = \int \frac{x\,dx}{x^4+2x^2+1+4}$

'×' & '÷' by 2

$$= \frac{1}{2}\int \frac{2x\,dx}{\left(x^2+1\right)^2+(2)^2} \qquad \text{('×' and '÷' by 2)}$$

Put $x^2+1 = 2Tan\theta \qquad 2xdx = 2\ Sec^2\theta d\theta$

$$= \frac{1}{2}\int \frac{2Sec^2\theta d\theta}{(2Tan\theta)^2+(2)^2}$$

$$= \frac{1}{2} 2 \int \frac{Sec^2\theta d\theta}{4Tan^2\theta+4}$$

$$= \int \frac{Sec^2\theta d\theta}{4\left(1+Tan^2\theta\right)} = \int \frac{Sec^2\theta d\theta}{4.\ Sec^2\theta}$$

$$= \frac{1}{4}\int 1\ d\theta = \frac{1}{4}\theta + c$$

$$= \frac{1}{4}Tan^{-1}\left(\frac{x^2+1}{2}\right) + c$$

19. $\int Cos\left(\sqrt{x}-\frac{x}{2}\right)\left(\frac{1}{\sqrt{x}}-1\right)dx$ **(Sargodha 2009)**

Put $\sqrt{x}-\frac{x}{2} = t$

$$\left(\frac{1}{2}x^{-1/2}-\frac{1}{2}\right)dx = dt$$

$$\frac{1}{2}\left(\frac{1}{\sqrt{x}}-1\right)dx = dt$$

$$\left(\frac{1}{\sqrt{x}}-1\right)dx = 2dt$$

$$= \int Cost\ (2dt) = 2\int Cost\ dt$$

$$= 2Sint + c$$

$$= 2Sint\left(\sqrt{x}-\frac{x}{2}\right)+c$$

20. $\int \frac{x+2}{\sqrt{x+3}}dx$

$$= \int \frac{x+2+1-1}{\sqrt{x+3}}dx$$

$$= \int \frac{x+3}{\sqrt{x+3}}dx - \int \frac{1}{\sqrt{x+3}}dx$$

$$= \int (x+3)^{1/2}dx - \int (x+3)^{-1/2}dx$$

$$= \frac{(x+3)^{1/2+1}}{\frac{1}{2}+1} - \frac{(x+3)^{-1/2}dx}{\frac{-1}{2}+1}$$

$$= \frac{(x+3)^{3/2}}{\frac{3}{2}} - \frac{(x+3)^{1/2}}{\frac{1}{2}} + c$$

$$= \frac{2}{3}(x+3)^{3/2} - 2\sqrt{x+3} + c$$

21. $\int \frac{\sqrt{2}}{Sinx + Cosx}dx$ **(Sargodha 2009)**

$$= \int \frac{\sqrt{2}}{\sqrt{2}\left(\frac{1}{\sqrt{2}}Sinx + Cosx.\frac{1}{\sqrt{2}}\right)}dx$$

$$= \int \frac{1}{Sinx.\frac{1}{\sqrt{2}} + Cosx.\frac{1}{\sqrt{2}}}dx$$

Put $\frac{1}{\sqrt{2}} = Sin\frac{\pi}{4}$ & $\frac{1}{\sqrt{2}} = Cos\frac{\pi}{4}$

$$=\int \frac{1}{Sinx\ Sin\frac{\pi}{4}+Cosx\ Cos\frac{\pi}{4}}dx$$

$$=\int \frac{1}{Cos\left(x-\frac{\pi}{4}\right)}dx=\int Sec\left(x-\frac{\pi}{4}\right)dx$$

$$=ln\left|Sec\left(x-\frac{\pi}{4}\right)+Tan\left(x-\frac{\pi}{4}\right)\right|+c$$

22. $$\int\frac{dx}{\frac{1}{2}Sinx+\frac{\sqrt{3}}{2}Cosx}dx$$

v.v.imp

Put $\frac{1}{2}=Cos\frac{\pi}{3}$

$\frac{\sqrt{3}}{2}=Sin\frac{\pi}{3}$

$$=\int\frac{dx}{Cos\frac{\pi}{3}Sinx+Sin\frac{\pi}{3}Cosx}$$

$$=\int\frac{dx}{Sinx\ Cos\frac{\pi}{3}+Cosx\ Sin\frac{\pi}{3}}=\int\frac{dx}{Sin\left(x+\frac{\pi}{3}\right)}$$

$$=\int Cosec\left(x+\frac{\pi}{3}\right)dx$$

$$=ln\left|Cosec\left(x+\frac{\pi}{3}\right)-Cot\left(x+\frac{\pi}{3}\right)\right|+c$$

Examples of 3.4

Example: $\int x^5 \ ln x\, dx$

$= \int (ln x)\ x^5\ dx$

$= (ln\ x).\dfrac{x^6}{6} - \int \dfrac{1}{x}.\dfrac{x^6}{6}\ dx$

$= (ln\ x).\dfrac{x^6}{6} - \dfrac{1}{6}\int x^5\ dx$

$= ln x\dfrac{x^6}{6} - \dfrac{1}{6}.\dfrac{x^6}{6} + c$

$= \dfrac{x^6}{6} ln x - \dfrac{x^6}{36} + c$

Example: $\int \sqrt{9+25x^2}\ dx$

$= \int \sqrt{(3)^2 + (5x)^2}\ dx$

$= \dfrac{5x}{2}\sqrt{9+25x^2} + \dfrac{9}{2} ln\left(5x + \sqrt{9+25x^2}\right) + c$

We used formula

$= \int \sqrt{a^2 + x^2}\ dx = \dfrac{x}{2}\sqrt{a^2+x^2} + \dfrac{a^2}{2} ln\left(x + \sqrt{a^2+x^2}\right) + c$

Exercise 3.4

1. i. $\int \underset{I}{x}\ \underset{II}{Sin x}\, dx = x(-Cos x) - \int 1.(-Cos x)\, dx$ **(Sargodha 2010,11)**

$= -x Cos x + \int Cos x\, dx = -x Cos x + Sin x + c$

ii. $\int ln x\, dx = \int \underset{I}{ln x}.\underset{II}{1}\, dx = ln x.x - \int \dfrac{1}{x}.x\ dx$ **(Sargodha 2009)**

$= x ln x - \int 1 dx = x ln x = x + c$

iii. $\int x\ ln x\, dx = \int \underset{I}{ln x}.\underset{II}{x}\, dx = ln x.\dfrac{x^2}{2} - \int \dfrac{1}{x}.\dfrac{x^2}{2}\, dx$

$= \dfrac{x^2}{2} ln x - \dfrac{1}{2}\int x\, dx = \dfrac{x^3}{2} ln x - \dfrac{1}{3}.\dfrac{x^3}{3} + c$

$= \dfrac{x^3}{3} ln x - \dfrac{x^3}{9} + c$

iv. $\int x^3\ ln x\, dx = \int \underset{I}{ln x}.\underset{II}{x^3}\, dx = ln x.\dfrac{x^3}{3} - \int \dfrac{1}{x}.\dfrac{x^3}{3}\, dx$

$$=\frac{x^3}{3}lnx-\frac{1}{3}\int x^2\,dx=\frac{x^3}{3}lnx-\frac{1}{3}.\frac{x^3}{3}+c$$

$$=\frac{x^3}{3}lnx-\frac{x^3}{9}+c$$

v. $\int x^3\ lnx\,dx=\int \underset{I}{lnx}.\underset{II}{x^3}\,dx=lnx.\frac{x^4}{4}-\int\frac{1}{x}.\frac{x^4}{4}dx$ **(Sargodha 2008)**

$$=\frac{x^4}{4}lnx-\frac{1}{4}\int x^3dx=\frac{x^4}{4}lnx-\frac{1}{4}.\frac{x^4}{4}+c$$

$$=\frac{x^4}{4}lnx-\frac{x^4}{16}+c$$

vi. $\int x^4\ lnx\,dx=\int \underset{I}{lnx}.\underset{II}{x^4}\,dx=lnx.\frac{x^5}{5}-\int\frac{1}{x}.\frac{x^5}{5}dx$

$$=\frac{x^5}{5}lnx-\frac{1}{5}\int x^4dx=\frac{x^5}{5}lnx-\frac{1}{5}.\frac{x^5}{5}+c$$

$$=\frac{x^5}{5}lnx-\frac{x^5}{25}+c$$

vii. $\int Tan^{-1}x\,dx=\int \underset{I}{Tan^{-1}x}.\underset{II}{1}\,dx$ **(Sargodha 2011)**

$$=Tan^{-1}x.x-\int\frac{1}{1+x^2}.xdx=xTan^{-1}x-\int\frac{x}{1+x^2}dx$$

$$=xTan^{-1}\frac{1}{2}-\int\frac{2x}{1+x^2}dx=xTan^{-1}x\frac{1}{2}ln\left|1+x^2\right|+c$$

viii. $\int \underset{I}{x^2}.\underset{II}{Sin}\,dx=x^2(-Cosx)-\int 2x(-Cosx)\,dx$

$$=-x^2Cosx+2\int \underset{I}{x}.\underset{II}{Cos}\,xdx$$

$$=-x^2Cosx+2\left[xSinx-\int 1.Sinxdx\right]$$

$$=-x^2Cosx+2xSinx-2\int Sinxdx$$

$$=-x^2Cosx+2xSinx+2Cosx+c$$

ix. $\int x^2Tan^{-1}xdx=\int \underset{I}{Tan^{-1}x}.\underset{II}{x^2}\,xdx=Tan^2x.\frac{x^3}{3}-\int\frac{1}{1+x^2}.\frac{x^3}{3}dx$

$$=\frac{x^3.Tan^{-1}x}{3}-\frac{1}{3}\int\frac{x^3}{1+x^2}dx$$

$$=\frac{x^3.Tan^{-1}x}{3}-\frac{1}{3}\int\left(x-\frac{x}{1+x^2}\right)dx$$

$$=\frac{x^3.Tan^{-1}x}{3}-\frac{1}{3}\left[\int x dx-\frac{1}{2}\int\frac{2x}{1+x^2}dx\right]$$

$$=\frac{x^3.Tan^{-1}x}{3}-\frac{1}{3}\left[\frac{x^2}{2}-\frac{1}{2}ln\left|1+x^2\right|\right]+c$$

$$=\frac{x^3.Tan^{-1}x}{3}-\frac{1}{6}\left(x^2-ln\left|1+x^2\right|\right)+c$$

x. $\int xTan^{-1}xdx=\int \underset{I}{Tan^{-1}}x.dx=Tan^{-1}x.\frac{x^2}{2}-\int\frac{1}{1+x^2}.\frac{x^2}{2}dx$ **(Sargodha 2009)**

$$=\frac{x^2Tan^{-1}x}{2}-\frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$$x^2+1\overset{1}{\sqrt{x^2}}$$

$$\underline{-x^2\pm1}$$

$$1$$

$$=\frac{x^2Tan^{-1}x}{2}-\frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2Tan^{-1}x}{2}-\frac{1}{2}\left[\int 1dx-\left(\frac{1}{1+x^2}\right)dx\right]$$

$$=\frac{x^2Tan^{-1}x}{2}-\frac{1}{2}\left(x-Tan^{-1}x\right)+c$$

xi. $\int x^3Tan^{-1}x\,dx=\int \underset{I}{Tan^{-1}}x.\underset{II}{x^3}dx=Tan^{-1}x.\frac{x^4}{4}-\int\frac{1}{1+x^2}.\frac{x^4}{4}dx$ **(Sgd 2008)**

$$=\frac{x^4Tan^{-1}x}{4}-\frac{1}{4}\int\frac{x^4}{x^2+1}dx$$

$$x^2+1\overset{x^2-1}{\sqrt{x^4}}$$

$$\underline{-x^4\pm x^2}$$

$$-x^2$$

$$\underline{\mp x^2\mp1}$$

$$+1$$

$$=\frac{x^4Tan^{-1}x}{4}-\frac{1}{4}\int\left(x^2-1+\frac{1}{1+x^2}\right)dx$$

$$=\frac{x^4Tan^{-1}x}{4}-\frac{1}{4}\left[\int x^2dx-\int 1dx+\int\frac{1}{1+x^2}dx\right]$$

$$=\frac{x^4Tan^{-1}x}{4}-\frac{1}{4}\left(\frac{x^3}{3}-x+Tan^{-1}x\right)+c$$

xii. $\int \underset{I}{x^3}\underset{II}{Cos}\,dx=x^3\,Sinx-\int 3x^2Sinxdx$

$$= x^3 Sinx - 3\int \underset{I}{x^2} \underset{II}{Sin}\, dx = x^3 Sinx - 3\left[x^2(-Cosx) - \int 2x(-Cosx)\, dx\right]$$

$$= x^3 Sinx + 3x^2 Cosx - 6\int \underset{I}{x} \underset{II}{Cosx}\, dx$$

$$= x^3 Sinx + 3x^2 Cosx - 6\left(xSinx - \int 1.Sinxdx\right)$$

$$= x^3 Sinx + 3x^2 Cosx - 6x\ Sinx + 6\int Sinxdx$$

$$= x^3 Sinx + 3x^2 Cosx - 6x\ Sinx - 6Cosx + c$$

xiii. $\int Sin^{-1}xdx = \int \underset{I}{Sin^{-1}}x.\underset{II}{1}\,dx$ **(Sargodha 2010)**

$$= Sinx^{-1}x.x - \int \frac{1}{\sqrt{1-x^2}}.xSinx^{-1}x - \int \frac{x}{\sqrt{1-x^2}}dx$$

$$= xSinx^{-1}x - \frac{1}{2}\int (1-x^2)^{-1/2} 2xdx$$

$$= xSinx^{-1}x - \frac{1}{2}\frac{(1+x^2)^{-1/2+1}}{-\frac{1}{2}+1} + c = xSinx^{-1}x - \frac{1}{2}\frac{(1-x^2)^{1/2}}{\frac{1}{2}} + c$$

$$= xSinx^{-1}x - \sqrt{1-x^2} + c$$

xiv. $\int x\,Sin^{-1}xdx = \int \underset{I}{Sin^{-1}}x.\underset{II}{x}\,dx = Sin^{-1}x\frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}}.\frac{x^2}{2}dx$

$$= \frac{x^2 Sin^{-1}x}{2} - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx = \frac{x^2 Sin^{-1}x}{2} + \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}}dx$$

$$= \frac{x^2 Sin^{-1}x}{2} + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}dx = \frac{x^2 Sin^{-1}x}{2} + \frac{1}{2}\int\left(\frac{1-x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}\right)dx$$

$$= \frac{x^2 Sin^{-1}x}{2} + \frac{1}{2}\left[\int \frac{\sqrt{1-x^2}\times\sqrt{1-x^2}}{\sqrt{1-x^2}}dx - \int\left(\frac{1}{\sqrt{1-x^2}}dx\right)\right]$$

$$= \frac{x^2 Sin^{-1}x}{2} + \frac{1}{2}\int\sqrt{1-x^2}\,dx - \frac{1}{2}Sin^{-1}x + c$$

$$= \frac{x^2 Sin^{-1}x}{2} + \frac{1}{2}\left[\int\frac{1}{2}Sin^{-1}x + \frac{x\sqrt{1-x^2}}{2}\right] - \frac{1}{2}Sin^{-1}x + c$$

$$= \frac{x^2 Sin^{-1}x}{2} + \frac{1}{4}Sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} - \frac{1}{2}Sin^{-1}x + c$$

$$= \frac{x^2 Sin^{-1}x}{2} + \frac{1}{4}Sin^{-1}x - \frac{1}{2}Sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + c$$

$$= \frac{x^2 Sin^{-1}x}{2} + \left(\frac{1}{4} - \frac{1}{2}\right)Sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + c$$

$$= \frac{x^2 Sin^{-1}x}{2} - \frac{1}{4}Sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + c$$

xv. $I=\int e^x Sinx\ Cosx dx=\frac{1}{2}\int e^x 2Sinx\ Cosx\ dx$

$I=\frac{1}{2}\int ex\ Sin2x\ dx=\frac{1}{2}\int Sin2x^x\ x.\ e^x\ dx=\frac{1}{2}\left[Sin2xe^x-\int 2Cos2xe^x\right]$ (1)

$=\frac{1}{2}e^x Sin2x-\frac{1}{2}\times 2\int e^x\ Cos2x\ dx$

$=\frac{1}{2}e^x Sin2x-\left[Cos2x\,e^x-\int(-2Sin2x)\ e^x dx\right]$

$I=\frac{1}{2}e^x Sin2x-e^x Cos2x-2\int e^x Sin2xdx$ (2)

Form (1) $I=\frac{1}{2}\int e^x Sin2x \Rightarrow 2I=\int e^x Sin2xdx$ put in (2)

$I=\frac{1}{2}e^x Sin2x-e^x Cos2x-2(2I)$

$I=\frac{1}{2}e^x Sin2x-e^x Cos2x-4I$

$I+4I=\frac{1}{2}e^x Sin2x-e^x Cos2x$

$5I=\frac{1}{2}e^x Sin2x-e^x Cos2x=e^x\left(\frac{1}{2}Sin2x-Cos2x\right)$

$I=\frac{1}{5}e^x\left(\frac{1}{2}Sin2x-Cos2x\right)+c$

$\frac{1}{5}e^x\left(\frac{1}{2}Sin2x(1-2Sin^2x)\right)+c$

$\frac{1}{5}e^x\left(\frac{1}{2}Sin2x-1+2Sin^2x\right)+c$

$\frac{1}{5}e^x\left(\frac{1}{2}Sin2x+2Sin^2x-1\right)+c$

xvi. $\int Sinx\ Cosx\ dx=\frac{1}{2}\int x\ 2Sinx\ Cosx\ dx$

$=\frac{1}{5}\int x\ Sin2x\ dx=\frac{1}{2}\left[x\left(\frac{-Cos2x}{2}\right)-\int 1.\left(\frac{-Cos2x}{2}\right)dx\right]$

$=\frac{-xCos2x}{4}+\frac{1}{4}\int Cos2xdx=\frac{-xCos2x}{4}+\frac{1}{4}\frac{Sin2x}{4}+c$

$=\frac{-xCos2x}{4}+\frac{Sin2x}{8}+c$

xvii. $\int Cosx^2 xdx=\int\frac{x\left(1+Cos^2\right)}{2}dx$

$=\frac{1}{2}\int(x+xCos2x)dx=\frac{1}{2}\int xdx+\frac{1}{2}\int x\ Cos2x\,dx$

$$=\frac{1}{2}.\frac{x^2}{2}+\frac{1}{2}\left(x\frac{xSin2x}{2}-\int 1.\frac{Sin2x}{2}\,dx\right)$$

$$=\frac{x^2}{4}+\frac{xSin2x}{4}-\frac{1}{4}\int Sin2xdx$$

$$=\frac{x^2}{4}+\frac{xSin2x}{4}-\frac{1}{4}\left(\frac{-Cos2x}{2}\right)+c$$

$$=\frac{1}{4}\left(x^2+x\ Sin2x+\frac{Cos2x}{2}\right)+c$$

xviii. $$\int xSin^2xdx=\int x\left(\frac{1-Cos^{2x}}{2}\right)dx$$

$$=\frac{1}{2}\int (x+x\ Cos2x)\,dx=\frac{1}{2}\int xdx-\frac{1}{2}\int \underset{I}{x}\underset{II}{Cos2x}\,dx$$

$$=\frac{1}{2}.\frac{x^2}{2}-\frac{1}{2}\left(\frac{xSin2x}{2}-\int 1.\frac{Sin2x}{2}\,dx\right)$$

$$=\frac{x^2}{4}-\frac{xSin2x}{4}+\frac{1}{4}\int Sin2xdx$$

$$=\frac{x^2}{4}-\frac{xSin2x}{4}+\frac{1}{4}\left(\frac{-Cos2x}{2}\right)+c$$

$$=\frac{x^2}{4}-\frac{xSin2x}{4}-\frac{1}{4}.\frac{Cos2x}{2}+c$$

$$=\frac{1}{4}\left(x^2-x\ Sin2x-\frac{Cos2x}{2}\right)+c$$

xix. $$\int (lnx)^2\,dx=\int \underset{I}{(lnx)^2}.\underset{II}{1}\,dx$$

$$=(lnx)^2.x-\int 2lnx.\frac{1}{\not{x}}.\not{x}\,dx$$

$$=x(lnx)^2-2\int \underset{II}{1}\,\underset{I}{lnx}\,dx$$

$$=x(lnx)^2-2\left[\int lnx.x-\int\frac{1}{\not{x}}.\not{x}\,dx\right]$$

$$=x(lnx)^2-2x\ lnx+2\int 1dx$$

$$=x(lnx)^2-2x\ lnx+2x+c$$

xx. $$\int ln\,(Tanx)\,Sec^2x\,dx$$

Put $Tanx=1$ $\quad Sec^2xdx+dt$

$$=\int ln(t)\,dt=\int 1.\,lnt\,dt=\int \underset{I}{lnt}.\underset{II}{1}\,dx$$

$$=lnt.t-\int\frac{1}{\not{t}}.\not{t}\,dt=t\ lnt-\int 1\,dt$$

$$=t\ lnt-t+c=Tanx\ ln(Tanx)-Tanx+c$$

xxi. $\int x.\frac{Sin^{-1}}{\sqrt{1-2x}}\,dx$

Put $Sin^{-1}x = t \Rightarrow \frac{1}{\sqrt{1-x^2}}dx = dt$

Or $x = Sint$

$= \int Sint.1\,dt - \int \underset{I}{t}.\underset{II}{Sint}\,dt$

$= \int Sint.1\,dt - \int \underset{I}{t}.\underset{II}{Sint}\,dt$

$= t(-Cost) - \int 1(-Cost)\,dt$

$= -tCost + Sint + c$

$= -Sin^{-1}x.\sqrt{1-x^2} + x + c$

Where $= -Sint = x$

$= Cos^2t = 1 - Sin^2t$

$= Cos^2t = 1 - x^2$

$= Cos^2t = \sqrt{1-x^2}$

2. i. $\int \boldsymbol{Tan^4xdx} = \int \boldsymbol{Tan^2x.\ Tan^2x}$

$= Tan^2x(Sec^2x - 1)dx = \int (Tan^2x\ Sec^2x - Tan^2x)\,dx$

$= \int Tan^2x\ Sec^2xdx - \int (Sec^2x - 1)\,dx$

$= \int (Tan)^2\ Sec^2dx - \int Sec^2xdx + \int 1dx$

$= \frac{Tan^3x}{3} - Tanx + x + c$

$\int \boldsymbol{Sec^4x\ dx} = \int \boldsymbol{Sec^2x\ Sec^2xdx}$

$= \int (1 + Tan^2x)\ Sec^2x\ dx = (Sec^2x + Tan^2x\ Sec^2x)\,dx$

$= \int Sec^2xdx + (Tanx)^2\ Sec^2x\ dx$

$= Tanx + \frac{Tan^3x}{3} + c$

iii. $\int \boldsymbol{e^x\ Sin2x\ Cosx\ dx} = \frac{1}{2}\int \boldsymbol{e^x\ 2Sin2x\ Cosxdx}$

$= \frac{1}{2}\int e^x\left[Sin(2x+x) + Sin(2x+x)\right]dx$

$= \frac{1}{2}\int e^x(Sin3x + Sinx)\,dx$

$= \frac{1}{2}\left[\int e^x \underset{I_1}{Sinx}\,dx + \int e^x\ \underset{I_2}{Sinx}\,dx\right] \quad (1)$

$I_1 \int \underset{II}{e^x}.\underset{I}{Sin}\ 3xdx = Sin3xe^x - \int 3Cos3x\ e^xdx$

$= e^x Sin3x - 3\int \underset{II}{e^x}.\underset{I}{Sin}\,3xdx = Sin3xe^x - 3\left[Cos3xe^x - \int \left(-3Sin3xe^x\,dx\right)\right]$

Use $2\,Sin\alpha Cos\beta = Sin(\alpha+\beta) + Sin(\alpha-\beta)$

$I_1 = e^x Sin3x\ Cos3x - 9\int e^x Sin3xdx$

$= e^x Sin3x - 3e^x Cos3x - 9I_1$

$I_1 + 9I_1 = e^x\left(Sin3x - 3Cos3x\right)$

$10I_1 = e^x\left(Sin3x - 3Cos3x\right)$

Now $I_2 = \int e^x Sinx\,dx = Sin\,xe^x - \int \underset{I}{Cosx}.\underset{II}{e^x}\,dx$

$= Sinxe^x - \left[Cosx\,e^x - \int (-Sinx)e^x dx\right]$

$I_2 = e^x Sinx - e^x Cosx - \int e^x Sinxdx$

$I_2 = e^x\left(Sinx - Cosx\right) - I_2$

$2I_2 = e^x\left(Sinx - Cosx\right)$

$I_2 = \frac{e^x}{2}\left(Sinx - Cosx\right)$

Put I_1 and I_2 in (1) we get so

$\int e^x Sin2xCosxdx = \frac{1}{2}\left[\frac{e^x}{10}\left(Sin3x - 3Cos3x\right) + \frac{e^x}{2}\left(Sin3x - Cosx\right)\right]$

$= \frac{e^x}{4}\left(\frac{1}{5}Sin3x - \frac{3}{5}Cos3x + Sinx - Cosx\right) + c$

iv. $\int Tan^3x\ Secxdx = \int Tan^2x\ .\ Tanx\ Secxdx$

$= \int \left(Sec^2x - 1\right) Secx\ Tanx\ dx$

$= \int Sec^2x\left(SecxTanx\right)dx - \int Secx\ Tanx\ dx$

$= \frac{Sec^3x}{3} - Secx + c$

v. $\int \underset{I}{x^3}\,.\,\underset{II}{e^{5x}}\,dx = x^3\frac{e^{5x}}{5} - \int 3x^2\frac{e^{5x}}{5}dx$

$= \frac{x^3e^{5x}}{5} - \frac{3}{5}\left[\int \underset{I}{x^2}\,\underset{II}{e^{5x}}\,dx\right]$

$= x^3e^{5x} - \frac{3}{5}\left[x^2\frac{e^{5x}}{5} - \int 2x\frac{e^{5x}}{5}dx\right]$

$= x^3e^{5x} - \frac{3x^2e^{5x}}{25} + \frac{6}{25}\int \underset{I}{x}\,\underset{I}{e^{5x}}\,dx$

$= x^3e^{5x} - \frac{3x^2e^{5x}}{25} + \frac{6}{25}\left[\frac{xe^{5x}}{5} - \int 1.\frac{e^{5x}}{5}dx\right]$

$$= x^3e^{5x} - \frac{3x^2e^{5x}}{25} + \frac{6xe^{5x}}{25} - \frac{6}{125}\int e^{5x}dx$$

$$= x^3e^{5x} - \frac{3x^2e^{5x}}{25} + \frac{6xe^{5x}}{25} - \frac{6}{125}.\frac{e^{5x}}{5}$$

$$= e^{5x}\left(x^3 \frac{3x^2}{25} + \frac{6x}{125} - \frac{6}{625}\right) + c$$

vi. $I = \int \underset{I}{e^x} . \underset{II}{Sin}\ 2xdx = e^{-x}\left(\frac{-Cos2x}{2}\right) - \int (-e)^2\left(\frac{-Cosx2x}{2}\right)dx$

$$= \frac{-e^xCos2x}{2} - \frac{1}{2}\int \underset{I}{e^x}\ \underset{II}{Cos}\ 2xdx$$

$$= \frac{-e^xCos2x}{2} - \frac{1}{2}\left[e^{-x}\frac{Sin2x}{2} - \int -e^{-x}\frac{Sin2x}{2}\ dx\right]$$

$$= \frac{-e^xCos2x}{2} - \frac{e^{-x}Sin2x}{4} - \frac{1}{4}\int e^{-x}Sin2xdx$$

$$= \frac{-e^{-x}}{2}\left(Cos2x\frac{Sin2x}{52}\right) - \frac{1}{4}I$$

$$I + \frac{1}{4}I = \frac{-e^{-x}}{2}\left(Cos2x\frac{Sin2x}{2}\right)$$

$$I\left(1 + \frac{1}{4}\right) = \frac{-e^{-x}}{2}\left(Cos2x\frac{Sin2x}{2}\right)$$

$$I\left(\frac{5}{4}\right) = \frac{-e^{-x}}{2}\left(Cos2x\frac{Sin2x}{2}\right)$$

$$I = \frac{-4^2}{2} \times \frac{e^{-x}}{2}\left(Cos2x\frac{Sin2x}{2}\right)$$

$$I = \frac{-2}{2}e^{-x}\left(Cos2x\frac{Sin2x}{2}\right) + c$$

vii. $I = \int \underset{I}{e^{2x}}\ \underset{II}{Cos}\ 3xdx$ **(Sargodha 2011)**

$$= e^{2x}\frac{Sin3x}{3} - \int 2e^{2x}.\frac{Sin3x}{3}dx$$

$$= \frac{e^{2x}Sin3x}{3} - \frac{2}{3}\int \underset{I}{e^{2x}}\ \underset{II}{Sin}\ 3xdx$$

$$= \frac{e^{2x}Sin3x}{3} - \frac{2}{3}\left[e^{2x}\left(\frac{-Cos3x}{3}\right) - \int 2e^{2x}\left(\frac{-Cos3x}{3}\right)dx\right]$$

$$= \frac{e^{2x}Sin3x}{3} + \frac{2}{3}e^{2x}Cos3x - \frac{4}{9}\int e^{2x}Cos3x\ dx$$

$$I = \frac{e^{2x}}{3}\left(Sin3x + \frac{2}{3}Cos3x\right) - \frac{4}{9}I$$

$$I\left(1+\frac{4}{9}\right)=\frac{e^{2x}}{3}\left(Sin3x+\frac{2}{3}Cos3x\right)$$

$$I\left(\frac{13}{9}\right)=\frac{e^{2x}}{3}\left(Sin3x+\frac{2}{3}Cos3x\right)$$

$$I=\frac{9^{3}}{13}\times\frac{e^{2x}}{3}\left(Sin3x+\frac{2}{3}Cos3x\right)$$

$$I=\frac{3}{13}\times e^{2x}\left(Sin3x+\frac{2}{3}Cos3x\right)+c$$

viii. $I=\int Cosec^{3}x\,dx=\int \underset{I}{Cosecx}\,\underset{II}{Cosec^{2}}\,xdx$

$$=Cosecx(-Cotx)-\int(-Cosecx\;Cotx)(-Cotx)\,dx$$

$$=-Cosecx\;Cotx-\int Cosecx\;Cot^{2}x\,dx$$

$$=-Cosecx\;Cotx-\int Cosecx\left(Cosec^{2}x-1\right)dx$$

$$I=-Cosecx\;Cotx-\int Cosec^{3}xdx+\int Cosecxdx$$

$$I=-Cosecx\;Cotx-I+ln|Cosecx-Cotx|+c$$

$$I+I=-Cosecx\;Cotx+ln|Cosecx-Cotx|+c$$

$$2I=-Cosecx\;Cotx+ln|Cosecx-Cotx|+c$$

$$I=\frac{-Cosecx\;Cotx}{2}+\frac{1}{2}ln|Cosecx-Cotx|+c$$

3. $\int e^{ax}\;Sinbx\;dx=\frac{1}{\sqrt{a^{2}+b^{2}}}\;e^{ax}\;Sin\left(bx-Tan^{-1}\frac{b}{a}\right)+c$

$$I=\int \underset{I}{e^{ax}}\;\underset{II}{Sinbx}\;dx=e^{ax}\frac{(-Cosbx)}{b}-\int ae^{ax}\frac{(-Cosbx)}{b}dx$$

$$=\frac{-ae^{ax}Cosbx}{b}+\frac{a}{b}\int \underset{I}{e^{ax}}\;\underset{II}{Cosbx}\;dx$$

$$=\frac{-ae^{ax}Cosbx}{b}+\frac{a}{b}\left[e^{ax}\frac{Sinbx}{b}-\int e^{ax}\frac{Sinbx}{b}dx\right]$$

$$=\frac{-ae^{ax}Cosbx}{b}+\frac{ae^{ax}Cosbx}{b^{2}}-\frac{a^{2}}{b^{2}}\int e^{ax}\;Sinbxdx$$

$$I=e^{ax}\left(\frac{-Cosbx}{b}+\frac{aSinbx}{b^{2}}\right)-\frac{a^{2}}{b^{2}}I$$

$$I+\frac{a^{2}}{b^{2}}I=e^{ax}\left(\frac{aCosbx}{b^{2}}+\frac{bSinbx}{b^{2}}\right)$$

$$I\left(1+\frac{a^{2}}{b^{2}}\right)=\frac{e^{ax}}{b^{2}}(aCosbx-bCosbx)$$

$$I\left(\frac{b^{2}+a^{2}}{b^{2}}\right)=\frac{e^{ax}}{b^{2}}(aCosbx-bCosbx)$$

$$I = \frac{b^2}{a^2+b^2} \times \frac{e^{ax}}{b^2}(aCosbx - bCoshx)$$

$$I = \frac{e^{ax}}{a^2+b^2}(aCosbx - bCosbx) \qquad (1)$$

Put $a = rCos\theta$ & $b = rSin\theta$

$$a^2 + b^2 = r^2Cos^2\theta + r^2Sin^2\theta = r^2\left(Cos^2\theta + Sin^2\theta\right) = r^2(1) = r^2$$

$$r = \sqrt{a^2+b^2} \ \& = \frac{rSin\theta}{rCos\theta} = \frac{a}{b} \Rightarrow Tan\theta = \frac{b}{a} \Rightarrow \theta = Tan^{-1}\frac{b}{a}$$

(1) become

$$I = \frac{e^{ax}}{a^2+b^2}(r\ Cos\theta\ Sinbx - rSin\theta\ Cosbx)$$

$$= \frac{e^{ax}}{a^2+b^2}.\ r\left(Sinbx\ Cos\theta - Cosbx\ Sin\theta\right)$$

$$= \frac{re^{ax}}{a^2+b^2}Sin(bx-\theta)$$

$$= \frac{\sqrt{a^2+b^2}\,e^{ax}}{\sqrt{a^2+b^2} \times \sqrt{a^2+b^2}}Sin\left(bx - Tan^{-1}\frac{b}{a}\right)$$

$$= \frac{1}{\sqrt{a^2+b^2}}e^{ax}Sin\left(bx - Tan^{-1}\frac{b}{a}\right)$$

Hence Proved

4. i. $I = \int \sqrt{a^4 - x^2}dx = \int \underset{I}{\left(a^2-x^2\right)^{1/2}} . \underset{II}{1}\ dx$

Note

Standard Subsitution

$\sqrt{a^2-x^2}$, $x = aSin\theta$

$\sqrt{a^2-x^2}$, $x = aTan\theta$

a^2-x^2, $x = aSec\theta$

$$= \left(a^2-x^2\right)^{1/2}.x - \int \frac{1}{2}\left(a^2-x^2\right)^{-1/2}(-2x).x\ dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{-x^2}{\sqrt{a^2-x^2}}dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}dx + \int \frac{a^2}{\sqrt{a^2-x^2}}dx$$

$$= x\sqrt{a^2-x^2} - \int \frac{\sqrt{a^2-x^2}\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}dx + a^2\int \frac{1}{\sqrt{a^2-x^2}}dx$$

$$I = x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,dx + a^2\,Sin^{-1}\frac{x}{a}$$

$$I + I = x\sqrt{a^2-x^2} + a^2\,Sin^{-1}\frac{x}{a}$$

$$I = x\sqrt{a^2-x^2} - 1 + a^2\,Sin^{-1}\frac{x}{a}$$

$$2I = x\sqrt{a^2 - x^2} + a^2 Sin^{-1}\frac{x}{a}$$

$$I = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} Sin^{-1}\frac{x}{a} + c$$

ii. $I = \int \sqrt{a^4 - x^2}\,dx$ **Put** $x = a\,Sec\theta$ $dx = aSec\theta\,Tan\theta\,d\theta$

$$= \int \sqrt{a^2 Sec^2\theta - a^2}\; aSec\theta\; Tan\theta\; d\theta$$

$$= \int \sqrt{a^2\left(Sec^2\theta - 1\right)}\; aSec\theta\; Tan\theta\; d\theta$$

$$= \int \sqrt{a^2 Sec^2\theta}\; aSec\theta\; Tan\theta\; d\theta = \int aTan\theta\,.\,aSec\theta\; Tan\theta\; d\theta$$

$$= a^2 \int \underset{I}{Tan\theta}\,.\,\underset{II}{Sec\theta}\; Tan\theta\, d\theta$$

$$= a^2\left[Tan\theta\,.\,Sec\theta - \int Sec^2\theta\,.\,Sec\theta\; d\theta\right]$$

$$= a^2\; Tan\theta Sec\theta - a^2\int Sec\theta d\theta - a^2\int Tan^2\theta\; Sec\theta d\theta$$

$$= a^2\sqrt{\frac{x^2}{a^2} - 1}.\frac{x}{a} - a^2\,ln\left|Sec\theta + Tan\theta\right| - I$$

$$I + I = a^2\sqrt{\frac{x^2 - a^2}{a}}.\frac{x}{a} - a^2 ln\left|Sec\theta + \sqrt{Sec^2\theta - 1}\right| + c$$

$$2I = a^2\,\frac{\sqrt{x^2 - a^2}}{a^2} - a^2 ln\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + c$$

$$I = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2} ln\left|\frac{x\sqrt{x^2 - a^2}}{a}\right| + c$$

iii. $I = \int \sqrt{4 - 5x^2}\,dx = \int\sqrt{(2)^2 - \left(\sqrt{5}x\right)^2}\,dx$ **(Sargodha 2011)**

We know that $\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} Sin^{-1}\frac{x}{a} + c$

$$\int \sqrt{(2)^2 - \sqrt{(5x)^2}}\,dx = \frac{\sqrt{5x}\times\sqrt{4 - 5x^2}}{2} + \frac{4}{2} Sin^{-1}\frac{\sqrt{5x}}{2} + c$$

$$\frac{\sqrt{5x}\times\sqrt{4 - 5x^2}}{2} + 2Sin^{-1}\frac{\sqrt{5x}}{2} + c$$

iv. $I = \int \sqrt{4 - 4x^2}\,dx$

We know that $\int \sqrt{a^2 - x^2}\,dx = \frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} Sin^{-1}\frac{x}{a} + c$

$$\int \sqrt{3 - 4x^2}\,dx = \sqrt{\left(\sqrt{3}\right)^2 - (2x)^2}\,dx = \frac{2x\sqrt{3 - 4x^2}}{2} + \frac{3}{2} Sin^{-1}\frac{2x}{\sqrt{3}} + c$$

v. $I=\int \sqrt{x^2+4}\,dx$ **(Sargodha 2008)**

Put $x = 2Tan\theta \Rightarrow dx = 2Sec^2\theta d\theta$

$= \int \sqrt{4Tan^2\theta + 4}\,.2Sec^2\theta d\theta$

$= \int \sqrt{4\left(1+Tan^2\theta\right)}.2Sec^2\theta d\theta$

$= \int \sqrt{4Sec^2\theta}.2Sec^2\theta d\theta = \int 2Sec\theta\ 2Sec^2\theta d\theta$

$= 4\int Sec\theta\left(1+Tan^2\theta\right)d\theta = 4\int Sec\theta\ Tan^2\theta d\theta$

$= 4\ ln\left|Sec\theta + Tan\theta\right| + 2(2I)$ (1)

$I = \int Sec\theta\, Tan^2\theta d\theta = \int \underset{II}{Sec\theta Tan\theta}\,.\int \underset{I}{Tan\theta} d\theta$

$= Tan\theta\, Sec\theta - \int Sec^2\theta\,.\,Sec\theta d\theta$

$= Tan\theta\, Sec\theta - \int \left(1+Tan^2\theta\right)Sec\theta d\theta$

$= Tan\theta\, Sec\theta - \int Sec\theta\ d\theta - \int Tan^2\theta\ Sec\theta\ d\theta$

$I = Tan\theta\ Sec\theta - ln\left|Sec\theta + Tan\theta\right|$ I

$2I = Tan\theta\ Sec\theta - ln\left|Sec\theta + Tan\theta\right|$

Put velue of 2I in (1)

$= 4ln\left|Sec\theta + Tan\theta\right| + 2Tan\theta\ Sec\theta - 2ln\left|Se\theta + Tan\theta\right| + c$

$= 2Tan\theta\ Sec\theta + 2ln\left|Sec\theta + Tan\theta\right| + c$

$= 2Tan\theta\sqrt{1+Tan^2\theta} + 2ln\left|\sqrt{1+Tan^2\theta} + Tan\theta\right| + c$

$= x\sqrt{1+\frac{x^2}{4}} + 2ln\left|\sqrt{1+\frac{x^2}{4}} + \frac{x}{2}\right| + c$

$= \frac{x\sqrt{4+x^2}}{2} + 2ln\left|\frac{x\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + c$

$= \frac{x\sqrt{4+x^2}}{2} + 2ln\left|\frac{x+\sqrt{x^2+4}}{2}\right| + c$

Alternate of (V)

$= \int \sqrt{x^2+4}\ dx = \int \sqrt{x^2+(2)^2}\ dx$

We know that

$= \int \sqrt{x^2+a^2}\ dx = \frac{x}{2}\int \sqrt{a^2+x^2} + \frac{a^2}{2}ln\left(x+\sqrt{a^2+x^2}\right) + c$

So $= \int \sqrt{x^2+4}\ dx = \frac{x}{2}\int \sqrt{x^2+4} + \frac{4}{2}ln\left(x+\sqrt{x^2+4}\right) + c$

Or $= \int \sqrt{x^2+4}\ dx = \frac{x}{2}\int \sqrt{x^2+4} + 2ln\left(x+\sqrt{x^2+4}\right) + c$

vi. $\int \underset{I}{x^2} \underset{II}{e^{ax}} dx = x^2 \frac{e^{ax}}{a} - \int 2x \frac{e^{ax}}{a} dx$

$= \frac{x^2 e^{ax}}{a} - \frac{2}{a} \int \underset{I}{x} \underset{II}{e^{ax}} dx$

$= \frac{x^2 e^{ax}}{a} - \frac{2}{a}\left[x \frac{e^{ax}}{a} - \int 1 . \frac{e^{ax}}{a} dx \right]$

$= \frac{x^2 e^{ax}}{a} - \frac{2xe^{ax}}{a^2} + \frac{2}{a^2} \int e^{ax} dx$

$= \frac{x^2 e^{ax}}{a} - \frac{2xe^{ax}}{a^2} + \frac{2}{a^2} . \frac{e^{ax}}{a} + c$

$= \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a^2} + \frac{2}{a^2} \right) + c$

5. i.

Rule I $\int e^x \left(f(x) + f'(x) \right) dx = e^x f(x) + c$

Rule II $\int e^{ax} \left(af(x) + f'(x) \right) dx = e^{ax} f(x) + c$

i. $\int e^x \left(\frac{1}{x} + lnx \right) dx = \int e^x \left(lnx + \frac{1}{x} \right) dx$ **(Sargodha 2011)**

Here $f(x) = lnx$ & $f'(x) = \frac{1}{x}$ So

$= e^x lnx + c$

ii. $\int e^x \left(Cosx + Sinx \right) dx = \int e^x \left(Sinx + Cosx \right) dx$ **(Sargodha 2012)**

Here $f(x) = Sinx$ & $f'(x) = Cosx$

$= e^x Sinx + c$

iii. $\int e^{ax} \left[aSec^{-1}x + \frac{1}{x\sqrt{x^2-1}} \right] dx$

Here $f(x) = Sec^{-1}x$ & $f'(x) = \frac{1}{x\sqrt{x^2-1}}$ So

$= e^{ax} Sec^{-1}x + c \int e^{ax} \left(af(x) + f'(x) \right) dx = e^{ax} f(x)$

iv. $\int e^{3x} \left(\frac{3Sinx - Cosx}{Sin^2 x} \right) dx = \int e^{3x} \left(\frac{3\,\not{Sin}x}{Sin^2 x} - \frac{Cosx}{Sin^2 x} \right) dx$

$= \int e^{3x} \left(3\frac{1}{Sinx} - \frac{Cosx}{Sinx} . \frac{1}{Sinx} \right) dx$

$= \int e^{3x} \left(3Cosecx + Cosecx\ Cotx \right) dx$

Here $f(x) = Cosecx$ & $f'(x) = -Cosecx\ Cotx$

Use $\int e^{ax} \left(af(x) + f'(x) \right) dx = e^{ax} f(x) + c$

$= e^{3x} Cosecx + c$

v. $\int e^{2x}(-Sinx+2Cosx)dx$ **(Sargodha 2010,11)**

$= \int e^{2x}(2Cosx - Sinx)dx$

Here $f(x) = Cosx$ & $f'(x) = -Sinx$

Use $\int e^{ax}(af(x)+f'(x))dx = e^{ax}f(x)+c$

$= e^{2x}Cosx + c$

vi. $\int \frac{xe^x}{(1+x)^2}dx = \int e^x\left(\frac{1+x-1}{(1+x)^2}\right)dx$

$= \int e^{2x}\left(\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2}\right)dx = \int e^x\left(\frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right)dx$

Here $f(x) = \frac{1}{(1+x)}$ Then $f'(x) = (-1)(1+x)^2 = \frac{-1}{(1+x)^2}$ So

Use $\int e^{ax}(af(x)+f'(x))dx = e^{ax}f(x)+c$

$= e^x\left(\frac{1}{1+x}\right)+c$

vii. $\int e^{-x}(Cosx - Sinx)dx$ **(Sargodha 2009,10)**

$= \int e^{-x}(-Sinx + Cosx)dx$

$= \int e^{-x}((-1)Sinx + Cosx)dx$

Use $\int e^{ax}(af(x)+f'(x))dx = e^{ax}f(x)+c$

$= e^x Sinx + c$

viii. $\int \frac{e^m Tan^{-1}x}{(1+x^2)^2}$

Put $= Tan^{-1}x = t$ $\quad \frac{1}{1+x^2}dx = dt$

$= \int e^t . dt = e^t + c$

$= e^{Tan^{-1}x} + c$

ix. $\int \frac{2x}{1+Sinx}dx$

$= \int \frac{2x}{1-Cos\left(\frac{\pi}{2}-x\right)}dx$

$$= \int \frac{2x}{2Sin^2 \frac{1}{2}\left(\frac{\pi}{2}-x\right)}$$

$$= \int \frac{x}{Sin^2\left(\frac{\pi}{4}-\frac{\pi}{2}\right)} dx = \int \underset{I}{x} \underset{II}{Cosec^2}\left(\frac{\pi}{4}-\frac{\pi}{2}\right) dx$$

$$= \left[\frac{-Cot\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\frac{-1}{2}}\right] - \int 1\left[\frac{-Cot\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\frac{-1}{2}}\right] dx$$

$$= 2xCot\left(\frac{\pi}{4}-\frac{x}{2}\right) - 2\int Cot\left(\frac{\pi}{4}-\frac{x}{2}\right) dx$$

$$= 2xCot\left(\frac{\pi}{4}-\frac{x}{2}\right) - 2\int \frac{Cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{Sin\left(\frac{\pi}{4}-\frac{x}{2}\right)} dx$$

$$= 2xCot\left(\frac{\pi}{4}-\frac{x}{2}\right) + 4\int \frac{\frac{-1}{2}Cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}{Sin\left(\frac{\pi}{4}-\frac{x}{2}\right)} dx$$

$$= 2xCot\left(\frac{\pi}{4}-\frac{x}{2}\right) - 4ln\left|\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + c$$

x. $$\int \frac{e^x(1+x)}{(2+x)^2} dx$$

$$= \int \frac{e^x(1+x+1-1)}{(2+x)^2}$$

$$= \int \frac{e^x(2+x-1)}{(2+x)^2}$$

$$= \int e^x\left[\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2}\right] dx$$

$$= \int e^x\left[\frac{1}{2+x} - \frac{1}{(2+x)^2}\right] dx$$

$$= \int e^x\left(f(x)+f'(x)\right) dx = e^x f(x)$$

$$= \int e^x\left(\frac{1}{2+x}\right) + c\,e^x f(x)$$

xi. $\int e^x\left(\frac{1-Sinx}{1-Cosx}\right)dx$

$$=\int e^x\left(\frac{1-2\ Sinx\frac{x}{2}Cos\frac{x}{2}}{2\ Sin^2\frac{x}{2}}\right)dx$$

$$=\int e^x\left[\frac{1}{2Sin^2\frac{x}{2}}-\frac{2\ Sin\frac{x}{2}Cos\frac{x}{2}}{2\ Sin^2\frac{x}{2}}\right]dx$$

$$=\int e^x\left[\frac{1}{2}Cosec^2-Cot\frac{x}{2}\right]dx$$

$$=\int e^x\left[-Cot\frac{x}{2}+\frac{1}{2}Cosec^2\frac{x}{2}\right]dx$$

$$=\int e^x\left(-Cot\frac{x}{2}\right)+c^*$$

Exercise 3.5

1. $= \int \frac{3x+1}{x^2-x-6}dx = \int \frac{3x+1}{x^2-3x+2x-6}dx$ **(Sargodha 2010)**

$= \int \frac{3x+1}{x(x-3)+2(x-3)}dx = \int \frac{3x+1}{(x-3)(x+2)}dx$

Suppose $\frac{3x+1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$ $\quad I$

'X' both sides by $(x-3)(x+2)$ *we get*

$3x+1 = A(x+2) + B(x-3) \quad II$

Put $x+2=0 \Rightarrow x=-2$ *in II*

$3(-2)+1 = A(-2+2)+b(-2-3) \Rightarrow -6+1 = 0-8B \Rightarrow -5=-5B \Rightarrow \boxed{B=1}$

Put $x-3=0 \Rightarrow x=3$ *in II*

$3(3+1) = A(3+2)+b(3-3) \Rightarrow 10 = 5A+10 \Rightarrow \boxed{A=2}$

Put values in I

$\int \frac{3x+1}{(x-3)(x+2)}dx = \int \left(\frac{2}{x-3} + \frac{1}{x+2}\right)dx$

$= 2\int \frac{1}{x-3}dx + \int \frac{1}{x+2}dx = ln|x-3| + ln|x+2| + c$

2. $= \int \frac{5x+8}{(x+3)(2x-1)}dx$ **(Sargodha 2008,11)**

Suppose $\int \frac{5x+8}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$ $\quad I$

'X' both sides by $(x+3)(2x-1)$ *we get*

$5x+8 = A(2x-1) + B(x+3) \quad II$

Put $x+3=0 \Rightarrow x=-3$ *in II*

$5(-3)+8 = A(2(-3)-1)+b(-3+3) \Rightarrow -15+8 = A(-6-1)+0 \Rightarrow -7=-7A$

$\Rightarrow \boxed{A=7}$

Put $2x-1=0 \Rightarrow 2x=1 \; \boxed{x=\frac{1}{2}}$ *in II*

$5\left(\frac{1}{2}\right)+8 = A(0)+B\left(\frac{1}{2}+3\right) \Rightarrow \frac{5}{2}+8 = B\left(\frac{7}{2}\right) \Rightarrow \frac{21}{2} = \frac{7}{2}B$

$\Rightarrow B = \frac{21}{2} \times \frac{2}{7} \Rightarrow \boxed{B=3}$

Put values in I

$\frac{5x+8}{(x+3)(2x-1)} = \frac{1}{x+3} + \frac{3}{2x-1} \Rightarrow \int \frac{5x+8}{(x+3)(2x-1)}dx = \int\left(\frac{1}{x+3} + \frac{3}{2x-1}\right)dx$

$$\int\frac{5x+8}{(x+3)(2x-1)}dx=\int\frac{1}{x+3}dx+3\int\frac{1}{2x-1}dx=ln|x+3|+\frac{3}{2}\int\frac{2}{2x-1}dx$$

$$=ln|x+3|+\frac{3}{2}ln|2x-1|+c$$

3. $$=\int\frac{x^2+3x-34}{x^2+2x-15}dx$$

$$=\int\left(1+\frac{x-19}{x^2+2x-15}\right)dx$$

$$x^2+2x+5\overline{\smash{)}x^2+3x-34}\quad 1$$

$$\frac{-x^2\pm 2x\mp 15}{x-19}$$

$$=\int 1dx+\int\frac{x-19}{x^2+5x-3x-15}dx$$

$$=x+\int\frac{x-19}{x(x+5)-3(x+5)}dx=x+\int\frac{x-19}{(x+5)(x-3)}dx$$

$$=x+1 \qquad (1)$$

Now $$I=\int\frac{x-19}{(x+5)(x-3)}dx$$

Suppose $$\frac{x-19}{(x+5)(x-3)}=\frac{A}{x+5}+\frac{B}{x-3} \qquad (2)$$

'X' both sides by $(x+5)(x-3)$ we get

$$x19=A(x-3)+B(x+5) \qquad (3)$$

Put $x-3=0 \Rightarrow x=3$ *in* (3)

$$3-19=A(3-3)+b(3+5)\Rightarrow -16=0+8B \Rightarrow \boxed{B=2}$$

Put $x+5=0 \Rightarrow x=-5$ *in* (3)

$$-5-19=A(-5-3)+B(-5+5) \Rightarrow -24=-8A+0 \Rightarrow \boxed{A=0}$$

Put in (2) $$\frac{x-19}{(x+5)(x-3)}=\frac{3}{x+5}+\frac{-2}{x-3}$$

$$\frac{x-19}{(x+5)(x-3)}=\frac{3}{x+5}+\frac{-2}{x-3} \Rightarrow \frac{x-19}{(x+5)(x-3)}dx=\int\left(\frac{3}{x+5}-\frac{2}{x-3}\right)dx$$

$$=3\int\frac{1}{x+5}dx-2\int\frac{1}{x-3}dx=3\,ln|x+5|-2ln|x-3|+c$$

4. $$\int\frac{(a-b)x}{(x-a)(x-b)}dx$$

Suppose $$\frac{(a-b)x}{(x-a)(x-b)}=\frac{A}{x-a}+\frac{B}{x-b} \qquad I$$

$'X'$ both side by $(x-a)(x-b)$ we get

$(a-b)x = A(x-b) + B(x-a)$ II

Put $x-a=0 \Rightarrow x=0$ in II

$(a-b)a = A(a-b)+B(a-a) \Rightarrow (a-b)a = A(a-b)+0$

$A=\frac{(a-b)a}{(a-b)} \Rightarrow \boxed{A=a}$

Put $x-b=0 \Rightarrow x=b$ in II

$(a-b)b = A(b-b)+B(b-a) \Rightarrow (a-b)b = 0-B(a-b) \Rightarrow B=\frac{(a-b)b}{(a-b)}$

$\Rightarrow \boxed{B=-b}$

Put values in I

$$\frac{(a-b)x}{(x-a)(x-b)} = \frac{a}{x-a} - \frac{b}{x-b} \Rightarrow \int\frac{(a-b)x}{(x-a)(x-b)}dx = \int\left(\frac{a}{x-a} - \frac{b}{x-b}\right)dx$$

$$= a\int \frac{1}{x-a}dx - b\int \frac{1}{x-b}dx = a\ln|x-a| - b\ln|x-b| + c$$

5. $\int \frac{3-x}{1-x-6x^2} = \int \frac{3-x}{1-3x+2x-6x^2}dx$ **(Sargodha 2009)**

$$= \int \frac{3-x}{(1-3x)+2x(1-3x)}dx = \int \frac{3-x}{(1-3x)(1+2x)}$$

Suppose $\frac{3-x}{(1-3x)(1+2x)} = \frac{A}{1-3x} + \frac{B}{1+2x}$ I $'X'$ by $(1-3x)(1+2x)$

$3-x = A(1+2x) + B(1-3x)$ II

Put $1+2x=0 \Rightarrow x=\frac{-1}{2}$ in II

$$3-\left(-\frac{1}{2}\right) = A(0)+B\left(1-3\left(-\frac{1}{2}\right)\right) \Rightarrow 3+\frac{1}{2} = B\left(1+\frac{3}{2}\right)$$

$$\frac{7}{2} = B\left(\frac{5}{2}\right) \Rightarrow B=\frac{7}{2}\times\frac{5}{2}=\frac{7}{2} \Rightarrow \boxed{B=\frac{7}{5}}$$

Put $1-3x=0 \Rightarrow x=\frac{1}{3}$ in II

$$3-\frac{1}{3} = A\left(1+2\left(\frac{1}{3}\right)\right)+B(0) \Rightarrow \frac{9-1}{3} = A\left(1+\frac{2}{3}\right)$$

$$\frac{8}{3} = A\frac{5}{3} \Rightarrow A=\frac{8}{3}\times\frac{3}{5} \Rightarrow \boxed{A=\frac{8}{5}}$$

Put value is I

$$\frac{3-x}{(1+2x)(1-3x)}=\frac{\frac{8}{5}}{1-3x}+\frac{\frac{7}{5}}{1+2x}$$

$$\Rightarrow \int\frac{3-x}{(1+2x)(1-3x)}dx=\frac{8}{5}\int\frac{1}{1-3x}dx\frac{7}{5}\int\frac{1}{1+2x}dx$$

$$=\frac{8}{5}\left(-\frac{1}{3}\right)\int\frac{-3}{1-3x}dx+\frac{7}{5}\left(\frac{1}{2}\right)\int\frac{12}{1+2x}dx$$

$$=\frac{-8}{15}\,ln|1-3x|+\frac{7}{10}ln\,|1+2x|+c$$

6. $\int\frac{2x}{x^2-a^2}dx=\int\frac{2x}{(x-a)(x+a)}dx$ **(Sargodha 2011)**

Suppose $\frac{2x}{(x-a)(x+a)}=\frac{A}{x-a}+\frac{B}{x+a}$ $\quad I$

'X' by $(x-a)(x+a)$ *both side we get*

$2x=A(x+a)+B(x-a)$ $\quad II$

Put $x+a=0 \Rightarrow x=-a$ *in II*

$2a=A(x+a)+B(0) \Rightarrow 2a=2A \Rightarrow \boxed{B=a}$

Put $x-a=0 \Rightarrow x=a$ *in II*

$2a=A(a+a)+B(0) \Rightarrow 2a=2A \Rightarrow \boxed{A=a}$

Put values in I

$$\frac{2x}{(x-a)(x+a)}=\frac{a}{x-a}+\frac{a}{x+a} \Rightarrow \int\left(\frac{a}{x-a}+\frac{a}{x+a}\right)dx$$

$$=a\left[\int\frac{1}{x-a}dx+\int\frac{1}{x+a}dx\right]=a\left[ln|x+a|+ln|x+a|\right]+c$$

$$=a\,ln(x-a)(x+a)+c=aln(x^2-a^2)+c$$

7. $\int\frac{1}{6x^2-5x-4}dx=\int\frac{1}{6x^2+8x-3x-4}dx=\int\frac{1}{2x(3x+4)-1(3x+4)}dx=\int\frac{1}{(3x+4)(2x-1)}dx$

Suppose $\frac{1}{(3x+4)(2x-1)}=\frac{A}{3x+4}+\frac{B}{2x-1}$ $\quad I$ **(Sargodha 2012)**

'X' both sides by $(3x+4)(2x-1)$ *we get*

$1=A(2x-1)+B(3x+4)$ $\quad II$

Put $2x-1=0 \Rightarrow x=\frac{1}{2}$ *in II*

Put

$$1=A(0)+B\left(3\left(\frac{1}{2}\right)+4\right) \Rightarrow 1=B\left(\frac{3}{2}+4\right) \Rightarrow 1=\frac{11}{2}B \Rightarrow \boxed{B=\frac{2}{11}}$$

$$3x+4=0 \qquad \Rightarrow \qquad x=\frac{-4}{3} \text{ in } II$$

$$I=A\left(2\left(-\frac{4}{3}\right)-1\right)+B(0) \;\Rightarrow\; 1=A\left(\frac{-11}{3}\right) \;\Rightarrow\; \boxed{A=\frac{-3}{11}}$$

$$\frac{1}{(3x+4)(2x-1)}=\frac{\frac{-3}{11}}{3x+4}+\frac{\frac{2}{11}}{2x-1}$$

$$\Rightarrow \qquad \int\frac{1}{(3x+4)(2x-1)}dx=\frac{-3}{11}\int\frac{1}{3x+4}dx+\frac{2}{11}\int\frac{1}{2x-1}dx$$

$$=-\frac{1}{11}\int\frac{3}{3x+4}dx+\frac{1}{11}\int\frac{2}{2x-1}dx$$

$$=-\frac{1}{11}ln|3x+4|+\frac{1}{11}ln|2x-1|+c$$

$$=-\frac{1}{11}\left[ln|2x-1|-ln|3x+4|\right]+c$$

$$=-\frac{1}{11}ln\left|\frac{2x-1}{3x+4}\right|c$$

8. $$\int\frac{1}{6x^2-5x-4}dx=\int\frac{1}{6x^2+8x-3x-4}dx=\int\frac{2x^3-3x^2-x-7}{2x^2-3x-2}dx$$

$$=\int\left(x+\frac{x-7}{2x^2-3x-2}\right)dx$$

$$\begin{array}{r} x \\ 2x^2-3x-2\overline{)2x^3-3x^2-x-7} \\ \underline{-2x^3\mp 3x^2\mp 2x} \\ x-7 \end{array}$$

$$=\int xdx+\int\frac{x-7}{2x^2-3x-2}dx$$

$$=\frac{x^2}{2} \qquad\qquad (1)$$

$$I=\int\frac{x-7}{2x^2-3x-2}dx=\int\frac{x-7}{2x^2-4x+1x-2}dx$$

$$=\int\frac{x-7}{2x(x-2)+1(x-2)}dx=\int\frac{x-7}{2x^2-4x+1x-2}dx$$

Let $\frac{x-7}{(x-2)(2x+1)}=\frac{A}{x-2}+\frac{B}{2x+1}$ (2)

'X' by $(x-2)(2x+1)$ we get

$x-7=A(2x+1)+B(x-2)$ (3)

Put $x-2=0 \Rightarrow x=2$ in (3)

$2-7=A(2(2)+1)+B(0) \Rightarrow -5=A(4+1) \Rightarrow -5=5A \Rightarrow \boxed{A=1}$

Put $2x+1=0 \Rightarrow x=-\frac{1}{2}$ in (3)

$-\frac{1}{2}-7=A(0)+B\left(\frac{-1}{2}-2\right) \Rightarrow \frac{-5}{2}=B\left(\frac{-5}{2}\right) \Rightarrow B=\frac{15^{3}}{2}\times\frac{2}{5} \Rightarrow \boxed{B=3}$

(2) becomes

$$\frac{x-7}{(x-2)(2x+1)}=\frac{-1}{x-2}+\frac{3}{2x+1}$$

$$\int\frac{x-7}{(x-2)(2x+1)}dx=-\int\frac{1}{x-2}dx+3\int\frac{1}{2x+1}dx$$

$$=-\int\frac{1}{x-2}dx+\frac{3}{2}\int\frac{1}{2x+1}dx$$

$$=-ln|x-2|+\frac{3}{2}ln|2x+1|+c$$

9. $\int\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx$

Let $\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}$ (1)

'X' by $(x-1)(x-2)(x-3)$ we get

$3x^2-12x+11=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-3)$ (2)

Put $x-1=0 \Rightarrow x=1$ in (2)

$3(1)^2-12(1)+11A(1-2)(1-3)+B(0)+C(0)$

$3-12+11=A(-1)(-2) \Rightarrow 2=2A \Rightarrow \boxed{A=1}$

Put $x-2=0 \Rightarrow x=2$ in (2)

$3(2)^2-12(2)+11A(0)+B(2-1)(2-3)+C(0)$

$3(4)-24+11=B(1)(-1) \Rightarrow 12=24+11 \Rightarrow -1=-B \Rightarrow \boxed{B=1}$

Put $x-3=0 \Rightarrow x=3$ *in* (2)

$3(3)^2-12(3)+11A(0)+B(0)+C(3-1)(3-2)$

$3(9)-36+11=C(2)(1) \Rightarrow 27=36+11=2C \Rightarrow 2=2C \Rightarrow \boxed{C=1}$

(1) *becomes* $\dfrac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\dfrac{1}{x-1}+\dfrac{1}{x-2}+\dfrac{1}{x-3}$

$\int\dfrac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx=\int\left(\dfrac{1}{x-1}+\dfrac{1}{x-2}+\dfrac{1}{x-3}\right)dx$

$=\int\dfrac{1}{x-1}dx+\int\dfrac{1}{x-2}dx+\int\dfrac{1}{x-3}dx$

$=ln|x+1|+n|x-2|+ln|x-3|$

$=ln(x-1)(x-2)(x-3)+c$

10. $\displaystyle\int\frac{2x-1}{x(x-1)(x-3)}dx$

Let $\dfrac{2x-1}{x(x-1)(x-3)}=\dfrac{A}{x}+\dfrac{B}{x-1}+\dfrac{C}{x-3}$ (1)

'X' by $x(x-1)(x-3)$ we get

$2x-1=A(x-1)(x-3)+Bx(x-3)+Cx(x-1)$ (2)

Put $x=0$ *in* (2)

$-1=A(0-1)(0-3) \Rightarrow -1=3A \Rightarrow \boxed{A=\dfrac{-1}{3}}$

Put $x-1=0 \Rightarrow x=1$ *in* (2)

$2(1)-1=A(0)+B(1)(1-3) \Rightarrow 2-1=B(-2) \Rightarrow 1=2B \Rightarrow \boxed{B=\dfrac{-1}{2}}$

Put $x-3=0 \Rightarrow x=3$ *in* (2)

$2(3)-1=A(3-1)(3-3)+B3(3-3)+C3(3-1)$

$6-1=0+0+3c(2) \Rightarrow 5=6c \Rightarrow \boxed{c=\dfrac{5}{6}}$

(1) *becomes* $\dfrac{2x-1}{x(x-1)(x-3)}=\dfrac{\frac{-1}{3}}{x}+\dfrac{\frac{-1}{2}}{x-1}+\dfrac{\frac{5}{6}}{x-3}$

$$\Rightarrow \int\frac{2x-1}{x(x-1)(x-3)}dx = \frac{-1}{3}\int\frac{1}{x}dx - \frac{1}{2}\int\frac{1}{x-1}dx + \frac{5}{6}\int\frac{1}{x-3}dx$$

$$= -\frac{1}{3}ln|x| - \frac{1}{2}ln|x-1| + \frac{5}{6}ln|x-3| + c$$

11. $$\int\frac{5x^2+9x+6}{(x^2-1)(2x+3)}dx = \int\frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)}dx$$

Let $$\frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3} \qquad (1)$$

$'X'$ by $x(x-1)(x+1)(2x+3)$ we get

$$= A(x+1)(2x+3) + B(x-1)(2x+3) + C(x-1)(x+1) \qquad (2)$$

Put $x-1=0 \quad \Rightarrow \quad x=1$ in (2)

$$5(1)+9(1)+6 = A(1+1)(2(1)+3)+0+0 \Rightarrow 5+9+6 = A(2)(5)$$

$$20 = 10A \Rightarrow \boxed{A=2}$$

Put $x+1=0 \quad \Rightarrow \quad x=-1$ in (2)

$$5(-1)^2 + 9(1) + 6 = A(0) + B(-1-1)(-2+3) + 0$$

$$5-9+6 = 0 + B(-2)(1) + 0 \Rightarrow 2 = -2B \Rightarrow \boxed{B=1}$$

Put $2x+3=0 \quad \Rightarrow \quad x = \frac{-1}{2}$ in (2)

$$5\left(\frac{-3}{2}\right)^2 + 9\left(\frac{-3}{1}\right) + 6 = A(0) + B(0) + C\left(-\frac{3}{2}-1\right)\left(-\frac{-3}{2}+1\right)$$

$$5\left(\frac{9}{4}\right) - \frac{27}{2} + 6 = C\left(\frac{-5}{2}\right)\left(\frac{-1}{2}\right) \Rightarrow \frac{45-54+24}{4} = \frac{5}{4}C \Rightarrow \frac{15}{4}\times\frac{4}{5} = C \Rightarrow \boxed{C=3}$$

(1) becomes

$$\frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)} = \frac{2}{x-1} + \frac{-1}{x+1} + \frac{3}{2x+3}$$

$$\Rightarrow \int\frac{5x^2+9x+6}{(x-1)(x+1)(2x+3)}dx = 2\int\frac{1}{x-1}dx - \int\frac{1}{x+1}dx + 3\int\frac{1}{2x+3}dx$$

$$= 2ln|x-1| - ln|x+1| + \frac{3}{2}\int\frac{2}{2x+3}dx$$

$$= 2ln|x-1| - ln|x+1| + \frac{3}{2}ln|2x+3| + c$$

12. $\int \frac{4+7x}{(x-1)^2(2x+3x)}dx$

Let $\frac{4+7x}{(x-1)^2(2x+3x)} = \frac{A}{(x+1)} + \frac{B}{(1+x)^2} + \frac{C}{(2+3x)}$ (1)

$'X'$ both sides by $(x+1)^2(2x+3)$ we get

$4+7x = A(1+x)(2+3x) + B(2+3x) + C(1+x)^2$ (2)

Put $1+x=0 \Rightarrow x=-1$ in (2)

$4+7(-1) = A(0) + B(2+3(-1)) + C(0) \Rightarrow 4-7 = B(2-3) \Rightarrow -3 = -B \Rightarrow \boxed{B=1}$

Put $2+3x=0 \Rightarrow x=\frac{-2}{3}$ in (2)

$4+7\left(\frac{-2}{3}\right) = A(0) + B(0) + C\left(1-\frac{2}{3}\right)^2 \Rightarrow 4-\frac{14}{3} = C\left(\frac{1}{3}\right)^2 \Rightarrow \frac{-2}{3} = \frac{C}{9}$

$\Rightarrow C = \frac{-2}{\not 3}\times \not 9^3 \Rightarrow \boxed{C=-6}$

Rearranging (2)

$4+7x = A(2+3x+2x+3x^2) + 2B + 3Bx + C + 2Cx + Cx^2$

$4+7x = 2A++5Ax+3Ax^2+2B+3Bx+C+2Cx+Cx^2$

$4+7x = (3A+C)x^2 + (5A+3B+2C)x + (2B+C)$

Comparing Co-efficient

$x^2; 0 = 3A+C \Rightarrow 0 = 3A-6 \Rightarrow 6 = 3A \Rightarrow \boxed{A=2}$

(1) becomes

$\frac{4+7x}{(1+x)^2(2+3x)} = \frac{2}{1+x} + \frac{1}{(1+x)^2} + \frac{-6}{2+3x}$

$\int \frac{4+7x}{(1+x)^2(2+3x)}dx = 2\int \frac{1}{1+x}dx + \int \frac{1}{(1+x)^2}dx + -6\int \frac{1}{2+3x}dx$

$= 2\int \frac{1}{1+x}dx + \int (1+x)^{-2}dx - 2\int \frac{3}{2+3x}dx$

$= 2ln|1+x| + \frac{(1+x)^{-2+1}}{-2+1} - 2ln|2+3x| + c$

$= 2ln|1+x| - \frac{1}{(1+x)} - 2ln|2+3x| + c$

13. $\int \frac{x^2}{(x-1)^2(x+1)}dx$ **(Sargodha 2009)**

Let $\frac{2x^2}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$ (1)

$'X'$ *both sides by* $(x-1)^2(x+1)$ *we get*

$2x^2 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$ (2)

Put $1-x=0 \Rightarrow x=1$ *in* (2)

$2(1)^2 = A(0)+B(1+1)+C(0) \Rightarrow 2=2B \Rightarrow \boxed{B=1}$

Put $1+x=0 \Rightarrow x=-1$ *in* (2)

$2(1)^2 = A(0)+B(1+1)+C(0) \Rightarrow 2=2B \Rightarrow \boxed{B=1}$

$2(-1)^2 = A(0)+B(0)+C(-1-1)^2 \Rightarrow 2(1)=C(-2)^2 \Rightarrow 2=4C \Rightarrow \boxed{C=\frac{1}{2}}$

Rearranging (2)

$2x^2 = Ax^2 - A + Bx + B + Cx^2 - 2Cx + C$

$2x^2 = (A+C)x^2 + (B-2c)x + (-A+B+C)$

Comparing Co-efficient

$x^2; 2 = A+C \Rightarrow 2 = A+\frac{1}{2} \Rightarrow A = 2-\frac{1}{2} \Rightarrow \boxed{A=\frac{3}{2}}$

(1) becomes

$$\frac{2x^2}{(x-1)^2(x+1)} = \frac{\frac{3}{2}}{x-1} + \frac{1}{(x-1)^2} + \frac{\frac{1}{2}}{x+1}$$

$$\Rightarrow \int \frac{2x^2}{(x-1)^2(x+1)}dx = \frac{3}{2}\int\frac{1}{x-1}dx + \int (x-1)^2 dx + \frac{1}{2}\int\frac{1}{x+1}dx$$

$$= \frac{3}{2}ln|x-1| + \frac{(x-1)^{-1}}{-1} + \frac{1}{2}ln|x+1| = \frac{3}{2}ln|x-1| - \frac{1}{(x-1)} + \frac{1}{2}ln|x+1| + c$$

14. $\int \frac{1}{(x-1)(x+1)^2}dx$

Let $\frac{1}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ *I*

$'X'\ by (x-1)(x+1)^2\ we\ get$

$$1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1) \qquad II$$

Put $1-x=0 \quad \Rightarrow \quad x=1\ in\ II$

$$1 = A(1+1)^2 + B(0) + C(0) \Rightarrow 1 = A(2)^2 \Rightarrow 1 = 4A \Rightarrow \boxed{A = \frac{1}{4}}$$

Put $1+x=0 \quad \Rightarrow \quad x=-1\ in\ II$

$$1 = A(0) + B(0) + C(-1-1) \Rightarrow 1 = 2C \Rightarrow 1 = 4A \Rightarrow \boxed{C = \frac{-1}{2}}$$

Rearranging II

$$1 = Ax^2 2Ax^2 + A + Bx^2 - B + Cx - C \Rightarrow 1 = (A+B)x^2 + (2A+C)x + (A-B-C)$$

x^2 ; Comparing Co-efficient

$$0 = A + B \Rightarrow 0 = \frac{1}{4} + B \Rightarrow \boxed{B = -\frac{1}{4}}$$

$$I\ become \frac{1}{(x-1)(x+1)^2} = \frac{\frac{1}{4}}{x-1} + \frac{\frac{-1}{4}}{x+1} + \frac{\frac{-1}{2}}{(x+1)^2}$$

$$\int \frac{1}{(x-1)(x+1)^2}dx = \frac{1}{4}\int \frac{1}{x+1}dx - \frac{1}{2}\int \frac{1}{(x+1)^2}dx$$

$$= \frac{1}{4}ln|x-1| - \frac{1}{4}ln|x+1| - \frac{1}{2}\int (x+1)^{-1}dx$$

$$= \frac{1}{4}ln|x-1| - \frac{1}{4}ln|x+1| - \frac{1}{2}\frac{(x+1)^{-2+1}}{-2+1} + c$$

$$= \frac{1}{4}ln|x-1| - \frac{1}{4}ln|x+1| - \frac{1}{2}\frac{(x+1)^{-1}}{1} + c$$

$$= \frac{1}{4}ln|x-1| - \frac{1}{4}ln|x+1| + \frac{1}{2}\frac{1}{(x+1)} + c$$

15. $\int \frac{x+4}{x^3 - 3x^2 + 4}dx \qquad x = -1 \quad x+1 = 0$

By synthetic division so

-1	1	-3	0	4
		-1	4	-4
	1	-4	4	0

$$x^3 - 3x^2 + 4 = (x+1)(x^2 - 4x + 4)$$

$$= (x+1)(x-2)^2$$

So $\int \frac{x+4}{x^3-3x^2+4}dx = \int \frac{1}{(x+1)(x-2)^2}dx$

Let $\frac{1}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$ (1)

$'X'$ $by(x+1)(x-2)^2$ *we get*

$2x^2 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$ (2)

Put $1+x=0 \Rightarrow x=-1$ *in* (2)

$1 = A(-1-2)^2 + B(0) + C(0) \Rightarrow 1 = A(-3)^2 \Rightarrow 1 = 9A \Rightarrow \boxed{A = \frac{1}{9}}$

Put $x-2=0 \Rightarrow x=2$ *in* (2)

$1 = A(0) + B(0) + C(2+1) \Rightarrow 1=1=3C \Rightarrow \boxed{C = \frac{1}{3}}$

Rearranging (2)

$1 = Ax^2 - 4Ax + 4A + Bx^2 + 2Bx + Bx - 2B + Cx + C$

Or $1 = Ax^2 - 4Ax + 4A + Bx^2 + 2Bx + Bx - 2B + Cx + C$

$1 = (A+B)x^2 + (-4A - 2B + B + C)x + (4A - 2B + C)$

Comparing Coefficient

$x^2; 0 = A + B \Rightarrow 0 = \frac{1}{9} + B \Rightarrow \boxed{B = -\frac{1}{9}}$

Put values in (1)

$$\frac{1}{(x+1)(x+2)^2} = \frac{\frac{1}{9}}{x+1} + \frac{\frac{-1}{9}}{x-2} + \frac{\frac{1}{3}}{(x-2)^2}$$

$$\int \frac{1}{(x+1)(x+2)^2}dx = \frac{1}{9}\int \frac{1}{x+1}dx + \int \frac{1}{x-2}dx + \frac{1}{3}\int (x-2)^{-2}dx$$

$$= \frac{1}{9}ln|x+1| - \frac{1}{9}ln|x-2| + \frac{1}{3}\frac{(x-2)^{-1}}{-1} + C$$

$$= \frac{1}{9}\left[ln\left|\frac{x+1}{x-2}\right|\right] - \frac{1}{3(x-2)} + C$$

16. $\int \frac{x^3-6x^2+25}{(x+1)^2(x-2)^2}dx$

Let $\frac{x^3-6x^2+25}{(x+1)^2-(x-2)^2} = \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x-2)}+\frac{D}{(x-2)^2}$ (1)

$'X'$ *both sides by* $(x+1)^2(x-2)^2$ *we get*

$x^3-6x^2+25 = A(x+1)(x-2)^2+B(x-2)^2+C(x+1)^2+D(x+1)^2$ (2)

Put $1+x=0 \Rightarrow x=-1$ *in* (2)

$(-1)^3+6(-1)^2+25 = A(0)+B(-1-2)^2+C(0)+D(0)$

Put $x-2=0 \Rightarrow x=2$ *in* (2)

$(-1)^3-6(-1)^2+25 = A(0)+B(-1-2)^2+C(0)+D(0)$

$-1-6+25 = B(-3)^2 \Rightarrow 18=9B \Rightarrow \boxed{B=2}$

Put $x-2=0 \Rightarrow x=2$ *in* (2)

$(2)^3-6(2)^2+25 = A(0)+B(0)+C(0)+D(2+1)^2$

$8-24+25 = D(3)^2 \Rightarrow 9=9D \Rightarrow \boxed{D=1}$

Rearranging (2)

$x^3-6x^2+25 = A(x+1)(x^2-4x+4)+B(x^2-4x+4)+C(x^2+2x+1)(x-2)+D(x^2+2x+1)$

$x^3-6x^2+25 = Ax^3-4Ax^2+\cancel{4Ax}+Ax^2-\cancel{4Ax}+4A+Bx^2-4Bx+4B+cx^3+\cancel{2cx^2}$
$+cx-\cancel{2cx^2}-4cx-2C+Dx^2+2Dx+D$

$x^3-6x^2+25 = (A+C)x^3+(-4A+A+B+D)x^2+(-4B+C-4C+2D)x+(4A+4B-2C+D)$

Comparing Coefficient

$x^3;\quad 1 = A+C$ (3)

$x^2;\quad -6 = -4A+A+B+D \Rightarrow -6=-3A+B+D$

Put values of B & D - 6 = -3A +2+1

$\Rightarrow -6=-3A \Rightarrow -9=-3A \Rightarrow \boxed{A=3}$

Put in (3) $1=3+C \Rightarrow \boxed{C=-2}$

(i) become

$$\frac{1}{(x+1)^2(x-1)^2} = \frac{3}{x+1}+\frac{2}{(x+1)^2}+\frac{-2}{x-2}+\frac{1}{(x-2)^2}$$

$$\int\frac{1}{(x+1)^2(x-1)^2}dx = 3\int\frac{1}{x+1}dx + 2\int(x+1)^{-2}dx - 2\int\frac{1}{x-2}dx + \int(x-2)^{-2}dx$$

$$= 3ln|x+1| + 2\frac{(x+1)^{-1}}{-1} - 2ln|x-2| + \frac{(x-2)^{-1}}{-1} + C$$

17. $$\int\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3}dx$$

Let $$\frac{x^3+22x^2+14x-17}{(x-3)-(x+2)^3} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3} \quad (1)$$

'X' both sides by $(x-3)(x+2)^3$ we get $x^3+22x^2+14x-17$

$$= A(x+2)^3 + B(x-3)(x+2)^2 + C(x-3)(x+2) + D(x-3) \quad (2)$$

Put $x-3=0 \quad \Rightarrow \quad x=3$ in (2)

$$(3)^3 + 22(3)^2 + 14(3) - 17 = A(3+2)^2 + B(0)^2 + C(0) + D(0)$$

$$27 + 22(9) + 42 - 17 = A(5)^3 \quad \Rightarrow \quad 27+198+42-17 = 125A$$

$$-50 = 125A \quad \Rightarrow \quad \boxed{A=2}$$

Put $x+2=0 \quad \Rightarrow \quad x=-2$ in (2)

$$(-2)^3 + 22(-2)^2 + 14(-2) - 17 = A(0) + B(0) + C(0) + D(-2-3)$$

$$-8 + 22(4) - 28 - 17 = -5D \quad \Rightarrow \quad 35 = -5D \quad \Rightarrow \quad \boxed{D=-7}$$

Rearranging (2)

$$x^3+22x^2+14x-17 = A(x^3+6x^2+12x+8) + B(x-3)(x^2+4x+4) + C(x^2+2x-3x-6) + D(x-3)$$

$$x^3+22x^2+14x-17 = Ax^3+6Ax^2+12Ax+8A+Bx^3+4Bx^2+4Bx-3Bx^2-12Bx -12B+Cx^2-Cx-6C+Dx-3D$$

Comparing Coefficient

x^3; $\quad 1 = A+B \quad \Rightarrow \quad 1 = 2+B \quad \Rightarrow \quad \boxed{B=-1}$

x^2; $\quad 22 = 6A+4B-3B+C \quad \Rightarrow \quad 2 = 6A+B+C \quad (3)$

$14 = 12A-8B-C+D$

x^2; $\quad 14 = 12A+4B-12B-C+D \quad \Rightarrow \quad 14 = 12(2)-8(-1)-C-7$

$\Rightarrow \quad 14 = 24+8-7-C \quad \Rightarrow \quad 14 = 25-C \quad \Rightarrow \quad 14-25-C \quad \Rightarrow \quad \boxed{C=11}$

Put values in (1)

$$\frac{x^3+22x^2+14x+17}{(x-3)(x+2)^3}=\frac{2}{x-3}+\frac{-1}{x+2}+\frac{11}{(x+2)^2}+\frac{-7}{(x+2)^2}$$

$$\int\frac{x^3+22x^2+14x+17}{(x-3)(x+2)^3}dx=2\int\frac{1}{x-3}dx-1\int\frac{1}{x+2}dx+11\int(x+2)^{-2}dx-7\int(x+2)^{-3}dx$$

$$=2\ ln|x-3|-ln|x+2|+11\frac{(x+2)^{-1}}{-1}-\frac{7(x+2)^{-2}}{-2}+C$$

$$=2\ ln|x-3|-ln|x+2|-\frac{11}{(x+2)}+\frac{7}{2}\frac{1}{(x+2)^2}+C$$

18. $\int\frac{x-2}{(x+1)(x^2+1)}dx$

Let $\frac{x-2}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$ (1)

$'X'\ both\,sides\ by\,(x+1)(x^2+1)^3\ We\ get$

$x-2=A(x^2+1)+(Bx+C)(x+1)$ (2)

Put $x+1=0 \Rightarrow x=-1\ in\ (2)$

$-1-2=A((-1)^2+1)+(Bx+C)(0) \Rightarrow -3=2A \Rightarrow \boxed{A=\frac{-3}{2}}$

Rearranging (2)

$x-2=Ax^3+A+Bx^2+Bx+Cx+C \Rightarrow x-2=(A+B)x^2+(B+C)x+(A+C)$

Comparing Coefficient

$x^2;\ 0=A+B \Rightarrow 0=\frac{3}{2}+B \Rightarrow \boxed{B=\frac{3}{2}}$

C =

$x^2;\ 1=A+B \Rightarrow 1=\frac{3}{2}+C \Rightarrow C=1-\frac{3}{2}=\frac{-1}{2} \Rightarrow \boxed{B=\frac{3}{2}}$

$Put\ values\ in\ (1)\ \frac{x-2}{(x+1)(x^2+1)}=\frac{\frac{-3}{2}}{x+1}+\frac{\frac{3}{2}x-\frac{1}{2}}{x^2+1}=\frac{\frac{3x-1}{2}}{x^2+1}$

$$\Rightarrow \int\frac{x-2}{(x+1)(x^2+1)}dx=\frac{-3}{2}\int\frac{1}{x+1}dx+\frac{1}{2}\int\frac{3x-1}{x^2+1}dx$$

$$=\frac{-3}{2}ln|x+1|+\frac{1}{2}\int\frac{1}{x+1}dx-\frac{1}{2}\int\frac{3x-1}{x^2+1}dx$$

$$=\frac{-3}{2}ln|x+1|+\frac{3}{4}\int\frac{2x}{x^2+1}dx-\frac{1}{2}\int\frac{1}{1+x^2}dx$$

$$=\frac{-3}{2}ln|x+1|+\frac{3}{4}ln|x^2+1|-\frac{1}{2}Tan^{-1}x+C$$

19. $$\int\frac{x}{(x-1)(x^2+1)}dx \qquad \text{I}$$

Now $$\frac{x}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

Multiply both sides by $(x-1)(x^2+1)$ we get

$$x=A(x^2+1)+(Bx+C)(x-1) \qquad \text{II}$$

Put $x-1=0 \Rightarrow x=1 \text{ in } II \Rightarrow 1=A(1^2+1)+0$

$\Rightarrow 1=2A \Rightarrow \boxed{A=\frac{1}{2}}$ Re-arrange II

$$x=Ax^2+A+Bx^2-Bx+Cx-C \Rightarrow x=(A+B)x^2+(-B+C)x+(A-C)$$

Comparing co-efficient

$x^2;0=A+B$ III

$x;1=-B+C$ IV

Constant; $0=A-C$ V

Put value of A in III

$0=\frac{1}{2}+B \Rightarrow \boxed{B=\frac{1}{2}}$

Put value of A in V

$0=\frac{1}{2}-C \Rightarrow \boxed{C=\frac{1}{2}}$

I become

$$\int\frac{x}{(x-1)(x^2+1)}dx=\int\left(\frac{\frac{1}{2}}{x-1}+\frac{\frac{-1}{2}x+\frac{1}{2}}{x^2+1}\right)dx$$

$$=\frac{1}{2}\int\frac{1}{x-1}dx+\int\frac{-x+1}{2(x^2+1)}dx$$

$$=\frac{1}{2}ln|x-1|+\frac{1}{2}\int\frac{-x+1}{x^2+1}dx$$

$$=\frac{1}{2}ln|x-1|-\frac{1}{4}\int\frac{2x^2-2}{x^2+1}dx$$

$$=\frac{1}{2}ln|x-1|-\frac{1}{4}\int\frac{2x}{x^2+1}dx-\frac{1}{4}\int\frac{-2}{1+x^2}dx$$

$$=\frac{1}{2}ln|x-1|-\frac{1}{4}ln|x^2+1|+\frac{1}{2}\int\frac{1}{1+x^2}dx$$

$$=\frac{1}{2}ln|x-1|-\frac{1}{4}ln|1+x^2|+\frac{1}{2}Tan^{-1}x+c$$

20. $\int\frac{9x-7}{(x+3)(x^2+1)}dx$ I

Now $\frac{9x-7}{(x+3)(x^2+1)}=\frac{A}{x+3}+\frac{Bx+C}{x^2+1}$

'X' by $(x+3)(x^2+1)$ II

$9x-7=A(x^2+1)+(Bx+C)(x+3)$

Put $x+3=0 \Rightarrow x=-3$ in II

$9(-3)-7=A((-3)^2+1)+(Bx+C)(0)$

$-27-7=A(10)+0$

$-34=10A \Rightarrow \boxed{A=\frac{-17}{5}}$

Re-arrange II

$9x-7=Ax^2+A+Bx^2+3Bx+Cx+3C$

Or $9x-7=(A+B)x^2+(3B+C)x+(A+3C)$

Comparing co-efficient

$x^2; 0=A+B$ III

$x; 9=3B+C$ IV

Constant ; $-7=A+3C$ V

Put value of A in III

$0=\frac{-17}{5}+B \Rightarrow \boxed{B=\frac{17}{5}}$

Put value of A in V

$0=-\frac{17}{5}+3C \Rightarrow 3C=-7+\frac{17}{5}$

$$-3C = \frac{-35+17}{5} = \frac{-18}{5}$$

$$\boxed{C = \frac{-6}{5}}$$ Put *in* I

$$\int \frac{9x-7}{(x+3)(x^2+1)}\,dx = \int\left(\frac{\frac{-17}{5}}{x+3} + \frac{\frac{17}{5}x - \frac{6}{5}}{x^2+1}\right)dx$$

$$= \frac{-17}{5}\int \frac{1}{x+3}\,dx + \int\left(\frac{17x-6}{5(x^2+1)}\right)dx$$

$$= \frac{-17}{5} ln|x+3| + \frac{17}{5}\int \frac{x - \frac{6}{17}}{x^2+1}dx$$

$$= \frac{-17}{5} ln|x+3| + \frac{17}{10}\int \frac{2x - \frac{17}{17}}{x^2+1}dx$$

$$= \frac{-17}{5} ln|x+3| + \frac{17}{10}\int \frac{2x}{x^2+1}dx + \frac{17}{10}\left(\frac{17}{10}\right)\int \frac{1}{1+x^2}dx$$

$$= \frac{-17}{5} ln|x+3| + \frac{17}{10} ln\left|1+x^2\right| - \frac{6}{5}Tan^{-1}x + C$$

21. $\displaystyle\int \frac{1+4x}{(x-3)(x^2+4)}dx$

Let $$\frac{1+4x}{(x-3)(x^2+4)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+4} \qquad (1)$$

'X' both sides by $(x-3)(x^2+4)$ we get

$$1+4x = A(x^2+4) + (Bx+C)(x-3) \qquad (2)$$

Put $x-3=0 \quad \Rightarrow \quad x = 3$ *in* (2)

$$1+4\left((3)^2+4\right) + (Bx+C)(0) \;\Rightarrow\; 13 = A(13) \;\Rightarrow\; A = \frac{13}{13} = 1 \;\boxed{A=1}$$

Rearranging $\quad 1+4x = Ax^2 + 4A + Bx^2 - 3Bx + Cx - 3C$

$$1+4x = (A+B)x^2 + (-3B+C)x + (4A-3C)$$

Comparing co-efficient

$x^2;\quad 0 = A+B \;\Rightarrow\; 0 = 1+B \;\Rightarrow\; \boxed{B=1}$

$x;\quad 4=-3B+C \Rightarrow 4=-3(-1)+C+4=3+C \Rightarrow \boxed{C=1}$

Put in (1) $\dfrac{1+4x}{(x-3)(x^2+4)}=\dfrac{1}{x-3}+\dfrac{-1(x)+1}{x^2+4}$

$$\int\frac{1+4x}{(x-3)(x^2+4)}dx=\int\frac{1}{x-3}dx+\int\frac{-1x+1}{x^2+4}dx$$

$$=ln|x-3|-\int\frac{x}{x^2+4}dx+\int\frac{1}{x^2+4}dx$$

$$=ln|x-3|-\frac{1}{2}\int\frac{2x}{x^2+4}dx+\int\frac{1}{x^2+(2)^2}dx$$

$$=ln|x-3|-\frac{1}{2}ln|x^2+4|+\frac{1}{2}Tan^{-1}\frac{x}{2}+c$$

22. $$\int\frac{12}{x^3+8}dx=\int\frac{12}{x^3+(2)^3}dx$$

$$=\int\frac{12}{(x+2)(x^2-2x+4)}dx$$

Let $$=\frac{12}{(x+2)(x^2-2x+4)}=\frac{A}{x+2}+\frac{Bx+C}{x^2-2x+4}\qquad \text{I}$$

'X' both sides by $(x+2)(x^2-2x+4)$ we get

$$12=A(x^2-2x+4)+(Bx+C)(x+2)\qquad \text{II}$$

Put in (1) $\dfrac{1+4x}{(x-3)(x^2+4)}=\dfrac{1}{x-3}+\dfrac{-1(x)+1}{x^2+4}$

Put $x+2=0 \Rightarrow x=-2$ we get

$12=A((-2)^2-2(-2)+4)+0 \Rightarrow 12=A(4+4+4)\Rightarrow 12=12A \Rightarrow \boxed{A=1}$

Rearrange II

$12=Ax^2-2Ax+4A+Bx^2+2Bx+Cx+2C$ or

$12=(A+B)x^2+(-2A+2B+C)x+(4A+2C)$

Comparing co-efficient

$x^2;\quad 0=A+B \Rightarrow 0=1+B \Rightarrow \boxed{B=1}$

$x;\quad 0=-2A+2b+C \Rightarrow 0=-2(1)+2(-1)+C$

$0=-2-2+C \Rightarrow 0=-4+C \Rightarrow \boxed{C=4}$

I become

$$\frac{12}{(x+2)(x^2-2x+4)}=\frac{1}{x+2}+\frac{-x+4}{x^2-2x+4}$$

$$\Rightarrow \int \frac{12}{(x+2)(x^2-2x+4)}dx=\int \frac{1}{x+2}dx-\int \frac{x}{x^2-2x+4}dx+4\int \frac{1}{x^2-2x+4}dx$$

$$=ln|x+2|-\frac{1}{2}\int \frac{2x}{x^2-2x+4}dx+4\int \frac{1}{x^2-2x+4}dx$$

$$=ln|x+2|-\frac{1}{2}\int \frac{2x-2+2}{x^2-2x+4}dx+4\int \frac{1}{x^2-2x+4}dx$$

$$=ln|x+2|-\frac{1}{2}\int \frac{2x-2}{x^2-2x+4}dx-\frac{1}{\cancel{2}}\times \cancel{2}\int \frac{1}{x^2-2x+4}dx+4\int \frac{1}{x^2+2x+4}dx$$

$$=ln|x+2|-\frac{1}{2}ln|x^2-2x+4|+3\int \frac{1}{x^2-2x+1+3}dx$$

$$=ln|x+2|-\frac{1}{2}ln|x^2-2x+4|+3\int \frac{1}{(x-1)^2+(\sqrt{3})^2}dx$$

$$=ln|x+2|-\frac{1}{2}ln|x^2-2x+4|+\cancel{3}\frac{1}{\cancel{\sqrt{3}}}Tan^{-1}\frac{x-1}{\sqrt{3}}+C$$

$$=ln|x+2|-\frac{1}{2}ln|x^2-2x+4|+\sqrt{3}Tan^{-1}\frac{x-1}{\sqrt{3}}+C$$

23. $$\int \frac{9x+6}{x^3-8}dx=\int \frac{9x+6}{x^3-(2)^3}dx$$

$$=\int \frac{9x+6}{(x-2)(x^2+2x+4)}dx \qquad \text{I}$$

Now $$=\frac{9x+6}{(x-2)(x^2+2x+4)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4}$$

'X' by $(x-2)(x^2+2x+4)$

$$9x+6=A(x^2+2x+4)+(Bx+C)(x-2) \qquad \text{II}$$

Put $x-2=0 \quad \Rightarrow \quad x=2$

$$9(2)+6=A(2^2+2(2)+4)+0$$

$$24=A(12) \quad \Rightarrow \quad \boxed{A=2}$$

Rearrange II

$$9x+6=Ax^2+2Ax+4A+Bx^2-2Bx+Cx-2C$$

$$9x+6=(A+B)x^2+(2A-2B+C)x+(4A-2C)$$

Comparing co-efficient

$x^2;\quad 0 = A + B \quad\quad$ III

$x;\quad 9 = 2A - 2B \quad\quad$ IV

Constant ; $6 = 4A - 2C \quad\quad$ V

Put value of A in III

$0 = 2 + B \quad \Rightarrow \quad \boxed{B = 2}$

Put value of A in IV

$6 = 2(4) - 2C \Rightarrow \quad 6 = 8 - 2C$

$-2 = -2C \quad \Rightarrow \quad \boxed{C = 1}$

I become

$$\int \frac{9x+6}{x^3-8}dx = \int \left(\frac{2}{x-2} + \frac{-2x+1}{x^2+2x+4}\right)dx$$

$$= 2\int \frac{1}{x-2}dx = \int \frac{2x-1+2-2}{x^2+2x+4}dx$$

$$= 2ln|x-2| - \int \frac{2x+2}{x^2+2x+4}dx - \int \frac{-3}{x^2+2x+4}dx$$

$$= 2ln|x-2| - ln|x^2+2x+4| + 3\int \frac{1}{x^3+2x+1+3}dx$$

$$= 2ln|x-2| - ln|x^2+2x+4| + 3\int \frac{1}{(x+1)^2 + (\sqrt{2})^2}dx$$

$$= 2ln|x-2| - ln|x^2+2x+4| + 3.\frac{1}{\sqrt{3}}Tan^{-1}\frac{x+1}{\sqrt{3}} + C$$

$$= 2ln|x-2| - ln|x^2+2x+4|\sqrt{3}Tan^{-1}\frac{x+1}{\sqrt{3}} + C$$

24. $\displaystyle\int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)}dx$

Let $\displaystyle\frac{2x^2+5x+3}{(x-1)^2(x^2+4)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+4} \quad\quad$ I

'X' by $(x-2)(x^2+2x+4)$

$2x^2+5x+3 = A(x-1)(x^2+4) + B(x^2+4) + (Cx+D)(x-1)^2 \quad\quad$ II

Put $x - 1 = 0 \quad \Rightarrow \quad x = -1$ in II

$2(1)^2 + 5(1) + 3 = 0 + B(1+4) + 0 \Rightarrow 2+5+3 = 5B \Rightarrow 10 = 5B \Rightarrow \boxed{B = 2}$

Rearrange II

$$2x^2+5x+3=A\left(x^2+4x-x^2-4\right)B\left(x^2+4\right)+\left(Cx+D\right)\left(x^2-2x-1\right)$$

$$2x^2+5x+3=Ax^3+4Ax-Ax^2-4A+Bx^2+4B+Cx^3-2Cx^2+Cx+Dx^2-2Dx+D$$

$$2x^2+5x+3=\left(A+C\right)x^3+\left(-A+B-2C+D\right)x^2+4A+\left(-2D\right)x+\left(-4A+4B+D\right)$$

Comparing co-efficient

x^3; $\quad 0=A+C$ $\qquad$ III

$\boxed{C=-A}$

x^2; $\quad 2=-A+B-2C+D$ $\qquad$ IV

x ; $\quad 5=4A+C-2D$ $\qquad$ V

Constant ; $3=-4A+4B+D$ $\qquad$ VI

From II C=-A $\qquad$ VI

IV become $\quad 2=-A+B-2\left(-A\right)+D$

$2=-A+2+2A+D$

$2-2=A+D$

$0=A+D$ $\qquad$ VII

Put $B=2$ *in* VI

$3=-4A+4(2)+D \quad\Rightarrow\quad 3=-4A+8+D$

$3-8=-4A+D \quad\Rightarrow\quad -5=-4A+D$ VII

Solve VI & VII $\qquad -5=-4A+D$

$\qquad\qquad\qquad\qquad \underline{-0=-A\pm D}$

$\qquad\qquad\qquad\qquad -5=-5A \quad\Rightarrow\quad \boxed{A=1}$

From III $\qquad 0=1+C \quad\Rightarrow\quad \boxed{C=-1}$

From VII $\qquad 0=1+D \quad\Rightarrow\quad \boxed{D=-1}$

(1) become

$$\frac{2x^2+5x+3}{(x-1)^2\left(x^2+4\right)}=\frac{1}{x-1}\frac{2}{(x-1)^2}+\frac{-x-1}{x^2+4}$$

$$\int\frac{2x^2+5x+3}{(x-1)^2\left(x^2+4\right)}dx=\int\frac{1}{x-1}dx+2\int(x-1)^{-2}dx-\int\frac{x}{x^2+4}dx-\int\frac{1}{x^2+4}dx$$

$$=ln\left|x-1\right|+2\frac{(x-1)^{-1}}{-1}-\frac{1}{2}\int\frac{2x}{x^2+4}dx-\int\frac{1}{x^2+(2)^2}dx$$

$$=ln\left|x-1\right|+\frac{2}{(x-1)}-\frac{1}{2}ln\left|x^2+4\right|-\frac{1}{2}Tan^{-1}\frac{x}{2}+C$$

25. $\int \frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx$ I

Now $\frac{2x^2-x-7}{(x+2)^2(x^2+x+1)} = \frac{A}{x+2} + \frac{Bx+C}{(x+4)^2} + \frac{Cx+D}{x^2+x+1}$

Multiply by $(x+2)^2(x^2+x+1)$

$2x^2-x-7 = A(x+2)(x^2+x+1) + B(x^2+x+1) + (Cx+D)(x+2)^2$ II

Put $x+2=0 \Rightarrow x=-2$ *in* II

$2(-2)^2-(-2)-7 = 0 + B((-2)^2-2+1)+0$

$8+2-7 = B(3) \Rightarrow 3=3B \Rightarrow \boxed{B=1}$

Rearrange II

$2x^2-x-7 = Ax^3+Ax^2+Ax+2Ax^2+2Ax+2A+Bx^2+Bx+B+Cx^2+4Cx^2$
$+4Cx+Dx^2+4Dx+4D$

Comparing co-efficient

x^3; $0=A+C$ III

x^2; $2=A+2A+B+4C+D$ IV

x ; $-1=A+2A+B+4C+4D$ V

Constant ; $-7=2A+B+4D$ VI

V – IV

$2=3A+B+4C+D$

$-1=-3A\pm B\pm 4C\pm 4D$

$3=-3 \Rightarrow \boxed{D=-1}$

Put value of B and D in VI

$-7=2A-3+4(-1)$

$-7=2A-3 \Rightarrow -4=2A \Rightarrow \boxed{A=-2}$

Put value of A in III

$0=-2+C \Rightarrow \boxed{A=-2}$

I become

$\int \frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}dx = \int \left(\frac{-2}{x+2} + \frac{1}{(x+2)^2} + \frac{2x-1}{x^2+x+1}\right)dx$

$=2\int \frac{1}{x+2}dx + \int (x+2)^{-2}dx + \int \frac{2x-1+1+1}{x^2+x+1}dx$

$$= -2ln|x-2| + \frac{(x+1)^{-2+1}}{-2+1} + \int \frac{2x+1}{x^2+x+1}dx + \int \frac{-2}{x^2+x+1+\frac{1}{4}-\frac{1}{4}}dx$$

$$= -2ln|x-2| + \frac{1}{(x+2)} + ln|x^2+x+1| - 2\int \frac{1}{x^2+x+1}dx + \int \frac{-2}{x^2+x+\frac{1}{4}+\frac{1}{4}}dx$$

$$= -2ln|x+2| - \frac{1}{(x+2)} + ln|x^2+x+1| - 2\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$= -2ln|x+2| - \frac{1}{(x+2)} + ln|x^2+x+1| - 2.\frac{1}{\frac{\sqrt{3}}{2}}Tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + C$$

$$= -2ln|x+2| - \frac{1}{(x+2)} + ln|x^2+x+1| - 2\times\frac{2}{\sqrt{3}}Tan^{-1}\left(\frac{2x+1}{2}\times\frac{2}{\sqrt{3}}\right) + C$$

$$= -2ln|x+2| - \frac{1}{(x+2)} + ln|x^2+x+1| - \frac{4}{\sqrt{3}}Tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

26. $$\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx$$

Let $= \frac{3x+1}{(4x^2+1)(x^2 1x+1)} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{x^2-x+1}$ I

'X' by $(4x^2+1)(x^2-x+1)$ we get

$3x+1 = (Ax+B)(x^2-x+1) + (Cx+D)(4x^2+1)$

or $3x+1 = Ax^3 + Ax^2 + Ax + Bx^2 - Bx + B + 4Cx^2 + Cx + 4Dx^2 + D$

or $3x+1 = (A+4C)x^3 + (-A+B+4D)x^2 + (A-B+C)x + (B+D)$

Comparing Co-efficient

x^3; $0 = A+4C$ II

x^2; $0 = -A+B+4D$ III

x ; $3 = A-B+D$ IV

Constant ; $1 = B+D$ V

Adding III & IV $0 = -A+B+4D$

$3 = A - B + C$

$3 = C + 4D \Rightarrow 9 = 3C + 12D$ VI

From II A = - 4C put in IV

$3 = -4C + B + C \Rightarrow 3 = -B - 3C$ VII

Add V & VII $\quad 1 = B + D$

$$\begin{array}{l} 3 = -B \quad -3C \\ \hline 4 = -3C + D \quad \text{VII} \end{array}$$

Put Add VI & VIII $\quad 9 = 3C + 12D$

$$\begin{array}{l} 4 = -3C + \quad D \\ \hline 13 = 13D \end{array} \Rightarrow \boxed{D=1}$$

From VIII 4 = - 3C + 1

$\Rightarrow \quad -3C = 4 - 1 = 3 \quad \Rightarrow \quad \boxed{C = -1}$

From V $\quad 1 = B + 1 \quad \Rightarrow \quad \boxed{B = 0}$ From V $\quad 0 = A + 4(-1)$

$\Rightarrow \quad 0 = A - 4 \boxed{A = 4}$

I become

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{4x+0}{4x^2+1} + \frac{-x+1}{x^2-x+1}$$

$$\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}dx = \int \frac{4x}{4x^2+1}dx - \int \frac{x}{x^2-x+1}dx + \int \frac{1}{x^2-x+1}dx$$

$$= \frac{1}{2}\int \frac{8x}{4x^2+1}dx - \frac{1}{2}\int \frac{2x}{x^2-x+1}dx + \int \frac{1}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln\left|4x^2+1\right| - \frac{1}{2}\int \frac{2x-1+1}{x^2-x+1}dx + \int \frac{1}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln\left|4x^2+1\right| - \frac{1}{2}\int \frac{2x-1}{x^2-x+1}dx - \frac{1}{2}\int \frac{1}{x^2-x+1}dx + \int \frac{1}{x^2-x+1}dx$$

$$= \frac{1}{2}\ln\left|4x^2+1\right| - \frac{1}{2}\left|x^2-x+1\right| + \frac{1}{2}\int \frac{1}{x^2-x+\frac{1}{4}+1-\frac{1}{4}}dx$$

$$= \frac{1}{2}\ln\left|4x^2+1\right| - \frac{1}{2}\left|x^2-x+1\right| + \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$= \frac{1}{2}\ln\left|4x^2+1\right| - \frac{1}{2}\left|x^2-x+1\right| + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}Tan^{-1}\left(\frac{\frac{2x-1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \frac{1}{2}\ln\left|4x^2+1\right| - \frac{1}{2}\left|x^2-x+1\right| + \frac{1}{\not{2}}\cdot\frac{\not{2}}{\sqrt{3}}Tan^{-1}\left(\frac{2x-1}{\not{2}} \times \frac{\not{2}}{\sqrt{3}}\right) + C$$

$$= \frac{1}{2}\ln\left|4x^2+1\right| - \frac{1}{2}\left|x^2-x+1\right| + \frac{1}{\sqrt{3}}Tan^{-1}\frac{2x-1}{\sqrt{3}} + C$$

27. $\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}dx$

Now $\frac{4x+1}{(x^2+4)(x^2+4x+5)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+4x+5}$

Multiply by $(x^2+4)(x^2+4x+5)$

$4x+1 = (Ax+B)(x^2+4x+5)+(Cx+D)(x^2+4)$

$4x+1 = Ax^3+4Ax^2+5Ax+Bx^2+4Bx+5B+Cx^3+4Cx+Dx^2+4D$

Or $4x+1 = (A+C)x^3+(4A+B+D)x^2+(5A+4B+4C)x+(5B+4D)$

Comparing co-efficient

$x^3; \quad 0 = A+C \qquad II$

$x^2; \quad 0 = 4A+B+D \quad III$

$x; \quad 4 = 5A+4B+4D \quad IV$

$4\times III - V$

$0 = 16A+4B+4D$

$\underline{-1 = \qquad \pm 5B \pm 4D} \quad VI$

$-1 = 16A - B$

$\Rightarrow b = 1+16A \qquad VII$

From II , $C = -A \qquad VIII$

Put VII, VIII in IV

$4 = 5A+4(1+16A)-4A$

$4 = 5A+4+16A-4A$

$4-4 = 17A \Rightarrow \boxed{0 = A}$

Put in II $0 = 0 + C \Rightarrow \boxed{C = 0}$

Put in VII $B = 1+16(0) \Rightarrow \boxed{B = 0}$

Put in V , $1 = 5(1)+4D \Rightarrow 1-5 = 4D \Rightarrow -4 = 4D \Rightarrow \boxed{D = -1}$

I Become $\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}dx \int\left(\frac{0(x)+1}{x^2+4} + \frac{0(x)-1}{x^2+4x+5}\right)dx$

$\int \frac{1}{x^2+(2)^2}dx - \int \frac{1}{x^2+4x+4+1}dx$

$= \frac{1}{2}Tan^{-1}\frac{x}{2} - \int \frac{1}{(x+2)^2+1}dx$

$= \frac{1}{2}Tan^{-1}\frac{x}{2} - Tan^{-1}(x+2)+C$

28. $\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)}dx$

Now $\frac{6a^2}{(x^2+a^2)(x^2+4a^2)} = \frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{x^2+a^2}$ I

'X' both sides by $(x^2+a^2)(x^2+4a^2)$

$6a^2 = (Ax+B)(x^2+4a^2)+(Cx+D)(x^2+a^2)$

Rearranging

$6a^2 = Ax^2 + 4a^2Ax + Bx^2 + 4a^2B + Cx^3 + Ca^2x + Dx^2 + Da^2$

$6a^2 = (A+C)x^2 + 4a^2A + a^2x + (B+D)x^2 + 4a^2B + Da^2$

Comparing co-efficient

x^3; $0 = A + C$ II

x^2; $0 = B + D$ III

x ; $0 = 4a^2A + Ca^2$ IV

Constant ; $6a^2 = 4a^2B + Da^2$ V

'X' IV by a^2 and solve with II

$0 = Aa^2 + \cancel{Ca^2}$ $\Rightarrow$ $\boxed{A=0}$

$$\frac{-0 = -4Aa^2 \pm \cancel{Ca^2}}{0 = -3Aa^2}$$ Put in II $\boxed{C=0}$

'X' IV by a^2 and solve with V

$$\frac{-6a^2 = -4Ba^2 \pm \cancel{Da^2}}{-6a^2 = -3Ba^2}$$

$\Rightarrow B = \frac{-6a^2}{-3Ba^2} = 2 \Rightarrow \boxed{B=2}$

Put of value of B in III $0 = 2 + D$ $\Rightarrow$ $\boxed{D=-2}$

I become

$$\frac{6a^2}{(x^2+a^2)(x^2+4a^2)} = \frac{0x+2}{x^2+a^2} + \frac{0x-2}{x^2+4a^2}$$

$$= 2.\frac{1}{a}Tan^{-1}\frac{x}{a} - \cancel{2}.\frac{1}{\cancel{2}a}Tan^{-1}\frac{x}{2a} + C$$

$$= \frac{2}{a}Tan^{-1}\frac{x}{a} - \frac{1}{a}Tan^{-1}\frac{x}{2a} + C$$

29. $\int \frac{2x^2-2}{x^4+x^2+1}dx$

$$= \int \frac{2x^2-2}{x^4+x^2+1+x^2-x^2}dx = \int \frac{2x^2-2}{x^4+2x^2+1-x^2}dx$$

$$= \int \frac{2x^2-2}{(x^2+1)^2-(x)^2}dx = \int \frac{2x^2-2}{(x^2+1+x)(x^2+1-x)}dx$$

Suppose $\frac{2x^2-2}{(x^2+x+1)(x^2-x+1)} = \frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1}$ I

'X' both sides by $(x^2+x+1)(x^2-x+1)$ we get

$$2x^2-2=(Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1)$$

Rearranging

$$2x^2-2=Ax^3-Ax^2+Ax+Bx^2-Bx+B+Cx^3+Cx^2+Cx+Dx^2+Dx+D$$

or $2x^2-2=(A+C)x^3+(-A+B+C+D)x^2+(A-B+C+D)x+(B+D)$

Comparing co-efficient

$x^3;\ 0=A+C$ II

$x^2;\ 2=-A+B+C+D$ III

$x\ ;\ 0=A-B+C+D$ IV

Constant; $-2=B+D$ V

Put V in II

$-2=-A+C+B+C$

$-2=-A+C+(-2)$

$-2=-A+C-2$

$-2+2=-A+C$

$-A+C=0$ VI

Solve II and VI

$0=A+C$

$\underline{0=-A+C}$

$0=2c^2 \Rightarrow \boxed{C=0}$

From II $0=A+0 \Rightarrow \boxed{A=0}$

Put II in IV $0=A+C-B+D$

$\Rightarrow\ 0=0-B+D$ VII

Solve V & VII $-2=\cancel{B}+D$

$\underline{0=-\cancel{B}+D}$

$-2=\ 2D \Rightarrow \boxed{D=1}$

From V $-2=B-1$

$\Rightarrow\ B=-2+1=-1$

I become

$$\frac{2x^2-2}{\left(x^2+x+1\right)\left(x^2-x+1\right)}=\frac{0.x-1}{x^2+x+1}+\frac{0.x-1}{x^2-x+1}$$

$$\Rightarrow \quad \int\frac{2x^2-2}{\left(x^2+x+1\right)\left(x^2-x+1\right)}dx=\int\frac{-1}{x^2+x+1}dx+\int\frac{-1}{x^2-x+1}dx$$

$$=-\int\frac{1}{x^2+x+1}-\int\frac{-1}{x^2-x+1}dx$$

$$=-\int\frac{1}{x^2+x+\frac{1}{4}+1-\frac{1}{4}}dx-\int\frac{-1}{x^2-x+\frac{1}{4}+1-\frac{1}{4}}dx$$

$$=-\int\frac{1}{\left(x+\frac{1}{4}\right)^2+\frac{3}{4}}dx-\int\frac{-1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

$$=-\int\frac{1}{\left(\frac{2x+1}{2}\right)^2+\left(\frac{\sqrt{3}}{4}\right)^2}dx-\int\frac{1}{\left(\frac{2x-1}{2}\right)^2+\left(\frac{\sqrt{3}}{4}\right)^2}dx$$

$$=-\frac{1}{\frac{\sqrt{3}}{2}}Tan^{-1}\frac{\frac{2x+1}{2}}{\frac{\sqrt{3}}{2}}-\frac{1}{\frac{\sqrt{3}}{2}}Tan^{-1}\frac{\frac{2x-1}{2}}{\frac{\sqrt{3}}{2}}+c$$

$$=-\frac{1}{\sqrt{3}}Tan^{-1}\frac{2x+1}{\not{2}}\times\frac{\not{2}}{\sqrt{3}}-\frac{2}{\sqrt{3}}Tan^{-1}\frac{2x+1}{\not{2}}\times\frac{\not{2}}{\sqrt{3}}$$

$$=-\frac{1}{\sqrt{3}}Tan^{-1}\frac{2x+1}{\not{2}}-\frac{2}{\sqrt{3}}Tan^{-1}\frac{2x-1}{\sqrt{3}}+C$$

30. $$\int\frac{3x-8}{\left(x^2-x+2\right)\left(x^2+x+2\right)}dx \qquad \text{I}$$

Now $$\frac{3x-8}{\left(x^2-x+2\right)\left(x^2+x+2\right)}=\frac{Ax+B}{x^2-x+2}+\frac{Cx+D}{x^2+x+2}$$

Multiply by $\left(x^2-x+2\right)\left(x^2+x+2\right)$

$$3x-8=\left(Ax+B\right)\left(x^2+x+2\right)+\left(Cx+D\right)\left(x^2-x+2\right)$$

$$3x-8=Ax^3+Ax^2+2Ax+Bx^2+Bx+2B+Cx^3-Cx^2+2Cx+Dx^2-Dx+2D$$

$$3x-8=\left(A+C\right)x^3+\left(A+B+C+D\right)x^2+\left(2A+B+2C-D\right)x+\left(2B+2D\right)$$

Comparing co-efficient

$x^3;\quad 0 = A + C \qquad$ II

$x^2;\quad 0 = A + B - C + D \qquad$ III

$x\ ;\quad 3 = 2A + B + 2C - D \qquad$ IV

Constant ; $-8 = 2B + 2D$

$\div$ by 2 $\quad -4 = B + D \qquad$ V

From III

$$0 = A + B - C + D$$

or $$0 = A - C + B + D$$

Put V $B + D = -4$

$$0 = A - C - 4$$

$\Rightarrow \quad A - C = 4 \qquad$ VI

II + VI

$$A + \cancel{C} = 0$$
$$A + \cancel{C} = 4$$
$$2A = 4 \quad \Rightarrow \boxed{A = 0}$$

V + VII

$$B + \cancel{D} = -4$$
$$B + \cancel{D} = 4$$
$$2B = -1 \quad \Rightarrow \boxed{B = \frac{-1}{2}}$$

Put value of A in II

$$0 = 2 + C \qquad \Rightarrow \qquad \boxed{C = -2}$$

Put value of B in VII

$$\frac{-1}{2} - D = 3$$

$$D = -3 - \frac{1}{2}$$

$$\boxed{D = \frac{-7}{2}}$$

Now from IV

$$3 = 2A + 2C + B - D$$
$$3 = 2(A + C) + B - D$$

Put A + C = 0

$$3 = 2(0) + B - D$$
$$B - D = 3 \qquad \text{VII}$$

I become

$$\int \frac{3x-8}{(x^2-x+2)(x^2+x+2)}dx = \int \left(\frac{2x-\frac{1}{2}}{x^2-x+2}+\frac{-2x-\frac{7}{2}}{x^2+x+2}\right)dx$$

$$= \int \left(\frac{2x-1+1+\frac{1}{2}}{x^2-x+2}dx - \int \frac{2x+1-1+\frac{7}{2}}{x^2+x+2}\right)dx$$

$$= \int \frac{2x-1}{x^2-x+2}dx + \int \frac{1-\frac{1}{2}}{x^2-x+\frac{1}{4}-\frac{1}{4}+2}dx + \int \frac{2x+1}{x^2+x+2}dx + \int \frac{\frac{7}{2}-1}{x^2+x+\frac{1}{4}-\frac{1}{4}+2}dx$$

$$= ln\left|x^2-x+2\right| + \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{7}{4}}dx + ln\left|x2+x+2\right| + \frac{5}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{7}{4}}dx$$

$$= ln\left|x^2-x+2\right| + \frac{1}{2}\int \frac{1}{\left(\frac{2x-1}{2}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}dx + ln\left|x^2+x+2\right| + \frac{5}{2}\int \frac{1}{\left(\frac{x2+1}{2}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}dx$$

$$= ln\left|x^2-x+2\right| + \frac{1}{2}.\frac{\sqrt{7}}{4}Tan^{-1}\left(\frac{\frac{2x-1}{2}}{\frac{\sqrt{7}}{2}}\right) + ln\left|x^2+x+2\right| + \frac{5}{2}.\frac{\sqrt{7}}{4}Tan^{-1}\left(\frac{\frac{2x+1}{2}}{\frac{\sqrt{7}}{2}}\right) + C$$

$$= ln\left|x^2-x+2\right| + \frac{1}{2}.\frac{2}{\sqrt{7}}Tan^{-1}\left(\frac{2x-1}{2}\times\frac{2}{\sqrt{7}}\right) + ln\left|x^2+x+2\right| + \frac{5}{2}.\frac{2}{\sqrt{7}}Tan^{-1}\frac{2x+1}{2}\times\frac{2}{\sqrt{7}} + C$$

$$= ln\left|x^2-x+2\right| + \frac{1}{\sqrt{7}}Tan^{-1}\frac{2x-1}{\sqrt{7}} + ln\left|x^2+x+2\right| + \frac{5}{\sqrt{7}}Tan^{-1}\frac{2x+1}{\sqrt{7}} + C$$

31 $$\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx \qquad \text{(I)}$$

Consider $\dfrac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)} = \dfrac{Ax+B}{(x^2+x+1)} + \dfrac{Cx+D}{(x^2+2x+3)}$

Multiplying both sides of $(i)(x^2+x+1)(x^2+2x+3)$ we have

$3x^3+4x^2+9x+5 = (Ax+B)(x^2+2x+3)+(Cx+F)(x^2+x+1)$

$= A(x^3+2x^2+3x)+B(x^2+2x+3)+C(x^3+x^2+x)+D(x^2+x+1)$

$$3x^3+4x^2+9x+5=(A+C)x^3+(2A+B+C+D)x^2+(3A+C+D)x+3B+D \quad ...(ii)$$

Equating Coefficients of x^3, x^2, x and constant terms of (ii) we have

$A+C=3$ (a)

$2A+B+C+D=4$ (b)

$3A+2B+C+D=9$ (c)

$3B+D=5$ (d)

Put C = 3 – A in (*b*) and (*c*) we get

$2A+B+3-A+D=4 \Rightarrow A+B+D=1$ .... (e)

$3A+2B+3-A+D=9 \Rightarrow 2A+2B+D=6$ .... (f)

Multiplying (e) by 2 and subtract from (f)

$2A+2B+D=6$

$\underline{2A+2B+2D=2}$

$D=4 \Rightarrow \boxed{D=-4}$

Since $3B+D=5 \Rightarrow 3B-4=5 \Rightarrow 3B=9 \Rightarrow \boxed{B=3}$

Putting values of B and D in (e) we have

$A+3-4=1 \Rightarrow A-1=1 \Rightarrow \boxed{A=2}$

Also $C=3-A=3-2=1 \Rightarrow \boxed{C=1}$

Hence (I) can be written as

$$\int \frac{3x^3+4x^2+9x+5}{(x^2+x+1)(x^2+2x+3)}dx=\int\left[\frac{2x+3}{(x^2+x+1)}+\frac{x-4}{x^2+2x+3}\right]dx$$

$$=\int \frac{2x+1}{x^2+x+1}dx+2\int\frac{1}{x^2+x+1}dx+\frac{1}{2}\int\frac{2x+3}{x^2+2x+3}dx-5\int\frac{1}{x^2+2x+3}dx$$

$$=ln(x^2+x+1)+2\int\frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}dx+\frac{1}{2}ln(x^2+2x+3)-5\int\frac{1}{x^2+2x+1+2}$$

$$=ln(x^2+x+1)+ln(x^2+2x+3)^{1/2}+2\int\frac{1}{\left(x+\frac{1}{2}\right)+\left(\frac{\sqrt{3}}{2}\right)^2}dx-5\int\frac{1}{(x+1)^2+(\sqrt{2})^2}dx$$

$$=ln(x^2+x+1)(x^2+2x+3)^{1/2}+2\times\frac{2}{\sqrt{3}}Tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)-\frac{5}{\sqrt{2}}Tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

$$=ln\left[(x^2+x+1)\sqrt{x^2+2x+3}\right]+\frac{4}{\sqrt{3}}Tan^{-1}\left(\frac{2x1}{\sqrt{3}}\right)-\frac{5}{\sqrt{2}}Tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

Exercise 3.6

1. $$\int_1^2 (x^2+1)dx = \int_1^2 x^2 dx + \int_2^2 1dx$$

$$= \left[\frac{x^3}{3}\right]_1^2 + [x]_1^2 = \frac{1}{3}\left[(x)^3 - (1)^3\right] + [2-1] = \frac{1}{3}[8-1]+1$$

$$= \frac{1}{3}(7)+1 = \frac{7}{3}+1\frac{7+3}{3} = \frac{10}{3}$$

2. $$\int_1^1 (x^{1/3}+1)dx = \int_{-1}^1 x^{1/3}dx + \int_{-1}^1 1dx$$

$$= \left[\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right]_{-1}^1 + [x]_{-1}^1 = \left[\frac{x^{4/3}}{\frac{4}{3}}\right]_{-1}^1 + [x]_{-1}^1 = \frac{3}{4}\left[x^{4/3}\right] + [x]_{-1}^1$$

$$= \frac{3}{4}\left[(1)^{4/3} - (-1)^{4/3}\right] + [1-(-1)] = \frac{4}{3}\left[1-(-1)^{3\times 4/3}\right] + [1+1]$$

$$= \frac{3}{4}[1-1] + 2 = 0+2 = 2$$

3. $$\int_{-2}^0 \frac{1}{(2x-1)^2}dx = \int_{-2}^0 (2x-1)^{-2}dx = \frac{1}{2}\int_{-2}^0 (2x-1)^{-2}dx$$ **(Sargodha 2010,11)**

$$= \frac{1}{2}\left[\frac{(2x-1)^{-2+1}}{-2+1}\right]_{-2}^0 = \frac{1}{2}\left[\frac{(2x-1)^{-1}}{-1}\right]_{-2}^0 = -\frac{1}{2}\left[\frac{1}{(2x-1)}\right]_{-2}^0$$

$$= -\frac{1}{2}\left[\frac{1}{2(0)-1}\right] = \frac{-1}{2}\left[\frac{1}{2(-2)-1}\right] = -\frac{1}{2}\left[\frac{1}{0-1} - \frac{1}{-4-1}\right]$$

$$= -\frac{1}{2}\left[-1-\left(-\frac{1}{5}\right)\right] = -\frac{1}{2}\left[-1+\frac{1}{5}\right] = -\frac{1}{2}\left[\frac{-5+1}{5}\right]$$

$$= \frac{1}{2}\left[\frac{-4}{5}\right] = \frac{2}{5}$$

4. $\int_{-6}^{2}\sqrt{3-x}\,dx = \int_{-6}^{2}(3-x)^{1/2}dx = -\int_{-6}^{2}(3-x)^{1/2}dx$

$$= -\left[\frac{(3-x)^{1/2+1}}{\frac{1}{2}+1}\right]_{-6}^{2} = -\left[\frac{(3-x)^{3/2}}{\frac{3}{2}}\right]_{-6}^{2} = -\frac{2}{3}\left[(3-2)^{3/2}-(3-(-6))^{3/2}\right]$$

$$= -\frac{2}{3}\left[(1)^{3/2}-(3+6)^{3/2}\right] = -\frac{2}{3}\left[1-(9)^{3/2}\right]$$

$$= -\frac{2}{3}\left[1-3^{2\times 3/2}\right] = \frac{-2}{3}\left[1-3^{3}\right] = \frac{-2}{3}\left[1-9\right]$$

$$= -\frac{2}{3}[-8] = \frac{16}{3}$$

5. $\int_{1}^{\sqrt{5}}\sqrt{(2t-1)^3}\,dt = \int_{1}^{\sqrt{5}}(2t-1)^{3\times 1/2}dt$

$$= \int_{1}^{\sqrt{5}}(2t-1)^{3/2}dt = \frac{1}{2}\int_{1}^{\sqrt{5}}(2t-1)^{3/2}2dt = \frac{1}{2}\left[\frac{(2t-1)^{3/2+1}}{\frac{3}{2}+1}\right]_{1}^{\sqrt{5}}$$

$$= \frac{1}{2}\left[\frac{(2t-1)^{5/2}}{\frac{5}{2}}\right]_{1}^{\sqrt{5}} = \frac{1}{2}\cdot\frac{2}{5}\left[(2(\sqrt{5})-1)^{5/2}-(2(1)-1)^{5/2}\right]$$

$$= \frac{1}{5}\left[(2(\sqrt{5}-1)^{5/2}-(1)^{5/2}\right] = \frac{1}{5}\left[(2\sqrt{5}-1)^{5/2}-1\right]$$

6. $\int_{2}^{\sqrt{5}}x\sqrt{x^2-1}\,dx = \int_{2}^{\sqrt{5}}(x^2-1)^{1/2}x\,dx$

$$= \frac{1}{2}\int_{2}^{\sqrt{5}}(x^2-1)^{1/2}2x\,dx = \frac{1}{2}\left[\frac{(x^2-1)^{1/2+1}}{\frac{1}{2}+1}\right]_{2}^{\sqrt{5}}$$

$$= \frac{1}{2}\left[\frac{(x^2-1)^{3/2}}{\frac{3}{2}}\right]_{2}^{\sqrt{5}} = \frac{1}{2}\cdot\frac{2}{3}\left[((\sqrt{5})^2-1)^{3/2}-((2)^2-1)^{3/2}\right]$$

$$=\frac{1}{3}\left[(5-1)^{3/2}-(4-1)^{3/2}\right]=\frac{1}{3}\left[(4)^{3/2}-(3)^{3/2}\right]$$

$$=\frac{1}{3}\left[2^{2\times 3/2}-3.3^{1/2}\right]=\frac{1}{3}\left[2^{3}-3\sqrt{3}\right]=\frac{1}{3}\left[8-3\sqrt{3}\right]$$

7. $$\int_1^2\frac{x}{x^2+2}dx=\frac{1}{2}\int_1^2\frac{2x}{x^2+2}dx$$ **(Sargodha 2008,09)**

$$=\frac{1}{2}\left[\ln\left|x^2+2\right|\right]_1^2=\left[\ln\left|(2)^2+2\right|-\ln(1)^2+2\right]$$

$$=\frac{1}{2}[\ln 6-\ln 3]=\frac{1}{2}\left[\ln\frac{6^2}{3}\right]=\frac{1}{2}\ln^2$$

8. $$\int_2^3\left(x-\frac{1}{x}\right)^2dx=\int_2^3\left(x^2+\frac{1}{x^2}-2x\frac{1}{x}\right)dx$$

$$=\int_2^3 x^2dx+\int_2^3 x^{-2}dx-2\int_2^3 1dx=\left[\frac{x^3}{3}\right]_2^3+\left[\frac{x^{-2+1}}{-2+1}\right]_2^3-2[x]_2^3$$

$$=\frac{1}{3}\left[(3)^3-(2)^3\right]+\left[\frac{(x^{-1})}{-1}\right]_2^3-2[3-2]$$

$$=\frac{1}{3}[27-8]-\left[\frac{1}{x}\right]_2^3-2(1)=\frac{1}{3}(19)-\left(\frac{1}{3}-\frac{1}{2}\right)-2$$

$$=\frac{19}{3}-\left(\frac{2-3}{6}\right)-2=\frac{19}{3}-\left(\frac{-1}{6}\right)-2=\frac{19}{3}+\frac{1}{6}-2=\frac{38+1}{6}-12$$

$$=\frac{27}{6}=\frac{9}{2}$$

9. $$\int_{-1}^1\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}dx$$

$$=\int_{-1}^1(x^2+x+1)^{1/2}\left(x+\frac{1}{2}\right)dx=\frac{1}{2}\int_{-1}^1(x^2+x+1)^{1/2}2\left(x+\frac{1}{2}\right)dx$$

$$=\frac{1}{2}\int(x^2+x+1)^{1/2}(2x+1)dx=\frac{1}{2}\left[\frac{(x^2+x+1)^{1/2}}{\frac{1}{2}+1}\right]_{-1}^1$$

$$=\frac{1}{2}\left[\frac{(x^2+x+1)^{3/2}}{\frac{3}{2}}\right]_{-1}^{1}=\frac{1}{2}\times\frac{2}{3}\left[(x^2+x+1)^{3/2}\right]_{-1}^{1}$$

$$=\frac{1}{3}\left[((1)^2+(1)+1)^{3/2}-((-1)^2+(-1)+1)^{3/2}\right]$$

$$=\frac{1}{3}\left[(1+1+1)^{3/2}-(1-1+1)^{3/2}\right]$$

$$=\frac{1}{3}\left[3^{3/2}-(1)^{3/2}\right]=\frac{1}{3}\left[3\sqrt{3}-1\right]$$

10. $\int_0^3\frac{dx}{x^2+9}=\int_0^3\frac{dx}{x^2+(3)^2}dx$ **(Sargodha 2010,11,12)**

$$=\left[\frac{1}{3}Tan^{-1}\frac{x}{3}\right]_0^3=\frac{1}{3}\left[Tan^{-1}\frac{3}{3}-Tan^{-1}\frac{0}{3}\right]=\frac{1}{3}\left[Tan^{-1}(1)-Tan^{-1}0\right]$$

$$=\frac{1}{3}\left(\frac{\pi}{4}-0\right)=\frac{\pi}{12}$$

11. $\int_{\pi/3}^{\pi/3}Cost\,dt=\left[S\text{int}\right]_{\pi/6}^{\pi/3}=Sin\frac{\pi}{3}-Sin\frac{\pi}{6}$ **(Sargodha 2009,10,11)**

$$=\frac{\sqrt{3}}{2}-\frac{1}{2}=\frac{\sqrt{3}-1}{2}$$

12. $\int_1^3\left(x+\frac{1}{x}\right)^{1/2}\left(1-\frac{1}{x^2}\right)dx$

Here $f(x)=x+\frac{1}{x}$ *then* $f(x)=1+(-1)x^{-2}=1-\frac{1}{x^2}$ *So*

$$=\left[\frac{\left(x+\frac{1}{x}\right)^{1/2+1}}{\frac{1}{2}+1}\right]_1^1=\left[\frac{\left(x+\frac{1}{x}\right)^{3/2}}{\frac{3}{2}}\right]_1^2=\frac{2}{3}\left[\left(2+\frac{1}{2}\right)^{3/2}-\left(1-\frac{1}{2}\right)^{3/2}\right]$$

$$=\left[\left(\frac{5}{2}\right)^{3/2}-(2)^{3/2}\right]=\frac{2}{3}\left[\frac{5\sqrt{5}}{2\sqrt{2}}-2\sqrt{2}\right]=\frac{2}{3}\left[\frac{5\sqrt{5}-4\times 2}{2\sqrt{2}}\right]$$

$$=\frac{2}{3}\left[\frac{5\sqrt{5}-8}{2\sqrt{2}}\right]=\frac{5\sqrt{5}-8}{3}$$

13. $\int_1^2 \ln x\, dx \int 1 \ln x\, dx = \int_0^2 \overset{\ln x}{\mathrm{I}} . \overset{1}{\mathrm{II}}\, dx$

$$=[\ln x \,.\, x]_1^2 - \int_1^2 \frac{1}{x}.x\, dx = [x \ln x]_1^2 - \int_1^2 1dx$$

$$=(2\ln 2 - 1\ln 1) - [x]_1^2 = (2\ln 2 - 1\times 0) - (2-1)$$

$$=2\ln 2 - 0 - 1 = 2\ln 2 - 1$$

14. $\int_0^2 \left(e^{x/2} - e^{-x/2}\right) dx = \int_0^2 e^{x/2} dx - \int_0^2 e^{-x/2} dx$

$$=\left[\frac{e^{x/2}}{\frac{1}{2}}\right]_0^2 - \left[\frac{e^{-x/2}}{\frac{-1}{2}}\right]_0^2 = 2\left[e^{x/2}\right]_0^2 + 2\left[e^{-x/2}\right]_0^2$$

$$=2\left[e^{2/2} - e^{0/2}\right] + 2\left[e^{-2/2} - e^{-0/2}\right] = 2\left[e^1 - e^0\right] + 2\left(e^{-1} - e^0\right)$$

$$=2(e-1) + 2\left(\frac{1}{e} - 1\right) = 2e + \frac{2}{e} - 2 = 2e - \frac{2}{e} - 4$$

$$=\left(e - \frac{1}{e}\right) - 4\left(\frac{e^2-1}{e}\right) - 4$$

15. $\int_0^{\pi/4} \frac{Cos\theta + Sin\theta}{1 + Cos 2\theta} d\theta$

$$\int_0^{\pi/4} \frac{Cos\theta + Sin\theta}{1Cos^2\theta} d\theta = \frac{1}{2}\int_0^{\pi/4}\left(\frac{Cos\theta}{Cos^2\theta} + \frac{Sin\theta}{Cos^2\theta}\right) d\theta$$

$$=\frac{1}{2}\int_0^{\pi/4}\left(\frac{1}{Cos\theta} + \frac{Sin\theta}{Cos\theta}\times\frac{1}{Cos\theta}\right) d\theta = \frac{1}{2}\int_0^{\pi/4}(Sec\theta + SecTan\theta) d\theta$$

$$=\frac{1}{2}\int_0^{\pi/4} Sec\theta\, d\theta + \frac{1}{2}\int_0^{\pi/4} Sec\theta Tan\theta d\theta$$

$$=\frac{1}{2}\left[\ln\left|Sec\theta + Tan\theta\right|\right]_0^{\pi/4} + \frac{1}{2}\left[Sec\theta\right]_0^{\pi/4}$$

Now $\boxed{Sec\frac{\pi}{4}=\frac{1}{Cos\frac{\pi}{4}}=\frac{1}{\frac{1}{\sqrt{2}}}=\sqrt{2}}$

$$=\frac{1}{2}\left[\ln\left(Sec\frac{\pi}{4}+Tan\frac{\pi}{4}\right)-\ln(Sec0+Tan0)\right]+\frac{1}{2}\left(Sec\frac{\pi}{4}-Sec0\right)$$

$$=\frac{1}{2}\left[\ln(\sqrt{2}+1)-\ln(1+0)\right]+\frac{1}{2}(\sqrt{2}-1)$$

$$=\frac{1}{2}\left[\ln(\sqrt{2}+1)-\ln(1)\right]+\frac{1}{2}(\sqrt{2}-1)$$

$$=\frac{1}{2}\left[\ln(\sqrt{2}+1)-0)\right]+\frac{1}{2}(\sqrt{2}-1)=\frac{1}{2}\left[\ln(\sqrt{2}+1)\right]+\frac{1}{2}(\sqrt{2}-1)$$

$$=\frac{1}{2}\left[\ln(\sqrt{2}+1)+(\sqrt{2}-1)\right]$$

16. $\int_0^{\pi/6} Cos^3\theta\, d\theta=\int_0^{\pi/6} Cos^2\theta\, d\theta$ $Cos^2\theta.Cos\theta$

$$\int_0^{\pi/6}(1-Sin^2\theta)Cos\theta=\int_0^{\pi/6}Cos\theta d\theta-\int_0^{\pi/6}Sin^2\theta Cos\theta d\theta$$

$$=[Sin\theta]_0^{\pi/6}-\left[\frac{Sin^3\theta}{3}\right]_0^{\pi/6}=\left(Sin\frac{\pi}{6}-Sin0\right)-\left(\frac{Sin^2\pi/6}{3}-\frac{Sin^3 0}{3}\right)$$

$$=\left(\frac{1}{2}-0\right)-\frac{1}{3}\left(\frac{1}{2}\right)^3-(0)^3=\left(\frac{1}{2}-0\right)-\frac{1}{3}\left(\frac{1}{8}\right)=\frac{1}{2}-\frac{1}{24}$$

$$=\frac{12-1}{24}=\frac{11}{24}$$

17. $\int_{\pi/6}^{\pi/4} Cos^2\theta Cot^2\theta d\theta$

$$=\int_{\pi/6}^{\pi/4}Cos^2\theta(Co\sec^2\theta-1)d\theta=\int_{\pi/6}^{\pi/4}(Cos^2\theta Co\sec^2\theta-Cos^2\theta)d\theta$$

$$=\int_{\pi/6}^{\pi/4}Cos^2\theta.\frac{1}{Sin^2\theta}d\theta-\int_{\pi/6}^{\pi/4}Cos^2\theta d\theta$$

$$= \int_{\pi/6}^{\pi/4} Cot^2\theta d\theta - \int_{\pi/6}^{\pi/4} \frac{1+Cos^2\theta}{2} d\theta$$

$$= \int_{\pi/6}^{\pi/4} (Cosec^2\theta - 1)d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/4} (1+Cos2\theta)d\theta$$

$$= \int_{\pi/6}^{\pi/4} (Cosec^2\theta d\theta - \int_{\pi/6}^{\pi/4} 1d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/4} Cos2\theta d\theta$$

$$= \left[-Cot\theta\right]_{\pi/6}^{\pi/4} - \left[\theta\right]_{\pi/6}^{\pi/4} - \frac{1}{2}\left[\frac{Sin2\theta}{2}\right]_{\pi/6}^{\pi/4}$$

$$= -Cot\frac{\pi}{4} + Cot\frac{\pi}{6} - \left(\frac{\pi}{4} - \frac{\pi}{6}\right) - \frac{1}{2}\left(Sin\cancel{2}\frac{\pi}{\cancel{4}_2} - Sin\frac{\pi}{\cancel{6}_3}\right)$$

$$= -1 + \sqrt{3} - \frac{3}{2}\left(\frac{\pi}{4} - \frac{\pi}{6}\right) - \frac{1}{4}\left(1 - \frac{\sqrt{3}}{2}\right)$$

$$= -1 + \sqrt{3} - \frac{3\pi}{8} + \frac{\cancel{3}\pi}{\cancel{12}_4} - \frac{1}{4} + \frac{\sqrt{3}}{8}$$

$$= \frac{-8 + 8\sqrt{3} - 3\pi + 2\pi - 2 + \sqrt{3}}{8} = \frac{9\sqrt{3} - \pi - 10}{8}$$

18. $$\int_0^{\pi/4} Cos^4 t d\theta = \int_0^{\pi/4} (Cos^2 - t)^2 dt = \int_0^{\pi/4} \left(\frac{1+Cos2t}{2}\right)^2 dt$$

$$= \int_0^{\pi/4} \frac{1 + 2Cos2t + Cos^2 2t}{4} dt \qquad 1 + Cos2t = 2Cos^2 t$$

$$= \frac{1}{4}\int_0^{\pi/4} \left(1 + 2Cos2t + \left(\frac{1+Cos4t}{2}\right)\right)dt \qquad 1 + Cos4t = 2Cos^2 2t$$

$$= \frac{1}{4}\int_0^{\pi/4} \frac{2 + 4Cos2t + 1 + Cos4t}{2} dt$$

$$= \frac{1}{8}\int_0^{\pi/4} (3 + 4Cos2t + Cos4t)dt$$

$$= \frac{1}{8}\left[3t + \frac{4Sin2t}{2} + \frac{Sin4t}{4}\right]_0^{\pi/4}$$

$$=\frac{1}{8}\left[3\frac{\pi}{4}+4\frac{Sin2\frac{\pi}{4_2}}{2}+\frac{Sin4\frac{\pi}{4}}{4}-3(0)-\frac{4Sin2(0)}{2}+\frac{Sin4(0)}{4}\right]$$

$$=\frac{1}{8}\left[\frac{3\pi}{4}+4\left(\frac{1}{2}\right)+\frac{0}{4}-0-\frac{0}{2}+\frac{0}{4}\right]=\frac{1}{8}\left(\frac{3\pi}{4}+2\right)$$

$$=\frac{1}{8}\left(\frac{3\pi+8}{4}\right)=\frac{3\pi+8}{32}$$

19. $$\int_0^{\pi/3} Cos^2\theta Sin\theta d\theta = -\int_0^{\pi/3} Cos^2\theta(-Sin\theta)d\theta$$

$$=-\left[\frac{Cos^3\theta}{3}\right]_0^{\pi/3}=-\frac{1}{3}\left[Cos^3\theta\right]_0^{\pi/3}=-\frac{1}{3}\left[Cos^3\frac{\pi}{3}-Cos^3 0\right]$$

$$=-\frac{1}{3}\left[\left(\frac{1}{3}\right)^3-(1)^3\right]=-\frac{1}{3}\left(\frac{1}{8}-1\right)=-\frac{1}{3}\left(\frac{-7}{8}\right)=\frac{7}{24}$$

20. $$\int_0^{\pi/4}(1+Cos^2\theta)Tan^2\theta d\theta$$

$$=\int_0^{\pi/4}(Tan^2\theta+Cos^2\theta Tan^2\theta)d\theta=\int_0^{\pi/4}\left(Sec^2\theta-1+Cos^2\theta\times\frac{Sin^2\theta}{Cos^2\theta}\right)d\theta$$

$$=\int_0^{\pi/4}Sec^2 d\theta-\int_0^{\pi/4}1d\theta+\int_0^{\pi/4}\left(\frac{1-Cos2\theta}{2}\right)d\theta$$

$$=\left[Tan\theta\right]_0^{\pi/4}-\left[\theta\right]_0^{\pi/4}+\frac{1}{2}\left[\theta-\frac{Sin\theta}{2}\right]_0^{\pi/4}$$

$$=Tan\frac{\pi}{4}-Tan\theta-\left(\frac{\pi}{4}-0\right)+\frac{1}{2}\left[\frac{\pi}{4}-\frac{Sin2\frac{\pi}{4}}{2}-0+\frac{Sin2(0)}{2}\right]$$

$$=1-0\left(\frac{\pi}{4}\right)+\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}-0+0\right)=1-\frac{\pi}{4}+\frac{\pi}{8}-\frac{1}{4}$$

$$=\frac{8-2\pi+\pi-2}{8}=\frac{6-\pi}{8}$$

21. $\int_0^{\pi/4} \frac{Sec\theta}{Sin\theta + Cos\theta} d\theta$ **(Sargodha 2009,10)**

$$= \int_0^{\pi/4} \frac{Sec^2\theta}{Sin\theta(Sin\theta + Cos\theta)} d\theta \int_0^{\pi/4} \frac{Sec^2\theta}{\frac{1}{Cos\theta}(Sin\theta + Cos\theta)}$$

$$= \int_0^{\pi/4} \frac{Sec^2\theta}{(Tan\theta + 1} d\theta = \left[\ln|Tan\theta + 1|\right]_0^{\pi/4}$$

$$= \ln\left(Tan\frac{\pi}{4} + 1\right) - \ln(Tan\theta + 1)$$

$$= \ln(1+1) - \ln(0+1) = \ln(2) - \ln(1) = \ln 2 - 0$$

$$= \ln(2)$$

22. $\int_{-1}^{5} |x - 3| dx$

$$= \int_{-1}^{5} -(x-3)dx + \int_{-1}^{5} (x-3)dx$$

$$= \left[-\left(\frac{x^2}{2} - 3x\right)\right]_{-1}^{3} + \left[\left(\frac{x^2}{2} - 3x\right)\right]_{3}^{5} = \left[-\left(\frac{x^2}{2} + 3x\right)\right]_{-1}^{3} + \left[\left(\frac{x^2}{2} - 3x\right)\right]_{3}^{5}$$

$$= \left(\frac{-3^2}{2} + 3(3) - \left(\frac{(-1)^2}{2}\right) + 3) - 1)\right) + \frac{5^2}{2} - 3(5) - \frac{3^2}{2} + 3(3)$$

$$= -\frac{9}{2} + 9 + \frac{1}{2} + 3 + \frac{25}{2} - 15 - \frac{9}{2} + 9$$

$$= \frac{-9 + 18 + 1 + 6 + 25 - 30 - 9 + 18}{2} = \frac{20}{2} = 10$$

23. $\int_{1/8}^{1} \left(\frac{x^{1/3} + 2}{x^{2/3}}\right) dx = \int_{1/8}^{1} (x^{1/3} + 2)^2 x^{-2/3} dx$ **(Sargodha 2008)**

$$= 3\int_{1/8}^{3} (x^{1/3} + 2)^2 \frac{1}{3} x^{-2/3} dx = 3\left[\frac{(x^{1/3} + 2)}{3}\right]_{1/8}^{1}$$

$$= \frac{3}{3}\left(((1)^{1/3} + 2) - \left(\left(\frac{1}{8}\right)^{1/3} + 2\right)^3\right)$$

$$=\left[(1+2)^3-\left(\left(\frac{1}{2}\right)^{3\times 1/3}+2\right)^3\right]=\left[3^3-\left(\frac{1}{2}+2\right)^3\right]$$

$$=27-\left(\frac{5}{2}\right)^3=27-\frac{125}{8}=\frac{216-125}{8}=\frac{91}{8}$$

24. $\displaystyle\int_1^3 \frac{x^2-2}{x+1}dx$

$$=\int_1^3\left(x-1-\frac{1}{x+1}\right)dx$$

$$=\int_1^3 xdx-\int_1^3 1dx-\int_1^3\frac{1}{x+1}dx$$

$$=\left[\frac{x^2}{2}\right]_1^3-[x]_1^3-\left[\ln|x+1|\right]_1^3$$

$$=\left(\frac{3^2}{2}-\frac{1^2}{2}\right)-(3-1)-(\ln(3+1)-\ln(1+1))$$

$$=\frac{9}{2}-\frac{1}{2}-2(\ln 4-\ln 2)$$

$$=\frac{9-1}{2}-2(\ln 2^2-\ln 2)=\frac{8^4}{2}-2(2\ln 2-\ln 2)$$

$$=4-2-\ln 2=2-\ln 2$$

25. $\displaystyle\int_2^3\frac{3x^2-2x+1}{(x-1)(x^2+1)}dx$

Let $\dfrac{3x^2-2x+1}{(x-1)(x^2+1)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2+1}$

'X' both sides by $(x-1)(x^2+1)$ we get

$3x^2-2x+1=A(x^2+1)+(Bx+C)(x-1)$

Put $x-1=0\Rightarrow x=1$ *in i*

$3(1)^2-2(1)+1=A((1)^2+1)+(Bx+c)(0)$

$3-1+1=A(2)+0\Rightarrow 2=2A\Rightarrow \boxed{A=1}$

Rearranging I

$3x^2-2x+1=Ax^2+A+Bx^2-Bx+Cx-C$

Comparing co-efficient

$x^2;\quad 3 = A + B \Rightarrow 3 = 1 + B \Rightarrow \boxed{B = 2}$

$x;\quad -2 = -B + C \Rightarrow -2 = 2 + C \Rightarrow C = -2 + 2 = 0\ \boxed{C = 0}$

$$\frac{3x^2 - 2x + 1}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{2x+0}{x^2+1}$$

$$\int_2^3 \frac{3x^2 - 2x + 1}{(x-1)(x^2+1)}dx = \int_2^3 \left(\frac{1}{x-1} + \frac{2x}{x^2+1}\right)dx$$

$$\int_2^3 \frac{1}{x-1}dx + \int_2^3 \frac{2x}{x^2+1}dx$$

$$= \left[\ln|x-1|\right]_2^3 + \left[\ln|x^2+1|\right]_2^3$$

$$\int_2^3 \frac{3x^2 + 2x + 1}{(x-1)(x^2+1)}dx = \ln(3-1) - \ln(2-1) + \ln(3^2+1) - \ln(2^2+1)$$

$$= \ln 2 - \ln(1) + \ln 10 - \ln 5$$

$$= \ln 2 - 0 + \ln 10 - \ln 5$$

$$= \ln 2 + \ln 10 - \ln 5 = \ln\frac{2 \times 10}{5}$$

$$= \ln\frac{20^{4}}{5} = \ln 4$$

26. $$\int_0^{\pi/4} \frac{Sinx - 1}{Cos^2 x}dx = \int_0^{\pi/4}\left(\frac{Sinx}{Cosx} \times \frac{1}{Cosx} - \frac{1}{Cos^2 x}\right)dx$$

$$\int_0^{\pi/4} Secx\ Tanx\,dx - \int_0^{\pi/4} Sec^2 x\,dx$$

$$= \left[Secx\right]_0^{\pi/4} - \left[Tanx\right]_0^{\pi/4} = \left(Sec\frac{\pi}{4} - Sec0\right) - \left(Tan\frac{\pi}{4} - Tan0\right)$$

$$= \sqrt{2} - 1 - (1 - 0) = \sqrt{2} - 1 - 1 = \sqrt{2} - 2$$

27. $$\int_0^{\pi/4} \frac{1}{1 + Sinx}dx = \int_0^{\pi/4} \frac{1 - Sinx}{Cos^2 x}dx = \int_0^{\pi/4}\left(\frac{1}{Cos^2 x} - \frac{Sinx}{Cos^2 x}\right)dx$$

$$\int_0^{\pi/4} (Sec^2 - TanxSecx)dx = \int_0^{\pi/4} Sec^2 x dx = \int_0^{\pi/4} SecxTanxdx$$

$$= \left[Tanx\right]_0^{\pi/4} - \left[Secx\right]_0^{\pi/4} = Tan\frac{\pi}{4} - Tan0\left(Sec\frac{\pi}{4} - Sec0\right)$$

$$= 1 - 0(\sqrt{2} - 1) = 1 - \sqrt{2} + 1 = 2 - \sqrt{2}$$

28. $\int_0^1 \frac{3x}{\sqrt{4-3x}}dx = \int_0^1 \frac{-3x}{\sqrt{4-3x}}dx$

$$= -\int_0^1 \frac{4-3x-4}{\sqrt{4-3x}}dx = -\int_0^1 \frac{4-3x}{\sqrt{4-3x}}dx + 4\int_0^1 \frac{1}{\sqrt{4-3x}}dx$$

$$= -\int_0^1 \frac{\sqrt{4-3x}\times\sqrt{4-3x}}{\sqrt{4-3x}}dx + 4\int_0^1 (4-3x)^{-1/2}dx$$

$$= -\left(\frac{-1}{3}\right)\int_0^1 (4-3x)^{1/2}(-3)dx + 4\left(-\frac{1}{3}\right)\int_0^1 (4-3x)^{-1/2}(-3)dx$$

$$= \frac{1}{3}\left[\frac{(4-3x)^{1/2+1}}{\frac{1}{2}+1}\right]_0^1 - \frac{4}{3}\left[\frac{(4-3x)^{1/2+1}}{\frac{-1}{2}+1}\right]_0^1$$

$$= \frac{1}{3}\left[\frac{(4-3x)^{3/2}}{\frac{3}{2}}\right]_0^1 - \frac{4}{3}\left[\frac{(4-3x)^{1/2}}{\frac{1}{2}}\right]_0^1$$

$$= \frac{1}{3}\times\frac{2}{3}\left[(4-3x)^{3/2}\right]_0^1 - \frac{4}{3}\times\frac{2}{1}\left[(4-3x)^{1/2}\right]_0^1$$

$$= \frac{2}{9}\left[(4-3(1))^{3/2} - (4-3(0))^{3/2}\right] - \frac{8}{3}\left[(4-3(1))^{1/2} - (4-3(0))^{1/2}\right]$$

$$= \frac{2}{9}\left[(4-3)^{3/2} - (4-0)^{3/2}\right] - \frac{8}{3}\left[(4-3)^{1/2} - (4-0)^{1/2}\right]$$

$$= \frac{2}{9}\left[(1)^{3/2} - (4)^{3/2}\right] - \frac{8}{3}\left[(1)^{1/2} - (4)^{1/2}\right]$$

$$= \frac{2}{9}\left[1 - 2^{2\times 3/2}\right] - \frac{8}{3}\left[1 - 2^{2\times 1/2}\right]$$

$$= \frac{2}{9}[1-8] - \frac{8}{3}[1-2] = \frac{2}{9}(-7) - \frac{8}{3}(-1)$$

$$= \frac{-14}{9} + \frac{8}{3} = \frac{-14+24}{9} = \frac{10}{9}$$

29. $\int_{\pi/6}^{\pi/2} \frac{Cosx}{Sinx(2+Sinx)}dx$ **(Sargodha 2011)**

Put $Sinx = t \Rightarrow Cosx\ dx = dt$

$$When \quad x=\frac{\pi}{2} \text{ then } t=Sin\frac{\pi}{2}=1$$

$$When \quad x=\frac{\pi}{6} \text{ then } t=Sin\frac{\pi}{6}=\frac{1}{2}$$

$$So \quad \int_{\pi/6}^{\pi/2}\frac{Cosxdx}{Sinx(2+Sinx)}=\int_{1/2}^{1}\frac{dt}{t(2+t)}$$

$$Let \quad \frac{1}{t(2+t)}=\frac{A}{t}+\frac{B}{2+t}$$

'X' both sides by t (2+t) we get

$$1=A(2+t)+Bt$$

$$Put \quad t=0$$

$$1=A(2+0)+0 \Rightarrow 1=2A \Rightarrow \boxed{A=\frac{1}{2}}$$

$$Put \quad 2+t=0 \Rightarrow t=-2 \text{ in } II$$

$$1=A(0)+B(-2) \Rightarrow \boxed{B=\frac{-1}{2}}$$

I become

$$\frac{1}{t(2+t)}=\frac{\frac{1}{2}}{t}+\frac{\frac{-1}{2}}{2+t}$$

$$\int_{1/2}^{1}\frac{1}{t(2+t)}dt=\int_{1/2}^{1}\left(\frac{\frac{1}{2}}{t}-\frac{\frac{-1}{2}}{2+t}\right)dt$$

$$=\frac{1}{2}\int_{1/2}^{1}\frac{1}{t}dt-\frac{1}{2}\int_{1/2}^{1}\frac{1}{2+t}dt=\frac{1}{2}\left[\ln t\right]_{1/2}^{1}-\frac{1}{2}\left[\ln(2+t)\right]_{1/2}^{1}$$

$$=\frac{1}{2}\left[\ln(1)-\left(\ln\frac{1}{2}-\ln(2+1)-\ln\left(2+\frac{1}{2}\right)\right)\right]$$

$$=\frac{1}{2}\left[0-\ln\frac{1}{2}-\ln 3+\ln\frac{5}{2}\right]$$

$$=\frac{1}{2}\left[-(\ln(1)-\ln 2)-\ln 3+\ln 5-\ln 2\right]$$

$$=\frac{1}{2}\left[-\ln(1)+\cancel{\ln 2}-\ln 3+\ln 5-\cancel{\ln 2}\right]$$

$$=\frac{1}{2}\left[0+\ln 5-\ln 3\right]=\frac{1}{2}\ln\frac{5}{3}$$

30. $\int_0^{\pi/2}\frac{Sinx}{(1+Cosx)(2+Cosx)}dx$

Put $Cosx = t \Rightarrow Sinx\, dx = dt$

When $x=0$ *then* $t = Cos0 = 1$

When $x=\frac{\pi}{2}$ *then* $t=Cos\frac{\pi}{2}=0$

$$=\int_1^0\frac{-dt}{(1+t)(2+t)}=-\int_1^0\frac{dt}{(1+t)(2+t)}$$

$$=\int_1^0\frac{+dt}{(1+t)(2+t)}\quad Take\ \frac{1}{(1+t)(2+t)}=\frac{A}{1+t}+\frac{B}{2+t}$$

'X' be $(1+t)(2+t)$ we get $1=A(2+t)+B(1+t)$

Put $2+t=0 \Rightarrow t=-2$ *then* $1=A(0)+(1-2)$

Put $1+t=0 \Rightarrow t=-1$ *then*

$1=A(2-1)+0 \Rightarrow \boxed{1=A}$ *So*

$$\frac{1}{(1+t)(2+t)}=\frac{1}{1+t}+\frac{-1}{2+t}$$

$$\Rightarrow \int_0^1\frac{1}{(1+t)(2+t)}dt=\int_0^1\left(\frac{1}{1+t}-\frac{1}{2+t}\right)dt=\int_0^1\frac{1}{1+t}dt-\int_0^1\frac{1}{2+t}dt$$

$$=\left[\ln|1+t|\right]_0^1-\left[\ln|2+t|\right]_0^1$$

$$=\ln(1+1)-\ln(1+0)-(\ln(2+1)-\ln(2+0))$$

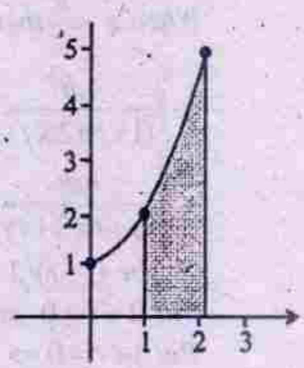
$$=\ln 2-\ln 1-\ln 3=2\ln 2-\ln 3$$

$$=\ln 2^2-\ln 3=\ln 4-\ln 3=\ln\frac{4}{3}$$

Exercise 3.7

1. $y = x^2 + 1 \quad x = 1 \text{ to } x = 2$ **(Sargodha 2007,09,10,11,12)**

x	0	1	2
y	1	2	5



$$\text{Area} = \int_1^2 (x^2+1)dx$$

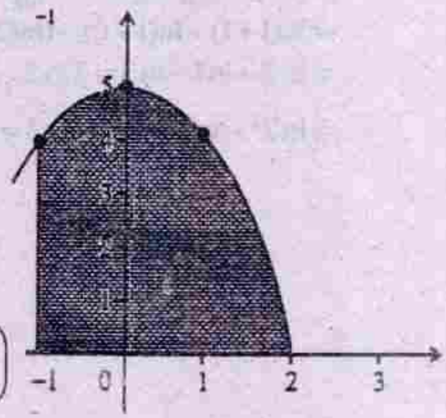
$$= \int_1^2 x^2 dx + \int_1^2 1 dx = \left[\frac{x^3}{3}\right]_1^2 + [x]_1^2$$

$$= \left(\frac{2^3}{3} - \frac{1^3}{3}\right) + (2-1)$$

$$= \frac{8}{3} - \frac{1}{3} + 1 = \frac{8-1+3}{3} = \frac{10}{3}$$

2. $y = 5 - x^2, \quad x = -1 \text{ to } x = 2$

x	-1	0	1	2
y	4	5	4	1



$$\text{Area} = \int_{-1}^2 (5-x^2)dx = \int_{-1}^2 5dx - \int_{-1}^2 x^2 dx$$

$$= 5\int_{-1}^2 1dx - \int_{-1}^2 x^2 dx = 5\left[\frac{x^3}{3}\right]_{-1}^2$$

$$= 5(2-(-1)) - \left(\frac{2^3}{3} - \frac{(-1)^3}{3}\right)$$

$$= 5(2+1) - \left(\frac{8}{3} - \frac{(-1)}{3}\right) = 5(3)\left(\frac{8}{3} + \frac{1}{3}\right)$$

$$= 15 - \left(\frac{8+1}{3}\right) = 15 - \left(\frac{9^3}{3}\right) = 15 - 3 = 12 Sq\ unit$$

3. $y = 3\sqrt{x}, \quad x = 1, x = 4$ (Sargodha 2008,10,11)

x	0	1	2	3	4	5
y	4	5	$3\sqrt{2}$	$3\sqrt{3}$	6	$3\sqrt{5}$

Area = $\int_1^4 3\sqrt{x}dx$

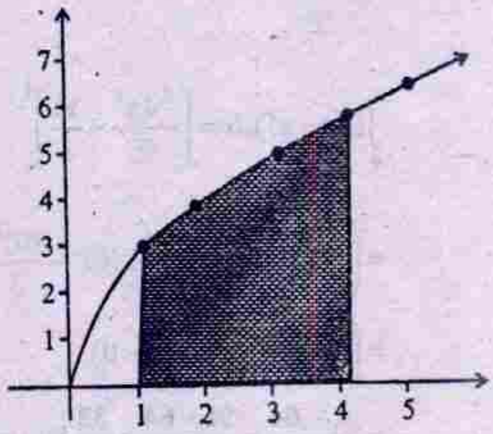
$= 3\int_1^4 (x)^{1/2} dx$

$= 3\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_1^4 = 3 \times \frac{2}{3}\left[x^{3/2}\right]_1^4$

$= 2\left[4^{3/2} - 1^{3/2}\right]$

$= 2\left[2^{2\times 3/2} - 1\right]$

$= 2[8-1]$



4. $y = Cosx$ (Sargodha 2008,09)

x	-90°	-60°	-30°	0	30°	60°	90°
y	0	+0.5	+0.8	1	0.8	0.5	0

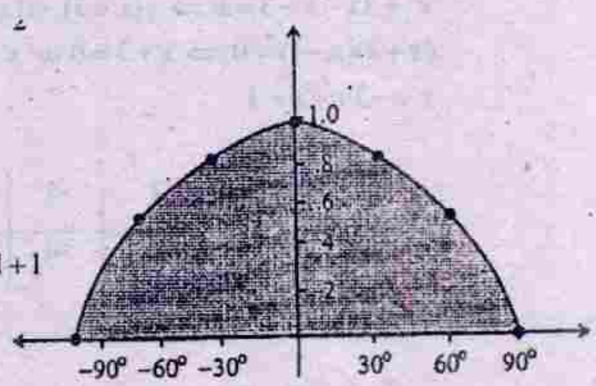
$x = \frac{-\pi}{2}$ to $x = \frac{\pi}{2}$

$x = -90^0$ to $x = 90^0$

Area = $\int_{-\pi/6}^{\pi/2} Cosx\, dx = [Sinx]_{-\pi/2}^{\pi/2}$

$= Sin\frac{\pi}{2} - Sin\left(\frac{-\pi}{2}\right) = 1 - (-1) = 1 + 1$

$= 2$ Sq units

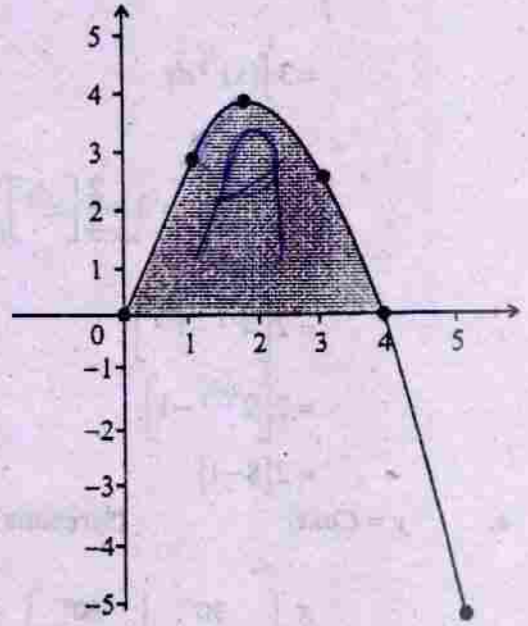


5. $y = 4x - x^2$

Put $y = 0 \Rightarrow 4x - x^2 = 0 \Rightarrow x(4 - x) = 0$

$x = 0$ *or* $4 - x = 0 \Rightarrow x = 0$ *or* $x = 4$

x	0	1	2	3	4	5
y	0	3	4	3	0	-5



$$\int_0^4 (4x - x^2)dx = \left[\frac{{}^2 4x^2}{\not 2} - \frac{x^3}{3}\right]_0^4$$

$$= \left(2(4)^2 - \frac{(4)^3}{3}\right) - \left(2(0)^2 - \frac{(0)^3}{3}\right)$$

$$= \left(2(16) - \frac{64}{3}\right) - (0 - 0)$$

$$= 32 - \frac{64}{3} = \frac{96 - 64}{3} = \frac{32}{3} \text{ Sq units}$$

6. $y = x^2 + 2x - 3$ **(Sargodha 2011)**

Put $y = 0 \Rightarrow x^2 + 2x - 3 = 0$

$x^2 + 3x - x - 3 = 0 \Rightarrow x(x + 3) - 1(x + 3) = 0$

$(x + 3)(x - 1) = 0 \Rightarrow x + 3 = 0$ *or* $x - 1 = 0$

$x = -3$ *or* $x = 1$

x	-3	-2	-1	0	1
y	0	-3	-4	-3	0

$$Area = \int_{-3}^{1}(x^2+2x-3)dx = \int_{-3}^{1}x^2dx + 2\int_{-3}^{1}xdx - 3\int_{-3}^{1}1dx$$

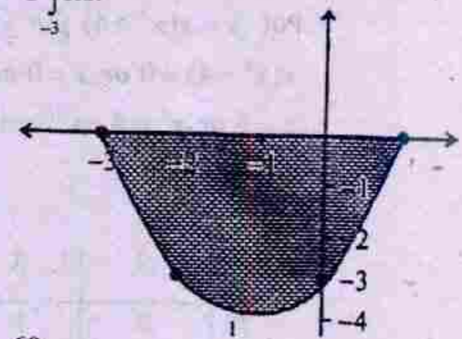
$$= \left[\frac{x^3}{3}\right]_{-3}^{1} + 2\left[\frac{x^2}{3}\right]_{-3}^{1} - 3[x]_{-3}^{1}$$

$$= \frac{1}{3} - \frac{(-3)^3}{3} + 2\left(\frac{(1)^2}{2} - \frac{(-3)^2}{2}\right) - 3(1-(-3))$$

$$= \frac{1}{3} - \left(\frac{-27}{3}\right) + 2\left(\frac{1}{2} - \frac{9}{2}\right) - 3(1+3)$$

$$= \frac{1}{3} + \frac{27}{3} + 1 - 9 - 12 = \frac{1}{3} + \frac{27}{3} - 20 = \frac{1+27-60}{3}$$

$$= \frac{32}{3} \text{ Sq units}$$



7. $y = x^3 + 1, \quad x = 2$

Put $y = 0 \Rightarrow x^3 + 1 = 0$

$x^3 + 1^3 = 0$ *or* $(x+1)(x^2 - x + 1) = 0$

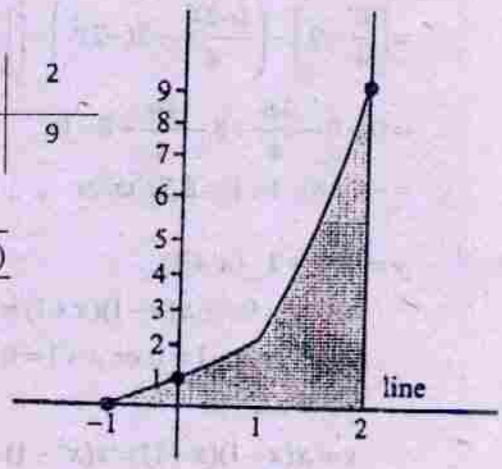
$x + 1 = 0$ *or* $x^2 - x + 1 = 0$

x	-1	0	1	2
y	0	1	2	9

$$x = -1 \text{ or } x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

$x = -1$ & $x = 2(line)$



$$Area = \int_{-1}^{2}(x^3+1)dx = \int_{-1}^{2}x^3dx + \int_{-1}^{2}1dx$$

$$= \left[\frac{x^4}{4}\right]_{-1}^{2} + [x]_{-1}^{2} = \left[\frac{2^4}{4} - \frac{(-1)^4}{4}\right] + (2-(-1)) = \frac{16}{4} - \frac{1}{4} + 2 + 1$$

$$= \frac{16}{4} - \frac{1}{4} + 3 = \frac{16-1+12}{4} = \frac{27}{4} \text{ Sq units}$$

8. $y = x^3 - 4x$

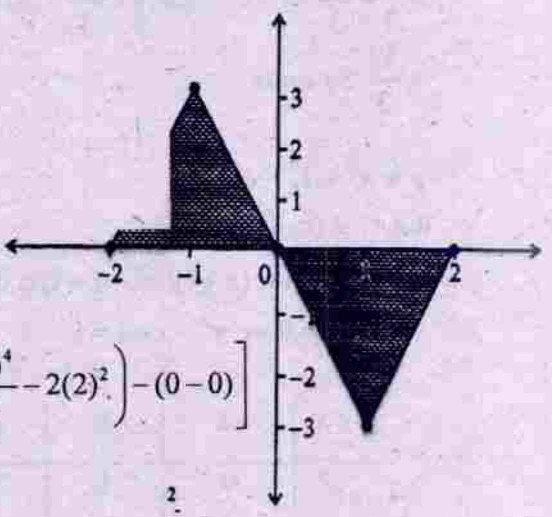
Put $y = x(x^2 - 4)$ put $y = 0 \Rightarrow$

$x(x^2 - 4) = 0$ or $x = 0$ or $x^2 - 4 = 0$

$x = 0$ or $x^2 = 4 \Rightarrow x = \pm 2$

$x = -2, 0, 2$

x	-2	-1	0	1	2
y	0	3	0	-3	0



Area = $\int_{-2}^{0}(x^3 - 4x)dx - \int_{0}^{2}(x^3 - 4x)dx$

$= \left[\frac{x^4}{4} - 4\frac{x^2}{2}\right]_{-2}^{0} - \left[\frac{x^4}{4} - 4\frac{x^2}{2}\right]_{0}^{2}$

$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} - \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$

$= \left(\frac{0}{4} - 0\right) - \left(\frac{(-2)^4}{4} - 2(-2)^2\right) - \left[\left(\frac{(2)^4}{4} - 2(2)^2\right) - (0 - 0)\right]$

$= 0 - 0 - \frac{16}{4} + 8 - \frac{16}{4} + 8 - 0$

$= -4 + 8 - 4 + 8 = 8\,Sq\ units$

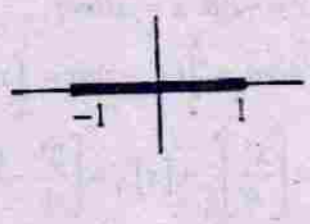
9. $y = x(x - 1\_(x + 1)$

Put $y = 0 \Rightarrow x(x-1)(x+1) = 0$

$x = 0$ or $x - 1 = 0$ or $x + 1 = 0$

$x = 0, 1, -1$

$y = x(x-1)(x+1) = x(x^2 - 1) = x^3 - x$



x	-1	0	1
y	0	0	0

$$\text{Now Area} = \int_{-1}^{0}(x^3 - x)dx - \int_{0}^{1}(x^3 - x)dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1}$$

$$= 0 - 0 - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right) - \left(\frac{(1)^4}{2} - \frac{(1)^2}{2}\right) - 0 - 0$$

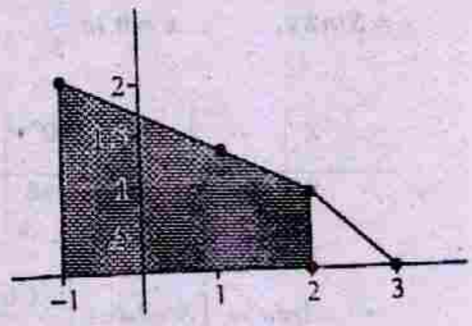
$$= \frac{1}{4} + \frac{1}{2} - \frac{1}{4} + \frac{1}{2} = \frac{-1+2-1+2}{4} = \frac{2}{4} = \frac{1}{2}\ Sq\ units$$

10. $y^2 = 3 - x \Rightarrow y = \sqrt{3 - x}$

$x = -1\ to\ x = 2$

x	-1	0	1	2	3
y	2	1.7	1.4	1	0

$$Area = \int_{-1}^{2}\sqrt{3 - x}dx = \int_{-1}^{2}(3 - x)^{1/2}dx$$



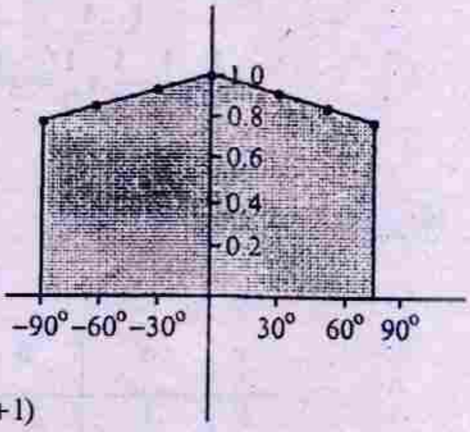
$$= -\int_{-1}^{2}(3-2)^{3/2} - (3-(-1))^{3/2} = -\frac{2}{3})1 - (3+1)^{3/2}) = -\frac{2}{3}(1 - 4^{3/2})$$

$$= -\frac{2}{3}(1 - 2^{2\times 3/2}) = -\frac{2}{3}(1 - 2^3) = -\frac{2}{3}(-7) = \frac{14}{3}Sq\ units$$

11. $y = Cos\frac{x}{2}, \qquad x = -\pi \ to \ \pi$

x	-90°	-60°	-30°	0	30°	60°	90°
y	0.70	0.86	0.96	1	0.96	0.8	0.7

$$Area = \int_{-\pi}^{\pi} Cos\frac{x}{2}dx = \left[\frac{Sin\frac{x}{2}}{\frac{1}{2}}\right]_{-\pi}^{\pi}$$



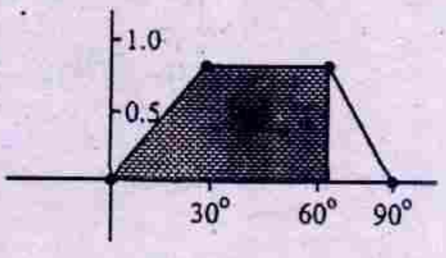
$$2 = \left(Sin\frac{\pi}{2} - Sin\frac{-\pi}{2}\right) = 2(1-(-1)) = 21(1+1)$$
$$= 4 \ Sq \ units$$

12. $y = Sin2x, \qquad x = 0 \ to \ \frac{\pi}{3}$

x	0	30°	60°	90°
y	0	0.86	0.86	

$$Area = \int_{0}^{\pi/3} Sin2dx = \left[\frac{-Cos2x}{2}\right]_{0}^{\pi/3}$$

$$= -\frac{1}{2}\left(Cos\frac{2\pi}{3} - Cos0\right)$$
$$= -\frac{1}{2}\left(-\frac{1}{2} - 1\right)$$
$$= -\frac{1}{2}\left(-\frac{3}{2}\right) = \frac{3}{4} \ Sq \ units$$



13. $y=\sqrt{2ax-x^2}$

Put $y=0 \Rightarrow \sqrt{2ax-x^2}=0 \Rightarrow 2ax-x^2=0$

$x(2a-x)=0 \Rightarrow x=0$ or $2a-x=0$

$x=0$ or $x=2a$

$$Area=\int_0^{2a}\sqrt{2ax-x^2}\,dx=\int_0^{2a}\sqrt{2ax-x^2-a^2+a^2}\,dx$$

$$=\int_0^{2a}\sqrt{-(-2ax+x^2+a^2)}\,dx=\int_0^{2a}\sqrt{-(a^2+x^2-2ax)}\,dx$$

$$=\int_0^{2a}\sqrt{a^2-(x-a)^2}\,dx \quad Put\ x-a=aSin\theta \quad dx=aCos\theta d\theta$$

When $x=0$ then $0-a=aSin\theta \Rightarrow Sin\theta=-1 \Rightarrow \theta=Sin^{-1}(-1)=\frac{-\pi}{2}$

When $x=2a$ then $2a=aSin\theta \Rightarrow Sin\theta=1 \Rightarrow \theta=Sin^{-1}(-1)=\frac{\pi}{2}$

$$=\int_{-\pi/2}^{\pi/2}\sqrt{a^2-a^2Sin^2\theta}\,aCos\theta d\theta=\int_{-\pi/2}^{\pi/2}\sqrt{a^2(1-\sin^2\theta)}\,aCos\theta d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\sqrt{a^2Cos^2\theta}\,aCos\theta d\theta=\int_{-\pi/2}^{\pi/2}aCos\theta.\ aCos\theta d\theta=\int_{-\pi/2}^{\pi/2}a^2Cos^2\theta d\theta$$

$$=\int_{-\pi/2}^{\pi/2}\left(\frac{1+Cos2\theta}{2}\right)d\theta=\frac{a^2}{2}\left[\theta+\frac{Sin^2\theta}{2}\right]_{-\pi/2}^{\pi/2}$$

$$=\frac{a^2}{2}\left[\frac{\pi}{2}+\frac{Sin\frac{2\pi}{2}}{2}-\left(\left(\frac{-\pi}{2}\right)-\frac{Sin2\left(\frac{-\pi}{2}\right)}{2}\right)\right]$$

$$=\frac{a^2}{2}\left[\frac{\pi}{2}+\frac{Sin\pi}{2}+\frac{\pi}{2}+\frac{Sin(-\pi)}{2}\right]=\frac{a^2}{2}\left[\frac{\pi}{2}+0+\frac{\pi}{2}+0\right]$$

$$=\frac{a^2}{2}\left(\frac{2\pi}{2}\right)=\frac{\pi a^2}{2}\ Sq.\ units$$

Exercise 3.8

first Part v.v.Imp

1.(i) $y = cx - 1$

$\frac{dy}{dx} = c(1) - 0 = c$

From (i) $cx = 1 + y \Rightarrow c = \frac{1+y}{x}$

So $\frac{dy}{dx} = \frac{1+y}{x} \Rightarrow x\frac{dy}{dx} = 1 + y$

(ii) $y^2 + y = c - \frac{1}{x} = c - x^{-1}$

Taking derivative

$2y\frac{dy}{dx} + \frac{dy}{dx} = 0 - (-1)x^{-2}$

$(2y+1)\frac{dy}{dx} = \frac{1}{x^2}$

$\Rightarrow \quad x^2(2y+1)\frac{dy}{dx} = 1$

$\Rightarrow \quad x^2(2y+1)\frac{dy}{dx} - 1 = 0$

(iii) $y^2 = e^{2x} + 2x + 1$

Taking derivative

$2y\frac{dy}{dx} = e^{2x}.\ 2 + 2$

$\div$ by 2

$y\frac{dy}{dx} = e^{2x} + 1$

$\Rightarrow y\frac{dy}{dx} - e^{2x} = 1$

(iv) $y = cex^2$

Taking derivative

$\frac{dy}{dx} = e.\ e^{2x}(2x)$

From I $ce^{2x} = y$ so

$$\frac{dy}{dx} = y.\,2x \Rightarrow \frac{1}{x}\frac{dy}{dx} = 2y$$

$$\Rightarrow \frac{1}{x}\frac{dy}{dx} - 2y = 0$$

(v) $y = Tan(e^x + c)$ **(Sargodha 2011,12)**

$$\frac{dy}{dx} = Sec^2(e^x + c)\frac{d}{dx}(e^x + c)$$

$$= (1 + Tan^2(e^x + c))(e^x)$$

Use I

$$= (1 + y^2)e^x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 + y^2}{e^{-x}}$$

2. $\frac{dy}{dx} = -y \Rightarrow dy = -ydx$ **(Sargodha 2008)**

$$\Rightarrow \frac{1}{y}dy = -dx$$

Taking integral both sides

$$\int \frac{1}{y}dy = -\int 1dx$$

$$\ln|y| = -x + \ln|c|$$

$$\ln|y| = -\ln|c| = -x$$

$$\ln\frac{y}{c} = -x$$

$$\Rightarrow \frac{y}{c} = e^{-x} \Rightarrow y = ce^{-x}$$

3. $ydx = xdy = 0$ **(Sargodha 2010)**

$$\Rightarrow ydx = -xdy$$

$$\frac{1}{x}dx = \frac{-1}{y}dy$$

Taking integral

$$\int \frac{1}{x}dx = -\int \frac{1}{y}dy$$

$$\ln|x| = -\ln|y| + \ln c$$

$\ln|x| + \ln|y| = \ln c$

$\ln(xy) = \ln c$

$\Rightarrow xy = c$

4. $\dfrac{dy}{dx} = \dfrac{1-x}{y}$

$\Rightarrow ydy = (1-x)dx$

Taking integral

$\int ydy = \int (1-x)dx$

$\dfrac{y^2}{2} = \left(x - \dfrac{x^2}{2}\right) + c_1$

X by 2

$y^2 = 2x - x^2 + 2c_1$ $\boxed{2c_1 = c}$

$\Rightarrow y^2 = 2x - x^2 + c$

$y^2 = x(2-x) + c$

5. $\dfrac{dy}{dx} = \dfrac{y}{x^2}$

$\Rightarrow \dfrac{1}{y}dy = \dfrac{1}{x^2}dx$

Taking integral

$\int \dfrac{1}{y}dy = \int x^2 dx$

$\ln|y| = \dfrac{x^{-2}+1}{-2+1} + \ln c$

$\ln|y| - \ln c = \dfrac{x^{-1}}{-1}$

$\ln\dfrac{y}{c} = -\dfrac{1}{x}$

$\dfrac{y}{c} = e^{-1/x} \Rightarrow y = ce^{-1/x}$

6. $Siny\ Cosec\,x\dfrac{dy}{dx} = 1$ **(Sargodha 2008,09)**

$\Rightarrow Siny\dfrac{1}{Sinx}dy = dx$

3.6, 3.8

$$Sinydy = Sinx\ dx$$

Taking integral

$$\int Siny\ dy = \int Sinx\,dx$$

$$-Cosy = -Cosx - C_1 \Rightarrow Cosy = Cosx + C_1$$

7. $\mathbf{xdy + y(x-1)dx = 0}$

$$xdy = -y(x-1)dx$$

$$\frac{1}{y}dy = -\left(\frac{x-1}{x}\right)dx$$

$$\frac{1}{y}dy = -\left(1-\frac{1}{x}\right)dx$$

Taking integral

$$\int\frac{1}{y}dy = -\int 1dx + \int\frac{1}{x}dx$$

$$\ln y = -x + \ln x + \ln c$$

$$\ln y - \ln x - \ln c = -x$$

$$\ln\frac{y}{cx} = -x \Rightarrow \frac{y}{cx} = e^{-x} \Rightarrow y = cxe^{-x}$$

8. $\mathbf{\frac{x^2+1}{y+1} = \frac{x}{y}\frac{dy}{dx}}$

$$\frac{x^2+1}{y+1} = \frac{1}{x}dx = \frac{1}{y}dy$$

$$\frac{x^2+1}{x}dx = \left(\frac{y+1}{y}\right)dy$$

Taking integral

$$\int\left(\frac{x^2}{x}+\frac{1}{x}\right)dx = \int\left(1+\frac{1}{y}\right)dy$$

$$\int\left(1+\frac{1}{y}\right)dy = \int\left(x+\frac{1}{x}\right)dx$$

$$y + \ln y = \frac{x^2}{2} + \ln x + \ln c$$

$$\ln e^y.\ y = \frac{x^2}{2} + \ln cx$$

$$\ln ye^y - \ln cx = \frac{x^2}{2}$$

$$\ln\frac{y^y}{cx} = \frac{x^2}{2} \Rightarrow \frac{ye^y}{cx} = e^{x^2/2} \Rightarrow ye^y = cxe^{x^2/2}$$

9. $$\frac{1}{x}\frac{dx}{dy} = \frac{1}{2}(1+y^2)$$

$$\Rightarrow\left(\frac{1}{1+y^2}\right)dy = \frac{1}{2}xdx$$

Taking integral

$$\int\frac{1}{(1+y^2)}dy = \frac{1}{2}\int xdx$$

$$Tan^{-1}y = \frac{1}{2}, \frac{x^2}{2} + c$$

$$Tan^{-1}y = \frac{x^2}{4} + c \text{ or } y = Tan\left(\frac{x^2}{4} + c\right)$$

10. $$2x^2y\frac{dx}{dy} = x^2 - 1$$

$$2xdy\left(\frac{x^2-1}{x^2}\right)dx$$

$$2ydy = (1-x^2)dx$$

Taking integral

$$2\int ydy = \int(1-x^2)dx$$

$$2.\frac{y^2}{2} = x - \frac{x^{-1}}{-1} + c$$

11. $$\frac{dx}{dy} + \frac{2xy}{2y+1} = x$$

$$\frac{dy}{d} = x - \frac{2xy}{2y+1}$$

$$\frac{dy}{dx} = \frac{2xy + x - 2xy}{2y+1}$$

$$\frac{dy}{dx} = \frac{x}{2y+1}$$

$(2y+1)dy = xdx$

Taking integral

$\int(2y+1)dy = \int x\,dx$

$\frac{2y^2}{2}1 = \frac{x^2}{2}+c$

$y^2+y = \frac{x^2}{2}+c$

$y(y+1) = \frac{x^2}{2}+c$

12. $(x^2-yx^2)\frac{dy}{dx}+y^2+xy^2=0$

$(x^2-yx^2)\frac{dy}{dx} = -(y^2+xy^2)$

$x^2(1-y)\frac{dy}{dx} = -y^2(1+x)$

$\frac{(1-y)}{y^2}dy = \frac{-(1+x)}{x^2}dx$

$-\left(\frac{1}{y^2}-\frac{2}{y^2}\right)dy = \left(\frac{1}{x^2}+\frac{x}{x^2}\right)dx$

$-\left(\frac{1}{y^2}-\frac{y}{y^2}\right)dy = \left(\frac{1}{x^2}+\frac{x}{x^2}\right)dx$

$-y^2+\frac{1}{y} = x^2+\frac{1}{x}$

Taking integral

$\int y^{-2}dy+\int\frac{1}{y}dy = \int x^2dx+\int\frac{1}{x}dx$

$-\frac{y^{-1}}{-1}+\ln|y| = \frac{x^{-1}}{-1}+\ln|x|+c$

$\Rightarrow \ln y+\frac{1}{y} = -\frac{1}{x}+\ln x+c$

13. $Sec^2xTanydx+Sec^2yTanxdy=0$ **(Sargodha 2011)**

$Sec^2yTanxdy = -TanySec^2xdx$

$$\frac{Sec^2 y}{Tany}dy = \frac{-Sec^2 x}{Tanx}dx$$

Taking integral

$$\int\frac{Sec^2 y}{Tany}dy = -\int\frac{Sec^2 x}{Tanx}dx$$

$$\ln(Tany) = -\ln(Tanx) + \ln c$$

$$\ln(Tany) = -\ln(Tanx) = \ln c$$

$$\ln TanxTany = \ln c$$

$$\Rightarrow TanxTany = \ln c$$

14. $$y - x\frac{dy}{dx} = 2\left(y^2 + \frac{dy}{dx}\right)$$

$$y - x\frac{dy}{dx} = 2y^2 + 2\frac{dy}{dx}$$

$$y - 2y^2 = x\frac{dy}{dx} + 2\frac{dy}{dx}$$

$$y-(1-2y) = (x+2)\frac{dy}{dx}$$

$$\left(\frac{1}{x+2}\right)dx = \frac{1}{y(1-2y)}dy$$

Solve $= \frac{1}{y(1-2y)} = \frac{A}{y} + \frac{B}{1-2y}$

$$\Rightarrow 1 = A(1-2y) + By$$

$$Put\ y = 0 \Rightarrow 1 = A(1-0) + 0 \Rightarrow \boxed{A=1}$$

$$Put\ 1-2y = 0 \quad \Rightarrow y = \frac{1}{2}$$

$$1 = A(0) + B\frac{1}{2} \Rightarrow \boxed{B=2}\ So$$

$$\left(\frac{1}{x+2}\right)dx = \left(\frac{1}{y} - \frac{2}{1-2y}\right)dy$$

Taking integral

$$\int\frac{1}{x+2}dx = \int\frac{1}{y}dy + \int\frac{1}{1-dy}dy$$

$$\int\frac{1}{x+2}dx=\int\frac{1}{y}dy-\int\frac{-2dy}{1-2y}$$
$$\ln|x+2|=\ln y-\ln|1-2y|-\ln$$
$$\ln c+\ln(x+2)=\ln y-\ln(1-2y)$$
$$\ln c(x+2)=\ln\frac{y}{1-2y}$$
$$c(x+2)=\frac{y}{1-2y}$$

15. $$1+Cosx\,Tany\frac{dy}{dx}=0$$
$$Cosx\ Tany\frac{dy}{dx}=-1$$
$$Tany\ dy=-\frac{1}{Cosx}dx$$
$$\int\frac{Siny}{Cosy}dy=-\int\frac{1}{Cosx}dx$$
$$\int\frac{-Siny}{Cosy}dy=\int\frac{1}{Cosx}dx$$
$$\ln|Cosy|=\int Secx\ dx$$
$$\ln|Cosy|=\ln|Secx+Tanx|+\ln c$$
$$\ln|Cosy|=\ln c(Secx+Tanx)$$
$$\Rightarrow Cosy=c(Secx+Tanx)$$

16. $$y-x\frac{dy}{dx}=3\left(1+x\frac{dy}{dx}\right)$$ **(Sargodha 2012)**
$$y-x\frac{dy}{dx}=3+3x\frac{dy}{dx}$$
$$y-3=x\frac{dy}{dx}+3x\frac{dy}{dx}$$
$$y-3=4x\frac{dy}{dx}$$
$$\frac{1}{x}dx=\frac{4}{y-3}dy$$

Taking Integral

$$\int\frac{1}{x}\,dx = 4\int\frac{1}{y-3}\,dy$$

$$\ln c + \ln|x| = 4\ln|y-3|$$

$$\ln cx = \ln(y-3)^4$$

$$(y-3)^4 = cx \Rightarrow y-3 = (cx)^{1/4}$$

$$y = 3e^{1/4}x^{1/4} = 3c_1x^{1/4}$$

17. $$\mathbf{Secx + Tany\frac{dy}{dx} = 0}$$

$$Tany\frac{dy}{dx} = -Secx$$

$$Tany\ dy = -Secx\ dx$$

Taking integral

$$\int\frac{-Siny}{Cosy}\,dy = \int Secx\ dx$$

$$\ln|Cosy| = \ln|Secx + Tanx| + \ln c$$

$$\ln(Cosy) = \ln c(Secx + Tanx)$$

$$\Rightarrow Cosy = c(Secx + Tanx)$$

18. $$\mathbf{(e^x + e^{-x})\frac{dy}{dx} = e^x - e^{-x}}$$

$$dy = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)dx$$

Taking integral

$$y = \int\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)dx = \ln\left|e^x + e^{-x}\right| + c$$

19. $$\mathbf{\frac{dy}{dx} - x = xy^2}$$

$$\frac{dy}{dx} = x + xy^2 = x(1+y^2)$$

$$\left(\frac{1}{1+y^2}\right)dy = x\,dx$$

Taking integral

$$\int \frac{1}{(1+y^2)}\,dy = \int x\,dx$$

$$Tan^{-1}y = \frac{x^2}{2} + c$$

$$Put\ \ y = \&\ x = 0$$

$$Tan^{-1}(1) = 0 + c \Rightarrow \boxed{\frac{\pi}{4} = c}$$

I become

$$Tan^{-1}y = \left(\frac{x^2}{2} + \frac{\pi}{4}\right)\ (particular\ Sol)$$

20. $$\frac{dx}{dt} = 2x \Rightarrow \frac{1}{x}\,dx = 2dt$$

Taking integral

$$\int \frac{1}{x}\,dx = 2\int 1\ dt$$

$$\ln x = 2t + \ln c$$

$$\ln x - \ln c = 2t$$

$$\ln \frac{x}{c} = 2t \Rightarrow \frac{x}{c} = e^{2t}$$

$$x = ce^{2t}\ put\ x = 4\ \&\ t = 0$$

$$4 = ce^{0} \Rightarrow \boxed{4 = c}$$

$$x = 4e^{2t}$$

21. $$\frac{ds}{dt} + 2st = 0$$

$$\frac{ds}{dt} = -2st \Rightarrow \frac{1}{s}\,ds = -2tdt$$

Taking integral

$$\int \frac{1}{s}\,ds = -2\int t\,dt$$

$$\ln s = \cancel{2}\frac{t^2}{\cancel{2}} + \ln c$$

$$\ln s - \ln c = -t^2$$

$$\ln \frac{s}{c} = -t^2 \Rightarrow \frac{s}{c} = e^{-t^2}$$

$s = ce^{-t^2}$

$Put\ S = 4e\ \&\ t = 0$

$4e = ce^0 \Rightarrow \boxed{4e = c}$

$S = 4e.\, e^{-t^2} = 4e^{-t^2+1}$

$S = 4e^{1-t^2}$

22. **Let p be numbers of bacteria then $\frac{dp}{dt} = kp$ (According to the given condition)**

$\frac{1}{p}dp = kdt$

Taking integral

$\int \frac{1}{p}dp = k\int 1dt$

$\ln p = kt + \ln c$

$\ln\frac{p}{c} = kt \Rightarrow \frac{p}{c} = e^{kt}$

$\Rightarrow p = ce^{kt}$

$Put\ p = 200,\ t = 0\ (Condition\ I)$

$200 = ce^0 \Rightarrow \boxed{c = 200}$

I become $p = 200e^{kt}$

$Condition\ II \Rightarrow p = 400\ when\ t = 2$

$So\ 400 = 200e^{2k} \Rightarrow \frac{400}{200} = e^{kt}$

$e^{2k} = 2 \Rightarrow \ln e^{2k} = \ln 2$

$\Rightarrow 2k\ \ln 2 \Rightarrow 2k(1) = \ln 2$

$k = \frac{1}{2}\ln 2\ II\ become$

$p = 200e^{\ln 2^{1/2}} = 200e^{1/2(\ln 2)}$

$p = 200e^{\ln 2^{1/2}} = 200.2^{1/2}$

$Put\ t = 4\ then$

$p = 200.2^{4/2} \Rightarrow p = 200.2^2$

$p = 200 \times 4 = 800$

23. Let v is velocity & g is acceleration of gravity then we know that

$$\frac{dv}{dt} = acceleration$$

$$\frac{dv}{dt} = -g \Rightarrow dv = -gdt$$

$$\Rightarrow \int dv = -g\int dt \Rightarrow v = -gt + c$$

$$Put\ v = 2450\ \&\ t = 0\ then$$

$$2450 = 0 + c_1 \Rightarrow c_1 = 2450$$

$$v = -gt + 2450$$

$$v = -980t + 2450$$

$$g = 9.8meter = 980cm\ so$$

$$\boxed{v = -980t + 2450}$$

Let h be the height

$$v = \frac{dh}{dt} = -980t + 2450$$

$$dh = (-980t + 2450)dt$$

$$\int dh = -980\int tdt + 2450\int 1dt$$

$$h = -980.\frac{t^2}{2} + 2450t + c_2$$

$$h = -490t^2 + 2450t + c_2$$

$$Put\ h = 0,\ t = 0$$

$$0 = 0 + 0 + c_2 \Rightarrow c_2 = 0$$

$$\boxed{h = 2450t - 290t^2}$$

From max height $v = 0$ then $0 = -980t + 2450$ from I

$$980t + 2450 \Rightarrow t = \frac{2450}{980} = \frac{5}{2}$$

Put in II

$$h = 2450\left(\frac{5}{2}\right) - 490\left(\frac{25}{4}\right)$$

$$= 6125 - 3062.5$$

$$h = 3062.5$$

So max height $= 3062.5$cm $\div$ by 100 to convert int meters

$$= 30.62m$$

TEST YOUR SKILLS Marks 100

OBJECTIVE

Q.No.1 Given below are a few possible answers to each statement of which one is correct, identify the correct one. (20)

1. The inverse process of differentiation is called
   (a) anti-derivative (b) integration
   (c) Both (a) & (b) (d) None of these
2. If $\phi'(x) = f(x)$, then $\phi(x)$ is called an
   (a) derivative (b) integral
   (c) differential coefficient (d) None of these
3. $\int (ax+b)^n dx =$ ______, if $n \neq -1$ then
   (a) $(ax+b)^{n+1}+c$ (b) $\dfrac{(ax+b)^{n+1}}{a(n+1)}+c$
   (c) $\dfrac{(ax+b)^{n+1}}{a}+c$ (d) $\dfrac{(ax+b)^{n}}{a(n+1)}+c$
4. $\int \text{Sec}^2(ax+b)dx =$ ______
   (a) $\text{Tan}(ax+b)+c$ (b) $\frac{1}{a}\text{Cos}(ax+b)+c$
   (c) $\frac{1}{a}\text{Tan}(ax+b)+c$ (d) $-\frac{1}{b}\text{Tan}(ax+b)$
5. $\int \text{Cosec}(ax+b)\text{Cot}(ax+b)dx =$ ______
   (a) $\frac{1}{a}\text{Sec}(ax+b)+c$ (b) $\frac{1}{a}\text{Cot}(ax+b)$
   (c) $-\frac{1}{a}\text{Cosec}(ax+b)+c$ (d) $-\frac{1}{a}\text{Cot}(ax+b)+c$
6. $\int \text{Sec}\, x\, dx =$ ______
   (a) $\ln|\text{Sec}\,x+\text{Cot}\,x|+c$ (b) $\ln|\text{Tan}\,x+\text{Cosec}\,x|+c$
   (c) $\ln|\text{Sec}\,x+\text{Tan}\,x|+c$ (d) $\ln|\text{Sin}\,x+\text{Cot}\,x|+c$
7. $\int \text{Cosec}\, x\, dx =$ ______
   (a) $\ln|\text{Cosec}\,x+\text{Cot}\,x|$ (b) $\ln|\text{Cosec}\,x-\text{Tan}\,x|+c$
   (c) $\ln|\text{Sin}\,x-\text{Cot}\,x|+c$ (d) $\ln|\text{Cosec}\,x-\text{Cot}\,x|+c$
8. $\int e^{ax+b}dx =$ ______
   (a) e^{ax+b} (b) $\frac{1}{a}e^{ax+b}+c$
   (c) $\frac{1}{b}e^{ax+b}+c$ (d) $\frac{1}{a}e^{ax-b}+c$

9. $\int \frac{ax+b}{ax^2+2bx+c}dx =$ ______

(a) $(ax^2+2bx+c)$ (b) $\frac{1}{2}\ln|ax^2+2bx+c|+c_1$

(c) $\ln|ax^2+2bx+c|+c_1$ (d) $\frac{(ax^2+2bx+c)}{2}$

10. $\int a^{x^2} x dx =$ ______

(a) $\frac{a^{x^2}}{\ln a}$ (b) $\frac{a^{x^2}}{2\ln a}+c$

(c) $a^{x^2}+c$ (d) $2a^{x^2}+c$

11. $\int \frac{1}{(1+x^2)Tan^{-1}x}dx =$ ______

(a) $\ln|1+x^2|+c$ (b) $(1+x^2)^2+c$

(c) $\text{Tan}^{-1}x+c$ (d) $\ln|\text{Tan}^{-1}x|+c$

12. $\int f(x)g'(x)dx =$ ______

(a) $f(x)g(x)$ (b) $f(x)g(x)-\int g(x)f'(x)dx+c$

(c) $f(x)g(x)+\int g(x)f'(x)dx+c$ (d) None of these

13. $\int \ln x\, dx =$ ______

(a) $x\ln x+c$ (b) $x\ln x-x+c$

(c) $x\ln x+x+c$ (d) $\ln x+x+c$

14. $\int e^x(\text{Sin}\,x-\text{Cos}\,x)dx =$ ______

(a) $e^x\text{Cos}\,x+c$ (b) $-e^x\text{Cos}\,x+c$

(c) $e^x\text{Sin}\,x+c$ (d) $-e^x\text{Sin}\,x+c$

15. If $a<c<b$, $\int_a^b f(x)dx =$ ______

(a) $\int_a^c f(x)dx$ (b) $\int_c^b f(x)dx$

(c) $\int_a^c f(x)dx-\int_c^b f(x)dx$ (d) $\int_a^c f(x)dx+\int_c^b f(x)dx$

16. $\int_0^5 f(x)dx =$ ______

(a) $\int_0^4 f(x)dx+\int_0^1 f(x)dx$ (b) $\int_0^2 f(x)dx+\int_2^5 f(x)dx$

(c) $\int_{5}^{0} f(x)dx$

(d) $\int_{0}^{4} f(x)dx + \int_{1}^{5} f(x)dx$

17. If $\int_{0}^{2} f(x)dx = 5$ then $3\int_{0}^{2} f(x)dx =$ ______

(a) 7
(b) 10
(c) 4
(d) 15

18. $\int_{0}^{\pi/3} \text{Sec}\,x \ \text{Tan}\,x\, dx =$ ______

(a) –1
(b) 0
(c) 1
(d) 2

19. $\int_{0}^{\sqrt{3}} \frac{dx}{1+x^2} =$ ______

(a) $\frac{\pi}{2}$
(b) $\frac{\pi}{4}$
(c) $\frac{\pi}{6}$
(d) $\frac{\pi}{3}$

20. The solution of $x\frac{dy}{dx} = 1 + y$ is

(a) $y = cx$
(b) $y = cx + 1$
(c) $y = cx - 1$
(d) None of these

SECTION I
SUBJECTIVE

Attempt any 25 (twenty five) short question from Question 2,3 and 4, at least 12 short questions from question 2, 12 short question from question 3 and 13 short question from question 4. All question carry equal marks. ***(25x2=50)***

Q.No. 2

i. What do you mean by the word "Integration"?

ii. Give a function $y = f(x)$ then distinguish between dy and δy

iii. Using differential find $\frac{dy}{dx}$: $x^4 + y^2 = xy^2$

iv. Find the approximate increase into the volume of a cube if the length of its each edge changes from 5 to 5.02.

v. Find the approximate increase in the area of a circular disc of its diameter is increased from 44 cm to 44.4 cm.

vi. Evaluate $\int (2x+3)^{1/2} dx$

vii. Evaluate $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$

viii. Evaluate $\int \text{Cosec}\, x dx$

ix. Evaluate $\int \text{Sec}\, x dx$

x. Evaluate $\int \frac{1}{x \ln x} dx$

xi. Evaluate $\int \frac{1}{(1+x^2)\,\text{Tan}^{-1} x} dx$

xii. Evaluate $\int \frac{\text{Cos}\, x}{\text{Sin}\, x \ln \text{Sin}\, x} dx$

Q.No. 3

i. Evaluate $\int x \,\text{Cos}\, x dx$

ii. Evaluate $\int x e^x dx$

iii. Evaluate $\int \text{Sin}^{-1} x dx$

iv. Evaluate $\int e^x \left(\frac{1}{x} + \ln x\right) dx$

v. Evaluate $\int e^x (\text{Cos}\, x + \text{Sin}\, x) dx$

vi. Evaluate $\int e^{-x} (\text{Cos}\, x - \text{Sin}\, x) dx$

vii. Evaluate $\int e^{2x} (-\text{Sin}\, x + 2\,\text{Cos}\, x) dx$

viii. Evaluate $\int e^{ax} \left(a\,\text{Sec}^{-1} x + \frac{1}{x\sqrt{x^2-1}}\right) dx$

ix. Evaluate $\int \frac{\text{Cot}\sqrt{x}}{\sqrt{x}} dx$

x. Evaluate $\int a^{x^2} x dx$

xi. Evaluate $\int \frac{e^{2x} + e^x}{e^x} dx$

xii. Evaluate $\int_{-1}^{3} (x^3 + 3x^2) dx$

Q.No. 4

i. Evaluate $\int_{0}^{3} \frac{dx}{x^2+9}$

ii. Evaluate $\int_{1}^{2} \frac{x}{x^2+2} dx$

iii. Evaluate $\int_{-1}^{2}(x+|x|)dx$

iv. Evaluate $\int \text{Sin}(a+b)x\,dx$

v. Solve $x\frac{dy}{dx}=1+y$

vi. $\frac{1}{x}\frac{dy}{dx}-2y=0$

vii. $ydx+xdy=0$

viii. $\frac{dy}{dx}=\frac{1-x}{y}$

ix. $(e^{x}+e^{-x})\frac{dy}{dx}=e^{x}-e^{-x}$

x. 1 + Cos x Tan x $\frac{dy}{dx}$ = 0

xi. Define differential equation and define order of differential equation..

xii. Evaluate $\int_{1}^{2}a^{x}dx$

xiii. Evaluate $\int_{1}^{2}(x^{2}+1)dx$

SECTION II

Attempt any 3 (three) questions. **(3x10=30)**

Q.No.5

(a) Solve $2x^{2}y\frac{dy}{dx}=x^{2}-1$ by differential equations

(b) Using differentials find $\frac{dy}{dx}$ when $\frac{y}{x}$ – lnx=lnc

Q.No.6

(a) Using differentials to approximate the value of $\cos 29^{0}$

(b) Find the area between the x-axis and the curve $y=\sqrt{2ax-x^{2}}$ when a >0.

Q.No.7

(a) Evaluate the indefinite integrals $\int\frac{\sqrt{y}(y+1)}{y}dy,(y>0)$

(b) Evaluate the definite integrals $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\cos x}{\sin x(2+\sin x)}dx$

Q.No.8

(a) Solve $\frac{dy}{dx}=\frac{3}{4}x^2+x-3,\ if\ y=0\ when\ x=2$

(b) Evaluate $\int \cos^3 x\sqrt{\sin x}\,dx, (\sin x>0)$

Q.No.9

(a) Evaluate $\int e^{2x}\left[-\sin x+2\cos x\right]dx$

(b) Evaluate $\int_{-1}^{2}(x+|x|)dx$

Previous Board Questions

1. Use differentials to find the value of $\frac{dy}{dx}$ if $x^2+2y^2=16$. (Lhr – 2008)
2. Find dy for $y=x^2$, $x=2$, $dx=0.01$. (Lhr – 2005)
3. Evaluate $\int \frac{x}{x+2}dx$. (Lhr – 2005)
4. Evaluate $\int (\sin x-\cos x)dx$. (Grw – 2005)
5. Evaluate $\int \frac{dx}{\sqrt{2x+x^2}}$ $(x>0)$. (Mirpur – 2009)
6. Evaluate $\int \frac{x^2-1}{(x-1)^2}dx$ $(x>1)$. (Multan – 2009)
7. Evaluate $\int \frac{\sqrt{2}}{\sin x+\cos x}dx$ (Faisalabad – 2009)
8. Evaluate $\int \operatorname{cosec} x\,dx$ (Faisalabad – 2009)
9. Evaluate $\int \sqrt{a^2-x^2}\,dx$. (Lahore – 2009)
10. Evaluate $\int \sqrt{1-\cos 2x}\,dx$.
    (Lhr — 2009, 2009, Faisalabad – 2009, Mirpur – 2009, Multan – 2009)
11. Evaluate $\int_{0}^{\frac{\pi}{2}} \sin^3 x\cos x\,dx$. (Multan – 2009)
12. Find the area bounded by the curve $y=x^3+3x^2$ and the x–axis.
    (Mirpur – 2009)
13. Evaluate $\int (\ln x)^2 dx$. (Lahore – 2010) Group – I

14. Evaluate $\int \frac{(a-b)x}{(x-a)(x-b)}dx\ (a > b)$ (Lahore – 2010) Group – I

15. Integrate $\int (2x+3)^{1/2}dx$ (Lahore – 2010) Group – II

16. Evaluate $\int lnx \cdot \frac{1}{x}dx$ (Lahore – 2010) Group – II

17. Find the area bounded by the curve $y = \cos x$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.
(Gujranwala – 2010)

18. Solve $\frac{dy}{dx} = \frac{3}{4}x^2 + x - 3$ if $y = 0$ where $x = 2$. (Gujranwala – 2010)

****

4.1 & 4.3 & 4.4

Introduction to Analytic Geometry

4

Definitions:

1. Co-ordinate system:
Draw the plane two mutually perpendicular lines intersect at0 origin divides plane in four equal parts. These lines are called axes and system is called co-ordinate system.

2. Translation:
Let xy-co-ordinate system be given and O'(h, k) is any point in plane. Through O' draw new perpendicular lines O'x and O'y parallel to Ox and Oy. New axes O'x and O'y are called translation of Ox and Oy.

3. Slop or Gradient:
The measure of steepness (ratio of rise to run) is termed as slope or gradient denoted by $m = Tan\alpha$

4. Trapezium:
A quadrilaterals having two parallel and two non-parallel sides.

5. Homogeneous Equation:
Equation $f(x, y) = 0$ is called homogeneous equation

Important Formulas

1. Distance = $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (point to point) (Sargodha 2010)

2. Ratio (divide internally) = $\left(\frac{k_1x_2 + k_2x_1}{k_1 + k_2}, \frac{k_1y_2 + k_2y_1}{k_1 + k_2}\right)$

3. Ration (Divide externally) = $\left(\frac{k_1x_2 - k_2x_1}{k_1 - k_2}, \frac{k_1y_2 - k_2y_1}{k_1 - k_2}\right)$

4. Mid Point = $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

5. Centroid = $\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

6. Equation of Translation $\left.\begin{array}{l} X = x - h \\ Y = y - k \end{array}\right\}$

7. Equation of Rotation $\left.\begin{array}{l} X = xCos\theta + ySin\theta \\ Y = yCos\theta - xSin\theta \end{array}\right\}$

8. Slope = $m = Tan\alpha$

9. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ (if two points are given)

10. $m = \dfrac{-a}{b}$ if line $(ax + by + c = 0)$ is given

11. Two lines are parallel if $\boxed{m_1 = m_2}$ *also* $\boxed{a_1b_2 - a_2b_1 = 0}$

12. Two lines are perpendicular $\boxed{m_1.m_2 = -1}$

13. Collinear ; slope of AB = slope of AC

14. Slope intercept from $y = mx + c$

15. Two intercept from $\dfrac{x}{a} + \dfrac{x}{b} = 1$

16. Equation of Line $(y - y_1) = m(x - x_1)$

17. Symmetric from $\dfrac{x - x_1}{Cos\alpha} = \dfrac{y - y_1}{Sin\alpha} = r$

18. Normal from $xCos\alpha + ySin\alpha = p$

19. Distance = d (from one point to line)= $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

20. Area of Triangle $\Delta = \dfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

21. $Tan\theta = \dfrac{m_2 - m_1}{1 + m_2 m_1}$

22. $m_1 + m_2 = \dfrac{-2h}{b}$ & $m_1 m_2 = \dfrac{a}{b}$

23. $Tan\theta = \dfrac{2\sqrt{h^2 - ab}}{a+b}$

24. $h^2 - ab = 0$ then lines are coincident

25. $a + b = 0 \; then \; \theta = 90^o$

26. Joint equation $ax^2 + 2hxy + by^2 = 0$

Exercise 4.1

1. **Objective Type:**

(i) $x > 0$ the right half plane ($x > 0$ mean x Positive)

(ii) ($x > 0$ and $y > 0$) first quadrant (Because both x and y positive in 1st quadrant)

(iii) $x = 0$ The $y-$axis (Because x is zero on $x-$axis)

(iv) $y = 0$ The $x-$axis (Because y is zero on $x-$axis)

(v) $x < 0$ and $y \geq 0$ the second quadrant (Because x negative and y positive in II quadrant).

(vi) $x = y$ point in the first and third quadrant.
(For example $(1,1),(2,2),(-3,-3),(-4,-4)$

(vii) $|x| = |y|$ first and third quadrant or second and fourth quadrant
(Because $\left[(1,1),(-1,-1), \; for \; I \; and \; III \; and \; (1,-1),(-1,1) \; for \; II \; and \; IV\right]$

(viii) $|x| \geq 3$ (on $x-axis$ less than equal to -3 and greater than equal to 3)
(Because $|x| \geq 3 \Rightarrow \pm x \geq 3, \; x \geq 3 \; and \; -x \geq 3, x \geq 3 \; and \; x \leq -3$)

(ix) $x > 2 \; and \; y = 2$ (In first quadrant x greater than 2 and y = 2).

(x) $x \; and \; y$ have opposite sign (The $II \; and \; IV$ quadrant) Example
$(2,-2) \; or \; (-2,2)$

2. (a) A (3, 1), B (−2, −4) **(Sargodha 2008, 10)**

$d = |AB| = \sqrt{(-2-3)^2 + (-4-1)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$

Mid point of $\overline{AB} = \left(\dfrac{3+(-2)}{2}, \dfrac{1+(-4)}{2}\right) = \left(\dfrac{3-2}{2}, \dfrac{1-4}{2}\right) = \left(\dfrac{1}{2}, \dfrac{-3}{2}\right)$

(b) $A(-8,3), B(2,-1)$

$$d = |AB| = \sqrt{(2-(-8))^2 + (-1-3)^2} = \sqrt{(2+8)^2 + (-1-3)^2}$$

Mid point of $\overline{AB} = \left(\frac{-8+2}{2}, \frac{3+(-1)}{2}\right) = \left(\frac{-6}{2}, \frac{2}{2}\right) = (-3,1)$

(c) $A\left(-\sqrt{5}, -\frac{1}{3}\right), B(-3\sqrt{5}, 5)$ **(Sargodha 2009, 11)**

$$d = |AB| = \sqrt{(-3\sqrt{5} - (-\sqrt{5}))^2 + \left(5 - \left(-\frac{1}{3}\right)\right)^2}$$

$$= \sqrt{(-3\sqrt{5} + \sqrt{5})^2 + \left(5 + \frac{1}{3}\right)^2} = \sqrt{(-2\sqrt{5})^2 + \left(\frac{16}{3}\right)^2}$$

$$= \sqrt{4 \times 5 = \frac{256}{9}} = \sqrt{20 + \frac{256}{9}} = \sqrt{\frac{180 + 256}{9}}$$

$$= \sqrt{\frac{436}{9}} = \frac{2\sqrt{109}}{3}$$

Mid point of $\overline{AB} = \left(\frac{-\sqrt{5} + (-3\sqrt{5})}{2}, \frac{-\frac{1}{3} + 5}{2}\right)$

$$= \left(\frac{-\sqrt{5} - 3\sqrt{5}}{2}, \frac{\frac{14}{3}}{2}\right) = \left(\frac{-4\sqrt{5}}{2}, \frac{14}{3} \times \frac{1}{2}\right)$$

$$= \left(-2\sqrt{5}, \frac{7}{3}\right)$$

3. **(a)** **Suppose O(0, 0) and Given** $A(\sqrt{176}, 7)$

Then $|OA| = \sqrt{(\sqrt{176} - 0)^2 + (7-0)^2} = \sqrt{176 + 49} = \sqrt{225} = 15$

Thus $A(\sqrt{176}, 7)$ is at 15 unit from origin.

(b) Suppose origin O(0, 0) and $A(10,-10)$

Then $|OA| = \sqrt{(10-0)^2+(-10-0)^2} = \sqrt{100+100} = \sqrt{200} = 10\sqrt{2}$

$A(10,-10)$ is NOT at 15 unit from origin.

(c) Suppose origin O(0, 0) and $A(1, 15)$

Then $|OA| = \sqrt{(1-0)^2+(15-0)^2} = \sqrt{1+225} = \sqrt{226}$

$A(1, 15)$ is NOT at 15 unit from origin.

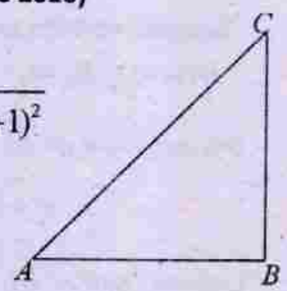
(d) $A\left(\frac{15}{2},\frac{15}{2}\right)$ and O (0, 0) then **(Sargodha 2010)**

Then $|OA| = \sqrt{\left(\frac{15}{2}-0\right)^2+\left(\frac{15}{2}-0\right)^2} = \sqrt{\frac{225}{4}+\frac{225}{4}} = \sqrt{\frac{450}{4}}$

$= \sqrt{\frac{225}{2}} = \frac{15}{\sqrt{2}}$ unit

A is NOT at 15 unit distance from origin.

4. (i) $A\ (0,2),\ B\ (\sqrt{3},-1),\ C\ (0,-2)$ **(Lahore 2010)**



$|AB| = \sqrt{(\sqrt{3}-0)^2+(-1-2)^2} = \sqrt{3+9} = \sqrt{12}$

$|BC| = \sqrt{(0-\sqrt{3})^2+(-2-(-1))^2} = \sqrt{(-\sqrt{3})^2+(-2+1)^2}$

$= \sqrt{3+1} = \sqrt{4} = 2$

$|AC| = \sqrt{(0-0)^2+(-2-2)^2}$

$= \sqrt{0+(-4)^2} = \sqrt{16} = 4$

Now $|AB|^2+|BC|^2 = (\sqrt{12})^2+(2)^2 = 12+4 = 16 = (4)^2 = |AC|^2$

PYTHAGORAS theorem satisfied So ABC is right triangle.

(ii) $A\ (3,1),\ B\ (-2,-3),\ C\ (2,\ 2)$

$|AB| = \sqrt{(-2-3)^2+(-3-1)^2} = \sqrt{(-5)^2+(-4)^2} = \sqrt{25+16} = \sqrt{41}$

$|BC| = \sqrt{(2-2)^2+(2+3)^2} = \sqrt{(4)^2+(5)^2} = \sqrt{16+25} = \sqrt{41}$

$= \sqrt{(2-(2))+(2-(-3))}$

$|CA| = \sqrt{(3-2)^2+(1-2)^2} = \sqrt{(1)^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$

Since $|AB| = |BC|$ So ABC is isosceles triangle.

(iii) **$A(5, 2)$, $B(-2, 3)$, $C(-3, -4)$, $D(4, -5)$**

$$|AB| = \sqrt{(-2-5)^2 + (3-2)^2} = \sqrt{(-7)^2 + (1)^2} = \sqrt{49+1} = \sqrt{50}$$

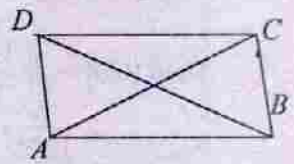
$$|BC| = \sqrt{(-3-(-2))^2 + (-4-3)^2} = \sqrt{(-3+2)^2 + (-7)^2} = \sqrt{1+49} = \sqrt{50}$$

$$|CD| = \sqrt{(4-(-3))^2 + (-5-(-4))^2} = \sqrt{(4+3)^2 + (-5+4)} = \sqrt{(7)^2 + (-1)^2} = \sqrt{50}$$

$$|AD| = \sqrt{(5-4)^2 + (2-(-5))^2} = \sqrt{(1)^2 + (7)^2} = \sqrt{1+49} = \sqrt{50}$$

$|AB| = |CD|$ and $|BC| = |AD|$ So ABCD is a parallelogram.

Now for Square Check diagonals



$$|AC| = \sqrt{(-3-5)^2 + (-4-2)^2} = \sqrt{(-8)^2 + (-6)^2}$$

$$= \sqrt{64+36} = \sqrt{100} = 10$$

$$|BD| = \sqrt{(4-(-2))^2 + (-5-3)^2} = \sqrt{(4+2)^2 + (-8)^2} = \sqrt{36+64} = \sqrt{100} = 10$$

Since $|AC| = |BD|$ So ABCD is also a Square.

5. **Suppose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and given mid points are $D(1, -1)$, $E(-4, -3)$, $F(-1, 1)$**

D is mid Point of $\overline{AB}$ so $1 = \dfrac{x_1 + x_2}{2}$ & $-1 = \dfrac{y_1 + y_2}{2}$

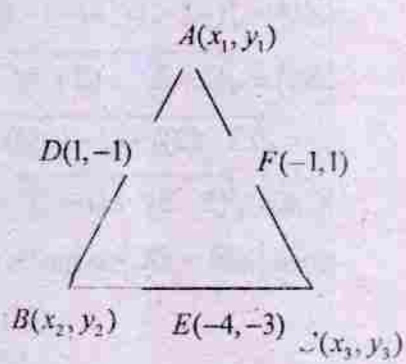
$$\Rightarrow x_1 + x_2 = 2 \quad I \quad \& \quad y_1 + y_2 = -2 \quad II$$

E is mid Point of $\overline{BC}$ so $-4 = \dfrac{x_2 + x_3}{2}$ & $\dfrac{y_2 + y_3}{2} = -3$

$$\Rightarrow x_2 + x_3 = -8 \quad III \quad \& \quad y_2 + y_3 = -6 \quad IV$$

F is mid Point of $\overline{AC}$ so $-1 = \dfrac{x_1 + x_3}{2}$ & $1 = \dfrac{y_1 + y_3}{2} = -3$

$$\Rightarrow x_1 + x_3 = -2 \quad V \quad \& \quad y_1 + y_3 = 2 \quad VI$$



Solving $I \;\&\; III$

$$\begin{array}{rl} & x_1 + \cancel{x_2} = 2 \\ I - III & \underline{-\cancel{x_2} \pm x_3 = \mp 8} \\ & x_1 - x_3 = 10 - (a) \end{array}$$

Solving $IV \;\&\; V$

$$IV + V \qquad \begin{array}{r} x_1 + \cancel{x_3} = -2 \\ x_1 - \cancel{x_3} = 10 \\ \hline 2x_1 = 8 \end{array}$$

$\boxed{x_1 = 4}$

Put $x_1 = 4$ *in I*

$x_2 = 2 - x_1 = 2 - 4 = -2$

$\boxed{x_2 = -2}$

Put $x_1 = 4$ *in V*

$x_3 = -2 - x_1 = -2 - 4 = -6$ $\boxed{x_3 = -6}$

Solving $II \,\&\, IV$

$II - IV$

$$\begin{array}{r} y_1 + \cancel{y_2} = -2 \\ \_\cancel{y_2} \pm y_3 = \mp 6 \\ \hline y_1 - y_3 = 4 - (b) \end{array}$$

Solving $b \,\&\, IV$

$b + IV$

$$\begin{array}{r} y_1 + \cancel{y_3} = 4 \\ \_y_1 + \cancel{y_3} = 2 \\ \hline 2y_1 = 6 \end{array}$$

$\boxed{y_1 = 3}$

Put $y_1 = 3$ *In I* $\mp$

Put $y_2 = -2 - y_1 = -2 - 3$

$\boxed{y_2 = -5}$

Put $y_1 = 3$ *in VI*

$y_3 = 2 - y_1 = -2 + 3$

$\boxed{y_3 = 1}$

6. $A(\sqrt{3},-1)$, $B(0,2)$, $C(h,-2)$

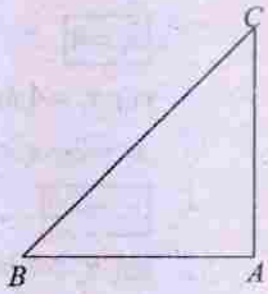
$|AB|=\sqrt{(0-\sqrt{3})^2+(2-(-1))^2}=\sqrt{(-\sqrt{3})^2+(2+1)^2}=\sqrt{3+9}=\sqrt{12}$

$|BC|=\sqrt{(h-0)^2+(-2-2)^2}=\sqrt{h^2+(-4)^2}=\sqrt{h^2+16}$

$|AC|=\sqrt{(-\sqrt{3}-h)^2+(-1-(-2))^2}$

$=\sqrt{3-2h\sqrt{3}+h^2+(-1+2)^2}$

$=\sqrt{h^2-2\sqrt{3}h+3+1}=\sqrt{h^2-2\sqrt{3}h+4}$



Now According to the given Condition

$|AB|^2+|AC|^2=|BC|^2$ So

$(\sqrt{12})^2+(\sqrt{h^2-2\sqrt{3}h+4})^2=(\sqrt{h^2+16})^2$

$12+h^2-2\sqrt{3}h+4=h^2+16 \Rightarrow h^2+16-12-h^2+2\sqrt{3}h-4=0$

$2\sqrt{3}h+16-16=0 \Rightarrow 2\sqrt{3}h=0 \quad \Rightarrow \boxed{h=0}$

7. $h=?$ $A(-1,h)$, $B(3,2)$, $C(7,3)$ are Collinear

Given points are collinear if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}=0 \quad \Rightarrow \begin{vmatrix} -1 & h & 1 \\ 3 & 2 & 1 \\ 7 & 3 & 1 \end{vmatrix}$$

Or $-1(2-3)-h(3-7)+1(9-14)=0$

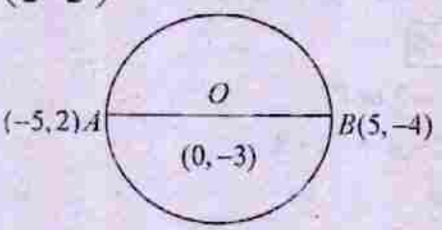
Or $-1(-1)-h(-4)+1(-5)=0 \quad \Rightarrow 1+4h-5=0$

$4h-4=0 \Rightarrow 4h=4 \Rightarrow \boxed{h=1}$

8. **Suppose centre is 0 and 0 is mid point of diameter so**

Centre $0=\left(\frac{-5+5}{2},\frac{-2-4}{2}\right)=\left(\frac{0}{2},\frac{-6}{2}\right)$

$=(0,-3)$



Radius $=|OA|=\sqrt{(-5-0)^2+(-2-(-3))^2}=\sqrt{(-5)^2+(-2+3)^2}$

$=\sqrt{25+1}=\sqrt{26}$

9. $A(h, 1)$, $B(2, 7)$, $C(-6, -7)$

According to the given Condition

$$|AB|^2 + |AC|^2 = |BC|^2 \text{———} I$$

Now $|AB| = \sqrt{(2-h)^2 + (7-1)^2} = \sqrt{4 - 4h + h^2 + 36} = \sqrt{40 - 4h + h^2}$

$|BC| = \sqrt{(-6-2)^2 + (-7-7)^2} = \sqrt{(-8)^2 + (-14)^2} = \sqrt{64 + 196} = \sqrt{260}$

$|AC| = \sqrt{(-6-h)^2 + (-7-1)^2} = \sqrt{36 + 12h + h^2 + 64} = \sqrt{h^2 + 12h + 100}$

Put in I

$(\sqrt{40 - 4h + h^2} + (\sqrt{h^2 + 12h + 100})^2 = (\sqrt{260})^2$

$40 - 4h + h^2 + h^2 + 12h + 100 = 260 \Rightarrow 2h^2 + 8h + 40 + 100 - 260 = 0$

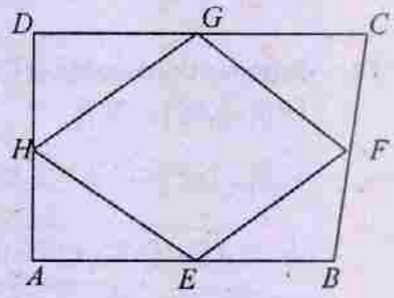
$2h^2 + 8h - 120 = 0 \div by\ 2,\ h^2 + 4h - 60 = 0$

$h^2 + 10h - 6h - 60 = 0 \quad \Rightarrow h(h+10) - 6(h+10) = 0$

$(h+10)(h-6) = 0 \quad \Rightarrow h + 10 = 0 \quad or \quad h - 6 = 0$

$h = -10 \ or \ h = 6$

10. $A(9, 3)$, $B(-7, 7)$, $C(-3, -7)$, $D(5, -5)$



Mid point of AB is E $\left(\frac{-7+9}{2}, \frac{7+3}{2}\right)$

$= E\left(\frac{2}{2}, \frac{10}{2}\right) = E(1, 5)$

Mid point of BC is F$\left(\frac{-7+(-3)}{2}, \frac{7+(-7)}{2}\right)$

$= F\left(\frac{-7-3}{2}, \frac{7-7}{2}\right) = F\left(\frac{-10}{2}, \frac{0}{2}\right) = F(-5, 0)$

Mid point of CD is G$\left(\frac{-3+5}{2}, \frac{-7+(-5)}{2}\right) = G\left(\frac{2}{2}, \frac{-7-5}{2}\right) = G\left(\frac{2}{2}, \frac{-12}{2}\right) = G(1, 6)$

Mid Point of AD is H$\left(\frac{9+5}{2}, \frac{3-5}{2}\right) = H\left(\frac{14}{2}, \frac{-2}{2}\right) = H(7, -1)$

Now for parallelogram

$|EF| = \sqrt{(-5-1)^2 + (0-5)^2} = \sqrt{(-6)^2 + (-5)^2} = \sqrt{36 + 25} = \sqrt{61}$

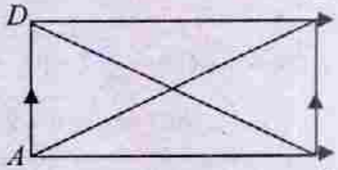
$|FG| = \sqrt{(1-(-5))^2 + (-6-0)^2} = \sqrt{(1+5)^2 + (-6)^2} = \sqrt{36 + 36} = \sqrt{72}$

$|GH| = \sqrt{(7-1)^2 + (-1+6)^2} = \sqrt{(6)^2 + (5)^2} = \sqrt{36 + 25} = \sqrt{61}$

$|HE| = \sqrt{(1-7)^2 + (-5-(-11))^2} = \sqrt{(-6)^2 + (5+1)^2} = \sqrt{36+36} = \sqrt{72}$

$|EF| = |GH|$ *and* $|FG| = |HE|$ *so*

EFGH is parallelogram



11. $A(-3, 0)$, $B(1, -2)$, $C(5, 0)$, $D(1, h)$

$|AB| = \sqrt{(1-(-3))^2 + (-2-2)^2} = \sqrt{(1+3)^2 + (-2)^2}$

$= \sqrt{(4)^2 + (-2)^2} = \sqrt{16+4} = \sqrt{20}$

$|CD| = \sqrt{(1-5)^2 + (h-0)^2} = \sqrt{(-4)^2 + h^2} = \sqrt{h^2+16}$

Given that ABCD is parallelogram so $|AB| = |CD|$

Or $\sqrt{20} = \sqrt{h^2+16} \Rightarrow h^2 + 16 = 20 \Rightarrow h^2 = 20 - 16 = 4 \Rightarrow \boxed{h = 2}$

To check square

$|AC| = \sqrt{(5-(-3))^2 + (0-0)^2} = \sqrt{(5+3)^2 + 0} = \sqrt{64} = 8$

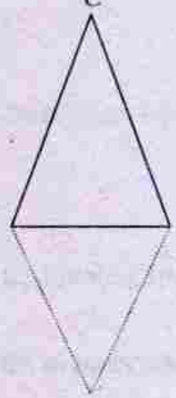
$|BC| = \sqrt{(1-1)^2 + (h-(-2))^2} = \sqrt{(1-1)^2 + (2+2)} = \sqrt{0 + (4)^2} = 4$

$|AC| \neq |BD|$ **So ABCD is not square**

12. Suppose third vertex is C (x, y), Given triangle is equilateral so

$|AB| = |AC| = |BC|$

$|AB| = |AC|$ ——— I & $|AC| = |BC|$ ——— II



$I \Rightarrow \sqrt{(3-(-3)^2 + (0-0)^2} = \sqrt{(x+3)^2 + (y-0)^2}$

So $\sqrt{36+0} = \sqrt{9 + 6x + x^2 + y^2}$

$\Rightarrow x^2 + y^2 + 6x + 9 = 36 \Rightarrow x^2 + y^2 + 6x + 9 - 36 = 0$

$x^2 + y^2 + 6x - 27 = 0$ ——— III

From II $|AC| = |BC|$

$\sqrt{(x-(-3))^2 + (y-0)^2} = \sqrt{(x-3)^2 + (y-0)^2}$

$\sqrt{9 + 6x + x^2 + y^2} = \sqrt{9 - 6x + x^2 + y^2}$

Squaring both sides

$\cancel{x^2} + \cancel{y^2} + 6x + 9 = \cancel{x^2} + \cancel{y^2} - 6x + 9$

$6x + 9 + 6x - 9 = 0 \Rightarrow 12x = 0 \Rightarrow x = 0$

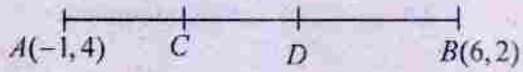
Put in III $(0)^2 + y^2 + 6(0) - 27 = 0 \Rightarrow y^2 = 27 \Rightarrow y = \sqrt{27}$

$$y = \pm 3\sqrt{3}$$

So two triangle are possible with
Vertex $C(0, 3\sqrt{3})$ & $C(0, -3\sqrt{3})$

13. Suppose points $C(x_1, y_1)$ & $D(x_2, y_2)$ trisect line $\overline{AB}$ (Guj 2010)

C divide $\overline{AB}$ in ration 1:2

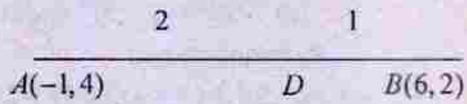


So co-ordinates are

$$x_1 = \frac{1(6)+2(-1)}{1+2} = \frac{6-2}{3} = \frac{4}{3}$$

$$y_1 = \frac{1(2)+2(4)}{1+2} = \frac{2+8}{3} = \frac{10}{3} \quad so\ C\left(\frac{4}{3}, \frac{10}{3}\right)$$

Now $D(x_2, y_2)$ divide AB in ratio 2 : 1



So co-ordinates are

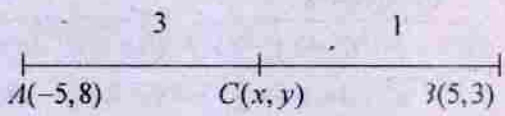
$$x_2 = \frac{2(6)+1(-1)}{1+2} = \frac{12-1}{3} = \frac{11}{3}$$

$$y_2 = \frac{2(2)+1(4)}{2+1} = \frac{4+4}{3} = \frac{8}{3} \qquad So\ D\left(\frac{11}{3}, \frac{8}{3}\right)$$

14. Suppose $C(x, y)$ is required point which divides AB in ratio.

Three fifth (3 : 2)

So co-ordinates are



$$x = \frac{3(5)+2(-5)}{3+2} = \frac{15-10}{5} = \frac{5}{5} = 1$$

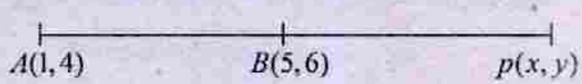
$$y = \frac{3(3)+2(8)}{3+2} = \frac{9+16}{5} = \frac{25}{5} = 5$$

$C(1, 5)$ is required point.

15. Suppose $p(x, y)$ is required point which lies (Sargodha 2011)

(i) On same side

B become mid point so



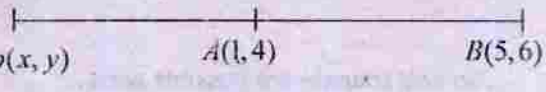
$$5 = \frac{1+x}{2}, \quad 6 = \frac{4+y}{2}$$

$$\Rightarrow \quad 10 = 1 + x \qquad 12 = 4 + y$$

$$x = 9 \qquad y = 8$$

So $p(9, 8)$ is required point.

(ii) **On opposite side**



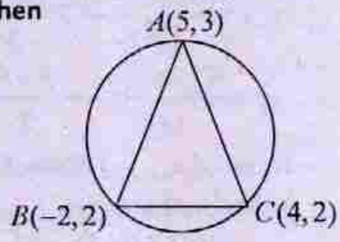
Now A divides $\overline{pB}$ in ratio 2 : 1

$$1=\frac{2(5)+1(x)}{2+1}=\frac{10+x}{3}\Rightarrow\frac{10+x}{3}=1\Rightarrow 10+x=3\Rightarrow x=7$$

$$4=\frac{2(6)+1(y)}{2+1}\Rightarrow 4=\frac{12+y}{3}\Rightarrow 12=12+y\Rightarrow y=0$$

So $p(-7,\ 0)$ is required point.

16. Suppose $P(x,y)$ is point equidistance from A,B and C then



$|PA|=|PB|\ \&\ |PB|=|PC|$

Now $|PA|=|PB|$

$$\sqrt{(x-5)^2+(y-3)^2}=\sqrt{(x-(-2))^2+|(y-2)^2}$$

By taking square

$$\Rightarrow (x-5)^2+(y-3)^2=(x+2)^2+(y-2)^2$$

$$x^2-10x+25+y^2-6y+9=x^2+4x+4+y^2-4y+4$$

$$or\ \ x^2+4x+y^2-4y+8-x^2+10x-25-y^2+6y-9=0$$

$$14x+2y-26=0\ \ or\ \ 7x+y-13=0 —— I$$

Now $|PB|=|PC|$

By squaring

$$\sqrt{(x-(-2))^2+(y-2)^2}=\sqrt{(x-4)^2+(y-2)^2}\Rightarrow(x+2)^2+(y-2)^2=(x-4)^2+(y-2)^2$$

$$x^2+4x+4+y^2-4y+4=x^2-8x+16+y^2-4y+4$$

$$or\ \ x^2+4x+4+y^2+4y-x^2+8x-16-y^2+4y-4=0$$

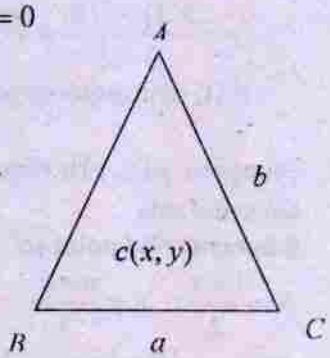
$$12x-12=0\ \Rightarrow 12x=12\ \Rightarrow x=1$$

Put in I $\quad 7(1)+y-13=0\ \Rightarrow y-6=0\ \Rightarrow y=6$

So required point is $P(1,\ 6)$

Now radius $=|PA|=\sqrt{(1-5)^2+(6-3)^2}=\sqrt{(-4)^2+(3)^2}$

$$=\sqrt{16+9}=\sqrt{25}=5$$



17. $A(4,-2)$, $B(-2,\ 4)$, $C(5,\ 5)$

Suppose centre is $C(x,y)$

$$|AB|=c=\sqrt{(-2-4)^2+(4+2)^2}=\sqrt{36+36}=\sqrt{72}=6\sqrt{2}$$

$$|BC|=a=\sqrt{5-(-2)^2+(5-4)^2}=\sqrt{49+1}=\sqrt{50}=5\sqrt{2}$$

$$|AC| = b = \sqrt{(5-4)^2 + (5-(-2))^2} = \sqrt{(1)^2 + (5+2)^2} = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$$

Now $x = \dfrac{ax_1 + bx_2 + cx_3}{a+b+c} = \dfrac{5\sqrt{2}(4) + 5\sqrt{2}(-2) + 6\sqrt{2}(5)}{5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}}$

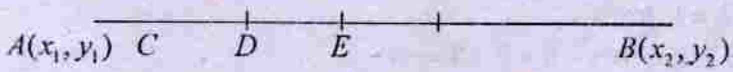
$$x = \frac{20\sqrt{2} - 10\sqrt{2} + 30\sqrt{2}}{16\sqrt{2}} = \frac{40\sqrt{2}}{16\sqrt{2}} = \frac{5}{2}$$

$$y = \frac{ay_1 + by_2 + cy_3}{a+b+c}$$

$$= \frac{5\sqrt{2}(-2) + 5\sqrt{2}(4) + 6\sqrt{2}(5)}{5\sqrt{2} + 5\sqrt{2} + 6\sqrt{2}} = \frac{-10\sqrt{2} + 20\sqrt{2} + 30\sqrt{2}}{16\sqrt{2}}$$

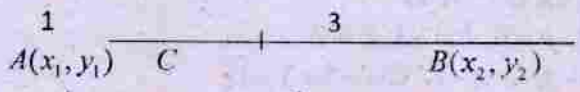
$= \dfrac{40\sqrt{2}}{16\sqrt{2}} = \dfrac{5}{2}$ So required in center is $= \left(\dfrac{5}{2}, \dfrac{5}{2}\right)$

18. (Sargodha 2012)



Case I; Suppose C divides

AB in ration 1 : 3 then

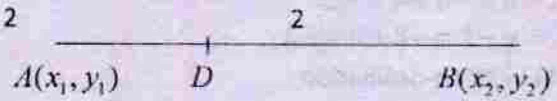


$$C\left(\frac{3x_1 + x_2}{1+3}, \frac{3y_1 + y_2}{1+3}\right) = C\left(\frac{3x_1 + x_2}{4}, \frac{3y_1 + y_2}{4}\right)$$

Case II;

Now D divides AB in ratio 2 : 2 so mid point

$$D\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



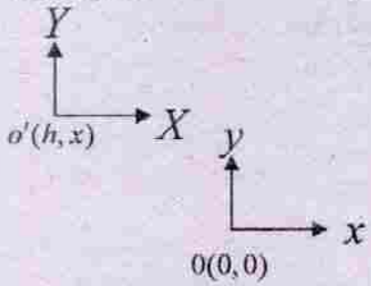
Case III;

Now E divides AB in ratio 3 : 1

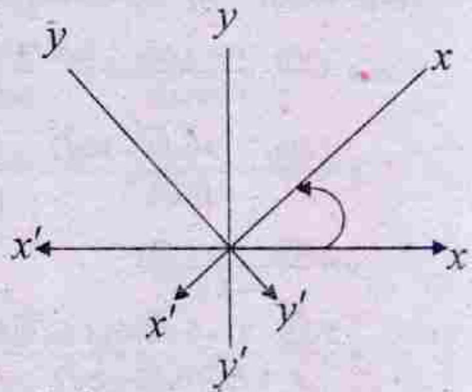
Then $E\left(\dfrac{x_1 + 3x_2}{1+3}, \dfrac{y_1 + 3y_2}{1+3}\right) = E\left(\dfrac{x_1 + 3x_2}{4}, \dfrac{y_1 + 3y_2}{4}\right)$

Exercise 4.2

For Translation



For Rotation



Relation between xy & xy co – ordinates

$X = x - h,\ Y = y - k$ and $X = xCos\theta + ySin\theta$

$Y = yCos\theta - xSin\theta$

For translation and rotation respectively.

1. (i) $P(3, 2), O'(1, 3)$ **(Lhr 2010, Sgd 2010)**

Here $x = 3, y = 2, h = 1, k = 3$

$X = x - h = 3 - 1 = 2$ & $Y = y - k = 2 - 3 = -1$

Answer is $P(2, -1)$ in XY co-ordination

(ii) $P(-2, 6), O'(-3, 2)$

$x = -2,\ y = 6,\ h = -3,\ k = 2$

$X = x - h = -2 - (-3) = -2 + 3 = 1$

$Y = y - k = 6 - 2 = 4$

$P(1, 4)$ in XY co-ordination

(iii) $P(-6, -8), O'(-4, -6)$ **(Sargodha 2011)**

$X = x - h = -6 + 4 = -2$

$Y = y - k = -8 + 6 = -2$

$P(1, 4)$ in XY co-ordination

(iv) $P\left(\frac{3}{2}, \frac{5}{2}\right); O'\left(-\frac{1}{2}, \frac{7}{2}\right)$

Here $x = \frac{3}{2},\ y = \frac{5}{2},\ h = -\frac{1}{2},\ x = \frac{7}{2}$

$$X = x - h = \frac{3}{2} - \left(-\frac{1}{2}\right) = \frac{3}{2} + \frac{1}{2} = \frac{3+1}{2} = \frac{4}{2} = 2$$

$$Y = y - k = \frac{5}{2} - \frac{7}{2} = \frac{5-7}{2} = \frac{-2}{2} = -1$$

Answer is $P(2, -1)$ in XY co-ordination

2. (i) change XY co-ordination into xy co-ordination into xy $P(8, 10)$; $O'(3, 4)$

Here $x = 8, y = 10, h = 3, k = 4$

$x = X + h = 8 + 3 = 11$ & $y = Y + k = 10 + 4 = 14$

Answer is $P(11, 14)$ in xy co-ordination

(ii) $P(-5, -3), O'(-2, -6)$

$X = -5, Y = -3, h = -2, k = -6$

$x = X + h = -5 - 2 = -7$

$y = Y + k = -3 - 6 = -9$

$P(-7, -9)$ in xy co-ordination

(iii) $P\left(-\frac{3}{4}, -\frac{7}{6}\right), O'\left(\frac{1}{4}, \frac{-1}{6}\right)$

Here $X = \frac{-3}{4}, Y = \frac{-7}{6}, h = \frac{1}{4}, k = \frac{-1}{6}$

$$x = X + h = -\frac{3}{4} + \frac{1}{4} = \frac{-3+1}{4} = \frac{-2}{4} = -\frac{1}{2}$$

$$y = Y + k = -\frac{7}{6} - \frac{1}{6} = \frac{-8}{6} = \frac{-4}{3}$$

So answer is $P\left(-\frac{1}{2}, \frac{-4}{3}\right)$ in xy co-ordination

(iv) $P(4, -3), O'(-2, 3)$

$X = 4, Y = -3, h = -2, k = 3$

$x = X + h = 4 - 2 = 2$

$y = Y + k = -3 + 3 = 0$

$P(2, 0)$ in xy co-ordination

3. (i) $P(5, 3); \theta = 45^o$ **(Sargodha 2011)**

Here $x = 5, y = 3, \theta = 45^o$

$$X = xCos\theta + ySin\theta = 5Cos45^o + 3Sin45^o = 5.\frac{1}{\sqrt{2}} + 3.\frac{1}{\sqrt{2}} = \frac{5+3}{\sqrt{2}} = \frac{8}{\sqrt{2}} = \frac{4\times 2}{\sqrt{2}} = 4\sqrt{2}$$

$$Y = yCos\theta - xSin\theta = 3Cos45^o - 5Sin45^o = 3.\frac{1}{\sqrt{2}} - 5.\frac{1}{\sqrt{2}} = \frac{3-5}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = \frac{-\sqrt{2}\times\sqrt{2}}{\sqrt{2}} = -\sqrt{2}$$

So answer is $P(4\sqrt{2}, -\sqrt{2})$ in XY co-ordinate.

(ii) $P(3,-7), \theta = 30^o, x = 3, y = -7$

$$X = xCos\theta + ySin\theta = 3Cos30^o + (-7)Sin30^o = 3.\frac{\sqrt{3}}{2} - 7.\frac{1}{2} = \frac{3\sqrt{3}-7}{2}$$

$$Y = yCos\theta - xSin\theta = -7Cos30^o - 3Sin30^o = -7.\frac{\sqrt{3}}{2} - 3.\frac{1}{2} = \frac{-7\sqrt{3}-3}{2}$$

So $P(X,Y) = P\left(\frac{3\sqrt{3}-7}{2}, \frac{-7\sqrt{3}-3}{2}\right)$ in XY co-ordinate.

(iii) $P(15, 10), \theta = Tan^{-1}\frac{1}{\sqrt{3}} \Rightarrow Tan\theta\frac{1}{\sqrt{3}}$

So $Sin\theta = \frac{1}{2}.\ Cos\theta = \frac{\sqrt{3}}{2}$

Now $X = xCos\theta + ySin\theta = 15Cos\theta + 10Sin\theta = 15\left(\frac{\sqrt{3}}{2}\right) + 10\left(\frac{1}{2}\right) = \frac{10\sqrt{3}+15}{2}$

$$Y = yCos\theta - xSin\theta = 10Cos\theta - 15Sin\theta = 10\left(\frac{\sqrt{3}}{2}\right) - 15\left(\frac{1}{2}\right) = \frac{10\sqrt{3}-15}{2}$$

So $P(x,y) = P\left(\frac{15\sqrt{3}+10}{2}, \frac{10\sqrt{3}-15}{2}\right)$ in XY co-ordinate.

4. (i) $P(-5, 3); \theta = 30^o$

Here $X = -5, Y = 3$

$$X = xCos\theta + ySin\theta \Rightarrow -5 = xCos30^o - xSin30^o$$

Or $-5 = x.\frac{\sqrt{3}}{2} + y.\frac{1}{2} \Rightarrow -10 = \sqrt{3}x + y$ ——— I

$Y = yCos\theta - xSin\theta \Rightarrow 3 = yCos30^o - xSin30^o \Rightarrow 3 = y.\frac{\sqrt{3}}{2} - x.\frac{1}{2} \Rightarrow 6 = \sqrt{3}y - x$ — II

Solve I & II $'X'$ by $\sqrt{3}$ we get

$$6\sqrt{3} = \sqrt{3}\times\sqrt{3}y - \sqrt{3}x$$

Or $6\sqrt{3} = 3y - \sqrt{3}x$ ——— III

Adding I & III

$$-10 = y + \sqrt{3}x$$

$$6\sqrt{3} = 3y - \sqrt{3}x$$

$$6\sqrt{3} - 10 = 4y$$

$$\Rightarrow y = \frac{6\sqrt{3}-10}{4} = \frac{2(3\sqrt{3}-5)}{4_2}$$

$$y = \frac{3\sqrt{3}-5}{2}$$

Put value of $y = \left(\frac{3\sqrt{3}-5}{2}\right)$ in I

$$-10 = \frac{3\sqrt{3}-5}{2} + \sqrt{3}x$$

Or $-10 = \frac{(X) by 2}{-20} = 3\sqrt{3} - \sqrt{\ } + 2\sqrt{3}x \Rightarrow -20+5-3\sqrt{3} = 2\sqrt{3}x$

Or $-15-3\sqrt{3} = 2\sqrt{3}x \Rightarrow x = \frac{-15-3\sqrt{3}}{2\sqrt{3}} = \frac{-5\times\sqrt{3}\times\sqrt{3}-3\sqrt{5}}{2\sqrt{3}}$

$$x = \frac{\sqrt{3}(-5\sqrt{3}-3)}{2\sqrt{3}} = \frac{-3-5\sqrt{3}}{2}$$

So $P(x,y) = P = \left(\frac{-3-5\sqrt{3}}{2}, \frac{3\sqrt{3}-5}{2}\right)$ in xy co-ordinate

(ii) $\mathbf{P(-7\sqrt{2}, 5\sqrt{2}), \theta = 45^o}$

$X = -7\sqrt{2},\ Y = 5\sqrt{2}$

We know that $X = xCos\theta + ySin\theta$ & $Y = yCos\theta - xSin\theta$

$-7\sqrt{2} = xCos45^o + ySin45^o$ $\quad$ $5\sqrt{2} = yCos45^o - xSin45^o$

$-7\sqrt{2} = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$ $\quad$ $5\sqrt{2} = \frac{y}{\sqrt{2}} - \frac{x}{\sqrt{2}}$

Multiply by $\sqrt{2}$ $\quad$ Multiply by $\sqrt{2}$

$-7(2) = x + y$ $\quad$ $5(2) = y - x$ or

$x + y = -14 \quad I$ $\quad$ $-x + y = 10 \quad II$

Solve I & II

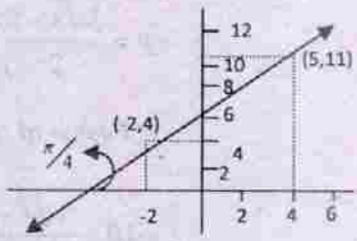
$$x + y = -14$$
$$\underline{x \pm y = 10}$$
$$2y = -4$$
$$\boxed{y = -2}$$

Put in $x - 2 = -14 \quad \Rightarrow \quad x = -12$

$P(-12, -2)$ *in xy* co-ordinate system.

Exercise 4.3

Note: $m = \text{slope} = \text{Tan}\,\alpha = \dfrac{y_2 - y_1}{x_2 - x_1}$



1. (i) $(-2,4),(5,11)$

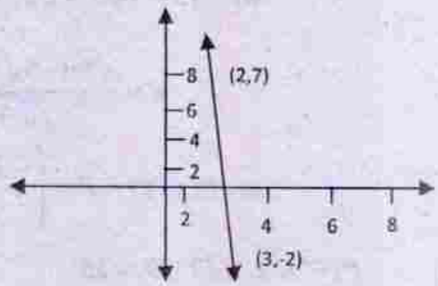
Here $x_1 = -2,\ y_1 = 4,\ x_2 = 5,\ y_2 = 11$

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11-4}{5-(-2)} = \frac{11-4}{5+2} = \frac{7}{7} = 1$$

Now $\quad Tan\alpha = m \quad \Rightarrow Tan\alpha = 1$

$$\Rightarrow \alpha = Tan^{-1}(1) = \frac{\pi}{4}$$

So inclination $= \alpha = \dfrac{\pi}{4}$



(ii) $(3,-2),\ (2,\ 7)$

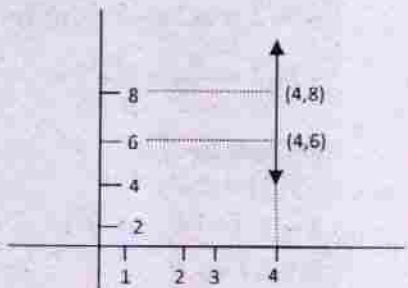
$x_1 = 3,\ y_1 = -2,\ x_2 = 2,\ y_2 = 7$

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-(-2)}{2-3}$$

$$= \frac{7+2}{-1} = \frac{9}{-1} = -9$$

Now $\quad Tan\alpha = m \quad \Rightarrow Tan\alpha = -9$

$$\alpha = \ \Rightarrow Tan^{-1}(-9) = 96''.34$$



(iii) $(4,6),\ (4,\ 8)$

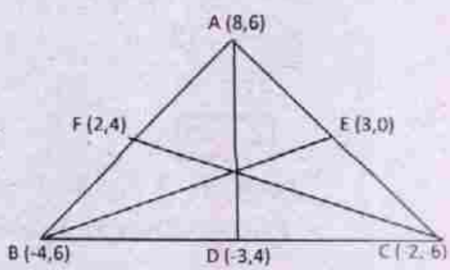
$x_1 = 4,\ y_1 = 6,\ x_2 = 4,\ y_2 = 8$

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-6}{4-4} = \frac{2}{0} = \infty$$

Now $\quad Tan\alpha = m \quad \Rightarrow Tan\alpha = \infty$

$$\alpha = Tan^{-1}(\infty) = \frac{\pi}{2} = 90^{o}$$

2. **Slope of sides**



$$\text{Slope of } AB = \frac{2-6}{-4-8} = \frac{-4}{-12} = \frac{1}{3}$$

$$\text{Slope of } BC = \frac{-6-2}{-2-(-4)} = \frac{-6-2}{-2+4} = \frac{-8}{2} = -4$$

$$\text{Slope of } CA = \frac{6-(-6)}{8-(-2)} = \frac{6+6}{8+2} = \frac{12}{10} = \frac{6}{5}$$

For medians

Mid point of BC is $D\left(\frac{-4+(-2)}{2},\frac{2+(-6)}{2}\right)=D\left(\frac{-4-2}{2},\frac{2-6}{2}\right)$

$=D\left(\frac{-6}{2},\frac{8}{2}\right)=D(-3,-2)$

Mid point of AC is $E\left(\frac{-2+8}{2},\frac{-6+6}{2}\right)=e\left(\frac{6}{2},\frac{0}{2}\right)=E(3,0)$

Mid point of AB is $F\left(\frac{-4+8}{2},\frac{2+6}{2}\right)=F\left(\frac{4}{2},\frac{8}{2}\right)=F(2,4)$

Slope of median $AD=\frac{4-6}{-3-8}=\frac{-2}{-11}=\frac{2}{11}$

Slope of median $BE=\frac{0-2}{3-(-4)}=\frac{-2}{3+4}=\frac{-2}{7}$

Slope of median $CF=\frac{4-(-6)}{2-(-2)}=\frac{4+6}{2+2}=\frac{10}{4}=\frac{5}{2}$

For Altitude

Altitude from A is perpendicular to side BC so

Slope of altitude from $A=-\frac{1}{-4}=\frac{1}{4}$

Altitude from B is $\perp$ are to side AC so

Slope of altitude from $B=-\frac{1}{\frac{6}{5}}=-\frac{5}{6}$

Altitude from C is $\perp$ are to side AB so

Slope of altitude from $C=-\frac{1}{\frac{1}{3}}=-3$

3. (a) **Pts are $A(-1,-3)$, $B(1,5)$, $C(2,9)$**

Slope of $AB=\frac{5-(-3)}{1-(-1)}=\frac{5+3}{1+1}=\frac{8}{2}=4$

Slope of $BC=\frac{9-5}{2-1}=\frac{4}{1}=4$

Slop AB = Slope of BC

So A, B, C lies on a line or collinear.

b) $A(4,-5), B(7,5), C(10,15)$

Slope of $\overline{AB} = \frac{5+5}{7-4} = \frac{10}{3}$

Slope of $BC = \frac{15-5}{10-7} = \frac{10}{3}$

Slop AB = Slope of BC

So A, B, C are collinear.

c) $A(-4,6), B(3,8), C(10,10)$

Slope of $AB = \frac{8-6}{3+4} = \frac{2}{7}$

Slope of $BC = \frac{10-8}{10-3} = \frac{2}{7}$

Slop AB = Slope of BC

So A, B, C are collinear.

d) $A(a,2b), B(c,a+b), C(2c-a,2a)$

Slope of $AB = \frac{a+b-2b}{c-a} = \frac{a-b}{c-a}$

Slope of $BC = \frac{2a-a-b}{2c-a-c} = \frac{a-b}{c-a}$

Slop AB = Slope of BC

So A, B, C are collinear.

4. $A(7,3), B(k,-6), C(-4,5), D(-6,4)$

Slope of $AB = \frac{-6-3}{k-7} = \frac{-9}{k-7}$

Slope of $CD = \frac{4-5}{-6+4} = \frac{-1}{-2} = \frac{1}{2}$

i) **For parallel**

Slope of AB = Slope of CD

$$\frac{-9}{k-7} = \frac{1}{2} \Rightarrow -9\times 2 = k-7 \Rightarrow -18 = k-7$$

$$k = -18+7 \Rightarrow \boxed{k=-11}$$

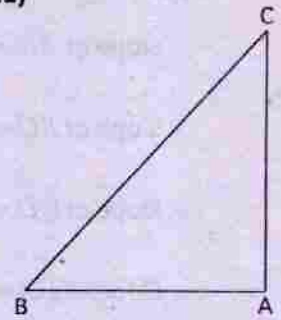
ii) **For perpendicular**

(Slope of AB) (Slope of CD) = −1

$$\left(\frac{-9}{k-7}\right)\left(\frac{1}{2}\right) = -1 \Rightarrow -9 = (-1)(2)(k-7)$$

$$-9 = -2k+14 \Rightarrow -2k = -9-14 = -23 \Rightarrow \boxed{k = \frac{23}{2}}$$

5. $A(6,1)$, $B(2,7)$, $C(-6,-7)$ (Sargodha 2011)



$m_1 =$ Slope of $AB = \frac{7-1}{2-6} = \frac{6}{-4} = -\frac{3}{2}$

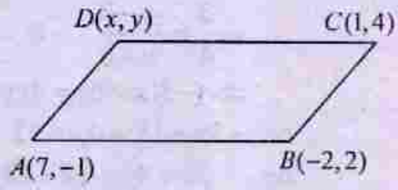
$m_2 =$ Slope of $BC = \frac{-7-7}{-6-2} = \frac{-14}{-8} = \frac{7}{4}$

$m_3 =$ Slope of $CA = \frac{1-(-7)}{6-(-6)} = \frac{1+7}{6+6} = \frac{8}{12} = \frac{2}{3}$

So $m_1 \times m_3 = \frac{-3}{2} \times \frac{2}{3} = -1$

Two sides are perpendicular so ABC is right triangle.

6. Suppose fourth point is $D(x,y)$ then



Slope of $AB = \frac{2-(-1)}{-2-7} = \frac{2+1}{-9} = \frac{3}{-9} = \frac{-1}{3}$

Slope of $BC = \frac{4-2}{1-(-2)} = \frac{2}{1+2} = \frac{2}{3}$

Slope of $CD = \frac{y-4}{x-1}$

Slope of $AD = \frac{y-(-1)}{x-7} = \frac{y+1}{x-7}$

Given figure is parallelogram so

Slope of AB = Slope of CD & Slope of BC = Slope of AD

$\frac{-1}{3} = \frac{y-4}{x-1}$ & $\frac{2}{3} = \frac{y+1}{x-7}$

or $-1(x-1) = 3(y-4)$ & $2(x-7) = 3(y+1)$

$-x+1 = 3y-12$ & $2x-14 = 3y+3$

or $x+3y-1-12=0$ & $2x-3y-14-3=0$

$x+3y-13=0$ I & $2x-3y-17=0$ II

Adding I & II

$$\begin{array}{r} x+3y-13=0 \\ 2x-3y-17=0 \\ \hline 3x-30=0 \end{array}$$

$\Rightarrow 3x = 30 \Rightarrow \boxed{x=10}$

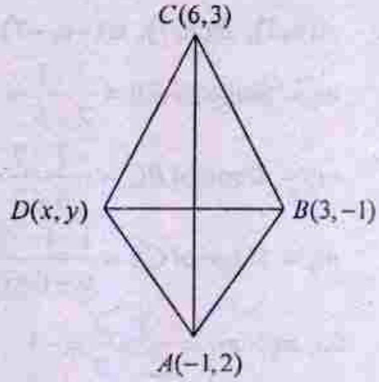
Put in $x+3y-13=0$

$10+3y-13=0 \Rightarrow 3y-3=0$

$3y=3 \Rightarrow y=1$

So required fourth pt is $D(10,1)$

7. Suppose fourth vertex is $D(x,y)$ then



Slope of $AB = \dfrac{-1-2}{3+1} = \dfrac{-3}{4}$

Slope of $BC = \dfrac{3+1}{6-3} = \dfrac{4}{3}$

Slope of $CD = \dfrac{y-3}{x-6}$

Slope of $DA = \dfrac{2-y}{-1-x}$

In rhombus opposite sides are parallel

So slope of AB = Slope of CD & Slope of BC = Slope of DA

$$-\frac{3}{4} = \frac{y-3}{x-6} \quad \& \quad \frac{4}{3} = \frac{2-y}{-1-x}$$

$\Rightarrow (-3(x-6) = 4(y-3) \quad \& \quad 4(-1-x) = 3(2-y)$

or $-3x+18 = 4y-12 \quad -4-4x = 6-3y$

or $3x+4y-12-18=0$ *or* $4x-3y+6+4=0$

or $3x+4y-30=0$ III *or* $4x-3y+10=0$

'X' by 3 & 'X' by 4

Adding I & II

$$9x + 12y - 90 = 0$$
$$16x - 12y + 40 = 0$$
$$25x - 50 = 0$$

$\Rightarrow 25x = 50 \Rightarrow \boxed{x=2}$

Put value of x *in* III

$3(2) + 4y - 30 = 0$

$6 + 4y - 30 = 0$

$4y - 24 = 0 \quad \Rightarrow \quad 4y = 0$

$4y = 24 \Rightarrow \boxed{y=6}$

Fourth vertex is $D(2,6)$

Now Slope of diagonal $AC = \dfrac{3-2}{6-(-1)} = \dfrac{1}{6+1} = \dfrac{1}{7}$

Slope of diagonal $BD = \dfrac{6-(-1)}{2-3} = \dfrac{6+1}{-1} = -7$

(Slope of AC)(Slope of BD) $= \dfrac{1}{7}(-7) = -1$

Hence diagonal AC & BD are $\perp$ ar

8. (i) **Give name $A(-3,-4), B(6,2)$ and $C(4,5), D(-2,-1)$**

Slope of line $AB = \dfrac{4-(-2)}{2-1} = \dfrac{4+2}{1} = 6$

Slope of line $CD = \dfrac{2-1}{-8-4} = \dfrac{1}{-12}$

Slope of line $AB \neq$ Slope of CD

So not parallel

And (Slope of line AB)(Slope of line CD) $= 6\left(-\dfrac{1}{2}\right) \neq -1$

So not perpendicular

So answer (iii) None

(ii) **Give name $A(1,-2), B(2,4)$ and $C(4,1), D(-8,2)$**

Slope of line $AB = \dfrac{2-4}{6-(-3)} = \dfrac{-2}{9}$

Slope of line $CD = \dfrac{-7-5}{-2-4} = \dfrac{-12}{-6} = 2$

Slope of line $AB \neq$ Slope of CD

So not parallel

And (Slope of line AB)(Slope of line CD) $= 2.\left(\dfrac{-2}{9}\right) \neq -1$

So not perpendicular

9. (a) **pt (7,-9)** **(Lahore 2010)**

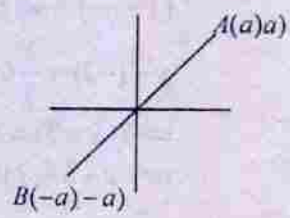
Equation of horizontal line is $y = y_1 \Rightarrow y = -9$

(b) **$(-5,2)$**

Equation of vertical line is $x = x_1 \Rightarrow x = -5$

(c) **Line bisecting first and third quadrant**

Co-ordinates in first $A(a,a)$ in third $B(-a,-a)$



Then $m = \dfrac{-a-a}{-a-a} = \dfrac{-2a}{-2a} = 1$

Now equation of line is $(y-y_1) = m(x-x_1)$

$$(y-a) = 1(x-a)$$

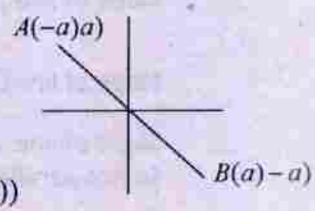
Through pt (a,a) & Slope 1 or $y - a = x - a$

$$y - a + a = x \Rightarrow \boxed{y = x}$$

(d) Line bisecting second and fourth quadrant

Co-ordinates are $A(-a,a)$ in third $B(a,-a)$

Then $m=\dfrac{-a-a}{a-(-a)}=\dfrac{-a-a}{a+a}=\dfrac{-2a}{2a}=-1$



Now equation of line is $y-y_1=m(x-x_1)$

$$y-a=-1(x+a)$$

Through pt $(-a,a)$ & $m=-1$ $\quad y-a=-1(x-(-a))$

$$y-a=-1(x+a)$$

$\Rightarrow \boxed{y=-x}$ $\quad y-a+a=-x$

10. (a) **pt $A(x_1,y_1)=(-6,5)$, $Slope=m=7$**

Equation of line is $(y-y_1)=m(x-x_1)$

$$y-5=7(x-(-6)) \Rightarrow y-5=7(x+6)$$

Or $\quad y-5=7x+42 \Rightarrow 7x+42-y+5=0$

$$7x-y+47=0$$

(b) **pt $Slope=m=0$ $\;put(x_1,y_1)=(8,-3)$**

Equation of line is $(y-y_1)=m(x-x_1) \Rightarrow y-(-3)=0(x-8)$

$\Rightarrow y+3=0$

(c) **pt $(x_1,y_1)=(-8,5), m=\infty$**

$(y-y_1)=m(x-x_1) \Rightarrow \dfrac{y-y_1}{m}=x-x_1 \;\Rightarrow \dfrac{4-5}{\infty}=x-(-8)$

$0=x+8 \;\Rightarrow x+8=0$

(d) **pt $A(-5,-3), B(9,-1)$**

Slope $\dfrac{-1-(-3)}{9-(-5)}=\dfrac{-1+3}{9+5}=\dfrac{2}{14}=\dfrac{1}{7}$

Equation of line through $(x_1,y_1)=(-5,-3)$ & $m=\dfrac{1}{7}$

$(y-y_1)=m(x-x_1)$

$y-(-3)=\dfrac{1}{7}(x-(-5)) \;\Rightarrow y+3=\dfrac{x+5}{7}$

$\Rightarrow 7(y+3)=x+5 \;\Rightarrow 7y+21=x+5$

$or \quad x+5-7y-21=0 \;\Rightarrow x-7y-16=0$

(e) **y-intercept mean** $x_1 = 0 (x_1, y_1) = (0, -7)$

$y - y_1 = m(x - x_1)$ & $m = -5$

$y - (-7) = -5(x - 0) \Rightarrow y + 7 = -5x$

$\Rightarrow 5x + y + 7 = 0$

(f) **x-intercept –9 mean** $y_1 = 0, (x_1, y_1) = (-9, 0) \quad m = 4$

$y - y_1 = m(x - x_1)$

$y - 0 = 4(x - (-9)) \Rightarrow y = 4(x + 9) = 4x + 36$

$\Rightarrow 4x - y + 36 = 0$

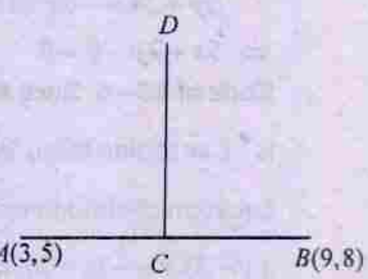
(g) **x-intercept** $= a = -3$ & **y-intercept** $= b = 4$

Equation of line having two intercept is

$$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{x}{-3} + \frac{y}{4} = 1$$

$$'X' \text{ by } -12 \Rightarrow -12\left(\frac{x}{-3}\right) + (-12)\left(\frac{y}{4}\right) = -12$$

$or \;\; 4x - 3y = -12$



11. **C is mid point of AB** **(Lahore 2010)**

So co-ordinates are $C\left(\frac{3+9}{2}, \frac{5+8}{2}\right) = \left(6, \frac{13}{2}\right)$

Slope of line $AB = \frac{8-5}{9-3} = \frac{3}{6} = \frac{1}{2}$

Slope of perpendicular bisector $CD = -2$

Equation of $\perp$ar bisector CD through $C\left(6, \frac{13}{2}\right)$ and slope -2

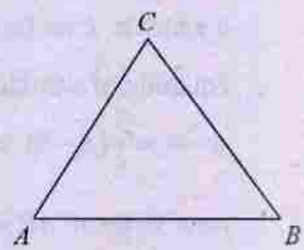
$y - y_1 = m(x - x_1)$

$y - \frac{13}{2} = -2(x - 6)$

$'X'$ by 2 $\;\; 2y - 13 = -4(x - 6)$

$2y - 13 = -4x + 24 \Rightarrow 4x + 2y - 13 - 24 = 0$

$4x + 2y - 37 = 0$



12. $A(-3, 2), B(5, 4), C(3, -8)$

Slope of side $AB = \frac{4-2}{5+3} = \frac{2}{8} = \frac{1}{4}$

Slope of side $BC = \frac{-8-4}{3-5} = \frac{-12}{-2} = 6$

Slope of side $CA = \frac{-8-2}{3+3} = \frac{-10}{6} = \frac{-5}{3}$

Equation of side AB is $\left(\text{through pt A \& Slope } \frac{1}{4}\right)$

$y-2 = \frac{1}{4}(x+3) \Rightarrow 4y-8 = x+3$

$\Rightarrow x-4y-11=0$

Equation of sides BC (through pt B)

$(y-4) = 6(x-5) \Rightarrow y-4 = 6x-30 \Rightarrow 6x-y-26=0$

Equation of side AC through C is

$(y+8) = \frac{-5}{3}(x-3)$

$3y+24 = -5x+15$

$\Rightarrow 5x+3y+9=0$

Slope of BC = 6 Since Altitude from A

is $\perp$ ar to side BC so Slope of altitude = $-\frac{1}{6}$

Equation of altitude from A is

$(y-2) = -\frac{1}{6}(x+3) \Rightarrow 6(y-2) = -(x+3)$

$6y-12 = -x-3 \Rightarrow x+6y-9=0$

Slope of $CA = \frac{-5}{3}$ Slope of altitude from

B which is $\perp ar$ to $CA = \frac{3}{5}$

Equation of altitude from B is

$y-4 = \frac{3}{5}(x-5) \Rightarrow 5y-20 = 3x-15 \Rightarrow 3x-5y+5=0$

Now Slope of $AB = \frac{1}{4}$ Slope of altitude from $C = -4$

Equation of altitude from C is

$(y+8) = -4(x-3) \Rightarrow y+8 = -4x+12$

$\Rightarrow 4x+y-4=0$

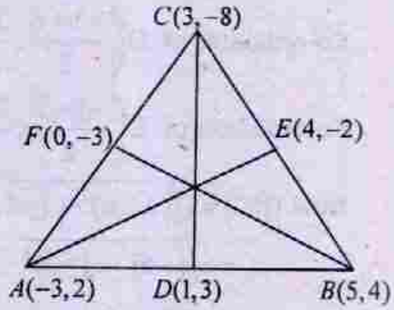
D,E,F are mid point of AB, BC & CA

So Co-ordinates of $D\left(\frac{-3+5}{2},\frac{2+4}{2}\right) D=(1,3)$

So Co-ordinates of $E\left(\frac{5+3}{2},\frac{4-8}{2}\right) E=(4,-2)$

So co-ordinates of $F\left(\frac{3-3}{2},\frac{-8+2}{2}\right) F=(0,-3)$

Medians are CD, AE, BF



Now Slope of $CD=\frac{3+8}{1-3}=\frac{11}{-2}$

Equation of median CD through C is

$y+8=\frac{11}{-2}(x-3)$

$\Rightarrow -2y-16=11x-33$

$\Rightarrow 11x+2y-17=0$

Slope of $AE=\frac{-2-2}{4+3}=-\frac{4}{7}$

Equation of median AE through A is

$y-2=-\frac{4}{7}(x+3) \Rightarrow 7y-14=-4x-12$

$\Rightarrow 4x+7y-14+12=0 \Rightarrow 4x+7y-2=0$

Slope of $BF=\frac{-3-4}{0-5}=\frac{-7}{-5}=\frac{7}{5}$

Equation of median BF through B is

$y-4=\frac{7}{5}(x-5) \Rightarrow 5y-20=7x-35$

$\Rightarrow 7x-5y-35+20=0 \Rightarrow 7x-5y-15=0$

13. **pt (−4, −6)**

Slope of given line $=-\frac{3}{2}$

Slope of required line $=\frac{2}{3}$

Equation of required line is

$y+6=\frac{2}{3}(x+4) \Rightarrow 3y+18=2x+8$

$\Rightarrow 2x-3y+18-18=0 \Rightarrow 2x-3y-10=0$

14. **pt $(11,-5)$ Slope $= -24$**

Slope of required line which is parallel $= -24$

Equation of required line is

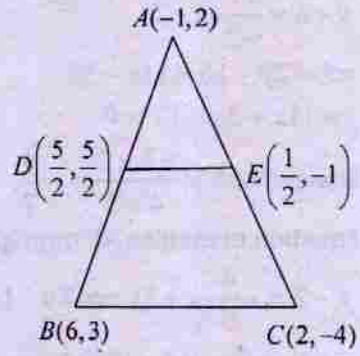
$y-(-5) = -24(x-11) \Rightarrow y+5 = -24+264$

$\Rightarrow y+5+24x-264 = 0 \Rightarrow 24x+y-259 = 0$

15. **As D and E are Mid points of $\overline{AB}$ and $\overline{AC}$ So**

Co-ordinates of $D\left(\frac{-1+6}{2},\frac{2+3}{2}\right) = \left(\frac{5}{2},\frac{5}{2}\right)$

Co-ordinates of $E\left(\frac{-1+2}{2},\frac{2-4}{2}\right) = \left(\frac{1}{2},-1\right)$



Now $|BC| = \sqrt{(2-6)^2+(-4-3)^2}$

$= \sqrt{(-4)^2+(-7)^2}$

$= \sqrt{65}$

$$|DE| = \sqrt{\left(\frac{1}{2}-\frac{5}{2}\right)^2+\left(-1-\frac{5}{2}\right)^2}$$

$$= \sqrt{\left(\frac{1-5}{2}\right)^2+\left(\frac{-2-5}{2}\right)^2}$$

$$= \sqrt{(-2)^2+\left(\frac{-7}{2}\right)^2} = \sqrt{4+\frac{49}{4}} = \sqrt{\frac{16+49}{4}}$$

$$= \sqrt{\frac{65}{4}} = \frac{1}{2}\sqrt{65}$$

$$\boxed{|DE| = \frac{1}{2}|BC|}$$

Also Slope $\overline{BC} = \frac{-4-3}{2-6} = \frac{-7}{-4} = \frac{7}{4}$

Slope $\overline{DE} = \dfrac{-1-\frac{5}{2}}{\frac{1}{2}-\frac{5}{2}} = \dfrac{\frac{-2-5}{2}}{\frac{1-5}{2}} = \frac{7}{\cancel{2}}\times\frac{\cancel{2}}{-4} = \frac{7}{4}$

Slope of BC = Slope of DE

So BC is parallel to DE.

Both result proved.

16. **Suppose l denoted liters and p denote price then**

$(l_1, p_1) = (560, 12.50), (l_2, p_2) = (700, 12),\ p = 12.25$

$$m = slope = \frac{p_2 - p_1}{l_2 - l_1} = \frac{12.00 - 12.5}{700 - 560} = \frac{-0.5}{140}$$

$m = \frac{-.5}{140} \times \frac{2}{2}$ Now equation of line through (560, 12.5) is

$p - p_1 = m(l - l_1)$

$p - 12.50 = \frac{-.5}{140}(l - 560)$

By cross multiplication

$140(p - 12.50) = -0.5(l - 560)$

$\frac{140}{-.5}(p - 12.50) = l - 560$

$-280(p - 12.5) + 560 = l$

$l = 560 - 280(p - 12.5)$

is required equation

Now at $p = 12.25$

$140(12.25 - 12.50) = -0.5(l - 560)$

$\Rightarrow 140(-0.25) = 0.5(l - 560) \Rightarrow -35 = -0.5(l - 560)$

$\Rightarrow l - 560 = \frac{-35}{-0.5} = 70 \ \Rightarrow l = 70 + 560 = 630$

Hence at p = 12.25. He can sell 630 liters.

17. **Let P = Population & t = Years**

Then $(p_1, t_1) = (60, 1961),\ (p_1, t_2) = (95, 1981)$

Now $m = slope = \frac{t_2 - t_1}{p_2 - p_1} = \frac{1981 - 1961}{95 - 60} = \frac{20}{35} = \frac{4}{7}$

Equation of line has

$(t - t_1) = m(p - p_1) \Rightarrow (t - 1961) = \frac{4}{7}(p - 60)$

Or $\frac{7}{4}(t - 1961) = p - 60 \Rightarrow \boxed{p = 60 + \frac{7}{4}(t - 1961)}$

(a) When $t = 1947$ **then** $p = 60 + \frac{7}{4}(1947 - 1961) \Rightarrow \boxed{p = 35.5\ Million}$

(b) When $t = 1997$ **then** $p = 60 + \frac{7}{4}(1997 - 1961) \Rightarrow \boxed{p = 123\ Million}$

18. **P denote price and t denote year**

$(p_1, t_1) = (1, 1980),\ (p_1, t_2) = (4, 1996)$

$$m = \frac{t_2 - t_1}{p_2 - p_1} = \frac{1996 - 1980}{4 - 1} = \frac{16}{3}$$

Equation is

$$(t - t_1) = m(p - p_1)$$

$$t - 1980 = \frac{16}{3}(p - 1) \Rightarrow 3t - 5940 = 16p - 16$$

$$16p = 3t - 5940 + 16 = 3t - 5924$$

$p = \frac{3}{16}t - \frac{5924}{16}$ is required equations.

At $t = 1990$

$$p = \frac{3(1990)}{16} - \frac{5924}{16} = \frac{5970 - 5924}{16} = \frac{46}{16} = 2.8 \text{ Million}$$

19. **We know that**

Freezing point of water $= (0, 32) \to (C_1, F_1)$

Boiling point of Water $= (100, 212) \to (C_2, F_2)$

C for Celsius and F for Fahrenheit

$$\text{Slope} = \frac{212 - 32}{100 - 0} = \frac{180}{100} = \frac{9}{5}$$

Equation of line is

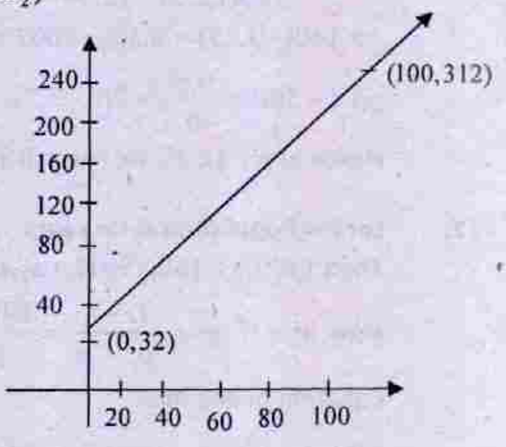
$$F - F_1 = m(C - C_1)$$

$$F - 32 = \frac{9}{5}(C - 0)$$

$$\Rightarrow\ 5F - 160 = 9c \Rightarrow 5F = 9c + 160$$

$$F = \frac{1}{5}(9c + 160)$$

Required equation of line.



20. **Let x represent score and y year**

$(x_1, y_1) = (592, 1998), (x_2, y_2) = (564, 2002)$

$$m = \frac{2002 - 1998}{564 - 592} = \frac{4}{-28} = -\frac{1}{7}$$

Equation of line

$$y - y_1 = m(x - x_1)$$

$\Rightarrow y-1998=-\frac{1}{7}(x-592)$

$\Rightarrow 7y-13986=-x+592$

$\Rightarrow 7y-13986+x-592=0$

$x+7y-14578=0$

$\Rightarrow x=-7y+14578$

Required equations

Now at $y=2006$

$x=-7(2006)+14578=-14042+14578$

$x=536$ Scare

21. (a) $2x-4y+11=0$

(i) **Slope intercept form**

$4y=2x+11$

$y=\frac{2}{4}x+\frac{11}{4}$ $\quad\because y=mx+c$

$y=\frac{1}{2}x+\frac{11}{4}$

(ii) **Two intercept form**

$2x-4y+11=0$

$2x-4y=-11$

$\div by\ -11$

$\frac{2x}{-11}-\frac{4y}{-11}=1$

$\frac{x}{\frac{-11}{2}}=\frac{y}{\frac{11}{4}}=1$ $\quad\because \frac{x}{a}+\frac{y}{b}=1$

(iii) **Normal Form**

$2x-4y+11=0$

$2x-4y=-11$

'÷' both sides by $\sqrt{(2)^2+(-4)^2}$

$\sqrt{4+16}=\sqrt{20}$

$\frac{2}{\sqrt{20}}x-\frac{4}{\sqrt{20}}y=\frac{-11}{\sqrt{20}} \Rightarrow \frac{-2}{\sqrt{20}}x+\frac{4}{\sqrt{20}}y=\frac{11}{\sqrt{20}}$

Where $\cos\alpha=\frac{-2}{\sqrt{20}}$, $\sin\alpha=\frac{4}{\sqrt{20}}$ and $p=\frac{11}{\sqrt{20}}$

$x\cos\alpha+y\sin\alpha=p$

$$Tan\alpha=\frac{Sin\alpha}{Cos\alpha}=\frac{\frac{4}{\sqrt{20}}}{\frac{-2}{\sqrt{20}}}$$

$$=\frac{4}{\sqrt{20}}\times\frac{\sqrt{20}}{-2}=-2$$

$$\alpha=\tan^{-1}(-2)=-63.4^o$$

Or $xCos116.5^o+ySin116.5^o=\frac{11}{2\sqrt{5}}$

(b)

(i) $\mathbf{4x+7y-2=0}$

$7y=-4x+2 \Rightarrow \boxed{y=\frac{-4}{7}x+\frac{2}{7}}$

is required slope intercept form.

(ii) $\mathbf{4x+7y-2=0}$

$4x+7y=2$

Divided by 2

$\frac{4}{2}x+\frac{7}{2}y=1 \quad or \quad 2x+\frac{7}{2y=1}$

Or $\boxed{\frac{x}{\frac{1}{2}}+\frac{y}{\frac{2}{7}}=1}$ is two intercept form

(iii) $\mathbf{4x+7y-2=0}$

Or $4x+7y=2$

Divide both side by $\sqrt{4^2+7^2}=\sqrt{65}$

$$\frac{4x}{\sqrt{65}}+\frac{7y}{\sqrt{65}}=\frac{2}{\sqrt{65}}$$

Put $Cos\alpha=\frac{4}{\sqrt{65}}$ & $Sin\alpha=\frac{7}{\sqrt{65}}$ & $p=\frac{2}{\sqrt{65}}$

$xCos\alpha+ySin\alpha=p$

Now $Tan\alpha=\frac{Sin\alpha}{Cos\alpha}=\frac{\frac{7}{\sqrt{65}}}{\frac{4}{\sqrt{65}}}=\frac{7}{4}$

$\alpha = Tan^{-1}(1.75) = 60.26''$

So $xCos60.26'' + ySin60.26'' = \frac{2}{\sqrt{65}}$

Where $p = \frac{2}{\sqrt{65}}$ is normal form.

(c) $\mathbf{15y - 8x + 3 = 0}$

(i) $15y - 8x + 3 = 0 \quad or \quad 15y = 8x - 3$

$y = \frac{8}{15}x - \frac{3}{15} \quad or \quad \boxed{y = \frac{8}{15}x - \frac{1}{5}}$

is required slope intercept form.

(ii) $\mathbf{15y - 8x + 3 = 0}$

$or \quad 15y - 8x = -3$

$-5y + \frac{8}{3}x = 1 \quad or \quad \boxed{\frac{x}{\frac{3}{8}} + \frac{y}{\frac{-1}{5}} = 1}$

is two intercept form.

(iii) $\mathbf{15y - 8x + 3 = 0}$

$or \quad 15y - 8x = -3$

Divide by $\sqrt{(15)^2 + (-8)^2} = \sqrt{225 + 64} = \sqrt{289}$

$\frac{15}{17}y - \frac{8}{17}x = \frac{-3}{17}$

$or \quad +\frac{8}{17}x - \frac{15}{17}y = +\frac{3}{17} I \quad 'X' \text{ by } (-1)$

Put $Cos\alpha = \frac{8}{17}, \; Sin\alpha = \frac{-15}{17}$

$Tan\alpha = \frac{Sin\alpha}{Cos\alpha} = \frac{\frac{15}{17}}{\frac{-8}{17}} = \frac{-15}{8}$

$\alpha = Tan^{-1}(-1.875) = 298.75''$

I become

$xCos(298.75'') + ySin(298.75'') = \frac{+3}{17}$

Where $p = \frac{+3}{17}$

is Normal form.

22. **Remember** Parallel $m_1 = m_2$

Perpendicular $m_1 m_2 = -1$

and Slope $= m = \dfrac{-a}{b}$

(a) $\mathbf{2x + y - 3 = 0,\ 4x + 2y + 5 = 0}$

$m_1 = \dfrac{-a}{b} = -2, \quad m_2 = \dfrac{-a}{b} = \dfrac{-4}{2} = -2$

$m_1 = m_2$ Parallel

(b) $\mathbf{3y = 2x + 5,\ \ 3x + 2y - 8 = 0}$

$2x - 3y 5 = 0, \quad m_2 = \dfrac{-3}{2}$

$m_1 = \dfrac{-2}{-3m_1} = \dfrac{-2}{-3} = \dfrac{2}{3}$

$m_1 m_2 = \left(\dfrac{2}{3}\right)\left(\dfrac{-3}{2}\right) = -1$ Perpendicular

(c) $\mathbf{4x + 2x - 1 = 0,\ \ x - 2y - 7 = 0}$

$2x + 4y - 1 = 0, \quad x - 2y - 7 = 0$

$m_1 = \dfrac{-2}{4} = \dfrac{-1}{2}, \quad m_2 = \dfrac{-1}{-2} = \dfrac{1}{2}$

$m_1 \neq m_2$ Neither Parallel nor perpendicular

(d) $\mathbf{4x - y + 2 = 0,\ \ 12x - 3y + 1 = 0}$

$m_1 = \dfrac{-4}{-1} = 4, \quad m_2 = \dfrac{-12}{-3} = 4$

$m_1 = m_2$ Parallel

(e) $\mathbf{12x + 35y - 7 = 0,\ \ 105x - 36y + 11 = 0}$

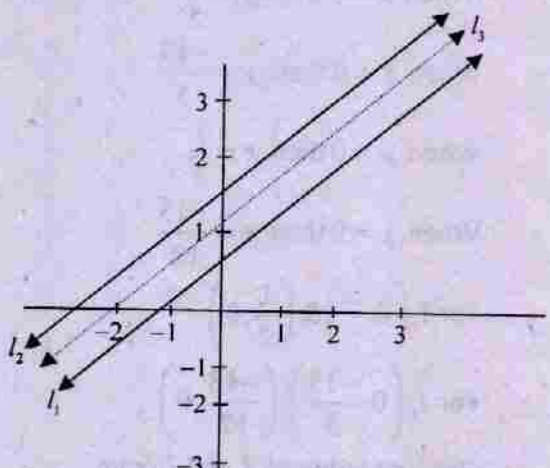
$m_1 = \dfrac{-12}{35}, \quad m_2 = \dfrac{-105}{-36}$

$m_2 = \dfrac{35}{12}$

$m_1 m_2 = \left(\dfrac{-12}{35}\right)\left(\dfrac{35}{12}\right) = -1$ Perpendicular

23. (a) $l_1; 3x-4y+3=0,$ (Lahore 2010)

$l_2; 3x-4y+7=0$



For l_1, when $x=0$ then $y=\frac{3}{4}$

For l_2, put $x=0$ then $y=\frac{7}{4}$

When $y=0$ then $x=-1$

When $y=0$ then $x=\frac{-7}{3}$

So $\left(0,\frac{3}{4}\right)$ & $(-1,0)$ *on* l_1

$\left(0,\frac{7}{4}\right),\left(\frac{-7}{3},0\right)$ *on* l_2

Mid point of $(-1,0)$ and

and Slope $=m=\frac{-a}{b}\left(\frac{-7}{3},0\right)=\left(\frac{-1-\frac{7}{3}}{2},\frac{0+0}{2}\right)$

$=\left(\frac{-10}{6},0\right)=\left(\frac{-5}{3},0\right)$

Slope of both given line $m=-\frac{a}{b}=\frac{3}{4}$

Equation of required midway

Line is $(y-y_1)=m(x-x_1)$

Or $y-0=\frac{3}{4}\left(x+\frac{5}{3}\right)$ *or* $4y=3x+5$

$\Rightarrow$ $\boxed{3x-4y+5=0}$

Now Distance between l_1 and $l_2=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}=\frac{|3(-1)-4(0)+7|}{\sqrt{(3)^2+(-4)^2}}=\boxed{\frac{4}{5}}$

Where $(x_1,y_1)=(-1,0)$ point of l_1 and $l_2=3x-4y+7$

(b) $l_1;\ 12x+5y-6=0$

$l_2;\ 12x+5y+13=0$

When $x=0$ then $y=\frac{6}{5}$

When $x=0$ then $y=\frac{-13}{5}$

When $y=0$ then $x=\frac{1}{2}$

When $y=0$ then $x=\frac{-13}{12}$

For $l_1\left(0,\frac{6}{5}\right)\&\left(\frac{1}{2},0\right)$

For $l_2\left(0,\frac{-13}{5}\right),\left(\frac{-13}{12},0\right)$

Distance between l_1 and l_2 from

$$p\left(\frac{1}{2},0\right) on\ l_1\ and\ l_2=12x+5y+13=\frac{\left|12\left(\frac{1}{2}\right)+5(0)+13\right|}{\sqrt{(12)^{12}+(5)^2}}$$

$$=\frac{|6+13|}{\sqrt{169}}=\frac{19}{13}$$

Now Mid Point of $\left(\frac{1}{2},0\right)$ and

$$\left(\frac{-13}{12},0\right)=\left(\frac{\frac{1}{2}-\frac{13}{12}}{2},\frac{0+0}{2}\right)=\left(\frac{-7}{24},0\right)$$

$$m=Slope=\frac{-a}{b}=\frac{-12}{5}$$

Required equation of Mid way line through

$\left(\frac{-7}{24},0\right)$ and $m=\frac{-12}{5}$ is

$$(y-y_1)=m(x-x_1)$$

$$\Rightarrow (y-0)=\frac{-12}{5}\left(x+\frac{7}{24}\right)\Rightarrow 5y=-12x-\frac{7}{2}$$

$$\Rightarrow \boxed{12x+5y+\frac{7}{2}=0}\ \&\ \boxed{d=\frac{19}{13}}$$

(c) $l_1;\ x+2y-5=0$

$l_2;\ 2x+4y=1$

When $x=0$ then $y=\frac{5}{2}$

When $x=0$ then $y=\frac{1}{4}$

When $y=0$ then $x=5$

When $y=0$ then $x=\frac{1}{2}$

$\left(0,\frac{5}{2}\right),(5,0)$

$\left(0,\frac{1}{4}\right),\left(\frac{1}{2},\right)0$

$d(from\ p(5,0)on\ l_1 and\ l_2)$

$=\frac{|2(5)+4(0)-1|}{\sqrt{(2)^2+(4)^2}}=\frac{9}{\sqrt{20}}$

$\boxed{d=\frac{9}{\sqrt{20}}}$ & $\boxed{d=\frac{9}{2\sqrt{5}}}$

Mid point of $(5,0)$ and $\left(\frac{1}{2},0\right)$

$=\left(\frac{5+\frac{1}{2}}{2},\frac{0+0}{2}\right)=\left(\frac{11}{4},0\right)$

$m=\frac{-1}{2}$

Required equation of line passing mid way is

$(y-y_1)=m(x-x_1)$

$(y-0)=-\frac{1}{2}x\left(x-\frac{11}{4}\right)$

$\Rightarrow 2y=-x+\frac{11}{4}$

$\Rightarrow \boxed{x+2y-\frac{11}{4}=0}$

24. **Pt$(-4,7)$**

$2x-7y+4=0$

$m=$ Slope of given line $=\dfrac{-a}{b}=\dfrac{2}{7}$

Slope of required line (Parallel) $=\dfrac{2}{7}$

Equation of line is

$(y-y_1)=m(x-x_1)$

$y-7=\dfrac{2}{7}(x+4) \Rightarrow 7y-49=2x+8$

$2x-7y+49+8=0 \Rightarrow 2x-7y+57=0$

25. **Pt$(5,-8), A(-15,-8), B(10,7)$**

Slope of AB (given line) $=\dfrac{7-(-8)}{10+15}=\dfrac{15}{25}=\dfrac{3}{5}$

Slope of required line which is perpendicular to given $=-\dfrac{5}{3}$

Equation of required line is

$y+8=-\dfrac{5}{3}(x-5) \Rightarrow 3y+24=-5x+25$

$\Rightarrow 3y+24+5x-25=0$

$5x+3y-1=0$

26. **Given line $2x-y+3=0$**

Any line $\perp ar$ to given is

$x+2y+c=0$

Note: Given line is

$2x-y+3=0$

$m=\dfrac{-a}{b}=2$

To find x intercept put $y=0 \Rightarrow x=-c$

To find y intercept put $x=0 \Rightarrow 2x+c=0$

Slope of $\perp ar$ line $=-\dfrac{1}{2} \Rightarrow y=\dfrac{-c}{2}$

Now $y=mx+c_1$

Given product $=(c)\left(-\dfrac{c}{2}\right)=3$

$y=-\frac{1}{2}x+c_1 \Rightarrow c^2=6 \Rightarrow c=\pm\sqrt{6}$

$2y=-x+2c_1$

Put value of c in 1

Or $\boxed{x+2y+c=0}$

$x+2y\pm\sqrt{6}=0$

Or $x+2y+\sqrt{6}=0$ & $x+2y-\sqrt{6}=0$

27. **Suppose required three vertices are $B(x_1,y_1), C(x_2,y_2), D(x_3,y_3)$ then**

Since E is mid point of AC so

$2=\frac{1+x_2}{2}$ & $1=\frac{4+y_2}{2}$

$\Rightarrow 4=1+x_2 \Rightarrow x_2=3$ & $2=4+y_2 \Rightarrow y_2=-2$ So $(x_2,y_2)=(3,-2)$

Slope of $AD=1=\frac{y_3-4}{x_3-1} \Rightarrow x_3-1=y_3-4 \Rightarrow x_3-y_3-1+4=0$

$x_3-y_3+3=0 \longrightarrow I$

Slope of $BC=1=\frac{-2-y_1}{3-x_1} \Rightarrow 3-x_1=-2-y_1 \Rightarrow -2-y_1+x_1-3=0$

$x_1-y_1-5=0 \longrightarrow II$

Slope of $AB=-\frac{1}{7}=\frac{y_1-4}{x_1-1} \Rightarrow -(x_1-1)=-7y_1-28 \Rightarrow -x_1+1=7y_1-28$

$\Rightarrow 7y_1-28+x_1-1=0 \Rightarrow x_1+7y_1-29=0 \longrightarrow III$

Slope of $DC=-\frac{1}{7}=\frac{-2-y_3}{3-x_3} \Rightarrow -(3-x_3)=-14-7y_3$

$-3+x_3=-14-7y_3$

$\Rightarrow x_3+7y_3-3+14=0$

$x_3+7y_3+11=0 \longrightarrow IV$

Solve I & IV

$IV-I$

$$\begin{array}{r} \cancel{x_3}+7y_3+11=0 \\ \underline{-\cancel{x_3}\mp y_3\pm 3=0} \\ 8y_3+8=0 \end{array}$$

$\Rightarrow 8y_3=-8$ $\boxed{y_3=-1}$

Put in I

$x_3-(-1)+3=0 \Rightarrow x_3+1+3=0$

$x_3+4=0 \Rightarrow \boxed{x_3=-4}$

Solve II & III

$x_1+7y_1-29=0$

$\underline{-x_1 \mp y_1 \mp 5=0}$

$8y_1-24=0$

$\Rightarrow 8y_1=24 \Rightarrow \boxed{y_1=3}$

Put in II

$x_1-3-5=0 \Rightarrow x_1-8=0$

$x_1=8$

Hence required co-ordinates are

$B(x_1,y_1)=B(8,3), C(x_2,y_2)=C(3,-2), D(x_3,y_3)=D(-4,-1)$

28. Above Line

Remember **IF sign y in given equation & our answer is same.**

Below Line

And IF signs are different

(a) $(5,8); 2x-3y+6=0$ **(Sargodha 2008, 11)**

Sign of co-efficient of $y=-3=-ve$

Now $2x-3y+6=2(5)-3(8)+6=10-24+6=-8=-ve$

So point is above the line

(b) $(-7,6); 4x+3y-9=0$ **(Sargodha 2008, 12)**

Sign of co-efficient of $y=+ve$

Now $4x+3y-9=4(-7)+3(6)-9=-28+18-9=-19=-ve$

Signs are different so point is below the line.

29. (a) $(0,0),(-4,7); 5x-7x+70=0$

Co-efficient of y in equation has sign $=-ve$

Put Pt$(0,0)$ so $5x-7y+70=5(0)-7(0)+70=70=+ve$

Pt is below line

Put pt$(-4,7)$ so $5x=7y+70=5(-4)-7(7)+70=-20-49+70$

$=-69+70=1++ve$

Pt $(-4,7)$ is below line

Both are below so both are n same side.

(b) $(0,0),(-4,7); 5x-7x+70=0$

Co-efficient of y in equation has sign $=-ve$

Put Pt$(0,0)$ so $5x-7y+70=5(0)-7(0)+70=70=+ve$

Pt is below line

Put pt $(-4,7)$ so $5x-7y+70=5(-4)-7(7)+70=-20-49+70$

$=-69+70=1++ve$

Pt $(-4,7)$ is below line

Both are below so both are n same side.

30. $(6,-1), 6x-4y+9=0$

Here $x_1=6,\ y_1=-1$

$a=6, b=-4, c=9$

$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

$$d=\frac{|6(6)-4(-1)+9|}{\sqrt{(6)^2+(-4)^2}}=\frac{|36+4+9|}{\sqrt{36+16}}=\frac{49}{\sqrt{52}}=\frac{49}{\sqrt{2\times2\times13}}=\frac{49}{2\sqrt{13}}$$

31. $A(5,3), B(-2,2), C(4,2)$

Here $x_1=5,\ y_1=3, x_2=-2, y_2=2, x_3=4, y_3=2$

$$\text{Area of triangle}=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}5 & 3 & 1\\ -2 & 2 & 1\\ 4 & 2 & 1\end{vmatrix}$$

$$=\frac{1}{2}[5(2-2)-3(-2-4)+1(-4-8)]=\frac{1}{2}[5(0)-3(-6)+1(-12)]$$

$$=\frac{1}{2}(0+18-12)=\frac{1}{2}(6)=3 \text{ Square unit}$$

32. $A(2,3), B(-1,1), C(4,-5)$

Here $x_1=2,\ y_1=3, x_2=-1, y_2=1, x_3=4, y_3=-5$

$$\text{Area of triangle}=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix}2 & 3 & 1\\ -1 & 1 & 1\\ 4 & -5 & 1\end{vmatrix}$$

$$=\frac{1}{2}[2(1-(-5))-3(-1-4)+1(5-4)]=\frac{1}{2}[2(1+5)-3(-5)+1(1)]$$

$$=\frac{1}{2}[12+15+1]=\frac{28}{2}=14\neq 0 \text{ Square unit}$$

So points are not collinear.

Exercise 4.4

1. (i) $x-2y+1=0 \quad I$ **(Sargodha 2010)**

$2x-y+2=0 \quad II$

Multiply II by 2 and solve

$4x-\cancel{2y}+4=0$

$\underline{{}_{-}x\mp\cancel{2y}\pm1=0}$

$3x+3=0$

$\Rightarrow 3x=-3 \Rightarrow \boxed{x=-1}$

Put value of x in I

$-\cancel{1}-2y+\cancel{1}=0 \Rightarrow -2y=0 \Rightarrow \boxed{y=0}$

$(-1,0)$ is point of intersection.

(ii) $3x+y+12=0 \quad I$ **(Sargodha 2009, 10)**

$x+2y-1=0 \quad II$

Multiply I by 2 and solve

$6x+\cancel{2y}+24=0$

$\underline{{}_{-}x\pm\cancel{2y}\mp1=0}$

$5x+25=0$

$5x+25=0$

$5x=-25 \Rightarrow \boxed{x=-5}$

Put value of x in II

$-5+2y-1=0 \Rightarrow 2y-6=0 \Rightarrow \boxed{y=3}$

So $(-5,3)$ is point of intersection.

(iii) $x+4y-12=0 \quad I$ **(Sargodha 2012)**

$x-3y+3=0 \quad II$

$x+4y-12=0$

$\underline{{}_{-}x\mp3y\pm1=0}$

$7y-15=0$

$\Rightarrow \boxed{y=\frac{15}{7}}$

Put in I

$x+4\left(\frac{15}{7}\right)-12=0$

$$x+\frac{60}{7}-12=0$$

$$x+\frac{60-84}{7}=0 \Rightarrow \boxed{x=\frac{24}{7}}$$

$\left(\frac{24}{7},\frac{15}{7}\right)$ is point of intersection.

2. (a) $2x+5y-8=0 \quad I$ **(Sargodha 2011)**

$3x-4y-6=0 \quad II$

Multiply I by 3 & II by 2 and solve

$$\cancel{6x}+15y-24=0$$
$$\underline{\cancel{6x}\mp 8y\mp 12=0}$$
$$23y-12=0$$

$$\boxed{y=\frac{12}{23}}$$

Put in I

$$2x+5\left(\frac{12}{23}\right)-8=0$$

$$\boxed{x=\frac{62}{23}}$$

So point of intersection is $\left(\frac{62}{23},\frac{12}{23}\right)$ Now slope of

$\left(\frac{62}{23},\frac{12}{23}\right)$ and $(2,-9)$ is

$$m=\frac{-9-\frac{12}{23}}{2-\frac{62}{23}}=\frac{\frac{-207-12}{23}}{\frac{46-62}{23}}=\frac{-219}{\cancel{23}}\times\frac{\cancel{23}}{-16}=\frac{219}{16}$$

Now equation of line through $(2,-9)$ and $m=\frac{219}{16}$

$$y-(-9)=\frac{219}{16}(x-2) \Rightarrow y+9=\frac{219}{16}(x-2)$$

$$16(y+9)=219(x-2) \Rightarrow 16y+144=219x-438$$

$$or\ 219x-16y=+144-438 \Rightarrow 219x-16y=+582$$

$$\Rightarrow 219x-16y-582=0$$

(b) $\mathbf{x - y - 4 = 0}$ **&** $\mathbf{7x + y + 20 = 0}$

Solving then $x - y - 4 = 0 \longrightarrow I$

Multiply I by 3 & II by 2 and solve

$$\frac{7x + y + 20 = 0}{8x + 16 = 0} \Rightarrow x = -2 \text{ Put value of x in I}$$

$$-2 - y - 4 = 0$$

$$\Rightarrow y = -6$$

$$x = -2 \ \& \ y = -6$$

Point of intersection is $(-2, -6)$

Given line is $6x + y - 14 = 0$

(i) **Slope of Given** $= -6 \Rightarrow m = Slope = -6$

Slope of required line which is parallel to given $= -6$

Equation of line through $(-2, -6)$ & $m = -6$

$$y - (-6) = -6(x - (-2)) \Rightarrow y + 6 = -6(x + 2)$$

$$\Rightarrow \quad y + 6 = -6x - 12 \Rightarrow 6x + y + 12 + 6 = 0$$

$$\Rightarrow \quad 6x + y + 18 = 0 \text{ (Required line)}$$

(ii) **Slope of Given** $= -6$

Slope of required line which is $\perp ar$ to given $= \frac{1}{6}$

Equation of line through $(-2, -6)$ & $m = \frac{1}{6}$

$$y - (-6) = \frac{1}{6}(x - (-2)) \Rightarrow y + 6 = \frac{1}{6}(x + 2)$$

$$\Rightarrow \quad 6y + 36 = x + 2 \Rightarrow x - 6y - 36 + 2 = 0$$

$$\Rightarrow \quad x - 6y - 34 = 0$$

(c) **Any line through intersection of**

Given lines $(x + 2y + 3) = 0$ *and* $3x + 4y + 7 = 0$ *is*

$(x + 2y + 3) + k(3x + 4y + 7) = 0$

or $x + 2y + 3 + 3kx + 4ky + 7k = 0$

or $x + 3kx + 2y + 4ky + 3 + 7k = 0$

$(3k + 1)x + (2 + 4k)y + 3 + 7k = 0$

To find x intercept put $y = 0 \Rightarrow (3k + 1)x + 3 + 7k = 0$

$$\Rightarrow x = \frac{-(3 + 7k)}{3k + 1}$$

To find y intercept put $x = 0$

$$\Rightarrow (2+4k)y+3+7k=0 \Rightarrow y=\frac{-(3+7k)}{2+4k}$$

Given both intercept are equal so

$$\frac{-(3+7k)}{3k+1}=\frac{-(3+7k)}{2+4k} \Rightarrow \frac{1}{3k+1}=\frac{\cancel{-(3+7k)}}{(2+4k)}\times\frac{1}{\cancel{(3+7k)}}$$

$$\Rightarrow 2+4k=(3k+1) \quad or \quad 2+4k=-3k+1$$

$$or\ 4k-3k=1-2 \Rightarrow \boxed{k=-1}$$

Put value of k in I

$$(x+2y+3)+(-1)(3x+4y+7)=0$$

$$or\ x+2y+3-3x-4y-7=0 \ or\ -2x-2y-4=0$$

$$or\ 2x+2y+4=0 \ \div by\ 2 \qquad x+y+2=0$$

3. **First we will find intersection of**

$16x-10y-33=0 \ I$

$12x+14y+20=0 \ II$

$'X'\ I$ by 14 and II by 10 we get and add

$$224x-\cancel{140y}-462=0$$
$$120x+\cancel{140y}+290=0$$
$$344x-172=0$$

$$\Rightarrow 344x=172 \Rightarrow x=\frac{172}{344}=\boxed{\frac{1}{2}}$$

Put in I $16\left(\frac{1}{2}\right)-10y-33=0 \Rightarrow 8-10y-33=0 \Rightarrow -10y-25=0$

$$10y=-25 \Rightarrow y=\frac{-25}{10} \Rightarrow y=\boxed{\frac{-5}{2}}$$

Point of intersection is $\left(\frac{1}{2},\frac{-5}{2}\right)$

Now we will find intersection of $x-y+4=0\ III$ & $x-7y+2=0\ IV$

$III-IV$

$$\cancel{x}-y+4=0$$
$$\_\ \cancel{x}\mp 7y\pm 2=0$$
$$6y+2=0$$

$$\Rightarrow 6y=-2 \Rightarrow y=\frac{-2}{6}=\frac{-1}{3}$$

Put value of y in $III x-\left(-\frac{1}{3}\right)+4=0 \Rightarrow x+\frac{1}{3}+4=0$

$x+\frac{1+12}{3}=0 \Rightarrow x=\frac{-13}{3}$ So point of intersection is

Slope through $\left(\frac{1}{2},\frac{-5}{2}\right)$ and $\left(\frac{-13}{3},\frac{-1}{3}\right)$

$$m=\frac{-\frac{1}{3}-\left(-\frac{5}{2}\right)}{\frac{-13}{3}-\frac{1}{2}}=\frac{-\frac{1}{3}+\frac{5}{2}}{\frac{-13}{2}-\frac{1}{2}}=\frac{\frac{-2+15}{6}}{\frac{-26-3}{6}}$$

$$m=\frac{\frac{13}{6}}{\frac{-29}{6}}=\frac{13}{6}\times\frac{6}{-29}=\frac{-13}{29}$$

Equation of required line through $\left(\frac{1}{2},\frac{-5}{2}\right)$ and $m=-\frac{13}{29}$

$$\left(x+\frac{5}{2}\right)=-\frac{13}{29}\left(x-\frac{1}{2}\right) \Rightarrow 29\left(y+\frac{5}{2}\right)=-13\left(x-\frac{1}{2}\right)$$

$$29y+\frac{145}{2}=-13x+\frac{13}{2} \Rightarrow 29y+13x+\frac{145}{2}-\frac{13}{2}=0$$

$$13x+29y+\frac{145-13}{2}=0 \Rightarrow 13x+29y+\frac{132}{2}=0$$

$$\Rightarrow 13x+29y+66=0$$

4. $y=m_1x+c_1,\ y=m_2x+c_2,\ y=m_3x+c_3$

Arranging them $m_1x-y+c_1=0,\ m_2x-y+c_2=0,\ m_3x-y+c_3=0$ they are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0 \Rightarrow \begin{vmatrix} m_1 & -1 & c_1 \\ m_2 & -1 & c_2 \\ m_3 & -1 & c_3 \end{vmatrix}$$

$$m_1(-c_3+c_2)-(-1)(m_2c_3-m_3c_2)+c_1(-m_2+m_3)=0$$

$$m_1(c_2-c_3)+m_2c_3-m_3c_2-m_2c_1+m_3c_1=0$$

$$m_1(c_2-c_3)+m_2(c_3-c_1)+m_3(c_1-c_2)=0$$

Required condition.

5. $2x-3y-1=0,\ 3x-y-5=0,\ 3x+py+8=0$ they meet at a point or concurrent if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 2 & -3 & -1 \\ 3 & -1 & -5 \\ 3 & p & 8 \end{vmatrix}$$

(Sargodha 2008, 09)

$$or \quad 2(-8+5p)-(-3)(24+15)+(-1)(3p+3)=0$$

$$or \quad 10p-16+3(39)-(3p+3)=0 \Rightarrow 10p-16+117-3p-3=0$$

$$or \quad 7p+98=0 \Rightarrow p=\frac{-98}{7}=-14 \Rightarrow \boxed{p=-14}$$

6. $4x-3y-8=0,\ 3x-4y-6=0,\ x-y-2=0$ they meet at a point or concurrent if

For Concurrency $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \ so$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 4 & -3 & -8 \\ 3 & -4 & -6 \\ 1 & -1 & -2 \end{vmatrix}$$

$$=4(8-6)-(-3)(-6+6)+(-8)(-3+4)=4(2)+3(0)-8(1)=8-8=0$$

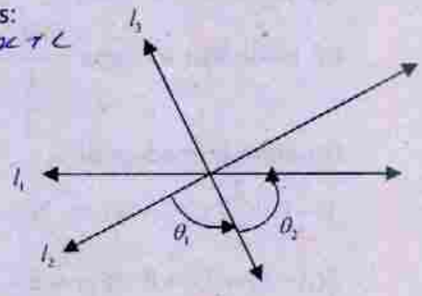
So line are concurrent.

Convert given equation into slope intercept forms:

$$3y=4x-8 \Rightarrow y=\frac{4}{3}x-\frac{8}{3} \quad m_1=\frac{4}{3}$$

$$4y=3x-6 \Rightarrow y=\frac{3}{4}x-\frac{6}{4} \quad m_2=\frac{3}{4}$$

$$y=x-2 \Rightarrow y=(1)x-2 \quad m_3=1$$



Now given condition prove if $\theta_1=\theta_2$

θ_1 is angle l_2 *to* l_3 *so*

$$\theta_1 = Tan^{-1}\left(\frac{m_3-m_2}{1+m_3m_2}\right) = Tan^{-1}\left(\frac{1-\frac{3}{4}}{1+1.\frac{3}{4}}\right) = Tan^{-1}\left(\frac{\frac{4-3}{4}}{\frac{4+3}{4}}\right)$$

$$= Tan^{-1}\left(\frac{\frac{1}{4}}{\frac{7}{4}}\right) = Tan^{-1}\left(\frac{1}{4}\times\frac{4}{7}\right) = Tan^{-1}\left(\frac{1}{7}\right)$$

θ_2 is angle from l_3 *to* l_1 *so*

$$\theta_2 = Tan^{-1}\left(\frac{m_1 - m_3}{1 + m_1 m_3}\right) = Tan^{-1}\left(\frac{\frac{4}{3} - 1}{1 + \frac{4}{3}.1}\right) = Tan^{-1}\frac{\frac{4-3}{3}}{\frac{3+4}{3}}$$

$$= Tan^{-1}\left(\frac{\frac{1}{3}}{\frac{7}{3}}\right) = Tan^{-1}\left(\frac{1}{3} \times \frac{3}{7}\right) = Tan^{-1}\left(\frac{1}{7}\right)$$

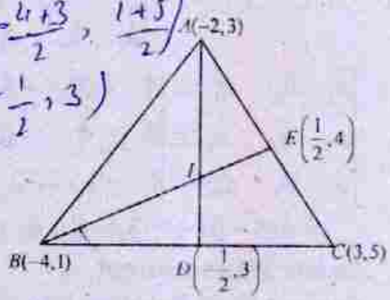
$\theta_1 = \theta_2$ Hence proved. Centriod: The Point where medians of as intersect each other.

7. $A(-2,3), B(-4,1), C(3,5)$

(i) **Centriod** is points of intersection of medians so first we find equation of medians then there meeting point (centroid).

AD is median so mid point of $\overline{BC} = \left(\frac{-4+3}{2}, \frac{1+5}{2}\right)$

$D = \left(-\frac{1}{2}, 3\right)$



Slope of $AD = \frac{3-3}{-\frac{1}{2}+2} = 0$

Equation of median AD thourgh $A(-2,3)$ is

$y - 3 = 0(x+2)$ $y - y_1 = m(x - x_1)$

$\Rightarrow y - 3 = 0 \Rightarrow y = 3$

BE is median so slope $= \frac{4-1}{\frac{1}{2}+4} = \frac{3}{\frac{9}{2}} = \frac{3}{1} \times \frac{2}{9_3} = \frac{2}{3}$

Equation of median BE is

$$y - 1 = \frac{2}{3}(x - (-4)) \Rightarrow 3(y-1) = 2(x+4) \Rightarrow put\ y = 3$$

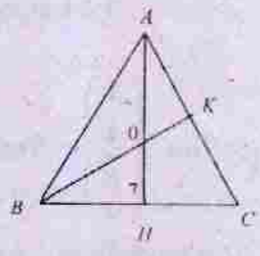
$$3(3-1) = 2x + 8 \Rightarrow 6 = 2x + 8 \Rightarrow 6 - 8 = 2x \Rightarrow -2 = 2x \Rightarrow \boxed{x = -1}$$

So centroid is $I\ (-1,3)$

(ii) **Orthocentre** The Point where altitudes of an intersect each other.

Orthrocentre is meeting point of angle bisectors so

Slope of $BC = \frac{5-1}{3-(-4)} = \frac{4}{3+4} = \frac{4}{7}$



Slope of angle bisector AH

Which is perpendicular to $BC = \frac{7}{4}$

Equation of angle AH is

$$(y-3) = -\frac{7}{4}(x - (-2)) \Rightarrow 4(y-3) = -7(x+2)$$

$4y-12=-7x-14 \Rightarrow 7x+4y-12+14=0$

$7x+4y+2=0$ ——— I

Slope of $AC=\dfrac{5-3}{3-(-2)}=\dfrac{2}{3+2}=\dfrac{2}{5}$

Slope of BK $(\perp ar\ to AC)=-\dfrac{5}{2}$

Equation of angle bisector BK is

$y-1=\dfrac{-5}{2}9x-(-4)) \Rightarrow 2(y-1)=--59x+4)$

$2y-2=-5x-20 \Rightarrow 5x+2y-2+20=0$

$10x+4y+36=0 \quad II$

$II-I$

$10x+\not{4y}+36=0$

$\underline{7x\pm\not{4y}\pm 2=0}$

$3x+34=0$

$\Rightarrow\ 3x=-34$

$\boxed{x=\dfrac{-34}{3}}$

Put value of $x=-\dfrac{34}{3}$

$7\left(-\dfrac{34}{3}\right)+4y+2=0$

$-\dfrac{238}{3}+4y+2=0$

$or\ 4y+2-\dfrac{238}{3}=0 \Rightarrow 4y+\dfrac{6-238}{3}=0 \Rightarrow 4y-\dfrac{232}{3}=0$

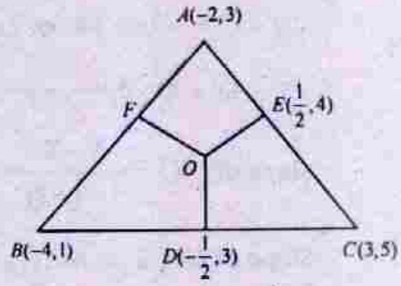
$4y=\dfrac{232}{3} \Rightarrow y=\dfrac{232}{3}\times\dfrac{1}{4}=\boxed{\dfrac{58}{3}}$

So orthocenter is $\left(\dfrac{-34}{3},\dfrac{58}{3}\right)$

(iii) **Circumcentre** The Point where same (right) perpendicular of bisector of intersect eacheachother

Meeting point of right bisector is called circumcentre

Slope of $BC=\dfrac{5-1}{3+4}=\dfrac{4}{7}$



Slope of right bisector $OD = -\frac{7}{4}$

Equation of OD is

$$y-3=-\frac{7}{4}\left(x-\left(-\frac{1}{2}\right)\right)$$

$$4(y-3)=-7\left(x+\frac{1}{3}\right) \Rightarrow 4y-12=-7x-\frac{7}{2}$$

$'X'$ both sides by 2 $8y-24=-14x-7 \Rightarrow 14x+8y-24+7=0$

$$14x+8y-17=0 \text{———} I$$

Slope of $AC=\frac{5-3}{3+2}=\frac{2}{5}$

Slope of $OE=\frac{-5}{2}$

Equation of OE is (Through pt E and Slope $\frac{-5}{2}$)

$$(y-4)=\frac{-5}{2}\left(x-\frac{1}{2}\right) \Rightarrow 2(y-4)=-5\left(x-\frac{1}{2}\right)$$

$$2y-8=5x+\frac{5}{2} \Rightarrow 4y-16=-10x+5('X' \text{ by } 2)$$

$$or\ 10x+4y-16-5=0 \Rightarrow 10+4y-21=0$$

$$'X' \text{ by } 2\ 20x+8y-42=0 \quad II$$

$$II-I$$

$$20x-\cancel{8y}-42=0$$

$$\underline{-14x \pm \cancel{8y} \mp 17=0}$$

$$6x-25=0$$

$$\Rightarrow \boxed{x=\frac{25}{6}}$$

Put value of x in I $\quad {}^{7}\cancel{14}\left(\frac{25}{\cancel{6}_3}\right)+8y-17$ or $\frac{175}{3}-17+8y=0$

$$8y+\frac{175-51}{3}=0 \Rightarrow 8y-\frac{124}{3}=0 \Rightarrow 8y=-\frac{124}{3}$$

$$y=\frac{\cancel{124}^{31}}{3\times \cancel{8}_2} \Rightarrow y=\frac{-31}{6}$$ So circumcentre is $\left(\frac{25}{6},\frac{-31}{6}\right)$

Now to check these points are collinear

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 3 & 1 \\ \frac{-34}{3} & \frac{58}{3} & 1 \\ \frac{25}{6} & \frac{-31}{6} & 1 \end{vmatrix}$$

$$= -1\left(\frac{58}{3}+\frac{31}{6}\right)-3\left(\frac{-34}{3}-\frac{25}{6}\right)+1\left[\left(\frac{-34}{3}\right)\left(\frac{-31}{6}\right)-\left(\frac{25}{6}\right)\left(\frac{58}{3}\right)\right]$$

$$= -\frac{58}{3}-\frac{31}{6}+34+\frac{25}{2}+\frac{1054}{18}-\frac{1450}{18}$$

$$= \frac{-348-93+612+225+1054-1450}{18} = \frac{1891-1891}{18} = \frac{0}{18}$$

$$= 0$$

Hence centroid, orthocenter and circumcentre are collinear (Yes).

8. $\mathbf{4x-3y-8=0,\ 3x-4y-6=0,\ x-y-2=0}$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 4 & -3 & -8 \\ 3 & -4 & -6 \\ 1 & -1 & -2 \end{vmatrix}$$

$$= 4(8-6)-(-6+6)+(-3+4) = 4(2)+3(0)-8(1) = 8-8 = 0$$

So given lines are concurrent

Now for point where they meet

$4x-3y-8=0$

$x-y-2=0$

$$\frac{x}{6-8} = \frac{y}{-8+8} = \frac{1}{-4+3} \Rightarrow \frac{x}{-2} = \frac{y}{0} = \frac{1}{-1} \Rightarrow x=-2,\ y=0$$

Point of concurrency is $(2,0)$

9. $\mathbf{x-2y-6=0,\ 3x-y+3=0,\ 2x+y-4=0}$

Solving I & II

$x-2y-6=0$

$3x-y+3=0$

$$\frac{x}{-6-6} = \frac{y}{-18-6} = \frac{1}{-1+6}$$

$$\frac{x}{-12} = \frac{y}{-21} = \frac{1}{5}$$

$$x = \frac{-12}{5},\ y = \frac{-21}{5}$$

Solving II & III

$3x-y+3=0$

$2x+y-4=0$

$$\frac{x}{4-3} = \frac{y}{6+12} = \frac{1}{3+2}$$

$$\frac{x}{1} = \frac{y}{18} = \frac{1}{5}$$

$$x = \frac{1}{5},\ y = \frac{18}{5}$$

Solving I & II

$x-2y-6=0$

$2x+y-4=0$

$$\frac{x}{8+6} = \frac{y}{-12+4} = \frac{1}{1+4}$$

$$\frac{x}{14} = \frac{y}{-8} = \frac{1}{5}$$

$$x = \frac{14}{5},\ y = \frac{-8}{5}$$

So vertices triangles are $A=\left(\frac{14}{5},\frac{-8}{5}\right)$, $B\left(\frac{1}{5},\frac{18}{5}\right)$, $C\left(\frac{-12}{5},\frac{-21}{5}\right)$.

$$m_1=\text{Slope of } AB=\frac{\frac{18}{5}-\left(\frac{-8}{5}\right)}{\frac{1}{5}-\frac{-14}{5}}=\frac{\frac{18+8}{5}}{\frac{1-14}{5}}=\frac{\frac{26}{5}}{\frac{-13}{5}}=\frac{26}{5}\times\frac{5}{-13}=-2$$

$$m_2=\text{Slope of } BC=\frac{\frac{-21}{5}-\frac{18}{5}}{\frac{-12}{5}-\frac{1}{5}}=\frac{\frac{-39}{5}}{\frac{-13}{5}}=\frac{-39}{-13}=3$$

$$m_3=\text{Slope of } CA=\frac{\frac{-8}{5}+\frac{21}{5}}{\frac{14}{5}+\frac{12}{5}}=\frac{\frac{13}{5}}{\frac{26}{5}}=\frac{13}{5}\times\frac{5}{26}=\frac{1}{2}$$

$$\theta_1=Tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)=Tan^{-1}\left(\frac{-2-3}{1+(-2)(3)}\right)=Tan^{-1}\left(\frac{-5}{1-6}\right)=Tan^{-1}\left(\frac{-5}{-5}\right)=Tan^{-1}(1)=45^o$$

$$\theta_2=Tan^{-1}\left(\frac{m_2-m_3}{1+m_2m_3}\right)=Tan^{-1}\left(\frac{3-\frac{1}{2}}{1+3.\frac{1}{2}}\right)=Tan^{-1}\left(\frac{\frac{6-1}{2}}{\frac{2+3}{2}}\right)=Tan^{-1}\left(\frac{\frac{5}{2}}{\frac{5}{2}}\right)=Tan^{-1}(1)=45^o$$

$$\theta_3=Tan^{-1}\left(\frac{m_3-m_1}{1+m_3m_1}\right)=Tan^{-1}\left(\frac{\frac{1}{2}-(-2)}{1+\frac{1}{2}(-2)}\right)=Tan^{-1}\left(\frac{\frac{1}{2}+2}{1-1}\right)=Tan^{-1}\left(\frac{\frac{5}{2}}{0}\right)=Tan^{-1}(\infty)=90^o$$

10. (a) l_1; (2,7), (7,10) **(Sargodha 2008)**

l_2; (1,1), (−5,3)

$$m_1=Slope\ of\ l_1=\frac{10-7}{7-2}=\frac{3}{5},\ m_2=Slope\ of\ l_2=\frac{3-1}{-5-1}=\frac{2}{-6}=-\frac{1}{3}$$

$$Angle=\theta=(l_1 to l_2)=Tan^{-1}\left(\frac{m_2-m_1}{1+m_2m_1}\right)=Tan^{-1}\left(\frac{-\frac{1}{3}-\frac{3}{5}}{1+\left(-\frac{1}{3}\right)\left(\frac{3}{5}\right)}\right)=Tan^{-1}\left(\frac{\frac{-5-9}{15}}{1-\frac{1}{5}}\right)$$

$$=Tan^{-1}\left(\frac{\frac{-14}{15}}{\frac{4}{5}}\right)=Tan^{-1}\left(-\frac{14}{15}\times\frac{5}{4}\right)=Tan^{-1}\left(-\frac{7}{6}\right)=Tan^{-1}(-1.16)$$

angle 180

$= -49°23' = 180° - 49°23' = 130°36'$

Acute angle $= Tan^{-1}\left(\frac{7}{6}\right) = Tan^{-1}(1.16) = 49°23'$

(b) $l_1; (3,-1), (5,7); l_2; (2,4), (-8,2)$

$m_1 = Slope\ of\ l_1 = \frac{7-(-1)}{5-3} = \frac{7+1}{2} = \frac{8}{2} = 4$

$m_2 = Slope\ of\ l_2 = \frac{2-4}{-8-2} = \frac{-2}{-10} = \frac{1}{5}$

$$\theta = Tan^{-1}\left(\frac{m_2 - m_1}{1+m_2m_1}\right) = Tan^{-1}\left(\frac{\frac{1}{5}-4}{1+\frac{1}{5}.4}\right) = Tan^{-1}\left(\frac{\frac{1-20}{5}}{\frac{5+4}{5}}\right) = Tan^{-1}\left(\frac{\frac{-19}{5}}{\frac{9}{5}}\right)$$

$$= Tan^{-1}\left(\frac{-19}{9}\right) = Tan^{-1}(-2.11) = -64°39' = 180° - 64°39' = 115°20'$$

Acute angle $= Tan^{-1}\left(\frac{19}{9}\right) = 64°39'$

(c) $l_1; (3,-7), (6,-4); l_2; (-1,2), (-6,-1)$ **(Gujrawala 2010)**

$m_1 = Slope\ of\ l_1 = \frac{-4-(-7)}{6-1} = \frac{-4+7}{6-1} = \frac{3}{5}$

$m_2 = Slope\ of\ l_2 = \frac{-1-2}{-6-(-1)} = \frac{-1-2}{-6+1} = \frac{3}{5}$

$$\theta = Tan^{-1}\left(\frac{m_2 - m_1}{1+m_2m_1}\right)$$

$$\theta = Tan^{-1}\frac{\frac{3}{5}-\frac{3}{5}}{1+\frac{3}{5}\times\frac{3}{5}}$$

$$\theta = Tan^{-1}\frac{0}{1+\frac{9}{25}} = Tan^{-1} = (0)$$

$\boxed{\theta = 0°}$

(d) $l_1; (-9,-1), (3,-5); l_2; (2,7), (-6,-7)$

$m_1 = Slope\ of\ l_1 = \frac{-5-(-1)}{3-(-9)} = \frac{-5+1}{3+9} = \frac{-4}{12} = \frac{-1}{3}$

$$m_2 = Slope\ of\ l_2 = \frac{-7-7}{-6-2} = \frac{-14}{-8} = \frac{7}{4}$$

$$\theta = Tan^{-1}\frac{m_2 - m_1}{1+m_2m_1} = Tan^{-1}\frac{\frac{7}{4}-\left(\frac{-1}{3}\right)}{1+\left(\frac{7}{4}\right)\left(\frac{-1}{3}\right)} = Tan^{-1}\frac{\frac{7}{4}+\frac{1}{3}}{1-\frac{7}{12}} = Tan^{-1}\frac{\frac{21+4}{12}}{\frac{12-7}{12}}$$

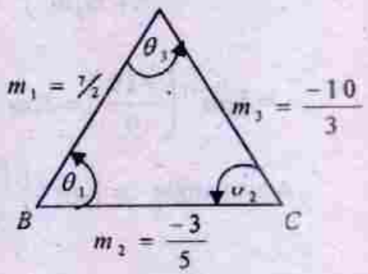
$$= Tan^{-1}\frac{\frac{25}{12}}{\frac{5}{12}} = Tan^{-1}(5) \Rightarrow \theta = 78.69^o$$

11. **(a)** $A(-2,11),\ B(-6,-3),\ C(4,-9)$

Slope of $AB = m_1 = \frac{-3-11}{-6+2} = \frac{-14}{-4} = \frac{7}{2}$

$m_2 = Slope\ of\ BC = \frac{-9+3}{4+6} = \frac{-6}{10} = \frac{-3}{5}$

$m_3 = Slope\ of\ CA = \frac{11+9}{-2-4} = \frac{20}{-6} = \frac{-10}{3}$



$$\theta_1 = Tan^{-1}\left(\frac{m_2-m_1}{1+m_2m_1}\right) = Tan^{-1}\left(\frac{\frac{7}{2}-\left(\frac{-3}{5}\right)}{1+\frac{7}{2}\left(\frac{-3}{5}\right)}\right)$$

$$= Tan^{-1}\left(\frac{\frac{7}{2}+\frac{3}{5}}{1-\frac{21}{10}}\right) = Tan^{-1}\left(\frac{\frac{35+6}{10}}{\frac{10-21}{10}}\right) = Tan^{-1}\left(\frac{41}{10}\times\frac{10}{-11}\right) = Tan^{-1}\left(\frac{-41}{11}\right)$$

$$= -74^o58' = \boxed{105^o}$$

Acute angle $= Tan^{-1}\left(\frac{7}{6}\right) = Tan^{-1}(1.16) = 49^o23'$

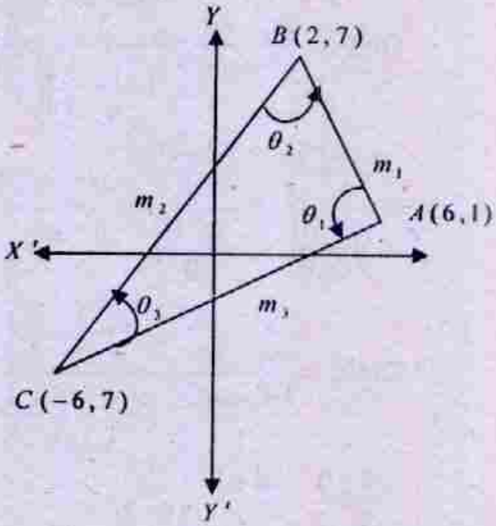
$$\theta_2 = Tan^{-1}\left(\frac{m_2-m_3}{1+m_2m_3}\right) = Tan^{-1}\left(\frac{\frac{-3}{5}-\left(\frac{-10}{3}\right)}{1+\left(\frac{-3}{5}\right)\left(\frac{-10}{3}\right)}\right) = Tan^{-1}\left(\frac{\frac{-3}{5}+\frac{10}{3}}{1+\frac{30}{15}}\right)$$

$$\theta_2 = Tan^{-1}\left(\frac{\frac{-9+50}{15}}{\frac{15+30}{15}}\right) = Tan^{-1}\left(\frac{41}{15}\times\frac{15}{45}\right)Tan^{-1}\left(\frac{41}{45}\right) = \boxed{42^o20'}$$

$$\theta_3 = Tan^{-1}\left(\frac{-41}{6}\times\frac{6}{-64}\right) = Tan^{-1}\left(\frac{41}{64}\right) = \boxed{32^o 38'}$$

So interior angles are

$\theta_1 = 105^o \quad \theta_2 = 42^o 20' \quad \theta_3 = 32^o 38'$



(b) $A(6,1),\ B(2,7),\ C(-6,-7)$

$$m_1 = \frac{7-1}{2-6} = \frac{6}{-4} = \frac{3}{2}$$

$$m_2 = \frac{-7-7}{-6-2} = \frac{-14}{-8} = \frac{7}{4}$$

$$m_3 = \frac{+1-(-7)}{6-(-6)} = \frac{8}{12} = \frac{2}{3}$$

θ_1, θ_2 *and* θ_3 are shown in the figure.

$$= \frac{\frac{2}{3}+\frac{2}{3}}{1-\frac{6}{6}} = \frac{\frac{4+9}{6}}{1-1} \text{ undefined}$$

$\Rightarrow \theta_1 = 90^o$

$$Tan\theta_2 = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{-\frac{3}{2}-\frac{7}{4}}{1+\left(-\frac{3}{2}\right)\left(\frac{7}{4}\right)} = \frac{\frac{-6-7}{4}}{\frac{8-21}{8}} = -\frac{13}{4}\times\left(-\frac{8}{13}\right) = 2$$

$\Rightarrow \theta_2 = 63.4^o$

$$Tan\theta_3 = \frac{m_2 - m_3}{1 + m_2 m_3} = \frac{\frac{7}{4}-\frac{2}{3}}{1+\left(\frac{7}{4}\right)\left(\frac{2}{3}\right)} = \frac{\frac{21-8}{12}}{\frac{12+14}{12}} = \frac{13}{26} = \frac{1}{2}$$

$\Rightarrow \theta_3 = 26.6^o$

(c) $A(2,-5),\ B(-4,-3),\ C(-1,5)$

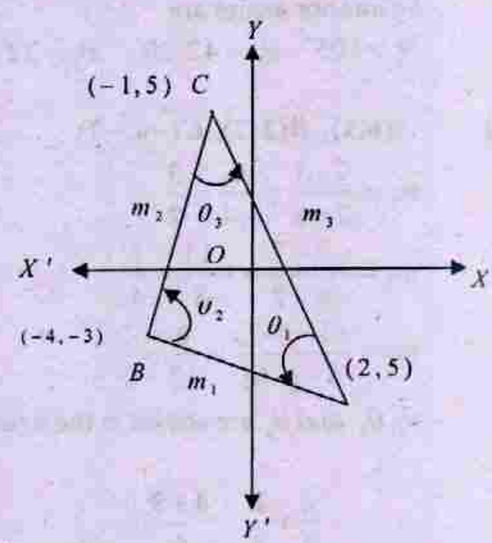
Let m_1, m_2, m_3 be the slopes of the sides AB, BC, and CA respectively. Then

$$m_1 = \frac{-3-(-5)}{-4-2} = \frac{-3+5}{-6} = -\frac{2}{6} = -\frac{1}{3}$$

$$m_2 = \frac{5-(-3)}{-1-(-4)} = \frac{5+3}{-1+4} = \frac{8}{3}$$

$$m_3 = \frac{5-(-5)}{-1-2} = \frac{5+5}{-1-2} = -\frac{10}{3}$$

θ_1, θ_2 *and* θ_3 are shown in the figure.



$$Tan\theta_1 = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{-\frac{1}{3} - \left(-\frac{10}{3}\right)}{1 + \left(-\frac{1}{3}\right)\left(-\frac{10}{3}\right)}$$

$$= \frac{-\frac{1}{3} + \frac{10}{3}}{1 + \frac{10}{9}} = \frac{\frac{-1+10}{3}}{\frac{9+10}{9}} = \frac{9}{3} \times \frac{9}{19} = \frac{27}{19}$$

$$Tan\theta_2 = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{\frac{8}{3} - \left(-\frac{1}{3}\right)}{1 + \left(\frac{8}{3}\right)\left(-\frac{1}{3}\right)}$$

$$= \frac{\frac{8}{3} + \frac{1}{3}}{1 - \frac{8}{9}} = \frac{\frac{8+1}{3}}{\frac{9-8}{9}} = \frac{9}{3} \times \frac{9}{1} = 27$$

$$Tan\theta_2 = 87.9^o$$

$$Tan\theta_3 = \frac{m_3 - m_2}{1 + m_3 m_2} = \frac{-\frac{10}{3} - \frac{8}{3}}{1 + \left(-\frac{10}{3}\right)\left(\frac{8}{3}\right)}$$

$$= \frac{\frac{-10-8}{3}}{1 - \frac{80}{9}} = \frac{\frac{-18}{3}}{\frac{9-80}{9}} = -6 \times \frac{9}{-71} = \frac{54}{71}$$

$$Tan\theta_3 = 37.2^o$$

(d) $\boldsymbol{A(2,8), B(-5,4), C(4,-9)}$

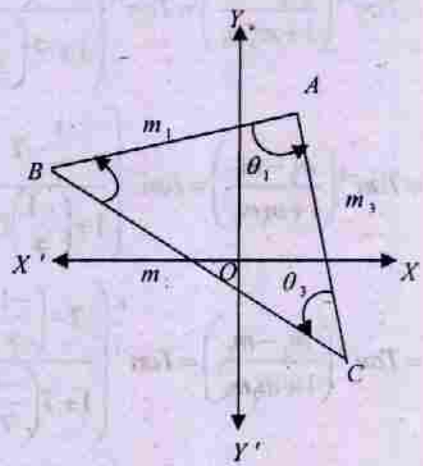
Let m_1, m_2, m_3 be the slopes of the sides AB, BC, and CA respectively. Then

$$m_1 = \frac{4-8}{-5-2} = \frac{-4}{-7} = \frac{4}{7}$$

$$m_2 = \frac{-9-4}{4-(-5)} = \frac{-9-4}{4+5} = \frac{-13}{9}$$

$$m_3=\frac{8-(-a)}{2-4}=\frac{17}{-2}=\frac{-17}{2}$$

θ_1,θ_2 *and* θ_3 are shown in the figure.



$$Tan\theta_1=\frac{m_3-m_1}{1+m_3m_1}=\frac{\frac{-17}{2}-\frac{4}{7}}{1+\left(\frac{-17}{2}\right)\left(\frac{4}{7}\right)}$$

$$=\frac{\frac{-119-8}{14}}{1-\frac{68}{14}}=\frac{\frac{-127}{14}}{\frac{14-68}{14}}=\frac{-127}{-54}=\frac{127}{54}$$

$\theta_1=67^o$

$$Tan\theta_2=\frac{m_1-m_2}{1+m_1m_2}=\frac{\frac{4}{7}-\left(\frac{-13}{9}\right)}{1+\left(\frac{4}{7}\right)\left(\frac{-13}{9}\right)}$$

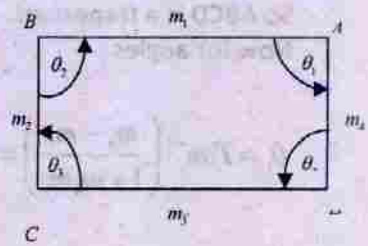
$$=\frac{\frac{4}{7}+\frac{13}{9}}{1-\frac{52}{63}}=\frac{\frac{36+91}{63}}{\frac{63-52}{63}}=\frac{127}{11}$$

$\Rightarrow\theta_2=85^o$

$$Tan\theta_3=\frac{m_2-m_3}{1+m_2m_3}=\frac{\frac{-13}{9}+\frac{17}{2}}{1+\left(\frac{-13}{9}\right)\left(\frac{-17}{2}\right)}$$

$$=\frac{\frac{-13}{9}+\frac{17}{2}}{1+\frac{221}{18}}=\frac{127}{239}\Rightarrow\theta_3=Tan^{-1}\frac{127}{239}\Rightarrow\theta_3=28^o$$

12. $A(5,2),\ B(-2,3),\ C(-3,-4),\ D(4,-5)$



Slope of $AB=m_1=\frac{3-2}{-2-5}=-\frac{1}{7}$

Slope of $BC=m_2=\frac{-4-3}{-3+2}=\frac{-7}{7}=7$

Slope of $CD=m_3=\frac{-5+4}{4+3}=-\frac{1}{7}$

Slope of $DA = m_4 = \dfrac{2+5}{5-4} = \dfrac{7}{1} = 7$

$$\theta_1 = Tan^{-1}\left(\frac{m_4 - m_1}{1+m_4m_1}\right) = Tan^{-1}\left(\frac{7-\left(\frac{-1}{7}\right)}{1+7\left(\frac{-1}{7}\right)}\right) = Tan^{-1}\left(\frac{7+\frac{1}{7}}{1-1}\right) = Tan^{-1}\left(\frac{7+\frac{1}{7}}{0}\right) = Tan^{-1}(\infty) = 90^o$$

$$\theta_2 = Tan^{-1}\left(\frac{m_1 - m_2}{1+m_1m_2}\right) = Tan^{-1}\left(\frac{\frac{1}{7}-7}{1+\left(\frac{-1}{7}\right)(7)}\right) = Tan^{-1}\left(\frac{\frac{-1-49}{7}}{1-1}\right) = Tan^{-1}\left(\frac{\frac{-50}{7}}{0}\right) = Tan^{-1} = (\infty) = 90^o$$

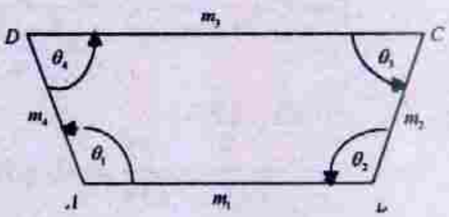
$$\theta_3 = Tan^{-1}\left(\frac{m_2 - m_3}{1+m_2m_3}\right) = Tan^{-1}\left(\frac{7-\left(\frac{-1}{7}\right)}{1+7\left(\frac{-1}{7}\right)}\right) = Tan^{-1}\left(\frac{7+\frac{1}{7}}{1-1}\right) = Tan^{-1}\left(\frac{7+\frac{1}{7}}{0}\right) = Tan^{-1} = (\infty) = 90^o$$

$$\theta_4 = Tan^{-1}\left(\frac{m_3 - m_4}{1+m_3m_4}\right) = Tan^{-1}\left(\frac{\frac{-1}{7}-7}{1+\left(\frac{-1}{7}\right)(7)}\right) = Tan^{-1}\left(\frac{\frac{-1}{7}-7}{1-1}\right) = Tan^{-1}\left(\frac{\frac{-1}{7}-7}{0}\right) = Tan^{-1} = (\infty) = 90^o$$

13. $\boldsymbol{A(-1,-1), B(-3,0), C(3,7), D(1,8)}$

Slope of $AB = m_1 = \dfrac{0-(-1)}{-3-(-1)} = \dfrac{1}{-3+1} = -\dfrac{1}{2}$

Slope of $BC = m_2 = \dfrac{7-0}{3-(-3)} = \dfrac{7}{3+3} = \dfrac{7}{6}$



Slope of $CD = m_3 = \dfrac{8-7}{1-3} = -\dfrac{1}{2}$

Slope of $DA = m_4 = \dfrac{-1-8}{-1-1} = \dfrac{-9}{-2} = \dfrac{9}{2}$

$m_1 = m_3$ *but* $m_2 \neq m_4$

Side AB is parallel to side CD but BC not parallel to DA.

So ABCD is a trapezium.

Now for angles

$$\theta_1 = Tan^{-1}\left(\frac{m_4 - m_1}{1+m_4m_1}\right) = Tan^{-1}\left(\frac{\frac{9}{2}-\left(-\frac{1}{2}\right)}{1+\left(\frac{9}{2}\right)\left(-\frac{1}{2}\right)}\right) = Tan^{-1}\left(\frac{\frac{9}{2}+\frac{1}{2}}{1-\frac{9}{4}}\right) = Tan^{-1}\left(\frac{\frac{9+1}{2}}{\frac{4-9}{4}}\right)$$

$$= Tan^{-1}\left(\frac{10}{2}\times\frac{4}{5}\right) = Tan^{-1}(4) = 75^o57'$$

$$\theta_2 = Tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1m_2}\right) = Tan^{-1}\left(\frac{-\frac{1}{2}-\frac{7}{6}}{1+\left(-\frac{1}{2}\right)\left(\frac{7}{6}\right)}\right) = Tan^{-1}\left(\frac{\frac{-3-7}{6}}{1-\frac{7}{12}}\right)$$

$$= Tan^{-1}\left(\frac{-10}{6}\times\frac{12}{5}\right) = Tan^{-1}(-4) = 104^o3'$$

$$\theta_3 = Tan^{-1}\left(\frac{m_2 - m_3}{1 + m_2m_3}\right) = Tan^{-1}\left(\frac{\frac{7}{6}+\frac{1}{2}}{1+\left(\frac{7}{6}\right)\left(-\frac{1}{2}\right)}\right) = Tan^{-1}\left(\frac{\frac{7+3}{6}}{\frac{6-7}{12}}\right)$$

$$= Tan^{-1}\left(\frac{10}{6}\times\frac{12}{5}\right) = Tan^{-1}(4) = 75^o57'$$

$$\theta_4 = Tan^{-1}\left(\frac{m_3 - m_4}{1 + m_3m_4}\right) = Tan^{-1}\left(\frac{-\frac{1}{2}-\frac{9}{2}}{1+\left(-\frac{1}{2}\right)\left(\frac{9}{2}\right)}\right) = Tan^{-1}\left(\frac{\frac{-1-9}{2}}{\frac{4-9}{4}}\right)$$

$$= Tan^{-1}\left(\frac{-10}{2}\times\frac{4}{-5}\right) = Tan^{-1}(4) = 75^o57'$$

(Sargodha 2011)

14. $7x - y - 10 = 0$ —— I, $10x + y - 41 = 0$ —— II, $30 + 2y + 3 = 0$ —— III

Solving I & II

$7x - y - 10 = 0$

$10x + y - 41 = 0$

$$\frac{x}{41+10} = \frac{y}{-100+287} = \frac{1}{7+10}$$

$$\frac{x}{51} = \frac{y}{187} = \frac{1}{17}$$

$$x = \frac{51}{17} = 3 \;\&\; y = \frac{187}{17}$$

$x = 3, \quad y = 11$

Solving II & III

$10x + y - 41 = 0$

$3x + 2y + 3 = 0$

$$\frac{x}{3+82} = \frac{y}{-30-123} = \frac{1}{20-3}$$

$$\frac{x}{85} = \frac{y}{-153} = \frac{1}{17}$$

$$x = \frac{85}{17} = 5 \quad y = \frac{153}{17} = -9$$

Solving III & I

$3x + 2y + 3 = 0$

$3x + 2y + 3 = 0$

$$\frac{x}{-3+20} = \frac{y}{-30-21} = \frac{1}{14+3}$$

$$\frac{x}{17} = \frac{y}{-51} = \frac{1}{17}$$

$$x = \frac{17}{17} = 1, \quad y = \frac{-51}{17} = -3$$

So vertices $A(3,11)$, $B(5,-9)$, $C(1,-3)$

Now Area of triangle $ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} 3 & 11 & 1 \\ 5 & -9 & 1 \\ 1 & -3 & 1 \end{vmatrix} = \frac{1}{2}[3(-9+3)-11(5-1)+1(-15+9)]$$

$$= \frac{1}{2}[3(-6)-11(4)+(-6)] = \frac{1}{2}(-18-44-6) = \frac{1}{2}(-68)$$

$= -34$ Sq units $= 34$ Sq units (always +ve)

15. Circumcentre

Meeting point of right bisector is called circumcentre

Slope of $BC = \frac{5-1}{3+4} = \frac{4}{7}$

Slope of right bisector $OD = -\frac{7}{4}$

Equation of OD is

$$y-3 = -\frac{7}{4}\left(x-\left(1\frac{1}{2}\right)\right)$$

$$4(y-3) = -7\left(x+\frac{1}{3}\right) \Rightarrow 4y-12 = -7x-\frac{7}{2}$$

$'X'$ both sides by 2 $8y-24 = -14x-7 \Rightarrow 14x+8y-24+7=0$

$14x+8y-17=0 \quad I$

Slope of $AC = \frac{5-3}{3+2} = \frac{2}{5}$

Slope of $OE = \frac{-5}{2}$

Equation of OE is (through point E & Slope $\frac{-5}{2}$)

$$(y-4) = \frac{-5}{2}\left(x-\frac{1}{2}\right) \Rightarrow 2(y-4) = -5\left(x-\frac{1}{2}\right)$$

$$2y-8 = 5x+\frac{5}{2} \Rightarrow 4y-16 = -10x+5('X' \; by \; 2)$$

$$or \; 10x+4y-16-5=0 \Rightarrow 10x+4y-21=0$$

$$'X' by \; 2 \qquad 20x+8y-42=0 \qquad II$$

$II - I$

$$\begin{array}{r} 20x - 8y - 42 = 0 \\ \underline{-14x \pm 8y \mp 17 = 0} \\ 6x - 25 = 0 \end{array}$$

$$\Rightarrow \boxed{x = \frac{25}{6}}$$

Put value of x in I $\quad {}^{7}14\left(\frac{25}{6_3}\right) + 8y - 17 = 0$ or $\frac{175}{3} - 17 + 8y = 0$

$$8 + \frac{175-51}{3} = 0 \Rightarrow 8y - \frac{124}{3} = 0 \Rightarrow 8y = -\frac{124}{3}$$

$$y = \frac{124^{31}}{3 \times 8_2} = 0 \Rightarrow y = \frac{-31}{6}$$ So circumcentre $\left(\frac{25}{6}, \frac{-31}{6}\right)$

16. (a) $x + 3y - 2 = 0$

$2x - y + 4 = 0$

$x - 11y + 14 = 0$

In matrix form

$$\begin{bmatrix} x & +3y & -2 \\ 2x & -y & +4 \\ x & -11y & +14 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Or $$\begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A \quad X = B$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$ for Concurrent

$$|A| = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$= 1(-14 + 44) - 3(28 - 4) + (-2)(-22 + 1)$

$= 1(30) - 3(24) - 2(-21)$

$= 30 - 72 + 42 = 72 - 72 = 0$

So lines are concurrent.

(b) $2x+3y+4=0$

$x-2y-3=0$

$3x+y-8=0$

An matrix form

$$\begin{bmatrix} 2x & 3y & 4 \\ x & -2y & -3 \\ 3x & y & -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Or $$\begin{bmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & -1 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A \quad X \ = B$

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & 1 & -8 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -2 & -3 \\ 3 & 1 & -8 \end{bmatrix} = 2(16+3)-3(-8+9)+4(1+6)$$

$= 2(19)-3(1)+4(7)$

$= 38-3+28 = 63$ (Not Concurrent)

(c) $3x-4y-2=0$

$x+2y-4=0$

$3x-2y+5=0$

In matrix form

$$\begin{bmatrix} 3x & -4y & -2 \\ x & 2y & -4 \\ 3x & -2y & 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} or \begin{bmatrix} 3 & -4 & -2 \\ 1 & 2 & -4 \\ 3 & -2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$A \quad X \ = B$

$$A = \begin{bmatrix} 3 & -4 & -2 \\ 1 & 2 & -4 \\ 3 & -2 & 5 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 3 & -4 & -2 \\ 1 & 2 & -4 \\ 3 & -2 & 5 \end{bmatrix} = 3(10-8)-(-4)(5+12)+(-2)(-2-6)$$

$= 6+68+16 = 90$ (Not Concurrent)

17. (a) $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$1\times x+0\times y+(-1)(1)=0$

$2\times x+0\times y+(1)(1)=0$

$0\times x+(-1)(y)+2(1)=0$ *or* $x+0-1=0$

$2x+0+1=0$

$0-y+2=0$

For Concurrency

$$A=\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$|A|=\begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix}=1(0+1)-0+(-1)(-2-0)$$

$=1-0+2=3\neq 0$

(b) $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x+y+2=0$

$2x+4y-3=0$

$3x+6y-5=0$

For Concurrent

$$A=\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$$

$$|A|=\begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}=1(-20+18)-1(-10+9)+2(12-12)$$

$=-2+1+0=-1$ (Not Concurrent)

Example 1 (4.5) Find angle $x^2 - xy - 6y^2 = 0$ **(Sargodha 2009, 10)**

Compare with $ax^2 - 2hxy - by^2 = 0$

$a_1 = 1,\ h = \frac{-1}{2},\ b = -6$

$$Tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{\sqrt{\left(\frac{-1}{2}\right)^2 - (1)(-6)}}{1-6} = \frac{2}{-5}\sqrt{\frac{1}{4}+6}$$

$$Tan\theta = \frac{2}{-5}\sqrt{\frac{25}{4}} = \frac{-2}{5} \times \frac{5}{2} = -1 \Rightarrow \theta = Tan^{-1}(-1) = 135^o$$

For Acute angle $= 180^o - 135^o = 45^o \Rightarrow \boxed{\theta = 45^o}$

Example 2 (4.5) Find a joint equation of straight lines through origin and perpendicular to the lines represented by $x^2 - xy - 6y^2 = 0$ **(Sargodha 2010)**

$x^2 + xy - 6y^2 = 0$

or $x^2 + 3xy - 2xy - 6y^2 = 0$

$x(x+3y) - 2y(x+3y) = 0$

$(x+3y)(x-2y) = 0$

Thus lines are $x + 3y = 0$ —— I and $x - 2y = 0$ —— II

Slope of $I = \frac{-1}{3} \Rightarrow$ Slope of perpendicular line $= 3$

$y = mx \Rightarrow y = 3x \Rightarrow \boxed{y - 3x = 0}$

Slope of $II = \frac{1}{2}$

Slope of perpendicular to $II = -2$

Any line perpendicular to II through $(0,0)$ is

$y = mx \Rightarrow y = -2x \Rightarrow \boxed{y - 2x = 0}$

Joint equation is $(y + 2x)(y - 3x) = 0$

$y^2 - 3xy + 2xy - 6x^2 = 0$

or $\boxed{y^2 - xy - 6x^2 = 0}$

Exercise 4.5

1. $10x^2 - 5y^2 = 0$ ——— I **(Lahore 2010, Gujrawala 2010)**

$or\ 10x^2 - 25xy + 2xy - 5y^2 = 0$

$or\ 5x(2x-5y) + y(2x-5y) = 0\ or\ (2x-5y)(5x+y) = 0$

$2x - 5y = 0\ \ or\ \ 5x + y = 0$

Are required lines

Now compare I by $ax^2 + 2hxy + by^2 = 0$

$$a = 10,\ 2h = -23 \Rightarrow h = \frac{-23}{2},\ b = -5$$

$$Tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{2\sqrt{\left(\frac{-23}{2}\right)^2 - (10)(-5)}}{10 + (-5)} = \frac{2\sqrt{\frac{529}{4} + 50}}{10 - 5}$$

$$= \frac{2\sqrt{\frac{529 + 200}{4}}}{5} = \frac{2}{5}\sqrt{\frac{729}{4}} = \frac{2}{5}.\frac{27}{2} = \frac{27}{5}$$

$$\theta = Tan^{-1}\left(\frac{27}{5}\right)$$

2. $3x^2 + 7xy + 2y^2 = 0$ **(Sargodha 2008, 2010)**

$3x^2 + 6xy + xy + 2y^2 = 0$

$3x(x + 2y) + y(x + 2y) = 0$

$(x + 2y)(3x + y) = 0$

$x + 2y = 0\ \ or\ \ 3x + y = 0$

Are required lines

Now compare $3x^2 + 7xy + 2y^2 = 0$

With $ax^2 + 2hxy + by^2 = 0$

$$a = 3,\ 2h = 7 \Rightarrow h = \frac{7}{2},\ b = 2$$

$$Tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{2\sqrt{\left(\frac{7}{2}\right)^2 - (3)(2)}}{3+2}$$

$$Tan\theta = \frac{2\sqrt{\frac{49}{4} - 6}}{5} = \frac{2}{5}\sqrt{\frac{49 - 24}{4}}$$

$$Tan\theta = \frac{2}{5}\sqrt{\frac{25}{4}} = \frac{2}{5}\times\frac{5}{2} = 1$$

$$\theta = Tan^{-1}(1) \Rightarrow \theta = \frac{\pi}{4} \text{ or } \boxed{\theta = 45''}$$

3. $\mathbf{9x^2 + 24xy + 16y^2 = 0}$

$$\text{or } (3x)^2 + 2(3x)(4y) + (4y)^2 = 0$$

$$(3x+4y)^2 = 0$$

$$(3x+4y)(3x+4y) = 0$$

$$3x+4y = 0 \text{ or } 3x+4y = 0$$

$\Rightarrow 3x+4y = 0$ (Coincident lines)

4. $\mathbf{2x^2 + 3xy - 5y^2 = 0}$ **(Lahore 2010)**

$$2x^2 - 2xy + 5xy - 5y^2 = 0$$

$$2x(x-y) + 5y(x-y) = 0$$

$$(x-y)(2x+5y) = 0$$

$$x - y = 0 \text{ or } 2x+5y = 0$$

Are required lines

Now compare $2x^2 + 3xy - 5y^2 = 0$

With $ax^2 + 2hxy + by^2 = 0$

$$a = 2,\ 2hx = 3 \Rightarrow h = \frac{3}{2},\ b = -5$$

$$Tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{2\sqrt{\left(\frac{3}{2}\right)^2 - (2)(-5)}}{2-5}$$

$$Tan\theta = \frac{2}{-3}\sqrt{\frac{2}{-3} + 10} = -\frac{2}{3}\sqrt{\frac{49}{4}}$$

$$Tan\theta = \left(\frac{-2}{3}\right)\left(\frac{7}{2}\right) = \frac{-7}{3}$$

$$\theta = Tan^{-1}\left(\frac{-7}{3}\right) = \boxed{113.2''}$$

5. $\mathbf{6x^2 - 19xy + 15y^2 = 0}$

$$6x^2 - 9xy - 10xy + 15y^2 = 0$$

$$3x(2x-3y) - 5y(2x-3y) = 0$$

$$(2x-3y)(3x-5y) = 0$$

$2x-3y=0 \quad or \quad 3x-5y=0$

Are required lines

Now compare $6x^2-19xy+15y^2=0$

With $ax^2+2hxy+by^2=0$

$a=6, 2h=-19 \Rightarrow h=\frac{-19}{2}, b=15$

$$Tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}=\frac{2\sqrt{\left(\frac{-19}{2}\right)^2-(6)(15)}}{6+15}$$

$$Tan\theta=\frac{2}{21}\sqrt{\frac{361}{4}-90}=\frac{2}{21}\sqrt{\frac{361-360}{4}}$$

$$Tan\theta=\frac{2}{21}\sqrt{\frac{1}{4}}=\frac{2}{21}\times\frac{1}{2}$$

$$Tan\theta=\frac{1}{21} \Rightarrow \theta=Tan^{-1}\left(\frac{1}{21}\right)$$

$$\boxed{\theta=2.73^o}$$

6. $\boldsymbol{x^2+2xy\ Sec\alpha+y^2=0}$

Divide by x^2 $1+2\frac{y}{x}\ Sec\alpha+\frac{y^2}{x^2}=0$

Or $\left(\frac{y}{x}\right)^2+2Sec\alpha\left(\frac{y}{x}+1=0\right)$

$a=1,\ b=2Sec\alpha,\ C=1$

$$\frac{y}{x}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\frac{y}{x}=\frac{-2Sec\alpha\pm\sqrt{(2Sec\alpha)^2-4(1)(1)}}{2(1)}$$

$$\frac{y}{x}=\frac{-2Sec\alpha\pm\sqrt{4Sec^2\alpha-4}}{2}$$

$$\frac{y}{x}=\frac{-2Sec\alpha\pm\sqrt{4(Sec^2\alpha-1)}}{2}$$

$$\frac{y}{x}=\frac{-2Sec\alpha\pm\sqrt{4(4Tan^2\alpha)}}{2}$$

$$\frac{y}{x}=\frac{-2Sec\alpha\pm 2Tan\alpha}{2}$$

$$\frac{y}{x}=\frac{2(-Sec\alpha\pm Tan\alpha)}{2}$$

$$y=(-Sec\alpha\pm Tan\alpha)x$$

$$y=(-Sec\alpha+Tan\alpha)x$$

and

$$y=(-Sec\alpha-Tan\alpha)x$$

Are required lines

Now compare $x^2+2Sec\alpha xy+y^2=0$

With $ax^2+2hxy+by^2=0$

$a=1,\ h=Sec\alpha,\ b=1$

$$Tan\theta=\frac{2\sqrt{h^2-ab}}{a+b}=\frac{2\sqrt{Sec^2\alpha-(1)(1)}}{1+1}$$

$$Tan\theta=\frac{2\sqrt{Sec^2\alpha-1}}{2}=\frac{2\sqrt{Tan^2\alpha}}{2}$$

$$Tan\theta=Tan\alpha\ \Rightarrow\ \boxed{\theta=\alpha}$$

7. $\boldsymbol{x^2+2xyTan\alpha-y^2=0}$

Compare with $ax^2+2hxy+by^2=0$

$a=1,\ 2h=-2Tan\alpha\ \Rightarrow h=-Tan\alpha,\ b=-1$

$$m_1+m_2=\frac{-2h}{b}=\frac{2Tan\alpha}{-1}=-2Tan\alpha$$

$$m_1m_2=\frac{a}{b}=\frac{1}{-1}=-1$$

Now any two equations through origin are

$y=m_1x\ \ \&\ \ y=m_2x$

Perpendicular to given line are

$$y=\frac{1}{m_1}x\ \ \&\ \ y=-\frac{1}{m_2}x$$

$\Rightarrow m_1y=-x\ \&\ m_2y=-x\ \Rightarrow m_1y+x=0\ \&\ m_2y+x=0$

Combined equation is

$$(m_1y+x)(m_2y=x)=0$$

$$m_1m_2y^2+m_1xy+m_2xy+x^2=0$$

$$m_1m_2y^2+(m_1+m_2)xy+x^2=0$$

Put values

$(-1)y^2 + (-2Tan\alpha)xy + x^2 = 0$

So $x_1^2 - 2Tan\alpha\, xy - y^2 = 0$

8. $ax^2 + 2hxy + by^2 = 0$ **(Sargodha 2009)**

$m_1 + m_2 = \frac{-2h}{b}$ & $m_1m_2 = \frac{a}{b}$

Any two equations through origin and $\perp ar$ to given

Equation are $y = \frac{1}{m_1}x$ & $y = \frac{1}{m_2}x$

$\Rightarrow m_1y + x = 0$ & $m_2y + x = 0$

Combined equation is $(m_1y + x)(m_2y + x) = 0$

$m_1m_2y^2 + m_1xy + m_2xy + x^2 = 0$

$m_1m_2y^2 + (m_1 + m_2)xy + x^2 = 0$

Put values $\frac{a}{b}y^2 + \left(\frac{-2h}{b}\right)xy + x^2 = 0$

$'X'$ by b $ay^2 - 2hxy + bx^2 = 0$

9. $10x^2 - xy - 21y^2 = 0$ & $x + y + 1$ —— III

$ax^2 + 2hxy + by^2 = 0$

$10x^2 - xy - 21y^2 = 0$

$10x^2 - 15xy + 14xy - 21y^2 = 0$

$5x(2x - 3y) + 7y(2x - 3y) = 0$

$(2x - 3y)(5x + 7y) = 0$

$2x - 3y = 0$ —— I & $5X + 7Y = 0$ —— II

Intersection point of I & II is $(0,0)$

Because both are passing through origin, To fin

Intersection point of I & III

$'X'$ III by 3 and add in I

$$\begin{array}{r} 2x - 3y \quad = 0 \\ 3x + 3y + 3 = 0 \\ \hline 5x + 3 = 0 \end{array}$$

$\Rightarrow x = \frac{-3}{5}$

Put in III $\frac{-3}{5} + y + 1 = 0$

$$y+1-\frac{3}{5}=0 \Rightarrow y+\frac{5-3}{5}=0$$

$$y+\frac{2}{5}=0 \Rightarrow y=\frac{-2}{5}$$

Intersection point $I \,\&\, III$ is $\left(\frac{-3}{5},\frac{-2}{5}\right)$

To solve $I \,\&\, III$ 'X' III by 7 & subtract

$$\begin{array}{r} 7x-7y+7=0 \\ \underline{-5x\pm 7y \quad\;\; =0} \\ 2x+7=0 \end{array}$$

$$\Rightarrow x=\frac{-7}{2}$$

Put in II $5\left(\frac{-7}{2}\right)+7y=0$

$$7y=\frac{35}{2} \Rightarrow y=\frac{35}{2}\times\frac{1}{7}=\frac{5}{2}$$

Intersection point of $II \,\&\, III$ is

$\left(\frac{-7}{2},\frac{5}{2}\right)$ Now to find area

$$\text{Area}=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$=\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{-3}{5} & \frac{-2}{5} & 1 \\ \frac{-7}{2} & \frac{5}{2} & 1 \end{vmatrix}$$

$$=\frac{1}{2}\left[0-0+1\left(\left(\frac{-3}{5}\right)\left(\frac{5}{2}\right)-\left(\frac{-2}{5}\right)\left(\frac{-7}{2}\right)\right)\right]$$

$$=\frac{1}{2}\left[\frac{-3}{2}-\frac{7}{5}\right]=\frac{1}{2}\left(\frac{-15-14}{10}\right)$$

$$=\frac{1}{2}\left(\frac{-29}{10}\right)=\frac{-29}{20}=\frac{29}{20} \text{ Sq. Units}$$

TEST YOUR SKILLS — Marks 100

OBJECTIVE

Q.No.1 Given below are a few possible answers to each statement of which one is correct, identify the correct one. **(20)**

1. In a plane two mutually perpendicular lines are called the
   (a) Radii (b) Coordinate axes
   (c) Medians (d) Altitudes
2. If (x, y) are the coordinates of a point P, then the second component of the ordered pair is called
   (a) x-coordinate (b) Abscissa
   (c) Ordinate (d) None of these
3. If $y = 0$ then the point $P(x, y)$ is on
   (a) Origin (b) y-axis
   (c) x-axis (d) 4th quadrant
4. The distance of a point $P(x, y)$ from origin is
   (a) $x + y$ (b) $x - y$
   (c) $\sqrt{x^2 + y^2}$ (d) $x^2 + y^2$
5. The distance of a point $P(-2, -3)$ from x-axis is
   (a) -2 (b) 2
   (c) -3 (d) 3
6. If distance of a point $P(x, y)$ from y-axis is 2 then
   (a) $2x = y$ (b) $2y = x$
   (c) $x = 2$ (d) $y = 2$
7. Let $A(x_1, y_1)$ & $B(x_2, y_2)$ be the two given points in a plane, the coordinates of the point dividing the line segment AB in the ratio $K_1 : K_2$ are
   (a) $\left(\dfrac{K_1x_1 + K_2x_2}{K_1 + K_2}, \dfrac{K_1y_1 + K_2y_2}{K_1 + K_2}\right)$ (b) $(K_1x_2 + K_2x_1, K_1y_2 + K_2y_1)$
   (c) $\left(\dfrac{K_1x_2 + K_2x_1}{K_1 - K_2}, \dfrac{K_1y_2 + K_2y_1}{K_1 - K_2}\right)$ (d) $\left(\dfrac{K_1x_2 + K_2x_1}{K_1 + K_2}, \dfrac{K_1y_2 + K_2y_1}{K_1 + K_2}\right)$
8. The centroid of a ΔABC is a point that divides each median in the ratio
   (a) 3 : 2 (b) 3 : 1
   (c) 1 : 2 (d) 2 : 1
9. If a line l is parallel to x-axis then inclination α is
   (a) $0°$ (b) $45°$
   (c) $90°$ (d) $180°$
10. If a non-vertical line l with inclination α then its slope is
    (a) Sin α (b) Cos α
    (c) Tan α (d) Cot α
11. Equation of a straight line passes through $P(a, b)$ and parallel to x-axis is
    (a) $x = a$ (b) $x = b$

(c) $y = b$ (d) $x + a = 0$

12. If $b < 0$ & $x = b$, then the line l is
    (a) On the right of y-axis (b) On the left of y-axis
    (c) Below the x-axis (d) Above the x-axis
13. $\dfrac{y - y_1}{\text{Sin}\,\alpha} = \dfrac{x - x_1}{\text{Cos}\,\alpha}$ is called
    (a) Intercepts form (b) Point-slope form
    (c) Normal form (d) Symmetric form
14: If $l : ax + by + c = 0$ & $P(x_1, y_1)$ also $a_1x + by_1 + c < 0$ then point P is
    (a) Above the line (b) Below the line
    (c) No result (d) None of these
15. Two non-parallel lines intersect each other at
    (a) More than one point (b) More than two points
    (c) Only one point (d) At least two points
16. Two non-parallel lines intersect each other at
    (a) Two points (b) More than one point
    (c) One and only one point (d) None of these
17. Altitudes of a triangle are
    (a) Collinear (b) Parallel
    (c) Concurrent (d) Mutually perpendicular
18. The distance d from the point $P(x_1, y_1)$ to live $l = ax + by + c = 0$ is
    (a) $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a+b}}$ (b) $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 - b^2}}$
    (c) $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$ (d) $\dfrac{|ax_1 + by_1|}{\sqrt{a^2 + b^2}}$
19. The lines represented by $ax^2 + 2hxy + by^2 = 0$, are different if
    (a) $h^2 - ab < 0$ (b) $h^2 - ab > 0$
    (c) $h^2 - ab = 0$ (d) $h^2 + ab = 0$
20. The line represented by $ax^2 + 2hxy + by^2 = 0$ a pair of orthogonal lines if
    (a) $a + h = 0$ (b) $a + b = 0$
    (c) $a - b = 0$ (d) $h + b = 0$

SECTION I

SUBJECTIVE

Attempt any 25 (twenty five) short question from Question 2,3 and 4, at least 12 short questions from question 2, 12 short question from question 3 and 13 short question from question 4. All question carry equal marks. ***(25x2=50)***

Q.No. 2

i. Show that the points $A(-1, 2)$, $B(7, 5)$ & $C(2, -6)$ are the vertices of a right triangle.

ii. Find the pt. which divide the join of $A(-6, 3)$ & $B(5, -2)$. Externally in the ratio of 2:3.

iii. Show that the pts $A(3, 1)$, $B(-2, -3)$ & $C(2, 2)$ are the vertices of an isosceles triangle.

iv. Find h such that the $A(-1, h)$, $B(3, 2)$ and $C(7, 3)$ are collinear.

v. Find the pt three-fifth of the way along the line segment from $A(-5, 8)$ to $B(5, 3)$.

vi. The coordinates of a pt P are $(-6, 9)$ the axes are translated through the pt. $0'(-3, 2)$. Find the coordinate of P referred to the new axes.

vii. The xy-coordinates axes are translated through the pt $O'(4, 6)$. The coordinates of the pt. P are $(2, -3)$ referred to the new axes. Find the coordinate of P referred to the original axes.

viii. The xy-coordinates are rotated about the origin through average of 30^o if the xy coordinates of a pt are $(5, 7)$. Find the xy-coordinate.

ix. By use of slopes show that the triangle with vertices $A(1, 1)$, $B(4, 5)$ and $C(12, -1)$ is a right angled triangle.

x. Find the eq. of the st. line if its slope is 2 & y-intercept = 5

xi. Find the eq. of the st. line if it is perpendicular to a line with slope -6 & $\frac{4}{3}$ is the y-intercept.

xii. Write the eq. of a st. line through the pt $(5, 1)$ and parallel to a line passing through the pts $(0, -1)$, $(7, -15)$.

Q.No. 3

i. Find an eq. of a line through the pts $(-2, 1)$ & $(6, -4)$.

ii. The length of perpendicular from origin to a line is 5 units and the inclination of this perpendicular is 120^o. Find the eq. of line.

iii. Find the pt of intersection of the lines $5x + 7y = 35$
$3x - 7y = 21$

iv. Find the dis take between // lines $l_1 = 2x - 5y + 13 = 0$
$l_2 = 2x - 5y + 6 = 0$

v. Find k so the line $A(7, 3)$, $B(K, -6)$ & the line $C(-4, 5)$, $D(-6, 4)$ are parallel.

vi. Find the eq. of the horizontal line through $(7, -9)$.

vii. Find the eq. of the vertical line through the pt $(-5, 3)$.

viii. Find the eq. of the line bisecting first and 3rd quadrant.

ix. Find the eq. of the line bisecting 2nd and 4th quadrant.

x. Find the eq. of a st. line through $(-8, 5)$ having slope undefined.

xi. Convert $4x + 7y - 2 = 0$ in intercept form.

xii. Check whether the two lines are parallel or perpendicular or neither.
$l_1 \equiv 2x + y - 3 = 0$
$l_2 \equiv 4x + 2y + 5 = 0$

Q.No. 4

i. Check whether the two lines are parallel or perpendicular or neither.
$l_1 \equiv 3y = 2x + 5$
$l_2 \equiv 3x + 2y - 8 = 0$

ii. Check whether the two lines are parallel or perpendicular or neither.

$l_1 \equiv 4y + 2x - 1 = 0$
$l_2 \equiv x - 2y - 7 = 0$

iii. Find whether the pt (5, 8) lies above or below the line $2x - 3y + 6 = 0$

iv. Find the distance from $R(6, -1)$ to the line $6x - 4y + 9 = 0$

v. Find the angle from l_1 to l_2 where:
$l_1 \equiv x - 2y - 6 = 0$
$l_2 \equiv 3x - y + 3 = 0$

vi. Find an eq. of each of the lines represented by
$20x^2 + 17xy - 24y^2 = 0$

vii. Find the measure of the angle between the lines represented by
$$x^2 - xy - 6y^2 = 0$$

viii. Distinguish between centroid and orthocenter.

ix. What is the angle between lines represented by
$$ax^2 + 2hxy + by^2 = 0$$

x. State the conditions represented by the lines of $ax^2 + 2hxy + by^2 = 0$ are real & imaginary coincident.

xi. Reduce $2x - 4y + 11 = 0$ in normal form.

xii. State the eq. representing the family of lines through the pt of intersection of $3x - 4y - 10 = 0$
$$x + 2y - 10 = 0$$

xiii. What is the symmetric form of $ax + by + c = 0$

SECTION II

Attempt any 3 (three) questions. **(3x10=30)**

Q.No.5

(a) Find the area of the region bounded by:
$10x^2 - xy - 21y^2 = 0$ and $x + y + 1 = 0$

(b) Find the condition that the lines $y = m_1x+c_1$; $y= m_2x+c_2$ and $y= m_3x+c_3$ are **concurrent.**

Q.No.6

(a) Find the angle from the line with slope $\frac{-7}{3}$ to the line with slope $\frac{5}{2}$

(b) Find the point of intersection of the lines $3x+y+12=0$ and $x+2y-1=0$

Q.No.7

(a) The three points A(7,–1) , B(–2, 2) and C (1,4) are consecutive vertices of a parallelogram. Find the fourth vertex.

(b) Find a joint equation of the lines through the origin and perpendicular to the lines: $x^2 - 2xy \tan\alpha - y^2=0$

Q.No.8

(a) Find h such that A(–1, h) B(3 ,2) and C(7, 3) are collinear.

(b) Show the points A(0,0), B(2,1), C(3,3), D(1,2) are the vertices of a rhombus.

Find its interior angles.

Q.No.9

(a) Find the point three – fifth of the way along the line segment from A(–5, 8) to B(5,3)

(b) Find an equation of each of the lines represented by
$20x^2 + 17xy - 24y^2 = 0$

Previous Board Questions

1. Find the distance between the points A (3, 1) and B (–2, –4). (Lahore – 2007)
2. Find the centroid of the triangle having vertices (–2, 3), (–4, 1), (3, 5) (Lahore – 2006)
3. Write down the translated coordinates of a point in plane by shifting origin at (h, k). (Multan – 2009)
4. Find the equations of lines represented by $6x^2 - 19xy + 15y^2 = 0$ (Lahore – 2009)
5. Show that the points A (–3, 6), B (3, 2) and C (6, 0) are collinear (Gujranwala 2007)
6. Find the distance from x – axis to y – 3 = 0. (Gujranwala – 2005)
7. Write intercepts form of equation of straight line. (Multan – 2009)
8. Write down the equation of line which cuts the x – axis at (2, 0) nd y – axis at (0, –4). (Mirpur – 2009)
9. Find the perpendicular bisector of the line segment joining the points A (3, 5) and B (9, 8). (Lahore – 2010) Group – I
10. Write the translated co-ordinates of a point (x, y) when origin in shifted at O′ (h, k). (Lahore – 2010) Group – I
11. Find the equation of horizontal line through (7, –9). (Lahore – 2010) Group – II
12. Find the angle from the line with slope $\frac{5}{2}$ to the line with slope $-\frac{7}{3}$. (Lahore – 2010) Group – II
13. Find the angle measured form the line l_1 to the line l_2 where:
    l_1 : joining (1, –7) and (6, –4)
    l_2 : joining (–1, 2) and (–6, –1) (Gujranwala – 2010)
14. Find the lines represented by the homogeneous equation $10x^2 - 23xy - 5y^2 = 0$. (Gujranwala – 2010)

Linear Inequalities and Linear Programming

5

Definitions:

1. Linear Inequalities:

Inequalities are expressed by following symbols <, >, ≤, ≥ with one or two variables are called linear inequalities.

2. Boundary of half plane:

$ax + by < c$ is called half **plane** region and line $ax + by = c$ is called. **Boundary of half plane.**

3. Left, Right, Upper, Lower Half Plane:

Vertical line divides the plane into left or right and non-vertical line divides into lower and upper half plane.

4. Vertex or Corner Point:

A point of a solution region where two of its boundary lines intersect is called vertex.

5. Non-Negative Constraints:

The variable used in the system of linear inequalities relating to the problem of every day life are non-negative and are called non-negative constraints.

6. Decision Variables:

The non-negative constraints play an important role for taking decision. So these variables are also called Decision Variables.

7. Feasible Region: (Sargodha 2010)

A region which is restricted to the first quadrant is called feasible region.

8. Feasible Solution:

Each point of feasible region is called feasible solution.

9. Optimal Solution:

The feasible solution which maximize or minimize the objective function is called the optimal solution.

10. Objective Function:

A function which is to be maximized or minimized is called an objective function.

11. Problem Constraints:

The system of linear inequalities involved in the problem concerned are called problem constraints.

Exercise 5.1

1. Graph the solution set of each of the following linear inequality in xy-plane:

(i) $2x + y \leq 6 \longrightarrow I$

Associated equation is

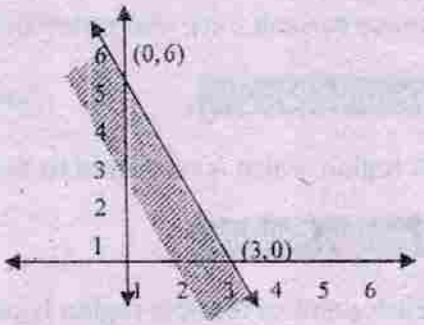
$2x + y = 6 \longrightarrow II$

Put $x = 0$ then $y = 6$ $(0,6)$

Put $y = 0$ then $x = 3$ $(3,0)$

Put $(0,0)$ *in* I

$0 \leq 6 \longrightarrow T$



(ii) $3x + 7y \geq 21 \longrightarrow I$

Associated equation is

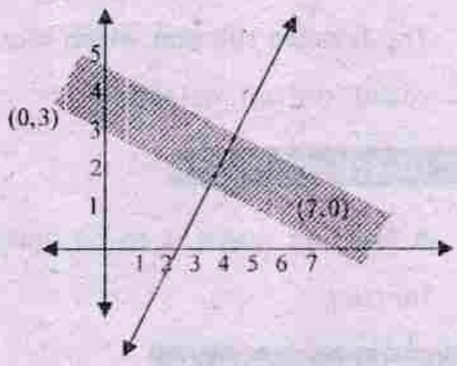
$3x + 7y = 21 \longrightarrow II$

Put $x = 0$ then $y = 3$ $(0,3)$

Put $y = 0$ then $x = 7$ $(7,0)$

Put $(0,0)$ *in equ* I

$0 \geq 21 \longrightarrow F$



(iii) $3x - 2y \geq 6 \longrightarrow I$ **(Sargodha 2010)**

Associated equation is

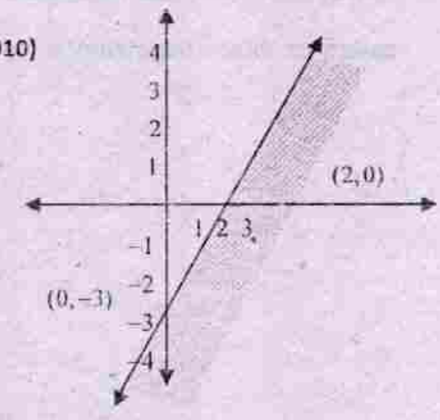
$3x - 2y = 6 \longrightarrow II$

Put $x = 0$ then $y = -3$ $(0,-3)$

Put $y = 0$ then $x = 2$ $(2,0)$

Put $(0,0)$ *in equ* I

$0 \geq 6 \longrightarrow F$



(iv) $5x-4y\le 20 \longrightarrow I$ (Lhr 2010, Guj 2010)

Associated equation is

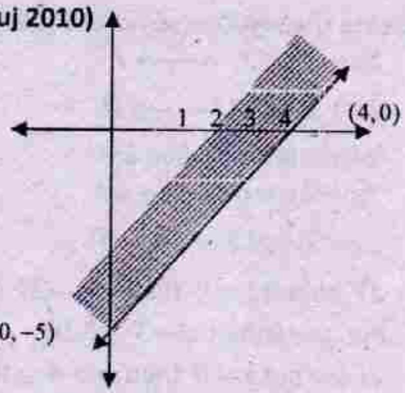
$l_1;\ 5x-4y=20 \longrightarrow II$

Put $x=0$ then $y=-5$ $(0,-5)$

Put $y=0$ then $x=4$ $(4,0)$

Put $(0,0)$ *in equ I*

$0\le 6 \longrightarrow T$



(v) $2x+1\ge 0 \longrightarrow I$ (Sgd 2009, 11, Lhr 2010)

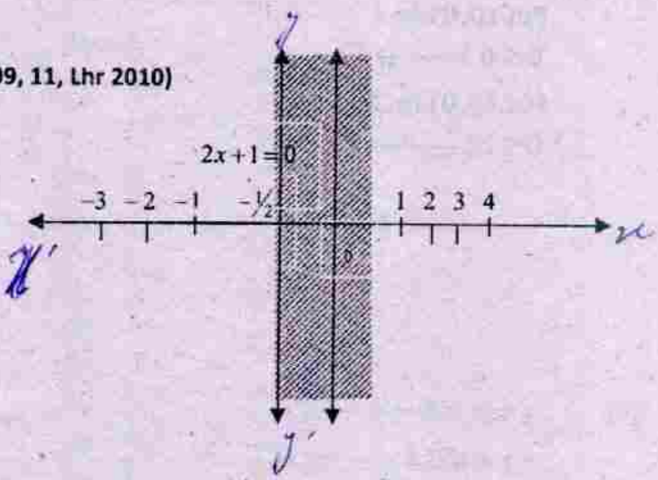
Associated equation is

$2x+1=0 \longrightarrow II$

$x=\frac{-1}{2}$

$y=-5$ $(0,-5)$

$1\ge 0 \longrightarrow T$



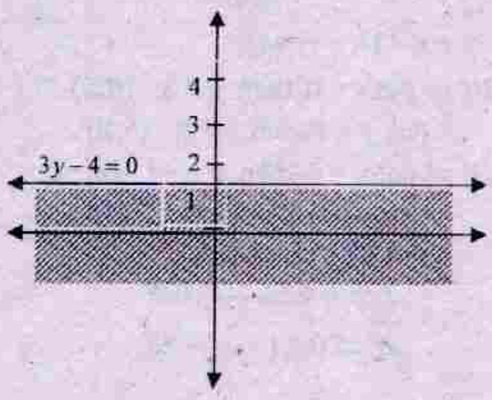
(vi) $3y-4\le 0 \longrightarrow I$

Associated equation is

$3y-4=0 \longrightarrow II$

$y=\frac{4}{3}$

$-4\le 0 \longrightarrow T$



2. Indicate the solution set of the following system of linear inequalities by shading:

(i) $2x-3y\le 6 \longrightarrow I$

$2x+3y\le 12 \longrightarrow II$

Associated equation are

$2x-3y=6 \longrightarrow III$

$2x+3y=12 \longrightarrow IV$

$IV \Rightarrow$ put $x=0$ then $y=-2$ $(0,-2)$

Put $y=0$ then $x=3$ $(3,0)$

$IV \Rightarrow$ put $x=0$ then $y=4$ $(0,4)$

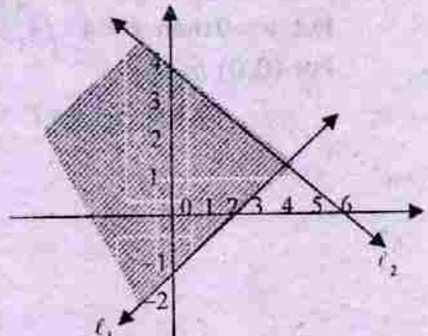
Put $y=0$ then $x=6$ $(6,0)$

Put $(0,0)$ *in I*

$0\le 6 \longrightarrow T$

Put $(0,0)$ *in II*

$0\le 12 \longrightarrow T$



(ii) $x+y\ge 5 \longrightarrow I$

$-y+x\le 1 \longrightarrow II$

Associated equation are

$x+y=5 \longrightarrow III$

$-y+x=1 \longrightarrow IV$

$III \Rightarrow$ put $x=0$ then $y=5$ $(0,5)$

Put $y=0$ then $x=5$ $(5,0)$

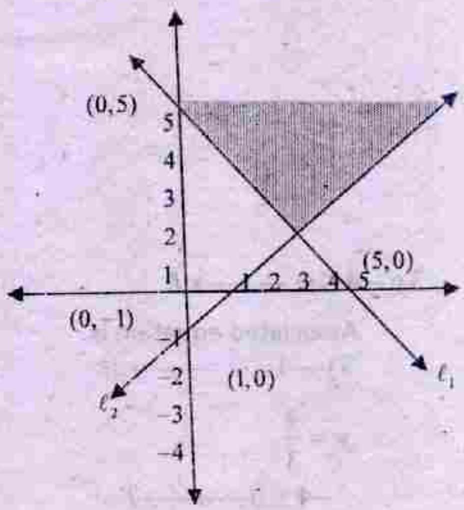
$IV \Rightarrow$ put $x=0$ then $y=-1$ $(0,-1)$

Put $y=0$ then $x=1$ $(1,0)$

Put $(0,0)$ *in I & II*

$I \Rightarrow 0\ge 5 \longrightarrow F$

$II \Rightarrow 0\le 1 \longrightarrow T$



(iii) $3x+7y \geq 21 \longrightarrow I$

$x-y \leq 2 \longrightarrow II$

Associated equation are

$l_1;\ 3x+7y=21 \longrightarrow III$

$l_2;\ x-y=2 \longrightarrow IV$

$III \Rightarrow$ put $x=0$ then $y=3$ $(0,3)$

Put $y=0$ then $x=7$ $(7,0)$

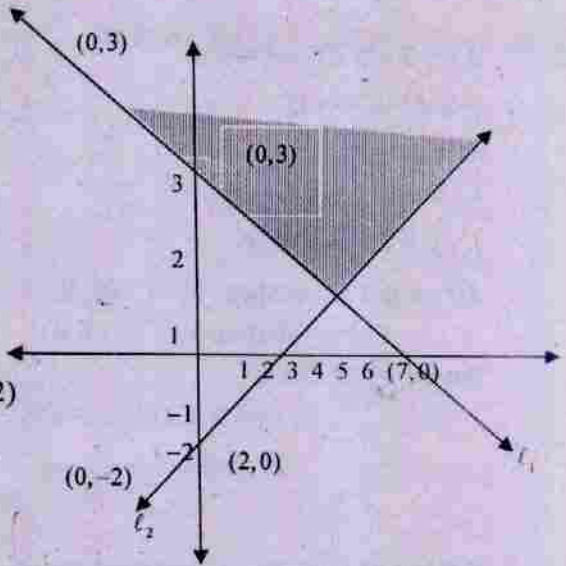
$IV \Rightarrow$ put $x=0$ then $y=-2$ $(0,-2)$

Put $y=0$ then $x=2$ $(2,0)$

Put $(0,0)$ *in I & II*

$I \Rightarrow 0 \geq 21 \longrightarrow F$

$II \Rightarrow 0 \leq 2 \longrightarrow T$



(iv) $4x-3y \leq 12 \longrightarrow I$

$x \geq \frac{-3}{2} \longrightarrow II$

Associated equation are

$l_1;\ 4x-3y=12 \longrightarrow III$

$l_2;\ x=\frac{-3}{2} \longrightarrow IV$

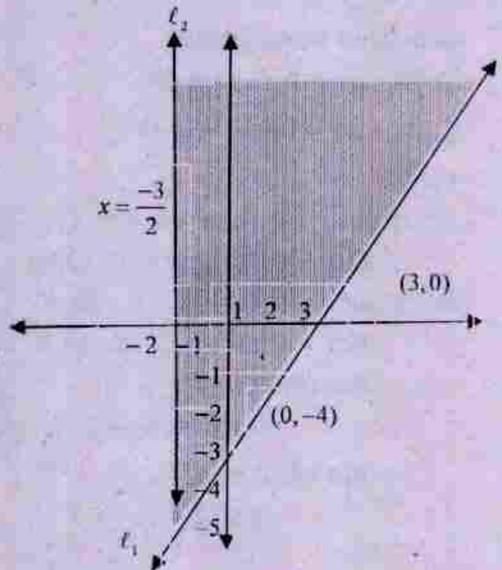
$III \Rightarrow$ put $x=0$ then $y=-4$ $(0,-4)$

Put $y=0$ then $x=3$ $(3,0)$

Put $(0,0)$ *in I & II*

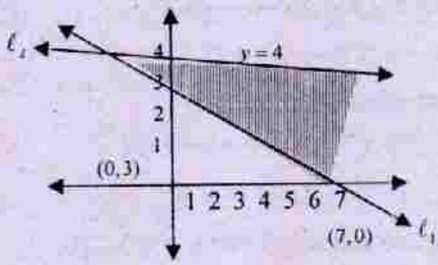
$0 \leq 12 \longrightarrow F$

$0 \geq \frac{-3}{2} \longrightarrow T$



(v) $3x + 7y \geq 21 \longrightarrow I$

$y \leq 4 \longrightarrow II$



Associated equation are

$l_1;\ 3x + 7y = 21 \longrightarrow III$

$l_2;\ y = 4 \longrightarrow IV$

$III \Rightarrow$ put $x = 0$ then $y = 3$ $(0,3)$

Put $y = 0$ then $x = 7$ $(7,0)$

Put $(0,0)$ *in* I & II

$I \Rightarrow 0 \geq 21 \longrightarrow F$

$II \Rightarrow 0 \leq 4 \longrightarrow T$

3. **Indicate the solution set of the following system of linear inequalities by shading:**

(i) $2x - 3y \leq 6 \longrightarrow I$

$2x + 3y \leq 12 \longrightarrow II$

Associated equation are

$l_1;\ 2x - 3y = 6 \longrightarrow III$

$l_2;\ 2x + 3y = 12 \longrightarrow IV$

$y > 0$

$III \Rightarrow$ put $x = 0$ then $y = -2$ $(0,-2)$

Put $y = 0$ then $x = 3$ $(3,0)$

$IV \Rightarrow$ put $x = 0$ then $y = 4$ $(0,4)$

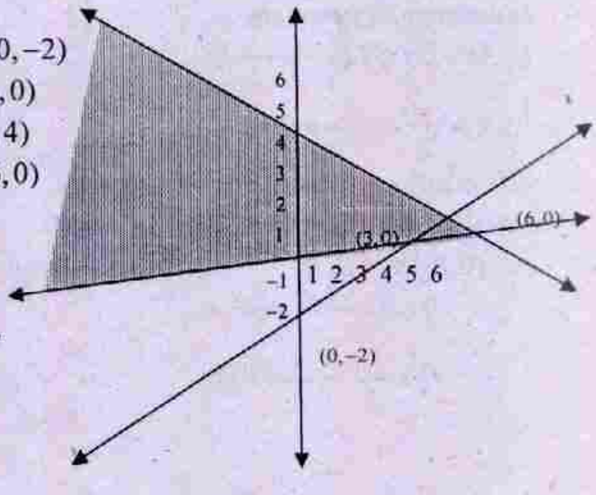
Put $y = 0$ then $x = 6$ $(6,0)$

Put $(0,0)$ *in* I

$0 \leq 6 \longrightarrow T$

Put $(0,0)$ *in* II

$0 \leq 12 \longrightarrow T$



(ii) $x + y \le 5 \longrightarrow I$

$y - 2x \le 2 \longrightarrow II$

$x \ge 0$

Associated equation are

$x + y = 5 \longrightarrow III$

$y - 2x = 2 \longrightarrow IV$

$III \Rightarrow$ put $x = 0$ then $y = 5$ $(0, 5)$

Put $y = 0$ then $x = 5$ $(5, 0)$

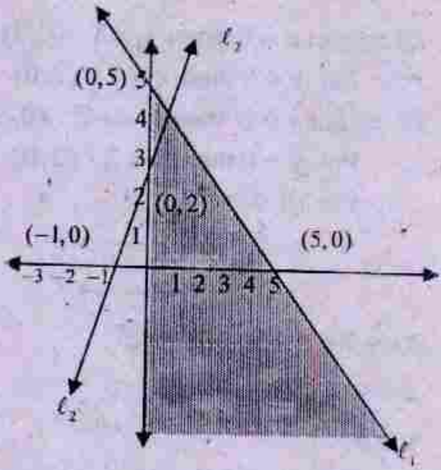
$IV \Rightarrow$ put $x = 0$ then $y = 2$ $(0, 2)$

Put $y = 0$ then $x = -1$ $(-1, 0)$

Put $(0,0)$ *in I & II*

$I \Rightarrow 0 \le 5 \longrightarrow T$

$II \Rightarrow 0 \le 2 \longrightarrow T$



(iii) $x + y \ge 5 \longrightarrow I$

$x - y \le 1 \longrightarrow II$

$y \ge 0$

Associated equation are

$l_1;\ x + y = 5 \longrightarrow III$

$l_2;\ x - y = 1 \longrightarrow IV$

$III \Rightarrow$ put $x = 0$ then $y = 5$ $(0, 5)$

Put $y = 0$ then $x = 5$ $(5, 0)$

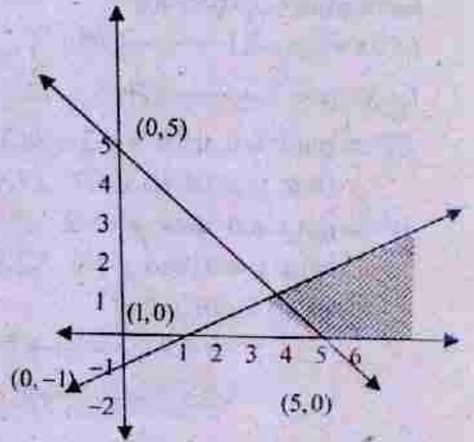
$IV \Rightarrow$ put $x = 0$ then $y = -1$ $(0, -1)$

Put $y = 0$ then $x = 1$ $(1, 0)$

Put $(0, 0)$ *in I & II*

$I \Rightarrow 0 \ge 5 \longrightarrow F$

$II \Rightarrow 0 \ge 1 \longrightarrow F$



(iv) $3x + 7y \le 21 \longrightarrow I$

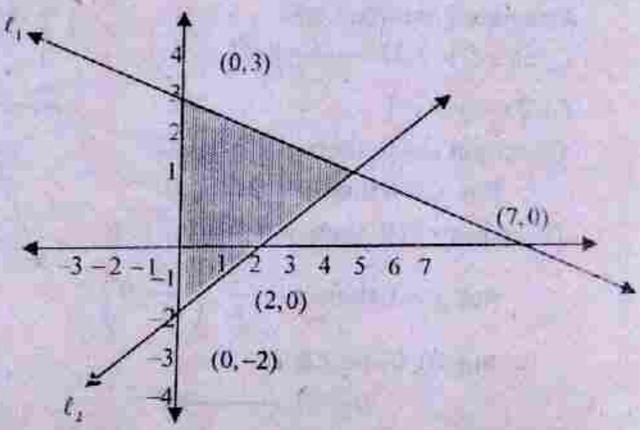
$x - y \le 2 \longrightarrow II$

$x \ge 0$

Associated equation are

$l_1;\ 3x + 7y = 21 \longrightarrow III$

$l_2;\ x - y = 2$



$III \Rightarrow$ put $x=0$ then $y=3$ $(0,3)$

Put $y=0$ then $x=7$ $(7,0)$

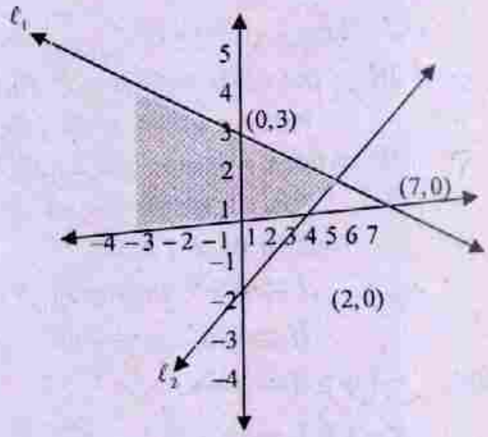
$IV \Rightarrow$ put $x=0$ then $y=-2$ $(0,-2)$

Put $y=0$ then $x=2$ $(2,0)$

Put $(0,0)$ *in I & II*

$I \Rightarrow 0 \le 21 \longrightarrow T$

$II \Rightarrow 0 \ge 2 \longrightarrow T$



(v) $\mathbf{3x+7y \le 21 \longrightarrow I}$

$\mathbf{x-y \le 2 \longrightarrow II}$

$\mathbf{y \ge 0}$

Associated equation are

$l_1;\ 3x+7y=21 \longrightarrow III$

$l_2;\ x-y=2 \longrightarrow IV$

$III \Rightarrow$ put $x=0$ then $y=3$ $(0,3)$

Put $y=0$ then $x=7$ $(7,0)$

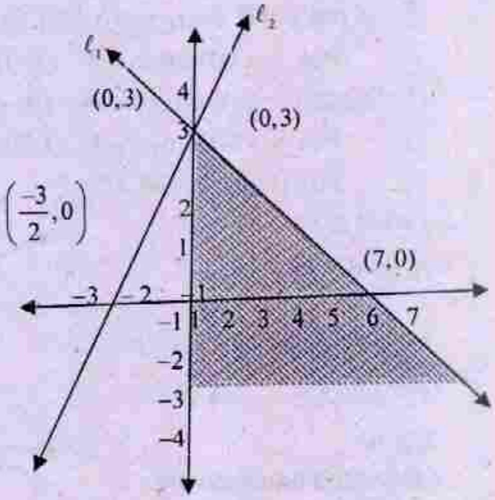
$IV \Rightarrow$ put $x=0$ then $y=-2$ $(0,-2)$

Put $y=0$ then $x=2$ $(2,0)$

Put $(0,0)$ *in I & II*

$0 \le 21 \longrightarrow T$

$0 \le 2 \longrightarrow T$



(vi) $\mathbf{3x+7y \le 21 \longrightarrow I}$

$\mathbf{2x-y \ge -3 \longrightarrow II}$

$\mathbf{x \ge 0}$

Associated equation are

$l_1;\ 3x+7y=21 \longrightarrow III$

$l_2;\ 2x-y=-3$

$III \Rightarrow$ put $x=0$ then $y=3$ $(0,3)$

Put $y=0$ then $x=7$ $(7,0)$

$IV \Rightarrow$ put $x=0$ then $y=3$ $(0,3)$

Put $y=0$ then $x=\frac{-3}{2}$, $\left(\frac{-3}{2},0\right)$

Put $(0,0)$ *in I & II*

$0 \le 21 \longrightarrow T$

$0 \geq -3 \longrightarrow T$

4. **Graph the solution region also find the corner points in each case.**

(i) $2x - 3y \leq 6 \longrightarrow I$

$2x + 3y \leq 12 \longrightarrow II$

$x \geq 0$

Associated equation are

$l_1;\ 2x - 3y = 6 \longrightarrow III$

$l_2;\ 2x + 3y = 12 \longrightarrow IV$

$III \Rightarrow$ put $x = 0$ then $y = -2$ $(0,-2)$

Put $y = 0$ then $x = 3$ $(3,0)$

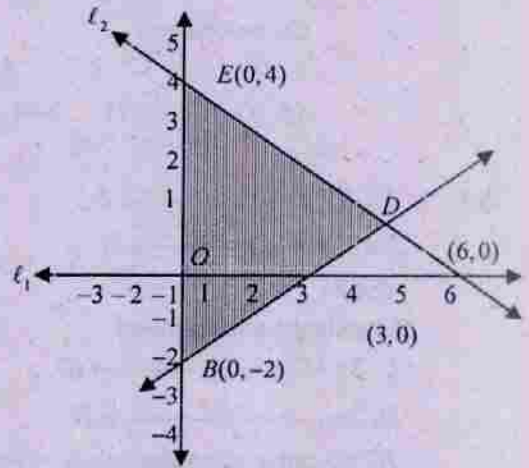
$IV \Rightarrow$ put $x = 0$ then $y = 4$ $(0,4)$

Put $y = 0$ then $x = 6$ $(6,0)$

Put $(0,0)$ in I & II

$0 \leq 6 \longrightarrow T$

$0 \leq 12 \longrightarrow T$



Corner Point : $B(0,-2)$, $D\left(\frac{9}{2},1\right)$, $E(0,4)$

(ii) $x + y \leq 5 \longrightarrow I$

$-2x + y \leq 2 \longrightarrow II$

$y \geq 0$

Associated equation are

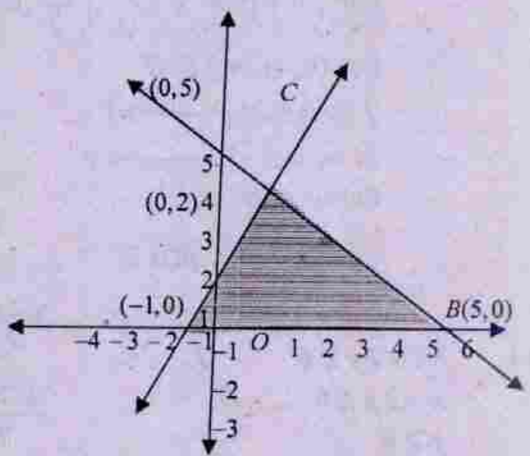
$l_1;\ x + y = 5 \longrightarrow III$

$l_2;\ -2x + y = 2 \longrightarrow IV$

$III \Rightarrow$ put $x = 0$ then $y = 5$ $(0,5)$

Put $y = 0$ then $x = 5$ $(5,0)$

Putting values of x in III



For values of Point c: $x + y = 5$ ——— l_1

$2x \pm y = +2$ ——— l_2

$3x \quad = 3$ $\boxed{\Rightarrow x = 1}$

$IV \Rightarrow$ put $x = 0$ then $y = 2$ $(0,2)$

Put $y = 0$ then $x = -1$ $(-1, 0)$

Put $(0,0)$ *in I & II*

$I \Rightarrow 0 \le 5 \longrightarrow T$

$II \Rightarrow 0 \le 1 \longrightarrow T$

Corner Points:

B	C	A
$(5,0)$	$(1,4)$	$(-1,0)$

(iii) $3x + 7y \le 21 \longrightarrow I$

$2x - y \le -3 \longrightarrow II$

$y \ge 0$

Associated equation are

$l_1;\ 3x + 7y = 21 \longrightarrow III$

$l_2;\ 2x - y = -3 \longrightarrow IV$

$III \Rightarrow$ put $x = 0$ then $y = 3$ $(0,3)$

Put $y = 0$ then $x = 7$ $(7,0)$

$IV \Rightarrow$ put $x = 0$ then $y = 3$ $(0,3)$

Put $y = 0$ then $x = \frac{-3}{2}, \left(\frac{-3}{2}, 0\right)$

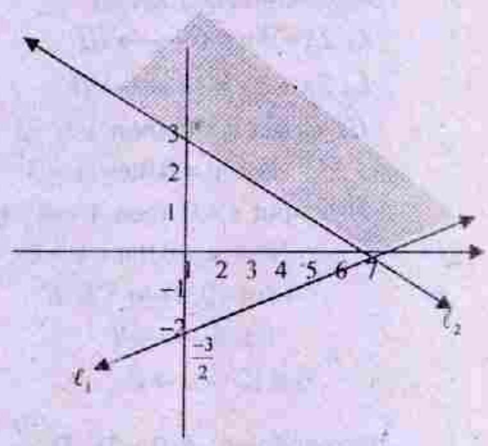
Put $(0,0)$ *in I & II*

$I \Rightarrow 0 \le 21 \longrightarrow T$

$II \Rightarrow 0 \le -3 \longrightarrow F$

Corner Point :

$A = \left(\frac{-3}{2}, 0\right),\ B(0,3)$



(iv) $3x + 2y \ge 6$ I

$x + 3y \le 6$ II

$y \ge 0$

Associated equation are

$l_1; 3x + 2y = 6$ III

$l_2;\ x + 3y = 6$ IV

Value of Point c:

$$\begin{array}{r} 9x + 6y = 18 \\ \pm 2x \pm 6y = \pm 12 \\ \hline 7x \qquad = 6 \end{array}$$

$x = \boxed{y = \frac{6}{7}}$

Putting value of x *in IV*

$\frac{6}{7} + 3y = 6 \Rightarrow \boxed{y = \frac{12}{7}}$

$III \Rightarrow$ put $x=0$ then $y=3$ $(0,3)$

Put $y=0$ then $x=2$ $(2,0)$

$IV \Rightarrow$ put $x=0$ then $y=2$ $(0,2)$

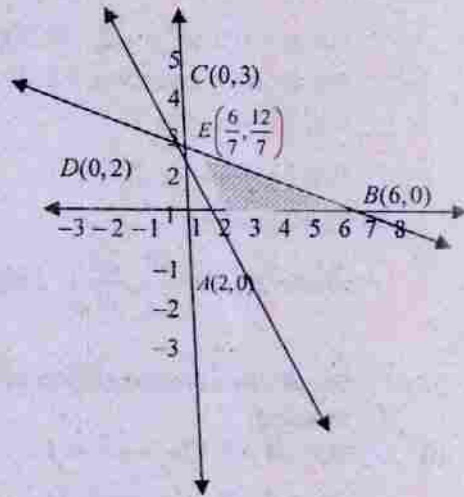
Put $y=0$ then $x=6$, $(6,0)$

$0\geq 6$ F

$0\leq 6$ T

Corner Points:

$A=(2,0)$, $B(6,0)$, $C=\left(\frac{6}{7},\frac{12}{7}\right)$



(v) $5x+7y\leq 35 \longrightarrow I$

$-x+3y\leq 3 \longrightarrow II$

$x\geq 0$

Associated equation are

$l_1; 5x+7y=35 \longrightarrow III$

$l_2; -x+3y=3 \longrightarrow IV$

$III \Rightarrow$ put $x=0$ then $y=5$ $(0,5)$

Put $y=0$ then $x=7$ $(7,0)$

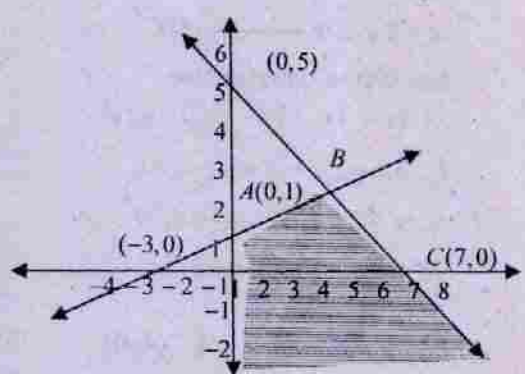
$IV \Rightarrow$ put $x=0$ then $y=1$ $(0,1)$

Put $y=0$ then $x=-3$, $(-3,0)$

$I \Rightarrow 0\leq 35 \longrightarrow T$

$II \Rightarrow 0\leq 3 \longrightarrow T$

Corner Points



(vi) $5x+7y\leq 35 \longrightarrow I$

$x-2y\leq 2 \longrightarrow II$

$x\geq 0$

Associated equation are

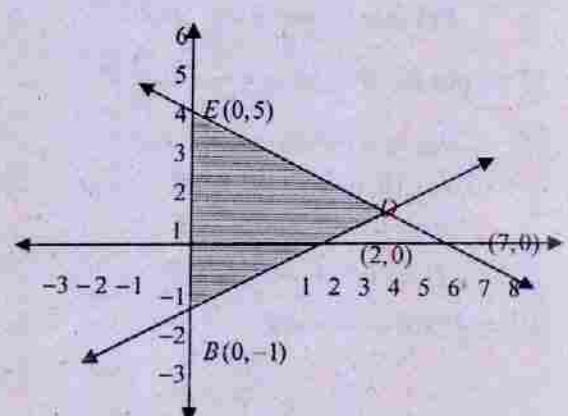
$l_1; 5x+7y=35 \longrightarrow III$

$l_2; x-2y=2 \longrightarrow IV$

$III \Rightarrow$ put $x=0$ then $y=5$ $(0,5)$

Put $y=0$ then $x=7$ $(7,0)$

$IV \Rightarrow$ put $x=0$ then $y=-1$ $(0,-1)$



Put $y=0$ then $x=2,\ (2,0)$

Put $(0,0)$ in equation I & II

$0 \le 35 \longrightarrow T$

$0 \le 2 \longrightarrow T$

Corner Points:

$B(0,-1),\ D\left(\frac{84}{17},\frac{25}{11}\right),\ E(0,5)$

5. Graph the solution region of the following system of linear inequalities by shading:

(i) $3x-4y \le 12 \longrightarrow I$

$3x+2y \ge 3 \longrightarrow II$

$x+2y \le 9 \longrightarrow III$

Associated equation are

$l_1;\ 3x-4y=12 \longrightarrow IV$

$l_2;\ 3x+2y=3 \longrightarrow V$

$l_3;\ x+2y=9 \longrightarrow VI$

$IV \Rightarrow$ put $x=0$ then

$y=-3\quad (0,-3)$

Put $y=0$ then $x=4\quad (4,0)$

$V \Rightarrow$ put $x=0$ then $y=\frac{3}{2},\left(0,\frac{3}{2}\right)$

Put $y=0$ then $x=1\quad (1,0)$

$VI \Rightarrow$ put $x=0$ then $y=\frac{9}{2},\left(0,\frac{9}{2}\right)$

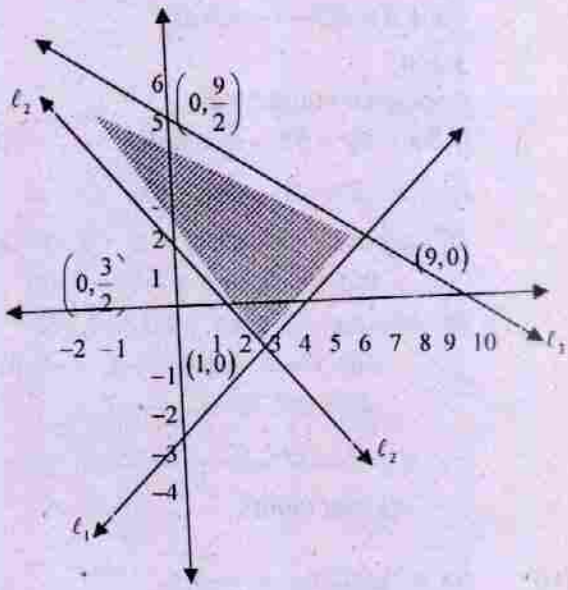
Put $y=0$ then $x=9\quad (9,0)$

Put $(0,0)$ in I, II & III

$I \Rightarrow 0 \le 12 \longrightarrow T$

$II \Rightarrow 0 \ge 3 \longrightarrow F$

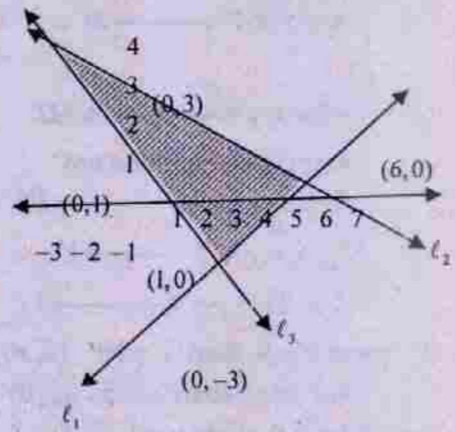
$III \Rightarrow 0 \le 9 \longrightarrow T$



(ii) $3x-4y\le 12 \longrightarrow I$

$x+2y\le 6 \longrightarrow II$

$x+y\ge 1 \longrightarrow III$



Associated equation are

$l_1;\ 3x-4y=12 \longrightarrow IV$

$l_2;\ x+2y=6 \longrightarrow V$

$l_3;\ x+y=1 \longrightarrow VI$

$IV \Rightarrow$ put $x=0$ then $y=-3$ $(0,-3)$

Put $y=0$ then $x=4$ $(4,0)$

$V \Rightarrow$ put $x=0$ then $y=3,\ (0,3)$

Put $y=0$ then $x=6$ $(6,0)$

$VI \Rightarrow$ put $x=0$ then $y=1,\ (0,1)$

Put $y=0$ then $x=1,$ $(1,0)$

Put $(0,0)$ *in* I, II & III

$I \Rightarrow 0\le 12 \longrightarrow T$

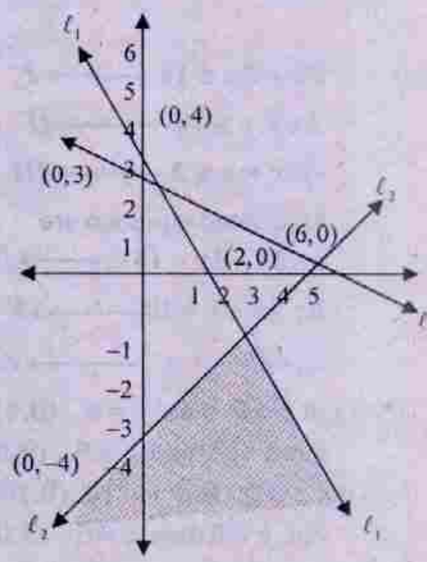
$II \Rightarrow 0\le 6 \longrightarrow T$

$III \Rightarrow 0\ge 1 \longrightarrow F$

(iii) $2x+y\le 4 \longrightarrow I$

$2x-3y\ge 12 \longrightarrow II$

$x+2y\le 6 \longrightarrow III$



Associated equation are

$l_1;\ 2x+y=4 \longrightarrow IV$

$l_2;\ 2x-3y=12 \longrightarrow V$

$l_3;\ x+2y=6 \longrightarrow VI$

$IV \Rightarrow$ put $x=0$ then $y=4$ $(0,4)$

Put $y=0$ then $x=2$ $(2,0)$

$V \Rightarrow$ put $x=0$ then $y=-4,\ (0,-4)$

Put $y=0$ then $x=6$ $(6,0)$

$VI \Rightarrow$ put $x=0$ then $y=3,\ (0,3)$

Put $y=0$ then $x=6,$ $(6,0)$

Put $(0,0)$ *in* I, II & III

$I \Rightarrow 0\le 4 \longrightarrow T$

$II \Rightarrow 0\ge 12 \longrightarrow F$

(iv) $2x+y\le 10 \longrightarrow I$

$x+y\le 7 \longrightarrow II$

$-2x+y\le 4 \longrightarrow III$

Associated equation are

$l_1;\ 2x+y=10 \longrightarrow IV$

$l_2;\ x+y=7 \longrightarrow V$

$l_3;-2x+y=4 \longrightarrow VI$

$IV \Rightarrow$ put $x=0$ then $y=10$ $(0,10)$

Put $y=0$ then $x=5$ $(5,0)$

$V \Rightarrow$ put $x=0$ then $y=1$, $(0,1)$

Put $y=0$ then $x=7$ $(7,0)$

$VI \Rightarrow$ put $x=0$ then $y=4$, $(0,4)$

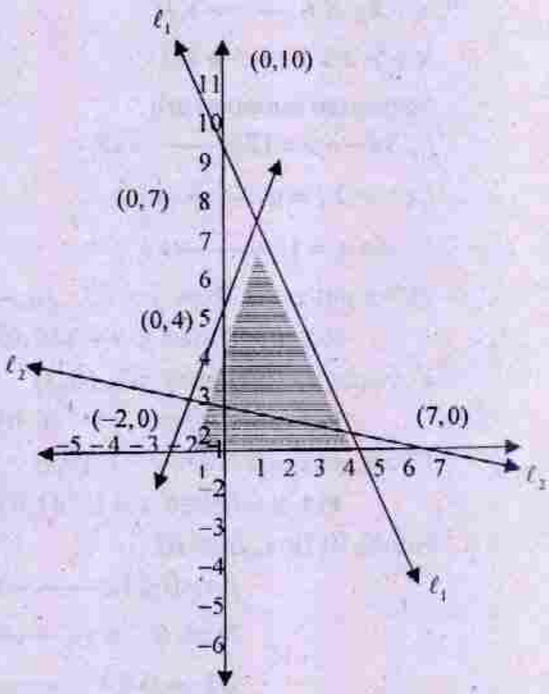
Put $y=0$ then $x=-2$, $(-2,0)$

Put $(0,0)$ *in* I, II & III

$I \Rightarrow 0\le 10 \longrightarrow T$

$II \Rightarrow 0\le 7 \longrightarrow T$

$III \Rightarrow 0\ge 4 \longrightarrow T$



(v) $2x+3y\le 18 \longrightarrow I$

$2x+y\le 10 \longrightarrow II$

$-2x+y\le 2 \longrightarrow III$

Associated equation are

$l_1;\ 2x+3y=18 \longrightarrow IV$

$l_2;\ 2x+y=10 \longrightarrow V$

$l_3;-2x+y=2 \longrightarrow VI$

$IV \Rightarrow$ put $x=0$ then $y=6$ $(0,6)$

Put $y=0$ then $x=9$ $(9,0)$

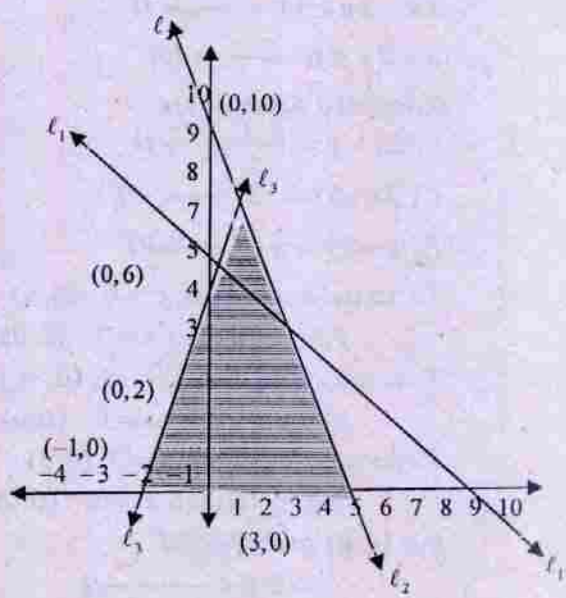
$V \Rightarrow$ put $x=0$ then $y=10$, $(0,10)$

Put $y=0$ then $x=5$ $(5,0)$

$VI \Rightarrow$ put $x=0$ then $y=2$, $(0,2)$

Put $y=0$ then $x=-1$, $(-1,0)$

Put $(0,0)$ *in* I, II & III



$$I \Rightarrow 0 \le 18 \longrightarrow T$$
$$II \Rightarrow 0 \le 10 \longrightarrow T$$

(vi) $3x - 2y \ge 3 \longrightarrow I$

$x + 4y \le 12 \longrightarrow II$

$3x + y \le 12 \longrightarrow III$

Associated equation are

$l_1;\ 3x - 2y = 3 \longrightarrow IV$

$l_2;\ x + 4y = 12 \longrightarrow V$

$l_3; 3x + y = 12 \longrightarrow VI$

$IV \Rightarrow$ put $x = 0$ then $y = \frac{-3}{2}, \left(0, \frac{-3}{2}\right)$

Put $y = 0$ then $x = 1$ $(1,0)$

$V \Rightarrow$ put $x = 0$ then $y = 3$, $(0,3)$

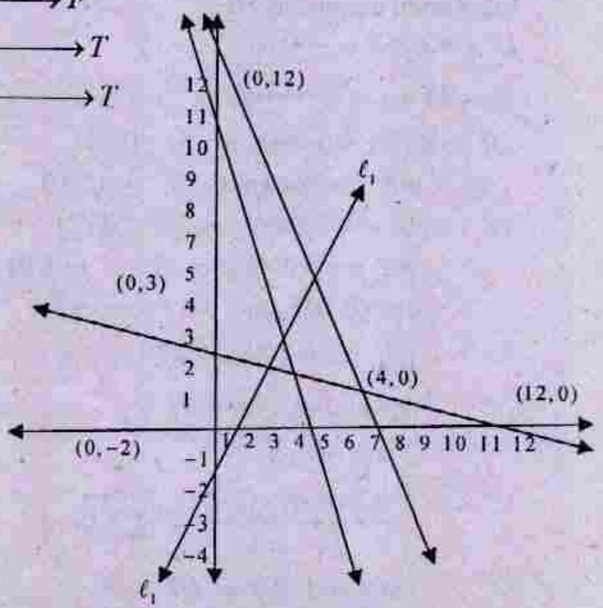
Put $y = 0$ then $x = 12$ $(12,0)$

$VI \Rightarrow$ put $x = 0$ then $y = 12$, $(0,12)$

Put $y = 0$ then $x = 4$, $(4,0)$

Put $(0,0)$ *in* I, II & III

$$I \Rightarrow 0 \ge 3 \longrightarrow F$$
$$II \Rightarrow 0 \le 12 \longrightarrow T$$
$$III \Rightarrow 0 \le 12 \longrightarrow T$$



Exercise 5.2

1. Graph the feasible region also find the corner points.

(i) $2x-3y\le 6 \longrightarrow I \quad , x\ge 0$ **(Sgd 2011)**

$2x+3y\le 12 \longrightarrow II \quad , y\ge 0$

Associated equations are

$l_1;\ 2x-3y=6 \longrightarrow III$

$l_2;\ 2x+3y=12 \longrightarrow IV$

$III \Rightarrow$ put $x=0$ then $y=-2,\ (0,-2)$

put $y=0$ then $x=3,\quad (3,0)$

$IV \Rightarrow$ put $x=0$ then $y=4,\quad (0,4)$

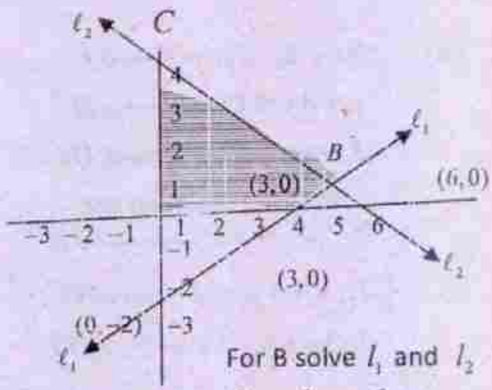
put $y=0$ then $x=6,\quad (6,0)$

put $(0,0)I \Rightarrow\ 0\le 6 \longrightarrow T$

$II \Rightarrow\ 0\le 12 \longrightarrow T$

Corner points are

$O(0,0),\ A(3,0),\ B\left(\frac{9}{2},1\right),\ (0,4)$



For B solve l_1 and l_2

$2x-3y=6$

$2x+3y=12$

$4x\ \ =18$

Put value of x in IV $\boxed{x=\frac{9}{2}}$

$2\left(\frac{9}{2}\right)+3y=12$

$3y=12-9=3 \Rightarrow \boxed{y=1}$

(ii) $x+y\le 5 \longrightarrow I \quad , x\ge 0$

$-2x+y\le 2 \longrightarrow II \quad , y\ge 0$

Associated equations are

$l_1;\ x+y=5 \longrightarrow III$

$l_2;\ -2x+y=2 \longrightarrow IV$

$III \Rightarrow$ put $x=0$ then $y=5,\ (0,5)$

put $y=0$ then $x=5,\quad (5,0)$

$IV \Rightarrow$ put $x=0$ then $y=2,\quad (0,2)$

put $y=0$ then $x=-1,\quad (-1,0)$

put $(0,0)I \Rightarrow\ 0\le 6 \longrightarrow T$

$II \Rightarrow\ 0\le 2 \longrightarrow T$

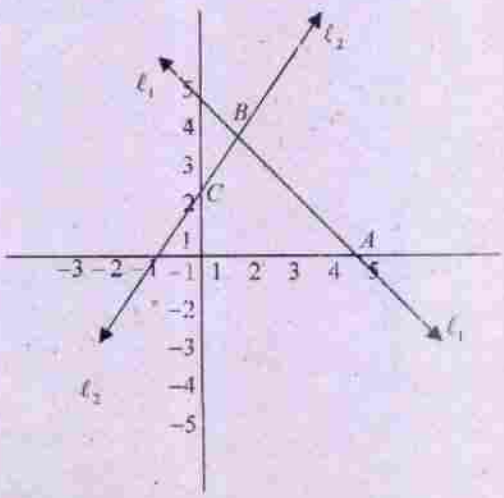
For B solve l_1 and l_2

$x+y=5$

$\mp 2x \pm y=\_2 \Rightarrow \boxed{x=1}$

$3x\ \ =3$

Put value of x in III



$1+y=5 \Rightarrow \boxed{y=4}$

Corner points are

$O(0,0),\ A(5,0),\ B(1,4),\ C(0,2)$

(iii) $x+y\le 5 \longrightarrow I \quad ,x\ge 0$ (Sgd 2010)

$-2x+y\le 2 \longrightarrow II \quad ,y\ge 0$

Associated equations are

$l_1;\ x+y=5 \longrightarrow III$

$l_2;\ -2x+y=2 \longrightarrow IV$

$III \Rightarrow$ put $x=0$ then $y=5,\ (0,5)$

put $y=0$ then $x=5,\ (5,0)$

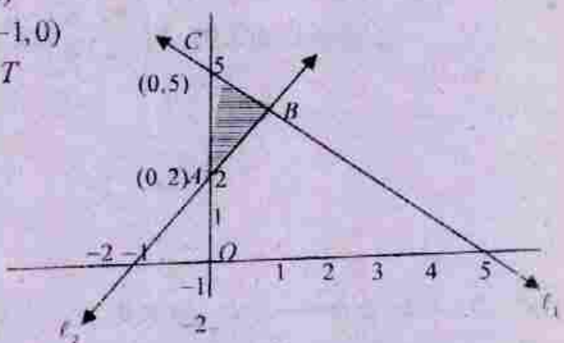
$IV \Rightarrow$ put $x=0$ then $y=2,\ (0,2)$

put $y=0$ then $x=-1,\ (-1,0)$

put $(0,0) I \Rightarrow 0\le 6 \longrightarrow T$

$II \Rightarrow 0\le 2 \longrightarrow F$

For B solve l_1 and l_2



$x-y=5$

$\underline{\pm 2x \pm y = \_2} \Rightarrow \boxed{x=1}$

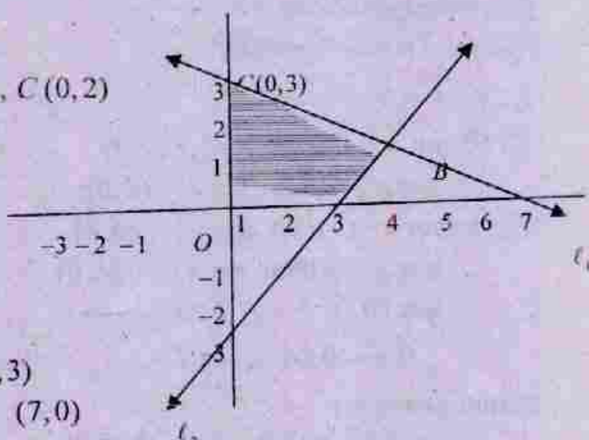
$3x \quad =3$

Put value of x in III

$1+y=5 \Rightarrow \boxed{y=4}$

Corner points are

$O(0,0),\ A(5,0),\ B(1,4),\ C(0,2)$



(iv) $3x+7y\le 21 \longrightarrow I \quad ,x\ge 0$

$x-y\le 3 \longrightarrow II \quad ,y\ge 0$

Associated equations are

$l_1;\ 3x+7y=21 \longrightarrow III$

$l_2;\ x-y=3 \longrightarrow IV$

$III \Rightarrow$ put $x=0$ then $y=3,\ (0,3)$

put $y=0$ then $x=7,\ (7,0)$

$IV \Rightarrow$ put $x=0$ then $y=-3,\ (0,-3)$

put $y=0$ then $x=3,\ (3,0)$

put $(0,0)\ I \Rightarrow 0\le 21 \longrightarrow T$

$$II \Rightarrow\ 0 \le 3 \longrightarrow T$$

For B solve l_1 and l_2

$$\begin{array}{r} 3x + 7y = 21 \\ \underline{_-3x \mp 3y = _-9} \\ 10y \quad = 12 \end{array} \Rightarrow \boxed{y = \frac{6}{5}}$$

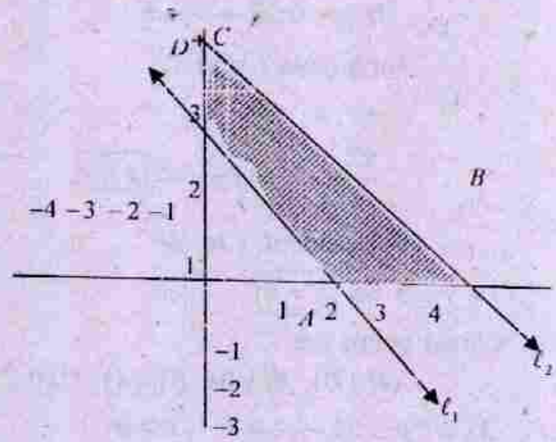
Put value of x *in* IV

$$x - \frac{6}{5} = 3$$

$$x = 3 + \frac{6}{5} = \frac{21}{5} \quad \Rightarrow \boxed{x = \frac{21}{5}}$$

Corner points are

$$O(0,0),\ A(3,0),\ B\left(\frac{21}{5}, \frac{6}{5}\right),\ C(0,3)$$



(v) $3x + 2y \ge 6 \longrightarrow I \quad , x \ge 0$

$x + y \le 4 \longrightarrow II \quad , y \ge 0$

Associated equations are

$l_1;\ 3x + 2y = 6 \longrightarrow III$

$l_2;\ x + y = 4 \longrightarrow IV$

$III \Rightarrow$ put $x = 0$ then $y = 3$, $(0,3)$

put $y = 0$ then $x = 2$, $(2,0)$

$IV \Rightarrow$ put $x = 0$ then $y = 4$, $(0,4)$

put $y = 0$ then $x = 4$, $(4,0)$

put $(0,0)$ $I \Rightarrow\ 0 \ge 6 \longrightarrow F$

$II \Rightarrow\ 0 \le 4 \longrightarrow T$

Corner points are

$$A(2,0),\ B(4,0),\ C(0,4),\ D(0,3)$$

(vi) $5x+7y \le 35 \longrightarrow I \quad , x \ge 0$ (Sgd 2010)

$x-2y \le 4 \longrightarrow II \quad , y \ge 0$

Associated equations are

$l_1;\ 5x+7y=35 \longrightarrow III$

$l_2;\ x-2y=4 \longrightarrow IV$

$III \Rightarrow$ put $x=0$ then $y=5$, $(0,5)$

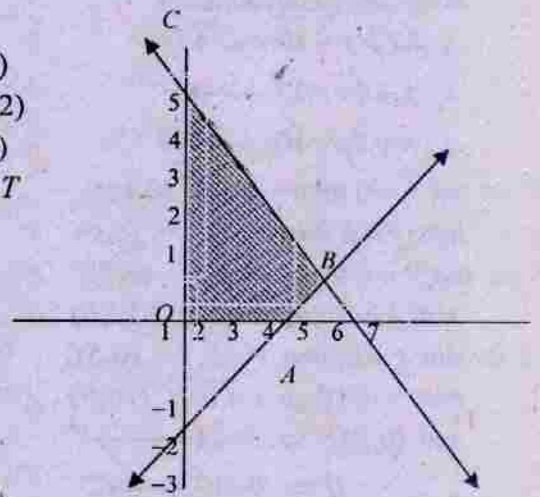
put $y=0$ then $x=7$, $(7,0)$

$IV \Rightarrow$ put $x=0$ then $y=-2$, $(0,-2)$

put $y=0$ then $x=4$, $(4,0)$

put $(0,0)$ $I \Rightarrow 0 \le 35 \longrightarrow T$

$II \Rightarrow 0 \le 4 \longrightarrow T$



For B solve l_1 and l_2

$$\begin{array}{r} \cancel{5x}+7y=35 \\ \underline{-\cancel{5x}\mp 10y=-20} \\ 17y=-15 \end{array}$$

$$\boxed{y=\frac{15}{17}}$$

Put value of y in

$IV\ \ x-2\left(\frac{15}{17}\right)=4$

$x=4+\frac{30}{17} \quad \Rightarrow \boxed{x=\frac{98}{17}}$

Corner points are

$$O(0,0),\ A(4,0),\ B\left(\frac{98}{17},\frac{15}{17}\right),\ C(0,5)$$

2. Graph the feasible region and also find the corner points in each case.

(i) $2x + y \le 10 \longrightarrow I$

$x + 4y \le 12 \longrightarrow II$

$x + 2y \le 10 \longrightarrow III$

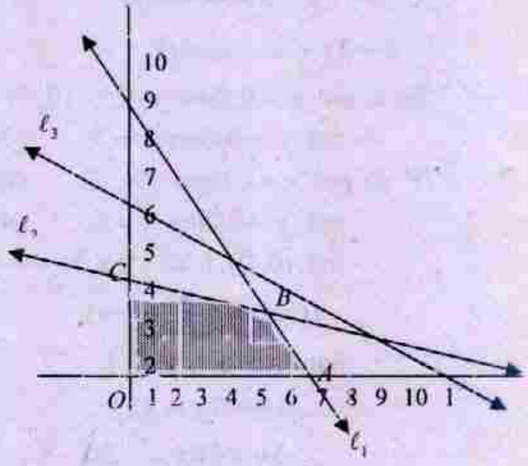
$x \ge 0,\ y \ge 0$

Associated equations are

$l_1;\ 2x + y = 10 \longrightarrow IV$

$l_2;\ x + 4y = 12 \longrightarrow V$

$l_3;\ x + 2y = 10 \longrightarrow VI$



$IV \Rightarrow$ put $x = 0$ then $y = 10$, $(0,10)$

put $y = 0$ then $x = 5$, $(5,0)$

$V \Rightarrow$ put $x = 0$ then $y = 3$, $(0,3)$

put $y = 0$ then $x = 12$, $(12,0)$

$VI \Rightarrow$ put $x = 0$ then $y = 5$, $(0,5)$

put $y = 0$ then $x = 10$, $(10,0)$

put $(0,0) I \Rightarrow 0 \le 10 \longrightarrow T$

$II \Rightarrow 0 \le 12 \longrightarrow T$

$III \Rightarrow 0 \le 10 \longrightarrow T$

For B solve l_3 and l_2

$$20 + y = 10 \longrightarrow \quad \cancel{2x} - y = 10$$

$$x + 4y = 12 \quad \underline{-\cancel{2x} \pm 8y = \_24} \Rightarrow \boxed{y = 2}$$

$$-7y \quad = -14$$

Put in IV

$2x + 3y = 18$

$2(x) + 3(2) = 18 \Rightarrow 2x = 18 - 6$

$2x = 12 \Rightarrow \boxed{x = 6}$

Corner points are

$O(0,0),\ A(5,0),\ B(6,2),\ C(0,3)$

(ii) $2x+3y \le 18 \longrightarrow I \qquad x \ge 0$

$2x+y \le 10 \longrightarrow II \qquad y \ge 0$

$x+4y \le 12 \longrightarrow III$

Associated equations are

$l_1;\ 2x+3y=18 \longrightarrow IV$

$l_2;\ 2x+y=10 \longrightarrow V$

$l_3;\ x+4y=12 \longrightarrow VI$

put $x=0$ then $y=6$, $(0,6)$

$IV \Rightarrow$ put $y=0$ then $x=9$, $(9,0)$

$V \Rightarrow$ put $x=0$ then $y=10$, $(0,10)$

put $y=0$ then $x=5$, $(5,0)$

$VI \Rightarrow$ put $x=0$ then $y=3$, $(0,3)$

put $y=0$ then $x=12$, $(12,0)$

put $(0,0) I \Rightarrow 0 \le 18 \longrightarrow T$

$II \Rightarrow 0 \le 10 \longrightarrow T$

$III \Rightarrow 0 \le 12 \longrightarrow T$

For B solve l_3 and l_2

$$\begin{array}{r} \cancel{2x} + y = 10 \\ \underline{-\cancel{2x} \pm 8y = -24} \\ -7y = -14 \end{array}$$

$\Rightarrow y=2$

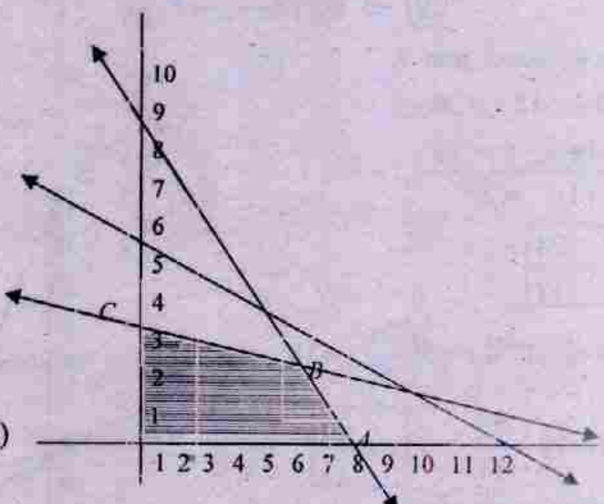
Put $y=2$ *in* V

$2x+2=10$

$2x=8 \Rightarrow \boxed{x=4}$

Corner points are

$O(0,0)$, $A(5,0)$, $B(4,2)$, $C(0,3)$



(iii) $2x+3y \le 18 \longrightarrow I \qquad x \ge 0 \quad y \ge 0$

$x+4y \le 12 \longrightarrow II$

$3x+y \le 12 \longrightarrow III$

Associated equations are

$l_1;\ 2x+3y=18 \longrightarrow IV$

$l_2;\ x+4y=12 \longrightarrow V$

$l_3;\ 3x+y=12 \longrightarrow VI$

$IV \Rightarrow$ put $x=0$ then $y=6$, $(0,6)$

put $y=0$ then $x=9$, $(9,0)$

$V \Rightarrow$ put $x=0$ then $y=3$, $(0,3)$

put $y=0$ then $x=12$, $(12,0)$

$VI \Rightarrow$ put $x=0$ then $y=12$, $(0,12)$

put $y=0$ then $x=4$, $(4,0)$

put $(0,0) I \Rightarrow 0 \le 18 \longrightarrow T$

$II \Rightarrow 0 \le 12 \longrightarrow T$

$III \Rightarrow 0 \le 12 \longrightarrow T$

For B solve l_1 and l_2 l_2 & l_3

$\cancel{3x}+12y=36$

$\underline{-\cancel{3x} \pm y = 12}$

$11y = 24$

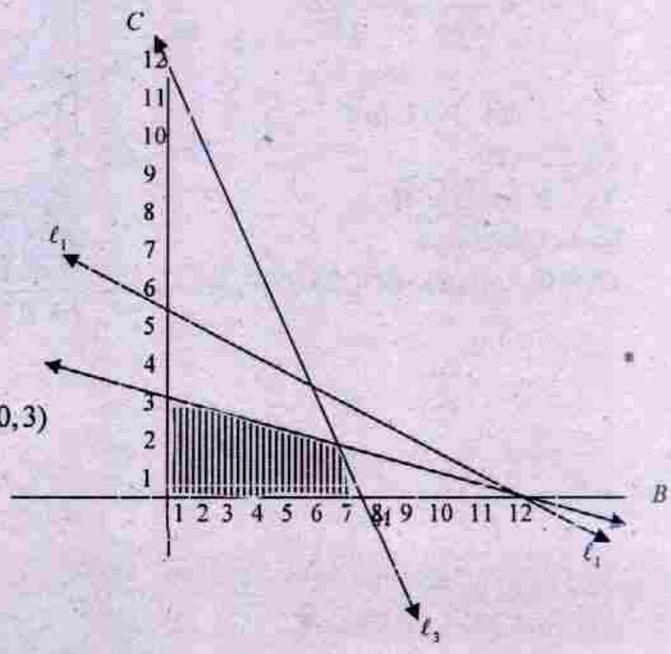
$\boxed{y=\frac{24}{11}}$

Put in IV $2x+3\left(\frac{24}{11}\right)=18$

$2x=18-\frac{72}{11} \Rightarrow \boxed{x=\frac{36}{11}}$

Corner points are

$O(0,0),\ A(4,0),\ B\left(\frac{36}{11},\frac{24}{11}\right),\ C(0,3)$



(iv) $x+2y \le 14 \longrightarrow I$ $x \ge 0$ $y \ge 0$

$3x+4y \le 36 \longrightarrow II$

$2x+y \le 10 \longrightarrow III$

Associated equations are

$l_1;\ x+2y=14$

$l_2;\ 3x+4y=36 \longrightarrow V$

$l_3;\ 2x+y=10 \longrightarrow VI$

$IV \Rightarrow$ put $x=0$ then $y=7$, $(0,7)$

put $y=0$ then $x=14$, $(14,0)$

$V \Rightarrow$ put $x=0$ then $y=9$, $(0,9)$

put $y=0$ then $x=12$, $(12,0)$

$VI \Rightarrow$ put $x=0$ then $y=10$, $(0,10)$

put $y=0$ then $x=5$, $(5,0)$

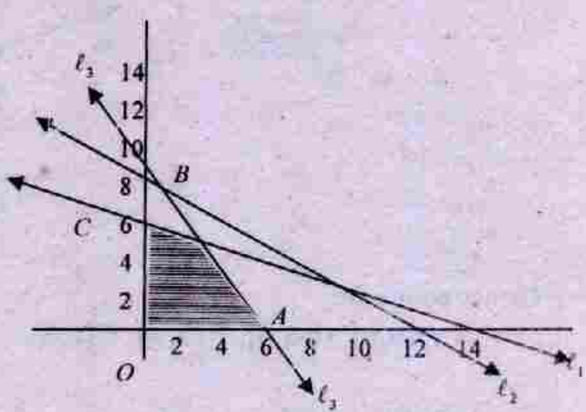
put $(0,0) I \Rightarrow 0 \le 14 \longrightarrow T$

$II \Rightarrow 0 \le 36 \longrightarrow T$

$III \Rightarrow 0 \le 10 \longrightarrow T$

For B solve l_1 and l_2

$$x + 2y = 14$$
$$\underline{-4x \pm 2y = -20}$$
$$-3x = -6$$
$$\boxed{x=2}$$



Put value of in x in IV

$2+2y=14$

$2y=12$

$\boxed{y=6}$

Corner points are

$O(0,0)$, $A(5,0)$, $B(2,6)$, $C(0,7)$

(v) $x+3y \le 15 \longrightarrow I$ $x \ge 0$

$2x+y \le 12 \longrightarrow II$ $y \ge 0$

$4x+3y \le 24 \longrightarrow III$

Associated equations are

$l_1;\ x+3y=15 \longrightarrow IV$

$l_2;\ 2x+y=12 \longrightarrow V$

$l_3;\ 4x+3y=24 \longrightarrow VI$

put $x=0$ then $y=5$, $(0,5)$

$IV \Rightarrow$ put $y=0$ then $15=x$, $(15,0)$

$V \rightarrow$ put $x=0$ then $y=12$, $(0,12)$

put $y=0$ then $x=6$, $(6,0)$

put $x=0$ then $y=8$, $(0,8)$

put $y=0$ then $x=6$, $(6,0)$

put $(0,0)I \Rightarrow 0 \le 15 \longrightarrow T$

$II \Rightarrow 0 \le 12 \longrightarrow T$

$III \Rightarrow 0 \le 24 \longrightarrow T$

For B solve l_1 and l_3

$$\begin{array}{r} x+\cancel{3y} = 15 \\ \underline{-4x \pm \cancel{3y} = -24} \\ -3x = -9 \end{array}$$

$\Rightarrow \boxed{x=3}$

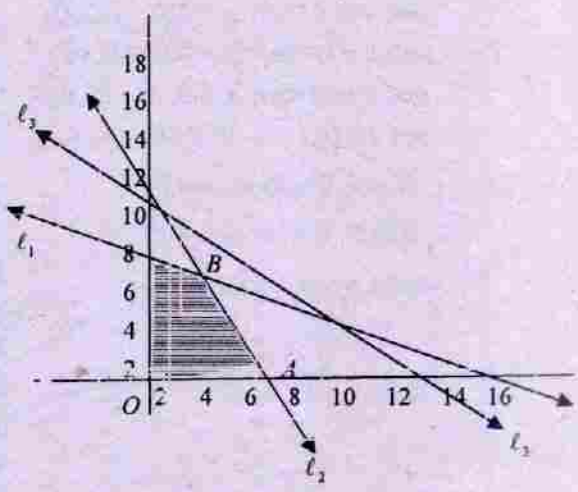
Put in IV

$x+3y=15$

$3+3y=15$

$3y=12$

$\boxed{y=4}$



Corner points are

$O(0,0)$, $A(6,0)$, $B(3,4)$, $C(0,5)$

(vi) $\mathbf{2x+y \le 20 \longrightarrow I}$

$\mathbf{8x+15y \le 120 \longrightarrow II}$

$\mathbf{x+y \le 11 \longrightarrow III}$

Associated equations are

$l_1;\ 2x+y=20 \longrightarrow IV$

$l_2;\ 8x+15y=120 \longrightarrow V$

$l_3;\ x+y=11 \longrightarrow VI$

$IV \Rightarrow$ put $y=0$ then $y=20$, $(0,20)$

put $y=0$ then $x=10$, $(10,0)$

$V \Rightarrow$ put $x=0$ then $y=8$, $(0,8)$

put $y=0$ then $x=15$, $(15,0)$

$VI \Rightarrow$ put $x=0$ then $y=11$, $(0,11)$

put $y=0$ then $x=11$, $(11,0)$

put $(0,0)I \Rightarrow 0 \le 20 \longrightarrow T$

$II \Rightarrow 0 \le 120 \longrightarrow T$

$III \Rightarrow 0 \le 11 \longrightarrow T$

For B solve l_1 and l_3

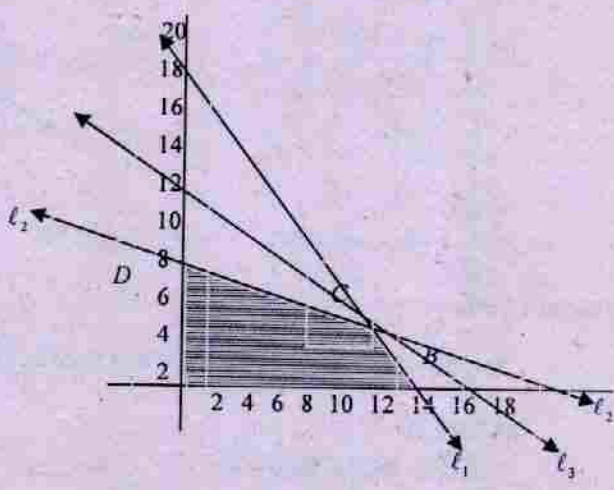
$$2x + \cancel{y} = 20$$
$$\underline{\_x \pm \cancel{y} = \_11}$$
$$x = 9$$
$$\Rightarrow \boxed{x=9}$$

$x + y = 11$

$9 + y = 11 \Rightarrow \boxed{y=2}$

For C solve l_3 and l_2

$$\cancel{8x} + 15y = 120$$
$$\underline{\_\cancel{8x} \pm 8y = \_88}$$
$$7y = 32$$
$$\boxed{y = \frac{32}{7}}$$



$x + y = 11$

$x = 11 - \frac{32}{7}$

$$\boxed{x = \frac{45}{7}}$$

Corner points are

$O(0,0)$, $A(10,0)$, $B(9,2)$, $C\left(\frac{45}{7}, \frac{32}{7}\right)$, $D(0,8)$

Example of 5.3 **(Sargodha 2009)**

Maximize and Minimize $f(x,y) = 4x + 5y$

$2x - 3y \le 6 \longrightarrow I$

$2x + y \ge 2 \longrightarrow II$

$2x + 3y \le 12 \longrightarrow III$ $\quad x \ge 0 \quad y \ge 0$

Associated equations are

$l_1;\ 2x-3y=6 \quad IV$

$l_2;\ 2x+y=2 \quad V$

$l_3;\ 2x+3y=12 \quad VI$

For $l_1;(0,-2),\ (3,0)$

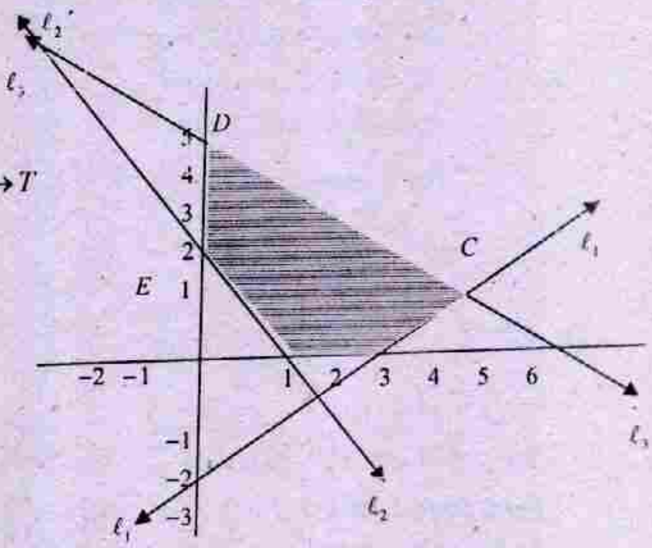
$l_2;\ (0,2),\ (1,0)$

$l_3;\ (0,4),\ (6,0)$

put $(0,0) I \Rightarrow 0\le 6 \longrightarrow T$

$II \Rightarrow 0\le 2 \longrightarrow F$

$III \Rightarrow 0\le 12 \longrightarrow T$



Solve l_1 and l_3

$$2x-3y=6$$
$$2x\pm 3y=12$$
$$4x \quad =18$$

$$\boxed{x=\frac{9}{2}}$$

$$2\left(\frac{9}{2}\right)-3y=6 \Rightarrow \quad -3y=-3 \Rightarrow \quad y=1$$

Corner Points

$$A(1,0),\ B(3,0),\ C\left(\frac{9}{2},1\right),\ D(0,4),\ E(0,2)$$

at $A(1,0)$

$f(1,0)=4(1)+5(0)=4$

at $B(3,0)$

$f(3,0)=4(3)+5(0)=12$

at $C\left(\frac{9}{2},1\right)$

$f\left(\frac{9}{2},1\right)=4\left(\frac{9}{2}\right)+5(1)=23$

at $E(0,2)$

$f(0,2)=4(0)+5(2)=10$

Max at $C\left(\frac{9}{2},1\right)$ & Min at $A(1,0)$

Exercise 5.3

1. **Maximize $f(x,y)=2x+5y$.** **(Sargodha 2010, 11)**

$2y-x\le 8;\ x-y\le 4;\ x\ge 0\ \&\ y\ge 0$

Sol. Let $2y-x\le 8 \longrightarrow (1)$

and $x-y\le 4 \longrightarrow (2)$

Associated equation are

$l_1;\ 2y-x=8 \longrightarrow (III)$, $l_2;\ x-y=4 \longrightarrow (IV)$

Put $x=0$ then $y=4$ $(0,4)$

Put $y=0$ then $x=-8$ $(-8,0)$

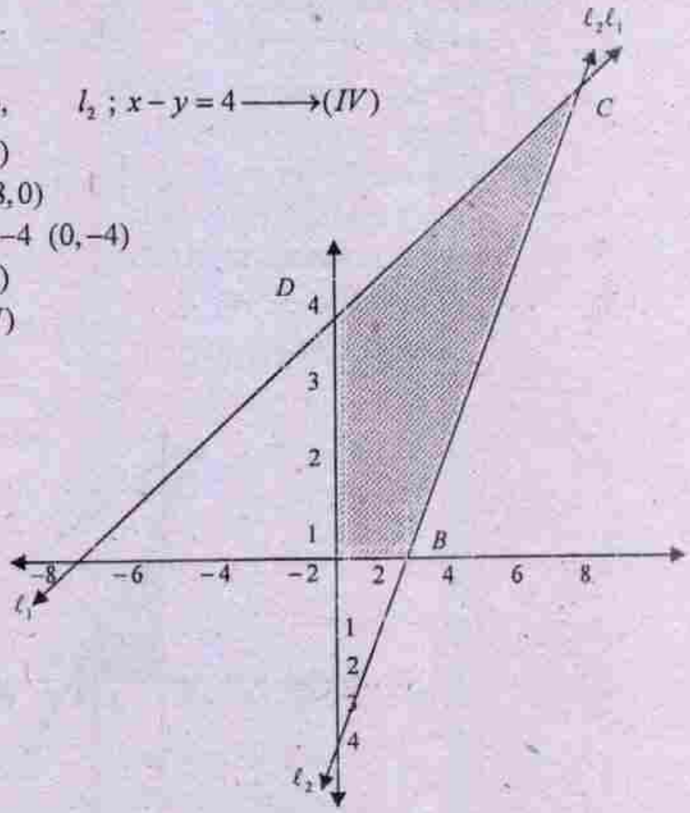
$III \Rightarrow$ Put $x=0$ then $y=-4$ $(0,-4)$

Put $y=0$ then $x=4$ $(4,0)$

Put $(0,0)$ in eq. (I) & (II)

$I \Rightarrow 0\le 8 \longrightarrow T$

$II \Rightarrow 0\le 4 \longrightarrow T$



Note: To find = C

Solve eq. (III) & (IV)

$$2y-x=8$$
$$-y+x=4$$
$$y=12$$

Put $y=12$ in (4)

$x-12=4$

$x=16$

$C(16,12)$

Corner points at feasible region are

$A(0,0),\ B(4,0),\ C(16,12),\ D(0,4)$

At A; $f(0,0)=2(0)+5(0)=0$

At B; $f(4,0)=2(4)+5(0)=8$

At C; $f(16,12)=2(16)+5(12)=92$

At D; $f(0,4)=2(0)+5(4)=20$

So f is maximum at $C(16,12)$

2. Maximize $f(x,y)=x+3y$. (Gujrawala 2010)

$2x+5y\le 30;\ 5x+4y\le 20;\ x\ge 0\ \&\ y\ge 0$

Sol. Let $2x+5y\le 30 \longrightarrow I$

and $5x+4y\le 20 \longrightarrow II$

Associated equation are

$l_1;\ 2x+5y=30 \longrightarrow III,\qquad l_2;\ 5x+4y=20 \longrightarrow IV$

$III \Rightarrow Put\ x=0\ then\ y=6,\ (0,6)$

$Put\ y=0\ then\ x=15,\ (15,0)$

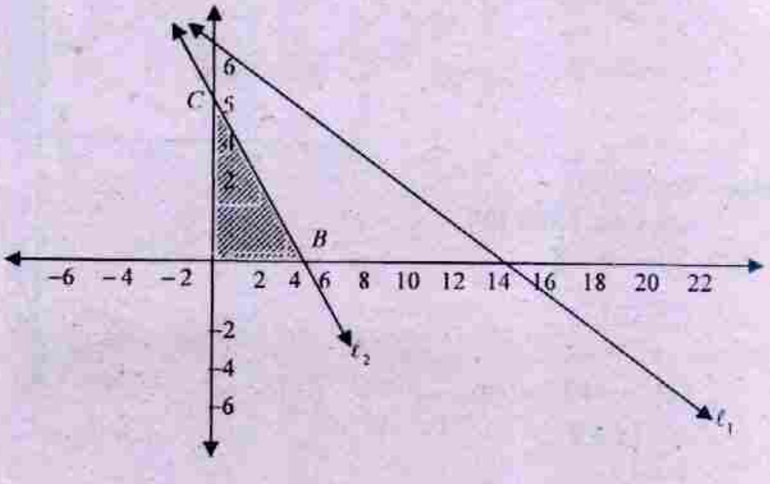
$IV \Rightarrow Put\ x=0\ then\ y=5,\ (0,5)$

$Put\ y=0\ then\ x=4,\ (4,0)$

Put $(0,0)$ *in eq.* $(I)\ \&\ (II)$

$I \Rightarrow 0\le 30 \longrightarrow T$

$II \Rightarrow 0\le 20 \longrightarrow T$



Corner points of feasible region are

$A(0,0),\ B(4,0),\ C(0,5)$

$At\ A,\ f(0,0)=0+3(0)=0$

$At\ B,\ f(4,0)=4+3(0)=4$

$At\ C,\ f(0,5)=0+3(5)=15$

So f is maximum at the corner point $C(0,5)$

3. Maximize $z = 2x + 3y$ $\quad x \geq 0$ & $y \geq 0$ (Lahore 2010)

$3x + 4y \leq 12;\ 2x + y \leq 4;\ 4x - y \leq 4$

Sol. Let $3x + 4y \leq 12 \longrightarrow I$

and $2x + y \leq 4 \longrightarrow II$

$4x - y \leq 4 \longrightarrow III$

Associated equation are

$l_1\ ;\ 3x + 4y = 12 \longrightarrow IV$

$l_2\ ;2x + y = 4 \longrightarrow V$

$l_3\ ;4x - y = 4 \longrightarrow VI$

From eq. (4), we get l_1 as $(0,3),(4,0)$ $IV \Rightarrow put\ x = 0\ then\ y = 3,\ (0,3)$

From eq. (5), we get l_2 as $(0,4),(2,0)$ $put\ y = 0\ then\ x = 4,\ (4,0)$

From eq. (6), we get l_3 as $(0,-4),(1,0)$ $V \Rightarrow x = 0\ then\ y = 4,\ (0,4)$

Put $(0,0)$ in eq. (1) & (2) & (3) $\quad put\ y = 0\ then\ x = 2,\ (2,0)$

$I \Rightarrow 0 \leq 12 \longrightarrow T$ $\quad VI \Rightarrow put\ x = 0\ then\ y = -4,\ (0,-4)$

$II \Rightarrow 0 \leq 4 \longrightarrow T$ $\quad put\ y = 0\ then\ x = 1,\ (1,0)$

$III \Rightarrow 0 \leq 4 \longrightarrow T$

Solve IV & V

$$8x + \cancel{4y} = 16$$

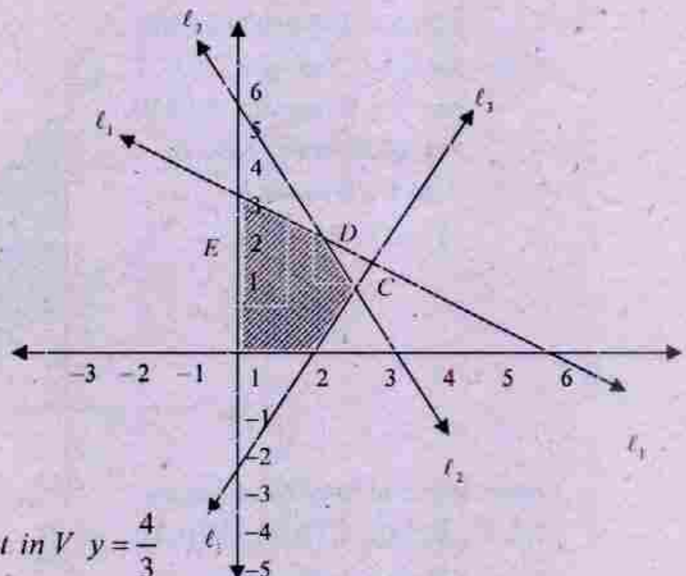
$$\underline{\pm 3x \pm \cancel{4y} = \pm 12}$$

$$5x \quad = 4 \Rightarrow \frac{4}{5}$$

put in V we get

$$D = \left(\frac{4}{5}, \frac{12}{5}\right) \& \ y = \frac{12}{5}$$

Again solve V & VI



$$2x + \cancel{y} = 4$$

$$\underline{4x \pm \cancel{y} = 4}$$

$$6x \quad = 8 \Rightarrow \boxed{x = \frac{4}{3}}\ put\ in\ V\ \ y = \frac{4}{3}$$

$$C = \left(\frac{4}{3}, \frac{4}{3}\right)$$

Corner points of feasible region are:

$$A(0,0),\ B(1,0),\ C\left(\frac{4}{3}, \frac{4}{3}\right),\ D\left(\frac{4}{5}, \frac{12}{5}\right),\ E(0,3)$$

At A, $f(0,0)=2(0)+3(0)=0$

At B, $f(1,0)=2(1)+3(0)=2$

At C, $f\left(\frac{4}{3},\frac{4}{3}\right)=2\left(\frac{4}{3}\right)+3\left(\frac{4}{3}\right)=\frac{20}{3}=6.6$

At D, $f\left(\frac{4}{5},\frac{12}{5}\right)=2\left(\frac{4}{5}\right)+3\left(\frac{12}{5}\right)=\frac{44}{5}=8.8$

At E, $f(0,3)=2(0)+3(3)=9$

So f is maximum at corner $E(0,3)$

4. **Maximize $z=2x+y$** **(Sargodha 2009, 11)**

$x+y\geq 3$; $7x+5y\leq 35$; $x\geq 0$ & $y\geq 0$.

Sol. Let $x+y\geq 3 \longrightarrow I$

and $7x+5y\leq 35 \longrightarrow II$

Associated equation are

l_1 ; $x+y=3 \longrightarrow III$

l_2 ; $7x+5y=35 \longrightarrow IV$

$III \Rightarrow$ *put* $x=0$ *then* $y=3$ $(0,3)$

put $y=0$ *then* $x=3$ $(3,0)$

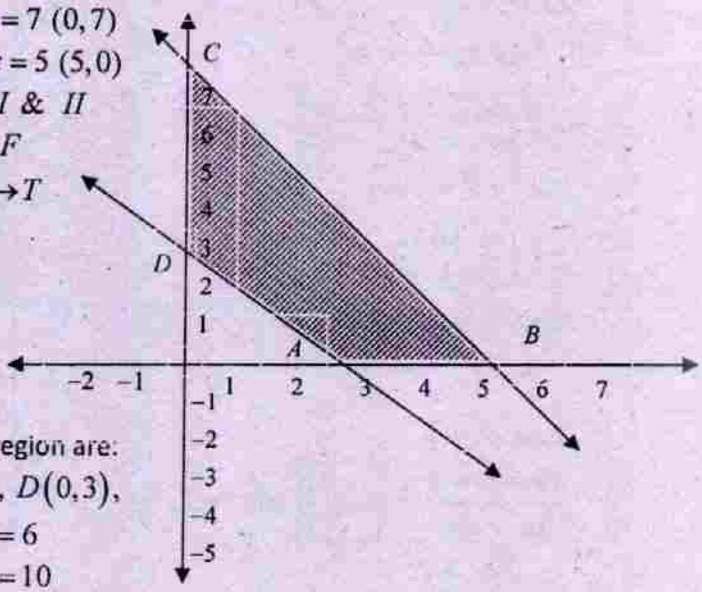
$IV \Rightarrow$ *put* $x=0$ *then* $y=7$ $(0,7)$

put $y=0$ *then* $x=5$ $(5,0)$

Put $(0,0)$ in eq. I & II

$I \Rightarrow 0\geq 0 \longrightarrow F$

$II \Rightarrow 0\leq 35 \longrightarrow T$



Corner points of feasible region are:

$A(3,0)$, $B(5,0)$, $C(0,7)$, $D(0,3)$,

At A, $f(3,0)=2(3)+0=6$

At B, $f(5,0)=2(5)+0=10$

At C, $f(0,7)=2(0)+7=7$

At D, $f(0,3)=2(0)+3=3$

So f is maximum at corner $B=(5,0)$

5. Maximize $f(x, y) = 2x + 3y$ (Sargodha 2011)

$2x + y \leq 8, \quad x + 2y \leq 14, \quad x \geq 0 \ \& \ y \geq 0$

Sol. Let $2x + y \leq 8 \longrightarrow I$

and $x + 2y \leq 14 \longrightarrow II$

Associated equation are

l_1 ; $2x + y = 8 \longrightarrow III$

l_2 ; $x + 2y = 14 \longrightarrow IV$

$III \Rightarrow put\ x = 0\ then\ y = 8,\ (0,8)$

$put\ y = 0\ then\ x = 4,\ (4,0)$

$IV \Rightarrow put\ x = 0\ then\ y = 7,\ (0,7)$

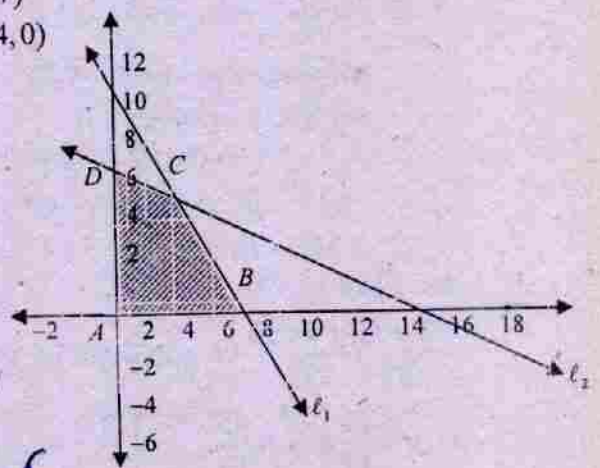
$put\ y = 0\ then\ x = 14,\ (14,0)$

Put $(0,0)$ in eq. I & II

$I \Rightarrow 0 \leq 8 \longrightarrow T$

$II \Rightarrow 0 \leq 14 \longrightarrow T$

So the graph is.



Note: Solve $2 \times III - IV$

$4x + \cancel{2y} = 16$

$\pm x \pm \cancel{2y} = \pm 14$

$3x = 2 \Rightarrow x = \dfrac{2}{3}$

$put\ x = \dfrac{2}{3}\ in\ III\ \ y = \dfrac{20}{3}$

So, $C = \left(\dfrac{2}{3}, \dfrac{20}{3}\right)$

Corner points of feasible region are:

$A(0,0),\ B(4,0),\ C\left(\dfrac{2}{3}, \dfrac{20}{3}\right),\ D(0,7),$

$At\ A,\ f(0,0) = 2(0) + 3(0) = 0$

$At\ B,\ f(4,0) = 2(4) + 3(0) = 8$

$At\ C,\ f\left(\dfrac{2}{3}, \dfrac{20}{3}\right) = 2\left(\dfrac{2}{3}\right) + 3\left(\dfrac{20}{2}\right) = 21.33$

$At\ D,\ f(0,7) = 2(0) + 3(7) = 21$

So f is maximum at corner $C = \left(\dfrac{2}{3}, \dfrac{20}{3}\right)$

6. $z = 3x + y$ **(Sargodha 2008, 12 Lahore 2010)**

$x + 6y \geq 9,\ \ 3x + 5y \geq 15$

Sol. Let $3x + 5y \geq 15 \longrightarrow I$

and $x + 6y \geq 9 \longrightarrow II$

Associated equation are

$l_1\ ;\ 3x + 5y = 15 \longrightarrow III$

$l_2\ ;\ x + 6y = 9 \longrightarrow IV$

$III \Rightarrow put\ x = 0\ then\ \ y = 3,\ (0,3)$

$put\ y = 0\ then\ \ x = 5,\ (5,0)$

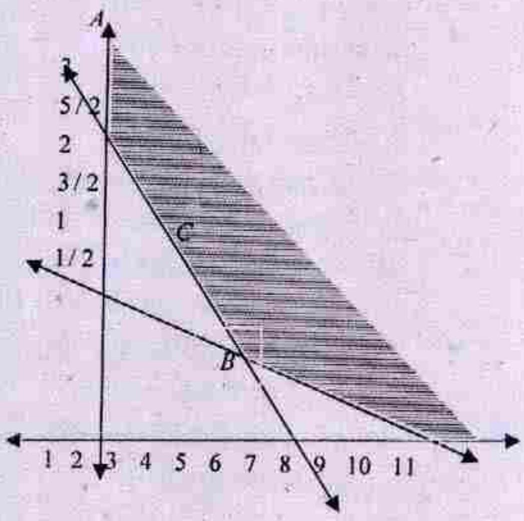
$IV \Rightarrow put\ x = 0\ then\ \ y = \frac{3}{2},\ (0, \frac{3}{2})$

$put\ y = 0\ then\ \ x = 9,\ (9,0)$

Put $(0,0)$ in eq. I & II

$I \Rightarrow 0 \geq 15 \longrightarrow F$

$II \Rightarrow 0 \geq 9 \longrightarrow F.$



$3 \times IV - III$

$$\begin{array}{l} \cancel{3x} + 18y = 27 \\ \underline{\cancel{\pm 3x} \pm 5y = \pm 15} \\ 13y = 12 \end{array} \Rightarrow put\ value\ of\ \ y\ in\ IV\ we\ get$$

$$\Rightarrow y = \frac{12}{13}\ \&\ x = \frac{45}{13}$$

So, $B = \left(\frac{45}{13}, \frac{12}{13}\right)$

Corner points of feasible region are:

$A = (0,3),\ B = \left(\frac{45}{13}, \frac{12}{13}\right),\ C = (9,0),$

$At\ A,\ f(0,3) = 3(0) + 3 = 3$

$At\ B,\ f\left(\frac{45}{13}, \frac{12}{13}\right) = 3\left(\frac{45}{13}\right) + \frac{12}{13} = 11.3$

$At\ C,\ f(9,0) = 3(9) + 0 = 27$

So f is minimum at corner $A = (0,3)$

7. **Each unit of food x costs Rs.25 and contains 2 units of protein and 4 unit of iron while each unit of food y costs Rs.30 and contains 3 units of protein and 2 unit of iron. Each animal must receive at least 12 units of protein and 16 units of iron each day. Find smallest possible cost to fed each animal.**

Sol. Function is $f(x,y)=25x+30y$

Let according to the given condition

$2x+3y\geq 12 \longrightarrow I \quad x\geq 0,\ y\geq 0$

$4x+2y\geq 16 \longrightarrow II$

Associated equation are

$l_1\ ;\ 2x+3y=12 \longrightarrow III$

$l_2\ ;4x+2y=16 \longrightarrow IV$

$III \Rightarrow put\ x=0\ then\ y=4,\ (0,4)$

$put\ y=0\ then\ x=6,\ (6,0)$

$IV \Rightarrow put\ x=0\ then\ y=8,\ (0,8)$

$put\ y=0\ then\ x=4,\ (4,0)$

Put $(0,0)$ in eq. I & II

$I \Rightarrow 0\geq 12 \longrightarrow F$

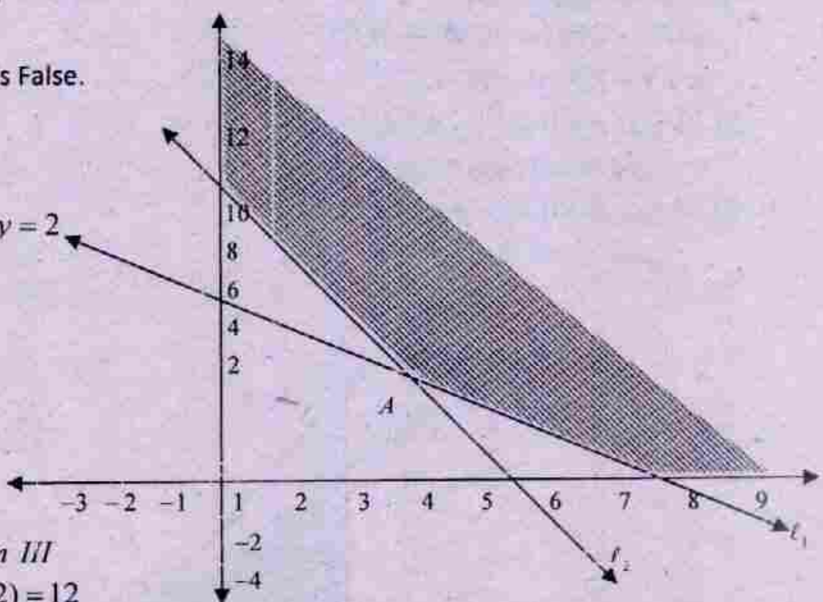
$II \Rightarrow 0\geq 16 \longrightarrow F$ is False.

$2\times III - IV$

$\cancel{4x}+6y = 24$

$\underline{-\cancel{4x}\pm 2y=-16} \Rightarrow y=2$

$4y \quad =8$



Put $\Rightarrow y=2\ in\ III$

$2x+3(2)=12$

$x=3$

We have $A(3,2)$

Corner points of feasible region are:

$A=(3,2),\ B=(6,0),\ C=(0,8)$

$At\ A,\ f(3,2)=25(3)+30(2)=135$

$At\ B,\ f(6,0)=25(6)+30(0)=150$

$At\ C,\ f(0,8)=25(0)+30(8)=240$

So the minimum cost is at the point $(3,2)$ where f is minimum. 3 units of food 'X' & 2 unit of 'Y' are used.

8. **A dealer wishes to purchase a number of fans and sewing machines. He has only Rs.5760 to invest and space of at most 20 items. A fan cost him Rs.360 and sewing machine costs Rs.240. His expectation is that he can sell a fan at a profit of Rs.22 and a sewing machine at a profit of Rs.18. How should he invest his money to maximize his profit.**

Sol. Let Fans $=x$ & Sewing Machines $=y$

$f(x,y)=22x+18y$

According to the given condition

$360x+240y\le 5760 \longrightarrow I$

$x+y\le 20 \longrightarrow II$

Associated equation are

$l_1\ ;\ 360x+240y=5760 \longrightarrow III$

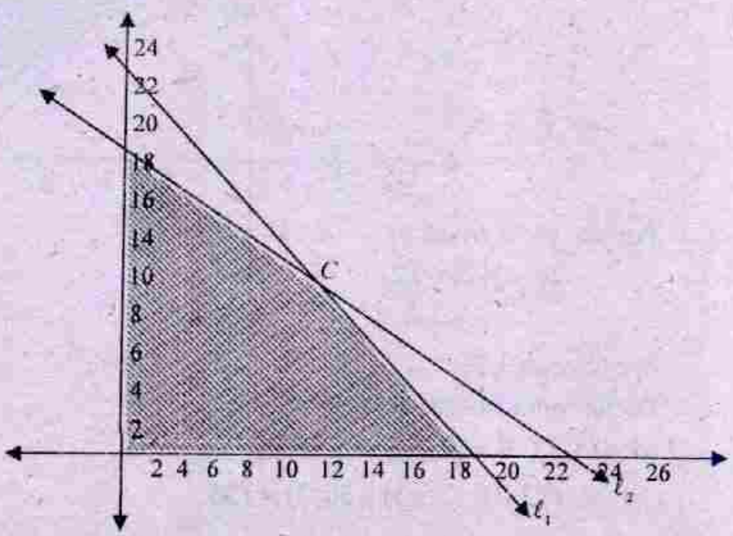
$l_2\ ;x+y=20 \longrightarrow IV$

$III \Rightarrow put\ \ x=0\ then\ y=24,\ (0,24)$

$put\ \ y=0\ then\ x=16,\ (16,0)$

$III \Rightarrow put\ \ x=0\ then\ y=20,\ (0,20)$

$put\ \ y=0\ then\ x=20,\ (20,0)$



Put $(0,0)$ in eq. I & II

$I \Rightarrow 0 \le 5760 \longrightarrow T$

$II \Rightarrow 0 \le 20 \longrightarrow T$

$360x + 240y = 5760$

$(\div)$ by $120 \Rightarrow 3x + 2y = 48 \longrightarrow V$

$$\begin{array}{r} 3x + 2y = 48 \\ -3x \pm 3y = 60 \\ \hline -y \quad = -12 \Rightarrow y = 12 \end{array}$$

Put in (IV) $x + 12 = 20$

$x = 8$ we get $C(8,12)$

Corner points of feasible region are:

$A = (0,0),\ B = (16,0),\ C = (8,12),\ D(0,20)$

$At\ A,\ f(0,0) = 22(0) + 18(0) = 0$

$At\ B,\ f(16,0) = 18(16) + 22(0) = 22(16) + 18(10)$

$At\ C,\ f(8,12) = 18(8) + 22(12) = 22(8) + 18(12)$

$At\ D,\ f(0,20) = 18(0) + 22(20) = 22(0) + 18$

So profit is maximum at corner $C = (8,2)$ where f is maximum. 8 fans and 12 sewing machines are purchased.

9. **A machine can produce product A by using 2 units of chemical and 1 unit of compound or can produce product B by using 1 unit of chemical and 2 units of compound. Only 800 units of chemical and 1000 units of compound are available. The profit per unit of A and B are Rs.30 and 20 respectively, maximize the profit function.**

Sol. Suppose x units of product A & y units of B

Thus $f(x, y) = 30x + 20y$

According to the given condition $\quad x \ge 0,\ \ y \ge 0$

$2x + y \le 800 \quad (1)$

$x + 2y \le 1000 \quad (2)$

Associated equation are

$l_1\ ;\ 2x + y = 800 \quad (3)$

$l_2\ ;\ x + 2y = 1000 \quad (4)$

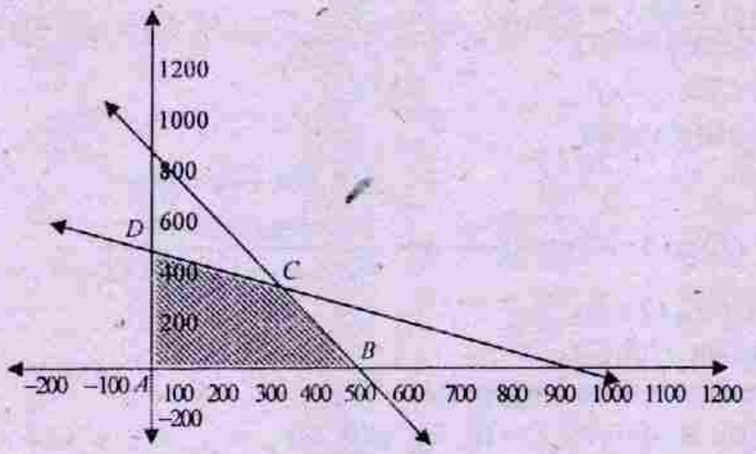
From eq. (3), we get l_1 as $(0,800), (400,0)$

From eq. (4), we get l_2 as $(0,500), (1000,0)$

Put $(0,0)$ in eq. (1) & (2)

We get $0 \le 800$ is True.

$0 \le 1000$ is True



For C solve l_1 and l_2

$$\begin{aligned} 4x + 2y &= 1600 \\ \underline{-x \pm 2y} &= -1000 \\ 3x &= 600 \end{aligned}$$

$\Rightarrow x = 200$

Put in (4)

$200 + 2y = 1000$

$y = 400$

$C(200, 400)$

Corner points of feasible region are:

$A = (0,0),\ B = (400,0),\ C = (200,400),\ D(0,500)$

$At\ A,\ f(0,0) = 30(0) + 20(0) = 0$

$At\ B,\ f(400,0) = 30(400) + 20(0) = 12000$

$At\ C,\ f(200,400) = 30(2000) + 20(400) = 6000 + 8000 = 14000$

$At\ D,\ f(0,500) = 30(0) + 20(500) = 1000$

So profit is maximum at corner $C = (200, 400)$ where f is maximum 200 of product A and 400 units of Product B are prepared.

TEST YOUR SKILLS — Marks 100

OBJECTIVE

Q.No.1 Given below are a few possible answers to each statement of which one is correct, identify the correct one. (20)

1. **An expression involving any of the symbols <, > or ≤, ≥ is called an**
   (a) Equation (b) Not inequality
   (c) Identity (d) Inequality
2. **$ax + b > 0$ is an**
   (a) Identity (b) Equation
   (c) Not inequality (d) Inequality
3. **$ax + b < 0$ is an**
   (a) Equation (b) Inequality
   (c) Not inequality (d) Identity
4. **$ax + by + c > 0$ is an**
   (a) Inequality (b) Identity
   (c) Equation (d) Not inequality
5. **$ax + by + c \le 0$ is an**
   (a) Equation (b) Identity
   (c) Inequality (d) None of these
6. **$ax + by <$ [illegible] is a linear inequality in**
   (a) [illegible]e variables (b) Two variables
   (c) One variable (d) None of these
7. **A non-vertical line divides the plane into two half planes**
   (a) Left & right (b) Upper & lower
   (c) Both (a) & (b) (d) None of these
8. **If $ax + by \le c$ then $ax + by = c$ is called an**
   (a) Identity (b) Associated equation
   (c) Both (a) & (b) (d) None of these
9. **A point (x, y) which satisfy a linear inequality in two variables x & y form its**
   (a) Domain (b) Range
   (c) Solution (d) Coefficients
10. **The graph of the inequality $ax + by < c$ is**
    (a) Circle (b) Parabola
    (c) Straight line (d) Half plane
11. **A point of a solution region where two of its boundary lines intersect, is called a**
    (a) Corner point (b) Vertex
    (c) Both (a) & (b) (d) None of these
12. **The system of linear inequality concerning the problem is named as problem**
    (a) Coefficients (b) Constraint
    (c) Decision variables (d) None of these

13. **The non-negative constraints are called:**
    (a) Coefficients (b) Solutions
    (c) Decision variables (d) None of these
14. **If the solution region is restricted to the 1st quadrant then it is called**
    (a) Feasible solution (b) Feasible region
    (c) Both (a) & (b) (d) None of these
15. **Each point of the feasible region is called**
    (a) Solution (b) Feasible solution
    (c) Both (a) & (b) (d) None of these
16. **A set consisting of all the feasible solutions of the system of linear in equalities is called a**
    (a) Solution set (b) Feasible solution set
    (c) Both (a) & (b) (d) None of these
17. **A function which is to be maximized or minimized is called an:**
    (a) Explicit function (b) Implicit function
    (c) Objective function (d) None of these
18. **The feasible solution which maximizes or minimizes the objective function is called the**
    (a) Feasible solution (b) Optimal solution
    (c) Both (a) & (b) (d) None of these
19. **The maximum & minimum values of the objective function occur in the feasible region at**
    (a) Any point (b) Corner points
    (c) Both (a) & (b) (d) None of these
20. **(2, 1) is one of the solutions of the inequality**
    (a) $x - y > 1$ (b) $3x + 5y < 6$
    (c) $2x + y < 7$ (d) None of these

SECTION I
SUBJECTIVE

Attempt any 25 (twenty five) short question from Question 2,3 and 4, at least 12 short questions from question 2, 12 short question from question 3 and 13 short question from question 4. All question carry equal marks. (25x2=50)

Q.No. 2

i. Write all the forms of a linear inequality in two variables.

ii. Define solution of a linear inequality in two variables. Give examples.

iii. Graph the inequality: (i) $x - 2y < 6$
(ii) $2x \geq -3$

iv. Define feasible solution.

v. Define objective function.

vi. What is linear programming?

vii. Define optimal solution.

viii. State theorem of linear programming.

ix. Define corner or vertex of solution region.

x. What is linear inequality?

xi. What is problem constraint?

xii. What is convex region?

Q.No. 3

i. Define feasible solution set and corner points.

ii. Minimize $z = 3x + y$ subject to the constraints
$3x + 5y \geq 15,\ x + 6y \geq 9,\ x \geq 0,\ y \geq 0$

iii. Indicate the solution set of the inequality $5x - 4y \leq 20$ in the xy-plane by shading.

iv. What is half planes

v. What is corresponding equation.

vi. Graph the inequality $x+y \geq 5$

vii. Graph the inequality $2x+1 \geq 0$

viii. Indicate the solution region of the following system of linear inequalities by shading. $x+y \leq 5$, $y-2x \leq 2$, $x \geq 0$

ix. Indicate the solution region of the following system of linear inequalities by shading. $x+y \geq 5$, $x-y \geq 1$, $y \geq 0$

x. Indicate the solution region of the following system of linear inequalities by shading. $3x+7y \leq 21$, $2x-y \geq -3$, $x \geq 0$

xi. What is fesible region ?

xii. What is decision variable.

Q.No. 4

i. Graph the solution set of each of the following linear inequality in xy-plane
$2x+y \leq 6$

ii. Graph the solution set of each of the following linear inequality in xy-plane
$5x+7 \geq 1$

iii. Graph the solution set of each of the following linear inequality in xy-plane
$5x-4y \leq 20$

iv. Graph the solution set of each of the following linear inequality in xy-plane
$3x-4 \leq 0$

v. Graph the solution set of each of the following linear inequality in xy-plane
$2x + 3y \leq 12$

vi. Graph the solution set of each of the following linear inequality in xy-plane
$2x + y \leq 10$

vii. Graph the feasible region of the following system of linear inequalities
$2x-3y \leq 6$, $2x+3y \leq 12$, $x \geq 0$, $y \geq 0$

viii. Graph the feasible region of the following system of linear inequalities

$x+y \leq 5$, $-2y+y \leq 2$, $x \geq 0$, $y \geq 0$

ix. Graph the feasible region of the following system of linear inequalities
$5x+7y \leq 35$, $x-2y \leq 4$, $x \geq 0$, $y \geq 0$

x. Find the corner points of
$x+2y \leq 14$, $3x+4y \leq 36$, $2x+y \leq 10$, $x \geq 0$, $y \geq 0$

xi. Find the corner points of
$2x-3y \leq 6$, $2x+3y \leq 12$, $x \geq 0$, $y \geq 0$

xii. Find the corner points of
$x+3y \leq 15$, $2x+y \leq 12$, $4x+3y \leq 24$, $x \geq 0$, $y \geq 0$

xiii. Find the minimum values of f and ϕ defined as;
$F(x,y) = 4x+5y$, $\phi(x,y)=4x+6y$

SECTION II

Attempt any 3 (three) questions. **(3x10=30)**

Q.No.5

(a) Maximize $f(x,y) = x+3y$, subject to the constraints , $2x+5y \leq 30$, $5x+4y \leq 20$, $x \geq 0$, $y \geq 0$

(b) Graph the feasible region of the following system of linear inequalities and Find the corner points of $x+y \leq 5$, $-2x+y \geq 2$, $x \geq 0$

Q.No.6

(a) Graph the solution region of the system of linear inequalities and find the corner points $5x+7y \leq 35$, $-x+3y \leq 3$, $x \geq 0$

(b) Maximize $f(x,y) = 2x+3y$, subject to the constraints , $2x+y \leq 8$, $x+2y \leq 14$, $x \geq 0$, $y \geq 0$

Q.No.7

(a) Find the minimum and maximum values of the function defined as;
$f(x,y) = 2x+3y$, subject to the constraints, $x-y \leq 2$, $x+y \leq 4$, $2x-y \leq 6$, $x \geq 0$

(b) Graph the solution region of the system of linear inequalities and find the corner points $3x+2y \geq 6$, $x+3y \leq 6$, $y \geq 0$

Q.No.8

(a) Maximize $z= 2x+3y$, subject to the constraints , $3x+4y \leq 12$, $2x+y \leq 4$, $4x-y \leq 4$, $x \geq 0$, $y \geq 0$

(b) Graph the feasible region of the following system of linear inequalities and Find the corner points of $2x+y \leq 20$, $8x+15y \leq 120$, $x+y \leq 11$, $x \geq 0$, $y \geq 0$

Q.No.9

(a) Find the minimum and maximum values of f and ϕ defined as;
$F(x,y) = 4x+5y$, $\phi(x,y)=4x+6y$

(b) Maximize $z= 2x+y$, subject to the constraints , $x+y \geq 3$, $7x+5y \leq 35$, $x \geq 0$, $y \geq 0$

Previous Board Questions

1. What is linear inequality? (Lhr – 2008)
2. What is feasible solution? (Grw – 2008)
3. What is convex region? (Lhr – 2008)
4. What is linear programming? (Grw – 2007, Lhr – 2008)
5. Define the associated equation of an inequality? (Lhr – 2008)
6. What is problem constraint?
7. What is an object function? (Lhr – 2007,08)
8. Shade the feasible region of inequality $5x - 4y \leq 20$. (Lahore – 2010) Group – I
9. Define feasible solution set and corner points. (Lahore – 2010) Group – I
10. Graph the solution set of the inequality: $2x + 1 \geq 0$ in xy – plane. (Lahore – 2010) Group – II
11. Minimize $z = 3x + y$ subject to the constraints $3x + 5y \geq 15$, $x + 6y \geq 9$, $x \geq 0$, $y \geq 0$ (Lahore – 2010) Group – II
12. State the theorem of linear programming. (Grw – 2010)
13. Indicate the solution set of the inequality $5x - 4y \leq 20$ in the xy-plane by shading. (Gujranwala – 2010)

Conic Sections

6

Definitions

1. Nappes:

Two parts of cone are called nappes.

2. Vertex:

Meeting point of two parts of cone is called vertex or apex.

3. Circle: **(Sargodha 2011)**

If cone is cut by a plane perpendicular to the axis of cone, then resulting section is circle.

We can also define circle as:

A locus of a pt which remains at a fixed distance from a certain point. The point is called centre of the circle and fixed distance is called radius of the circle.

4. Ellipse:

If the cone is cut by a plane and the cutting plane is slightly tilted and cuts only one nappe of cone then resulting section is an ellipse.

5. Hyperbola: **(Sargodha 2011)**

If the cone is cut by a plane and the cutting plane is parallel to the axis of cone and intersect both its nappes, then curve of intersection is Hyperbola.

6. Point circle: **(Sargodha 2008)**

If the plane passes through vertex of cone, the intersection is a single point or point circle or if $r = 0$.

7. Parametric equations: **(Sargodha 2010)**

$x = r\text{Cos}\theta$, $y = r\text{Sin}\theta$ are parametric equations of circle.

8. Tangent:

A line that touch the curve without cutting through it.

9. Normal:

A line perpendicular to Tangent is called normal.

10. Tangential distance:

Length of tangent is called tangential distance and its formula is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

11. Chord of contact:

The line joining points of contact of chord.

12. Conic section:

$\frac{|PF|}{|PM|} = e$ (+*ve* constant) is called conic section if $e < 1$ then Ellipse, if $e = 1$ then parabola, if $e > 1$ then hyperbola.

13. Eccentricity:

The number e is called eccentricity.

14. Equation of parabola

$y^2 = 4ax$ is called **equation of parabola**, $F(a, 0)$ is **focus**, A line at equal distance from vertex opposite to focus is **directrix**. The point where axis meet the parabola is called **vertex** (0, 0). The line through focus and perpendicular to directrix is called **axis of parabola** and focal chard perpendicular to axis is called **latusrectum.**

15. Vertices

For ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$ points A, A' are called **vertices** and $AA' = 2a$ is called **Major axis**, B, B' are **covertices** and $BB' = 2b$ is **minor axis.**

16. Transverse or Focal

The line segment $AA' = 2a$ is called **Transverse or Focal** of Hyperbola and BB' (line segment) is called **conjugate *ax*.**

17. Central Conics:

Ellipse and hyperbola are called central conics.

18. Degenerate Conic:

Under certain condition equatic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ not represent as conic. In such case this is called degenerate conic.

19. Parabola: **(Sargodha 2008, 11)**

If the intersecting plane is parallel to a generator of the cone but cuts one nape only is called parabola.

Important Formulae

1. Equation of circle in standard form $(x - h)^2 + (y - k)^2 = r^2$ (Centre (h, k), radius-r) **(Sargodha 2011)**
2. If centre is at origin then equation of circle $x^2 + y^2 = r^2$ **(Sargodha 2008)**
3. General equation of circle $x^2 + y^2 + 2gx + 2fy + c = 0$ where centre $= (-g, -f)$ and radius $= \sqrt{g^2 + f^2 - c}$
4. For circle equation of tangent $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ at (x_1, y_1)
5. For circle equation of Normal at (x_1, y_1) is $(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$
6. Equation of parabola $y^2 = 4ax$ and at (x_1, y_1) is $y^2 = 4ax_1$ whose vertex is at origin.
7. Standard equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If $a > b$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ if $a > b$ **(Sgd 2008)**
8. Standard equation of Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x, y) $\boxed{\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1}$
9. $y = mx + \frac{a}{m}$ is tangent to $y^2 = 4ax$
10. $y = mx \pm \sqrt{a^2m^2 + b^2}$ Tangent of Ellipse
11. $y = mx \pm \sqrt{a^2m^2 - b^2}$ Tangent of hyperbola.

Exercise 6.1

1. (a) Centre at $(5,-2)$, $r=4$

$$(x-h)^2+(y-k)^2=r^2$$
$$(x-5)^2+(y-(-2)^2=(4)^2$$
$$x^2+10x+25+(y+2)^2=16$$
$$x^2+10x+25+y^2+4y+4-16=0$$
$$x^2+y^2-10x+4y+13=0$$

(b) Centre $(\sqrt{2},-3\sqrt{3})$, $r=2\sqrt{2}$ **(Sargodha 2011)**

$$(x-h)^2+(y-k)^2=r^2$$
$$(x-\sqrt{2})^2+(y-(-3\sqrt{3}))^2=(2\sqrt{2})^2$$
$$x^2-2\sqrt{2x}+2+(y+3\sqrt{3})^2=4\times 2$$
$$x^2-2\sqrt{2x}+2+y^2+6\sqrt{3y}+(9\times 3)=8$$
$$x^2-2\sqrt{2x}+2+y^2+6\sqrt{3y}+27-8=0$$
$$x^2+y^2-2\sqrt{2x}+6\sqrt{3y}+21=0$$

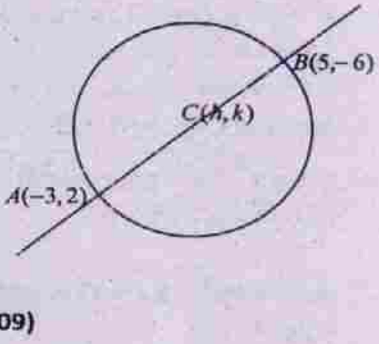
(c) Diameter at $(-3, 2)$, $(5,-6)$

Centre is (h, k) so

$$(h, k)=\left(\frac{-3+5}{2},\frac{-6+2}{2}\right)$$
$$(h, k)=(1,-2)$$

Radius $=r=|AC|=\sqrt{(h-(-3))^2+(k-2)^2}$

$$r=\sqrt{(h+3)^2+(k-2)^2}$$
$$r=\sqrt{(1+3)^2+(-2-2)^2}=\sqrt{16+16}=\sqrt{32}$$
$$(x-h)^2+(y-k)^2=r^2$$
$$(x-1)^2+(y-(-2))^2=(\sqrt{32})^2$$
$$(x-1)^2+(y+2)^2=32$$
$$x^2-2x+1+y^2+4y+4-32=0$$
$$x^2+y^2-2x+4y-27=0$$



2. (a) Find centre and radius **(Lhr 2010, Sgd 2009)**

$$x^2+y^2+12x-10y=0$$

Compare with

$$x^2+y^2+2gx+2fy+c=0$$
$$2g=12 \Rightarrow g=6 \;\&\; c=0,\ 2f=-10 \Rightarrow f=-5$$

Centre is $(-g,-f)=(-6, 5)$

Radius $= r = \sqrt{g^2 + f^2 - c}$

$r = \sqrt{(6)^2 + (-5)^2 - 0} = \sqrt{36 + 25} = \sqrt{61}$

(b) **Find centre and radius** **(Sgd 2008, 11)**

$5x^2 + 5y^2 + 14x + 12y - 10 = 0$

$\div$ *by* 5

$x^2 + y^2 + \frac{14}{5}x + \frac{12}{5}y - 2 = 0$

Compare with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$2g = \frac{14}{5} \Rightarrow g = \frac{7}{5}, c = -2$

$2f = \frac{12}{5} \Rightarrow f = \frac{6}{5}$

Centre is $(-g, -f) = \left(-\frac{7}{5}, \frac{-6}{5}\right)$

$r = \sqrt{g^2 + f^2 - c} = \sqrt{\left(\frac{-7}{5}\right)^2 + \left(\frac{-6}{5}\right)^2 - (-2)}$

$r = \sqrt{\frac{49}{25} + \frac{36}{25} + 2} = \sqrt{\frac{49 + 36 + 50}{25}}$

$r = \sqrt{\frac{135}{25}} = \sqrt{\frac{27}{5}}$

(c) $\mathbf{x^2 + y^2 - 6x + 4y + 13 = 0}$ **(Sgd 2011)**

Compare with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$2g = -6 \Rightarrow g = -3$

$2f = 4 \Rightarrow f = 2, \; c = 13$

Centre is $(-g, -f) = (3, -2)$

$r = \sqrt{g^2 + f^2 - c} = \sqrt{(-3)^2 + (2)^2 - 13}$

$r = \sqrt{9 + 4 - 13} = 0$

(d) $\mathbf{4x^2 + 4y^2 - 8x + 12y - 25 = 0}$ **(Sgd 2007, 10)**

$\div$ *by* 4

$x^2 + y^2 - 2x + 3y - \frac{25}{4} = 0$

Compare with

$x^2 + y^2 + 2gx + 2fy + c = 0$

$2g = -2 \Rightarrow g = -1$

$2f = 3 \Rightarrow f = \frac{3}{2},\ c = \frac{-25}{4}$

Centre is $(-g,-f) = \left(1, \frac{-3}{2}\right)$

$r = \sqrt{g^2 + f^2 - c} = \sqrt{(-1)^2 + \left(\frac{3}{2}\right)^2 + \frac{25}{4}}$

$r = \sqrt{1 + \frac{9}{4} + \frac{25}{4}} = \sqrt{\frac{4+9+25}{4}}$

$r = \sqrt{\frac{38}{4}} = \sqrt{\frac{19}{2}}$

3. (a) **Write equation of circle passing through** **(Sargodha 2008)**
$A(4,5)$, $B(-4,-3)$, $C(8,-3)$

Sol: is clear from figure that

$|OA|^2 = |OB|^2$

$(h-4)^2 + (k-5)^2 = (h+4)^2 + (k+3)^2$

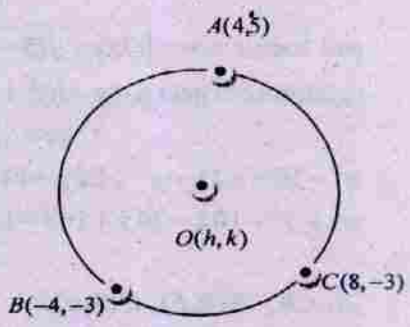
Or

$h^2 - 8h + 16 + k^2 - 10k + 25 = h^2 + 8h + 16 + k^2 + 6k + 9$

$-8h - 10k + 16 + 25 = 8h + 6k + 25$

$8h + 6k - 16 = 0$

$\div\ by\ 16 \Rightarrow h + k - 1 = 0$



Also $|OB|^2 = |OC|^2$

$(h+4)^2 + (k+3)^2 = (h-8)^2 + (k+3)^2$

$h^2 + 8h + 16 = h^2 - 16h + 64$

$8h + 16h = 64 - 16$

$24h = 48 \Rightarrow h = 2$

Put in I $\quad h + k - 1 = 0$

$2 + k - 1 = 0$

$\Rightarrow k + 1 = 0 \Rightarrow k = -1$

$r = |OA| = \sqrt{(h-4)^2 + (k-5)^2}$

$r = \sqrt{(2-4)^2 + (-1-5)^2}$

$r = \sqrt{4+36} = \sqrt{40}$

Now equation of circle is

$(x-h)^2 + (y-k)^2 = r^2$

$(x-2)^2 + (y+1)^2 = (\sqrt{40})^2$

$x^2 - 4x + 4 + y^2 + 2y + 1 - 40 = 0$

$x^2 + y^2 - 4x + 2y - 35 = 0$

(b) $A(-7,7),\ B(5,-1),\ C(10,0)$

It is clear from figure that

$|OA|^2 = |OB|^2$

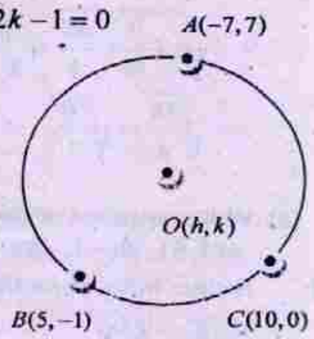
$(h+7)^2 + (k-7)^2 = (h-5)^2 + (k+1)^2$

$h^2 + 14h + 49 + k^2 - 14k + 49 = h^2 - 10h + 25 + k^2 + 2k + 1$

Or

$h^2 + 14h + 49 + k^2 - 14k + 49 - h^2 + 10h - 25 - k^2 - 2k - 1 = 0$

$24h - 16k + 72 = 0 \quad or \quad 6h - 4k + 18 = 0$



Or $3h - 2k + 9 = 0 \quad I \quad Also \ |OB|^2 = |OC|^2$

$(h-5)^2 + (k+1)^2 = (h-10)^2 + (k-0)^2$

$h^2 - 10h + 25 + k^2 + 2k + 1 = h^2 - 20h + 100 + k^2$

Or $h^2 - 10h + 25 + k^2 + 2k + 1 - h^2 + 20h - 100 - k^2 = 0$

$10h + 2k - 74 = 0 \quad or \quad 5h + k - 37 = 0 \quad II$

$k = 37 - 5h \quad III$

Put III in I

$3h - 2(37 - 5h) + 9 = 0$

$3h - 74 + 10h + 9 = 0 \Rightarrow 13h - 65 = 0 \Rightarrow 13h = 65 \Rightarrow h = 5$

Put value of $h = 5$ in III $\quad k = 37 - 5(5)$

$k = 37 - 25 \Rightarrow k = 12$

Centre $= O(h,k) = (5,12)$

and radius $= r = |OE| = \sqrt{(5-10)^2 + (12-0)^2} = \sqrt{25+144} = 13$

Equation of circle is $(x-h)^2 + (y-k)^2 = r^2$

$(x-5)^2 + (y-12)^2 = (13)^2 = 169$

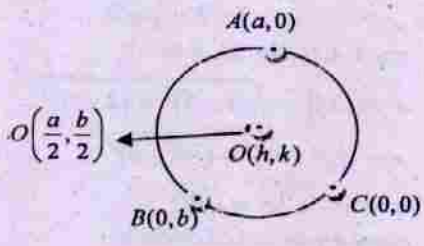
$x^2 - 10x + 25 + y^2 - 24y + 144 - 169 = 0$

$x^2 + y^2 - 10x - 24y + 169 - 169 = 0 \Rightarrow x^2 + y^2 - 10x - 24y = 0$

c) $A(a,0),\ B(0,b),\ C(0,0)$

It is clear from figure that

$|OA|^2 = |OC|^2$



$(h-a)^2 + (k-0)^2 = (h-0)^2 + (k-0)^2$

$h^2 - 2ah + a^2 + k^2 = h^2 + k^2 = h^2 - 2ah + a^2 + k^2 - h^2 - k^2 = 0 \Rightarrow 2ah = a^2 \Rightarrow h = \frac{9}{2}$

Also $|OB|^2 = |OC|^2$

$(h-0)^2 + (k-b)^2 = (h-0)^2 + (k-0)^2 \Rightarrow h^2 + k^2 - 2bk + b^2 = h^2 + k^2$

$$h^2+k^2-2bk+b^2-h^2-k^2=0 \Rightarrow 2bk=b^2=k=\frac{b}{2}$$

$$r=|OC|=\sqrt{\left(\frac{a}{2}-0\right)^2+\left(\frac{b}{2}-0\right)^2}=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\sqrt{\frac{a^2+b^2}{4}}=\frac{\sqrt{a^2+b^2}}{2}$$

Equation of circle is

$$(x-h)^2=(y-k)^2=r^2 \Rightarrow \left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\left(\frac{\sqrt{a^2+b^2}}{2}\right)^2$$

$$x^2+\frac{a^2}{4}-2\frac{a}{2}x+y^2+\frac{b}{4}-2\frac{b}{2}y=\frac{a^2+b^2}{4}$$

$$x^2+y^2-ax-by+\frac{a^2+b^2}{4}-\frac{a^2+b^2}{4}=0 \Rightarrow x^2+y^2-ax-by=0$$

d) $A(5,6),\ B(-3,2),\ C(3,-4)$

It is clear from figure that

$$|OA|^2=|OB|^2$$

$$(h-5)^2+(k-6)^2=(h+3)^2+(k-2)^2$$

$$h^2-10h+25+k^2-12k+36=h^2+6h+9+k^2-4k+4$$

$$or\ h^2-10x+k^2-12k+61-h^2-k^2-6h+4k-13=0$$

$$-16h-8k+48=0 \div by\ 8$$

$$-2h-k+6=0 \Rightarrow k=-2h+6 \longrightarrow I$$

$$|OB|^2=|OC|^2$$

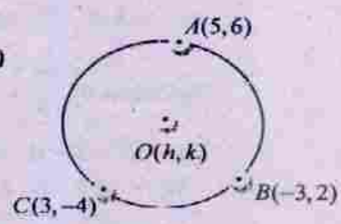
$$(h+3)^2+(k-2)^2=(h-3)^2+(k+4)^2$$

$$h^2+6h+9+k^2-4k+4=h^2-6h+9+k^2+8k+16$$

$$h^2+6h+k^2-4k+13=h^2-6h+k^2+8k+25$$

$$h^2+6h+k^2-4k+13-h^2+6h-k^2-8k-25=0$$

$$12h-12k-12=0 \Rightarrow h-k-1=0 \longrightarrow II$$



Put I in II

$$h-(-2h+6)-1=0 \Rightarrow h+2h-6-1=0$$

$$3h-7=0 \Rightarrow h=\frac{7}{3}$$

Put value of h in $I \Rightarrow k=-2\left(\frac{7}{3}\right)+6$

$$k=\frac{4}{3}$$

$$r=|OB|=\sqrt{\left(\frac{7}{3}+3\right)^2+\left(\frac{4}{3}-2\right)^2}$$

$$r=\sqrt{\frac{256+4}{9}}=\frac{\sqrt{260}}{3}$$

Equation of circle is

$(x-h)^2+(y-k)^2=r^2 \Rightarrow \left(x-\frac{7}{3}\right)^2+\left(y-\frac{4}{3}\right)^2=\left(\frac{\sqrt{260}}{3}\right)^2$

$x^2-\frac{14}{3}x+\frac{49}{9}+y^2-\frac{8}{3}y+\frac{16}{9}=\frac{260}{9}$

$x^2+y^2-\frac{14}{3}x-\frac{8}{3}y-\frac{195}{9}=0 \quad or \quad x^2+y^2-\frac{14}{3}x-\frac{8}{3}y-\frac{65}{3}=0$

Multiply by 3

$3x^2+3y^2-14x-8y-65=0$

$3(x^2+y^2)-14x-8y-65=0$

4. (a) Find equation of circle passing through $A(3,-1)$, $B(0,1)$, and Centre at $4x-3y-3=0$

Since $C(h,k)$ lies on line so

$4h-3k-3=0 \longrightarrow I$

Also $|AC|^2=|BC|^2$

$(h-3)^2+(k+1)^2=(h-0)^2+(k-1)^2$

$h^2-6h+9+k^2+2k+1=h^2+k^2-2k+1$

$-6h+2k+10=-2k+1$

$-6h+4k+9=0 \longrightarrow II$

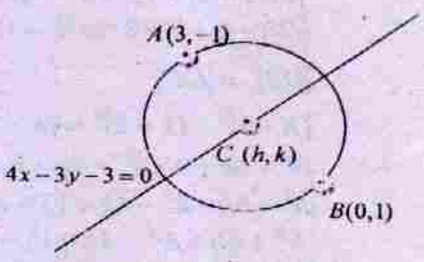
Multiply I by 4 and II x 3 then add

$16h-12k-12=0$

$-18h+12k+27=0$

$-2h+15=0$

$\Rightarrow -2h=-15 \Rightarrow h=\frac{15}{2}$



I become $4\left(\frac{15}{2}\right)-3k-3=0$

$30-3k-3=0 \Rightarrow -3k+27=0$

$-3k=-27 \Rightarrow k=9$

$r=|AC|=\sqrt{(h-3)^2+(k+1)^2}$

$r=\sqrt{\left(\frac{15}{2}-3\right)^2+(9+1)^2}$

$r=\sqrt{\left(\frac{9}{2}\right)^2+(10)^2}=\sqrt{\frac{81}{4}+100}, r=\sqrt{\frac{481}{4}}$

$(x-h)^2+(y-x)^2=r^2$

$\left(x-\frac{15}{2}\right)^2+(y-9)^2=\left(\sqrt{\frac{481}{4}}\right)^2$

$$x^2-15x+\frac{225}{4}+y^2-18y+81=\frac{481}{4}$$

$$x^2+y^2-15x-18y+\frac{225-481+324}{4}=0$$

$$x^2+y^2-15x-18y+\frac{68}{4}=0$$

$$\Rightarrow\ x^2+y^2-15x-18y+17=0$$

(b) **Circle passing through $A(-3,1)$ with radius 2 and centre at $2x-3y+3=0$**

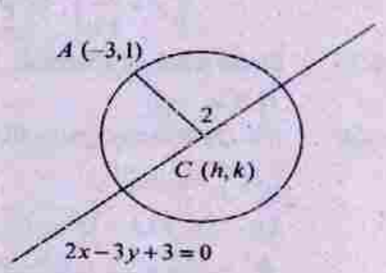
Centre $C(h,k)$

Lies on $2x-3y+3=0$

So $2h-3k+3=0 \longrightarrow I$

Also $|AC|=2$

$\Rightarrow\ |AC|^2=4$



$(h+3)^2+(k-1)^2=4$

$h^2+6h+9+k^2-2k+1-4=0$

$h^2+k^2+6h-2k+6=0 \longrightarrow II$

From I $2h=3k-3\ \Rightarrow\ h=\frac{3k-3}{2}$

Put in II

$$\left(\frac{3k-3}{2}\right)^2+k^2+6\left(\frac{3k-3}{2}\right)-2k+6=0$$

$$\frac{9k^2-18k+9}{4}+k^2+9k-9-2k+6=0$$

'×' by 4

$$9k^2-18k+9+4k^2+36k-36-8k+24=0$$

$$13k^2+10k-3=0$$

$$13k^2+13k-3k-3=0$$

$$13k(k+1)-3(k+1)=0$$

$$(k+1)(13k-3)=0$$

$$k+1=0\quad or\quad 13k-3=0$$

$$k=-1\quad or\quad k=\frac{3}{13}$$

When $k=-1$ then $h=\frac{3(-1)-3}{2}=-3$

When $k=\frac{3}{13}$ then $h=\frac{3\left(\frac{3}{13}\right)-3}{2}$

$$h=\frac{-30}{13}\times\frac{1}{2}\ \Rightarrow\ h=\frac{-15}{13}$$

we have $c_1(-3,-1)$

$$c_2\left(\frac{-15}{13},\frac{3}{13}\right)$$

Equation I $(c_1(-3,-1),\ r=2)$

$$(x-h)^2+(y-k)^2=r^2$$
$$(x+3)^2+(y+1)^2=(2)^2$$
$$x^2+6x+9+y^2+2y+1=4$$
$$x^2+y^2+6x+2y+6=0$$

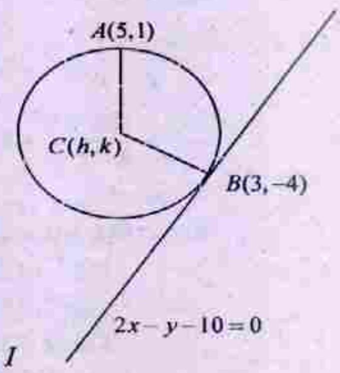
Equation II at $c_2\left(\frac{-15}{13},\frac{3}{13}\right), r=2$

$$\left(x+\frac{15}{13}\right)^2+\left(y-\frac{3}{13}\right)^2=(2)^2$$

(c) **Circle passing through A (5, 1) and tangent to the line $2x-y-10=0$ at B (3, −4)** **(Sargodha 2010)**

Sol: It is clear from figure that

$$|AC|^2=|BC|^2$$
$$(h-5)^2+(k-1)^2=(h-3)^2+(k+4)^2$$
$$h^2-10h+25+k^2-2k+1=h^2-6h+9+k^2+8k+16$$
$$-10h-2k+26=-6h+8k+25$$
$$or\quad -6h+8k+25+10h+2k-26=0$$
$$4h+10k-1=0\longrightarrow I$$



Slope of $2x-y-10=\frac{-a}{b}=\frac{-2}{-1}=2$

Slope of $BC=\frac{k+4}{h-3}$

Because both are $\perp$ are so

$$(2)\left(\frac{k+4}{h-3}\right)=-1 \qquad 2k+8=-h+3$$
$$or\quad h+2k+5=0 \quad II \quad 5\times II-I$$

$$5h+10k+25=0$$
$$\underline{\ _-4h\pm10k\mp1=0\ }$$
$$h+26=0$$
$$\Rightarrow h=-26$$

Put in II

$$-26+2k+5=0$$
$$2k-21=0\Rightarrow k=\frac{21}{2}$$

Centre $\left(-26,\frac{21}{2}\right)$

$$r=|AC|=\sqrt{(h-5)^2+(k-1)^2}$$

$$=\sqrt{(-26-5)^2+\left(\frac{21}{2}-1\right)^2}$$

$$=\sqrt{(31)^2+\left(\frac{19}{2}\right)^2}=\sqrt{961+\frac{361}{4}}=\sqrt{\frac{3844+361}{2}}$$

$$=\sqrt{\frac{4205}{2}}$$

Equation of circle is

$$(x-h)^2+(y-k)^2=r^2$$

$$(x+26)^2+\left(y-\frac{21}{2}\right)^2=\left(\frac{\sqrt{4205}}{2}\right)^2$$

$$x^2+52x+676+y^2-21y+\frac{441}{4}-\frac{4205}{4}=0$$

$$x^2+y^2+52x-21y+676+\frac{441}{4}-\frac{4205}{4}=0$$

$$x^2+y^2+52x-21y+\left(-\frac{1016}{4}\right)=0$$

$$x^2+y^2+52x-21y-265=0$$

(d) **Circle passing through $A\ (1,4)$, $B(-1,8)$ and tangent to $x+3y-3=0$**

It is clear from figure that

$$|AC|^2=|BC|^2$$

$$(h-1)^2+(k-4)^2=(h+1)^2+(k-8)^2$$

$$\cancel{h^2}-2h+\cancel{1}-\cancel{k^2}-8k+16=\cancel{h^2}+2h+\cancel{1}+\cancel{k^2}-16k+64$$

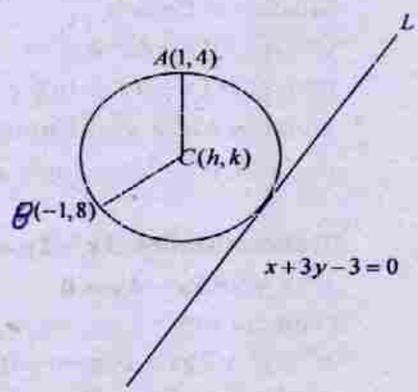
$$-2h-8k+16=2h-16k+64$$

$$2h-16k+64+2h+8k-16=0$$

$$4h-8k+48=0$$

$$or\quad h-2k+12=0$$

$$\Rightarrow\quad h=2k-12\longrightarrow I$$



Now $|AC|=|CL|$ where L is tangent line

$$\sqrt{(h-1)^2+(k-4)^2}=\frac{|h+3k-31|}{\sqrt{(1)^2+(3)^2}}$$

$$\sqrt{h^2-2h+1+k^2-8k+16}=\frac{|h+3k-3|}{\sqrt{10}}$$

Squaring both sides urgent

$$h^2-2h+k^2-8k+17=\frac{h^2+9k^2+9+6hk-18k-6h}{10}$$

or $10h^2+10k^2-20h-80k+170=h^2+9k^2+6hk-18k-6h+9$

or $10h^2+10k^2-20h-80k+170-h^2-9k^2-6hk+18k+6h-9=0$

$9h^2+k^2-14h-62k+161-6hk=0$

Put I in II

$9(2k-12)^2+k^2-14(2k-12)-62k+161-6k(2k-12)=0$

$36k^2-432k+1296+k^2-28k+168-62k+161-12k^2+72k=0$

$25k^2-450k+1625=0$

$\div$ by 25

$k^2-18k+65=0$

or $k^2-13k-5k+65=0$

$k(k-13)-5(k-13)=0$

or $(k-13)(k-5)=0$

$\Rightarrow k-13=0$ or $k-5=0$ $(Use-I)$

When $k=13$ then $h=2\ (13)-12=14$

When $k=5$ then $h=2\ (5)-12=-2$

$C_1\ (-2,5),\ C_2\ (14,13)\ r_1=|AC_1|=\sqrt{(1+2)^2+(4-5)^2}=\sqrt{10}$

$r_2=|AC_2|=\sqrt{(1-14)^2+(4-13)^2}=\sqrt{250}$

Equation of Circle I is

$(x+2)^2+(y-5)^2=(\sqrt{10})^2$

$(x+2)^2+(y-5)^2=10$

Equation of Circle II is

$(x-14)^2+(y-13)^2=(\sqrt{250})^2$

$(x-14)^2+(y-13)^2=250$

5. **Write equation of circle lies in II quadrant and tangent to the both axis with radius a because Circle is in 2nd quadrant and have radius a so $C(-a,a),\ r=a$**

Equation of Circle is

$(x-h)^2+(y-k)^2=r^2$

$(x+a)^2+(y-a)^2=(a)^2$

$x^2+2ax+a^2+y^2-2ay+a^2=a^2$

or $x^2+y^2+2ax-2ay+a^2=0$

6. **(i) Show that line $3x-2y=0$ is tangent to circle $x^2+y^2+6x-4y=0$**

$x^2+y^2+6x-4y=0$

Compare with

$x^2+y^2+2gx+2fy+c=0$

$g=3,\ f=-2,\ c=0$

$C(-g,-f)=(-3,2)$

$r=\sqrt{g^2+f^2-c}=\sqrt{(-3)^2+(2)^2-0}$

$r=\sqrt{9+4}=\sqrt{13}$

Distance of centre from line

$$=\frac{|3(-3)-2(2)+0|}{\sqrt{(3)^2+(-2)^2}}$$

$$=\frac{|-9-4|}{\sqrt{13}}=\frac{13}{\sqrt{13}}=\sqrt{13}=r$$

Hence line is tangent

Now for line

(ii) $2x+3y-13=0$

Compare $x^2+y^2+6x-4y=0$ with $x^2+y^2+2gx+2fy+C$

We get $g=3,\ f=-2,\ C=0$

$C(-3,2),\ r=\sqrt{g^2+f^2-C}=\sqrt{9+4-0}$

$=\sqrt{13}$

$C(-3,2),\ r=\sqrt{13}$

Distance of line $(2x+3y-13=0)$

From Circle is $=\dfrac{|2(-3)+3(2)-13|}{\sqrt{(2)^2+(3)^2}}$

$$=\frac{|-6+6-13|}{\sqrt{4+9}}=\frac{13}{\sqrt{13}}=\sqrt{13}=r$$

Hence $2x+3y-13=0$ is tangent to Circle.

7. Show that $x^2+y^2+2x-2y-7=0$ and $x^2+y^2-6x+4y+9=0$ touch externally.

$c_1;\ x^2+y^2+2x-2y-7=0$

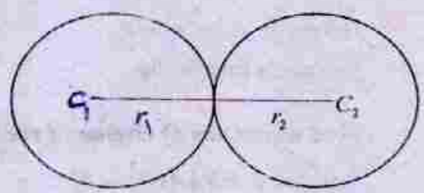
Compare with

$x^2+y^2+2gx+2fy+c=0$

$g=1,\ f=-1,\ c=-7$

$c_1(-1,1),\ r_1=\sqrt{g^2+f^2-c}=\sqrt{1+1+7}$

$r_1=3$



For Circle II

$x^2+y^2-6x+4y+9=0$

Compare with

$x^2+y^2+2gx+2fy+c=0$

$g=-3,\ f=2,\ c=9$

$c_2=(3,-2),\ r=\sqrt{g^2+f^2-c}$

$r_2=\sqrt{9+4-9}=2$

$|c_1c_2|=\sqrt{(-1-3)^2+(1+2)^2}$

$=\sqrt{25}=5$
$=3+2$
$=r_1+r_2$

Hence touch externally

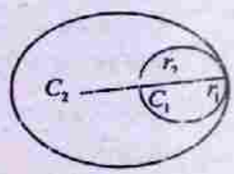
8. Show that circles (Sargodha 2007)

$x^2+y^2+2x-8=0 \longrightarrow I$

$x^2+y^2-6x+6y-46=0 \longrightarrow II$

Touch internally

Circle I Compare with



$x^2+y^2+2gx+2fy+c=0$ we get

$g=1,\ f=0,\ c=-8$

$r_1=\sqrt{g^2+f^2-c}=\sqrt{1+0-(-8)}=3=\sqrt{1+8}=\sqrt{9}$

$C_1(-g,-f)=(-1,0)$

Circle II

$x^2+y^2-6x+6y-46=0$

Compare with

$x^2+y^2+2gx+2fy+c=0$

$g=-3,\ f=3,\ c=-46$

$C_2(-g,-f)=(3,-3)$

$r_2=\sqrt{g^2+f^2-c}=\sqrt{9+9+46}$

$r_2=\sqrt{64}=8$

$|c_2c_1|=\sqrt{(3+1)^2+(-3-0)^2}=\sqrt{25}$
$=5$

And $r_2-r_1=8-3=5$

Hence $|c_1c_2|=r_2-r_1$

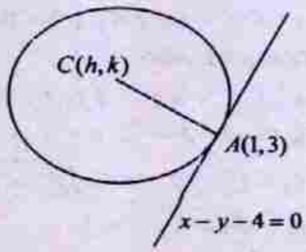
So touch internally.

9. Find equation of circles of radius 2 and tangent to line $x-y-4=0$ at $(1,-3)$

$|AC|=2 \Rightarrow |AC|^2=4$

$(h-1)^2+(k+3)^2=4$

$h^2-2h+1+k^2+6k+9=4$



$h^2-2h+1+k^2+6k+9-4=0$

$h^2+k^2-2h+6k+6=0 \longrightarrow I$

Slope of $AC=\dfrac{k+3}{h-1}$

Slope of $x-y-4=0=\dfrac{-a}{b}=\dfrac{-1}{-1}=1$

$m_1 m_2 = -1$ For Perpendicular

$$\left(\frac{k+3}{h-1}\right)(1) = -1$$

$$k+3 = -h+1 \Rightarrow -h-k-2=0$$

$$or \quad h = -k-2 \qquad II$$

Put II in I

$$(-k-2)^2 + k^2 - 2(-k-2) + 6k + 6 = 0$$

$$k^2 + 4k + 4 + k^2 + 2k + 4 + 6k + 6 = 0$$

$$2k^2 + 12k + 14 = 0$$

$\div$ *by* 2

$$k^2 + 6k + 7 = 0$$

$$k = \frac{-6 \pm \sqrt{36-28}}{2} = \frac{-6 \pm \sqrt{8}}{2}$$

$$k = \frac{-6 \pm 2\sqrt{2}}{2} = \frac{\cancel{2}(-3 \pm \sqrt{2})}{\cancel{2}}$$

$$k = -3 + \sqrt{2}, \ \ k = -3 - \sqrt{2}$$

When $k = -3 + \sqrt{2}$ *then* $h = -(-3 + \sqrt{2}) - 2$

$$h = 3 - \sqrt{2} - 2 = 1 - \sqrt{2}$$

When $k = -3 - \sqrt{2}$ *then* $h = -(-3 - \sqrt{2}) - 2$

$$h = 3 + \sqrt{2} - 2 = 1 + \sqrt{2}$$

$$C_1(1+\sqrt{2}, -3-\sqrt{2}), r = 2$$

$$C_2(1-\sqrt{2}, -3+\sqrt{2}), r = 2$$

Equation of circle I & II are

$$(x-1-\sqrt{2})^2 + (y+3+\sqrt{2})^2 = (2)^2$$

$$(x-1+\sqrt{2})^2 + (y+3-\sqrt{2})^2 = (2)^2$$

Exercise 6.2

1. (i) $x^2 + y^2 = 25 \quad at \ P(4,3)$ **(Sargodha 2011)**

Take derivative w.r.t x

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow x + y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Slope of Tangent $= \frac{dy}{dx}$ at $P(4,3) = \frac{-4}{3}$

Equation of Tangent is

$$(y-3) = \frac{-4}{3}(x-4)$$

$$or \quad 3y - 9 = -4x + 16$$

$$\Rightarrow \quad 4x + 3y - 25 = 0$$

$$\Rightarrow \quad 4x + 3y = 25$$

Slope of normal $= \frac{4}{3}$

Equation of normal

$$(y-3) = \frac{4}{3}(x-4)$$

$$4y - 12 = 3x - 12$$

$$3x - 4y = 0$$

$y - y_1 = m(x - x_1)$

For point $Q\ (5Cos\theta,\ 5Sin\theta)$

Slope of tangent $= \frac{dy}{dx}$ at $Q = \frac{-5Cos\theta}{5Sin\theta} = \frac{-Cos\theta}{Sin\theta}$

Equation of tangent is

$$(y - 5Sin\theta) = \frac{-Cos\theta}{Sin\theta}(x - 5Cos\theta)$$

$$ySin\theta - 5Sin^2\theta = -xCos\theta + 5Cos^2\theta$$

$$xCos\theta + ySin\theta = 5(Cos^2\theta + Sin^2\theta)$$

$$xCos\theta + ySin\theta = 5$$

Slope of Normal $= \frac{Sin\theta}{Cos\theta}$

Equation of normal

$$(y - 5Sin\theta) = \frac{Sin\theta}{Cos\theta}(x - 5Cos\theta)$$

$$yCos\theta - 5Cos\theta Sin\theta = xSin\theta - Cos\theta Sin\theta$$

$$xSin\theta - yCos\theta - \cancel{5Cos\theta Sin\theta + 5Cos\theta Sin\theta} = 0$$

$$xSin\theta - yCos\theta = 0$$

(ii) $\mathbf{3x^2 + 3y^2 + 5x - 13y + 2 = 0}$

Take derivative at $P\left(1, \frac{10}{3}\right) w.r.t.x$

$$6x + 6y\frac{dy}{dx} + 5 - 13\frac{dy}{dx} = 0$$

$$(6y - 13)\frac{dy}{dx} + (6x + 5) = 0$$

$$\frac{dy}{dx} = -\frac{(6x+5)}{(6y-13)}$$

Slope of Tangent $= \frac{dy}{dx}$ at $P = \frac{-6(1) - 5}{6\left(\frac{10}{3}\right) - 13}$

$$= \frac{-6-5}{20-13} = \frac{-11}{7}$$

Equation of tangent is

$$\left(y - \frac{10}{3}\right) = -\frac{11}{7}(x - 1)$$

$$or \;\; 7y - \frac{70}{3} = -11x + 11$$

$$11x + 7y = 11 + \frac{70}{3}$$

$$11x + 7y = \frac{103}{3} \;\; or \;\; 33x + 21y = 103$$

Slope of normal $= \frac{7}{11}$

Equation of normal at $\left(1, \frac{10}{3}\right)$

is $\left(y - \frac{10}{3}\right) = \frac{7}{11}(x - 1)$

$$11y - \frac{110}{3} = 7x - 7$$

$$or \;\; \frac{33y - 110}{3} = 7x - 7$$

$\Rightarrow 33y - 110 = 21x - 21$

$21x - 33y + 110 - 21 = 0$

$21x - 33y + 89 = 0$

4. $\mathbf{4x^2 + 4y^2 - 16x + 24y - 117 = 0}$

Abscissa $= x = -4$

Put $x = -4$ then

$4(-4)^2 + 4y^2 - 16(-4) + 24y - 117 = 0$

$64 + 4y^2 + 64 + 24y - 117 = 0$

$4y^2 + 24y + 11 = 0$

$4y^2 + 22y + 2y + 11 = 0$

$2y(2y + 11) + 1(2y + 11) = 0$

$(2y + 11)(2y + 1) = 0$

$2y + 11 = 0 \quad or \quad 2y + 1 = 0$

$y = \dfrac{-11}{2} \quad or \quad y = \dfrac{-1}{2}$

$P_1\left(-4, \dfrac{-11}{2}\right), \left(-4, \dfrac{-1}{2}\right)$

Take derivative of I

$4.2x + 4.2y\dfrac{dy}{dx} - 16.1 + 24\dfrac{dy}{dx} = 0$

$\div\ by\ 8$

$x + y\dfrac{dy}{dx} - 2 + 3\dfrac{dy}{dx} = 0$

$(y + 3)\dfrac{dy}{dx} = 2 - x$

$\dfrac{dy}{dx} = \dfrac{2 - x}{y + 3}$

Slope of tangent $= \dfrac{dy}{dx}$ $at\ P_1 = \dfrac{2 - (-4)}{\dfrac{-11}{2} + 3}$

$\dfrac{2 + 4}{\dfrac{-5}{2}} = 6 \times \left(\dfrac{2}{-5}\right) = \dfrac{-12}{5}$

Slope of Tangent $= \dfrac{-12}{5}$

Slope of Normal $= \frac{5}{12}$

Equation of tangent

$$\left(y+\frac{11}{2}\right)=\frac{-12}{5}(x+4)$$

$$\frac{2y+11}{2}=\frac{-12x-48}{5}$$

$$or \quad 10y+55=-24x-96$$

$$\boxed{24x+10y+151=0}$$

Equation of normal

$$\left(y+\frac{11}{2}\right)=\frac{5}{12}(x+4)$$

$$\frac{2y+11}{2}=\frac{5x+20}{12}$$

$$24y+132=10x+40$$

$$\boxed{10x-24y-92=0}$$

Slope of tangent $=\frac{dy}{dx}$ at $P_2=\dfrac{2-(-4)}{\frac{-1}{2}+3}$

$$=\frac{2+4}{\frac{5}{2}}=6\times\frac{2}{5}=\frac{12}{5}$$

Slope of tangent $=\frac{12}{5}$

Slope of Normal $=\frac{-5}{12}$

Equation of Normal is $=\left(y+\frac{1}{2}\right)=\frac{-5}{12}(x+4)$

3. (i) $x^2+y^2=81$ **or** $x^2+y^2-81=0$ **(Sargodha 2008, 10, Lhr 2010)**

Put $(5,6)$ in left member of equation

$$(5)^2+(6)^2-81=25+36-81=-20<0=-Ve$$

Inside circle

(ii) $2x^2+2y^2+12x-8y+1=0$ **(Gujrawala 2010)**

Put $(5,6)$ in left member of equation

$$2(5)^2+2(6)^2+12(5)-8(6)+1$$

$$50+72+60-48+1$$
$$135=+Ve$$

Outside circle.

4. Find length of tangent from $p(-5,4)$ to the circle

$$5x^2+5y^2-10x+15y-131=0$$
$$5x^2+5y^2-10x+15y-131=0$$
$$\div \text{ by } 5$$
$$x^2+y^2-2x+3y-\frac{131}{5}=0$$

Compare with

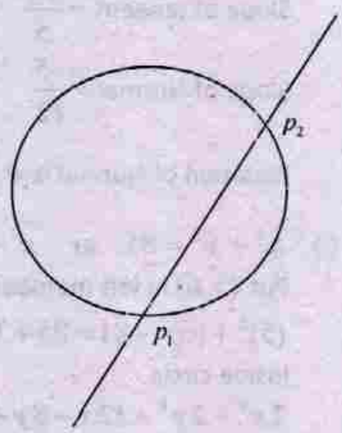
$$x^2+y^2+2gx+2fy+c=0$$
$$g=-1,\ f=\frac{3}{2},\ c=\frac{-131}{5}\quad x_1=-5,\quad y_1=4$$

Length of tangent

$$=\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$$
$$=\sqrt{(-5)^2+(4)^2+2(-1)(-5)+2\left(\frac{3}{2}\right)4-\frac{131}{5}}$$
$$=\sqrt{25+16+10+12-\frac{131}{5}}=\sqrt{\frac{184}{5}}$$

5. Find length of the chord cut off from the line $2x+3y=13$ by the circle $x^2+y^2=26$ **(Sargodha 2010)**

Sol: Given line is



$$|p_1p_2=?|$$
$$2x+3y=13$$
$$\text{or } 2x=13-3y$$
$$\text{or } x=\frac{13-3y}{2}$$

Put in $x^2+y^2=26$

$$\left(\frac{13-3y}{2}\right)^2+y^2=26$$
$$\frac{169+9y^2-78y}{4}+y^2=26$$

'×' by 4

$169+9y^2-78y+4y^2=104$

$or\ 13y^2-78y+65=0$

$13y^2-13y-65y+65=0$

$13y(y-1)-65(y-1)=0$

$(y-1)(13y-65)=0$

$y=1 \quad or \quad y=\dfrac{65}{13}$

When $y=1$ *then* $x=\dfrac{13-3(1)}{2}=5$

When $y=\dfrac{65}{13}$ *then* $x=\dfrac{13-3\left(\dfrac{65}{13}\right)}{2}$

$x=\left(\dfrac{169-195}{13}\right)\times\dfrac{1}{2}=\left(\dfrac{-26}{13}\right)\left(\dfrac{1}{2}\right)=-1$

$P_1=\left(-1,\dfrac{65}{13}\right),\ P_2(5,1)$

are point of contact.

Length of chord $=|P_1P_2|$

$=\sqrt{(5+1)^2+\left(1-\dfrac{65}{13}\right)^2}$

$=\sqrt{(6)^2+\left(\dfrac{-52}{13}\right)^2}$

$=\sqrt{36+\dfrac{2704}{169}}=\sqrt{\dfrac{6084+2704}{169}}$

$=\sqrt{\dfrac{8788}{169}}=\sqrt{52}=2\sqrt{13}$

6. **Find points of contact**

$x+2y=6 \longrightarrow I$

$x^2+y^2-2x-2y-39=0 \longrightarrow II$

From I $x+2y=6$

$x=6-2y$

II become

$(6-2y)^2+y^2-2(6-2y)-2y-39=0$

$36+4y^2-24y+y^2-12+4y-2y-39=0$

$5y^2-22y-15=0$

$5y^2-25y+3y-15=0$

$5y(y-5)+3(y-5)=0$

$(y-5)(5y+3)=0$

$y-5=0 \quad or \quad 5y+3=0$

$y=5 \quad or \quad y=\frac{-3}{5}$

When $y=5$ *then* $x=6-2(5)=-4$

When $y=\frac{-3}{5}$ *then* $x=6-2\left(\frac{-3}{5}\right)$

$x=6+\frac{6}{5}=\frac{36}{5}$

Points of contacts are $p_1(-4,5)$ & $p_2\left(\frac{36}{5},\frac{-3}{5}\right)$

7. Find equation of Tangent to circle $x^2+y^2=2$

(i) Parallel to line $x-2y+1=0$ **(Sargodha 2008, 09)**

Sol: Given circle is

$x^2+y^2=2=(\sqrt{2})^2$

$r=a=\sqrt{2}$

Slope of line $=\frac{-a}{b}=\frac{-1}{-2}=\frac{1}{2}$

$m=\frac{1}{2}$ (Because Parallel)

Equation of tangent is

$y=mx\pm a\sqrt{1+m^2}$

$y=\frac{1}{2}x\pm\sqrt{2}\sqrt{1+\left(\frac{1}{2}\right)^2}$

$y=\frac{x}{2}\pm\sqrt{2}\sqrt{1+\frac{1}{4}}$

$y=\frac{x}{2}\pm\frac{\sqrt{2}\sqrt{5}}{2}$

'x' *by* 2

$2y = x \pm \sqrt{10}$

$x - 2y + \sqrt{10} = 0$

$x - 2y - \sqrt{10} = 0$

(ii) **Perpendicular to line $3x + 2y = 6$**

Slope of line $= \dfrac{-a}{b} = -\dfrac{3}{2}$

Slope of tangent ($\perp ar$ to geven)= $m = \dfrac{2}{3}$

$r = a = \sqrt{2}$ as in (i)

$y = mx \pm a\sqrt{1+m^2}$

$y = \dfrac{2}{3}x \pm \sqrt{2}\sqrt{1+\dfrac{4}{9}}$

$y = \dfrac{2}{3}x \pm \sqrt{2}\sqrt{\dfrac{13}{2}}$

'×' by 3

$3y = 2x \pm \sqrt{2}\sqrt{13}$

$3y = 2x \pm \sqrt{26}$

$2x - 3y + \sqrt{26} = 0$

$2x - 3y - \sqrt{26} = 0$

8. (i) **Find equation of the tangent drawn from $(0,5)$ to $x^2 + y^2 = 16$** **(Sgd 2008, 10)**

Sol: Equation of Circle is $x^2 + y^2 = 16$

$or\ x^2 + y^2 = (4)^2,\ \boxed{a = 4}$

Equation of tangent is

$y = mx + a\sqrt{1+m^2}$

$at\ (0,5)\ \&\ a = 4$

$5 = m(0) + 4\sqrt{1+m^2}$

$5 = 4\sqrt{1+m^2} \Rightarrow 25 = 16\,(1+m^2)$

$1 + m^2 = \dfrac{25}{16} \Rightarrow m^2 = \dfrac{25}{16} - 1 = \dfrac{9}{16}$

$m = \pm\dfrac{3}{4}$

Put $m=\pm\frac{3}{4}$ and $a=4$ in I

$y=\pm\frac{3}{4}x+4\sqrt{1+\frac{9}{16}}$

$y=\pm\frac{3}{4}x+4\sqrt{\frac{25}{16}}$

$y=\pm\frac{3}{4}x+4.\frac{5}{4}$

'×' by 4

$4y=\pm 3x+20$

$4y=3x+20 \quad or \quad 4y=-3x+20$

$3x-4y+20=0 \quad or \quad 3x+4y-20=0$

(ii) $(-1,2),\ x^2+y^2+4x+2y=0$

Compare with

$x^2+y^2+2gx+2fy+c=0$

$g=2,\ f=1,\ c=0$

$r=a=\sqrt{g^2+f^2-c}=\sqrt{(2)^2+(1)^2-0}$

$a=\sqrt{5} \quad Centre(-g,-f)=(-2,-1)$

Let m be slope of tangent drawn from point $(-1,2)$ to given circle then equation of tangent I s

$y-y_1=m(x-x_1)$

$(y-2)=m(x+1)$

$or\ y-2=mx+m$

$mx-y+m+2=0$ ______ I

Distance of line from

Center $(-2,-1)=r$

$\Rightarrow \frac{|m(-2)-(-1)+m+2|}{\sqrt{m^2+1}}=\sqrt{5} \Rightarrow \frac{|-m+3|}{\sqrt{m^2+1}}=\sqrt{5}$

Squaring both sides

$\frac{m^2-6m+9}{m^2+1}=5$

$m^2-6m+9=5m^2+5$

$or\ 5m^2+5-m^2+6m-9=0$

$4m^2+6m-4=0$

$4m^2+8m-2m-4=0$

$4m(m+2)-2(m+2)=0$

$(m+2)(4m-2)=0$

$m+2=0 \quad or \quad 4m-2=0$

$m=-2 \quad or \quad m=\frac{1}{2}$

Put $m=-2$ in I

$-2x-y-2+2=0 \Rightarrow \boxed{2x+y=0}$

Put $m=\frac{1}{2}$ in I

$\frac{1}{2}x-y+\frac{1}{2}+2=0$

'×' by 2 $\quad x-2y+1+4=0$

$\boxed{x-2y+5=0}$

(iii) $\mathbf{(-7,-2)}$ ***to*** $\mathbf{(x+1)^2+(y-2)^2=26=(\sqrt{26})^2}$

$(x-h)^2+(y-k)^2=r^2$

Center $(-1,2)$, $r=\sqrt{26}$

By comparing with

Suppose equation of tangent is

$(y-y_1)=m(x-x_1) \quad at\ (-7,-2)$

$(y+2)=m(x+7)$

$y+2=mx+m7 \Rightarrow mx-y+7m-2=0$

Distance of tangent from

$C(-1,2)=r$

$\Rightarrow \frac{|m(-1)-2+7m-2|}{\sqrt{m^2+(-1)^2}}=\sqrt{26}$

$\frac{|6m-4|}{\sqrt{m^2+1}}=\sqrt{26}$

By squaring both sides

$\frac{36m^2-48m+16}{m^2+1}=26$

$or\ \ 36m^2-48m+16=26m^2+26$

$36m^2-26m^2-48m+16-26=0$

$10m^2 - 48m - 10 = 0$

$10m^2 - 50m + 2m - 10 = 0$

$10m(m-5) + 2(m-5) = 0$

$(m-5)(10m = 2) = 0$

$m - 5 = 0 \;\; or \;\; m = \dfrac{-2}{10}$

$m = 5 \;\; or \;\; m = \dfrac{-1}{5}$

When $m = 5$ then I become

$5x - y + 7m - 2 = 0$

$5x - y + 7(5) - 2 = 0$

$\boxed{5x - y + 33 = 0}$

When $m = \dfrac{-1}{5}$ then I

become

$\left(-\dfrac{1}{5}\right)x - y + 7\left(-\dfrac{1}{5}\right) - 2 = 0$

'×' by 5

$-x - 5y - 7 - 12 = 0$

$-x - 5y - 17 = 0$

$\boxed{x + 5y + 17 = 0}$

For point of contact

Solve $x + 5y + 17 = 0 \longrightarrow I$

$(x+1)^2 + (y-2)^2 = 26 \longrightarrow II$

$I \Rightarrow x = -5y - 17 \longrightarrow III$

Put III in II

$(-5y - 17 + 1)^2 + (y-2)^2 = 26$

$(-5y - 16)^2 + (y-2)^2 = 26$

$25y^2 + 160y + 256 + y^2 - 4y + 4 - 26 = 0$

$26y^2 + 156y + 234 = 0$

Divide by 26

$y^2 + 6y + 9 = 0 \;\Rightarrow\; (y+3)^2 = 0$

$\Rightarrow \; y = -3$

Put in $III \Rightarrow x = -5(-3) - 17 = 15 - 17 = -2$

So $(-2,-3)$ is point of contact

Case II

Solve $5x-y+33=0$ $\quad IV$

and $(x+1)^2+(y-2)^2=26$ $\quad II$

From IV $\quad y=5x+33$

Put V in II,

$(x+1)^2+(5x+33-2)^2=26$

$x^2+2x+1+(5x+31)^2-26=0$

$x^2+2x+1+25x^2+310x+961-26=0$

$26x^2+312x+936=0$

$\div$ by 26

$x^2+12x+36=0 \Rightarrow (x+6)^2=0 \Rightarrow x=-6$

V become $\quad y=5(-6)+33 \Rightarrow y=3$

$(-6,\ 3)$ is point of contact.

9. **Find an equation of the chord of contact of the tangent drawn from $(4,5)$ to $2x^2+2y^2-8x+12y+21=0$**

$\div$ by 2

$x^2+y^2-4x+6y+\frac{21}{2}=0$

Compare with

$x^2+y^2+2gx+2fy+c=0$

$g=-2,\ f=3,\ c=\frac{21}{2},\ x_1=4,\ y_1=5$

Required equation is

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

$4x+5y-2(x+4)+3(y+5)+\frac{21}{2}=0$

or $\boxed{4x+16y+35=0}$

Exercise 6.3

1. Prove that normal lines of a circle pass through the centre of the circle?

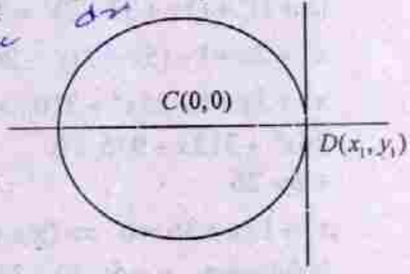
Let $x^2 + y^2 = r^2$ is equation of circle with centre $(0,0)$

For Slope

Take derivative w.r.t

$$2x + 2y\frac{dy}{dx} = 0$$

$$or\ x + y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$



$$\text{Slope of tangent} = m = \frac{dy}{dx}(x_1, y_1) = -\frac{x_1}{y_1}$$

$$\text{Slope of Normal} = \frac{y_1}{x_1}$$

Equation of normal through (x_1, y_1)

is $(y - y_1) = \frac{y_1}{x_1}(x - x_1)$

or $x_1y - x_1y_1 = xy_1 - x_1y_1$

$x_1y = xy_1$

Put center $(0,0)$

$x_1(0) = 0(y_1) \Rightarrow 0 = 0$

Hence Normal lines pass through center.

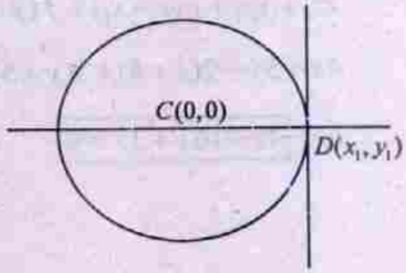
2. Prove that straight line drawn from the center of a circle perpendicular to a tangent pass through the point of tangency.

Let $x^2 + y^2 = r^2$ is

equation of circle with centre

$$2x + 2y\frac{dy}{dx} = 0$$

$$or\ x + y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$



$$\text{Slope of tangent} = m = \frac{dy}{dx}(x_1, y_1) = \frac{-x_1}{y_1}$$

$$\text{Slope o f perpendicular line} = \frac{1}{m} = \frac{y_1}{x_1}$$

Equation of perpendicular line is through $(0,0)$ is

$$(y-0)=\frac{y_1}{x_1}(x-0)$$

$$\Rightarrow x_1 y = x y_1$$

Put (x_1, y_1)

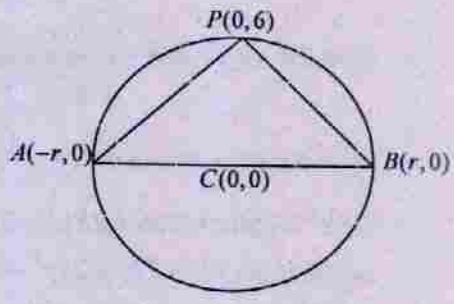
$$x_1 y_1 = x_1 y_1$$

Hence perpendicular line pass through point of tangency (x_1, y_1)

3. Prove that Mid point of the hypotenuse of a right triangle is the circum center of the triangle.

Let $x^2+y^2=r^2$ I

is equation of circle with center (0, 0) and $P(a,b)$ is any point of circle.



$P(a,b)$ lies on I

So $a^2+b^2=r^2$

$$|OA|=\sqrt{(-r-0)^2+(0-0)^2}=r$$

$$|OB|=\sqrt{(r-0)^2+(0-0)^2}=r$$

$$|OP|=\sqrt{(a-0)^2+(b-0)^2}=\sqrt{a^2+b^2}$$

$$=\sqrt{r^2}=r \quad (Use\ II)$$

$|OA|=|OB|=|OP|$ and O is circumcentre.

4. Prove that the perpendicular dropped from a point of circle on a diameter is a mean proportional between the segment into which it divides the diameter.

Let equation or circle is

Let $x^2+y^2=r^2$

$P(a,b)$ lies on it

So $a^2+b^2=r^2$

Or $b^2=r^2-a^2$

Now $|PQ|=\sqrt{(a-a)^2+(b-0)^2}=b^2$

$$|AQ|=|AO|+|OQ|=r+a$$

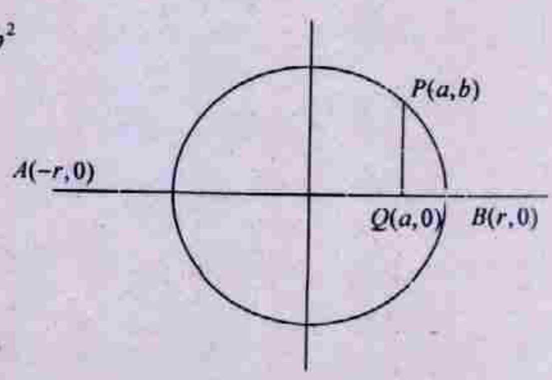
$$|BQ|=|OB|-|OQ|=r-a$$

$$|AQ|.|BQ|=(r+a)(r-a)$$

$$=r^2-a^2$$

$$|AQ|.|BQ|=b^2 \quad (Use\ I)$$

$$=|PQ|^2$$



$|AQ|.|BQ| = |PQ|.|PQ|$

∴ Hence proved.

Example 2 of (6.4) **(Sargodha 2007)**

Find an equation of the parabola whose focus is $F(-3,4)$ and directrix is

$3x-4y+5=0$

Let $P\ (x,y)$ be any point then by definition.

$|PF| = |PM|$ or $|PF|^2 = |PM|^2$

$$(x+3)^2+(y-4)^2 = \left(\frac{|3x-4y+5|}{\sqrt{(3)^2+(-4)^2}}\right)^2$$

$$x^2+6x+9+y^2-8y+16 = \frac{(3x-4y+5)^2}{25}$$

$$25x^2+150x+225+25y^2-200y+400 = 9x^2+16y^2+25-24xy+30x-40y$$

$$or\ 25x^2+150x+225+25y^2-220y+400-9x^2-16y^2-25+24xy-30x+40y=0$$

$$or\ 16x^2+24xy+9y^2+120x-160y+600=0$$

Is required equation.

Exercise 6.4

1. **Find the focus, vertex and directrix of the parabola.**

(i) $y^2 = 8x$ **(Sgd 2010)**

Compare with

$y^2 = 4ax$

$\Rightarrow 4a = 8 \Rightarrow a = 2$

Focus: $F(a,0) = F(2,0)$

Vertex; $V(0,0) = V(0,0)$

DTX; $x = -a \Rightarrow x = -2$

(ii) $x^2 = -16y$

Compare with

$x^2 = -4ay$

$\Rightarrow 4a = 16 \Rightarrow a = 4$

Focus: $F(0,-a) = F(0,-4)$

Vertex; $V(0,0) = V(0,0)$

DTX; $y = a \Rightarrow y = 4$

(iii) $x^2 = 5y$

Compare with

$x^2 = 4ay$

$\Rightarrow 4a = 5 \Rightarrow a = \dfrac{5}{4}$

Focus: $F(0,a) = F\left(0,\dfrac{5}{4}\right)$

Vertex; $V(0,0) = V(0,0)$

DTX; $y = -a \Rightarrow y = \dfrac{-5}{4}$

(iv) $y^2 = -12x$

Compare with

$y^2 = -4ax$

$\Rightarrow 4a = 12 \Rightarrow a = 3$

Focus: $F(-a,0) = F(-3,0)$

Vertex; $V(0,0) = V(0,0)$

DTX; $x = a \Rightarrow x = 3$

(v) $x^2 = 4(y-1)$

Put $x = X$ & $y - 1 = Y$

$X^2 = 4Y$

Compare with

$X^2 = 4aY$

$4a = 4 \Rightarrow a = 1$

Focus: $F(0,a) = F(0,1) \Rightarrow X = 0, Y = 1$

So $\boxed{F(0,2)}$ $\Rightarrow x = 0,\ y - 1 = 0$

$\Rightarrow x = 0,\ y = 2$

Vertex: $V(0,0) \Rightarrow X = 0, Y = 1$

So $\boxed{V(0,1)}$ $\Rightarrow x = 0,\ y - 1 = 0$

$y = 1$

DTX; $Y = -a \Rightarrow Y = -1$

$\Rightarrow y - 1 = -1$

$\Rightarrow \boxed{y = 0}$

(vi) $y^2 = -8(x-3)$ **(Sgd 2011)**

Put $y = Y$ & $x - 3 = X$

$Y^2 = -8X$

Compare with

$Y^2 = -8X \Rightarrow 4a = 8 \Rightarrow a = 2$

Focus:

$F(-a,0) = F(-2,0) \Rightarrow X = -2, Y = 0$

$x - 3 = -2,\ y = 0$

$x = 1,\ y = 0$ So $\boxed{F(1,0)}$

Vertex: $V(0,0) \Rightarrow X = 0, Y = 0$

So $x - 3 = 0,\ y = 0$

$x = 3,\ y = 0$ $\boxed{V(3,0)}$

DTX; $X = a \Rightarrow x - 3 = 0 \Rightarrow \boxed{x = 5}$

(vii) $(x-1)^2 = 8(y+2)$ **(Sgd 2009,10)**

Put $x - 1 = X$ & $y + 2 = Y$

Then $X^2 = 8Y^2$

Compare with

$X^2 = 4aY \Rightarrow 4a = 8 \Rightarrow a = 2$

Focus;

$X = 0,\ Y = 2$

$x - 1 = 0,\ y + 2 = 2$

$x = 1,\ y = 0 \Rightarrow \boxed{F(1,0)}$

Vertex; $V(0,0) \Rightarrow X = 0,\ Y = 0$

$x - 1 = 0,\ y + 2 = 0$

$\boxed{V(1,-2)}$ $x = 1,\ y = -2$

DTX; $Y = -a \Rightarrow y + 2 = -2$

$\boxed{y=-4}$

(viii) $y=6x^2-1$ **(Sgd 2010)**

or $y+1=6x^2$

$\Rightarrow x^2=\frac{1}{6}(y+1)$

Put $x=X$ & $y+1=Y$

$X^2=\frac{1}{6}Y$

Compare with

$X^2=4aY \Rightarrow 4a=\frac{1}{6} \Rightarrow a=\frac{1}{24}$

Focus; $F(0,a)=F\left(0,\frac{1}{24}\right)$

$X=0,\ Y=\frac{1}{24}$

$x=0,\ y+1=\frac{1}{24} \Rightarrow y=\frac{1}{24}-1$

$y=\frac{-23}{24}$

$\boxed{F\left(0,\frac{-23}{24}\right)}$

$V(0,0) \Rightarrow X=0,\ Y=0$

$\boxed{V(0,-1)}$ $x=0,\ y+1=0$

$y=-1$

DTX; $Y=-a \Rightarrow y+1=-\frac{1}{24}$

$y=-1-\frac{1}{24}$ $\boxed{y=\frac{-25}{24}}$

(ix) $x+8-y^2+2y=0$ **(Sgd 2009)**

$x+8=y^2-2y$

Add '1' both sides

$x+8+1=y^2-2y+1$

$x+9=(y-1)^2$

Or $(y-1)^2=(x+9)$

Put $y-1=Y$ & $x+9=X$

$Y^2=X$

Compare with

$Y^2=4aX \Rightarrow 4a=1 \Rightarrow a=\frac{1}{4}$

$F(a,0)=F\left(\frac{1}{4},0\right)$

$X=\frac{1}{4},\ Y=0$

$x+9=\frac{1}{4},\ y-1=0$

$x=\frac{1}{4}-9 \Rightarrow x=\frac{-35}{4},\ y=1$

$\boxed{F\left(\frac{-35}{4},1\right)}=F(-8.7,1)$

$V(0,0) \Rightarrow X=0,\ Y=0$

$x+9=0,\ y-1=0$

$\boxed{V(-9,1)}$ $x=-9,\ y=1$

DTX; $X=-a \Rightarrow X=\frac{-1}{4}$

or $x+9=\frac{-1}{4} \Rightarrow x=\frac{-9-1}{4}$

$\boxed{x=\frac{-37}{4}} \Rightarrow \boxed{x=-9.2}$

(x) $x^2-4x-8y+4=0$

$x^2-4x+4=8y$

$(x-2)^2=8y$

Put $x-2=X$ & $y=Y$

$X^2=8Y$

Compare with

$X^2=4aY \Rightarrow 4a=8 \Rightarrow \boxed{a=2}$

$F(0,a)=F(0,2)$

$X=0,\ Y=2$

$x-2=0,\ y=2 \Rightarrow x=2,\ y=2$ $\boxed{F(2,2)}$

$V(0,0) \Rightarrow X=0,\ Y=0$

$x-2=0$ *or* $y=0$

$x=0$ *or* $y=0$

$\boxed{V(2,0)}$

DTX; $Y=-a \Rightarrow \boxed{y=-2}$

2. (i) $F(-3,1)$, DTX; $x=3 \Rightarrow x-3=0$ **(Gujrawala 2010)**

Suppose $P(x,y)$ any point on parabola, then by definition

$|PF|^2=|PM|^2$

$$\left[\sqrt{(x+3)^2+(y-1)^2}\right]^2=\left[\frac{|x-3|}{\sqrt{(1)^2+(0)^2}}\right]^2$$

$$x^2+6x+9+y^2-2y+1=\frac{x^2-6x+9}{1}$$

$6x+y^2-2y+1=-6x$

or $6x+y^2-2y+1+6x=0$

or $y^2-2y+12x+1=0$

is required equation of parabola.

(ii) $F(2,5)$, DTX is $y=1$ **(Sgd 2009, 11)**

Take point $P(x,y)$ or $y-1=0$

Then by definition

$|PF|^2=|PM|^2$

$$\left[\sqrt{(x-2)^2+(y-5)^2}\right]^2=\left[\frac{|y-1|}{\sqrt{(0)^2+(1)^2}}\right]^2$$

$$(x-2)^2+(y-5)^2=\frac{|y-1|^2}{1}$$

$x^2-4x+4+y^2-10y+25=y^2-2y+1$

$x^2-4x+4-10y+25+2y-1=0$

$x^2-4x+4-8y+28=0$

is required equation of parabola.

(iii) $F(-3,1)$, DTX; $M; x-2y-3=0$

Take point $P(x,y)$

Then by definition

$|PF|^2=|PM|^2$

$$\left[\sqrt{(x+3)^2+(y-1)^2}\right]^2=\left[\frac{|x-2y-3|}{\sqrt{(1)^2+(-2)^2}}\right]^2$$

$$(x+3)^2+(y-1)^2=\frac{|x-2y-3|^2}{(\sqrt{5})^2}$$

$$x^2+6x+9+y^2-2y+1=\frac{x^2+4y^2+9-4xy-6x+12y}{5}$$

$5(x^2+y^2+6x-2y+10)=x^2+4y^2-4xy-6x+12y+9$

$5x^2+5y^2+30x-10y+50-x^2-4y^2+4xy+6x-12y-9=0$

$4x^2+y^2+36x-22y+4xy+41=0$

is required equation of parabola.

(iv) $F(1,2), V(3,2)\ V(h,x)$

$a=|FV|$

$a=\sqrt{(3-1)^2+(2-2)^2}$

$a=\sqrt{4+0}=0$

Equation of parabola

$(y-k)^2=-4a(x-h) \qquad h=3$

$(y-2)^2=-4(2)(x-3) \qquad k=2$

$(y-2)^2=-8(x-3) \qquad a=2$

(v) $F(-1,0), V(-1,2)\ h=-1,\ x=2$

$a=|VF|$

$a=\sqrt{(-1+1)^2+(2-0)^2}$

$a=\sqrt{0+4}=2$

Equation of parabola

$(x-h)^2=-4a(y-k)$

$(x+1)^2=-4(2)(y-2)$

$(x+1)^2=-8(y-2)$

(vi) $F(2,2), DTX;\ X=-2 \Rightarrow X+2=0$

Take $P(x,y)$ then

$|PF|^2=|PM|^2$

$$\left[\sqrt{(x-2)^2+(y-2)^2}\right]^2=\left[\frac{|x+2|}{\sqrt{(1)^2+(0)^2}}\right]^2$$

$(x-2)^2(y-2)^2=|x+2|^2$

$x^2-4x+4+y^2-4y+4=x^2+4x+4$

$-4x+4+y^2-4y-4x=0$

$y^2-4y-8x+4=0$

$y^2-4y+4=8x$

$(y-2)^2=8x$

Is Equation of parabola

(vii) $DTX;\ y=3, V(2,2)\ h=2,\ k=2$

Equation of parabola is

$(x-h)^2=-4a(y-k)$

$(x-2)^2=-4a(y-2)$

Put $x-2=X,\ y-2=y$

$X^2=-4aY$

$DTX;\ Y=a$

$y-2=a$ put $y=3$

$3-2=a \Rightarrow \boxed{a=1}$

$3-2=a \Rightarrow \boxed{a=1}$

I become

$(x-2)^2=-4(1)(y-2) \Rightarrow (x-2)^2=-4(y-2)$

(viii) $DTX;\ y=1$

Length of latusrectum $=4a=8$

and open downward $\boxed{a=2}$

Equation of down word is

$(x-h)^2=-4a(y-k)$

$(x-h)^2=-4(2)(y-k)$

$(x-h)^2=-8(y-k)$

Put $x-h=X$ & $y-k=Y$

$X^2=-8y$

$DTX;\ Y=a$

$y-k=2$

$\boxed{put\ y=1\ given}$ $1-k=2 \Rightarrow \boxed{k=-1}$

I become

$(x-h)^2=-8(y+1)$

(ix) Axis $y=0$ through $(2,1)$ *and* $(11,-2)$

Let equation of required parabola be

$(y-k)^2=4a(x-h) \quad I$

Axis $y=0$ so vertex of parabola

Lies on x-axis $\Rightarrow \boxed{k=0}$

So $I \Rightarrow y^2=4a(x-h) \quad II$

Put $(2,1)$ *in II*

$1^2=4a(2-h)$

$or\ 1=8a-4ah \quad III$

Put $(11,-2)$ *in II*

$(-2)^2=4a(11-h)$

$4=4a(11-h)$

$1=a11-ah \quad IV$

'X' $IV\times4$ and subtract from III

$$\begin{array}{l} 4=44a-\cancel{4ah} \\ \underline{-1=-8a\mp\cancel{4ah}} \\ 3=36a \Rightarrow \boxed{a=\frac{1}{12}} \end{array}$$

III become $1=8\left(\frac{1}{12}\right)-4\left(\frac{1}{12}\right)h$

$1=\frac{2}{3}-\frac{1}{3}h \Rightarrow \frac{-1}{3}h=1-\frac{2}{3}$

$-\frac{1}{3}h=\frac{1}{3} \Rightarrow \boxed{h=-1}$

II become $a=\frac{1}{12}$

$y^2=4\left(\frac{1}{12}\right)(x+1)$

$\boxed{y^2=\frac{1}{3}(x+1)}$

(x) **Axis parallel to y-axis the points** $(0,3),(3,4),(4,11)$ **lies on graph.**

Equation of parabola is

$(x-h)^2=4a(y-k) \quad I$

Put $(0,3) \Rightarrow x=0,\ y=3$ *in I*

$h^2=4a(3-k)$

$h^2=12a-4ak \quad II$

Put $(3,4) \Rightarrow x=3,\ y=4$ *in I*

$(3-h)^2=4a(4-k)$

$9-6h+h^2=16a-4ak \quad III$

Put $(4,11) \Rightarrow x=4,\ y=11$ *in I*

$(4-h)^2=4a(11-k)$

$16-8h+h^2=44a-4ak \quad IV$

$IV-III$

$16-8h+\cancel{h^2}=44a-\cancel{4ak}$

$\underline{\ \ 9\mp 6h\pm\cancel{h^2}=\ \ 16a\mp\cancel{4ak}}$

$7-2h,=28a \quad V$

$III-II$

$9-6h+\cancel{h^2}=16a-\cancel{4ak}$

$\underline{\ \ \ \ \ \ \ \ \cancel{h^2}=\ \ 12a\mp\cancel{4ak}}$

$9-6h,=4a \quad VI$

$V\times 3$

$21-6h=84a \quad VII$

$VII-VI$

$21-\cancel{6h}=84a$

$\underline{\ \ 9\mp\cancel{6h}=\ \ 4a}$

$12=80a \Rightarrow a=\frac{12}{80}$

or $\boxed{a=\frac{3}{20}}$

Put value of $a=\frac{3}{20}$ *in VI*

$9-6h=4\left(\frac{3}{20}\right)$

$9-6h=4\left(\frac{3}{20}\right)$

$9-6h=\frac{3}{5} \Rightarrow 6h=9-\frac{3}{5}$

$6h=\frac{42}{5} \Rightarrow \boxed{h=\frac{7}{5}}$

Put value in *II*

$h^2=12a-4ax$

$\left(\frac{7}{5}\right)^2=12\left(\frac{3}{20}\right)-4\left(\frac{3}{20}\right)k$

$\frac{49}{25}=\frac{18}{10}-\frac{49}{25}=\frac{450-490}{250}$

$\frac{3}{5}k=\frac{-40}{250}$

$\Rightarrow \frac{3}{5}k=\frac{-40}{250} \Rightarrow k=\frac{-4}{253}\times\frac{5}{3}$

$\boxed{k=\frac{-4}{15}}$ put value in *I*

$\left(x-\frac{7}{5}\right)^2=4\left(\frac{3}{20}\right)\left(y+\frac{4}{15}\right)$

$\boxed{\left(x-\frac{7}{5}\right)^2+\frac{3}{5}\left(y+\frac{4}{15}\right)}$

1. Find an equation of parabola having focus at origin and directrix parallel to

(i) x-axis (ii) y-axis

(i) For parallel to x-axis (DTX)

Case-I

DTX ; $y=a$ *and* $F(0,0)$

Or $M; y-a=0,\ P(x,y)$ any point.

$|PF|^2=|PM|^2$

$$\left[\sqrt{(x-0)^2+(y-0)^2}\right]^2=\left[\frac{|y-a|}{\sqrt{(0)^2+(1)^2}}\right]^2$$

$x^2+y^2=y^2-2ay+a^2$

$\boxed{x^2+2ay-a^2=0}$

Case-II

$y=-a$ (also 11 to x-axis)

$M;\ y+a=0,\ F(0,0), P(x,y)$

$|PF|^2=|PM|^2$

$$\left[\sqrt{(x-0)^2+(y-0)^2}\right]^2=\left[\frac{|x+a|}{\sqrt{(1)^2+(0)^2}}\right]^2$$

$\cancel{x^2}+y^2=\cancel{x^2}+2ax+a^2$

$\boxed{y^2-2ax-a^2}=0$

(ii) **For DTX parallel to y-axis**

Case-I

$M;\ x=-a,\ F(0,0),\ P(x,y)$ any point

$|PF|^2=|PM|^2$

$$\left[\sqrt{(x-0)^2+(y-0)^2}\right]^2=\left[\frac{|x+a|}{\sqrt{1^2+0^2}}\right]^2$$

$x^2+y^2=x^2+2ax+a^2 \Rightarrow \boxed{y^2-2ax-a^2=0}$

Case-II

$M;\ x=a,\ F(0,0),\ P(x,y)$

$|PF|^2=|PM|^2$

$$\left[\sqrt{(x-0)^2+(y-0)^2}\right]^2=\left[\frac{|x+a|}{\sqrt{1^2+0^2}}\right]^2$$

$x^2+y^2=x^2-2ax+a^2 \Rightarrow \boxed{y^2+2ax-a^2=0}$

2. **Show that an equation of parabola with $F(aCos\alpha, aSin\alpha)$, Driectrix**

$xCos\alpha+ySin\alpha+a=0$ *is* $(xSin\alpha-yCos\alpha)^2=4a(xCos\alpha+ySin\alpha)$

$F(aCos\alpha, aSin\alpha) M; xCos\alpha+ySin\alpha+a=0$

Suppose $P(x,y)$ is any point on parabola then

$|PF|^2=|PM|^2$

$$\left[\sqrt{(x-aCos\alpha)^2+(y-aSin\alpha)^2}\right]^2=\left[\frac{|xCos\alpha+ySin\alpha+a|}{\sqrt{Cos^2\alpha+Sin^2\alpha}}\right]^2$$

$x^2+a^2Cos^2\alpha-2axCos\alpha+y^2+a^2Sin^2\alpha-2aySin\alpha=x^2Cos^2\alpha+y^2Sin^2\alpha$
$+a^2+2xyCos\alpha Sin\alpha+2axCos\alpha+2aySin\alpha$

$x^2+y^2+a^2(Cos^2\alpha+Sin^2\alpha)-2axCos\alpha-2aySin\alpha-x^2Cos^2\alpha-y^2Sin^2\alpha$
$-a^2=2xyCos\alpha Sin\alpha+2axCos\alpha+2aySin\alpha$

$x^2-x^2Cos^2\alpha+y^2-y^2Sin^2\alpha+\cancel{a^2}-\cancel{a^2}=2axCos\alpha+2axCos\alpha+2aySin\alpha$

$+2aySin\alpha+2xyCos\alpha Sin\alpha$

$x^2(1-Cos^2\alpha)+y^2(1-Sin^2\alpha)=4axCos\alpha+4aySin\alpha+2xyCos\alpha Sin\alpha$

$x^2Sin^2\alpha+y^2Cos^2-2xyCos\alpha Sin\alpha=4a(xCos\alpha+ySin\alpha)$

$(xSin\alpha-yCos\alpha)^2=4a(xCos\alpha+ySin\alpha)$

3. **Show that the ordinate at any point P of the parabola is a mean proportional between length of latusrectum and abscission of P.**

We know that

$y^2=4ax;\ y=\pm\sqrt{4ax}$

(ordinate of P) $=\pm\sqrt{(\text{Length of latusrectum})\times(\text{abscission of P})}$

4. **A Comet has parabolic orbit with earth at focus when comet is 150,000Km from earth line joining comet and earth make an angle of 30^o How close will the comet to earth.**

Let E (earth) is at origin.

Then

$\frac{x}{|PE|}=Cos30^o$

$x=|PE|=Cos30^o$

$x=\frac{\sqrt{3}}{2}(150,000)$

$x=129900 \quad But\ x=-2a \Rightarrow 129900=-2a \Rightarrow a=-64950$

or $a=64950Km\ (omit\ -ve)$

5. **Find an equation of parabola formed by the cables of a suspension bridge whose span am and vertical height of tower bm.**

Let equation of

Parabola is $x^2=4a'y \quad I$

$P\left(\frac{a}{2},b\right)$ lies on parabola so

$\left(\frac{a}{2}\right)^2=4a'b \Rightarrow \frac{a^2}{4}=4a'b$

$a'=\frac{a^2}{4}\times\frac{1}{4b} \Rightarrow a'=\frac{a^2}{16b} \quad II$

Put II in I

$x^2=4\left(\frac{a^2}{16b}\right)y \Rightarrow \boxed{x^2=\frac{a^2 4b}{}y}$ is required equation.

6. **A parabola arch has 100m base and height 25m. Find height at 30m from centre.**

Equation of parabola is

$x^2=-4ay \quad I$

$at\ P(50,25) \Rightarrow x=50,\ y=25$

$(50)^2=-4a(25) \Rightarrow 2500-100a$

$a=-25$

For $x=30,\ a=-25,\ y=?$

I become

$(30)^2=-4(-25)y$

$900=100y \Rightarrow \boxed{y=9m}$

7. Show that tangent at any point P of a parabola makes equal angles which the line PF and line through P and parallel to x-axis.

Let equation of parabola is

$DTX\ \ y^2=4ax$

$m_1=0$ (Because line is parabola to x-axis)

For $m_2,\ 2y\dfrac{dy}{dx}=4a$ (Take derivative of I)

$\dfrac{dy}{dx}=\dfrac{4a}{2y} \Rightarrow \dfrac{dy}{dx}=\dfrac{2a}{y}$, $m_2=\dfrac{dy}{dx}P(x_1,y_1)=\dfrac{2a}{y_1}$, $m_3=\dfrac{y_1-0}{x_1-a}=\dfrac{y_1}{x_1-a}$

$$Tan\theta_1=\frac{m_2-m_1}{1+m_2m_1}=\frac{\frac{2a}{y_1}-0}{1+\left(\frac{2a}{y_1}\right)(0)}$$

$$Tan\theta_1=\frac{2a}{y_1}\Rightarrow \theta_1=Tan^{-1}\frac{2a}{y_1}$$

$$Tan\theta_2=\frac{m_3-m_2}{1+m_3m_2}$$

$$Tan\theta_2=\frac{\left(\frac{y_1}{x_1-a}\right)-\left(\frac{2a}{y_1}\right)}{1+\left(\frac{y_1}{x_1-a}\right)\left(\frac{2a}{y_1}\right)}$$

$$Tan\theta_2=\frac{\frac{y_1^2-2ax_1+2a^2}{(x_1-a)(y_1)}}{\frac{y_1x_1-ay_1+2ay}{y_1(x_1-a)}}$$

Put $y_1^2=4ax_1$ (Because P is on parabola)

$$Tan\theta_2=\frac{\frac{4ax_1-2ax_1+2a^2}{y_1(x_1-a)}}{\frac{y_1x_1-ay_1+2ay}{y_1(x_1-a)}}$$

$$Tan\theta_2=\frac{2a\,\cancel{(x_1+a)}}{y_1\,\cancel{(x_1+a)}}$$

$$\theta_2=Tan^{-1}\left(\frac{2a}{y_1}\right)$$

Compare *II & III* , $\boxed{\theta_1=\theta_2}$

Exercise 6.5

1. (i) Foci $(\pm 3, 0)$

Length of Minor Axis = 10

Here $c=3$ *and* $2b=10 \Rightarrow b=5$

Now $c^2=a^2-b^2 \Rightarrow a=a^2-25 \Rightarrow a^2=34$

Thus requied equation of the

Ellipse is $\dfrac{x^2}{34}+\dfrac{y^2}{25}=1$

From (1)

$a^2=34 \Rightarrow a=\pm\sqrt{34}$

$b^2=25 \Rightarrow a=\pm 5$

$\therefore$ Vertices of the ellipe on the x-axis are $(\pm\sqrt{34},0)$ and co-vertices are $(0,\pm 5)$ and graph of the ellipse.

(ii) Foci $(0,-1)$, & $(0,5)$

Length of Major Axis = 6

Here centre of the ellipse

$=\left(\dfrac{0+0}{2},\dfrac{-1-5}{2}\right)=(0,-3)$ and c is the distance from the centre to each focus. So

$C=\sqrt{(0-0)^2+(-3+1)^2}=\sqrt{0+4}=2 \Rightarrow \boxed{c=2}$

Also given that $2a=6$

$\Rightarrow a=3$

Now using $c^2=a^2-b^2$

$\Rightarrow 4=9-b^2 \Rightarrow b^2=9-4 \Rightarrow b^2=5$

From the foci we see that major axis is along the y-axis.

Thus required equation of the ellipse is

$\dfrac{(y+3)^2}{9}+\dfrac{x^2}{5}=1$

From (1) $a^2=9 \Rightarrow a=\pm 3$

$b^2=5 \Rightarrow b=\pm\sqrt{5}$

$\therefore$ vertices are $(0,\pm 3)$

i.e., $(0,0)$, $(0,-6)$ and convert ices are $(\sqrt{5},-3)$, $(\sqrt{-5},3)$ and the graph is.

(iii) Foci $(\pm 3\sqrt{3},0)$ **Vertices** $(\pm 6,0)$

Here $c=3\sqrt{3}$ *and* $\boxed{a=6}$

Now using $c^2=a^2-b^2$

$\Rightarrow 27=36-b^2 \Rightarrow \boxed{b^2=9}$

Then required equation of the ellipse is

$$\frac{x^2}{36} = \frac{y^2}{9} = 1 \quad (1)$$

Here $a^2 = 36 \Rightarrow a = \pm 6$

$\therefore$ Vertices $(6,0), (-6,0)$

and $b^2 = 9 \Rightarrow b = \pm 3$

So co-vertices are $(0,3), (0,-3)$

and the graph of (1) is

(iv) **Vertices $(-1,1), (5,1)$**

Foci $(4,1), (0,1)$

Here Mid point of the Foci

= Centre of the ellipse

$$= \left(\frac{4+0}{2}, \frac{1+1}{2}\right) = (2,1)$$

Distance between the vertices

$$= 2a = \sqrt{(5+1)^2 + (1-1)^2} = \sqrt{36} = 6$$

$$\Rightarrow 2a = 6 \Rightarrow \boxed{a=3}$$

Distance between the Foci $= 2c$

$$= \sqrt{(4-0)^2 + (1-1)^2} = \sqrt{16} = 4$$

$$\Rightarrow 2c = 4 \Rightarrow \boxed{c=2}$$

Now using $c^2 = a^2 - b^2 \Rightarrow b^2 = a^2 - c^2$

$$\Rightarrow b^2 = 9 - 4 = 5 \Rightarrow \boxed{b^2 = 5}$$

Thus required equation of the ellipse

$$\frac{(x-2)^2}{9} + \frac{(y-1)}{5} = 1$$

$\therefore$ Axis is 11 to the x-axis

Graph is

(v) **Foci $(\pm\sqrt{5}, 0)$ and through $\left(\frac{3}{2}, \sqrt{3}\right)$**

Here Foci are $(\sqrt{5},0)$ & $(-\sqrt{5},0)$

Centre is $= \left(\frac{\sqrt{5}-\sqrt{5}}{2}, \frac{0+0}{2}\right) = (0,0)$

and $c = \sqrt{5}$

Now using $c^2 = a^2 - b^2 \Rightarrow 5 = a^2 - b^2$

$$\Rightarrow \boxed{b^2 = a^2 - 5} \quad (i)$$

From the foci we see that Major Axis of the Ellipse is along the x-axis so the required equation is of the form.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{a^2-5}=1-(1)$ $\qquad b^2=a^2-5$

If thus ellipse passes through $\left(\frac{3}{2},\sqrt{3}\right)$

Then $\frac{9}{4a^2}+\frac{3}{a^2-5}=1$

$\Rightarrow 9(a^2-5)+12a^2=4a^2(a^2-5)$

$\Rightarrow 9a^2-45+12a^2=4a^4-20a^2$

$21a^2-45+4a^4+20a^2=0$

$-4a^4+41a^2-45=0$

$\Rightarrow 4a^4-41a^2+45=0$

Which is quadratic is a^2

$a^2=\frac{41\pm\sqrt{1681-720}}{8}=\frac{41\pm31}{8}$

$a^2=\frac{41+31}{8},\frac{41-31}{8}=\frac{72}{8},\frac{10}{8}$

$a^2=9,\frac{5}{4}$

For $a^2=9$

From (i) $b^2=9-5 \Rightarrow \boxed{b^2=4}$

For $a^2=\frac{5}{4}$

From (i) $b^2=\frac{5}{4}-5=\frac{5-20}{4}=\frac{-15}{4}<0$

$i.e.\ b^2=\frac{-15}{4}<0$ Neglecting

(which is not possible)

$\therefore$ Required equation is

$\frac{x^2}{9}+\frac{y^2}{4}=1$

$a^2=9 \Rightarrow a=\pm3$

$b^2=4 \Rightarrow b=\pm2$

& graph is

(vi) **Vertices** $(0,\pm5)$, $\quad e=\frac{3}{5}$ **(Sargodha 2009)**

Here $\boxed{a=5}$ $\quad e=\frac{3}{5}$, $\quad centre\ (0,0)$

Now $\because c=ae \Rightarrow c=5\left(\frac{3}{5}\right)=3$

$c=3$

Again using $c^2=a^2-b^2$

$\Rightarrow 9=25-b^2 \Rightarrow b^2=25-9 \Rightarrow \boxed{b^2=9}$

From the vertices $(0,5)$, $(0,-5)$ we see that axis of the ellipse is along y-axis.

$\therefore$ Required equation of the ellipse is

$$\frac{x^2}{9}+\frac{y^2}{25}=1$$

$$i.e.\ \frac{y^2}{25}+\frac{x^2}{9}=1 \quad (1)$$

From (1) $a^2=25 \Rightarrow a=\pm 5$

$\therefore$ vertices are $(0,5)$, $(0,-5)$

and $b^2=9 \Rightarrow b=\pm 3$

$\therefore$ co-vertices $(3,0)$, $(-3,0)$

Thus graph of the ellipse (1) is

(vii) **Centre $(0,0)$ *Focus* $(0,-3)$ *Vertex* $(0,4)$**

Here $c=3$, $a=4$

Now using $c^2=a^2-b^2$

$\Rightarrow b^2=a^2-c^2 \Rightarrow b^2=16-6=7$

$\boxed{b^2=7}$

Thus required equation of the ellipse

$$\frac{x^2}{16}+\frac{y^2}{7}=1 \quad (1)$$

$\because a^2=16 \Rightarrow a=\pm 4$

$\therefore$ Vertices $(0,4)$, $(0,-4)$

and $b^2=7 \Rightarrow b=\pm\sqrt{7}$

$\therefore$ Co-vertices $(\sqrt{7},0)$, $(-\sqrt{7},0)$

Centre $(0,0)$, Foci $(0,\pm 3)$,

So graph is

(viii) **Centre $(2,2)$**

$2a=8 \Rightarrow a=4$ // to $y-axis$

and $2b=6 \Rightarrow b=3$ //to $x-axis$

Required equation is

$$\frac{(y-2)^2}{16}+\frac{(x-2)^2}{9}=1$$

Vertices are $(2,2\pm 4)$

i.e. $(2,6)$, $(2,-2)$

Co-vertices are $(2\pm 3,2)$

i.e. $(5,2),(-1,2)$, *centre* $(2,2)$

Thus graph of (1) is

(xi) $C(0,0),\ (2,3),\ (6,1)$

Let equation of the ellipse is

$$\frac{x^2}{d_1^2}+\frac{y^2}{d_2^2}=1 \quad (1)$$

The ellipse passes through

$(2,3)$ *and* $(6,1)$

$$\therefore \frac{4}{d_1^2}+\frac{9}{d_2^2}=1 \quad (2)$$

$$\therefore \frac{36}{d_1^2}+\frac{1}{d_2^2}=1 \quad (3)$$

By multiplying equation (2) by 9 and then subtracting (3) from it.

$$\frac{36}{d_1^2}+\frac{81}{d_2^2}=9$$

$$\frac{36}{d_1^2}+\frac{1}{d_2^2}=-1$$

$$\frac{80}{d_2^2}=8 \Rightarrow 8d_2^2=80$$

$$\Rightarrow \boxed{d_2^2=10}$$

Putting $d_2^2=10$ *in* (2) *we get*

$$\frac{4}{d_1^2}+\frac{9}{10}=1 \Rightarrow \frac{4}{d_1^2}=1-\frac{9}{10}$$

$$\Rightarrow \frac{4}{d_1^2}=\frac{1}{10} \Rightarrow \boxed{d_1^2=40}$$

Thus required equation of the ellipse is

$$\frac{x^2}{40}+\frac{y^2}{10}=1$$

Here $C(0,0)$

$a^2=40 \Rightarrow a=2\sqrt{10}$

$\therefore$ Vertices $(\pm2\sqrt{10},0)$

and $b^2=10 \Rightarrow b=\pm\sqrt{10}$

$\therefore$ co-vertices $(0,\pm\sqrt{10})$

Points on the ellipse are $(2,3)$ & $(6,1)$

Thus graph is

(x) **Centre** $C(0,0),\ (3,1),\ (4,0)$

Let equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (1)$$

As the points $(3,1)$ *and* $(4,0)$

Lie on (1)

$\therefore \frac{9}{a^2}+\frac{1}{b^2}=1 \quad (2)$

and $\frac{16}{a^2}+0=1 \Rightarrow \frac{16}{a^2}=1$

$\Rightarrow \boxed{a^2=16}$

Putting $a^2=16$ in (2) we get

$\frac{9}{16}+\frac{1}{b^2}=1 \Rightarrow \frac{1}{b^2}=1-\frac{9}{16}=\frac{7}{16}$

$7b^2=16 \Rightarrow \boxed{b^2=\frac{16}{7}}$

$\therefore$ Required equation of the ellipse is

$\frac{x^2}{16}+\frac{y^2}{\frac{16}{7}}=1$

$\Rightarrow \frac{x^2}{16}+\frac{7y^2}{16}=1$

$\because a^2=16 \Rightarrow a=\pm 4$

So vertices $(\pm 4,0)$

$b^2=\frac{16}{7} \Rightarrow 6=\pm\frac{4}{\sqrt{7}}$

Co-vertices $\left(0,\frac{4}{\sqrt{7}}\right)$

Graph is

2. **(i)** $x^2+4y^2=16$ **(Sargodha 2010, 11)**

$\Rightarrow \frac{x^2}{16}+\frac{4y^2}{16}=\frac{16}{16}$

$\Rightarrow \frac{x^2}{16}+\frac{y^2}{4}=1 \quad (1)$

Here $a^2=16 \Rightarrow a=4$

$b^2=4 \Rightarrow b=2$

and $c^2=a^2-b^2=16-4 \Rightarrow c^2=12 \Rightarrow c=2\sqrt{3}$

Now Center is (0,0)

Foci are $(\pm 2\sqrt{3},0)$

Eccentricity: $e=\frac{c}{a}=\frac{2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}$

Vertices: $(\pm 4,0)$

Directrices: $x=\pm\frac{c}{e^2}=\pm\frac{a}{e^2}$

$\Rightarrow x = \pm\frac{a}{e} = \pm\frac{4}{\frac{\sqrt{3}}{2}}$

$\Rightarrow x = \pm\frac{8}{\sqrt{3}}$

(ii) $9x^2 + y^2 = 18$

$\Rightarrow \frac{9x^2}{18} + \frac{y^2}{18} = \frac{18}{18}$

$\frac{x^2}{2} + \frac{y^2}{18} = 1$ (1)

Here $a^2 = 18 \Rightarrow a = 3\sqrt{2}$

$b^2 = 2 \Rightarrow b = \sqrt{2}$

and $c^2 = a^2 - b^2 = 18 - 2 = 16$

$c = 4, e = \frac{c}{a} = \frac{4}{3\sqrt{2}}$

Now centre: (0,0)

Foci: $(0, \pm c) = (0, \pm 4)$

Eccentricity: $e = \frac{c}{a} = \frac{4}{3\sqrt{2}}$

$\Rightarrow e = \frac{4}{3\sqrt{2}}$

Vertices: $(0, \pm a) = (0, \pm 3\sqrt{2})$

Directrices: $y = \pm\frac{c}{e^2} = \pm\frac{ae}{e^2}$

$\Rightarrow y = \pm\frac{a}{e} \Rightarrow y = \pm\frac{3\sqrt{2}}{\frac{4}{3\sqrt{2}}} = \pm\frac{18}{4}$

$\Rightarrow y = \pm\frac{9}{2}$

3. (iii) $25x^2 + 9y^2 = 225$ **(Sargodha 2008, 11 Gujrawala 2010)**

$\Rightarrow \frac{25x^2}{225} + \frac{9y^2}{225} = \frac{225}{225}$

$\frac{x^2}{9} + \frac{y^2}{25} = 1$ (1)

Here $a^2 = 25 \Rightarrow a = 5$

$b^2 = 9 \Rightarrow b = 3$

Using $c^2 = a^2 - b^2$

$\Rightarrow c^2 = 25 - 9 = 16 \Rightarrow c = 4$

Now centre of the ellipse (1) is (0,0)

Foci : $(0, \pm 4)$

Eccentricity: $e=\frac{c}{a}=\frac{4}{5}$

Vertices: $(0,\pm a)=(0,\pm 5)$

Directices: $y=\pm\frac{c}{e^2}=\pm\frac{ae}{e^2}$

$\Rightarrow y=\pm\frac{a}{e}=\pm\frac{5}{\frac{4}{5}}=\pm\frac{25}{4}$

$\Rightarrow y=\pm\frac{25}{4}$

(iv) $\frac{(2x-1)^2}{4}+\frac{(y+2)^2}{16}=1 \qquad (1)$

$\Rightarrow \frac{x^2}{4}+\frac{y^2}{16}=1 \qquad (1) \quad X=2x-1 \quad Y=y+2$

$a^2=16 \Rightarrow a=4,\ b^2=4 \ \Rightarrow b=2$

and $c^2=a^2-b^2=16-4=12 \Rightarrow c=2\sqrt{3}$

For centre $X=0,\ Y=0$

$\Rightarrow 2x-1=0 \ \Rightarrow y+2=0$

$x=\frac{1}{2} \qquad y=-2$

$\therefore$ Centre is $\left(\frac{1}{2},-2\right)$

For Foci: $X=0 \quad Y=\pm c$

$\Rightarrow 2x-1=0 \quad y+2=\pm 2\sqrt{3}$

$x=\frac{1}{2} \quad y=-2\pm 2\sqrt{3}$

$\therefore$ Foci: $\left(\frac{1}{2},-2\pm 2\sqrt{3}\right)$

Eccentricity $e=\frac{c}{a}=\frac{2\sqrt{3}}{4}$

$\Rightarrow e=\frac{\sqrt{3}}{2}$

Directrices $Y=\pm\frac{c}{e^2}=\pm\frac{ae}{e^2}=\pm\frac{a}{e}$

$\Rightarrow y+2=\pm\frac{4}{\frac{\sqrt{3}}{2}} \Rightarrow y=-2\pm\frac{8}{\sqrt{8}}$

(v) $x^2+16x+4y^2-16y+76=0$ **(Sargodha 2010)**

$\Rightarrow x^2-16x+64+4(y^2-4y)+76=64$

$\Rightarrow (x+8)^2+4(y^2-4y+4-4)=64-76$

$(x+8)^2+4(y-2)^2-16=-12$

$(x+8)^2+4(y-2)^2=-12+16$

$(x+8)^2 4(y-2)^2=4$

$\Rightarrow \frac{(x+8)^2}{4}+\frac{4(y-2)^2}{4}=\frac{4}{4}$

$\frac{(x+8)^2}{4}+\frac{4(y-2)^2}{4}=1$

Which is of the form

$\frac{X^2}{4}+\frac{Y^2}{4}=1$

Where $X=x+8,\ Y=y-2$

$a^2=4 \Rightarrow a=2$

$b^2=1 \Rightarrow b=1$

Now using $c^2=a^2-b^2$

$=4-1=3$

$c=\sqrt{3}$

For centre $X=0,\ Y=0$

$\Rightarrow x+8=0,\quad y-2=0$

$x=-8,\ y=2$

$\therefore$ Required centre of the ellipse is $(-8,2)$

For Foci:

$X=\pm c,\ Y=0$

$\Rightarrow x+8=\pm\sqrt{3},\ y-2=0$

$x=-8\pm\sqrt{3},\ y-2=0$

$\therefore$ Foci $(-8\pm\sqrt{3},2)$

For Eccentricity $e=\frac{c}{a}=\frac{\sqrt{3}}{2}$

For Vertices

$X=\pm a,\ Y=0$

$\Rightarrow x+8=\pm 2,\ y-2=0$

$x=-8\pm 2=-6,\ (-10,2)\ \boxed{y=2}$

$\therefore$ Vertices are $(-6,2),\ (-10,2)$

Directrices $X=\pm\frac{c}{e^2}=\pm\frac{\sqrt{3}}{\frac{3}{4}}$

$\Rightarrow x+8=\pm\frac{4\sqrt{3}}{3}=\pm\frac{4}{\sqrt{3}}$

$\Rightarrow x=-8\pm\frac{4}{\sqrt{3}}$

(vi) $25x^2+4y^2-250x-16y+514=0$

$\Rightarrow 25x^2-250x+4y^2-16y=-541$

$25(x^2-10x)+4(y^2-4y)=-541$

$25(x^2-10x+25-25)+4(y^2-4y+4-4=-541$

$25\left[(x-5)^2-25\right]+\left[(y-2)^2-4\right]=-541$

$25(x-5)^2-625+4(y-2)^2-16=-541$

$25(x-5)^2+4(y-2)^2=-541+625+16$

$25(x-5)^2+4(y-2)^2=100$

$\frac{25(x-5)^2}{100}+\frac{4(y-2)^2}{100}=\frac{100}{100}$

$\frac{(x-5)^2}{4}+\frac{(y-2)^2}{25}=1$ (1)

Which is of the form:

$\frac{X^2}{4}+\frac{Y^2}{25}=1$ (2)

Where $X=x-5,\ Y=y-2$

$a^2=25 \Rightarrow a=5$

$b^2=4 \Rightarrow b=2$

Now using $c^2=a^2-b^2$

$\Rightarrow c^2=25-4=21 \Rightarrow c=\sqrt{21}$

Now for centre

$X=0$ *and* $Y=0$

$\Rightarrow x-5=0 \quad y-2=0$

$x=5 \quad y=2$

$\therefore$ Centre of the Ellipse (5, 2)

For Foci:

$X=0$ *and* $Y=\pm c$

$\Rightarrow x-5=0 \quad y-2=\pm\sqrt{21}$

$x=5 \quad y=2\pm\sqrt{21}$

$\therefore$ Foci: $(5,\ 2\pm\sqrt{21}$

Eccentricity $e=\frac{c}{a}=\frac{\sqrt{21}}{5}$

For Vertices

$X=0$ *and* $Y=\pm a$

$\Rightarrow x-5=0 \quad y-2=\pm 5$

$x=5 \quad y=2\pm 5=7,\ -3$

$\therefore$ Vertices are

$(5,-3,\ (5,7)$

Directrices: $Y = \pm\frac{c}{e^2}$

$$\Rightarrow y-2=\pm\frac{\frac{\sqrt{21}}{21}}{\frac{21}{25}}=\frac{25\sqrt{21}}{21}$$

$$=1\pm\frac{25}{\sqrt{21}}$$

4. $0<c<a,\ F(-c,0),\ F'(c,0),\ P(x,y)$

Given that $|PF|+|PF|=2a$

$$\Rightarrow \sqrt{(x+c)^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2} \qquad (1)$$

Squaring both sides of (1)

$$(x+c)^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2+c^2-2cx+y^2$$

$$\Rightarrow 2cx+2cx-4a^2=-4a\sqrt{(x+c)^2+y^2}$$

$$4cx-4a^2=-4a\sqrt{x^2-2cx+c^2+y^2}$$

$$4(cx-a^2)=-4a\sqrt{x^2+y^2+c^2-2cx}$$

$$\Rightarrow a^2-cx=a\sqrt{x^2+y^2+c^2-2cx}$$

Again squaring both sides of (2) we set.

$$a^4+c^2x^2-2a^2cx=a^2x^2+a^2y^2+a^2c^2-2a^2cx$$

$$\Rightarrow c^2x^2-a^2x^2-a^2y^2=a^2c^2-a^4$$

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$$

Dividing both sides by $a^2(a^2-c^2)$

$$\frac{(a^2-c^2)x^2}{a^2(a^2-c^2)}+\frac{a^2y^2}{a^2(a^2-c^2)}=\frac{a^2(a^2-c^2)}{a^2(a^2-c^2)}$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

i.e, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Where $a^2-c^2=b^2$

Which is an ellipse.

5. $P(x,y)\quad (0,0),\ (1,1),2$

We name the given points

$O(0,0)$ & $A(1,1)$

Now given that

$|OP|+|AP|=2$

i.e., $\sqrt{(x-0)^2+(y-0)^2}+\sqrt{(x-1)^2+(y-1)^2}=2$

$$\Rightarrow \sqrt{(x-1)^2+(y-1)^2}=2-\sqrt{x^2+y^2}$$

Squaring both sides of (1) we have

$x^2-2x=1+y^2-2y+1=4=x^2+y^2-4\sqrt{x^2+y^2}$

$-2x-2y-2=-4\sqrt{x^2+y^2}$

$\Rightarrow x+y+1=2\sqrt{x^2+y^2}$

or $2\sqrt{x^2+y^2}=x+y+1$

Again squaring both sides of (2)

$4(x^2+y^2)=x^2+y^2=1+2xy+2x=2y$

$4x^2+4y^2-x^2-y^2-2xy-2x-2y-1=0$

$3x^2+3y^2-2xy-2x-2y-1=0$

$3x^2-2xy\_3y^2-2x-2y-1=0$

6. **Latusrectum: The focal chord perpendicular to the Major Axis is called latusrectum of the Ellipse.**

Let us consider the Ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (1)\ a>b$

Suppose LL' be the Focus is $F(c,0)$

Then the point

L *is* $L(e,y_1)$ when Focus is $F(c,0)$

$\because L(e,y_1)$ lies on (1)

$\therefore \frac{c^2}{a^2}+\frac{y_1}{b^2}=1 \quad y_1^2=b^2\left(1-\frac{c^2}{a^2}\right)$

$\Rightarrow y_1^2=b^2\left(\frac{a^2-c^2}{a^2}\right)$

$y_1^2=b^2\left(\frac{b^2}{a^2}\right)$

$\because a^2-c^2=b^2$

$\Rightarrow y_1^2=\frac{b^4}{a^2}\Rightarrow y_1=\pm\frac{b^2}{a^2}$

$\because$ The Point L and L' are

$L\left(c,\frac{b^2}{a}\right)$ *and* $L'\left(c,\frac{-b^2}{a}\right)$

Now $|LL'|=\sqrt{(c-c)^2+\left(\frac{b^2}{a}+\frac{b^2}{a}\right)^2}=\sqrt{\left(\frac{b^2+b^2}{a}\right)}=\frac{2b^2}{a}$

Thus $|LL'|=\frac{2b^2}{a}$ Hence proved.

7. **Given $2a=4\sqrt{2}$**

$\Rightarrow a=2\sqrt{2}\Rightarrow \boxed{a^2=8}$

Also given that $2c = 2b$

$\Rightarrow c = b$

Now using $c^2 = a^2 - b^2$

$\Rightarrow b^2 = a^2 - b^2 \Rightarrow 2b^2 = a^2$

$\Rightarrow 2b^2 = 8 \Rightarrow \boxed{b^2 = 4}$

Thus required equation of the Ellipse is $\frac{x^2}{8} + \frac{y^2}{4} = 1$

8. **Let the sum be at F these we have**

$a - c = 17 \quad (1)$

$a + c = 183 \quad (2)$

Adding (1) and (2)

$2a = 200 \Rightarrow \boxed{a = 100}$

Putting $a = 100$ *in* (2) *we get*

$100 + c = 183 \Rightarrow c = 83$

Now using $c^2 = a^2 - b^2$

$\Rightarrow b^2 = a^2 - c^2 = (100)^2 - (83)^2$

$= 10000 - 6889 = 3111$

$\boxed{b^2 = 3111}$

$\therefore$ The equation is

$$\frac{x^2}{100^2} + \frac{y^2}{3111} = 1$$

9. **Here**

$2a = 90 \Rightarrow \boxed{a = 45}$

and $b = 30$

$\therefore$ Equation of the ellipse is

$$\frac{x^2}{(45)^2} + \frac{y^2}{(30)^2} = 1$$

At the height $20\sqrt{2m}$ *Let* x_1 be the distance from the centre then the point $(x_1 20\sqrt{2})$ lies

on the ellipse (1).

$$\therefore \frac{x_1^2}{(45)^2} + \frac{(20\sqrt{2})^2}{(30)^2} = 1$$

$$\frac{x_1^2}{2025} + \frac{800}{900} = 1 \quad \Rightarrow \frac{x_1^2}{2025} = 1 - \frac{8}{9}$$

$$\Rightarrow \frac{x_1^2}{2025} = \frac{1}{9} \quad \Rightarrow 9x_1^2 = 2025$$

$\Rightarrow x_1^2 = 225 \quad \Rightarrow x_1 = \pm 15$

$\Rightarrow x_1 = 15m$ (Neglecting Negative value of x_1)

$\therefore$ Required distance from the centre = 15m.

10. Let the earth be at F

Given that

$2a = 768,806km$

$\Rightarrow \boxed{a = 384403\,km}$

$2b = 767,746\,km$

$b = 383873\,km$

Using $c^2 = a^2 - b^2$

$\Rightarrow c^2 = (a-b)(a+b)$

$\Rightarrow c^2 = 530(768276)$

$c^2 = 407186280$

$c = 20178.86$

Now required greatest distance

$= a + c = 404582\,km \quad (Approx)$

and Least distance $= a - c$

$= 364224\,km\ (Approx)$

HYPERBOLA **(Sargodha 2010, Gujrawala 2010)**

Let $e > 1$ and F be a fixed point and L be a line not containing F. Also Let (x, y) be a point in the plane and $|PM|$ be the perpendicular distance of P from L. The set of all points $P(x, y)$

Such that $\frac{|PF|}{|PM|} = e > 1$ is called a Hyperbola.

F is the Focus, L is the Directrix and $e > 1$ is the Eccentricity of the Hyperbola point $B(0,-6)$ and $B'(0,6)$ is called Conjugate of the hyporbola.

The mid point $(0,0)$ of AA' is called Centre.

Exercise 6.6

1. (i) **Centre** $(0,0)$, **Focus** $(6,0)$, **Vertex** $(4,0)$, **(Sargodha 2011)**

There $c=6, \quad a=4$

Now using $c^2=a^2+b^2$

$\Rightarrow 36=16+b^2 \Rightarrow b^2=36-16$

$\Rightarrow \boxed{b^2=20}$

Also x-axis is the transverse

Axis of the hyperbola $y=\pm\frac{\sqrt{5}}{2}x$

$\therefore$ Required equation is

$\frac{x^2}{16}-\frac{y^2}{20}=1$

(ii) **Foci** $(\pm5,0)$ **Vertex** $(3,0)$, **(Sargodha 2009)**

Here $c=5, \quad a=3$

Now using $c^2=a^2+b^2 \Rightarrow b^2=c^2-a^2$

$\Rightarrow b^2=25-9=16 \Rightarrow b=4$

$\therefore$ Equation of the hyperbola

$\frac{x^2}{9}-\frac{y^2}{16}=1$

Asymptotes are

$y=\pm\frac{4}{3}x$

Centre (0,0)

Transverse Axis is x-axis

(iii) **Foci** $(2\pm5\sqrt{2},-7)$

$F(2+5\sqrt{2},-7),\ F(2-5\sqrt{2},-7)$

Mid point of Foci is the centre

$\therefore$ Centre $=\left(\frac{2+5\sqrt{2}+2-5\sqrt{2}}{2},\frac{-7-7}{2}\right)=(2,-7)$

Given that $2a=10 \Rightarrow a=5$

Now $|FF'|=2c=\sqrt{(2+5\sqrt{2}-2+5\sqrt{2})^2+(-7+7)^2}$

$\Rightarrow 2c=10\sqrt{2} \Rightarrow c=5\sqrt{2}$

Using $c^2=a^2+b^2$

$\Rightarrow 50=25+b^2$

$\Rightarrow b^2=25$

Transverse Axis is along the horizontal line

$y=-7$

and $a = 5$ so vertices are

$(2 \pm 5, -7) \Rightarrow (7, -7)$ and $(-3, -7)$

(iv) **Foci $(0, \pm 9)$ Directrices $y = \pm 4$**

Transverse Axis is y-axis

$c = 9 \Rightarrow ae = 9 \quad (1)$

and $\frac{a}{e} = 4 \Rightarrow e = \frac{a}{4} \quad (2)$

Using (2) in (1) we have

$a\left(\frac{a}{4}\right) = 9 \Rightarrow a^2 = 36 \Rightarrow a = 6$

From (2) $e = \frac{6}{4} = \frac{3}{2}$

Now using $c^2 = a^2 + b^2$

$\Rightarrow 81 - 36 + b^2 \Rightarrow b^2 = 81 - 36$

$b^2 = 45$

Now required equation of the hyperbola

$\frac{y^2}{36} - \frac{x^2}{45} = 1$

Foci $(0, 9), (0, -9)$

Vertices $(0, 6), (0, -6)$

Centre (0,0) and graph is.

(v) **Centre (2,2) Horizontal transverse axis**

$2a = 6 \Rightarrow a = 3, \ e = 2$

$c = ae = 3(2) \Rightarrow c = 6$

Now using $c^2 = a^2 + b^2$

$\Rightarrow 36 = 9 + b^2 \Rightarrow b^2 = 27$

Thus required equation of the hyperbola is

$\frac{(x-2)^2}{9} - \frac{(y-2)^2}{27} = 1$

Which is of the form

$\frac{x^2}{9} - \frac{y^2}{27} = 1$

Where $X = x - 2$, $Y = y - 2$

Centre $X = 0$, $Y = 0$

$\Rightarrow x - 2 = 0$, $y - 2 = 0$

$\Rightarrow x = 2$, $y = 0$

$\therefore$ Centre is (2,2)

Foci $(\pm c, 0)$

i.e., $X = \pm c \qquad Y = 0$

$x - 2 = \pm b \qquad y - 2 = 0$

$x = 2 \pm 5 = 8, -4 \qquad y = 2$

$\therefore$ Foci $(-4, 2)$, $(8, 2)$

Vertices $(\pm a, 0)$

i.e, $X = \pm a \qquad Y = 0$

$x = 2 \pm 3 \qquad y - 2 = 0$

$x = 2 \pm 3 = 5 - 1$

$\therefore$ Vertices are $(-1, 2)$. $(5, 2)$

Thus graph is

(vi) **Vertices $(2, \pm 3)$ i.e, $(2, 3)$, $(2, -3)$ point on the hyperbola $(0, 5)$**

Center of the hyperbola is the mid point of the vertices $= \left(\frac{2+2}{2}, \frac{3-3}{2}\right) = (2, 0)$

Now vertices $A(2, 3)$, $A'(2, -3)$

$2a = |AA'| = \sqrt{(2-2)^2 + (3+3)^2} = 6$

$2a = 6 \Rightarrow a = 3$

From the vertices transverse

Axis is vertical so the equation is

$\frac{y^2}{9} - \frac{(x-2)^2}{b^2} = 1 \quad (1)$

$\because$ The point (0, 5) lies on (1)

$\therefore \frac{25}{9} - \frac{4}{b^2} = 1 \Rightarrow \frac{4}{b^2} = \frac{25}{9} - 1$

$\Rightarrow \frac{4}{b^2} = \frac{16}{9} \Rightarrow 16b^2 = 36 \Rightarrow b^2 = \frac{36}{16}$

$\Rightarrow b^2 = \frac{9}{4}$

Thus equation (1) becomes

$\frac{y^2}{9} - \frac{(x-2)^2}{\frac{9}{4}} = 1$

Which of the form

$\frac{Y^2}{9} - \frac{X^2}{\frac{9}{4}} = 1$

Where $Y = y$, $X = x - 2$

$a^2 = 9, \quad b^2 = \frac{9}{4}$

$c^2 = a^2 + b^2 = 9 + \frac{9}{4} = \frac{45}{4}$

$c = \frac{3\sqrt{5}}{2}$

Foci $(0, \pm c)$

i.e., $X=0$, $Y=\pm c$

$x-2=0$ $\quad y=\pm\frac{3\sqrt{5}}{2}$

$x=2$

$\therefore$ Foci are $\left(2,\frac{3\sqrt{5}}{2}\right),\left(2,\frac{-3\sqrt{5}}{2}\right)$

Centre (2,0)

Vertices (2,3), (2, −3)

(vii) Foci $(5,-2)$, $(5,4)$

One vertex (5,3)

Transverse Axis is parallel to the y-axis

Center Mid point of FF'

$=\left(\frac{5+5}{2},\frac{-2+4}{2}\right)$

$a=$ Length between the centre & the vertex (5,3)

$=\sqrt{(5-5)^2+(3-1)^2}=2$

$\boxed{a=2}$

$2c=|FF'|=\sqrt{(5-5)^2+(4+2)^2}$

$2c=6 \Rightarrow \boxed{c=3}$

Using $c^2=a^2+b^2$

$9=4+b^2 \Rightarrow b^2=5$

$\boxed{b^2=5}$

Now required equation of the hyperbola is

$\frac{(y-1)^2}{4}-\frac{(x-5)^2}{5}=1$

Which is of the form

$\frac{Y^2}{4}-\frac{X^2}{5}=1$

Where $X=x-5$, $Y=y-1$

$a^2=4,\ \ b^2=5$

Centre (5,1)

Vertices $(0,\pm a)$

i.e., $X=0\ \ Y=\pm a$

$x-5=0$, $y-1=\pm 3$

$x=5\ \ \ y=1\pm 3=4,-2$

$\therefore$ Foci are $(5,-2)$, $(5,4)$

Now graph of the hyperbola is

2. (i) $x^2 - y^2 = 9$

$\Rightarrow \frac{x^2}{9} - \frac{y^2}{9} = 1 \qquad (1)$

Here $a^2 = 9 \Rightarrow a = 3$ Transverse Axis is along X-axis.

$b^2 = 9 \Rightarrow b = 3$

Using

$c^2 = a^2 + b^2 = 9 + 9 = 18 \Rightarrow c = 3\sqrt{2}$

Now centre of (1) is (0,0)

Foci $(\pm c, 0) \Rightarrow (\pm 3\sqrt{2}, 0)$

Eccentricity $e = \frac{c}{a} = \frac{3\sqrt{2}}{3} = \sqrt{2}$

Vertices $(\pm c, 0) \Rightarrow (\pm 3, 0)$

Directrices $x = \pm \frac{c}{e^2} = \pm \frac{3\sqrt{2}}{2}$

$\Rightarrow x = \pm \frac{3}{\sqrt{2}}$

(ii) $\frac{x^2}{4} - \frac{y^2}{9} = 1$ **(Sargodha 2008, 11)**

Here

$a^2 = 4 \Rightarrow a = 2$ Transverse Axis is along the x-axis

$b^2 = 9 \Rightarrow b = 3$

Using $c^2 = a^2 + b^2 = 4 + 9 = 13$

$c = \sqrt{13}$

Now centre is (0,0)

Foci are $(\pm 2, 0)$

Eccentricity $e = \frac{c}{a} = \frac{\sqrt{13}}{2}$

Vertices $(\pm 2, 0)$

Directrices $x = \pm \frac{c}{e^2} = \pm \frac{\sqrt{13}}{\frac{13}{4}}$

$\Rightarrow x = \pm \frac{4\sqrt{13}}{13} = \pm \frac{4}{\sqrt{13}}$

(iii) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

Transverse Axis is along y-axis

Here $a^2 = 16 \Rightarrow a = 4$

$b^2 = 9 \Rightarrow b = 3$

Using $c^2 = a^2 + b^2 \Rightarrow c^2 = 16 + 9$

$c^2 = 25 \Rightarrow c = 5$

Now centre of (1) is (0,0)

Foci are $(0, \pm 5)$

Eccentricity $c = \frac{c}{a} = \frac{5}{4}$

Vertices $(0, \pm 4)$

Directrices $y = \pm \frac{c}{e^2} = \pm \frac{5}{\frac{25}{16}}$

$\Rightarrow y = \pm 5 \times \frac{16}{25} = \pm \frac{16}{5}$

(iv) $\frac{y^2}{4} - \frac{x^2}{1} = 1$

Transverse Axis is along y-axis

Here $a^2 = 4 \Rightarrow a = 2$

$b^2 = 41 \Rightarrow b = 1$

Using $c^2 = a^2 + b^2 \Rightarrow c^2 = 4 + 1 = 5$

$\Rightarrow c = \sqrt{5}$

Now centre (0,0)

Foci are $(0, \pm\sqrt{5})$

Eccentricity $e = \frac{e}{a} = \frac{5}{4}$

Vertices are $(0, \pm 2)$

Equation of directrices

$y = \pm \frac{c}{e^2} = \frac{\sqrt{5}}{\frac{5}{2}} = \frac{4\sqrt{5}}{5}$

$y = \pm \frac{4}{\sqrt{5}}$

(v) $\frac{(x-1)^2}{2} - \frac{(y-1)^2}{9} = 1$ (1)

(1) is of the form

$\frac{X^2}{2} - \frac{Y^2}{9} =$ (2)

Where $X = x - 1$ $Y = y - 1$

and $a^2 = 2 \Rightarrow a = \sqrt{2}$ $b^2 = a \Rightarrow b = 3$

For Centre (0,0) $c = \sqrt{2+9} = \sqrt{11}$

$X = 0$ $Y = 0$ Transverse Axis is 11 to x-axis

$\Rightarrow x - 1 = 0$ $y - a = 0$

$x = 1$ $y = 1$

$\therefore$ Centre (1,1)

For Foci $(\pm c,0)$

$X=\pm\sqrt{11},\ Y=0$

$x-1=\pm\sqrt{11},\ y-a=0$

$x=\pm\sqrt{11},\ y=1$

$\therefore$ Foci are $(1\pm\sqrt{11},1)$

For Eccentricity

$e=\dfrac{c}{a}\ \Rightarrow\ e=\dfrac{\sqrt{11}}{\sqrt{2}}$

$\Rightarrow\ e=\sqrt{\dfrac{11}{2}}$

For Vertices $(\pm a,0)$

$X=\pm\sqrt{2},\ Y=0$

$x-1=\pm\sqrt{2},\ y-1=0$

$x=1\pm\sqrt{2},\ y=1$

$\therefore$ Vertices are $(1\pm\sqrt{2},1)$

Equation of directrices

$X=\pm\dfrac{c}{e^2}=\pm\dfrac{ae}{e^2}=\pm\dfrac{a}{e}$

$\Rightarrow x-1=\pm\dfrac{\sqrt{2}}{\sqrt{\dfrac{11}{2}}}\Rightarrow x=1\pm\dfrac{2}{\sqrt{11}}$

(vi) $\dfrac{(y+2)^2}{9}-\dfrac{(x-2)^2}{16}=1$ (1)

Which is of the form

$\dfrac{Y^2}{9}-\dfrac{X^2}{16}=$ (2)

Where $Y=y+2$, $X=x-2$

$a^2=7\ \Rightarrow\ a=3$

$b^2=16\ \Rightarrow\ b=4$

Transverse is parallel to the y-axis

Now using $c^2=a^2+b^2$

$\Rightarrow c^2=a+16=25\Rightarrow c=5$

For centre of (1)

$X=0$, $Y=0$

$x-2=0$, $y+2=0$

$x=2$, $y=-2$

$\therefore$ Centre is $(2,-2)$

For Foci $(0,\pm c)$

$\Rightarrow X=0 \qquad Y=\pm 5$

$\Rightarrow x-2=0 \qquad y+2=\pm 5$

$x=2 \qquad y=-2\pm 5$

$y=-2+5, -2-5$

$y=3, -7$

$\therefore$ Foci are

$(2,3), (2,-7)$

Eccentricity $e=\dfrac{c}{a}$

$\Rightarrow e=\dfrac{5}{3}$

For Vertices $(0,\pm a)$

$\Rightarrow X=0 \qquad Y=\pm a$

$\Rightarrow x-2=0 \qquad y-2=\pm 3$

$x=2 \qquad y=-2=\pm 3$

$y=-2+3, -2-3$

$y=1, -5$

$\therefore$ Vertices are $(2,1),(2,-5)$

Equation of Directrices

$Y=\pm\dfrac{c}{a} \Rightarrow y+2=\pm\dfrac{5}{\frac{25}{9}}$

$\Rightarrow y+2=\pm\dfrac{45}{25}$

$y=-2\pm\dfrac{9}{5}$

(vii) $9x^2-12x-y^2-2y+2=0$ **(Gujrawala 2010)**

$\Rightarrow 9\left(\dfrac{9x^2}{9}-\dfrac{12x}{9}\right)-(y^2-2y)=-2$

$9\left(x^2-\dfrac{4}{3}x\right)-(y^2-2y)=-2$

$9\left(x^2-\dfrac{4}{3}+\dfrac{4}{9}-\dfrac{4}{9}\right)-(y^2+2y+1-1)=-2$

$9\left[\left(x^2-\dfrac{4}{3}\right)-\dfrac{4}{9}\right]-\left[(y+1)^2-1\right]=-2$

$9\left(x-\dfrac{2}{3}\right)^2-4-(y+1)^2+1=-2$

$9\left(x-\dfrac{2}{3}\right)^2-(y+1)^2=-2-1+4$

$$9\left(x-\frac{2}{3}\right)^2-(y+1)^2=1$$

$$\frac{\left(x-\frac{2}{3}\right)^2}{\frac{1}{9}}-\frac{(y+1)^2}{1}=1$$

Which is of the form

$$\frac{X^2}{\frac{1}{9}}-\frac{Y^2}{1}=1$$

When $X=x-\frac{2}{3}$, $Y=y+1$

$a^2=\frac{1}{9} \Rightarrow a=\frac{1}{3}$

$b^2=1 \Rightarrow b=1$

Using $c^2=a^2+b^2=\frac{1}{9}+1=\frac{10}{9}$

$c=\frac{\sqrt{10}}{3}$

Now for centre of (1)

$X=0 \qquad Y=0$

$\Rightarrow x-\frac{2}{3}=0 \qquad y+1=0$

$x=\frac{2}{3} \qquad y=-1$

$\therefore$ Centre is $\left(\frac{2}{3},-1\right)$

For Foci

$X=\pm c \qquad Y=0$

$\Rightarrow x-\frac{2}{3}=\pm\frac{\sqrt{10}}{3} \qquad y+1=0$

$x=\frac{2}{3}\pm\frac{\sqrt{10}}{3} \qquad y=-1$

$\therefore$ Required Foci $\left(\frac{2}{3}\pm\frac{\sqrt{10}}{3},1\right)$

Eccentricity $e=\frac{c}{a}$

$$\Rightarrow c=\frac{\frac{\sqrt{10}}{3}}{\frac{1}{3}} \Rightarrow c=\sqrt{10}$$

Vertices

$X=\pm a \qquad Y=0$

$\Rightarrow x-\frac{2}{3}=\pm\frac{1}{3} \qquad y+1=0$

$x=\frac{2}{3}\pm\frac{1}{3} \qquad y=-1$

$x=1,\frac{1}{3}$

$\therefore$ Foci are $\left(\frac{1}{3},-1\right)$ & $(1,-1)$

Equation of Directrices

$$X=\pm\frac{c}{e^2} \Rightarrow x-\frac{2}{3}=\pm\frac{\frac{\sqrt{10}}{3}}{\frac{10}{1}}$$

$$x=\frac{2}{3}\pm\frac{1}{3\sqrt{10}}$$

(viii) $4x^2+12y-x^2+4x+1=0$

$4(y^2+3y)-(x^2-4x)=-1$

$4\left[y^2+3y+\frac{9}{4}-\frac{9}{4}\right]-\left[x^2-4x+4-4\right]=-1$

$4\left[\left(y+\frac{3}{2}\right)^2-\frac{9}{4}\right]-\left[(x-2)^2-4\right]=-1$

$4\left(y+\frac{3}{2}\right)^2-9-(x-2)^2+4=-1$

$4\left(y+\frac{3}{2}\right)^2-(x-2)^2=4$

$$\Rightarrow\frac{\left(y+\frac{3}{2}\right)^2}{1}-\frac{(x-2)^2}{4}=1$$

Which is of the form

$$\frac{X^2}{1}-\frac{Y^2}{4}=1$$

When $Y=y+\frac{2}{3}$, $X=x-2$

$a^2=1 \Rightarrow a=1$ **Transverse Axis is along** $X=0$

$b^2=4 \Rightarrow b=2$ $\Rightarrow x-2=0$

Using $c^2=a^2+b^2$ *i.e,* along $x=2$

$\Rightarrow c^2=1+4 \Rightarrow c^2=5 \Rightarrow c=\sqrt{5}$

For Centre

$X=0 \qquad Y=0$

$\Rightarrow \; x-2=0 \qquad y+\frac{3}{2}=0$

$x=2 \qquad y=-\frac{3}{2}$

$\therefore$ Centre is $\left(2,\frac{-3}{3}\right)$

For Foci

$X=0 \qquad Y=\pm c$

$\Rightarrow x-2=0 \qquad y+\frac{3}{2}=\pm\sqrt{5}$

$x=2 \qquad y=-\frac{3}{2}\pm\sqrt{5}$

$\therefore$ Required Foci $\left(2,\frac{-3}{2}+\sqrt{5}\right)$

Eccentricity $e=\frac{c}{a} \Rightarrow c=\frac{\sqrt{5}}{1}$

$\Rightarrow e=\sqrt{5}$

Vertices

$X=0 \qquad Y=\pm a$

$\Rightarrow x-2 \qquad y+\frac{3}{2}=\pm 1$

$x=2 \qquad y=\frac{3}{2}\pm 1$

$y=\frac{-3}{2}+1,\ \frac{-3}{2}-1=\frac{-1}{2},\frac{-5}{2}$

$\therefore$ vertices are $\left(2,\frac{-5}{2}\right),\left(2,\frac{-1}{2}\right)$

Equation of Directrices

$Y=\pm\frac{c}{e^2}$

$\Rightarrow y+\frac{3}{2}=\pm\frac{\sqrt{5}}{5}$

$\Rightarrow y=\frac{-3}{2}\pm\frac{1}{\sqrt{5}}$

(xi) $x^2-y^2+8x-2y-10=0$

$x^2+8x-y^2-2y=10$

$x^2+8x+16-16-(y^2+2y+1-1)=10$

$(x+4)^2-(y+1)^2=25$

$\Rightarrow \frac{(x+4)^2}{25}-\frac{(y+1)^2}{25}=1$

Which is of the form

$\frac{X^2}{25}-\frac{Y^2}{25}=1$

Where $X=x+4,\ Y=y+1$

$a^2=25 \Rightarrow a=5$

$b^2=25 \Rightarrow b=5$

Using $c^2=a^2+b^2=25+25=0$

$c=5\sqrt{2}$

For Centre

$X=0 \qquad Y=0$

$\Rightarrow x+4=0 \qquad y+1=0$

$x=-4 \qquad y=-1$

$\therefore$ Centre $(-4,-1)$

For Foci

$X=\pm c \qquad Y=0$

$x+4=\pm 5\sqrt{2} \qquad y+1=0$

$x=-4\pm 5\sqrt{2} \qquad y=-1$

$\therefore$ Foci are $(-4\pm 5\sqrt{2},-1)$

Eccentricity $e=\frac{c}{a} \Rightarrow e=\frac{5\sqrt{2}}{5}$

$\Rightarrow e=\sqrt{2}$

For vertices

$X=\pm a \qquad Y=0$

$x+4=\pm 5 \qquad y+1=0$

$x+4=\pm 5 \qquad y+1=0$

$x=-4\pm 5 \qquad y=-1$

$x=1,-9$

$\therefore$ vertices are $(1,-1),(-9,-1)$

Directrices

$X=\pm\frac{c}{e^2}$

$\Rightarrow x+4=\pm\frac{5\sqrt{2}}{2} \Rightarrow y=-4\pm\sqrt{\frac{5}{2}}$

(x) $9x^2-y^2-36x-6y+18=0$

$9x^2-36x-y^2-6y+18=0$

$9(x^2-4x)-(y^2+6y)=-18$

$9(x^2-4x+4-4))-(y^2+6y+9-9)=-18$

$9\left[(x-2)^2-4\right]-\left[(y+3)^2-9\right]=-18$

$\Rightarrow 9(x-2)^2-36-(y+3)^2+9=-18$

$$9(x-2)^2-(y+3)^2=-18+27$$
$$9(x-2)^2-(y+3)^2=9$$
$$\Rightarrow \frac{(x-2)^2}{1}-\frac{(y+3)^2}{9}=1 \quad (1)$$

Which is of the form

$$\frac{X^2}{1}-\frac{Y^2}{9}=1 \quad (2)$$

Where $X=x-2,\ \ Y=y+3$

$a^2=1 \ \Rightarrow a=1$

$b^2=9 \ \Rightarrow b=3$

Using $c^2=a^2+b^2 \ \Rightarrow c^2=1+9=10$

$\Rightarrow c=\sqrt{10}$

Now for centre

$X=0 \qquad Y=0$

$\Rightarrow x-2=0 \qquad y+3=0$

$x=2 \qquad y=-3$

$\therefore$ Required center is $(2,-3)$

For Foci

$X=\pm c \qquad Y=0$

$x-2=\pm\sqrt{10} \qquad y+3=0$

$x=2\pm\sqrt{10} \qquad y=-3$

$\therefore$ Foci are $(2\pm\sqrt{10},-3)$

Eccentricity $e=\frac{c}{a} \Rightarrow e=\frac{\sqrt{10}}{1}$

$\Rightarrow e=\sqrt{10}$

For vertices

$X=\pm a \qquad Y=0$

$x-2=\pm 1 \qquad y+3=0$

$x=2\pm 1 \qquad y=-3$

$x=3,1$

$\therefore$ Vertices are $(1,-3),\ (3,-3)$

Directrices $X=\pm\frac{c}{e^2}$

$\Rightarrow x-2=\pm\frac{\sqrt{10}}{10} \Rightarrow x=2\pm\frac{1}{\sqrt{10}}$

3. $0<a<c$

$F(-c,0),\ F'(c,0),\ P(x,y)$

Given that

$|PF|-|PF|=\pm 2a$

$\Rightarrow \sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a$

$\Rightarrow \sqrt{(x+c)^x+y^2}=\pm 2a+\sqrt{(x-c)^2+y^2}$

Squaring both sides of the above (1)

Equation we have

$(x+c)^2+y^2=4a^2+(x-c)^2+y^2$

$\pm 4a\sqrt{(x-c)^2+y^2}$

$x^2+2cx+c^2+y^2=4a^2+x^2-2cx+c^2+y^2$

$4cx-4a^2=\pm 4a\sqrt{(x-c)^2+y^2}$

$\Rightarrow cx-a^2=\pm a\sqrt{(x-c)^2+y^2}$

Squaring both sides of (2)

$c^2x^2-2a^2cx+a^4=a^2\left[x^2-2cx+c^2+y^2\right]$

$c^2x^2-2a^2cx+a^4=a^2x^2-2a^2cx+a^2c^2+a^2y^2$

$\Rightarrow c^2x^2+a^2-a^2x^2-a^2y^2=a^2c^2$

$(c^2-a^2)x^2-a^2y^2=a^2c^2-a^4$

$=a^2(c^2-a^2)$

$$\frac{(c^2-a^2)x^2}{a^2(c^2-a^2)}-\frac{a^2y^2}{a^2(c^2-a^2)}=\frac{a^2(c^2-a^2)}{a^2(c^2-a^2)}$$

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1$$

Hence proved.

4. $F(-5,-5)$, $F'(5,5)$

$A(-3\sqrt{2},-3\sqrt{2})$, $A'(3\sqrt{2},3\sqrt{2})$

Let $P(x,y)$ by any point on the hyperbola

Now $2a=|AA'|=\sqrt{(3\sqrt{2}+3\sqrt{2})^2+(3\sqrt{2}+3\sqrt{2})^2}$

$\Rightarrow 2a=\sqrt{(6\sqrt{2})^2+(6\sqrt{2})^2}=\sqrt{72+72}=\sqrt{144}$

$\Rightarrow 2a=12$

Using $|PF|-|PF'|=\pm 2a$

$\Rightarrow |PF|=\pm 2a+|PF'|$

$\sqrt{(x+5)^2+(y+5)^2}=\pm 12+\sqrt{(x-5)^2+(y-5)^2}$

Squaring both sides of the above

$(x+5)^2+(y+5)^2=144+(x-5)^2+(y-5)^2\pm 24\sqrt{(x-5)^2+(y-5)^2}$

$\Rightarrow x^2+y^2+10x+10y=50=144+x^2+y^2-10x-10y+50\pm 24\sqrt{(x-5)^2+(y-5)^2}$

$\Rightarrow 20x+20y=144\pm 24\sqrt{(x-5)^2+(y-5)^2}$

$5x+5y=36\pm 6\sqrt{(x-5)^2+(y-5)^2}$

$$5x+5y-36=\pm 6\sqrt{(x-5)^2+(y-5)^2}$$

$$\Rightarrow \pm 6\sqrt{(x-5)^2+(y-5)^2}=5x+5y-36$$

Squaring both sides of the above

$$36\left[x^2+y^2-10x-10y+50\right]=25x^2+25y^2+1296+50x-360x-360y$$

$$\Rightarrow 36x^2+36y^2-360x-360y+1800=25x^2+25y^2+1296+50xy-360x-360y$$

$$\Rightarrow 36x^2+25x^2+36y^2-25y^2-50xy+1800-1296=0$$

Which is the required equation of the hyperbola.

5. **Given points (2,2), (10,2)**

Let $P(x,y)$ be any point on the hyperbola. Then given that

$$\sqrt{(x-2)^2+(y-2)^2}-\sqrt{(x-10)^2+(y-2)^2}=6$$

$$\Rightarrow \sqrt{(x-2)^2+(y-2)^2}=6+\sqrt{(x-10)^2+(y-2)^2}$$

Squaring both sides we have

$$(x-2)^2+(y-2)^2=36+(x-10)^2+(y-2)^2+12\sqrt{(x-10)^2+(y-2)^2}$$

$$\Rightarrow x^2+y^2-4x-4y+8=36+x^2+y^2-20x-4y+104+12\sqrt{(x-10)^2+(y-2)^2}$$

$$\Rightarrow -4x+20x+8-36-104=12\sqrt{x^2+y^2-20x-4y+104}$$

$$\Rightarrow 16x-132=12\sqrt{x^2+y^2-20x-4y+104}$$

$$\Rightarrow 4x-33=3\sqrt{x^2+y^2-20x-4y+104}$$

Squaring both sides

$$(4x-33)^2=9(x^2+y^2-20x-4y+104)$$

$$16x^2+1089-264x=9x^2+9y^2-180x-36y+936$$

$$\Rightarrow 7x^2-9y^2-84x+36y+153=0$$

Which is the required equation

6. **Let two listening F_1 and F_2 hear the sound of enemy gun after t and t – 1 second respectively here listening posts are 1400m apart.**

i.e., $2c=1400 \Rightarrow \boxed{c=700}$

If P is the position of enemy gun.

Given that sound travels at 1080ft/sec

So we have

$$|PF_1|-|PF_2|=2a$$

$$\Rightarrow 1080t-(1080)(t-t)=2a$$

$$1080t-1080t+1080=2a$$

$$\Rightarrow 2a=1080 \Rightarrow \boxed{a=540}$$

Now using $c^2=a^2+b^2$

$$\Rightarrow b^2=c^2-a^2$$

$$=(700)^2-(540)^2$$

$$=490000-291600$$

$$b^2=198400$$

6 sec
7 d

Thus equation of hyperbola is

$$\frac{x^2}{(540)^2}-\frac{y^2}{198400}=1$$

$$\Rightarrow \frac{x^2}{291600}-\frac{y^2}{198400}=1$$

Tangent and Normals:

Line	Curve	Condition of Tangent	Equation of Tangent
$Y=mx+c$	$y^2=4ax$	$c=\frac{a}{m}$	$y=mx+\frac{a}{m}$
$Y=mx+c$	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$c=\pm\sqrt{a^2m^2+b^2}$	$y=mx\pm\sqrt{a^2m^2+b^2}$
$Y=mx+c$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$c=\pm\sqrt{a^2m^2-b^2}$	$y=mx\pm\sqrt{a^2m^2-b^2}$

Given Curve
and point say $P(x_1,y_1)$

Slope of tangent at $P(x_1,y_1)=m=\left.\frac{dy}{dx}\right]_{(x_1,y_1)}$

Then slope of normal $=-\frac{1}{m}$

Equation of tangent is $y-y_1=m(x-x_1)$

Equation of normal is $y-y_1=-\frac{1}{m}(x-x_1)$

Exercise 6.7

1. (i) $y^2 = 4ax$ at $(at^2, 2at)$

$y^2 = 4ax$ (1)

Differentiating both sides of (1) w.r.t.x

$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$

$\Rightarrow 2y\frac{dy}{dx} = 4a \quad \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$

Slope of tangent to (1) at $(at^2, 2at)$

$= m = \left.\frac{dy}{dx}\right]_{(at^2, 2at)} = \frac{2a}{2at} = \frac{1}{t}$

Now equation of tangent at $(at^2, 2at)$

$y - 2at = \frac{1}{t}(x - at^2)$

$\Rightarrow yt - 2at^2 = x - at^2$

$\Rightarrow \boxed{yt = x + at^2}$

and equation of normal at $(at^2, 2at)$

$y - 2at = -t(x - at^2)$

$y - 2at = -tx + at^3$

$\Rightarrow \boxed{y = -tx + 2at + at^3}$

(ii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(aCos\theta,\ bSin\theta)$

$\Rightarrow b^2x^2 + a^2y^2 = a^2y^2$

Differentiating (1) w.r.t.x we get.

$\frac{d}{dx}(b^2x^2 + a^2y^2) = \frac{d}{dx}(a^2b^2)$

$\Rightarrow 2b^2x^2 + 2a^2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{b^2x}{a^2y}$

Slope of Tangent at $(aCos\theta, bSin\theta) = m$

$= -\frac{b^2.aCos\theta}{a^2.bSin\theta} = -\frac{bCos\theta}{aSin\theta}$

And Slope of normal to (1) at $(aCos\theta, bSin\theta)$

$= m' = -\frac{1}{m} = \frac{aSin\theta}{bCos\theta}$

Now equation of tangent to (1)

At $(aCos\theta, bSin\theta)$ becomes

$$Y - bSin\theta = -\frac{bCos\theta}{aSin\theta}(x - aCos\theta)$$

$$\Rightarrow YaSin\theta, abSin^2\theta = -xbCos\theta + abCos^2\theta$$

$$\Rightarrow xbCos\theta, + yaSin\theta = abCos^2\theta + abSin^2\theta$$

$$= ab(Cos^2\theta + abSin^2\theta)$$

$$\Rightarrow xbCos\theta, + yaSin\theta = ab$$

$$\Rightarrow \frac{xbCos\theta}{ab} + \frac{yaSin\theta}{ab} = \frac{ab}{ab}$$

$$\Rightarrow \frac{x}{a}Cos\theta + \frac{y}{b}Sin\theta = 1$$

Equation of normal to (1) at $(aCos\theta, bSin\theta)$

$$Y - bSin\theta = \frac{aSin\theta}{bCos\theta}(x - aCos\theta)$$

$$\Rightarrow \frac{Y - bSin\theta}{aSin\theta} = \frac{x - aCos\theta}{bCos\theta}$$

$$\Rightarrow \frac{Y}{aSin\theta} - \frac{b}{a} = \frac{x}{bCos\theta} - \frac{a}{b}$$

$$\Rightarrow \frac{Y}{aSin\theta} - \frac{x}{bCos\theta} - \frac{a}{b} + \frac{b}{a}$$

$$\Rightarrow \frac{y}{a}Cosec\,\theta - \frac{x}{b}Sec\theta = \frac{-a^2 + b^2}{ab}$$

$$\Rightarrow \frac{x}{a}Sec\theta - \frac{y}{a}Cosec\,\theta = \frac{a^2 - b^2}{ab}$$

$$\Rightarrow axSec\theta - byCosec\,\theta = a^2 - b^2$$

(iii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(aSec\theta,\ bTan\theta)$ **(Gujrawala 2010)**

$$\Rightarrow b^2x^2 + a^2y^2 = a^2b^2$$

Differentiating both sides of (1) w.r.t.x

$$2b^2x - 2a^2y\frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

Slope of tangent to (1) $(aSec\theta, bTan\theta)$

$$= m = \left.\frac{dy}{dx}\right]_{(aSec, bTan\theta)} = \frac{b^2.aSec\theta}{a^2.Tan\theta}$$

$$\Rightarrow m = \frac{bSec\theta}{aTan\theta}$$

Slope of the normal to (1) at $(aSec\theta, bTan\theta)$

$$= m' = -\frac{1}{m} = -\frac{aTan\theta}{bSec\theta}$$

$\therefore$ Equation of tangent to (1) at $(aSec\theta, bTan\theta)$

$$Y - bTan\theta = \frac{bSec\theta}{aTan\theta}(x - aSec\theta)$$

$\Rightarrow aYTan\theta - abTan^2\theta = bxSec\theta - abSec^2\theta$

$\Rightarrow bxSec\theta - aYTan\theta = abSec^2\theta - abTan^2\theta$

$= ab\left[Sec^2\theta - abTan^2\theta\right]$

$= ab\left[1 + Tan^2\theta - Tan^2\theta\right]$

$bxSec\theta - ayTan\theta = ab$

$\dfrac{bxSec\theta}{ab} - \dfrac{ayTan\theta}{ab} = \dfrac{ab}{ab}$

$\dfrac{x}{a}Sec\theta - \dfrac{y}{b}Tan\theta = 1$

Equation of normal to (1)

At $(aSec\theta, bTan\theta)$ becomes

$Y - bTan\theta = \dfrac{bSec\theta}{aTan\theta}(x - aSec\theta)$

$\Rightarrow \dfrac{y - bTan\theta}{aTan\theta} = -\left(\dfrac{x - aSec\theta}{bSec\theta}\right)$

$\Rightarrow \dfrac{y}{aTan\theta} - \dfrac{bTan\theta}{aTan\theta} = -\dfrac{x}{bSec\theta} + \dfrac{aSec\theta}{bSec\theta}$

$\Rightarrow \dfrac{x}{bSec\theta} + \dfrac{y}{aTan\theta} = \dfrac{aSec\theta}{bSec\theta} + \dfrac{bTan\theta}{aTan\theta}$

$\Rightarrow \dfrac{x}{b}Cos\theta + \dfrac{y}{a}Cot\theta = \dfrac{a}{b} + \dfrac{b}{a}$

$\Rightarrow \dfrac{x}{b}Cos\theta + \dfrac{y}{a}Cot\theta = \dfrac{a^2 + b^2}{ab}$

$\Rightarrow axCos\theta + byCot\theta = a^2 + b^2$

2. (i) $3x^2 = -16y$ (1)

When $y = -3$

From (1) $3x^2 = -16(-3) \Rightarrow 3x^2 = 48$

$\Rightarrow x^2 = 16 \Rightarrow x = \pm 4$

Thus we have to find equations of tangents at $(4, -3)$ & $(-4, -3)$

Differentiate both sides of (1) w.r.t.x

$\dfrac{d}{dx}(3x^2) = \dfrac{d}{dx}(-16y)$

$\Rightarrow 6x = -16\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = -\dfrac{3x}{8}$

At the point $(4, -3)$

$\left.\dfrac{dy}{dx}\right]_{(4,-3)} = \dfrac{-3(4)}{8} = \dfrac{-3}{2}$

Now equation of tangent to (1) at $(4, -3)$ with slope $\dfrac{-3}{2}$

become $y-(-3)=\frac{-3}{2}(x-4)$

$\Rightarrow 2(y+3)=-3x+12$

$\Rightarrow -3x+12=2y+6$

$\Rightarrow -3x-2y+6=0$

$\Rightarrow 3x+2y-6=0$

At the point $(-4,-3)$

$\left.\frac{dy}{dx}\right]_{(4,-3)}=\frac{-3(-4)}{8}=\frac{3}{2}$

Now equation of the tangent to (1) at $(-4,-3)$ with slope $\frac{3}{2}$ becomes

$y+3=\frac{3}{2}(x+4)$

$\Rightarrow 2y+6=3x+12$

$\Rightarrow 3x-2y+6=0$

(ii) $3x^2-7y^2=20$ **(1)**

Put $y=-1$ in (1) we get

$3x^2-7=20 \Rightarrow 3x^2=27 \Rightarrow x^2=9$

$\Rightarrow x=\pm 3$

Thus we have to find equations of tangents at the points $(3,-1)$ and $(-3,-1)$

Now differentiating both sides of (1) w.r.t.x we have

$\frac{d}{dx}(3x^2-7y^2)\frac{d}{dx}(20)$

$3\frac{d}{dx}(x^2)-7\frac{d}{dx}(y^2)=0$

$3(2x)-7(2)y\frac{dy}{dx}=0$

$\Rightarrow \frac{dy}{dx}=\frac{6y}{14y} \Rightarrow \frac{dy}{dx}=\frac{3x}{7y}$

At the point $(3,-1)$

Slope of the tangent to (1) at $(3,-1)$

$=\left.\frac{dy}{dx}\right]_{(3,-1)}=\frac{3(3)}{7(-1)}=-\frac{9}{7}$

Thus equation of the tangent at $(3,-1)$ becomes

$y+1=-\frac{9}{7}(x-3)$

$\Rightarrow 7y+7=-9x+27 \Rightarrow \boxed{9x+7y-20=0}$

At the point $(-3,-1)$

$=\left.\frac{dy}{dx}\right]_{(-3,-1)}=\frac{3(-3)}{7(-1)}=\frac{9}{7}$

The equation of tangent at $(-3,-1)$

Becomes $y+1=\frac{9}{7}(x+3)$

$\Rightarrow 7y+7=9x+27$

$\Rightarrow 9x-7y+20=0$

(iii) $\mathbf{3x^2-7y^2+2x-y-48=0}$

Putting $x=4$ in (1) we have

$48-7y^2+8-4-48=0$

$\Rightarrow 7y^2+y-8=0$

$\Rightarrow 7y^2-7y+8y-8=0$

$\Rightarrow 7y(y-1)+8(y-1)=0$

$\Rightarrow (y-1)(7y+8)=0$

$\Rightarrow y-1=0 \text{ or } 7y+8=0$

$y=1 \quad 7y=-8$

$y=\frac{-8}{7}$

Thus we have to find equations of tangent at $(4,1)$ & $\left(4,\frac{-8}{7}\right)$

Now differentiating both sides of (1) w.r.t 'x' we have

$\frac{d}{dx}(3x^2-7y^2+2x-y-48)=\frac{d}{dx}(0)$

$\Rightarrow 6x-14y\frac{dy}{dx}+2-\frac{dy}{dx}=0$

$(14y+1)\frac{dy}{dx}=6x+2$

$\Rightarrow \frac{dy}{dx}=\frac{6x+2}{14y+1}$

Slope of tangent to (1) at $(4,1)$

$=\left.\frac{dy}{dx}\right]_{(4,1)}=\frac{24+2}{14+1}=\frac{26}{15}$

Thus equation of tangent of $(4,1)$ with slope $\frac{26}{15}$ becomes

$y-1=\frac{26}{15}(x-4)$

$\Rightarrow 15(y-1)=26(x-4)$

$\Rightarrow 15y-15=26x-104$

$\Rightarrow 26x-15y-89=0$

Slope of tangent at $\left(4,\frac{-8}{7}\right)$

$$=\left[\frac{dy}{dx}\right]_{(4,-8/7)}=\frac{6(4)+2}{14\left(\frac{-8}{7}\right)+1}$$

$$=\frac{24+2}{-16+1}=-\frac{26}{15}$$

Thus equation of tangent to (1)

at $\left(4,\frac{-8}{7}\right)$ becomes

$$y+\frac{8}{7}=\frac{-26}{15}(x-4)$$

$$\Rightarrow 15\left(y+\frac{8}{7}\right)=-26(x-4)$$

$$15y+\frac{120}{7}=-26x+104$$

$$26x+15y=104-\frac{120}{7}$$

$$=\frac{728-120}{7}=\frac{608}{7}$$

$$26x+15y-\frac{608}{7}=0$$

$$\Rightarrow 13x+\frac{15}{y}y-\frac{304}{7}=0$$

3. (i) $x^2+y^2=25$ $\quad(7,-1)$

$\Rightarrow x^2+y^2=5^2$ $\quad$ (1)

Here $a=5$

Equations of tangents to (1) from any points are of the form

$y=mx\pm a\sqrt{1+m^2}$ $\quad \forall m\in R$ $\quad$ (2)

Put $a=5$ in (2) we have

$y=mx\pm 5\sqrt{1+m^2}$ $\quad$ (3)

As (3) passes through $(7,-1)$

$\therefore -1=7m\pm 5\sqrt{1+m^2}$ $\quad$ (A)

$-1=7m=\pm 5\sqrt{1+m^2}$ $\quad$ (4)

Squaring both sides of (4) we get

$$1+49m^2+14m=25(1+m^2)$$

$$49m^2+14m+1=25+25m^2$$

$$49m^2+14m+1-25-25m^2=0$$

$$24m^2+14m-24=0$$

$$\Rightarrow 12m^2+7m-12=0$$

$\Rightarrow 12m^2+16m-9m-12=0$

$\Rightarrow 4m(3m+4)-3(3m+4)=0$

$\Rightarrow (3m+4)(4m-3)=0$

$\Rightarrow 3m+4=0$ *or* $4m-3=0$

$m=\frac{-4}{3}$ $\quad m=\frac{3}{4}$

Now $m=\frac{-4}{3}$ satisfies the equation.

$-1=7m+5\sqrt{1+m^2}$

These the equation of tangent is

$y=mx+5\sqrt{1+m^2}$

i.e., $y=\frac{-4}{3}x+5\sqrt{1+\left(\frac{-4}{3}\right)^2}$

$=\frac{-4x}{3}+\frac{5(5)}{3}$

$\Rightarrow 3y=-4x+25$

$\Rightarrow \boxed{4x+3y-25=0}$

and $=\frac{3}{4}$ satisfies the equation.

$-1=7m-5\sqrt{1+m^2}$

Thus the equation of tangent is

$y=mx-5\sqrt{1+m^2}$

i.e., $y=\frac{3}{4}x-5\sqrt{1+\frac{9}{16}}$

$y=\frac{3}{4}x-\frac{25}{4} \Rightarrow 4y=3x-25$

$\Rightarrow \boxed{3x-4y-25=0}$

(ii) $\quad y^2=12x \quad (1) \qquad$ ***through*** $(1,4)$

Here $4a=12 \Rightarrow \boxed{a=3}$

Equations of tangents to (1) are of the form

$y=mx+\frac{3}{m} \quad (2) \qquad \forall m\in R$

of (2) passes through (1,4)

Then $4=m+\frac{2}{m} \Rightarrow m^2+4m$

$\Rightarrow m^2-4m+3=0$

$\Rightarrow m^2-m-3m+3=0$

$m(m-1)-3(m-1)=0$

$(m-1)(m-3)=0$

$\Rightarrow m-1=0 \quad or \quad m-3=0$

$m=1 \quad or \quad m=3$

For $m=1$

Equation of tangent (2) becomes

$y=x+3 \Rightarrow \boxed{x-y+3=0}$

For $m=3$

Equation of tangent (2) becomes

$y=3x+\frac{3}{3} \Rightarrow \boxed{3x-y+1=0}$

(iii) $\mathbf{x^2 - 2y^2 = 2}$

$\frac{x^2}{2}-\frac{2y^2}{2}=\frac{2}{2}$

$\frac{x^2}{2}-\frac{y^2}{1}1=1 \quad (1)$

Here $a^2=2, b^2=1$

Equation of tangents to (1) are of the

From $y=mx\pm\sqrt{a^2m^2-b^2}$

i.e., $y=mx\pm\sqrt{2m^2-1} \quad (2)$

If (2) passes through $(1,-2)$

Then $-2=m\pm\sqrt{2m^2-1}$

$\Rightarrow -2-m=\pm\sqrt{2m^2-1} \quad (3)$

Squaring both sides of (3) we have

$4+m^2+4m=2m^2-1$

$\Rightarrow 2m^2-m^2-4m-1-4=0$

$\Rightarrow m^2-4m-5=0$

$\Rightarrow (m+1)(m-5)=0$

$\Rightarrow m+1=0 \quad m-5=0$

$m=-1$

Now $\boxed{m=-1}$ satisfies the equation

$-2-m=-\sqrt{2m^2-1}$

Equation of tangent is

$Y=mx-\sqrt{2m^2-1}$

i.e., $Y=-x-\sqrt{2-1} \Rightarrow y=x-1$

$\Rightarrow x+y+1=0$

and $m=5$ equation of tangent is

$-2-m=-\sqrt{2m^2-1}$

Thus for $m=5$ equation of tangent is

$y = 5x - \sqrt{2(5)^2 - 1}$

$\Rightarrow y = 5x - 7 \Rightarrow \boxed{5x - y - 7 = 0}$

4. $y^2 = 8x \qquad (1)$

$2x + 3y - 10 = 0 \qquad (2)$

Differentiating both sides of (1) w.r.t 'x' we have

$\frac{d}{dx}(y^2) = \frac{d}{dx}(8x)$

$\Rightarrow 2y\frac{dy}{dx} = 8 \quad \Rightarrow \frac{dy}{dx} = \frac{4}{y}$

Thus Slope of tangent to (1) $= \frac{4}{y}$

and Slope of normal to (1) $= \frac{-y}{4}$

Slope of line (2) $= \frac{-2}{3}$

Since normal to (1) is parallel to line (2)

$\therefore \frac{-y}{4} = \frac{-2}{3} \Rightarrow \boxed{y = \frac{8}{3}}$

Putting $y = \frac{8}{3}$ in (1) we get

$\left(\frac{8}{3}\right)^2 = 8x \Rightarrow 8x = \frac{64}{9}$

$\Rightarrow \boxed{y = \frac{8}{3}}$

Now slope of normal to (1)

at $\left(\frac{8}{9}, \frac{8}{3}\right) = -\frac{\frac{8}{3}}{4} = -\frac{8}{3} \times \frac{1}{4} = -\frac{2}{3}$

Now required equation of normal is

$y - \frac{8}{3} = -\frac{2}{3}\left(x - \frac{8}{9}\right)$

$y - \frac{8}{3} = -\frac{2}{3}x + \frac{16}{27}$

$\Rightarrow 27y - 72 = -18x + 16$

$\Rightarrow 18x + 27y - 88 = 0$

5. $\frac{x^2}{4} + \frac{y^2}{1} = 1 \qquad (1)$

$2x - 4y + 5 = 0 \qquad (2)$

From (1) $a^2 = 4,\ b^2 = 1$

And slope of line (2) $\frac{2}{4}=\frac{1}{2}$

As the tangents are parallel (2)

$\therefore$ Slope of tangent to (1) 11 to (2) $=\frac{1}{2}$

Now required equations of tangents are

$y=mx\pm\sqrt{a^2m^2+b^2}$

$\Rightarrow y=\frac{1}{2}x\pm\sqrt{4\left(\frac{1}{4}\right)+1}$

$y=\frac{1}{2}x\pm\sqrt{2} \Rightarrow 2y=x\pm2\sqrt{2}$

$\Rightarrow x-2y\pm2\sqrt{2}=0$

i.e., $x-2y+2\sqrt{2}=0$

$x-2y-2\sqrt{2}=0$

6. $9x^2-4y^2=36$

$\Rightarrow \frac{9x^2}{36}-\frac{4y^2}{36}=\frac{36}{36}$

$\Rightarrow \frac{x^2}{4}-\frac{y^2}{9}=1 \quad (1)$

$5x-2y+7=0 \quad (2)$

From (1) $a^2=4,\ b^2=9$

Slope of the line (2) $=\frac{5}{2}$

Now Slope of tangent parallel to (2) $=\frac{5}{2}$

Thus required equations of tangents to (1) and parallel to (2) are

$y=\frac{5}{2}x\pm\sqrt{4\left(\frac{25}{4}\right)-9}$

$y=\frac{5}{2}x\pm4 \Rightarrow 2y=5x\pm8$

$\Rightarrow 5x-2y\pm8=0$

$\Rightarrow 5x-2y+8=0\ ,\ 5x-2y-8=0$

7. (i) $x^2=80y \quad (1)$

$x^2+y^2=81 \quad (2)$

Let $y=mx+c \quad (3)$

Be common tangent to (1) and (2)

Using (3) in (1) we have

$x^2=80(mx+c)$

$\Rightarrow x^2-80mx-80c=0 \quad (4)$

If (3) is tangent to (1) then (4) has equal roots.

$\Rightarrow$ Discriminant of (4) = 0

$\Rightarrow (-80m)^2 - 4(1) - (-80c) = 0$

$\Rightarrow 6400m^2 + 320c = 0$

$\Rightarrow 320c = -6400m^2$

$\Rightarrow c = -\dfrac{6400m^2}{320}$

$\Rightarrow c = -20m^2$ (5)

If (3) is tangent to (2) then

$c^2 = 81(1+m^2)$ (6)

Using (5) in (6) we have

$\Rightarrow 400m^4 - 81m^2 - 81 = 0$ (7)

$\Rightarrow 400m^4 - 225m^2 + 144m^2 - 81 = 0$

$\Rightarrow 25m^2(16m^2 - 9) + 9(16m^2 - 9) = 0$

$\Rightarrow (16m^2 - 9) + (25m^2 + 9) = 0$

$\Rightarrow 16m^2 - 9 = 0 \quad or \quad 25m^2 + 9 = 0$

$\Rightarrow 16m^2 = 9$

$m^2 = \dfrac{9}{16}$

$\Rightarrow m = \pm\dfrac{3}{4}$

$25m^2 = -9$

$m^2 = \dfrac{-9}{25}$

Neglecting Negative value of m^2

($\because m^2$ is not negative)

Using $m = \pm\dfrac{3}{4}$ in (5) we have

$c = -20\left(\dfrac{9}{16}\right) = -\dfrac{45}{4}$

Thus the required equations of common tangents become

$y = \pm\dfrac{3}{4}x - \dfrac{45}{4}$

$\Rightarrow 4y = \pm 3x - 45$

$\Rightarrow \pm 3x - 4y - 45 = 0$

(ii) $x^2 = 16x$, $4a = 16$ (1)

$x^2 = 2y \Rightarrow \boxed{a = 4}$ (2)

Let $y = mx + c$ (3)

Be the equation of common tangents to (1) and (2).

Now if (3) is tangents (1)

Then $c=\frac{a}{m} \Rightarrow c=\frac{4}{m}$ (4)

Using (4) in (3) we have

$y=mx+\frac{4}{m}$ (5)

Using (5) in (2) we have

$x^2=2\left(mx+\frac{4}{m}\right)$

$x^2=2mx+\frac{8}{m}$

$\Rightarrow x^2-2mx-\frac{8}{m}=0$

Now (3) is tangent of roots of (6) are equal then

Discriminant of (6) = 0

$\Rightarrow (-2m)^2-4(1)\left(\frac{-8}{m}\right)=0$

$\Rightarrow 4m^2+\frac{32}{m}=0$

$\Rightarrow m^2+\frac{8}{m}=0$

$\Rightarrow m^3+8=0 \Rightarrow m^3+2^3=0$

$\Rightarrow (m+2)(m^2-2m+4)=0$

$\Rightarrow m+2=0$ or m^2-2m+4

$\Rightarrow \boxed{m=-2}$ (Neglecting complex roots) $m=\frac{2\pm\sqrt{4-16}}{2}$

Putting $m=-2$ in (4) we have

$c=\frac{4}{-2} \Rightarrow \boxed{c=-2}$

Thus required equation of common tangent is

$y=-2x-2$

$\Rightarrow \boxed{2x+y+2=0}$

(iii) $y^2=16x$ (1)

$x^2=2y$ (2)

$y=mx+c$ (3)

Be the common tangent to (1) and (2)

If (3) is tangent to (1) then

$c=\frac{a}{m} \Rightarrow c=\frac{4}{m}$ (4)

Using (3) in (2) we got

$x^2=2(mx+c)$

$\Rightarrow x^2-2mx-2c=0$ (5)

If (3) is tangent to (2) then roots of (5) are equal

$\Rightarrow$ Discriminant of (5) =0

$\Rightarrow (-2m)^2 - 4(1)(-2c) = 0$

$4m^2 + 8c = 0$ (6)

Using (4) in (6) we get

$4m^2 + 8\left(\dfrac{4}{m}\right) = 0$

$\Rightarrow m^2 + \dfrac{8}{m} = 0 \Rightarrow m^3 + 8 = 0$

$\Rightarrow (m+2)(m^2 - 2m + 4) = 0$

$\Rightarrow m + 2 = 0 \quad or \quad m^2 - 2m + 4 = 0$

$\boxed{m = -2}$ $\because m^2 - 2m4 = 0$ gives imaginary values

Putting $m = -2$ in (4) we have

$c = \dfrac{4}{-2} \Rightarrow \boxed{c = -2}$

Thus required equation of common tangent becomes

$y = -2x - 2$

$\Rightarrow 2x + y + 2 = 0$

8. (i) $\dfrac{x^2}{18} + \dfrac{y^2}{8} = 1$ (1)

$\dfrac{x^2}{3} + \dfrac{y^2}{3} = 1$ (2)

By multiplying equation (2) by $\dfrac{1}{6}$ and then subtracting if from (1) we have

$$\begin{array}{l} \dfrac{x^2}{18} + \dfrac{y^2}{8} = 1 \\ \underline{\pm\dfrac{x^2}{18} \mp \dfrac{y^2}{8} = -\dfrac{1}{6}} \\ \dfrac{y^2}{8} + \dfrac{y^2}{19} = 1 - \dfrac{1}{6} \end{array}$$

$\Rightarrow \dfrac{9y^2 + 4y^2}{72} = \dfrac{6-1}{6}$

$\Rightarrow \dfrac{13y^2}{72} = \dfrac{5}{6} \Rightarrow \dfrac{13y^2}{12} = 5$

$\Rightarrow 13y^2 = 60 \Rightarrow y^2 = \dfrac{60}{13}$

$\Rightarrow y = \pm\sqrt{\dfrac{60}{13}}$

Putting $y^2 = \dfrac{60}{13}$ in (2) we get

$$\frac{x^2}{3}-\frac{\frac{60}{13}}{3}=1 \Rightarrow \frac{x^2}{3}=1+\frac{60}{13}\times\frac{1}{3}$$

$$\Rightarrow \frac{x^2}{3}=1+\frac{20}{13}=\frac{13+20}{13}=\frac{33}{13}$$

$$\Rightarrow x^2=\frac{99}{13} \Rightarrow x=\pm\sqrt{\frac{99}{13}}$$

Thus required points of intersection of given conics

$$\left(\pm\sqrt{\frac{99}{13}},\pm\sqrt{\frac{60}{13}}\right)$$

(ii) $x^2+y^2=8$ (1)

$x^2-y^2=1$ (2)

$2x^2=9$

By adding

$$\Rightarrow x^2=\frac{9}{2} \Rightarrow x=\pm\sqrt{\frac{9}{2}}$$

Putting $x^2=\frac{9}{2}$ in (1) we get

$$\frac{9}{2}+y^2=8$$

$$y^2=8-\frac{9}{2} \Rightarrow y^2=\frac{7}{2}$$

$$\Rightarrow y=\pm\sqrt{\frac{7}{2}}$$

Thus required points are

$$\left(\pm\sqrt{\frac{9}{2}},\pm\sqrt{\frac{7}{2}}\right)$$

(iii) $3x^2-4y^2=12$ (1)

$-2x^2+3y^2=7$ (2)

Multiplying equation (1) by 2 and equation (2) by 3 we have

$6x^2-8y^2=24$ (3)

$-2x^2+3y^2=7$ (4)

$y^2=5$ By adding

$$\Rightarrow y=\pm\sqrt{45}$$

Putting $y^2=45$ in (1) we have

$$3x^2-4(45)=12$$

$$\Rightarrow x^2-60=4 \Rightarrow x^2=64$$

$$\Rightarrow x=\pm 8$$

Thus required points of intersection of given conics are

$(\pm 8, \pm\sqrt{45})$

(iv) $3x^2 + 5y^2 = 60$ (1)

$9x^2 + y^2 = 124$ (2)

By multiplying equation (1) by 3 and then subtracting (2) from it we have

$9x^2 + 15y^2 = 18$

$-9x^2 \pm y^2 = \_124$

Putting $y^2 = 4$ in (1) we get

$3x^2 + 20 = 60 \Rightarrow 3x^2 = 40$

$\Rightarrow x^2 = \frac{40}{3} \Rightarrow x = \pm\sqrt{\frac{40}{3}}$

Thus the point of intersection of the given conics are

$\left(\pm\sqrt{\frac{40}{3}}, \pm 2\right)$

(v) $4x^2 + y^2 = 16$ (1)

$x^2 + y^2 + y + 8 = 0$ (2)

By multiplying equation (2) by 4 then subtracting (1) from it we have

$4x^2 + 4y^2 + 4y + 32 = 0$

$-4x^2 \pm y^2 = \_16$

$3y^2 + 4y + 32 = -16$

$3y^2 + 4y + 48 = 0$

$y = \frac{-4 \pm \sqrt{16 - 4(3)(48)}}{6}$

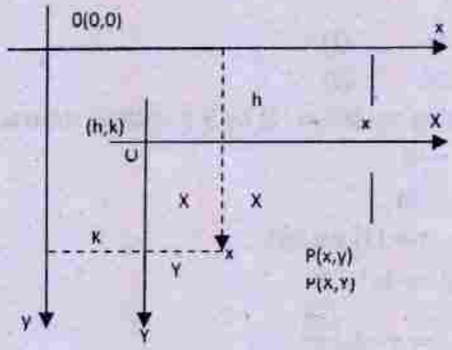
$y = \frac{-4 \pm \sqrt{16 - 576}}{6} = \frac{-4 \pm \sqrt{-560}}{6}$

$y = \frac{-4 \pm \sqrt{-560}i}{6}$

At the value of y are complex (Imaginary)

So no real points of intersection of given conics exist.

Translation of Axes:



If a point p has coordinates (x, y) referred to the xy-system and has coordinates (x, y) referred to the translated axes OX', OY' through $O'(h, x)$

Then $\left.\begin{array}{ll} x = X + h, & y = Y + k \\ X = x - h, & Y = y - k \end{array}\right]$

Rotation of Axes:

Let xy-coordinate system by given. We rotate Ox, Oy about the origin through an angle $\theta(0 < \theta < 90^0)$ so that new axes are OX and OY as shown in the figure. Let a point P has coordinate (x,y) referred to the xy-coordinates (X,Y) referred to XY-coordinate system. We have to find XY coordinates in terms in terms of the given coordinates x,y. Let α be the measure of the inclination of OP. From P, draw PM perpendicular to Ox and PM' perpendicular to OX. Let $|OP| = r$

From ΔOPM, we have

$$\left.\begin{array}{l} OM = x = rCos\alpha \\ MP = y = rSin\alpha \end{array}\right] \quad (1)$$

From ΔOPM,

$OM' = X = rCos(\alpha - \theta)$

$\Rightarrow \boxed{X = rCos\theta + ySin\theta}$ (2) using (1)

Also $MP' = Y = rSin(\alpha - 0)$

$\Rightarrow Y = rSin\alpha Cos\theta - rCos\alpha Sin\theta$

$\Rightarrow \boxed{Y = rCos\theta - xSin\theta}$ (3) using (1)

Thus

$(X, Y) = (xCos\theta + ySin\theta,\ yCos\theta - xSin\theta)$

are the coordinates of P referred to the new axes OX and OY.

(I) $x = X + h, \quad y = Y + x$

$X = x - h, \quad Y = y - k$

$X = xCos0 + ySin0$

$Y = yCos0 - xSin0$

Exercise 6.8

1. (i) $x^2+16y-16=0$ (1) $O'(0,1)$

Equation of transformation are

$x=X+0$; $y=Y+1$

Putting $x=X$ and $y=Y+1$ in (1)

We have

$X^2+16(Y+1)-16=0$

$\Rightarrow X^2+16Y+16-16=0$

$\Rightarrow X^2+16Y=0$

(ii) $4x^2+y^2+16x-10y+37=0$

$O'(-2,5)$

Equations of transformation are

$x=X-2$ *and* $y=Y+5$

Putting $x=X-2$ & $y=Y+5$ in (1)

We have

$4(X-2)^2+(Y+5)^2+16(x-2)-10(Y+5)+37=0$

$\Rightarrow 4(X^2-4X+4)+Y^2+10Y+25+16X-32-10Y-50+37=0$

$\Rightarrow 4X^2-16X+16+Y^2+16X-20=0$

$\Rightarrow 4X^2+Y^2-4=0$

Which is the required equation.

(iii) $9x^2+4y^2+18x-16y-11=0$ (1)

$O'(-1,2)$

Equations of transformation are

$x=X-1$ *and* $y=Y+2$

Putting $x=X-1$ & $y=Y+2$ in (1) we have

$9(X-1)^2+4(Y+2)+18(X-1)-16(Y+2)-11=0$

$\Rightarrow 9(X^2-2X+1)+4(Y^2+4Y+4)+18X-18-16Y-32-11=0$

$\Rightarrow 9X^2-18X+9+4Y^2+16Y+16+18X-16Y-61=0$

$\Rightarrow 9X^2+4Y^2-36=0$

(iv) $x^2-y^2+4x+8y-11=0$ (1) $O'(-2,4)$

Equations of transformation are

$x=X-2$; $y=Y+4$

Putting $x=X-2$; $y=Y+4$ in (1)

We have

$(X-2)^2-(Y+4)^2+4(X-2)+8(Y+4)-11=0$

$\Rightarrow X^2-4X+4Y^2-8Y-16+4X-8+9y+32-11=0$

$\Rightarrow X^2-Y^2+1=0$ Required equation

(v) $9x^2-4y^2+36x+8y-4=0$ (1)

Equations of transformation are

$x=X-2$; $y=Y+1$

Putting $x=X-2$; $y=Y+1$ in (1)

We have

$9(X-2)^2-4(Y+1)^2+36(X-2)+8(Y+1)-4=0$

$\Rightarrow 9(X^2-4X+4)-4(Y^2+2Y+1)+36X-72+8Y+8-4=0$

$\Rightarrow 9X^2-36X+36-4Y^2-8Y-4+36X-68=0$

$\Rightarrow 9X^2-4Y^2-36=0$

Which is the required equation.

2. (i) $3x^2-2y^2+24x+12y+24=0$

Let the coordinates of the new origin be (h,k). Then equations of transformation are

$x=X+h$; $y+Y+k$

Now putting $x=X+h$ *and* $y=Y+k$ in (1) we have

$3(X+h)^2-2((Y-k)^2+24(X+h)+12(Y+k)+24=0$

$\Rightarrow 3(X^2+2hX+h^2)-2(Y^2+2kY+k^2)+24X+24h+12Y+12k+24=0$

$\Rightarrow 3X^2+6hX+3h^2-2Y^2-4kY-2k^2+24X+24h+12Y+12k+24=0$

$\Rightarrow 3X^2-2Y^2+6(h+4)X-4(K-3)Y+3h^2-2k^2+24h+12k+24=0$

Now we put $h+4=0 \Rightarrow h=-4$

and $k-3=0 \Rightarrow k=3$ in (2)

Then $3X^2-2Y^2+3(-4)^2-2(3)^2+24(-4)+12(3)+24=0$

$\Rightarrow 3X^2-2Y^2+48-18-96+36+24=0$

$\Rightarrow 3X^2-2Y^2-6=0$

Which is the new transformed equation with new origin $(-4,3)$

(ii) $25x^2+9y^2+50x-36y-164=0$

Suppose coordinates of the new origin be (h,k) Then equation of transformation are

$x=X+h$ *and* $y=Y+k$

Putting these values of $x \& y$ in (1) we have

$25(X+h)^2+9(Y+k)^2+50(X+h)-36(Y-k)-164=0$

$\Rightarrow 25(X^2+2hX+h^2)+9(Y^2+2kY+k^2)+50X+50h-36Y-36k-164=0$

$\Rightarrow 25X^2+50hX+25h^2+9Y^2+18kY+9k^2+50X+50h-36Y-36k-164=0$

$\Rightarrow 25X^2+9Y^2+50hX+50X+18kY-36Y+25h^2+9k^2+50h-36k-164=0$

$\Rightarrow 25X^2+9Y^2+(50h+50)X+(18k-36)Y+25h^2+9k^2+50h-36k-164=0$

For the removal of first degree terms we put

$50h+50=0$ *and* $189k-36=0$

$h=-1$ $k=2$

So the new origin is $(-1,2)$

Now putting $h=-1,\ k=2$ in (2)

$25X^2+9Y^2+(0)X+(0)Y+25+36-50-72-164=0$

$\Rightarrow 25X^2+9Y^2-225=0$

(iii) $x^2-y^2-6x+2y+7=0$ (1)

Suppose (h,k) be the coordinates of the new origin. Then equations of Transformation are

$x=X+h$ and $y=Y+k$

Putting $x=X+h$ and $y=Y+k$ in (1)

We have

$(X+h)^2-(Y+k^2)-6(X+h)+2*Y+k)+7=0$

$\Rightarrow X^2+2hX+h^2-Y^2-2kY-k^2-6X-6h+2y+2k+7=0$

$\Rightarrow X^2-Y^2+2hX-6X-2kY+2Y+h^2-k^2-6h+2k+7=0$

$\Rightarrow X^2-Y^2+(2h-6)X-(2k-2)Y+h^2-k^2-6h+2k+7=0$ (2)

Now for the first degree terms to be removed we put

$2h-6=0$ and $2k-2=0$

$\Rightarrow h=3 \qquad k=1$

So the required now origin is (3, 1)

Putting $h=3$ & $k=1$ in (2) we have

$X^2-Y^2-1=0$ Required equation.

3. (i) $xy=1$ (1) $\theta=45^o$

Equations of transformation are

$x=XCos45^o-YSin45^o=X\left(\frac{1}{\sqrt{2}}\right)-Y\left(\frac{1}{\sqrt{2}}\right)$

$\Rightarrow x=\frac{X-Y}{\sqrt{2}}$

and $y=YCos\theta+XSin\theta=YCos45^o+XSin45^o$

$y=Y\left(\frac{1}{\sqrt{2}}\right)+X\left(\frac{1}{\sqrt{2}}\right)$

Using (i) and (ii) in (1) we have

$\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)=1$

$\Rightarrow X^2-Y^2=2$ is the required equation.

(ii) $7x^2-8xy+y^2-9=0$ (1) $\theta=Tan^{-1}(2)$

$\theta=Tan^{-1}(2) \Rightarrow Tan\theta=2 \quad \Rightarrow Tan\theta=\frac{2}{1}$

As θ is in quadrant I

So $Sin\theta=\frac{2}{\sqrt{2}},\ Cos\theta=\frac{1}{\sqrt{2}}$

Now equations of transformation are

$x = XCos\theta - YSin\theta = X\left(\frac{1}{\sqrt{5}}\right) - Y\left(\frac{2}{\sqrt{5}}\right)$

$\Rightarrow x = \frac{X-2Y}{\sqrt{5}}$ (i)

and $Y = XCos\theta - YSin\theta = X\left(\frac{2}{\sqrt{5}}\right) + Y\left(\frac{1}{\sqrt{5}}\right)$

$\Rightarrow Y = \frac{2X+Y}{\sqrt{5}}$ (ii)

Using (i) & (ii) in (1) we have

$7\left(\frac{X-2y}{\sqrt{5}}\right)^2 - 8\left(\frac{X-2Y}{\sqrt{5}}\right)\left(\frac{X-2Y}{\sqrt{5}}\right) + \left(\frac{2X+Y}{\sqrt{5}}\right)^2 - 9 = 0$

$7\left(\frac{X^2-4XY-4Y^2}{5}\right)^2 - 8\left(\frac{2X^2-3XY-2Y^2}{5}\right) + \frac{4X^2+Y^2+5XY}{5} - 9 = 0$

$\Rightarrow 7(X^2-4XY+4Y^2) - 8(2X^2-3XY-2Y^2) + 4X^2 + Y^2 + 4XY - 45 = 0$

$\Rightarrow 7X^2 - 28XY + 28Y^2 - 16X^2 + 24XY + 16Y^2 + 4X^2 + Y^2 + 4XY - 45 = 0$

$\Rightarrow -5X^2 + 45Y^2 - 45 = 0$

$\Rightarrow X^2 - 9Y^2 + 9 = 0$

(iii) $\mathbf{9x^2 + 12xy + 4y^2 - x - y = 0}$ **(1)**

$\theta = Tan^{-1}\frac{2}{3} \Rightarrow Tan\theta = \frac{2}{3}$

$\Rightarrow \frac{Sin\theta}{Cos\theta} = \frac{2}{3} \Rightarrow 3Sin\theta = 2Cos\theta$

$\Rightarrow 9Sin^2\theta = 4Cos^2\theta \Rightarrow 9Sin^2\theta = 4(1 - Sin^2\theta)$

$\Rightarrow 9Sin^2\theta = 4 - 4Sin^2\theta \Rightarrow 13Sin^2\theta = 4$

$(\because \theta \Rightarrow Sin^2\theta = \frac{4}{13} \Rightarrow Sin\theta = \frac{2}{\sqrt{13}}$

($\because \theta$ is in quadrant I)

$\& \ Cos^2\theta = 1 - Sin^2\theta = 1 - \frac{4}{13} = \frac{9}{13}$

$Cos\theta = \frac{3}{\sqrt{13}}$

Now equation of transformation are

$x = XCos\theta - YSin\theta = X\left(\frac{3}{\sqrt{13}}\right) - Y\left(\frac{2}{\sqrt{13}}\right)$

$\Rightarrow x = \frac{3X-2Y}{\sqrt{13}}$

$y = XSin\theta + YCos\theta$

$\Rightarrow y - X\left(\frac{2}{\sqrt{13}}\right) + Y\left(\frac{2}{\sqrt{13}}\right) \Rightarrow y = \frac{2X+3Y}{\sqrt{13}}$

Putting there values of x and y in (1)

$$9\left(\frac{3X-2Y}{\sqrt{13}}\right)^2+12\left(\frac{3X-2Y}{\sqrt{13}}\right)\left(\frac{2X+3Y}{\sqrt{13}}\right)+\left(\frac{2X+3Y}{\sqrt{13}}\right)^2-\frac{3X-2Y}{\sqrt{13}}-\frac{2X+3Y}{\sqrt{13}}=0$$

$$\Rightarrow\frac{9}{13}(9X^2+4Y^2-12XY)+12(6X^2-6Y^2+5XY)+4(4X^2+9Y^2+12XY)-\sqrt{13}(5X-Y)=0$$

$$\Rightarrow 81X^2+36Y^2-108XY+72X^2-72Y^2+60XY+16X^2+36Y^2+48XY-5\sqrt{13}X-\sqrt{13}Y=0$$

$$\Rightarrow 169X^2-5\sqrt{13}X-\sqrt{13}Y=0$$

$$\Rightarrow (13)(\sqrt{13})^2X^2-5\sqrt{13}X-\sqrt{13}Y=0$$

$$\Rightarrow 13(\sqrt{13}X^2-5X-Y=0$$

(iv) $x^2-2xy+y^2-2\sqrt{2}x-2\sqrt{2}y+2=0$ (1)

$\theta=45^\circ$

Equation of transformation are

$$x=XCos\theta-YSin\theta$$

$$\Rightarrow x=XCos45^\circ-YSin45^\circ=X\left(\frac{1}{\sqrt{2}}\right)-Y\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow\frac{X-Y}{\sqrt{2}}$$

and $y=XSin\theta+YCos\theta=XSin45^\circ+YCos45^\circ$

$$\Rightarrow y=X\left(\frac{1}{\sqrt{2}}\right)+Y\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow\frac{X+Y}{\sqrt{2}}$$

Using (i) and (ii) in (1) we have

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2+2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)+\left(\frac{X+Y}{\sqrt{2}}\right)^2-\sqrt{2}\left(\frac{X-Y}{\sqrt{2}}\right)+2=0$$

$$\Rightarrow\frac{X^2-2XY+Y^2}{2}-2\left(\frac{X^2-Y^2}{2}\right)+\frac{X^2+2XY+Y^2}{2}-2X+\cancel{2Y}-2X-\cancel{2Y}+2=0$$

$$\Rightarrow X^2-2XY+Y^2-2X^2+2Y^2+X^2+2XY+Y^2-8X+4=0$$

$\Rightarrow Y^2-2X+1=0$ which is the required equation.

4. (i) $2x^2+6xy\_10y^2-11=0$ (1)

Let the axes be rotated through an angle θ. Then equations of transformation are

$x=XCos\theta-YSin\theta$ (i)

$y=XSin\theta+YCos\theta$ (ii)

Using (i) and (ii) in (1) we get.

$$2(XCos\theta-YSin\theta)^2+6(XCos\theta-YSin\theta)(XSin\theta+YCos\theta)+10(XSin\theta+YCos\theta)^2-11=0$$

$$\Rightarrow 2(X^2Cos^2\theta-2XYCos\theta Sin\theta+Y^2Sin^2\theta+6(X^2Cos\theta Sin\theta+XYCos^2\theta-XYSin^2\theta-$$
$$Y^2Sin\theta Cos\theta)+10(X^2Sin^2\theta+Y^2Cos^2\theta+2XYSin\theta Cos\theta)-11=0$$

$$\Rightarrow 2X^2Cos^2\theta-4XYCos\theta Sin\theta+2Y^2Sin^2\theta+6X^2Cos\theta Sin\theta+6XYCos^2\theta-6XYSin^2\theta-$$

$6Y^2Sin\theta Cos\theta + 10X^2Sin^2\theta + 10Y^2Cos^2\theta + 20XYSin\theta Cos\theta - 11 = 0$

$\Rightarrow X^2(2Cos^2\theta + 6Cos\theta Sin\theta + 10Sin^2\theta) + XY(-4Cos\theta Sin\theta + 6Cos^2\theta - 6Sin^2\theta +$

$20Sin\theta Cos\theta) + Y^2(2Sin^2\theta - 6Sin\theta Cos\theta + 10Cos^2\theta) - 11 = 0$

The equation (2) will be free from product term XY if

$-Cos\theta Sin\theta + 6Cos^2\theta - 6Sin^2\theta + 20Sin\theta Cos\theta = 0$

$\Rightarrow 6Cos^2\theta - 6Sin^2\theta + 16Sin\theta Cos\theta = 0$

$\Rightarrow 3Cos^2\theta - 3Sin^2\theta + 8Sin\theta Cos\theta = 0$

$\Rightarrow 3\frac{Cos^2\theta}{Cos^2\theta} - \frac{3Sin^2\theta}{Cos^2\theta} + \frac{8Sin\theta Cos\theta}{Cos^2\theta} = 0$

$\Rightarrow 3 - 3Tan^2\theta + 8Tan\theta = 0$

$\Rightarrow 3Tan^2\theta - 8Tan\theta - 3 = 0$

$\Rightarrow 3Tan^2\theta - 9Tan\theta + Tan\theta - 3 = 0$

$3Tan\theta(Tan\theta - 3)(Tan\theta - 3) = 0$

$\Rightarrow (Tan\theta - 3)(3Tan\theta + 1) = 0$

$\Rightarrow Tan\theta - 3 = 0 \quad or \quad 3Tan\theta + 1 = 0$

$Tan\theta = 3 \quad or \quad Tan\theta = -\frac{1}{3}$

As θ is in First Quadrant.

So $Tan\theta = -\frac{1}{3}$ is not admissible value.

$\Rightarrow Tan\theta = 3 \Rightarrow \frac{Sin\theta}{Cos\theta} = 3$

$\Rightarrow Sin\theta = 3Cos\theta \Rightarrow Sin^2\theta = 9Cos^2\theta$

$\Rightarrow Sin^2\theta = 9(1 - Sin^2\theta)$

$\Rightarrow Sin^2\theta = 9 - 9Sin^2\theta \Rightarrow 10Sin^2\theta = 9$

$\Rightarrow Sin^2\theta = \frac{9}{10} \Rightarrow \boxed{Sin\theta = \frac{3}{\sqrt{10}}}$

From $\frac{\frac{2}{\sqrt{10}}}{Cos\theta} = 3 \Rightarrow 3Cos\theta = \frac{3}{\sqrt{10}}$

$\Rightarrow Cos\theta = \frac{3}{3\sqrt{10}} \Rightarrow \boxed{Cos\theta = \frac{1}{\sqrt{10}}}$

Putting $Sin\theta = \frac{3}{\sqrt{10}}$, $Cos\theta = \frac{1}{\sqrt{10}}$ in (2)

$X^2\left[2\left(\frac{1}{10}\right) + 6\left(\frac{1}{\sqrt{10}}\right)\left(\frac{3}{\sqrt{10}}\right) + 10\left(\frac{9}{10}\right)\right] + XY(0) + Y^2\left[2\left(\frac{9}{10}\right) - 6\left(\frac{3}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right) + 10\left(\frac{1}{10}\right)\right] - 11 = 0$

$\Rightarrow X^2\left[\frac{2}{10} + \frac{18}{10} + \frac{90}{10}\right] + Y^2\left[\frac{18}{10} - \frac{18}{10} + \frac{10}{10}\right] - 11 = 0 \Rightarrow 11X^2 + Y^2 - 11 = 0$

Which is the required equation

(ii) $xy+4x-3y-10=0 \quad (1)$

Let the axes be rotated through an angle θ Then equations of transformation are

$x = XCos\theta - YSin\theta \quad (i)$

$y = XSin\theta + YCos\theta \quad (ii)$

Using (i) and (ii) in (1) we have

$(XCos\theta - YSin\theta)(XSin\theta + YCos\theta) + 4(XCos\theta - YSin\theta) - 3(XSin\theta + YCos\theta) - 10 = 0$

$\Rightarrow X^2Cos\theta Sin\theta + XYCos^\theta XYSin^2\theta - Y^2Sin\theta Cos\theta + 4XCos\theta - 4XCos\theta - 4YSin\theta - 3XSin\theta -$
$3YCos\theta - 10 = 0$

$\Rightarrow X^2Cos\theta Sin\theta - Y^2Sin\theta Cos\theta + XY(Cos^2\theta - Sin^2\theta)X(4Cos\theta - 3Sin\theta) - Y(4Sin\theta +$
$3Cos\theta) - 10 = 0$

Now the equation (2) will be free from product term if

$Cos^2\theta - Sin^2\theta = 0 \Rightarrow Sin^2\theta = Cos^2\theta$

$\Rightarrow Tan^2\theta = 1 \Rightarrow Tan\theta = \pm 1$

$Tan\theta = 1 \quad \because Tan\theta = -1$ is not admissible

$\Rightarrow \boxed{\theta = 45^\circ}$

Puttng $\theta = 45°$ in (2) we have

$$X^2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) - Y^2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + XY(0) + X\left(\frac{4}{\sqrt{2}} - \frac{3}{\sqrt{2}}\right) - y\left(\frac{4}{\sqrt{2}} + \frac{3}{\sqrt{2}}\right) - 10 = 0$$

$$\Rightarrow \frac{X^2}{2} - \frac{Y^2}{2} + \frac{X}{\sqrt{2}} - Y\left(\frac{7}{\sqrt{2}}\right) - 10 = 0$$

$$\Rightarrow X^2 - Y^2 + \sqrt{2}X - 7\sqrt{2}Y - 10 = 0$$

Which is the required equation.

(iii) $5x^2 - 6xy + 5y^2 - 8 = 0 \quad (1)$

Let the axes be rotated through an angle θ then equations of transformation are

$x = XCos\theta - YSin\theta \quad (i)$

$y = XSin\theta + YCos\theta \quad (ii)$

Using (i) and (ii) in (1) we have

$5(XCos\theta - YSin\theta)^2 - 6(XCos\theta - YSin\theta)(XSin\theta + YCos\theta) + 5(XSin\theta + YCos\theta)^2 - 8 = 0$

$\Rightarrow 5(X^2Cos^2\theta - 2XYCos\theta Sin\theta + Y^2Sin^2\theta) - 6(X^2Cos\theta Sin\theta + XYCos^2\theta - XYSin^2\theta -$
$Y^2Sin\theta Cos\theta) + 5(X^2Sin^2\theta + Y^2Cos^2\theta + 2XYSin\theta Cos\theta) - 8 = 0$

$\Rightarrow X^2(5Cos^2\theta - 6Cos\theta Sin\theta + 55Sin^2\theta) + XY(-10Cos\theta Sin\theta - 6Cos^2\theta + 6Sin^2\theta +$
$10Cos\theta Sin\theta) + Y^2(5Sin^2\theta + 6Sin\theta Cos\theta + 5Cos^2\theta) - 8 = 0 \quad (2)$

Now equation (2) will be free from product term XY if

$-10Cos\theta Sin\theta - 6Cos^2\theta + 6Sin^2\theta + 10Cos\theta Sin\theta = 0$

$\Rightarrow 6Sin^2\theta = 6Cos^2\theta \Rightarrow Sin^2\theta = Cos^2\theta$

$\Rightarrow Tan\theta = 1 \Rightarrow Tan\theta = \pm 1$

$\Rightarrow Tan\theta = 1 \quad \because Tan\theta = -1$ is not admissible $\Rightarrow \boxed{\theta = 45''}$

Put $\theta = 15''$ in (1) we have

$$x^2\left[5\left(\frac{1}{\sqrt{2}}\right)^2-6\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)+5\left(\frac{1}{\sqrt{2}}\right)^2\right]$$

$$=XY(0)+Y^2\left[5\left(\frac{1}{\sqrt{2}}\right)^2 6\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)+5\left(\frac{1}{\sqrt{2}}\right)^2\right]-8=9$$

$$\Rightarrow X^2\left[\frac{5}{2}-\frac{6}{2}+\frac{5}{2}\right]+Y^2\left[\frac{5}{2}+\frac{6}{2}+\frac{5}{2}\right]-8=0 \Rightarrow X^2\left(\frac{4}{2}\right)+Y^2\left(\frac{16}{2}\right)-8=0$$

$\Rightarrow 2X^2+8Y^2-8=0 \Rightarrow X^2+4Y^2-8-0$ Which is the required equation.

The general Equation of Sound Degree:

$$Ax^2+By^2+Gx^2+Fy+C=0$$

The most general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy=c=0 \qquad (1)$$

represents a conic.

Here is called the discriminant (1) represents.

(i) An ellipse or a circle if $h^2-ab<0$

(ii) A parabola if $h^2-ab=0$

(iii) A hyperbola if $h^2-ab>0$

If the axes are rotated about the origin through an angle $\theta(\theta<\theta<90^\circ)$ where θ is given by $Tan2\theta=\frac{2h}{a-b}$

If $a=b$ or $a=0=b$ then the axes are rotated through an angle 45^o.

Equation of transformation are

$x=XCos\theta-YSin\theta \qquad (i)$

$y=XSin\theta+YCos\theta \qquad (ii)$

Using (i) and (ii) in (1) we have the equation of the form

$AX^2+BY^2+2GX+2FY+C=0$

Solving (i) and (ii) for X, Y we find

$X=xCos\theta+ySin\theta$

$Y=-xSin\theta+yCos\theta$

Under certain conditions equation

$ax^2+2hxy+by^2+2gx+2fy+c=0$

May not represent any conic. In such a case we say (1) represents a degenerate conic one such degenerate conic is a pair of straight lines represented by (1) if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

Exercise 6.9

1. (i) $4x^2-4xy+y^2-6=0$

$\Rightarrow 4x^2+2(-2)xy+y^2=6=0 \qquad (1)$

Here $a=4,\ h=-2\ \&\ b=2$

In order to remove the term involving xy the angle through which axes be rotated is given by.

$$Tan2\theta=\frac{2h}{a-b}$$

$$\Rightarrow Tan2\theta=\frac{2(-2)}{4-1}=\frac{-4}{3}\Rightarrow\frac{2Tan\theta}{1-Tan^2\theta}=\frac{-4}{3}$$

$$\Rightarrow\frac{Tan\theta}{1-Tan^2\theta}=-\frac{2}{3}\Rightarrow 3Tan\theta=-2(1-Tan^2\theta)$$

$$\Rightarrow 3Tan\theta=-2+2Tan^2\theta$$

$$\Rightarrow 2Tan^2\theta=-3Tan\theta-2=0$$

$$\Rightarrow 2Tan^2\theta-4Tan\theta+Tan\theta-2=0$$

$$2Tan\theta(4Tan\theta-2)+1(Tan\theta-2)=0$$

$$(Tan\theta-2)(2Tan\theta+1)=0$$

$$\Rightarrow Tan\theta-2=0\quad or\quad 2Tan\theta+1=0$$

$$Tan\theta=2\qquad Tan\theta=-\frac{1}{2}$$

$Cot\theta=\frac{1}{2}$ (Not admissible) $\because\theta$ is in I quadrant.

Now $Sec^2\theta=1+Tan^2\theta=1+4=5$

$Sec^2\theta=\sqrt{5}\quad\Rightarrow\boxed{Cot\theta=\frac{1}{\sqrt{5}}}\quad\because\theta\ is\ in\ I$ quadrant

and $Cosec^2\theta=1+Cot^2\theta=1+\frac{1}{4}=\frac{5}{4}$

$Cosec^2\theta=\frac{\sqrt{5}}{2}\qquad\because 0<\theta<90^\circ$

$$\Rightarrow\boxed{Sin\theta=\frac{2}{\sqrt{5}}}$$

Now equation of transformation are

$$x=XCos\theta-YSin\theta=\frac{X-2Y}{\sqrt{5}}\qquad(3)$$

$$x=XSin\theta+YCos\theta=\frac{2X+Y}{\sqrt{5}}\qquad(4)$$

Putting these values $x\ \&\ y$ in (1) we have

$$4\left(\frac{X-2Y}{\sqrt{5}}\right)^2-4\left(\frac{X-2Y}{\sqrt{5}}\right)\left(\frac{X+2Y}{\sqrt{5}}\right)+\left(\frac{X+2Y}{\sqrt{5}}\right)^2-6=0$$

$$\frac{4}{5}(X^2-4XY+4Y^2)-\frac{4}{5}(2X^2-3XY-2Y^2)+\frac{1}{5}(4X^2+4XY+Y^2)-6=0$$

$$\Rightarrow\ 4X^2-16XY+16Y^2-8X^2+12XY+8Y^2+4X^2+4XY+Y^2-30=0$$

$$\Rightarrow\ 25Y^2-30=0\ \ \Rightarrow 5Y^2-6=0\ \ \Rightarrow Y^2=\frac{6}{5}$$

$\Rightarrow Y=\pm\sqrt{\frac{6}{5}}$ Which represents a pair of straight lines.

To find equation in xy plane we have from (3) and (4)

$X-2Y=\sqrt{5}x$ (5)

$2X-Y=-\sqrt{5}y$ (6)

By multiplying equation (5) by (2) and then subtracting (6) from it we have

$$2X-4Y=2\sqrt{5}x$$
$$\underline{-2X\pm Y=-\sqrt{5}y}$$
$$-5Y=2\sqrt{5}x-\sqrt{5}y$$

$$\Rightarrow 5Y=\sqrt{5}y-2\sqrt{5}x=\sqrt{5}(y-2x)$$

$$Y=\frac{1}{\sqrt{5}}(y-2x)$$

$$\pm\sqrt{\frac{6}{5}}=\frac{1}{\sqrt{5}}(y-2x)\qquad \because Y=\pm\sqrt{\frac{6}{5}}$$

$$\Rightarrow \pm\sqrt{6}=y-2x$$

$$\Rightarrow 2x-y\pm\sqrt{6}=0$$

Pair of lines

$2x-y+\sqrt{6}=0$, $2x-y-\sqrt{6}=0$

(ii) $\mathbf{x^2-2xy+y^2=8x-8y=0}$

$x^2+2(-1)xy+y^2-8x-8y=0$ (1)

$a=1,\ \ b=1,\ \ h=-1$

If θ is the angle of rotation to remove the xy term. Then

$$Tan2\theta=\frac{2h}{a-b}$$

$$\Rightarrow Tan2\theta=\frac{2(-1)}{1-1}=-\frac{2}{0}\qquad\Rightarrow 2\theta=90^o\ \ \Rightarrow\theta=45^o$$

Equation of transformation are

$$x=XCos45^o-YSin45^o=X\left(\frac{1}{\sqrt{2}}\right)-Y\left(\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow x=\frac{X-Y}{\sqrt{2}}\qquad (i)$$

$$y=XSin\theta+YCos\theta=XSin45^o+YCos45^o$$

$$\Rightarrow y=\frac{X+Y}{\sqrt{2}}\qquad (ii)$$

Using (i) and (ii) in (1) we have

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2-2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)+\left(\frac{X+Y}{\sqrt{2}}\right)^2-8\left(\frac{X+Y}{\sqrt{2}}\right)=0$$

$$\Rightarrow\left(\frac{X-Y}{\sqrt{2}}-\frac{X+Y}{\sqrt{2}}\right)^2-8\left(\frac{X-Y}{\sqrt{2}}-\frac{X-Y}{\sqrt{2}}\right)=0$$

$$\left(\frac{X-Y-X-Y}{\sqrt{2}}\right)^2-8\left(\frac{X-Y+X+Y}{\sqrt{2}}\right)=0$$

$$\frac{4Y^2}{2}-\frac{8(2X)}{\sqrt{2}}=0$$

$$\Rightarrow Y^2-\frac{8}{\sqrt{2}}X=0 \quad\Rightarrow Y^2=4\sqrt{2}X$$

$$\Rightarrow Y^2=4\sqrt{2} \qquad (2)$$

Which represents parabola.

Now from (i) and (ii) we have

$$X-Y=\sqrt{2}x \qquad (3)$$

$$X+Y=\sqrt{2}y \qquad (4)$$

Adding (3) & (4) we have

$$2X=\sqrt{2}x+\sqrt{2}Y=\sqrt{2}(x+y) \quad\Rightarrow X=\frac{1}{\sqrt{2}}(x+y)$$

Subtracting (4) from (3) we have

$$-2Y=\sqrt{2}x-\sqrt{2}y\Rightarrow 2Y=\sqrt{2}y-\sqrt{2}$$

Elements of the parabola are

Foci:

$$X=\sqrt{2} \qquad Y=0$$

$$\Rightarrow\frac{1}{\sqrt{2}}(x+y)=\sqrt{2} \qquad \frac{1}{\sqrt{2}}(y-x)=0$$

$$x+y=2 \qquad y-x=0$$

Now Solving

$$\begin{array}{r} x+y=2 \\ -x+y=0 \\ \hline 2y=2 \end{array}$$

Putting $y=1\ in\ x+y=2$ we have

$$x+1=2 \Rightarrow x=1$$

Thus Focus in xy plane is (1,1)

Vertex:

$$X=0 \qquad Y=0$$

$$\Rightarrow\frac{1}{\sqrt{2}}(x+y)=0 \qquad \frac{1}{\sqrt{2}}(y-x)=0$$

$$x+y=0 \qquad x-y=0$$

Solving $x+y=0$ & $x-y=0$

We have $x=0 \quad y=0$

Thus vertex : (0 ,0)

Axis of parabola Y = 0

Directrix $X=-\sqrt{2}$

$\Rightarrow \frac{1}{\sqrt{2}}(x+y)=-\sqrt{2}$

$\Rightarrow x+y=-2$

$\Rightarrow x+y+2=0$

(iii) $x^2+2xy+y^2+2\sqrt{2}x-2\sqrt{2}y+2=0$ (1)

$a=1=b,\ 2h=2 \Rightarrow h=1$

If θ is the angle of rotation to remove xy term

Then $Tan^2\theta=\frac{2h}{a-b}=\frac{2}{1-1}=\frac{2}{0}$

$\Rightarrow 2\theta=90^o \Rightarrow \boxed{\theta=45^o}$

Equation of transformation are

$x=XCos\theta-YSin\theta \Rightarrow x=\frac{X-Y}{\sqrt{2}}$ (i)

$y=XSin\theta+YCos\theta \Rightarrow y=\frac{X+Y}{\sqrt{2}}$ (ii)

Using (i) and (ii) in (1) we have

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2+2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)+\left(\frac{X-Y}{\sqrt{2}}\right)+2\sqrt{2}\left(\frac{X-Y}{\sqrt{2}}\right)-2\sqrt{2}\left(\frac{X+Y}{\sqrt{2}}\right)+\left(\frac{X+Y}{\sqrt{2}}\right)^2$$

$$\Rightarrow \left(\frac{X-Y}{\sqrt{2}}+\frac{X+Y}{\sqrt{2}}\right)^2+2X-2Y-2X-2Y+2=0$$

$$\Rightarrow \left(\frac{2X}{\sqrt{2}}\right)^2-4Y+2=0 \quad \Rightarrow \frac{4X^2}{2}-4Y+2=0$$

$\Rightarrow 2X^2-4Y+2=0 \Rightarrow X^2-2Y+1=0$

$\Rightarrow X^2=2Y-1$

Or $X^2=2\left(Y-\frac{1}{2}\right)$ (2)

Which represents a parabola

Now from (i) and (ii)

$X-Y=\sqrt{2}x$ (iii)

$X+Y=\sqrt{2}y$

$2X=\sqrt{2}(x+y)$

$\Rightarrow \boxed{X=\frac{x+y}{\sqrt{2}}}$

Also subtracting (iv) from (iii) we get

$-2Y = \sqrt{2}(x-y) \Rightarrow \boxed{Y = \frac{x-y}{\sqrt{2}}}$

Elements of the parabola (2)

From (2) $4a = 2 \Rightarrow \boxed{a = \frac{1}{2}}$

Vertex:

$X = 0 \qquad Y - \frac{1}{2} = 0$

i.e., $X = 0 \qquad Y = \frac{1}{2}$

$\frac{x+y}{\sqrt{2}} = 0 \qquad \frac{y-x}{\sqrt{2}} = \frac{1}{2}$

From

$$\begin{array}{r} x+y = 0 \\ -x+y = \frac{1}{\sqrt{2}} \\ \hline 2y = \frac{1}{\sqrt{2}} \end{array} \qquad \text{by adding}$$

$\boxed{y = \frac{1}{2\sqrt{2}}}$

Putting $y = \frac{1}{2\sqrt{2}}$ *in* $x + y = 0$

We have $x + \frac{1}{2\sqrt{2}} = 0 \quad \Rightarrow \quad x = -\frac{1}{2\sqrt{2}}$

Thus vertex the parabola in xy-system is $\left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

Foci:

$\Rightarrow X = 0 \qquad Y - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow \frac{x+y}{\sqrt{2}} \qquad Y = \frac{1}{2} + \frac{1}{2} 1 = 1$

$\Rightarrow x + y = 0 \qquad \frac{y-x}{\sqrt{2}} = 1$

$\Rightarrow x + y = 0 \qquad -x + y = \sqrt{2}$

Now by adding

$$\begin{array}{r} x+y = 0 \\ -x+y = \sqrt{2} \\ \hline 2y = \sqrt{2} \end{array} \Rightarrow y = \frac{1}{\sqrt{2}}$$

Putting $y=\frac{1}{\sqrt{2}}$ in $x+y=0$

We have $x+\frac{1}{\sqrt{2}}=0 \Rightarrow \quad x=-\frac{1}{\sqrt{2}}$

Thus Focus in xy system is $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

Axis $X=0 \Rightarrow \quad \frac{x+y}{\sqrt{2}}=0$

$\Rightarrow \quad x+y=0$

Directrix: $Y-\frac{1}{2}=-\frac{1}{2}$

$Y=-\frac{1}{2}+\frac{1}{2} \quad \Rightarrow Y=0$

$\Rightarrow \frac{x-y}{\sqrt{2}}=0 \quad \Rightarrow x-y=0$

(iv) $\mathbf{x^2+xy+y^2-4=0}$ $\mathbf{a=1,\quad 2h=1,\quad b=1}$

Here $a=1,\quad 2h=1,\quad b=1$

If θ is the angle of rotation to remove the term involving xy.

The $Tan2\theta=\frac{2h}{a-b}=\frac{1}{1-1}=\frac{1}{0}$

$\Rightarrow 2\theta=90^o \Rightarrow \theta=45^o$

Equation of transformation are

$x=XCos\theta-YSin\theta \quad \Rightarrow x=\frac{X-Y}{\sqrt{2}} \Rightarrow X-Y=\sqrt{2}x$

$X-Y=\sqrt{2}x \qquad (i)$

$y=\frac{X+Y}{\sqrt{2}} \Rightarrow X+Y=\sqrt{2}y \quad (ii)$

Adding (i) and (ii) we have

$2X=\sqrt{2}(x+y) \quad \Rightarrow \boxed{X=\frac{x+y}{\sqrt{2}}}$

Subtracting (ii) from (i) we have

$2Y=\sqrt{2}x-\sqrt{2}y \Rightarrow \boxed{Y=\frac{y-x}{\sqrt{2}}}$

Putting the value of x & y in (1) we have

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2-2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right)+\left(\frac{X+Y}{\sqrt{2}}\right)^2-4=0$$

$$\frac{X^2-2XY+Y^2}{2}+\frac{X^2-Y^2}{2}+\frac{X^2+Y^2+2XY}{2}-4=0$$

$$\Rightarrow X^2-2XY+Y^2+X^2+Y^2+2XY-8=0$$

$$3X^2+Y^2-8=0 \Rightarrow 3X^2+Y^2=0$$

$\Rightarrow \dfrac{3X^2}{8}+\dfrac{Y^2}{8}=1$

$\Rightarrow \dfrac{X^2}{\frac{8}{3}}+\dfrac{Y^2}{8}=1 \qquad (2)$

Which represents and Ellipse

Elements of the Ellipse

Centre $X=0 \qquad Y=0$

$\Rightarrow \dfrac{x+y}{\sqrt{2}}=0 \qquad \dfrac{x-y}{\sqrt{2}}=0$

$x+y=0 \qquad x-y=0$

Solving $x+y=0$ & $x-y=0$ we have

$x=0 \qquad y=0$

$\therefore$ Centre (0, 0)

$a^2=8 \Rightarrow a=2\sqrt{2}$

$b^2=\dfrac{8}{3} \Rightarrow b=\dfrac{2\sqrt{2}}{\sqrt{3}}$

Vertices

$X=0 \qquad Y=\pm 2\sqrt{2}$

$\Rightarrow \dfrac{x+y}{\sqrt{2}}=0 \qquad \dfrac{x-y}{\sqrt{2}}=\pm 2\sqrt{2}$

$x+y=0 \qquad -x+y=4, \quad -x+y=-4$

Solving $x+y=0$ & $-x+y=-4$

$2y=4 \Rightarrow y=2, \quad x=2$

One vertex is $(-2,2)$

Solving $x+y=0$ and $-x+y=-4$

$2y=-4 \Rightarrow y=-2$ then $x=2$

Thus vertices are $(-2,2),(2,-2)$

Equation of major axis

$X=0 \Rightarrow \dfrac{x+y}{\sqrt{2}}=0 \Rightarrow x+y=0$

Equation of minor axis:

$Y=0 \Rightarrow \dfrac{x-y}{\sqrt{2}}=0 \Rightarrow x-y=0$

Now $c^2=a^2-b^2=8-\dfrac{8}{3}=\dfrac{24-8}{3}=\dfrac{16}{3}$

$\Rightarrow c=\dfrac{4}{\sqrt{3}}$

Eccentricity $\Rightarrow e=\frac{c}{a}=\frac{\frac{4}{\sqrt{3}}}{2\sqrt{2}}$

$$e=\frac{4}{\sqrt{3}}\times=\frac{1}{2\sqrt{2}}=\frac{6}{\sqrt{6}}$$

Foci

$X=0 \quad Y=\pm\frac{4}{\sqrt{3}}$

$\Rightarrow\frac{x+y}{\sqrt{2}}=0 \qquad \frac{y-x}{\sqrt{2}}=\pm\frac{4}{\sqrt{3}}$

$x+y=0 \qquad -x+y=\pm\frac{4\sqrt{2}}{\sqrt{3}}$

Solving $\quad x+y=0 \qquad -x+y=\pm\frac{4\sqrt{2}}{\sqrt{3}}$

$y=\frac{2\sqrt{2}}{\sqrt{3}} \qquad x=\frac{-2\sqrt{2}}{\sqrt{3}}$

Solving $\quad x+y=0 \ \ \& \ -x+y=\frac{-4\sqrt{2}}{\sqrt{3}}$

$y=\frac{-2\sqrt{2}}{\sqrt{3}} \qquad x=\frac{2\sqrt{2}}{\sqrt{3}}$

Thus required foci are

$\left(\frac{2\sqrt{2}}{\sqrt{3}},\frac{-2\sqrt{2}}{\sqrt{3}}\right)$ and $\left(\frac{-2\sqrt{2}}{\sqrt{3}},\frac{2\sqrt{2}}{\sqrt{3}}\right)$

(v) $\quad 7x^2-6\sqrt{3}xy+13y^2-16=0 \qquad$ (I)

Let θ be the angle of rotation to remove the term involving the product xy then

$Tan2\theta=\frac{-6\sqrt{3}}{7-13}=\frac{-6\sqrt{3}}{-6}=\sqrt{3}$

$2\theta=60^o \Rightarrow \boxed{\theta=30^o}$

Now equation of transformation are

$x=XCos30^o-YSin30^o=\frac{\sqrt{3}X-Y}{2}$

$y=XSin30^o+YCos30^o=\frac{X+\sqrt{3}Y}{2}$

$\Rightarrow x=\frac{\sqrt{3}X-Y}{2}\Rightarrow\sqrt{3}X-Y=2x \quad (i)$

$\Rightarrow y=\frac{X+\sqrt{3}Y}{2}\Rightarrow X+\sqrt{3}Y=2y \quad (ii)$

Putting values of x & y in (1) we get

$$\Rightarrow 7\left(\frac{\sqrt{3}X-Y}{2}\right)^2-6\sqrt{3}\left(\frac{\sqrt{3}X-Y}{2}\right)\left(\frac{\sqrt{3}X+Y}{2}\right)+13\left(\frac{\sqrt{3}X+Y}{2}\right)^2-16=0$$

$$\Rightarrow 7\left(\frac{3X^2-2\sqrt{3}XY+y^2}{2}\right)^2-\frac{6\sqrt{3}}{4}(\sqrt{3}X^2+2XY-\sqrt{3}Y^2)+13\left(\frac{X^2+3Y^2+2\sqrt{3}XY}{2}\right)-16=0$$

$$\Rightarrow 21X^2-14\sqrt{3}XY+7Y^2-18X^2-12\sqrt{3}XY+8Y^2+13X^2+39Y^2+26\sqrt{3}XY-64=0$$

$$\Rightarrow 16X^2+64Y^2-64=0$$

$$16X^2+64Y^2=64$$

$$\frac{16X^2}{64}+\frac{64Y^2}{64}=\frac{64}{64}$$

$$\frac{X^2}{4}+\frac{Y^2}{1}=1 \qquad (2)$$

The equation (2) represent an Ellipse

From (2)

$a^2=4 \Rightarrow a=2$

$b^2=1 \Rightarrow b=1$

Using $c^2=a^2-b^2=4-1=3 \Rightarrow c=\sqrt{3}$

Now multiplying equation (i) by $\sqrt{3}$ and then adding in (ii) we have

$$3X-\sqrt{3}Y=2\sqrt{3}x$$

$$\underline{X+\sqrt{3}Y=2y}$$

$$4X= \quad 2(\sqrt{3}x+y) \quad \Rightarrow X=\frac{\sqrt{3}x+y}{2}$$

Now from (i) $Y=\sqrt{3}X-2x$

$$\Rightarrow Y=\sqrt{3}\left(\frac{\sqrt{3}x+y}{2}\right)-2x=\frac{3x+\sqrt{3}y-4x}{2}$$

$$\Rightarrow Y=\frac{\sqrt{3}y-x}{2}$$

Centre $X=0 \qquad Y=0$

$\Rightarrow \sqrt{3}x+y=0 \quad \sqrt{3}y-x=0$

$\Rightarrow x=0 \quad y=0$

$\Rightarrow$ Centre in xy system is (0,0)

Foci $X=\pm\sqrt{3}$, $Y=0$

$\Rightarrow \sqrt{3}x+y=\pm 2\sqrt{3} \quad \sqrt{3}y-x=0$

Put $x=\sqrt{3}y$ in $\sqrt{3}x+y=2\sqrt{3}$ we have

$$\sqrt{3}(\sqrt{3}y)+y=2\sqrt{3} \Rightarrow 4y=2\sqrt{3} \Rightarrow y=\frac{\sqrt{3}}{2}$$

and $x=\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)=\Rightarrow x=\frac{-3}{2}$

So the foci of the ellipse in xy-system

are $\left(\frac{2}{3},\frac{\sqrt{3}}{2}\right)$ $\left(\frac{-3}{2},\frac{-\sqrt{3}}{2}\right)$

Vertices

$X=\pm 2,\quad y=0$

$\Rightarrow \frac{\sqrt{3}x+y}{2}=\pm 2\quad \frac{\sqrt{3}y-x}{2}=0$

$\Rightarrow \sqrt{3}x+y=\pm 4\quad x=\sqrt{3}y$

Putting $x=\sqrt{3}y$ *in* $\sqrt{3}x+y=4$ we get.

$\sqrt{3}(\sqrt{3}y)+y=4\quad \Rightarrow 4y=4\quad \Rightarrow y=1$

and $x=\sqrt{3}(-1)\Rightarrow x=-\sqrt{3}$

Thus vertices of the ellipse in xy plane are $(\sqrt{3},1),(-\sqrt{3},-1)$

Major Axis of (2) is $Y=0$

i.e., $\frac{\sqrt{3}x+y}{2}=0\Rightarrow \sqrt{3}x+y=0$

Eccentricity: $e=\frac{c}{a}$

i.e., $e=\frac{\sqrt{3}}{2}$

Directrices of (2) are $X=\pm\frac{c}{e^2}$

$\Rightarrow \frac{\sqrt{3}x+y}{2}=\pm\frac{\sqrt{3}}{\frac{3}{4}}=\pm\frac{4\sqrt{3}}{3}=\pm\frac{4}{\sqrt{3}}$

$\Rightarrow \sqrt{3}x+y=\pm\frac{8}{\sqrt{3}}\Rightarrow 3x+\sqrt{3}=\pm 8$

Thus the equations of directrices in xy plane are $3x+\sqrt{3}=\pm 8$

(vi) $4x^2-4xy+7y^2+12x+6y-9=0$ (1)

If θ is the angle of rotation to eliminate the xy-term. Then

$Tan2\theta=\frac{2h}{a-b}=\frac{-4}{4-7}=\frac{-4}{-3}=\frac{4}{3}$

$\Rightarrow \frac{2Tan\theta}{1-Tan^2\theta}=\frac{4}{3}\Rightarrow \frac{Tan\theta}{1-Tan^2\theta}=\frac{2}{3}$

$\Rightarrow 3Tan\theta=2(1-Tan^2\theta)$

$\Rightarrow 3Tan\theta=2-2Tan^2\theta$

$\Rightarrow 2Tan^2\theta+3Tan\theta-2=0$

$\Rightarrow 2Tan^2\theta+4Tan\theta-Tan\theta-2=0$

$\Rightarrow 2Tan\theta(Tan\theta+2)-1(Tan\theta+2)=0$

$\Rightarrow (Tan\theta+2)(2Tan\theta-1)=0$

$\Rightarrow Tan\theta+2=0 \quad or \quad 2Tan\theta-1=0$

$Tan\theta=-2 \qquad Tan\theta=\dfrac{1}{2}$

Since $0<\theta<90°$

$\therefore Tan\theta=-2$ is not admissible.

Now $Tan\theta=\dfrac{1}{2} \quad \Rightarrow \dfrac{Sin\theta}{Cos\theta}=\dfrac{1}{2}$

$\Rightarrow 2Sin\theta=Cos\theta \Rightarrow 4Sin^2\theta=Cos^2\theta$

$4Sin^2\theta=1-Sin^2\theta \Rightarrow 5Sin^2\theta=1$

$\boxed{Sin\theta=\dfrac{1}{\sqrt{5}}}$ and $\boxed{Cos\theta=\dfrac{2}{\sqrt{5}}}$

Equations of transformation are

$x=XCos\theta-YSin\theta=\dfrac{2X-Y}{\sqrt{5}}$

$y=XSin\theta+YCos\theta=\dfrac{X+2Y}{\sqrt{5}}$

$\Rightarrow x=\dfrac{2X-Y}{\sqrt{5}} \Rightarrow 2X-Y=\sqrt{5}x \qquad (i)$

$y=\dfrac{X+2Y}{\sqrt{5}} \Rightarrow X+2Y=\sqrt{5}y \qquad (ii)$

Putting the values of x & y in (1) we have

$$4\left(\frac{2X-Y}{\sqrt{5}}\right)^2-\left(\frac{2X-Y}{\sqrt{5}}\right)\left(\frac{X+2Y}{\sqrt{5}}\right)+7\left(\frac{X+2Y}{\sqrt{5}}\right)^2+12\left(\frac{2X-Y}{\sqrt{5}}\right)+6\left(\frac{X+2Y}{\sqrt{5}}\right)-9=0$$

$$\Rightarrow 4\left(\frac{4X^2-4XY+Y^2}{5}\right)-4\left(\frac{2X^2+3XY-2Y^2}{5}\right)+7\left(\frac{X^2+4XY+4Y^2}{5}\right)+\frac{12}{\sqrt{5}}(2X-Y)+$$
$$\frac{6}{\sqrt{5}}(X+2Y)-9=0$$

$$\Rightarrow 16X^2-16XY+4Y^2=8X^2-12XY+8Y^2+7X^2+28XY+28Y^2+12\sqrt{5}(2X-Y)+$$
$$6\sqrt{5}(X+2Y)-45=0$$

$\Rightarrow 15X^2+40Y^2+24\sqrt{5}X-12\sqrt{5}+6\sqrt{5}X+12\sqrt{5}Y-45=0$

$15X^2+40Y^2+30\sqrt{5}X-45=0$

$\Rightarrow X^2+2\sqrt{5}X+\dfrac{8}{3}Y^2-3=0$

$\Rightarrow X^2+2\sqrt{5}X+5+\dfrac{8}{3}Y^2-3+5$

$(X+\sqrt{5})^2+\dfrac{8}{3}Y^2=8$

$$\Rightarrow \frac{(X+\sqrt{5})^2}{8}+\frac{Y^2}{3}=1 \qquad (2)$$

Which is an Ellipse

From (2)

$a^2=8 \Rightarrow a=2\sqrt{2}$

$b^2=3 \Rightarrow b=\sqrt{3}$

From (i) and (ii)

$X=\frac{2x+y}{\sqrt{5}}$ & $Y=\frac{2y-x}{\sqrt{5}}$

Using $c^2=a^2-b^2$

$\Rightarrow c^2=8-3=5 \Rightarrow c=\sqrt{5}$

Eccentricity

$e=\frac{c}{a}=\frac{\sqrt{5}}{2\sqrt{2}}$

$e=\frac{1}{2}\sqrt{\frac{5}{2}}=\sqrt{\frac{5}{8}}$

Centre of (2) is

$X+\sqrt{5}=0$ & $Y=0$

$\Rightarrow X=-\sqrt{5} \quad \frac{2y-x}{\sqrt{5}}=0$

i.e., $\frac{y+2x}{\sqrt{5}}=-\sqrt{5} \qquad x=2y$

$\Rightarrow 2x+y=-5$

Using $x=2y$ *in* $2x+y=-5$

$2(2y)+y=-5 \Rightarrow 5y=-5 \Rightarrow y=-1$

and $x=2(-1) \Rightarrow x=-2$

$\therefore$ Centre of the ellipse in xy plane is $(-2,-1)$

Foci of (2) are

$X+\sqrt{5}=\pm\sqrt{5}$ & $Y=0$

$\Rightarrow X=-\sqrt{5}\pm\sqrt{5} \qquad 2y-x=0$

$x=2y$

$2x+y=-5\pm 5$

$2x+y=0,\ \ 2x+y=-10$

Using $x=2y$ *in* $2x+y=-10$

$x=0 \qquad y=0$

Putting $x=2y$ *in* $2x+y=-10$ we get

$2(2y)+y=-10 \Rightarrow 5y=-10 \Rightarrow \boxed{y=-2}$

and $x=2(-2) \Rightarrow \boxed{x=-4}$

Thus foci of ellipse (2) in xy-system are (0,0) , $(-4,-2)$

Vertices are

$X+\sqrt{5}=\pm 2\sqrt{2}$, $Y=0$

i.e., $X=-\sqrt{5}\pm 2\sqrt{2}$, $\frac{2y-x}{\sqrt{5}}=0$

$\Rightarrow \frac{2x-y}{\sqrt{5}}=-\sqrt{5}\pm 2\sqrt{2}$, $x=2y$

Putting $x=2y$ *in* $2x+y=-5+2\sqrt{10}$

We get $2(2y)+y=-5+2\sqrt{10} \Rightarrow 5y=-5+2\sqrt{10}$

$y=-1+\frac{2\sqrt{10}}{\sqrt{5}}=-1+\sqrt{\frac{8}{5}}$ and

$x=2\left(-1+\sqrt{\frac{8}{5}}\right)=-2+\sqrt{\frac{32}{5}}$

Again putting $x=2y$ *in* $2x+y=-5-2\sqrt{10}$

$5y=-5-2\sqrt{10} \Rightarrow y=-1-\sqrt{\frac{8}{5}}$ and

$x=2\left(-1-\sqrt{\frac{8}{5}}\right)=-2-\sqrt{\frac{32}{5}}$

Thus foci of the ellipse in xy plane are

$\left(-2+\sqrt{\frac{32}{5}}, -1+\sqrt{\frac{8}{5}}\right), \left(-2-\sqrt{\frac{32}{5}}, -1-\sqrt{\frac{8}{5}}\right)$

Major axis of ellipse (2) is

$Y=0 \Rightarrow \frac{2y-x}{\sqrt{5}}=0 \Rightarrow x-2y=0$

$\therefore$ Major axis in xy-plane is $x-2y=0$

Minor axis of ellipse (2) is

$X=0 \Rightarrow \frac{2x+y}{\sqrt{5}}=0 \Rightarrow 2x+y=0$

$\therefore$ Major axis is in xy-plane is

$2x+y=0$

Directrices

$X+\sqrt{5}=\pm\frac{c}{e^2}=\pm\frac{\sqrt{5}}{\left(\sqrt{\frac{5}{8}}\right)^2}=\frac{\pm\sqrt{5}}{\frac{5}{2}}$

$\Rightarrow \frac{2x+y}{\sqrt{5}}+\sqrt{5}=\pm\sqrt{5}.\frac{8}{5}=\pm\frac{8}{\sqrt{5}}$

$\Rightarrow 2x+y+5=\pm 8$

$\Rightarrow 2x+y=-5\pm 8$

$2x+y=3, \quad 2x+y=-13$

Are the directrices in the xy-plane.

TEST YOUR SKILLS — Marks 100

OBJECTIVE

Q.No.1 Given below are a few possible answers to each statement of which one is correct, identify the correct one. (20)

1. **The size of the circle depends on how near the plane is the vertex of the**
   (a) Triangle (b) Rectangle
   (c) Cone (d) None of these
2. **The Greek mathematician Apollonius discovered many intersecting properties of the**
   (a) Straight line (b) Plane
   (c) Conic sections (d) None of these
3. **The theory of conics plays an important role in modern**
   (a) Physics (b) Calculus
   (c) Space mechanics (d) None of these
4. **If $c(h, k)$ is the centre and r is the radius of a circle then equation of the circle is**
   (a) $(x+h)^2+(y+k)^2=r^2$ (b) $(x-h)^2+(y-k)^2=r^2$
   (c) $(x-h)+(y-k)=r$ (d) $(x+h)+(y+k)=r$
5. **A circle with radius $r = 1$ is called**
   (a) Point circle (b) Standard form
   (c) General form (d) Unit circle
6. **A circle $x^2+y^2+2gx+2fy+c=0$ is centred at**
   (a) (g, f) (b) (f, c)
   (c) $(-g, -f)$ (d) (g, c)
7. **Centre of the circle $x^2+y^2-6x+4y+13=0$ is**
   (a) $(3, 2)$ (b) $(3, -2)$
   (c) $(-3, -2)$ (d) $(-3, 2)$
8. **A chord which contain the centre of the circle is called a**
   (a) Radius (b) Chord
   (c) Diameter (d) None of these
9. **The line $y = mx + c$ is tangent to the circle $x^2+y^2=a^2$ if $c =$**
   (a) $\pm m\sqrt{1+a^2}$ (b) $\pm m\sqrt{1-a^2}$
   (c) $\pm a\sqrt{1-m^2}$ (d) $\pm a\sqrt{1+m^2}$
10. **The equation of normal to the circle $x^2+y^2=a^2$ at $P(x_1, y_1)$ is**

(a) $xx_1 - yy_1 = a^2$ (b) $xx_1 = yy_1$

(c) $xx_1 + yy_1 = a^2$ (d) $xy_1 = yx_1$

11. For a conic the fixed line *L* is called a

(a) Eccentricity (b) Directrix

(c) Focus (d) Vertex

12. Parametric equations of the parabola $y^2 = 4ax$ are

(a) $x = at,\ y = 2a$ (b) $x = at^2,\ y = 2a$

(c) $x = at,\ y = 2at$ (d) $x = at^2,\ y = 2at$

13. If equation of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then directrices are

(a) $x = \pm\frac{c}{e}$ (b) $x = \pm\frac{c}{e^2}$

(c) $y = \pm\frac{c}{e}$ (d) $y = \pm\frac{c}{e^2}$

14. If equation of ellipse is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ then vertices are

(a) $(0, \pm a)$ (b) $(+a, 0)$

(c) $(\pm a, b)$ (d) $(a, \pm b)$

15. If equation of ellipse is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ then centre is

(a) (0, 0) (b) (a, b)

(c) (b, a) (d) $(-a, -b)$

16. End points of the major axis of an ellipse are called its

(a) Centre (b) Foci

(c) Vertices (d) Co-vertices

17. If equation of hyperbola is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ then foci are

(a) $(0, \pm c)$ (b) $(\pm c, 0)$

(c) $(\pm a, 0)$ (d) $(0, \pm b)$

18. If the equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then vertices are

(a) $(\pm a, 0)$ (b) $(0, \pm a)$

(c) $(0, \pm b)$ (d) $(\pm b, 0)$

19. **A second degree equation: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents parabola if**
    (a) $h^2 - ab = 0$ (b) $h^2 - ab > 0$
    (c) $h^2 - ab < 0$ (d) None of these

20. **If the distance of any point on the curve from any of the two lines approaches zero then it is called**
    (a) Axis (b) Directrics
    (c) Asymptotes (d) None of these

SECTION I
SUBJECTIVE

Attempt any 25 (twenty five) short question from Question 2,3 and 4, at least 12 short questions from question 2, 12 short question from question 3 and 13 short question from question 4. All question carry equal marks. (25x2=50)

Q. No. 2

i. What is the general form of eq. of a circle? Also find its centre and radius.

ii. Find the eq. of a circle whose centre is at (5, −2) & radius is 4.

iii. Find centre and radius of the circle

$$4x^2 + 4y^2 - 8x + 12y - 25 = 0$$

iv. Write the eq. of tangent and normal to the circle $x2 + y2 + 2gx + 2fy + c = 0$ at the pt$(x1, y1)$

v. What is the condition that the pt. (x_1, y_1) lies outside, on or inside the circle?
$x^2 + y^2 + 2gx + 2fy + c = 0$

vi. Determine whether the pt. (−5, 6) lies outside, on or inside the circle $x^2 + y^2 + 4x - 6y - 12 = 0$

vii. Write the condition that the line $y = mx + c$ should be tangent to the circle $x^2 + y^2 = a^2$. Also write the eq. of tangent.

viii. Find the length of the tangent from the pt. $P(-5, 10)$ to the circle $5x^2 + 5y^2 + 14x + 12y - 10 = 0$

ix. Check the position of the pt(−5, 6) with respect to the circle $x^2 + y^2 = 81$

x. Define conic section & what the value of e when it will represent parallel, ellipse hyperbola.

xi. What are the parametric eqs of the parabola $y^2 = 4ax$

xii. Under what condition the eq. $ax^2 + by^2 + 2gx + 2fy + c = 0$ represents a parabola?

Q.No. 3

i. Find focus, directrix of the parabola: $x^2 = -16y$

ii. Find the eq. of parabola whose focus is $F(-3, 1)$ & directrix $x = 3$

iii. Find an eq. of the parabola having focus at the origin & directrix parallel to (i) x–axiis, (ii) y-axis

iv. Show that the ordinate at any pt. p of the parabola is a mean proportional between the length of latus sectum and the abscissa of p.

v. Find an eq. of the ellipse having centre at (0, 0), focus (0, –3) and one vertex at (0, 4).

vi. Prove that length of L.R. of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{2b^2}{a}$

vii. Define hyperbola.

viii. What are parametric eqs of an ellipse? and What is the parametric eq. of a hyperbola?

ix. What are the eqs of the directrices of an ellipse when major axis is along x-axis?

x. What are the eqs of the directrices of a hyperbola when transverse axis is along x-axis?

xi. Find an eq. of the hyperbola where foci are $(\pm 4, 0)$ & vertex are $(\pm 2, 0)$.

xii. Find eccentricity, vertices & foci of the parabola: $\frac{y^2}{16}-\frac{x^2}{49}=1$

Q.No. 4

i. Find the eq. of the hyperbola whose focus (6, 0), vertex (4, 0) centre (0, 0).

ii. Write the eq. of tangent & normal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the pt (x_1, y_1).

iii. Write the eq. of tangent & normal to hyperbola to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=$ at the pt (x_1, y_1).

iv. Find the condition that we line $y = mx + c$ should be tangent to the parabola $y^2 = 4ax$ & also find eq. of tangent.

v. Find the condition that the line $y = mx + c$ should be tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Also find eq. of tangent.

vi. Find the condition that the line $y = mx + c$ should be tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Also find eq. of tangent.

vii. Find the conditions that the 2nd degree eq. of the form $Ax^2+By^2+Gx+Fy+C=0$

represent a circle, ellipse, hyperbola or a parabola.

viii. Find the conditions that the most general eq. of 2nd degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ represent a circle or ellipse, parabola, hyperbola.

ix. Find vertex & centre of the ellipse $9x^2+y^2=18$

x. Find eccentricity & directrices of $x^2+4y^2=16$

xi. Find centre & axis of the ellipse $\frac{(y+2)^2}{9}+\frac{(x-2)^2}{16}=1$

xii. Find semi major & minor axis of the ellipse $4x^2+7y^2=36$

xiii. Find eccentricity of the hyperbola $25x^2-16y^2=400$

SECTION II

Attempt any 3 (three) questions. **(3x10=30)**

Q.No.5

(a) Show the circles, $x^2+y^2+2x-8=0$ and $x^2+y^2-6x+6y-46=0$ touch internally

(b) Find the co-ordinates of the points of intersection of the line $2x+y=5$ and the circle $x^2+y^2+2x-9=0$. Also find the length of the intercepted chord.

Q.No.6

(a) Find the length of the chord cut off from the line $2x+3y=13$ by the circle $x^2+y^2=26$

(b) Prove that perpendicular dropped from the centre of the circle on a chord bisects the chord.

Q.No.7

(a) Find equation of the circles of radius 2 and tangent to the line $x-y-4=0$ at $A(1,-3)$

(b) Prove that normal lines of a circle pass through the centre of the circle.

Q.No.8

(a) A parabolic arch has a 100 m base and height 25 m. find the height of the arch at the point 30m from the centre of the base.

(b) Prove that the lactusrectum of the ellipse. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *is* $\frac{2b^2}{a}$.

Q.No.9

(a) Find any point on a hyperbola the difference of its distances from the points (2.2) and (10.2) is 6. Find an equation of the hyperbola.

(b) Find equation of the tangent and normal to $y^2=4ax$

Previous Board Questions

1. Define a circle?
2. What is the centre of the circle? (Lhr – 2008)
3. Find an equation of circle with centre at (5, – 2) and radius 4?
   (Lhr – 2007, 2008, 2009)
4. Find an equation of circle with centre at $(\sqrt{2}, -3\sqrt{3})$ and radius $2\sqrt{2}$.
   (Fsd – 2009)
5. Find the centre and radius.
   $5x^2 + 5y^2 + 14x - 12y - 10 = 0$
   (Lhr – 2009, Fsd – 2009)
6. What are the parametric equations of circle $x^2 + y^2 = r^2$.?
   (Mtn – 2009)
7. Find an equation of circle with ends of diameter at (–3, 2) and (5, – 6).?
   (Grw – 2005, Lhr – 2008, Mtn – 2009)
8. Write standard equation of the parabola with x – axis as its axis?
   (Mirpur – 2009)
9. Find the vertex and focus of parabola $x^2 - 4x - 8y + 4 = 0$?
   (Lhr – 2006, 2008, Fsd – 2009)
10. What do you mean by major and minor axes of ellipse?
    (Mtn – 2009)
11. Define hyperbola?
    (Lhr – 2008)
12. Find the centre and equations of directrices of the hyperbola
    $\frac{y^2}{16} - \frac{x^2}{9} = 1.$
    (Mirpur – 2009)
13. Find the equation of hyperbola with centre (0, 0), focus (6, 0) and vertex (4, 0)?
    (Lahore – 2007, 2008)
14. Check the position of point (5, 6) with respect to circle $x^2 + y^2 = 81$.
    (Lahore – 2010) Group – I
15. Find the equation of parabola having focus at (0, 0) and directrix $y = 2$.
    (Lahore – 2010) Group – I
16. Find the equation of the normal to the circle $x^2 + y^2 = 25$ at (4, 3).
    (Lahore – 2010) Group – II
17. Prove analytically that the line joining the centre of a circle to the mid point of its chord is perpendicular to the chord.
    (Lahore – 2010) Group – II
18. Check the position of the point (5, 6) with respect to the circle $x^2 + y^2 = 64$.
    (Gujranwala – 2010)
19. Find an equation of parabola with focus at (–3, 1) and directrix $x = -3$.
    (Gujranwala – 2010)

Vectors

7

Definitions

1. **Scalar:** A physical quantity which is defined only by its magnitude.
2. **Vector:** A physical quantity defined by its magnitude and direction also.
3. **Magnitude or Length or Norm:** Absolute value of vector is called magnitude or length or Norm $|\overrightarrow{AB}|$
4. **Unit Vector:** A vector whose magnitude is unity or 1, $\hat{v} = \frac{\underline{v}}{|\underline{v}|}$
5. **Equal Vectors:** Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal if they have same magnitude and direction $|\overrightarrow{AB}| = |\overrightarrow{CD}|$
6. **Parallel Vectors:** Two vectors are parallel if and only if they are non-zero scalar multiple of each other $\underline{a} = \lambda \underline{b}$
7. **Triangular Law:** If $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are two sides of triangle then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ is called triangular law.
8. **Position Vector:** The vector of whose initial point is the origin O terminal point is P.
9. **Zero Vector:** If magnitude of a vector is zero then it is called zero vector.
10. **Direction angles and Direction Cosines:** Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ be non-zero vector and α, β, γ are angle formed between $\underline{r}$ and $\underline{i}, \underline{j}, \underline{k}$ respectively then α, β, γ are Direction angles and $\cos\alpha, \cos\beta, \cos\gamma$ are Direction Cosines.
11. **Scalar or Dot Product:** If $\underline{u}$ and $\underline{v}$ are non-zero vectors in a plane with same initial line then their dot product is $\underline{u}.\underline{V} = |\underline{u}|\ |\underline{v}| \cos\theta$
12. **Cross or Vector Product:** If u & v are non zero vectors then

$\underline{u} \times \underline{v} = (|\underline{u}|\ |\underline{v}| \sin\theta)\ \hat{n}$ **(Sargodha 2008)**

Important Formulae

1. Vector = $\overrightarrow{AB}$ or $\underline{U}$
2. Scalar = AB or U or $|\overrightarrow{AB}|$
3. Magnitude = $|\overrightarrow{AB}|$
4. Unit Vector = $\hat{\underline{v}} = \dfrac{\underline{v}}{|\underline{v}|}$
5. Triangular Law of addition $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
6. Equal Vectors $|\overleftrightarrow{AB}| = |\overrightarrow{CD}|$
7. Position Vector = $\overrightarrow{OP}$
8. Ratio Formula $\underline{r} = \dfrac{p\underline{a} + q\underline{b}}{p+q}$
9. Direction angles α, β, γ
10. Direction Cosines $\cos\alpha = \dfrac{x}{r}$, $\cos\beta = \dfrac{y}{r}$, $\cos\gamma = \dfrac{z}{r}$
11. Triples can be direction angle if $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
12. Scalar product or Dot product of $\underline{u}$ and $\underline{v}$ $\underline{u}.\underline{V} = |\underline{u}|\ |\underline{v}| \cos\theta$
13. Perpendicular $\underline{u}.\underline{v} = 0$
14. Parallel $\underline{u} = \lambda\underline{v}$ or $\underline{u} \times \underline{v} = 0$
15. Vector product or Cross Product = $\underline{u} \times \underline{v} = |\underline{u}|\ |\underline{v}| \sin\theta\hat{n}$
16. $\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$, where $\underline{u} = u_1\underline{i} + u_2\underline{j} + u_3\underline{k}$, $\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$
17. Area of parallel gram ABCD = $|\overrightarrow{AB} \times \overrightarrow{AC}|$
18. Area of Triangle ABC = $\dfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$
19. Volume of parallelepiped = $\underline{u}.\underline{v} \times \underline{w}$
20. $\underline{u}, \underline{v}, \underline{w}$ are coplanar if $\underline{u}.\underline{v} \times \underline{w} = 0$
21. Volume of Tetrahedron = $\dfrac{1}{6}(\underline{u}.\underline{v} \times \underline{w})$
22. Work done = $\underline{F}.\underline{D}$
23. Moment of Force = $\underline{r} \times \underline{F}$
24. $\underline{u}.\underline{v} \times \underline{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$
25. $\underline{u}.\underline{v} \times \underline{w} = \underline{v}.\underline{w} \times \underline{u} = \underline{w}.\underline{u} \times \underline{v}$

Complete.

Exercise 7.1

1. (i) $P(2,3), Q(6,-2)$

$\overrightarrow{PQ}=(6-2)\underline{i}+(-2-3)\underline{j}=4\underline{i}-5\underline{j}$

(ii) $P(0,5), Q(-1,-6)$

$\overrightarrow{PQ}=(-1-0)\underline{i}+(-6-5)\underline{j}=-\underline{i}-11\underline{j}$

2. (i) $\underline{u}=2\underline{i}-7\underline{i}$

$|\underline{u}|=\sqrt{(2)^2+(-7)^2}=\sqrt{4+49}=\sqrt{53}$

(ii) $\underline{u}=\underline{i}+\underline{j}$

$\Rightarrow |\underline{u}|=\sqrt{(1)^2+(1)^2}=\sqrt{1+1}=\sqrt{2}$

(iii) $\underline{u}=[3,-4]$ **(Sargodha 2009)**

$=3\underline{i}-4\underline{j}$

Then $|\underline{u}|=\sqrt{(3)^2+(-4)^2}=\sqrt{9+16}=\sqrt{25}=5\,(always\ +ve)$

3. (i) $\underline{u}=2\underline{i}-7\underline{j},\ \underline{v}=\underline{i}-6\underline{j},\ \underline{w}=-\underline{i}+\underline{j}$ **(Sargodha 2011)**

$\underline{u}+\underline{v}-\underline{w}=(2\underline{i}-7\underline{j})+(\underline{i}-6\underline{j})-(-\underline{i}+\underline{j})$

$=4\underline{i}-14\underline{j}$

(ii) $2\underline{u}-3\underline{v}+4\underline{w}=2(2\underline{i}-7\underline{j})-3(\underline{i}-6\underline{j})+(-\underline{i}+\underline{j})$

$4\underline{i}-14\underline{j}-3\underline{i}+18\underline{j}-\underline{i}+\underline{j}=0\underline{i}+5\underline{j}$

(iii) $\frac{1}{2}\underline{u}+\frac{1}{2}\underline{v}+\frac{1}{2}\underline{w}=\frac{1}{2}(\underline{u}+\underline{v}+\underline{w})$

$=\frac{1}{2}(2\underline{i}-7\underline{j}+\underline{i}-6\underline{j}-\underline{i}+\underline{j})=\frac{1}{2}(2\underline{i}-12\underline{j})=\underline{i}-6\underline{j}$

4. $A(1,-1),\ B(2,0),\ C(-1,3),\ D(-2,2)$

$\overrightarrow{AB}=(2,-1)\underline{i}+(0-(-1))\underline{j}=i+\underline{j}$

$\overrightarrow{CD}=(-2,+1)\underline{i}+(2-3)\underline{j}=i-\underline{j}$

$\overrightarrow{AB}+\overrightarrow{CD}=(\underline{i}+\underline{j})+(-\underline{i}-\underline{j})=\underline{i}+\underline{j}-\underline{i}-\underline{j}=0\underline{i}+0\underline{i}=\underline{O}$

5. $\overrightarrow{AB}=4i-2j,\ B(-2,5),\ \overrightarrow{AO}=?,\ O(0,0)\ (origin)$ **(Sargodha 2010)**

$\overrightarrow{AO}=\overrightarrow{AB}+\overrightarrow{BO}$ and $\overrightarrow{BO}=(0+2)\underline{i}+(0-5)\underline{j}$

$4\underline{i}-2\underline{j}+2\underline{i}-5\underline{j} \qquad =2\underline{i}-5\underline{j}$

$= 6\underline{i} - 7\underline{j}$

6. (i) $\underline{v} = 2\underline{i} - \underline{j}$ **(Sargodha 2009, Lahore 2010)**

$|\underline{v}| = \sqrt{(2)^2 + (-1)^2} = \sqrt{4+1} = \sqrt{5}$

Then $\dfrac{\underline{v}}{|\underline{v}|} = \dfrac{2\underline{i} - \underline{j}}{\sqrt{5}} = \dfrac{1}{\sqrt{5}}(2\underline{i} - \underline{i})$

(ii) $\underline{v} = \dfrac{1}{2}\underline{i} + \dfrac{\sqrt{3}}{2}\underline{j}$ **(Sargodha 2008,10, Lahore 2010)**

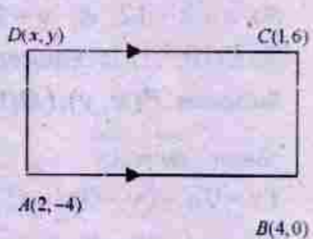
Then $|\underline{v}| = \sqrt{\left(\dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2} = \sqrt{\dfrac{1}{4} + \dfrac{3}{4}} = \sqrt{\dfrac{4}{4}} = 1$

$\dfrac{\underline{v}}{|\underline{v}|} = \dfrac{\frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}}{1} = \dfrac{1}{2}\underline{i} + \dfrac{\sqrt{3}}{2}\underline{j}$

(iii) $\underline{v} = \dfrac{\sqrt{3}}{2}\underline{i} - \dfrac{1}{2}\underline{j}$

Then $|\underline{v}| = \sqrt{\left(-\dfrac{\sqrt{3}}{2}\right)^2 + \left(-\dfrac{1}{2}\right)^2} = \sqrt{\dfrac{3}{4} + \dfrac{1}{4}} = \sqrt{\dfrac{4}{4}} = 1$

$\hat{\underline{v}} = \dfrac{\underline{v}}{|\underline{v}|} = \dfrac{\frac{-\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}}{1} = \dfrac{\sqrt{3}}{2}\underline{i} - \dfrac{1}{2}\underline{j}$



7. (i) **Suppose $D(x, y)$ then in ABCD**

Parallelogram $\overrightarrow{AB}$ is parallel.

to $\overrightarrow{DC}$ So

$\overrightarrow{AB} = \overrightarrow{DC}$ where

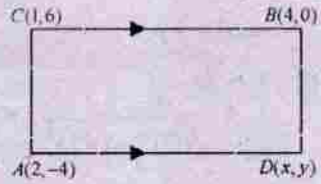
So $2\underline{i} + 4\underline{j} = (1-x)\underline{i} + (6-y)\underline{j}$ $\left[\begin{array}{l} AB = (4-2)\underline{i} + (0-(-4))\underline{j} = 2\underline{i} + 4\underline{j} \\ DC = (1-x)\underline{i} + (6-y)\underline{j} \end{array}\right.$

$\Rightarrow 2 = (1-x)$ & $4 = (6-y)$

or $2 - 1 = x$ $\quad -2 = -y$

$\Rightarrow x = -1$ & $y = 2$

So $O(-1, 2)$ is required point.



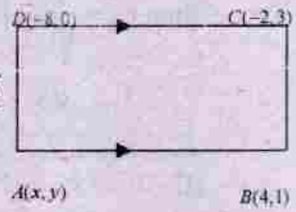
(ii) **ADBC is Parallelogram so $\overrightarrow{AD}$ is**

$\|$ to $\overrightarrow{CB}$ so $\overrightarrow{AD} = \overrightarrow{CB}$

So $(x-2)\underline{i}+(y+4)\underline{j}=(4-1)\underline{i}+(0-6)\underline{j}$

$(x-2)\underline{i}+(y+4)\underline{j}=3\underline{j}=3\underline{i}=6\underline{j} \Rightarrow x-2=3 \;\&\; y+4=-4$

$So\ D(5,-10) \quad \Rightarrow x=5 \;\&\; y=-10$



8. (i) **Suppose $A(x,y)$, then ABCD is**

Parallelogram is required point

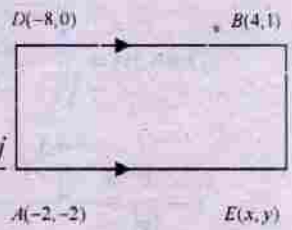
so $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$ so $\overrightarrow{AB}=\overrightarrow{DC}$

$(4-x)\underline{i}+(1-y)\underline{j}=(-2+8)\underline{i}+(3-0)\underline{j} \Rightarrow (4-x)\underline{i}+(1-y)\underline{j}=6\underline{i}+3\underline{j}$

$\Rightarrow 4-x=6 \;\&\; 1-y=3 \Rightarrow -x=6-4 \;\&\; -y=3-1$

$\Rightarrow -x=2 \;\&\; -y=2 \Rightarrow x=-2 \;\&\; y=-2$

So $A(-2,-2)$ is required point.



(ii) **Suppose $E(x,y)$ is required point** $\overrightarrow{AE}$ *is* $\|$

to $\overrightarrow{DB}$ so $\overrightarrow{AE}=\overrightarrow{DB}$

$(x-(-2))\underline{i}+(y-(-2))\underline{j}=(4-(-8))\underline{i}+(1-0)\underline{j}$

$(x+2)\underline{i}+(y+2)\underline{j}=(4+8)\underline{i}+\underline{j} \Rightarrow (x+2)\underline{i}+(y+2)\underline{j}=12\underline{i}+\underline{j}$

So $x+2=12 \;\&\; y+2=1$ $So\ x=10 \;\&\; y=-1$

So $E(10,-1)$ is required point.

9. **Suppose $P(x,y)$, ($O(0,0)$ *origin*) $A=(-3,7)$, $B(1,0)$**

Given $\overrightarrow{OP}=\overrightarrow{AB}$

$(x-0)\underline{i}+(y-0)\underline{j}=(1+3)\underline{i}+(0-7)\underline{j}$

$x\underline{i}+y\underline{j}=4\underline{i}-7\underline{j} \Rightarrow x=4 \;\&\; y=-7$

So point is $P(4,-7)$

10. $\overrightarrow{AB}=A(0,0),\ B(a,o),\ C(b,c),\ D(b-a,c)$ **(Sargodha 2010)**

$\overrightarrow{DC}=(b-(b-a))\underline{i}+(c-c)\underline{j}=(\not{b}-\not{b}+a)\underline{i}+0\underline{j}$

$=a\underline{i} \quad \overrightarrow{AB}=\overrightarrow{DC}$ *so* $\overrightarrow{AB}$ *is parallel to* $\overrightarrow{DC}$

Now $\overrightarrow{AD}=(b-a-0)\underline{i}+(c-0)\underline{j}=(b-a)\underline{i}+c\underline{j}$

$$\overrightarrow{BC} = (b-a)\underline{i} + (c-0)\underline{j} = (b-a)\underline{i} + c\underline{j}$$

$\overrightarrow{AD} = \overrightarrow{BC}$ so $\overrightarrow{AD}$ is parallel to $\overrightarrow{BC}$

$\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$ & $\overrightarrow{BC}$ is parallel to $\overrightarrow{AD}$ so ABCD is parallelogram.

11. **Suppose $A(x,y)$ and given $B(1,2)$, $C(-2,5)$, $D(4,11)$ so $\overrightarrow{AB} = \overrightarrow{DC}$**

$(1-x)\underline{i} + (2-y)\underline{j} = (4-(-2))\underline{i} + (11-5)\underline{j}$

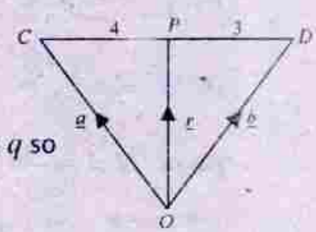
$(1-x)\underline{i} + (2-y)\underline{j} = (4+2)\underline{i} + 6\underline{j} \Rightarrow (1-x)\underline{i} + (2-y)\underline{j} = 6\underline{i} + 6\underline{j}$

$\Rightarrow 1-x = 6$ & $2-y = 6$ $\Rightarrow -x = 6-1$ & $-y = 6-2$

$-x = 5$ & $-y = 4 \Rightarrow x = -5$ & $y = -4$

So $A(-5,-4)$ is required point.

12. **Suppose**



$\underline{a}$ is p.v of C so $a = 2\underline{i} - 3\underline{j}$

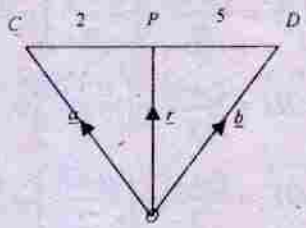
$\underline{b}$ is p.v of D so $b = 3\underline{i} + 2\underline{j}$

$\underline{r}$ is p.v of point P which divides it into ratio 4 : 3 or $p:q$ so

$$\underline{r} = \frac{q\underline{a} + p\underline{b}}{p+q} = \frac{3(2\underline{i} - 3\underline{j}) + 4(3\underline{i} + 2\underline{j})}{4+3}$$

$$\underline{r} = \frac{6\underline{i} - 9\underline{j} + 12\underline{i} + 8\underline{j}}{7} = \frac{18}{7}\underline{i} - \frac{1}{7}\underline{j}$$

13. ***$\underline{a}$ is p.v of C so $\underline{a} = 5\underline{j}$***



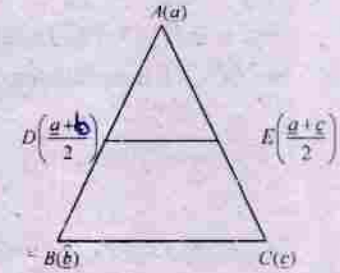
$\underline{b}$ is p.v of D so $\underline{b} = 4\underline{i} + \underline{j}$

$p:q = 2:5$

$$\underline{r} = \frac{q\underline{a} + b p}{p+q} = \frac{5(5\underline{j}) + 2(4\underline{i} + \underline{j})}{5+3}$$

$$\underline{r} = \frac{25\underline{j} + 8\underline{i} + 2\underline{j}}{7} = \frac{8}{7}\underline{i} + \frac{27}{7}\underline{j}$$

14. **Suppose that $\underline{a}$, $\underline{b}$, $\underline{c}$ are position vectors of A, B, C respectively** **(Sgd 2010)**



Then pv of $D = \dfrac{\underline{a} + \underline{b}}{2}$

$p.v$ of $E = \dfrac{\underline{a} + \underline{c}}{2}$

$\vec{BC} = \vec{OC} - \vec{OB} = \underline{c} - \underline{b}$ & $\vec{DE} = \vec{OE} - \vec{OD} = \frac{\underline{a}+\underline{c}}{2} - \left(\frac{\underline{a}+\underline{b}}{2}\right)$

$= \frac{1}{2}(\underline{a} + \underline{c} - \underline{a} - \underline{b})$

$\vec{DE} = \frac{1}{2}(\underline{c} - \underline{b}) = \frac{1}{2}\vec{BC}$

So $\vec{DE} = \frac{1}{2}\vec{BC}$ Hence proved.

15. Suppose that a, b, c, d are position vectors of A, B, C,D respectively then

(Fsd 2011, Sgd 2012)

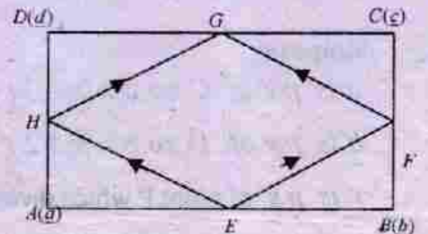
a+b+c+d

$pv \text{ of } E = \frac{\underline{a}+\underline{b}}{2}$

$pv \text{ of } F = \frac{\underline{b}+\underline{c}}{2}$

$pv \text{ of } G = \frac{\underline{c}+\underline{d}}{2}$

$pv \text{ of } H = \frac{\underline{a}+\underline{d}}{2}$



$\vec{EF} = \vec{OF} - \vec{OE} = \frac{\underline{b}+\underline{c}}{2} - \left(\frac{\underline{a}+\underline{b}}{2}\right) = \frac{\underline{b}+\underline{c}-\underline{a}-\underline{b}}{2} = \frac{\underline{c}-\underline{a}}{2}$

$\vec{FG} = \vec{OG} - \vec{OF} = \frac{\underline{c}+\underline{d}}{2} - \left(\frac{\underline{b}+\underline{c}}{2}\right) = \frac{\underline{c}+\underline{d}-\underline{b}-\underline{c}}{2} = \frac{\underline{d}-\underline{b}}{2}$

$\vec{HG} = \vec{OG} - \vec{OH} = \frac{\underline{c}+\underline{d}}{2} - \left(\frac{\underline{a}+\underline{b}}{2}\right) = \frac{\underline{c}+\underline{d}-\underline{a}-\underline{d}}{2} = \frac{\underline{c}-\underline{a}}{2}$

$\vec{EH} = \vec{OH} - \vec{OE} = \frac{\underline{a}+\underline{d}}{2} - \left(\frac{\underline{a}+\underline{b}}{2}\right) = \frac{\underline{a}+\underline{d}-\underline{a}-\underline{b}}{2} = \frac{\underline{d}-\underline{b}}{2}$

$\vec{EF} = \vec{HG} \Rightarrow \vec{EF}$ is parallel to $\vec{HG}$

$\vec{FG} = \vec{EH} \Rightarrow \vec{FG}$ is parallel to $\vec{EH}$

So $EFGH$ is parallelogram.

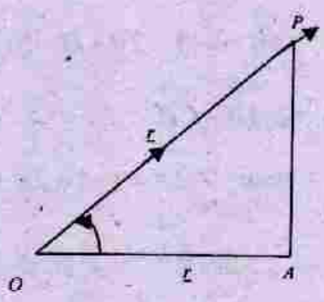
Theorem: Prove that $Cos^2\alpha + Cos^2\beta + Cos^2\gamma = 1$ **(Sargodha 2009,11)**

Proof: Suppose a vector $\underline{r}$ s.t

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} \ \& \left|\underline{r}\right| = \sqrt{x^2 + y^2 + z^2}$$

$$or\ r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2 \longrightarrow 1$$



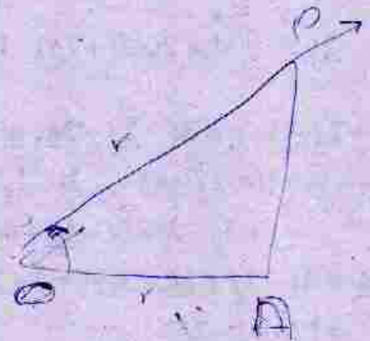
In a right triangle OAP

$$Cos\alpha = \frac{x}{\underline{\gamma}} \text{ Similarly}$$

$$Cos\beta = \frac{y}{\underline{\gamma}} \ \& \ Cos\gamma = \frac{z}{\underline{\gamma}}$$

So $Cos^2\alpha + Cos^2\beta + Cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = \frac{x^2 + y^2 + z^2}{r^2} = \frac{r^2}{r^2} = 1$

Hence proved that $Cos^2\alpha + Cos^2\beta + Cos^2\gamma = 1$



Exercise 7.2

1. (i) $A(2,5),\ B(-1,1),\ C(2,-6)$

$\overrightarrow{AB} = (-1-2)\underline{i} + (1-5)\underline{j} = -3\underline{i} - 4\underline{j}$

(ii) $2\overrightarrow{AB} - \overrightarrow{CB}$

Now $\overrightarrow{AB} = (-1-2)\underline{i} + (1-5)\underline{j} = -3\underline{i} - 4\underline{j}$.

$\overrightarrow{CB} = (-1-2)\underline{i} + (1+6)\underline{j} = -3\underline{i} + 7\underline{j}$

Then $2\overrightarrow{AB} - \overrightarrow{CB} = 2(-3\underline{i} - 4\underline{j}) - (3\underline{i} + 7\underline{j})$

$= -6\underline{i} - 8\underline{j} + 3\underline{i} - 7\underline{j} = 3\underline{i} - 15\underline{j}$

(iii) $2\overrightarrow{CB} - 2\overrightarrow{CA}$

Now $\overrightarrow{CB} = (-1-2)\underline{i} + (1+6)\underline{j} = -3\underline{i} + 7\underline{j}$

$\overrightarrow{CA} = (2-2)\underline{i} + (5+6)\underline{j} = 0\underline{i} + 11\underline{j}$

$2\overrightarrow{CB} - \overrightarrow{CA} = 2(-3\underline{i} + 7\underline{j}) - 11\underline{j} = -6\underline{i} + 14\underline{j} - 11\underline{j} = 6\underline{i} + 3\underline{j}$

2. (i) $\underline{u} = \underline{i} + 2\underline{j} - \underline{k},\ \underline{v} = 3\underline{i} - 2\underline{j} + 2\underline{k},\ \underline{w} = 5\underline{i} - \underline{j} + 3\underline{k}$

$\underline{u} + 2\underline{v} + \underline{w} = (\underline{i} + 2\underline{j} - \underline{k}) + 2(3\underline{i} - 2\underline{j} + 2\underline{k}) + (5\underline{i} - \underline{j} + 3\underline{k})$

$= \underline{i} + 2\underline{j} - \underline{k} + 6\underline{i} - 4\underline{j} + 4\underline{k} + 5\underline{i} - \underline{j} + 3\underline{k} = 12\underline{i} - 3\underline{j} + 6\underline{k}$

(ii) $\underline{v} - 3\underline{w} = 3\underline{i} - 2\underline{j} + 2\underline{k} - 3(5\underline{i} - \underline{j} + 3\underline{k}) = 3\underline{i} - 2\underline{j} + 2\underline{k} - 15\underline{i} + 3\underline{j} - 9\underline{k}$ (Sgd 2009)

$= 12\underline{i} + \underline{j} - 7\underline{k}$

(iii) $3\underline{v} + \underline{w} = 3(3\underline{i} - 2\underline{j} + 2\underline{k}) + 5\underline{i} - \underline{j} + 3\underline{k} = 9\underline{i} - 6\underline{j} + 6\underline{k} + 5\underline{i} - \underline{j} + 3\underline{k}$

$3\underline{v} + \underline{w} = 14\underline{i} - 7\underline{j} + 9\underline{k}$

$|3v + w| = \sqrt{(14)^2 + (-7)^2 + (9)^2} = \sqrt{196 + 49 + 81} = \sqrt{326}$

3. (i) $\underline{v} = 2\underline{i} + 3\underline{j} + 4\underline{k}$

Magnitude $= |\underline{v}| = \sqrt{(2)^2 + (3)^2 + (4)^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$

Direction of Cosines are $Cos\alpha = \dfrac{2}{\sqrt{29}}, Cos\beta = \dfrac{3}{\sqrt{29}}, Cos\gamma = \dfrac{4}{\sqrt{29}}$

(ii) $\underline{v} = \underline{i} - \underline{j} - \underline{k}$

Magnitude $= |\underline{v}| = \sqrt{(1)^2 + (-1)^2 + (-1)^2} = \sqrt{1+1+1} = \sqrt{3}$

Direction of Cosines are $Cos\alpha = \frac{1}{\sqrt{3}}, Cos\beta = \frac{-1}{\sqrt{3}}, Cos\gamma = \frac{-1}{\sqrt{3}}$

(iii) $\underline{v} = 4\underline{i} - 5\underline{j}$

Magnitude $= |\underline{v}| = \sqrt{(4)^2 + (-5)^2} = \sqrt{16+25} = \sqrt{41}$

Direction of Cosines are $Cos\alpha = \frac{4}{\sqrt{41}}, Cos\beta = \frac{-5}{\sqrt{41}}$

4. $|\alpha\underline{i} + (\alpha+1)\underline{j} + 2\underline{k}| = 3$ $\quad \alpha^2 + \alpha^2 + 1 + 2\alpha + 4$ $\rightarrow 2\alpha^2 + 2\alpha + 5$

or $\sqrt{\alpha^2 + (\alpha+1)^2 + (2)^2} = 3 \Rightarrow \sqrt{2\alpha^2 + 2\alpha + 5} = 3$

Squaring both side

$2\alpha^2 + 2\alpha + 5 = 9$

$2\alpha^2 + 2\alpha + 5 - 9 = 0$ or $2\alpha^2 + 2\alpha - 4 = 0$ $\div$ by $\alpha^2 + \alpha - 2 = 0$

$\alpha^2 + 2\alpha - \alpha - 2 = 0$ or $\alpha(\alpha+2) - 1(\alpha+2) = 0$ or $(\alpha+2)(\alpha-1) = 0 = 0$

$\alpha + 2 = 0$ or $\alpha - 1 = 0 \Rightarrow \alpha = -2$ or $\alpha = 1$

5. $\underline{v} = \underline{i} + 2\underline{j} - \underline{k}$

Then $|\underline{v}| = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{1+4+1} = \sqrt{6}$

Unit vertices $\frac{\underline{v}}{|\underline{v}|} = \frac{\underline{i} + 2\underline{j} - \underline{k}}{\sqrt{6}} = \frac{1}{\sqrt{6}}\underline{i} + \frac{2}{\sqrt{6}}\underline{j} - \frac{1}{\sqrt{6}}\underline{k}$

6. $\underline{a} = 3\underline{i} - \underline{j} - 4\underline{k}$, $\underline{b} = 2\underline{i} - 4\underline{j} - 3\underline{k}$, $\underline{c} = \underline{i} + 2\underline{j} - \underline{k}$, **(Sgd 2008, Lahore 2010)**

$3\underline{a} - 2\underline{b} + 4\underline{c} = 3(3\underline{i} - \underline{j} - 4\underline{k}) - 2(2\underline{i} - 4\underline{j} - 3\underline{k}) + 4(\underline{i} + 2\underline{j} - \underline{k})$

$= 9\underline{i} - 3\underline{j} - 12\underline{k} + 4\underline{i} + 8\underline{j} + 6\underline{k} + 4\underline{i} + 8\underline{j} - 4\underline{k}$

$= 17\underline{i} + 13\underline{j} - 10\underline{k}$

$|3\underline{a} - 2\underline{b} + 4\underline{c}| = \sqrt{(17)^2 + (13)^2 + (-10)^2} = \sqrt{289+169+100} = \sqrt{558}$

Unit vector$= \frac{3\underline{a} - 2\underline{b} + 4\underline{c}}{|3\underline{a} - 2\underline{b} + 4\underline{c}|} = \frac{17\underline{i} + 13\underline{j} - 10\underline{k}}{\sqrt{558}}$

$= \frac{17}{\sqrt{558}}\underline{i} + \frac{13}{\sqrt{558}}\underline{j} - \frac{10}{\sqrt{558}}\underline{k}$

7. (i) **magnitude = 4 & $\underline{v} = 2\underline{i} - 3\underline{j} + 6\underline{k}$** **(Sargodha 2008, 09)**

$$|\underline{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{4+9+36} = \sqrt{49} = 7$$

$$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{2\underline{i} - 3\underline{j} + 6\underline{k}}{7}$$

$$\text{Required vector} = 4 \times \frac{(2\underline{i} - 3\underline{j} + 6\underline{k})}{7} = \frac{8}{7}\underline{i} - \frac{12}{7}\underline{j} + \frac{24}{7}\underline{k}$$

(ii) **Magnitude = 2 & $\underline{v} = -\underline{i} + \underline{j} + \underline{k}$**

$$|\underline{v}| = \sqrt{(-1)^2 + (1)^2 + (1)^2} = \sqrt{1+1+1} = \sqrt{3}$$

$$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{-\underline{i} + \underline{j} + \underline{k}}{\sqrt{3}}$$

$$\text{Required vector} = \frac{2 \times (-\underline{i} + \underline{j} + \underline{k})}{\sqrt{3}} = \frac{-2}{\sqrt{3}}\underline{i} + \frac{2}{\sqrt{3}}\underline{j} + \frac{2}{\sqrt{3}}\underline{k}$$

8. $\overrightarrow{AB} = \underline{u} = 2\underline{i} + 3\underline{j} + 4\underline{k},\ \overrightarrow{BC} = \underline{v} = -\underline{i} + 3\underline{j} - \underline{k},\ \overrightarrow{AC} = \underline{w} = \underline{i} + 6\underline{j} + z\underline{k},$

(Guj 2010, Sgd 2011)

If $\underline{u}, \underline{v}, \underline{w}$ represent sides of triangle then $\underline{u} + \underline{v} = \underline{w}$

or $(2\underline{i} + 3\underline{j} + 4\underline{k}) + (-\underline{i} + 3\underline{j} - \underline{k}) = (\underline{i} + 6\underline{j} + z\underline{k})$

or $2\underline{i} + 3\underline{j} + 4\underline{k} - \underline{i} + 3\underline{j} - \underline{k} = \underline{i} + 6\underline{j} + z\underline{k}$

or $\underline{i} + 6\underline{j} + 3\underline{k} = \underline{i} + 6\underline{j} + z\underline{k} \Rightarrow z = 3$

9. Position vector of A is $\overrightarrow{OA} = 2\underline{i} - \underline{j} + \underline{k}$ A, B, C and D

Position vector of B is $\overrightarrow{OB} = 3\underline{i} + \underline{j}$

Position vector of C is $\overrightarrow{OC} = 2\underline{i} + 4\underline{j} - 2\underline{k}$

Position vector of D is $\overrightarrow{OD} = -\underline{i} - 2\underline{j} + \underline{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\underline{i} + \underline{j} - (2\underline{i} - \underline{j} + \underline{k}) = 3\underline{i} + \underline{j} - 2\underline{i} + \underline{j} - \underline{k} = \underline{i} + 2\underline{j} - \underline{k}$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC} = -\underline{i} - 2\underline{j} + \underline{k} - (2\underline{i} + 4\underline{j} - 2\underline{k})$$

$$\overrightarrow{CD} = -\underline{i} - 2\underline{j} + \underline{k} - 2\underline{i} - 4\underline{j} + 2\underline{k}$$

$$\overrightarrow{CD}=3\underline{i}-6\underline{j}+3\underline{k}=-3(\underline{i}+2\underline{j}-\underline{k})$$

$$\overrightarrow{CD}=-3\,\overrightarrow{AB}$$

Hence $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$

10. (i) $v=2\underline{i}-4\underline{j}+4\underline{k}$

$$=|\underline{v}|=\sqrt{(2)^2+(-4)^2+(4)^2}=\sqrt{4+16+16}=\sqrt{36}=6$$

$$\hat{\underline{v}}=\frac{\underline{v}}{|\underline{v}|}=\frac{2\underline{i}-4\underline{j}+4\underline{k}}{6}$$

Required vector $2\times\left(\frac{2\underline{i}-4\underline{j}+4\underline{k}}{6_3}\right)=\left(\frac{2\underline{i}-4\underline{j}+4\underline{k}}{3}\right)$ Same direction

$$=\frac{-(2\underline{i}-4\underline{j}+4\underline{k})}{3}=\frac{-2\underline{i}+4\underline{j}-4\underline{k}}{3}$$ opposite direction

(ii) $\underline{v}=c\underline{w}$

$$(\underline{i}-3\underline{j}+4\underline{k})=c(a\underline{i}+9\underline{j}-12\underline{k})$$

$$i-3j+4k=ac\underline{i}+9c\underline{j}-12c\underline{k}$$

$$ac=1\longrightarrow I\ \&\ -3=9c \qquad \Rightarrow c=\frac{-1}{3}$$

Put c *in* I

$$a\left(\frac{-1}{3}\right)=1 \quad\Rightarrow a=-3\ ,\ c=-\frac{1}{3}$$

(iii) $\underline{v}=\underline{i}-2\underline{j}+3\underline{k}$

$$|\underline{v}|=\sqrt{(1)^2+(-2)^2+(3)^2}=\sqrt{1+4+9}=\sqrt{14}$$

$$\hat{\underline{v}}=\frac{\underline{v}}{|\underline{v}|}=\frac{i-2j+3\underline{k}}{\sqrt{14}}$$

Required vector in opposite direction

$$=\frac{5\times(-1)(\underline{i}-2\underline{j}+3\underline{k})}{\sqrt{14}}=\frac{-5}{\sqrt{14}}\underline{i}+\frac{10}{\sqrt{14}}\underline{j}-\frac{15}{\sqrt{14}}\underline{k}$$

(iv) $\underline{u}=c\underline{v}$

$$3\underline{i}-\underline{j}+4\underline{k}=C(a\underline{i}+b\underline{j}-2\underline{k})$$

$$3\underline{i}-\underline{j}+4\underline{k}=ac\underline{i}+bc\underline{j}-2c\underline{k}$$

$ac = 3$ —— I , $bc = -1$ —— II , $-2c = 4$ —— III

Solving III $c = -2$

Put value of c in I

$a(-2) = 3 \Rightarrow a = \dfrac{-3}{2}$

Put value of c in II

$b(-2) = -1 \Rightarrow b = \dfrac{1}{2}$

11. (i) $\underline{v} = 3\underline{i} - \underline{j} + 2\underline{k}$

$= |\underline{v}| = \sqrt{(3)^2 + (-1)^2 + (2)^2} = \sqrt{9+1+4} = \sqrt{14}$

Direction Cosines are $Cos\alpha = \dfrac{3}{\sqrt{14}}, Cos\beta = \dfrac{-1}{\sqrt{14}}, Cos\gamma = \dfrac{2}{\sqrt{14}}$

(ii) $\underline{v} = 6\underline{i} - 2\underline{j} + \underline{k}$ **(Gujrawala 2010)**

$|\underline{v}| = \sqrt{(6)^2 + (-2)^2 + (1)^2} = \sqrt{36+4+1} = \sqrt{41}$

Direction Cosines are $Cos\alpha = \dfrac{6}{\sqrt{41}}, Cos\beta = \dfrac{-2}{\sqrt{41}}, Cos\gamma = \dfrac{1}{\sqrt{41}}$

(iii) $\overrightarrow{PQ} = (1-2)\underline{i} + (3-1)\underline{j} + (1-5)\underline{k} = -\underline{i} + 2\underline{j} - 4\underline{k}$ $P(2,1,5)$ $Q(1,3,1)$

$\left|\overrightarrow{PQ}\right| = \sqrt{(-1)^2 + (2)^2 + (-4)^2} = \sqrt{1+4+16} = \sqrt{21}$

Direction Cosines $\overline{PQ}$ are $Cos\alpha = \dfrac{-1}{\sqrt{21}}, Cos\beta = \dfrac{2}{\sqrt{21}}, Cos\gamma = \dfrac{-4}{\sqrt{21}}$

12. (i) Here $\alpha = 45^o, \beta = 45^o, \gamma = 60^o$

Then $Cos^2\alpha + Cos^2\beta + Cos^2\gamma = Cos^2 45^o + Cos^2 45^o + Cos^2 60^o$

$$= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{2+2+1}{4} = \frac{5}{4} \neq 1$$

So given triples are not direction angles.

(ii) **Here** $\alpha = 30^o, \beta = 45^o, \gamma = 60^o$

Then $Cos^2\alpha + Cos^2\beta + Cos^2\gamma = Cos^2 30^o + Cos^2 45^o + Cos^2 60^o$

$$=\left(\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{2}\right)^2$$

$$=\frac{3}{4}+\frac{1}{2}+\frac{1}{4}=\frac{3+2+1}{4}=\frac{6}{4}\quad=\frac{3}{2}\neq 1$$

So given triples are not directions angles.

(iii) **Here** $\alpha=45^o,\beta=60^o,\gamma=60^o$

Then $Cos^2\alpha+Cos^2\beta+Cos^2\gamma=Cos^2 45^o+Cos^2 60^o+Cos^2 60^o$

$$=\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2$$

$$=\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=\frac{2+1+1}{4}=\frac{4}{4}=1$$

Here given triples is directions angles.

	30°	60°	45°
Sin	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$
Cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
Tan			1

Exercise 7.3

Prove that $\underline{u}.\underline{v} = \underline{v}.\underline{u}$ **(Sargodha 2008)**

$\underline{u} \cdot \underline{v} = |\underline{u}||\underline{v}| \cos\theta$

$= |\underline{v}||\underline{u}| \cos(-\theta)$

$= |\underline{v}||\underline{u}| \cos\theta$

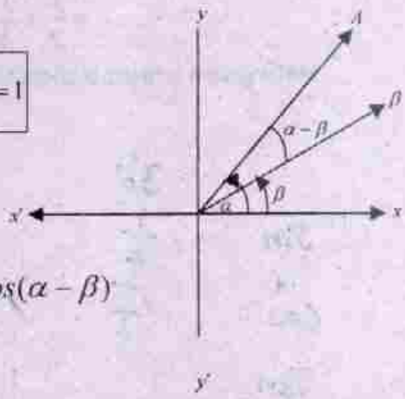
$\underline{u}.\underline{v} = \underline{v}.\underline{u}$ Hence proved.

Example No.9: Prove that $Cos(\alpha-\beta) = Cos\alpha\ Cos\beta + Sin\alpha\ Sin\beta$ **(Sgd 2011, Lhr 2010)**

Proof: Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be two unit vectors.

Then $\overrightarrow{OA} = Cos\alpha\underline{i} + Sin\alpha\underline{j}$

and $\overrightarrow{OB} = Cos\beta\underline{i} + Sin\beta\underline{j}$ Note $|\overrightarrow{OA}| = |\overrightarrow{OB}| = 1$



We know that

$$\overrightarrow{OA} \cdot \overrightarrow{OB} = |\overrightarrow{OA}||\overrightarrow{OB}|Cos(\alpha-\beta)$$

or $(Cos\alpha\underline{i} + Sin\alpha\underline{j}).(Cos\beta\underline{i} + Sin\beta\underline{j}) = |1||1|Cos(\alpha-\beta)$

$Cos\alpha\,Cos\beta + Sin\alpha Sin\beta = Cos(\alpha-\beta)$

or $Cos(\alpha-\beta) = Cos\alpha\ Cos\beta + Sin\alpha\,Sin\beta$

Hence proved.

1. (i) $\underline{u} = 3\underline{i} + \underline{j} - \underline{k}$ & $\underline{v} = 2\underline{i} - \underline{j} + \underline{k}$ **(Sargodha 2008,11)**

$\underline{u}.\underline{v} = (3\underline{i} + \underline{j} - \underline{k}) \cdot (2\underline{i} - \underline{j} + \underline{k}) = 3(2) + (1)(-1) + (-1)(1)$

$= 6 - 1 - 1 = 4$

$|\underline{u}| = \sqrt{(3)^2 + (1)^2 + (-1)^2} = \sqrt{9+1+1} = \sqrt{11}$

$|\underline{v}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{4+1+1} = \sqrt{6}$

$$Cos\theta = \frac{\underline{u}.\underline{v}}{|\underline{u}|.|\underline{v}|} = \frac{4}{\sqrt{11}\sqrt{6}} = \frac{4}{\sqrt{11\times6}} = \frac{4}{\sqrt{66}}$$

(ii) $\underline{u} = \underline{i} - 3\underline{j} + 4\underline{k}$, $\underline{v} = 4\underline{i} - \underline{j} + 3\underline{k}$

$\underline{u}.\underline{v} = (\underline{i} - 3\underline{j} + 4\underline{k}) \cdot (4\underline{i} - \underline{j} + 3\underline{k}) = (1)(4) + (-3)(-1) + 4(3)$

$= 4 + 3 + 12 = 19$

$|\underline{u}| = \sqrt{(1)^2 + (-3)^2 + (4)^2} = \sqrt{1+9+16} = \sqrt{26}$

$|\underline{v}| = \sqrt{(4)^2 + (-1)^2 + (3)^2} = \sqrt{16+1+9} = \sqrt{26}$

$$Cos\theta = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}|\cdot|\underline{v}|} = \frac{19}{\sqrt{26}\sqrt{26}} = \frac{19}{(\sqrt{26})^2} = \frac{19}{26}$$

(iii) $\underline{u} = [-3,5] = -3\underline{i} + 5\underline{j}$ & $\underline{v} = [6,-2] = 6\underline{i} - 2\underline{j}$ **(Gujrawala 2010)**

$$\underline{u}\cdot\underline{v} = (-3\underline{i} + 5\underline{j})\cdot(6\underline{i} - 2\underline{j}) = (-3)(6) + (5)(-2) = -18 - 10 = -28$$

$$|\underline{u}| = \sqrt{(-3)^2 + (5)^2} = \sqrt{9+25} = \sqrt{34}$$

$$|\underline{v}| = \sqrt{(6)^2 + (-2)^2} = \sqrt{36+4} = \sqrt{40}$$

$$Cos\theta = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}|\cdot|\underline{v}|} = \frac{-28}{\sqrt{34}\sqrt{40}} = \frac{-28}{\sqrt{7\times2\times2\times2\times2\times5}}$$

$$= \frac{-28}{2\times2\sqrt{17\times5}} = \frac{-_2 8}{4\sqrt{85}} = \frac{-7}{\sqrt{85}}$$

(iv) $\underline{u} = [2,-3,1] = 2\underline{i} - 3\underline{j} + \underline{k}$, $\underline{v} = [2,4,1] = 2\underline{i} + 4\underline{j} + \underline{k}$

$$\underline{u}\cdot\underline{v} = (2\underline{i} - 3\underline{j} + \underline{k})\cdot(2\underline{i} + 4\underline{j} + \underline{k}) = 2(2) + (-3)(4) = +(1)(1) = 4 - 12 + 1 = -7$$

$$|\underline{u}| = \sqrt{(2)^2 + (-3)^2 + (1)^2} = \sqrt{4+9+1} = \sqrt{14}$$

$$|\underline{v}| = \sqrt{(2)^2 + (4)^2 + (1)^2} = \sqrt{4+16+1} = \sqrt{21}$$

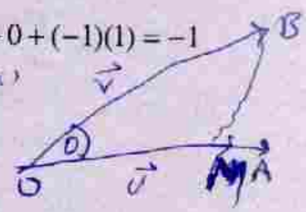
$$Cos\theta = \frac{\underline{u}\cdot\underline{v}}{|\underline{u}|\cdot|\underline{v}|} = \frac{-7}{\sqrt{14}\sqrt{21}} = \frac{-7}{\sqrt{14\times21}} = \frac{-7}{\sqrt{2\times7\times7\times3}}$$

$$= \frac{-7}{7\sqrt{2\times3}} = \frac{-7}{7\sqrt{6}} = -\frac{1}{\sqrt{6}}$$

2. (i) $\underline{a} = \underline{i} - \underline{k}$, $\underline{b} = \underline{j} + \underline{k}$ **(Sargodha 2010,11)**

$$|\underline{a}| = \sqrt{(1)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2} \quad \underline{a}.\underline{b} = (\underline{i} - \underline{k}).(\underline{j} + \underline{k}) = 0 + 0 + (-1)(1) = -1$$

$$|\underline{b}| = \sqrt{(1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$$



Projection of $\underline{a}$ along $\underline{b} = \frac{a.b}{|\underline{b}|} = \frac{-1}{\sqrt{2}}$

Projection of $\underline{b}$ along $\underline{a} = \frac{a.b}{|\underline{a}|} = \frac{-1}{\sqrt{2}}$

(ii) $\underline{a} = 3\underline{i} + \underline{j} - \underline{k}$, $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$ **(Sargodha 2009,12)**

$$\underline{a}.\underline{b} = (3\underline{i} + j - \underline{k}).(-2\underline{i} - \underline{j} + \underline{k}) = 3(-2) + (1)(-1) + (-1)(1) = -6 - 1 - 1 = -8$$

$$|\underline{a}| = \sqrt{(3)^2 + (1)^2 + (-1)^2} = \sqrt{9+1+1} = \sqrt{11}$$

$$|\underline{b}| = \sqrt{(-2)^2 + (-1)^2 + (1)^2} = \sqrt{4+1+1} = \sqrt{6}$$

Projection of $\underline{a}$ along $\underline{b} = \frac{a.b}{|\underline{b}|} = \frac{-8}{\sqrt{6}}$

Projection of $\underline{b}$ along $\underline{a} = \frac{a.b}{|\underline{a}|} = \frac{-8}{\sqrt{11}}$

3. (i) $\underline{u} = 2\alpha\underline{i} + \underline{j} - \underline{k},\ \underline{v} = \underline{i} + \alpha\underline{j} + 4\underline{k}$, $\underline{u}$ & $\underline{v}$ **are perpendicular if** $\underline{u}.\underline{v} = 0$

Or $(2\alpha\underline{i} + \underline{j} - \underline{k})(\underline{i} + \alpha\underline{j} + 4\underline{k}) = 0$ **(Sgd 2010, Lhr 2010)**

Or $(2\alpha)(1) + (1)(\alpha) + (-1)(4) = 0 \Rightarrow 2\alpha + \alpha - 4 = 0$

$3\alpha - 4 = 0 \Rightarrow 3\alpha = 4 \Rightarrow \alpha = \frac{4}{3}$

(ii) $\underline{u} = \alpha\underline{i} + 2\alpha\underline{j} - \underline{k},\ \underline{v} = \underline{i} + \alpha\underline{j} + 3\underline{k}$, $\underline{u}$ & $\underline{v}$ **are perpendicular if** $\underline{u}.\underline{v} = 0$

Or $(\alpha\underline{i} + 2\alpha\underline{j} - \underline{k}).(\underline{i} + \alpha\underline{j} + 3\underline{k}) = 0$

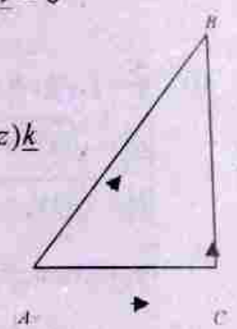
$(\alpha)(1) + (2\alpha)(\alpha) + (-1)(3) = 0 \Rightarrow \alpha + 2\alpha^2 - 3 = 0$

$2\alpha^2 + \alpha - 3 = 0$ or $2\alpha^2 + 3\alpha - 2\alpha - 3 = 0$

Or $\alpha(2\alpha + 3) - 1(2\alpha + 3) = 0 \Rightarrow (2\alpha + 3)(\alpha - 1) = 0$

$2\alpha + 3 = 0$ or $\alpha - 1 = 0 \Rightarrow \alpha = \frac{-3}{2}$ or $\alpha = 1$

4. $A(1,-1,0),\ B(-2,2,1)\ C(0,2,z)$ **are perpendicular if** $\underline{u}.\underline{v} = 0$



Now $\vec{AB} = (-2-1)\underline{i} + (2+1)\underline{j} + (1-0)\underline{k} = -3\underline{i} + 3\underline{j} + \underline{k}$

$\vec{CB} = (-2-0)\underline{i} + (2-2)\underline{j} + (i-z)\underline{k} = -2\underline{i} + 0\underline{j} + (1-z)\underline{k}$

$\vec{AC} = (0-1)\underline{i} + (2+1)\underline{j} + (z-0)\underline{k} = -\underline{i} + 3\underline{j} + z\underline{k}$

Since right angle is at C so

$\vec{AC}.\vec{CB} = 0$

Or $(-2\underline{i} + (1-z)\underline{k}).(-\underline{i} + 3\underline{j} + z\underline{k}) = 0$

$2(1) + (1-z)(z) = 0$ or $2 - z^2 + z = 0$

'X' by -1

$z^2 - z - 2 = 0$ or $z^2 - 2z + z - 2 = 0$

$z(z-2) + 1(z-2) = 0$ or $(z-2)(z+1) = 0$

$z - 2 = 0$ or $z + 1 = 0 \Rightarrow z = 2$ or $z = -1$

5. Suppose $\underline{v} = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$ **(Sargodha 2011)**

Then $\underline{v}\,.\,\underline{i} = 0 \Rightarrow (v_1\underline{i} + v_2\underline{j} + v_3\underline{k})\,.\underline{i} = 0 \Rightarrow v_1 = 0$

$\underline{v}\,.\,\underline{j} = 0 \Rightarrow (v_1\underline{i} + v_2\underline{j} + v_3\underline{k})\,.\underline{j} = 0 \Rightarrow v_2 = 0$

$\underline{v}\,.\,\underline{k} = 0 \Rightarrow (v_1\underline{i} + v_2\underline{j} + v_3\underline{k})\,.\underline{k} = 0 \Rightarrow v_3 = 0$

So $\underline{v} = 0\underline{i} + 0\underline{j} + 0\underline{k} + 0$ (Null Vector)

6. (i) $a = 3\underline{i} - 2\underline{j} + \underline{k},\ b = \underline{i} - 3\underline{j} + 5\underline{k},\ c = 2\underline{i} + \underline{j} - 4\underline{k},$ **(Sargodha 2009)**

Now $\underline{b} + \underline{c} = \underline{i} - 3\underline{j} + 5\underline{k} + 2\underline{i} + \underline{j} - 4\underline{k} = 3\underline{i} - 2\underline{j} + \underline{k},$

$\underline{b} + \underline{c} = a$

So $\underline{a}, \underline{b}, \underline{c}$ are sides of triangle

Now $\underline{a}\,.\,\underline{c} = (3\underline{i} - 2\underline{j} + \underline{k}).\,(2\underline{i} + \underline{j} - 4\underline{k})$

$= 3(2) + (-2)(1) + (1)(-4) = 6 - 2 - 4 = 6 - 6 = 0$

Side $\underline{a}$ and $\underline{c}$ are perpendicular and $\underline{a}, \underline{b}, \underline{c}$ are sides of triangle so given sides are of right triangle.

(ii) **$P(1,3,2)$, $Q(4,1,4)$, $R\ (6,5,5)$**

$\overrightarrow{PQ} = (4-1)\underline{i} + (1-3)\underline{j} + (4-2)\underline{k} = 3\underline{i} - 2\underline{j} + 2\underline{k}$

$\overrightarrow{QR} = (6-4)\underline{i} + (5-1)\underline{j} + (5-4)\underline{k} = 2\underline{i} + 4\underline{j} + \underline{k}$

$\overrightarrow{PR} = (6-1)\underline{i} + (5-3)\underline{j} + (5-2)\underline{k} = 5\underline{i} + 2\underline{j} + 3\underline{k}$

$\overrightarrow{PQ} + \overrightarrow{QR} = 3\underline{i} - 2\underline{j} + 2\underline{k} + 2\underline{i} + 4\underline{j} + \underline{k}$

$= 5\underline{i} + 2\underline{j} + 3\underline{k} = \overrightarrow{AC}$

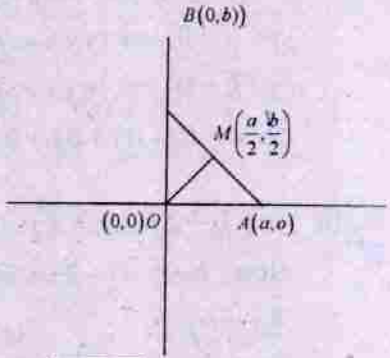
Hence A , B, C is triangle.

Also $\overrightarrow{PQ}\,.\,\overrightarrow{QR} = (3\underline{i} - 2\underline{j} + 2\underline{k}).\,(2\underline{i} + 4\underline{j} + \underline{k})$

$= 3(2) + (-2)(4) + 2(1) = 6 - 8 + 2 = 8 - 8 = 0$

Both conditions are satisfied so PQR are vertices of right triangle.

7. **M is mid point of hypotenuse**



Now Co-ordinate of M are M $\left(\frac{a}{2},\frac{b}{2}\right)$

$$\overrightarrow{AM}=\left(\frac{a}{2}-a\right)\underline{i}+\left(\frac{b}{2}-0\right)\underline{j}=-\frac{a}{2}\underline{i}+\frac{b}{2}\underline{j}$$

$$\overrightarrow{OM}=\left(\frac{a}{2}-0\right)\underline{i}+\left(\frac{b}{2}-0\right)\underline{j}=\frac{a}{2}\underline{i}+\frac{b}{2}\underline{j}$$

$$\overrightarrow{BM}=\left(\frac{a}{2}-0\right)\underline{i}+\left(\frac{b}{2}-b\right)\underline{j}=\frac{a}{2}\underline{i}-\frac{b}{2}\underline{j}$$

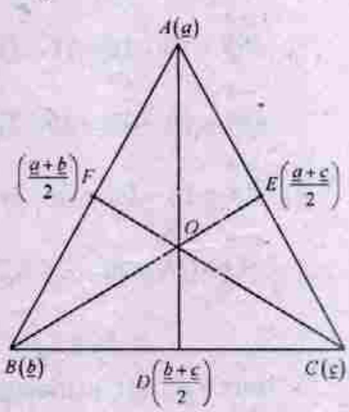
$$\left|\overrightarrow{AM}\right|=\sqrt{\left(-\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2}=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\sqrt{\frac{a^2+b^2}{4}}=\sqrt{\frac{a^2+b^2}{2}}$$

$$\left|\overrightarrow{BM}\right|=\sqrt{\left(\frac{a}{2}\right)^2+\left(-\frac{b}{2}\right)^2}=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\sqrt{\frac{a^2+b^2}{4}}=\sqrt{\frac{a^2+b^2}{2}}$$

$$\left|\overrightarrow{OM}\right|=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2}=\sqrt{\frac{a^2}{4}+\frac{b^2}{4}}=\sqrt{\frac{a^2+b^2}{4}}=\sqrt{\frac{a^2+b^2}{2}}$$

$\left|\overrightarrow{AM}\right|=\left|\overrightarrow{AM}\right|=\left|\overrightarrow{BM}\right|$ Hence proved.

8. **Suppose $\underline{a}$, $\underline{b}$, $\underline{c}$ are position vectors of A, B, C then**



Position vector of D, $\overrightarrow{OD}=\frac{\underline{b}-\underline{c}}{2}$

Position vector of E, $\overrightarrow{OE}=\frac{\underline{a}-\underline{c}}{2}$

Position vector of F, $\overrightarrow{OF}=\frac{\underline{a}+\underline{b}}{2}$

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\underline{b}-\underline{c}$, $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\underline{c}-\underline{b}$, $\overrightarrow{CA}=\overrightarrow{OA}-\overrightarrow{OC}=\underline{a}-\underline{c}$

Now $\overrightarrow{OD}$ is perpendicular to $\overrightarrow{BC}$ so $\overrightarrow{OD}\cdot\overrightarrow{BC}=0$

Or $\left(\frac{\underline{b}+\underline{c}}{2}\right)(\underline{c}-\underline{b})=0 \Rightarrow (\underline{c}+\underline{b})(\underline{c}-\underline{b})=0 \Rightarrow c^2-b^2=0$ (I)

$\overrightarrow{OE}$ is $\perp$ ar to $\overrightarrow{CA}$ so $\overrightarrow{OE}\cdot\overrightarrow{CA}=0$

$\left(\frac{\underline{a}+\underline{c}}{2}\right)(\underline{a}-\underline{c})=0 \Rightarrow (\underline{a}+\underline{c})(\underline{a}-\underline{c})=0 \Rightarrow a^2-c^2=0$ (II)

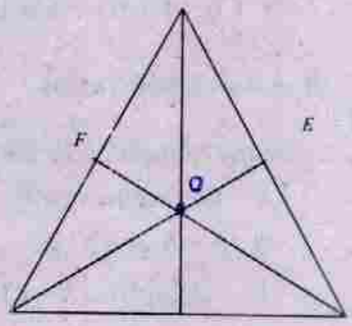
Adding I & II

$$\cancel{c}^2 - b^2 = 0$$
$$a^2 - \cancel{c}^2 = 0$$
$$\overline{a^2 - b^2 = 0}$$

$Or\ (\underline{a}-\underline{b})(\underline{a}+\underline{b})=0\ \ or\ \ (\underline{a}-\underline{b})\left(\dfrac{\underline{a}+\underline{b}}{2}\right)=0$

$\Rightarrow \overrightarrow{AB}$ *is perpendicular to* $\overrightarrow{OF}$ Hence proved.

9. $\underline{a}, \underline{b}, \underline{c}$ **are p.v of A, B, C** **(Sgd 2010, Fsd 2011)**



AD is parallel to A so $\overrightarrow{AD}=\pi\underline{a}$

$\overrightarrow{OA}=\underline{a}\quad \overrightarrow{OB}=\underline{b}\quad \overrightarrow{OC}=\underline{c}$

$\overrightarrow{AD}$ *is* $\perp$*ar to* $\overrightarrow{BC}$ *so*

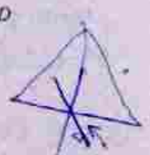
$\underline{a}\,.\,(\underline{c}-\underline{b})=0 \Rightarrow \underline{a}.(\underline{c}-\underline{b})=0$

$or\ \ \underline{a}\,.\,\underline{c}-\underline{a}.\underline{b}=0\ \Rightarrow \underline{a}\,.\,\underline{c}=\underline{a}\,.\,\underline{b} \longrightarrow I$

$\overrightarrow{BE}$ *is perpendicular to* $\overrightarrow{OB}$ *so* $\overrightarrow{BE}=\underline{b}$

Altitude $\overrightarrow{BE}$ *is to* $\overrightarrow{CA}$ *so* $\underline{b}\,.\,(\underline{a}-\underline{c})=0$

$\Rightarrow \underline{b}\,.\,\underline{a}-\underline{b}\,.\underline{c}=0 \Rightarrow \underline{b}\,.\,\underline{a}=\underline{b}\,.\,\underline{c} \Rightarrow \underline{a}\,.\,\underline{b}=\underline{b}\,.\,\underline{c} \longrightarrow II$



Comparing I & II $\ \underline{a}\,.\underline{c}=\underline{b}.\underline{c} \Rightarrow \underline{b}\,.\,\underline{c}-\underline{a}\,.\,\underline{c}=0$

$(\underline{b}-\underline{a}).\underline{c}=0$ *so* $\overrightarrow{AB}$ *is* $\perp$ *ar* to Altitude $\overrightarrow{CF}$

Hence Proved.

10. **Take any point** $P(x,y)$ **on semicircle. Also** $A(-a,0)$ **and** $B(a,0)$ **are ends of Diameter.**

Now $\overrightarrow{AP}\,.\,\overrightarrow{BP}=\left[(x-(-a))\underline{i}+(y-0)\underline{j}\right].\left[(x-a)\underline{i}+(y-0)\underline{j}\right]$

$\qquad =\left[(x+a)\underline{i}+y\underline{j}\right].\left[(x-a)\underline{i}+y\underline{j}\right]=(x+a)(x-a)+y^2$

$=x^2-a^2+y^2=x^2+y^2-a^2$

We know in circle $x^2+y^2=r^2=a^2$ *so*

$\overrightarrow{AP}\,.\,\overrightarrow{BP}=\cancel{a}^2-\cancel{a}^2=0$ So $\overrightarrow{AP}$. *is perpendicular to* $\overrightarrow{BP}$

Hence proved.

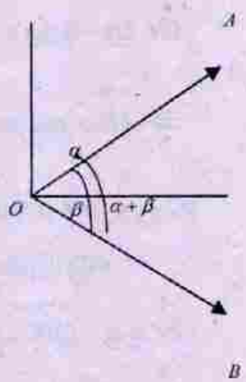
Angle between $\overrightarrow{AP}$ & $\overrightarrow{BP}$ is of 90°

11. $Cos(\alpha+\beta) = Cos\alpha Cos\beta - Sin\alpha Sin\beta$ Suppose $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are unit vectors.

then $\overrightarrow{OA} = Cos\alpha\,\underline{i} + Sin\alpha\,\underline{j}$

$\overrightarrow{OB} = Cos\beta\,\underline{i} + Sin\beta\,\underline{j}$

$\overrightarrow{OA}\,.\,\overrightarrow{OB} = \left|\overrightarrow{OA}\right|.\left|\overrightarrow{OB}\right| Cos(\alpha+\beta)$



$(Cos\alpha\,\underline{i} + Sin\alpha\,\underline{j})\,.\,(Cos\beta\,\underline{i} - Sin\beta\,\underline{j})$

$= 1.\,Cos(\alpha+\beta)$

$Cos\alpha Cos\beta - Sin\alpha Sin\beta = Cos(\alpha+\beta)$

Or $Cos(\alpha+\beta) = Cos\alpha Cos\beta - Sin\alpha Sin\alpha\beta$

12. (i) $b = c\ CosA + aCos$

In any triangle $\overrightarrow{AB} + \overrightarrow{BA} + \overrightarrow{CA} = 0$ or $\underline{a}+\underline{b}+\underline{c}$ or $\underline{b} = \underline{a}+\underline{c}$

'X' Dot product by $\underline{b}$

$-\underline{b}\,.\,\underline{b} = \underline{b}\,.\underline{a} + \underline{b}\,.\,\underline{c}$

$-\underline{b}^2 = |\underline{b}||\underline{a}|\ Cos(\pi - c) + |\underline{b}||\underline{c}|\ Cos(\pi - A)$ $\qquad$ $Note: Cos(\pi - A) = -CosA$

$-\underline{b}^2 = ab(-Cosc) + bc(-CosA) = -abCosc - bcCosA$

$\div$ both sides by $-b$

$b = cCosA + aCosc$ Hence Proved.

(ii) $c = a\ CosB + bCosA$

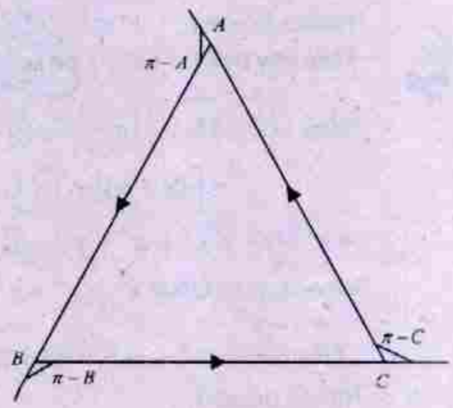
In any triangle $\overrightarrow{AB} + \overrightarrow{BA} + \overrightarrow{CA} = 0$

or $\underline{a}+\underline{b}+\underline{c} = 0$

or $-\underline{c} = \underline{a}+\underline{b}$

Taking Dot product by c

$-\underline{c}\,.\,\underline{c} = \underline{c}\,.\,\underline{a} + \underline{c}\,.\underline{b}$



$-c^2 = |\underline{c}||\underline{a}| Cos(\pi - B) + cbCos(\pi - A)$

$-c^2 = ca(-CosB) + cb(-CosA)$

$-c^2 = -a\ cCosB - bc\ CosA$

$\div$ by $-c$

$c = aCosB + bCosA$

(iii) We know that in any triangle $\vec{AB}+\vec{BC}+\vec{CA}=0$ (Sargodha 2009)

$or\ \underline{a}+\underline{b}+\underline{c}=0$

$or\ -\underline{b}=(\underline{a}+\underline{c}) \qquad (I)$

Taking dot product by $-\underline{b}$

$-\underline{b}.(\underline{b})=(\underline{a}+\underline{c}).(-\underline{b})$

$\underline{b}.\underline{b}=(\underline{a}+\underline{c}).(\underline{a}+\underline{c}) \quad Use-I$

$b^2=\underline{a}.\underline{a}+\underline{a}.\underline{c}+\underline{c}.\underline{a}+\underline{c}.\underline{c}$

$b^2=a^2+2a.c+c^2$

$b^2=a^2+c^2+2\,|a||c|Cos(\pi-B)$

$b^2=a^2+c^2+2ac(-Cos\beta)$

$b^2=a^2+c^2-2acCos\beta$

(iv) $c^2=a^2+b^2-2ab\ CosC$

We know that in any triangle $\vec{AB}+\vec{BC}+\vec{CA}=0$

$or\ \underline{a}+\underline{b}+\underline{c}=0$

$-\underline{c}=\underline{a}+\underline{b} \qquad (I)$

Taking dot product by $-\underline{c}$

$-\underline{c}.(\underline{c})=(\underline{a}+\underline{b}).(-\underline{c})$

$c^2=(\underline{a}+\underline{b})(\underline{a}+\underline{b}) \quad Use-I$

$c^2=\underline{a}.\underline{a}+\underline{a}.\underline{b}+\underline{b}.\underline{a}+\underline{b}.\underline{b}$

$c^2=a^2+2\underline{a}.\underline{b}+b^2 \qquad (\underline{a}.\underline{b}=\underline{b}.\underline{a})$

$c^2=a^2+b^2+2\,|\underline{a}||\underline{b}|Cos(\pi-c)$

$c^2=a^2+b^2+2ab(-CosC)$

$c^2=a^2+b^2-2abCosC$

Hence proved.

i·i , j·j , k·k = 1
i·j , j·k , k·i = 0
i×i , j×j , k×k = 0
i×j , j×k , k×i = 1

Exercise 7.4

1. (i) $\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ & $\underline{b} = \underline{i} - \underline{j} + \underline{k}$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \underline{i}(1-1) - \underline{j}(2+1) + \underline{k}(-2-1) = 3\underline{j} - 3\underline{k}$$

$$\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \underline{i}(1-1) - \underline{j}(-1-2) + \underline{k}(1+2) = 3\underline{j} + 3\underline{k}$$

Now $\underline{a}.(\underline{a}\times\underline{b}) = (2\underline{i} + \underline{j} - \underline{k}).(-3\underline{j} - 3\underline{k}) = (1)(3) + (-1)(-3) = -3 + 3 = 0$

So $\underline{a}$ & $\underline{a}\times\underline{b}$ are perpendicular

$\underline{b}.(\underline{a}\times\underline{b}) = (\underline{i} - \underline{j} + \underline{k}).(-3\underline{j} - 3\underline{k}) = (-1)(-3) + (1)(-3) = 3 - 3 = 0$

$\underline{b}$ & $\underline{a}\times\underline{b}$ are perpendicular

$\underline{b}.(\underline{b}\times\underline{a}) = (\underline{i} - \underline{j} + \underline{k}).(3\underline{j} + 3\underline{k}) = (1)(3) + (-1)(3) = 3 - 3 = 0$

$\underline{b}$ & $\underline{b}\times\underline{a}$ are perpendicular

$\underline{a}.(\underline{b}\times\underline{a}) = (2\underline{i} + \underline{j} - \underline{k}).(3\underline{j} + 3\underline{k}) = 3 - 3 = 0$

$\underline{a}$ & $\underline{b}\times\underline{a}$ are perpendicular.

(ii) $\underline{a} = \underline{i} + \underline{j}$ & $\underline{b} = \underline{i} - \underline{j}$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(-1-1) = -2\underline{k}$$

$$\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(1+1) = 2\underline{k}$$

$\underline{a}.(\underline{a}\times\underline{b}) = (\underline{i} + \underline{j}).(-2\underline{k})\ (-2\underline{k}) = -2(0) + -2(0) = 0$

$\underline{b}.(\underline{a}\times\underline{b}) = (\underline{i} - \underline{j})(2\underline{k}), = 2(0) + 2(0) = 0$

$\underline{a}.(\underline{b}\times\underline{a}) = (\underline{i} + \underline{j}).\ 2\underline{k}, = 2(0) + 2(0) = 0$

$\underline{b}.(\underline{b}\times\underline{a}) = (\underline{i} - \underline{j}).\ 2\underline{k} = 2(0) - 2(0) = 0 - 0 = 0$

Hence all above are $\perp$ar so proved.

(iii) $\underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}$ & $\underline{b} = \underline{i} + \underline{j}$ **(Lahore 2010)**

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \underline{i}(0-1) - \underline{j}(0-1) + \underline{k}(3+2) = -\underline{i} + \underline{j} + 5\underline{k}$$

$$\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{vmatrix} = \underline{i}(1+0) - \underline{j}(1-0) + \underline{k}(-2-3) = \underline{i} - \underline{j} - 5\underline{k}$$

i. $\underline{a}\cdot(\underline{a}\times\underline{b}) = (3\underline{i} - 2\underline{j} + \underline{k}).(-\underline{i} + \underline{j} + 5\underline{k}) = -3 - 2 + 5 = 0$

Hence $\underline{a}$ & $\underline{a}\times\underline{b}$ are perpendicular

ii. $\underline{a}\cdot(\underline{b}\times\underline{a}) = (3\underline{i} - 2\underline{j} + \underline{k}).(\underline{i} - \underline{j} - 5\underline{k}) = 3 + 2 - 5 = 0$

Hence $\underline{a}$ & $\underline{b}\times\underline{a}$ are perpendicular

iii. $\underline{b}\cdot(\underline{a}\times\underline{b}) = (\underline{i} + \underline{j}).(-\underline{i} + \underline{j} + 5\underline{k}) = -1 + 1 + 0 = 0$

Hence $\underline{b}$ & $\underline{a}\times\underline{b}$ are perpendicular

iv. $\underline{b}\cdot(\underline{b}\times\underline{a}) = (\underline{i} + \underline{j}).(\underline{i} - \underline{j} - 5\underline{k}) = 1 - 1 - 0 = 0$

Hence $\underline{b}$ & $\underline{b}\times\underline{a}$ are perpendicular

(iv) $\underline{a} = -4\underline{i} + \underline{j} - 2\underline{k}$ & $\underline{b} = 2\underline{i} + \underline{j} + \underline{k}$ **(Sargodha 2009)**

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix} = \underline{i}(1+2) - \underline{j}(-4+4) + \underline{k}(-4-2) = 3\underline{i} - 0\underline{j} - 6\underline{k}$$

$$\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 1 \\ -4 & 1 & -2 \end{vmatrix} = \underline{i}(-2-1) - \underline{j}(-4+4) + \underline{k}(2+4) = 3\underline{i} - 0\underline{j} + 6\underline{k}$$

i. $\underline{a}\cdot(\underline{a}\times\underline{b}) = (-4\underline{i} + \underline{j} - 2\underline{k}).(3\underline{i} - 0\underline{j} - 6\underline{k}) = -12 - 0 + 12 = 0$

Hence $\underline{a}$ & $\underline{a}\times\underline{b}$ are perpendicular

ii. $\underline{b}\cdot(\underline{a}\times\underline{b}) = (2\underline{i} + \underline{j} + \underline{k}).(3\underline{i} - 0\underline{j} - 6\underline{k}) = 6 - 0 - 6 = 0$

Hence $\underline{b}$ & $\underline{a}\times\underline{b}$ are perpendicular

iii. $\underline{a}\cdot(\underline{b}\times\underline{a}) = (2\underline{i} + \underline{j} + \underline{k}).(-3\underline{i} - 0\underline{j} + 6\underline{k}) = -6 - 0 + 6 = 0$

Hence $\underline{a}$ & $\underline{b}\times\underline{a}$ are perpendicular

iv. $\underline{b}\cdot(\underline{b}\times\underline{a}) = (2\underline{i} + \underline{j} + \underline{k}).(-3\underline{i} - 0\underline{j} + 6\underline{k}) = -6 - 0 + 6 = 0$

Hence $\underline{b}$ & $\underline{b}\times\underline{a}$ are perpendicular

2. (i) $\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}$, $\underline{b} = 4\underline{i} + 3\underline{j} - \underline{k}$ **(Sargodha 2011, Guj 2010)**

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = \underline{i}(6+9) - \underline{j}(-2+12) + \underline{k}(6+24) = 15\underline{i} - 10\underline{j} + 30\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(15)^2 + (-10)^2 + (30)^2} = \sqrt{225+100+900} = \sqrt{1225} = 35$$

Required unit vector $= \dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \dfrac{15\underline{i} - 10\underline{j} + 30\underline{k}}{35} = \dfrac{3}{7}\underline{i} - \dfrac{2}{7}\underline{j} + \dfrac{6}{7}\underline{k}$

Question no (3)

Now $|\underline{a}| = \sqrt{(2)^2 + (-6)^2 + (-3)^2} = \sqrt{4+36+9} = \sqrt{49} = 7$

$|\underline{b}| = \sqrt{(4)^2 + (3)^2 + (-1)^2} = \sqrt{16+9+1} = \sqrt{26}$

base x height

$$Sin\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|} = \frac{35}{7\sqrt{26}} = \frac{5}{\sqrt{26}}$$

$\dfrac{|\vec{U}|\,|V|\sin\theta}{|\vec{U} \times \vec{V}|}$

(ii) $\underline{a} = -\underline{i} - \underline{j} - \underline{k}$, $\underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k}$

Area of Pargram $= |\vec{U} \times \vec{V}|$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix} = \underline{i}(-4-3) - \underline{j}(-4+2) + \underline{k}(3+2) = -7\underline{i} + 2\underline{j} + 5\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(-7)^2 + (2)^2 + (5)^2} = \sqrt{49+4+25} = \sqrt{78}$$

Required unit vector $= \dfrac{-7\underline{i} + 2\underline{j} + 5\underline{k}}{\sqrt{78}} = \dfrac{-7}{\sqrt{78}}\underline{i} + \dfrac{2}{\sqrt{78}}\underline{j} + \dfrac{5}{\sqrt{78}}\underline{k}$

Now $|\underline{a}| = \sqrt{(-1)^2 + (-1)^2 + (-1)^2} = \sqrt{3}$

$|\underline{b}| = \sqrt{(2)^2 + (-3)^2 + (4)^2} = \sqrt{4+9+16} = \sqrt{29}$

$$Sin\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|} = \frac{\sqrt{78}}{\sqrt{3}\sqrt{29}} = \sqrt{\frac{26}{29}}$$

(iii) $\underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k}$, $\underline{b} = -\underline{i} + \underline{j} - 2\underline{k}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix} = \underline{i}(4-4) - \underline{j}(-4+4) + \underline{k}(2-2) = 0\underline{i} - 0\underline{j} + 0$$

$= 0$ or *Null Vector*

(iv) $\underline{a} = \underline{i} + \underline{j},\ \underline{b} = \underline{i} - \underline{j}$ **(Sargodha 2012)**

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = \underline{i}(0+0) - \underline{j}(0-0) + \underline{k}(-1-1) = 0\underline{i} - 0\underline{j} - 2\underline{k}$$

$$|\underline{a} \times \underline{b}| = \sqrt{(0)^2 + (0)^2 + (-2)^2} = \sqrt{4} = 2$$

Required unit vector $= \dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \dfrac{-2\underline{k}}{2} = -\underline{k}$

Now $|\underline{a}| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

$$|\underline{b}| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$$

$$Sin\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}||\underline{b}|} = \frac{2}{\sqrt{2} \times \sqrt{2}} = \frac{2}{2} = 1$$

3. (i) $P(0,0,0),\ Q(2,3,2),\ R(-1,1,4)$

$$\overrightarrow{PQ} = (2-0)\underline{i} + (3-0)\underline{j} + (2-0)\underline{k} = 2\underline{i} + 3\underline{j} + 2\underline{k}$$

Area of $\Delta = \frac{1}{2}|\vec{u} \times \vec{v}|$

$$\overrightarrow{PR} = (-1-0)\underline{i} + (1-0)\underline{j} + (4-0)\underline{k} = -\underline{i} + \underline{j} + 4\underline{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 3 & 2 \\ -1 & 1 & 4 \end{vmatrix} = \underline{i}(12-2) - \underline{j}(8+2) + \underline{k}(2+3) = 10\underline{i} - 10\underline{j} + 5\underline{k}$$

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{(10)^2 + (10)^2 + (5)^2} = \sqrt{100+100+25} = \sqrt{225} = 15$$

Now Area of triangle $PQR = \dfrac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$

$$= \frac{1}{2}(15) = \frac{15}{2} \text{ Sq. Units.}$$

(ii) $P(1,-1,-1),\ Q(2,0,-1),\ R(0,2,1)$

$$\overrightarrow{PQ} = (2-1)\underline{i} + (0+1)\underline{j} + (-1+1)\underline{k} = \underline{i} + \underline{j}$$

$$\overrightarrow{PR} = (0-1)\underline{i} + (2+1)\underline{j} + (1+1)\underline{k} = -\underline{i} + 3\underline{j} + 2\underline{k}$$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & 1 & 0\\ -1 & 3 & 2\end{vmatrix}=\underline{i}(2-0)-\underline{j}(2-0)+\underline{k}(3+1)=2\underline{i}-2\underline{j}+4\underline{k}$$

$$\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\sqrt{(2)^2+(-2)^2+(4)^2}=\sqrt{4+4+16}=\sqrt{24}=\sqrt{2\times2\times6}=2\sqrt{6}$$

Now Area of triangle $PQR=\frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|=\frac{2\sqrt{6}}{2}=\sqrt{6}$ Sq. Units.

4. (i) $A(0,0,0)$, $B(1,2,3)$, $C(2,-1,1)$ $D,(3,1,4)$ **(Sargodha 2012)**

$$\overrightarrow{AB}=(1-0)\underline{i}+(2-0)\underline{j}+(3-0)\underline{k}=\underline{i}+2\underline{j}+3\underline{k}$$

$$\overrightarrow{AC}=(2-0)\underline{i}+(-1-0)\underline{j}+(1-0)\underline{k}=-2\underline{i}-\underline{j}+\underline{k}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & 2 & 3\\ 2 & -1 & 1\end{vmatrix}=\underline{i}(2+3)-\underline{j}(1-6)+\underline{k}(-1-4)=5\underline{i}+5\underline{j}-5\underline{k}$$

$$\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{(5)^2+(5)^2+(-5)^2}=\sqrt{25+25+25}=\sqrt{75}=\sqrt{5\times5\times3}=5\sqrt{3}$$

Area of Parallelogram $=\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=5\sqrt{3}$ Sq. Units.

(ii) $A(1,2,-1)$, $B(4,2,-3)$, $C(6,-5,2)$ $D,(9,-5,0)$

$$\overrightarrow{AB}=(4-1)\underline{i}+(2-2)\underline{j}+(-3+1)\underline{k}=3\underline{i}-2\underline{k}$$

$$\overrightarrow{AC}=(6-1)\underline{i}+(-5-2)\underline{j}+(2+1)\underline{k}=5\underline{i}-7\underline{j}+3\underline{k}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 3 & 0 & -2\\ 5 & -7 & 3\end{vmatrix}=\underline{i}(0-14)-\underline{j}(9+10)+\underline{k}(-21-0)=-14\underline{i}-19\underline{j}-21\underline{k}$$

$$\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{(-14)^2+(-19)^2+(-21)^2}=\sqrt{196+361+441}=\sqrt{998}$$

Area of Parallelogram $ABCD=\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{998}$ Sq. Units.

(iii) $A(-1,1,1)$, $B(-1,2,2)$, $C(-3,4,-5)$ $D(-3,5,-4)$

$$\overrightarrow{AB}=(-1+1)\underline{i}+(2-1)\underline{j}+(2-1)\underline{k}=0\underline{i}+\underline{j}+\underline{k}$$

$$\overrightarrow{AC} = (-3+1)\underline{i} + (4-1)\underline{j} + (-5-1)\underline{k} = -2\underline{i} + 3\underline{j} - 6\underline{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{vmatrix} = \underline{i}(-6-3) - \underline{j}(0+2) + \underline{k}(0+2) = -9\underline{i} - 2\underline{j} + 2\underline{k}$$

Area of Parallelogram $ABCD = \sqrt{(-9)^2 + (-2)^2 + (2)^2} = \sqrt{89}$

5. (i) $\underline{u} = 5\underline{i} - \underline{j} + \underline{k},\ \underline{v} = \underline{j} - 5\underline{k},\ \underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k},$

$\underline{u}.\ \underline{v} = (5\underline{i} - \underline{j} + \underline{k})(\underline{j} - 5\underline{k}) = (-1)(1) + (1)(-5) = -1 - 5 = -6 \neq 0$

Not Perpendicular

$\underline{v}.\ \underline{w} = (\underline{j} - 5\underline{k})(-15\underline{i} + 3\underline{j} - 3\underline{k}) = (3) + (-5)(-3) = 3 + 15 = 18 \neq 0$

Not perpendicular

$\underline{u}.\ \underline{w} = (5\underline{i} - \underline{j} + \underline{k}).\ (-15\underline{i} + 3\underline{j} - 3\underline{k}) = 5(-15) + (-1)(3) + 1(-3) = -75 - 3 - 3 = -81 \neq 0$

Not Perpendicular

Now $\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k} = -3(5\underline{i} - \underline{j} + \underline{k})$

$\underline{w} = -3\underline{u}$ so $\underline{u}$ & $\underline{w}$ are parallel

(ii) $\underline{u} = \underline{i} + 2\underline{j} - \underline{k},\ \underline{v} = -\underline{i} + \underline{j} + \underline{k},\ \underline{w} = \frac{-\pi}{2}\underline{i} + \pi\underline{j} + \frac{\pi}{2}\underline{k},$

For parallel

$$\underline{w} = \frac{-\pi}{2}\underline{i} - \pi\underline{j} + \frac{\pi}{2}\underline{k} = \frac{-\pi\underline{i} + 2\pi\underline{j} + \pi\underline{k}}{2} = \frac{-\pi}{2}(\underline{i} + 2\underline{j} - \underline{k})$$

$\underline{w} = \frac{-\pi}{2}\underline{u}$ so $\underline{u}$ & $\underline{w}$ are parallel.

Now for perpendicular

$\underline{u}.\ \underline{v} = (\underline{i} + 2\underline{j} - \underline{k})(-\underline{i} + \underline{j} + \underline{k}) = 1(-1) + 2(1) + (-1)(1) = -1 + 2 - 1 = 0$

For $\underline{u}$ & $\underline{w}$ no need to check for perpendicular because they are parallel.

6. $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = 0$ **(Sargodha 2007, 11)**

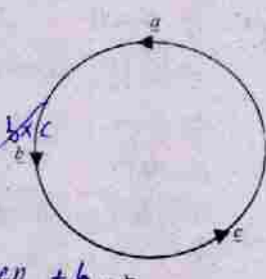
L.H.S $= \underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b})$

$= \underline{a} \times \underline{b} + \underline{a} \times \underline{c} + \underline{b} \times \underline{c} + \underline{b} \times \underline{a} + \underline{c} \times \underline{a} + \underline{c} \times \underline{b}$

$= \underline{c} - \underline{b} + \underline{a} - \underline{c} + \underline{b} - \underline{a} = 0$

Note $= \underline{a} \times \underline{b} = \underline{c}$

But $= \underline{b} \times \underline{a} = -\underline{c}$



7. $\underline{a}+\underline{b}+\underline{c}=0$ **(Sargodha 2010)**

Take cross product with $\underline{a}$

$\underline{a}\times(\underline{a}+\underline{b}+\underline{c})=\underline{a}\times 0$

$\underline{a}\times\underline{a}+\underline{a}\times\underline{b}+\underline{a}\times\underline{c}=0 \Rightarrow 0+\underline{a}\times\underline{b}+\underline{a}\times\underline{c}=0$

$\underline{a}\times\underline{b}=-\underline{a}\times\underline{c} \Rightarrow \underline{a}\times\underline{b}=\underline{a}\times\underline{c}$ (1)

Take cross product with $\underline{b}$

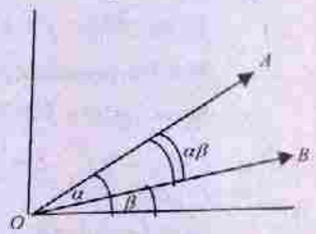
$\underline{b}\times(\underline{a}+\underline{b}+\underline{c})=\underline{b}\times 0 \Rightarrow \underline{b}\times\underline{a}+\underline{b}\times\underline{b}+\underline{b}\times\underline{c}=0$

$\underline{b}\times\underline{a}+0+\underline{b}\times\underline{c} \Rightarrow -\underline{a}\times\underline{b}+\underline{b}\times\underline{c}=0$

$\underline{b}\times\underline{c}=\underline{a}\times\underline{b}$ (II)

Combining I and II then $\underline{a}\times\underline{b}=\underline{b}\times\underline{c}=\underline{c}\times\underline{a}$

8. $\boldsymbol{Sin(\alpha-\beta)=Sin\alpha Cos\beta-Cos\alpha Sin\beta}$ **(Lahore 2010, Sargodha 2012)**



Suppose two unit vectors $\overrightarrow{OA}$ & $\overrightarrow{OB}$

$\overrightarrow{OA}=Cos\alpha\underline{i}+Sin\alpha\underline{j}$

$\overrightarrow{OB}=Cos\beta\underline{i}+Sin\beta\underline{j}$

We know that $\overrightarrow{OB}\times\overrightarrow{OA}=\left|\overrightarrow{OB}\right|\left|\overrightarrow{OA}\right|Sin(\alpha-\beta)\underline{k}$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ Cos\beta & Sin\beta & 0 \\ Cos\alpha & Sin\alpha & 0 \end{vmatrix}=1.1\ Sin(\alpha-\beta)\underline{k} \quad \left(\begin{array}{c}\left|\overrightarrow{OA}\right|=\left|\overrightarrow{OB}\right| \\ \text{Because they are} \\ \text{unit Vector}\end{array}\right)$$

$\underline{i}(0-0)-\underline{j}(0-0)+\underline{k}(Sin\alpha Cos\beta-Cos\alpha Sin\beta)=Sin(\alpha-\beta)\underline{k}$

$(Sin\alpha Cos\beta-Cos\alpha Sin\beta)\underline{k}=Sin(\alpha-\beta)\underline{k}$

Hence Sin $Sin(\alpha-\beta)\underline{k}=Sin\alpha Cos\beta-Cos\alpha Sin\beta$

9. $\underline{a}\cdot\underline{b}=0$

$\Rightarrow|\underline{a}||\underline{b}|\ Cos\theta=0$

$\Rightarrow Cos\theta=0 \Rightarrow \theta=Cos^{-1}(0)$

$\theta=90^o$

So $\underline{a}$ & $\underline{b}$ are perpendicular

$\underline{a} \times \underline{b} = 0$

$\Rightarrow |\underline{a}|\,|\underline{b}|\; Sin\theta = 0 \;\Rightarrow Sin\theta = 0$

$\Rightarrow \theta = Sin^{-1}(0) = 0$

$\theta = 0, \pi$

So $\underline{a}$ & $\underline{b}$ are parallel.

At the same time parallel and perpendicular not possible so one vector should be Zero or null.

Example 7 of (7.5) **(Sargodha 2008)**

The Constant forces $2i - 5j + 6k$ & $-i + 2j + k$ act on a body which is displaced from position $P(4, -3, 2)$ *to* $Q(6, 1, -3)$, Find work done.

Sol. $\underline{F_1} = 2\underline{i} - 5\underline{j} + 6\underline{k},\ \underline{F_2} = -\underline{i} + 2\underline{j} - \underline{k},$

$F = \underline{F_1} + \underline{F_2} = 2\underline{i} - 5\underline{j} + 6\underline{k} - \underline{i} + 2\underline{j} - \underline{k} = \underline{i} - 3\underline{j} + 5\underline{k}$

Displacement $= \overrightarrow{PQ} = (6-4)\underline{i} + (1+3)\underline{j} + (-3+2)\underline{k} = 2\underline{i} + 4\underline{j}$

$\underline{d} = 2\underline{i} + 4\underline{j} - \underline{k}$

Work done $\underline{F} \cdot \underline{d} = (\underline{i} - 3\underline{j} + 5\underline{k}).(2\underline{i} + 4\underline{j} - \underline{k})$

$= 2 - 12 - 5 = -15 = 15nt.$ (always +ve)

Example 2 of (7.5) **(Sargodha 2008)**

Prove that $A(-3, 5, -4)$, $B(-1, 1, 1)$, $C(-1, 2, 2)$, $D(-3, 4, -5)$ are Coplanar

Sol. $\overrightarrow{AB} = (-1+3)\underline{i} + (1-5)\underline{j} + (1+4)\underline{k} = 2\underline{i} - 4\underline{j} + 5\underline{k}$

$\overrightarrow{AC} = (-1+3)\underline{i} + (2-5)\underline{j} + (2+4)\underline{k} = 2\underline{i} - 3\underline{j} + 6\underline{k}$

$\overrightarrow{AD} = (-3+3)\underline{i} + (4-5)\underline{j} + (-5+4)\underline{k} = 0\underline{i} - \underline{j} - \underline{k}$

$$\overrightarrow{AB}.\overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} 2 & -4 & 5 \\ 2 & -3 & 6 \\ 0 & -1 & -1 \end{vmatrix} = 2(3+6) - (-4)(-2-0) + 5(-2+0) = 18 - 8 - 10 = 0$$

Hence Coplanar.

Unseen of (7.5) **(Sargodha 2010)**

Find volume of parallelepiped $\underline{u} = 2\underline{i} + \underline{j} - \underline{k}$, $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$, $\underline{w} = \underline{i} - 7\underline{j} - 4\underline{k}$,

Volume of parallelepiped $u \cdot v \times w = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -2 & 3 \\ 1 & -7 & -4 \end{vmatrix}$

$$= 2(8+21) - 1(-4-3) + (-1)(-7+2)$$
$$= 58 + 7 + 5 = 70$$

Example 5 of (7.5) **(Sargodha 2008)**

Proved that

$A(-6\underline{i} + 3\underline{j} + 2\underline{k})$ $B(3\underline{i} - 2\underline{j} + 4\underline{k})$, $C(5\underline{i} + 7\underline{j} + 3\underline{k})$, $D(-13\underline{i} + 17\underline{j} - \underline{k})$ are coplanar

$\overrightarrow{OA} = -6\underline{i} + 3\underline{j} + 2\underline{k}$ $\overrightarrow{OB} = 3\underline{i} - 2\underline{j} + 4\underline{k}$

$\overrightarrow{OC} = 5\underline{i} + 7\underline{j} + 3\underline{k}$ $\overrightarrow{OD} = -13\underline{i} + 17\underline{j} + \underline{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\underline{i} - 2\underline{j} + 4\underline{k} + 6\underline{i} - 3\underline{j} - 2\underline{k} = 9\underline{i} - 5\underline{j} + \underline{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 5\underline{i} + 7\underline{j} + 3\underline{k} + 6\underline{i} - 3\underline{j} - 2\underline{k} = 11\underline{i} + 4\underline{j} + \underline{k}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{AD} = -13\underline{i} + 17\underline{j} - \underline{k} + 6\underline{i} - 3\underline{j} - 2\underline{k} = -7\underline{i} + 14\underline{j} - 3\underline{k}$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} 9 & -5 & 2 \\ 11 & 4 & 1 \\ -7 & 14 & -3 \end{vmatrix} = 9(-12-14) - (-5)(-33+7) + 2(154+28)$$
$$= -234 - 130 + 364 = 0$$

Hence coplanar.

Exercise 7.5

Theorem 01:

Prove that $\underline{u}\ .\ \underline{v}\times\underline{w}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Proof Suppose $\underline{u}=a_1\underline{i}+b_1\underline{j}+c_1\underline{k}.\ \underline{v}=a_2\underline{i}+b_2\underline{j}+c_2\underline{k}.\ \underline{w}=a_3\underline{i}+b_3\underline{j}+c_3\underline{k}.$

Then $\underline{v}\times\underline{w}=\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\underline{i}(b_2c_3-c_2b_3)-\underline{a}(a_2c_3-a_3c_2)+\underline{k}(a_2b_3-a_3b_2)$

Now $\underline{u}\ .\ \underline{v}\times\underline{w}=(a_1\underline{i}+b_1\underline{j}+c_1\underline{k}).\left[(b_2c_3-c_2b_3)\underline{i}-(a_2c_3-a_3c_2)\underline{j}+(a_2b_3-a_3b_2)\underline{k})\right]$

$$=a_1(b_2c_3-c_2b_3)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2) \qquad (I)$$

Now $=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=a_1(b_2c_3-c_2b_3)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2) \qquad (II)$

Comparing I & II so

$$\underline{u}\ .\ \underline{v}\times\underline{w}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Theorem 02:

Prove that $\underline{u}\ .\ \underline{v}\times\underline{w}=\underline{v}\ .\ \underline{w}\times\underline{u}=\underline{w}\ .\ \underline{u}\times\underline{v}$

$$\underline{u}\ .\ \underline{v}\times\underline{w}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$=-\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ Interchanging } R_1 \text{ and } R_2$$

$$=\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix} \text{ Interchanging } R_2 \text{ and } R_3$$

$$\underline{u}\ .\ \underline{v}\times\underline{w}=\underline{v}\ .\ \underline{w}\times\underline{u}\longrightarrow I$$

Now $\underline{v} \cdot \underline{w} \times \underline{u} = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix}$

$= -\begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$ Interchanging R_1 and R_2

$= (-)(-)\begin{vmatrix} a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$ Interchanging R_2 and R_3

$\underline{v} \cdot \underline{w} \times \underline{u} = \underline{w} \cdot \underline{u} \times \underline{v} \longrightarrow II$

Comparing I & II we have $\underline{u} \cdot \underline{v} \times \underline{w} = \underline{v} \cdot \underline{w} \times \underline{u} = \underline{w} \cdot \underline{u} \times \underline{v}$

1. (i) $\underline{u} = 3\underline{i} + 2\underline{k},\ \underline{v} = \underline{i} + 2\underline{j} + \underline{k},\ \underline{w} = -\underline{j} + 4\underline{k}$

Volume of parallelepiped $= \underline{u} . \underline{v} \times \underline{w}$

$$= \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix}$$

$= 3(8+1) - 0 + 2(-1-0) = 27 - 2 = 25$

(ii) $\underline{u} = \underline{i} - 4\underline{j} - \underline{k},\ \underline{v} = \underline{i} - \underline{j} - 2\underline{k},\ \underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$

Volume of parallelepiped $= \underline{u} . \underline{v} \times \underline{w}$

$$= \begin{vmatrix} 1 & -4 & -1 \\ 1 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

$= 1(-1-6) - (-4)(1+4) + (-1)(-3+2)$

$= -7 + 20 + 1 = 14$

(iii) $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k},\ \underline{v} = 2\underline{i} - \underline{j} - \underline{k},\ \underline{w} = \underline{j} + \underline{k}$

Volume of parallelepiped $= \underline{u} . \underline{v} \times \underline{w}$

$$= \begin{vmatrix} 1 & -2 & 3 \\ 2 & -1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$= 1(-1+1) - (-2)(2+0) + 3(2-0) = 0 + 4 + 6 = 10$

2. $\underline{a} = 3\underline{i} - \underline{j} + 5\underline{k},\ \underline{b} = 4\underline{i} + 3\underline{j} - 2\underline{k},\ \underline{c} = 2\underline{i} + 5\underline{j} + \underline{k}$

$$\underline{a}.\underline{b}\times\underline{c} = \begin{vmatrix} 3 & -1 & 5 \\ 4 & 3 & -2 \\ 2 & 5 & 1 \end{vmatrix} = 3(3+10)+1(4+4)+5(20-6) = 39+8+70 = 117$$

$$\underline{b}.\underline{c}\times\underline{a} = \begin{vmatrix} 4 & 3 & -2 \\ 2 & -5 & 5 \\ 3 & -1 & 5 \end{vmatrix} = 4(25+1)-3(10-3)-2(-2-15) = 104-21+34 = 117$$

$$\underline{c}.\underline{a}\times\underline{b} = \begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix} = 2(2-15)-5(-6-20)+1(9+4) = -26+130+13 = 117$$

Hence $\underline{a}.\ \underline{b}\times\underline{c} = \underline{b}.\ \underline{c}\times\underline{a} = \underline{a}\times\underline{b}$

3. $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k},\ v = -2i + 3\underline{j} - 4\underline{k},\ \underline{w} = i - 3\underline{j} + 5\underline{k}$ **(Sgd 2008,09, Guj 2010)**

$$\underline{u}.\underline{v}\times\underline{w} = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 1(15-12)-(-2)(-10+4)+3(6-3) = 3-12+9 = 12-12 = 0$$

So given vectors are coplanar.

4. (i) **Find the constant α such that**

$\underline{i} - \underline{j} + \underline{k},\ \underline{i} - 2\underline{j} - 3\underline{k}$ *and* $3\underline{i} - \alpha\,\underline{j} + 5\underline{k}$ are coplanar

According to Given condition.

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix} = 0 \Rightarrow 1(-10-3\alpha)-1(-1)(5+9)+1(-\alpha+6) = 0$$

$$-10-3\alpha+14-\alpha+6 = 0$$

$$or\quad -4\alpha+10 = 0 \Rightarrow 4\alpha = 10 \Rightarrow \alpha = \frac{10}{4} \Rightarrow \alpha = \frac{5}{2}$$

(ii) $\underline{u} = \underline{i} - 2\alpha\,\underline{j} - \underline{k},\ \underline{v} = \underline{i} - \underline{j} + 2\underline{k},\ \underline{w} = \alpha\underline{i} - 1\underline{j} + \underline{k}$

$\underline{u}.\underline{v}.\underline{w}$ are coplanar if $\underline{u}.\underline{v}\times\underline{w} = 0$

$$So\ \begin{vmatrix} 1 & -2\alpha & -1 \\ 1 & -1 & 2 \\ \alpha & -1 & 1 \end{vmatrix} = 1(-1+2)-(-2\alpha)(1-2\alpha)+(-1)(-1+\alpha) = 0$$

$$or\quad -1+2+2\alpha-4\alpha^2+1-\alpha = 0$$

$\Rightarrow -4\alpha^2+\alpha+2 = 0$ Multiply by (-1) $\Rightarrow\quad 4\alpha^2-\alpha-2 = 0$

Using quadratic formula

$$\alpha = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(4)(-2)}}{2(4)} = \frac{1 \pm \sqrt{1+32}}{8} = \frac{1 \pm \sqrt{33}}{8}$$

5. (i) $\mathbf{2\underline{i} \times 2\underline{j}.\underline{k} = 4(\underline{i} \times \underline{j}).\underline{k} = 4\underline{k}\ .\underline{k} = 4(1) = 4}$

(ii) $3\underline{j}.\underline{k} \times \underline{i} = 4\underline{j}.\underline{j} = 3(1) = 3$

(iii) $\left[\underline{k}\,\underline{i}\,\underline{j}\right] = \underline{k}.\underline{i} \times \underline{j} = \underline{k}.\underline{k} = 1$

(iv) $\left[\underline{i}\,\underline{j}\,\underline{k}\right] = \underline{i}.\underline{i} \times \underline{k} = \underline{i} = \underline{i}.(-\underline{j}) = -\underline{i}.\underline{j} = 0$

(b) $\underline{u}.\underline{v} \times \underline{w} + \underline{v}.\underline{w} \times \underline{u} + \underline{w}.\underline{u} \times \underline{v} = 3\underline{u}.\underline{v} \times \underline{w}$

Now $\underline{u}\ .\ \underline{v} \times \underline{w} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $\quad \underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$, $\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$, $\underline{w} = a_3\underline{i} + b_3\underline{j} + c_3\underline{k}$

$$= -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{Interchanging } R_1 \text{ and } R_2$$

$$= (-)(-)\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \end{vmatrix} \quad \text{Interchanging } R_2 \text{ and } R_3$$

$= \underline{v}.\underline{w} \times \underline{u} \Rightarrow \underline{u}.\underline{v} \times \underline{w} = \underline{v}.\underline{w} \times \underline{u}$ $\quad (I)$

Similarly we can prove $= \underline{w}.\underline{u} \times \underline{v} = \underline{u}.\underline{v} \times \underline{w}$ $\quad (II)$

$= \underline{u}.\underline{v} \times \underline{w} = \underline{v}.\underline{w} \times \underline{u} = \underline{w}.\underline{u} \times \underline{v}$

Now L.H.S $= \underline{u}.\underline{v} \times \underline{w} + \underline{v}.\underline{w} \times \underline{u} + \underline{w}.\underline{u} \times \underline{v}$

$= \underline{u}.\underline{v} \times \underline{w} + \underline{u}.\underline{v} \times \underline{w} + \underline{u}.\underline{v} \times \underline{w}$

$= 3\underline{u}.\underline{v} \times \underline{w} =$ R.H.S Hence proved.

$= 2(\Delta ABC)(\text{height})$

$= \frac{1}{3}\left[\frac{1}{2}|\vec{u} \times \vec{v}| \times h\right]$

$= \frac{1}{6}\left[\vec{u} \times \vec{v} \cdot \vec{h}\right]$

6. (i) $\mathbf{A(0,1,2)\quad B(3,2,1)\quad C(1,2,1),\quad D(5,5,6)}$

$\overrightarrow{AB} = (3-0)\underline{i} + (2-1)\underline{j} + (1-2)\underline{k} = 3\underline{i} + \underline{j} - \underline{k}$

$\overrightarrow{AC} = (1-0)\underline{i} + (2-1)\underline{j} + (1-2)\underline{k} = \underline{i} + \underline{j} - \underline{k}$

$\overrightarrow{AD} = (5-0)\underline{i} + (5-1)\underline{j} + (6-2)\underline{k} = 5\underline{i} + 4\underline{j} + 4\underline{k}$

$$\vec{AB}.\vec{AC}\times\vec{AD}=\begin{vmatrix}3 & 1 & -1\\ 1 & 1 & -1\\ 5 & 4 & 4\end{vmatrix}=3(4+4)-1(4+5)+(-1)(4-5)$$

$$=24-9+1=16$$

Volume of tetrahedron $=\frac{1}{6}\left[\overline{AB}.\overline{AC}\times\overline{AD}\right]=\frac{16}{6}=\frac{8}{3}$ Sq. Units.

(ii) $A(2,1,8)$ $B(3,2,9)$ $C(2,1,4)$, $D(3,3,10)$ **(Sargodha 2009 Lahore 2010)**

$$\vec{AB}=(3-2)\underline{i}+(2-1)\underline{j}+(9-8)\underline{k}=\underline{i}+\underline{j}+\underline{k}$$

$$\vec{AC}=(2-2)\underline{i}+(1-1)\underline{j}+(4-8)\underline{k}=0\underline{i}+0\underline{j}-4\underline{k}$$

$$\vec{AD}=(3-2)\underline{i}+(3-1)\underline{j}+(10-8)\underline{k}=\underline{i}+2\underline{j}+2\underline{k}$$

$$\vec{AB}.\vec{AC}\times\vec{AD}=\begin{vmatrix}1 & 1 & 1\\ 0 & 0 & -4\\ 1 & 2 & 2\end{vmatrix}=1(0+8)-1(0+4)+1(0-0)=8-4+0=4$$

Volume of tetrahedron $=\frac{1}{6}\left[\vec{AB}.\vec{AC}\times\vec{AD}\right]=\frac{1}{6}(4)=\frac{2}{3}$ Sq. Units.

7. $P_1(3,1,-2)$, $P_2(2,4,6)$, $\underline{F}=4\underline{i}+3\underline{j}+5\underline{k}$

$$d=\vec{P_1P_2}=(2-3)\underline{i}+(4-1)\underline{j}+(6+2)\underline{k}=-\underline{i}+3\underline{j}+8\underline{k}$$

Work done= $\underline{F}.\underline{d}=(4\underline{i}+3\underline{j}+5\underline{k}).(-\underline{i}+3\underline{j}+8\underline{k})$

$$=4(-1)+3(3)+5(8)=-4+9+40=45N$$

8. $A(1,2,3)$, $B(5,4,1)$, $\underline{F}_1=4\underline{i}+\underline{j}-3\underline{k}$, $\underline{F}_2=3\underline{i}-\underline{j}-\underline{k}$

$$\underline{F}=\underline{F}_1+\underline{F}_2=4\underline{i}+\underline{j}-3\underline{k}+3\underline{i}-\underline{j}-\underline{k}$$

$$=7\underline{i}-4\underline{k}$$

$$\underline{d}=\vec{AB}=(5-1)\underline{i}+(4-2)\underline{j}+(1-3)\underline{k}=4\underline{i}+2\underline{j}-2\underline{k}$$

Work done $\underline{F}.\underline{d}=(7\underline{i}-4\underline{k}).(4\underline{i}+2\underline{j}-2\underline{k})$

$$=7(4)+(-4)(-2)=28+8=36N$$

9. $A(5,-5,-7)$ $B(6,2,-2)$

$$\underline{F}_1=10\underline{i}-\underline{j}+11\underline{k}\quad \underline{F}_2=4\underline{i}+5\underline{j}+9\underline{k}\quad \underline{F}_3=-2\underline{i}+\underline{j}-9\underline{k}$$

$$\underline{F}=\underline{F}_1+\underline{F}_2+\underline{F}_3=10\underline{i}-\underline{j}+11\underline{k}+4\underline{i}+5\underline{j}+9\underline{k}-2\underline{i}+\underline{j}-9\underline{k}=12\underline{i}+5\underline{j}+11\underline{k}$$

$\underline{d} = \overrightarrow{AB} = (5-1)\underline{i} + (4-2)\underline{j} + (1-3)\underline{k} = 4\underline{i} + 2\underline{j} - 2\underline{k}$ $(6-5)\underline{i} + (2+5)\underline{j} + (-2+7)\underline{k}$

$\underline{d} = \overrightarrow{AB} = \underline{i} + 7\underline{j} + 5\underline{k}$

Work done $\underline{F}.\underline{d}$

$= (12\underline{i} + 5\underline{j} + 11\underline{k}).(\underline{i} + 7\underline{j} + 5\underline{k})$

$= 12(1) + 5(7) + 11(5) = 12 + 35 + 55$

$= 102 \neq 67$ (given in book)

10. Let $\underline{u} = 2\underline{i} - 2\underline{j} + \underline{k}$ $A(1,2,3)$ $B(5,3,7)$

$|\underline{u}| = \sqrt{(2)^2 + (-2)^2 + (1)} = \sqrt{4+4+1} = \sqrt{9} = 3$

$\hat{\underline{u}} = \dfrac{\underline{u}}{|\underline{u}|} = \dfrac{2\underline{i} - 2\underline{j} + \underline{k}}{3}$ Now $\underline{F} = 6.\hat{\underline{u}} = \dfrac{^2\cancel{6} \times (2\underline{i} - 2\underline{j} + \underline{k})}{\cancel{3}}$

$\underline{F} = 4\underline{i} - 4\underline{j} + 2\underline{k}$

$\underline{d} = \overrightarrow{AB} = (5-1)\underline{i} + (3-2)\underline{j} + (7-3)\underline{k} = 4\underline{i} + \underline{j} + 4\underline{k}$

Work done $\underline{F}.\underline{d}$

$= (4\underline{i} - 4\underline{j} + 2\underline{k}).(4\underline{i} + \underline{j} + 4\underline{k})$

$= 4(4) + -4(1) + 2(4)$

$= 16 - 4 + 8 = 20$

11. $\underline{F} = 3\underline{i} + 2\underline{i} - 4\underline{k}$ $A(1,-1,2)$ $B(2,-1,3)$ **(Sargodha 2009)**

$\underline{r} = \overrightarrow{BA} = (1-2)\underline{i} - (-1+1)\underline{j} + (2-3)\underline{k} = -\underline{i} - \underline{k}$

Moment of Force $= M = \underline{r} \times \underline{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix}$$

$= \underline{i}(0+2) - \underline{j}(4+3) + \underline{k}(-2-0) = 2\underline{i} - 7\underline{j} - 2\underline{k}$

12. $\underline{F} = 4\underline{i} - 3\underline{k}$ $A(2,-2,5)$ $B(1,-3,1)$

$\underline{r} = \overrightarrow{BA} = (2-1)\underline{i} - (-2+3)\underline{j} + (5-1)\underline{k} = \underline{i} + \underline{j} + 4\underline{k}$

Moment of Force $= M = \underline{r} \times \underline{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix}$$

$= \underline{i}(-3-0) - \underline{j}(-3-16) + \underline{k}(0-4) = -3\underline{i} + 19\underline{j} - 4\underline{k}$

13. $\underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$ $A(1,-2,1)$ $B(2,0,-2)$ **(Sargodha 2011)**

$\underline{r} = \overrightarrow{BA} = (1-2)\underline{i} + (-2-0)\underline{j} + (1+2)\underline{k} = -\underline{i} - 2\underline{j} + 3\underline{k}$

Moment of Force $= M = \underline{r} \times \underline{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$$

$= \underline{i}(6-3) - \underline{j}(3-6) + \underline{k}(-1+4)$

$= 3\underline{i} + 3\underline{j} + 3\underline{k}$

14. $\underline{F_1} = \underline{i} - 2\underline{j}$, $\underline{F_2} = 3\underline{i} + 2\underline{j} - \underline{k}$, $\underline{F_3} = 5\underline{i} + 2\underline{k}$, $A(1,1,1)$ $P(2,0,1)$

$\underline{F} = \underline{F_1} + \underline{F_2} + \underline{F_3}$

$= \underline{i} - 2\underline{i} + 3\underline{i} + 2\underline{j} - \underline{k} + 5\underline{i} + 2\underline{k} = 9\underline{i} + \underline{k}$

$\underline{r} = \overrightarrow{AP} = (2-1)\underline{i} + (0-1)\underline{j} + (1-1)\underline{k} = \underline{i} - \underline{j}$

Moment $= M = \underline{r} \times \underline{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 9 & 0 & 1 \end{vmatrix}$$

$= \underline{i}(-1-0) - \underline{j}(1-0) + \underline{k}(0+9)$

$= -\underline{i} - \underline{j} + 9\underline{k}$

15. $\underline{F} = 7\underline{i} + 4\underline{j} - 3\underline{k}$ $P(1,-2,3)$ $Q(2,1,1)$ **(Sargodha 2008)**

$\underline{r} = \overrightarrow{QP} = (1-2)\underline{i} + (-2-1)\underline{j} + (3-1)\underline{k} = -\underline{i} - 3\underline{j} + 2\underline{k}$

Moment of Force $= M = \underline{r} \times \underline{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -3 & 2 \\ 7 & 4 & -3 \end{vmatrix}$$

$= \underline{i}(9-8) - \underline{j}(3-14) + \underline{k}(-4+21)$

$= \underline{i} + 11\underline{j} + 17\underline{k}$

TEST YOUR SKILLS — Marks 100

OBJECTIVE

Q.No.1 Given below are a few possible answers to each statement of which one is correct, identify the correct one. (20)

1. A physical quantity that can be specified by a number along with unit is called a
   (a) Vector (b) Scalar
   (c) Constant (d) Variable
2. A physical quantity that possess both magnitude and direction is called a
   (a) Scalar (b) Variable
   (c) Constant (d) Vector
3. A unit vector is defined as a vector whose magnitude is
   (a) Zero (b) Unity
   (c) More than 1 (d) Less than 1
4. If terminal point B of vector $\overline{AB}$ coincides with its initial point A then it is called
   (a) Unit vector (b) Zero vector
   (c) Null vector (d) Both (b) & (c)
5. The vector, whose initial point is the origin O a[illegible] whose terminal point is P *i.e.,* $\overline{OP}$ is called
   (a) Unit vector (b) Null vector
   (c) Position vector (d) Scalar
6. Magnitude of the vector $x\underline{i} + y\underline{j}$ is
   (a) $\sqrt{x+y}$ (b) $\sqrt{x^2-y^2}$
   (c) x^2-y^2 (d) $\sqrt{x^2+y^2}$
7. If $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are direction cosines of a vector $\underline{a}$ then
   (a) $\text{Cos}\,\alpha\,\text{Cos}\,\gamma$ (b) $\text{Cos}\,\alpha + \text{Cos}\,\beta + \text{Cos}\,\gamma = 0$
   (c) $\text{Cos}\,\alpha + \text{Cos}\,\beta + \text{Cos}\,\gamma = 1$ (d) $\text{Cos}^2\alpha + \text{Cos}^2\beta + \text{Cos}^2\gamma = 1$
8. For two vector $\underline{a}$ & $\underline{b}$: $\underline{a}\,.\,\underline{b}$ =
   (a) $a + b$ (b) $\underline{a} - \underline{b}$
   (c) ab (d) $\underline{b}\,.\,\underline{a}$
9. For unit vector $\underline{\hat{k}}$, $\underline{\hat{k}}\,.\,\underline{\hat{k}}$ =
   (a) 0 (b) −1
   (c) 2 (d) 1
10. If the angle θ between the vectors $\underline{u}$ & $\underline{v}$ is π then the vectors are
    (a) Perpendicular (b) Collinear
    (c) Parallel (d) Both (b) & (c)
11. If $\underline{i}$ & $\underline{j}$ are two unit vectors then $\underline{i} \times \underline{j}$ =
    (a) 0 (b) 1
    (c) −1 (d) $\underline{k}$
12. Two non-zero vectors $\underline{a}$ & $\underline{b}$ are parallel if $\underline{a} \times \underline{b}$ =
    (a) 1 (b) −1

(c) 0 (d) $-\underline{b} \times \underline{a}$

13. **The zero vector is regarded to be parallel to:**
    (a) Every vector (b) In some cases
    (c) Both (a) & (b) (d) None of these
14. **If $\underline{a}$ & $\underline{b}$ are vectors along two adjacent sides of a parallelogram then area of a parallelogram is**
    (a) $|\underline{a}| . |\underline{b}|$ (b) $\underline{a} . \underline{b}$
    (c) $\frac{1}{2}|\underline{a} \times \underline{b}|$ (d) $|\underline{a} \times \underline{b}|$
15. **If $\underline{a}$, $\underline{b}$ & $\underline{c}$ are three non-zero vectors then scalar triple product of these vectors is**
    (a) $\underline{a} . \underline{b} \times \underline{c}$ (b) $\underline{a} \times \underline{b} . \underline{c}$
    (c) Both (a) & (b) (d) $\underline{a} . \underline{b} . \underline{c}$
16. **If $\underline{a}$, $\underline{b}$ & $\underline{c}$ are three non-zero vectors then vector triple product of these vectors is**
    (a) $\underline{a} . \underline{b} \times \underline{c}$ (b) $\underline{a} \times \underline{b} . \underline{c}$
    (c) $\underline{a} \times \underline{b} \times \underline{c}$ (d) Both (a) & (b)
17. **If three edges of a tetrahedron given as vector & $\underline{a}$, $\underline{b}$ & $\underline{c}$ then the volume is**
    (a) $\frac{1}{3}(\underline{a} . \underline{b} \times \underline{c})$ (b) $\frac{1}{2}(\underline{a} . \underline{b} \times \underline{c})$
    (c) $\frac{1}{6}(\underline{a} . \underline{b} \times \underline{c})$ (d) $\frac{1}{4}(\underline{a} . \underline{b} \times \underline{c})$
18. **Moment of force $\overline{F}$ about a point is given by**
    (a) Dot product (b) Cross product
    (c) Both (a) & (b) (d) None of these
19. **The vectors lying in the same plane are called**
    (a) Collinear vectors (b) Perpendicular vectors
    (c) Coplanar vectors (d) Parallel vectors
20. **Moment of a force about a point is:**
    (a) Vector quantity (b) Scalar quantity
    (c) Zero (d) None of these

SECTION I
SUBJECTIVE

Attempt any 25 (twenty five) short question from Question 2,3 and 4, at least 12 short questions from question 2, 12 short question from question 3 and 13 short question from question 4. All question carry equal marks. ***(25x2=50)***

Q.No. 2

i. Find a unit vector in the direction of the vector $\underline{v} = \underline{i} + 2\underline{j} - \underline{k}$
ii. Find a vector of magnitude 4 & is parallel to $2\underline{i} - 3\underline{j} + 6\underline{k}$
iii. Find α so that $|\alpha\underline{i} + (\alpha + 1)\underline{j} + 2\underline{k}| = 3$
iv. What are direction cosines of $\underline{v} = 2\underline{i} + 3\underline{j} + 4\underline{k}$
v. Find two vectors of length 2 parallel to the vector $\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$

vi. Find the constant a so that the vectors $\underline{v} = i - 3j + 4\underline{k}$ and $\underline{w} = a\underline{i} + 9j - 12\underline{k}$ are parallel.

vii. Find a and b so that the vectors $3i - \underline{j} + 4\underline{k}$ and $\underline{ai} + \underline{bj} - 2\underline{k}$ are parallel.

viii. Check whether the triple (i) $45^\circ, 45^\circ, 60^\circ$, (ii) $45^\circ, 60^\circ, 60^\circ$ be the direction angles of a single vector.

ix. Find a vector of length 5 in the direction opposite that of $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$

x. Find a scalar 'α' so that the vectors $2\underline{i} + \alpha\underline{j} + 5\underline{k}$ and $3\underline{i} + \underline{j} + \alpha\underline{k}$ are perpendicular.

xi. If $\underline{u} = a_1\underline{i} + b_1\underline{j} + c_1\underline{k}$

$\underline{v} = a_2\underline{i} + b_2\underline{j} + c_2\underline{k}$

Prove that $\underline{u} \cdot \underline{v} = (a_1)(a_2) + (b_1)(b_2) + (c_1)(c_2)$

xii. Find the angle between $\underline{u} = 2\underline{i} - \underline{j} + \underline{k}$

$\underline{v} = -\underline{i} + \underline{j}$

Q.No. 3

i. Calculate the projection of $\underline{a}$ along $\underline{b}$ $\underline{a} = \underline{i} - \underline{k}$, $\underline{b} = \underline{j} + \underline{k}$ & the projection of $\underline{b}$ along $\underline{a}$.

ii. If $\underline{v}$ is a vector such that $\underline{v} \cdot \underline{i} = 0$, $\underline{v} \cdot \underline{j} = 0$, $\underline{v} \cdot \underline{k} = 0$ find $\underline{v}$.

iii. Find a vector perpendicular to each of the vector $\underline{a} = 2\underline{i} + \underline{j} + \underline{k}$ and $b = 4\underline{i} + 2\underline{j} - \underline{k}$

iv. Find a unit vector perpendicular to $\underline{a}$ & $\underline{b}$

$\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k}$, $\underline{b} = 4\underline{i} + 3\underline{j} - \underline{k}$

v. Prove that $\underline{a} \times (\underline{b} + \underline{c}) + \underline{b} \times (\underline{c} + \underline{a}) + \underline{c} \times (\underline{a} + \underline{b}) = 0$

vi. Find the volume of the parallelepiped determined by $\underline{u} = \underline{i} + 2\underline{j} - \underline{k}$

$\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$

$\underline{w} = \underline{i} - 7\underline{j} - 4\underline{k}$

vii. Find α so that $\alpha\underline{i} + \underline{j}$, $\underline{i} + \underline{j} + 3\underline{k}$ & $2\underline{i} + \underline{j} - 2\underline{k}$ are coplanar.

viii. If $\underline{a} = 3\underline{i} - \underline{j} + 5\underline{k}$, $\underline{b} = 4\underline{i} + 3\underline{j} - 2\underline{k}$ & $\underline{c} = 2\underline{i} + 5\underline{j} + \underline{k}$ find $\underline{a} \cdot \underline{b} \times \underline{c}$

ix. Prove that vectors $\underline{i} - 2\underline{j} + 3\underline{k}$, $-2\underline{i} + 3\underline{j} - 4\underline{k}$ & $\underline{i} - 3\underline{j} + 5\underline{k}$ are coplanar.

x. Find the value of (i) $2\underline{i} \times 2\underline{j} \cdot \underline{k}$, (ii) $3\underline{j} \cdot \underline{k} \times \underline{i}$, (iii) $[\underline{k}\ \underline{i}\ \underline{j}]$, (iv) $[\underline{i}\ \underline{j}\ \underline{k}]$

xi. Find work done by $\underline{F} = 2\underline{i} + 4\underline{j}$ if its pts of application to a body moves if from $A(1, 1)$ to $B(4, 6)$

xii. A force $F = 7\underline{i} + 4\underline{j} - 3\underline{k}$ is applied at $P(1, -2, 3)$

Find its moments about the pt $Q(2, 1, 1)$

Q.No. 4

i. Find a vector from A to the origin where $\overrightarrow{AB} = 4i - 2j$ & $B = (-2, 5)$

ii. If $\underline{u} = 2\underline{i} + 3\underline{j} + \underline{k}$

$\underline{v} = 4\underline{i} + 6\underline{j} + 2\underline{k}$

$\underline{w} = -6\underline{i} - 9\underline{j} - 3\underline{k}$

(a) Find $|\underline{u} - \underline{v} - \underline{w}|$

(b) Show that $\underline{u}$, $\underline{v}$ & $\underline{w}$ are parallel to each other.

iii. If $\overrightarrow{OC} = 2\underline{i} - 3\underline{j}$ and $\overrightarrow{OD} = 3\underline{i} + 2\underline{j}$

Then find a position vector of a pt which divide it in the ratio of 4 : 3

iv. If $\overrightarrow{OE} = 5\underline{i}$, $\overrightarrow{OF} = 4\underline{i} + \underline{i}$

Then find position vector of a pt which divide it in the ratio of 2 : 5.

v. Define direction cosines & direction angles of a vector.

vi. What is the geometrical interpretation of $\underline{u} \cdot (\underline{v} \times \underline{w})$ (scalar triple product)

vii. If $\underline{a} \times \underline{b} = 0$ & $\underline{a} \cdot \underline{b} = 0$

What conversion can be drawn about $\underline{a}$ or $\underline{b}$?

viii. Define scalar & cross products of two vectors.

ix. What is geo metrical interpretation of $\underline{a} \times \underline{b}$.

x. What is null vector?

xi. What are equal vectors?

xii. What is position vector of a point?

xiii. Find α, so that $|\alpha\, \underline{i} + (\alpha+1)\, \underline{j} + 2\underline{k}| = 3.$?

SECTION II

Attempt any 3 (three) questions. **(3x10=30)**

Q.No.5

(a) If force F=7i+4j−3k is applied at P(1,−2,3). Find its moment about the point Q(2,1,1)

(b) Prove that $\cos^2\alpha + \text{Cos}^2\beta + \cos^2\gamma = 1$

Q.No.6

(a) Find the value of α, so that α i+j, i+j+3k and 2i+j−2k are coplanar.

(b) Find the vector from the point A to the origin where $\overline{AB}$ =4i−2j and B is the point (−2, 5)

Q.No.7

(a) If a+b+c=0, then prove that axb=bxc=cxa.

(b) Find a unit vector in the direction of the vector $v = \frac{1}{2}i + \frac{\sqrt{3}}{2}j$

Q.No.8

(a) Prove that $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

(b) If u=2i+3j+4k, v=−i+3j−k and w=i+6j+zk represent the sides of a triangle. Find the value of z.

Q.No.9

(a) Prove that in triangle ABC. $b^2=c^2+a^2-2ca\cos B$

(b) Find area of triangle , determined by the point P, Q and R.

P(1, −1, −1) : Q(2, 0, −1): R(0, 2, 1)

Previous Board Questions

1. What is null vector? (Lhr – 2000)
2. What are equal vectors?
3. What is position vector of a point? (Mirpur – 2009)
4. Find α, so that $|\alpha \underline{i} + (\alpha+1)\underline{j} + 2\underline{k}| = 3$.? (Guj – 2007, Lhr – 2008, Fsd – 2009)
5. Show that the diagonals of a parallelogram bisect each other? (Lhr – 2007)
6. What are direction angles of a vector? (Mirpur – 2009)
7. Find the value of $[\hat{k}, \hat{i}, \hat{j}]$? (Lhr – 2006)
8. Define scalar product? (Lhr – 2008)
9. Find a vector of length 5 in the direction opposite to $\underline{v} = \underline{i} - 2\underline{j} + 3\underline{k}$? (Mtn – 2009)
10. Find the value of $2\hat{i} \times 3\hat{j} . 4\hat{k}$? (Grw – 2005)
11. Show that $\underline{i} . \underline{j} \times \underline{k} = 1$? (Grw – 2007)
12. Find if $a\underline{i} + \underline{j} . \underline{i} + \underline{j} + 3k, 2\underline{i} + \underline{j} - 2\underline{k}$? (Mtn – 2009, Mirpur – 2009)
13. What are coplaner vectors?
14. What is unit vector?
15. Find the vector perpendicular to the plane of $\underline{i} + \underline{j}$ and $\underline{i} - \underline{j}$. (Faisalabad – 2009)
16. Find a unit vector in the direction of vector $2\underline{i} - \underline{j}$. (Lahore – 2010) Group – I
17. Find $\underline{a} \times \underline{b}$ when $\underline{a} = [2, 3, 1]$ and $\underline{b} = [1, 0, 2]$. (Lahore – 2010) Group – I
18. Write a unit vector in the direction of the vector $-\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$. (Lahore – 2010) Group – II
19. Prove that (vectorially) that $\cos(\alpha - \beta) = \cos\beta + \sin\alpha \sin\beta$. (Lahore – 2010) Group – II
20. Find the direction cosines for the vector $\underline{a} = 6\underline{i} - 2\underline{j} + \underline{k}$. (Gujranwala – 2010)
21. Find the cosine of the angle θ between $\underline{u}$ and $\underline{v}$ where $\underline{u} = [-3, 5]$, $\underline{v} = [6, -2]$. (Gujranwala – 2010)