Analytical Expression Of Genuine Tripartite Quantum Discord For Symmetrical X-states

The study of classical and quantum correlations in bipartite and multipartite systems is crucial for the development of quantum information theory. Among the quantifiers adopted in tripartite systems, the genuine tripartite quantum discord (GTQD), estimating the amount of quantum correlations shared among all the subsystems, plays a key role since it represents the natural extension of quantum discord used in bipartite systems. In this paper, we derive an analytical expression of GTQD for three-qubit systems characterized by a subclass of symmetrical X-states. Our approach has been tested on both GHZ and maximally mixed states reproducing the expected results. Furthermore, we believe that the procedure here developed constitutes a valid guideline to investigate quantum correlations in form of discord in more general multipartite systems.

I INTRODUCTION

Quantum correlations are assuming increasing relevance, since they can be exploited to improve our ability to perform many informational and computational tasks (Datta et al., 2005; Datta and Vidal, 2007; Datta et al., 2008; Lanyon et al., 2008). Therefore, the problem of their characterization and quantification has become a significant topic of research. Traditionally, the most used form of quantum correlation is entanglement, and the development of quantum information theory is fundamentally due to its implementation in information and communication protocols (Nielsen and Chuang, 2000; Horodecki et al., 2009).

A form of quantum correlation other than entanglement is quantum discord (QD) (Ollivier and Zurek, 2001; Henderson and Vedral, 2001; Luo, 2008a), which can be expressed in terms of the difference between the total and the classical correlations for a system when one of its subparties is subject to an unobserved measure process. Such a quantity, however, significantly depends upon both the subsystem chosen and the measurement performed on it: in particular, if the measurement is carefully selected, we can minimize its “disturbing effect” on the system (Luo, 2008b). This choice corresponds to the minimization of QD firstly over a set of possible measurements (on a fixed subsystem), typically projective von Neumann measurements (Luo, 2008b; Ollivier and Zurek, 2001), and secondly over all possible subsystem on which the local measurement can be performed. Recent efforts in the study of the optimization processes have led to analytical expression for quantum discord in some particular (Luo, 2008a) and more general states (Girolami and Adesso, 2011; Javad Akhtarshenas et al., 2013) in systems composed of two qubits.

Both entanglement and QD have widely been analyzed and used in bipartite systems, while their extension to multipartite systems is still discussed and tackled with different approaches (Bennett et al., 2011; Rulli and Sarandy, 2011; Xu, 2013; Chakrabarty et al., 2011; Giorgi et al., 2011). For instance, Vinjanampathy et al. (Vinjanampathy and Rau, 2012) proposed a method to evaluate analytically quantum discord for a n-partite system of qubits in some special cases, but they treated the whole system as a bipartite one (each subparty containing

1111 or n-11n-1italic_n - 1 qubits, respectively). On the other hand, Giorgi et al. (Giorgi et al., 2011) defined, for a n-partite system, genuine n-partite correlations, which can be divided into total, classical or quantum. These kinds of correlations are shared between all the n parties which form the system, i.e. they cannot be accounted for considering any of the possible subsystems. The quantum part of genuine correlations is quantified by genuine n-partite quantum discord. The approach of Ref. (Giorgi et al., 2011) represents a natural extension of the concept of QD as introduced for bipartite systems, and this is the reason why we will follow it in the present work. However, it requires massive numerical optimization procedures over a number of parameters, thus making the calculations very demanding (Huang, 2014). Therefore, the application of such a criterion is not easily amenable.

This justifies the scarce number of works investigating the time evolution of quantum correlations in multipartite system coupled to noisy environments. Specifically only few cases have been considered: two level systems undergoing random telegraph noise (Buscemi and Bordone, 2013; Maziero and Zimmer, 2012) and quantum phase transitions in spin systems (Grimsmo et al., 2012; Cai and Abliz, 2013a; Qiu et al., 2014).

The purpose of this paper is to derive an analytical expression for the genuine tripartite quantum discord (GTQD) for a class of three qubits systems. In detail, we will focus on those systems described by X-states, which play a relevant role in a large number of physical systems and allow for easy calculations of certain entanglement measures (Weinstein, 2009, 2010). X-states have been widely investigated also in bipartite systems, where an analytical expression for QD has been proposed in (Ali et al., 2010). However, this approach has been questioned, since it is not always providing the correct result (Lu et al., 2011; Chen et al., 2011; Huang, 2013).

The paper is organized as follows. In Sec. II we introduce the genuine quantifiers for correlations in multipartite quantum systems. In Sec. III we introduce the expression for a symmetrical tripartite X-state and derive some constraints on its defining parameters. In Sec. IV we estimate all the quantities required to compute GTQD, and in particular we describe the optimization procedures (both numerical and analytical) appearing in the expression of GTQD. Sec. V concerns the comparison between our results on GTQD and others already present in the literature and, finally, in Sec. VI we draw conclusions.

II QUANTIFIERS FOR GENUINE TRIPARTITE CORRELATIONS

Here we illustrate the correlation measures adopted in this work to quantify tripartite quantum discord and entanglement.

II.1 Tripartite Quantum Discord

In a tripartite system, described by a state =A,B,Csubscript\rho=\rho_A,B,Citalic_ = italic_ start_POSTSUBSCRIPT italic_A , italic_B , italic_C end_POSTSUBSCRIPT, the tripartite quantum mutual information is obtained as a generalization of the quantum mutual information for bipartite systems:

T()=S(A)+S(B)+S(C)-S(),subscriptsubscriptsubscriptT(\rho)=S(\rho_A)+S(\rho_B)+S(\rho_C)-S(\rho),italic_T (italic_) = italic_S (italic_ start_POSTSUBSCRIPT italic_A end_POSTSUBSCRIPT) + italic_S (italic_ start_POSTSUBSCRIPT italic_B end_POSTSUBSCRIPT) + italic_S (italic_ start_POSTSUBSCRIPT italic_C end_POSTSUBSCRIPT) - italic_S (italic_) , (1)

and represents the total amount of correlations encoded in this system. Here S()=-Tr[log2()]Trdelimited-[]subscript2S(\rho)=-\mathrmTr[\rho\log_2(\rho)]italic_S (italic_) = - roman_Tr [italic_ roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (italic_)] is the von Neumann entropy, and isubscript\rho_iitalic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT (i=A,B,C)(i=A,B,C)(italic_i = italic_A , italic_B , italic_C) is the reduced density matrix for the subsystem iitalic_i. Following the same procedure used in the literature for bipartite systems (Ollivier and Zurek, 2001), Giorgi et al. (Giorgi et al., 2011) define the tripartite classical correlations in the system as the quantum version (of a classical analogue) of the mutual information derived from the Bayes' rule:

J()=maxi,j,kA,B,C[S(i)-S(i|j)+S(k)-S(k|ij)],delimited-[]subscriptsubscriptconditionalsubscriptsubscriptconditionalJ(\rho)=\underseti,j,k\in\A,B,C\\max[S(\rho_i)-S(\rho_j)+S(\rho_k%)-S(\rho_ij)],italic_J (italic_) = start_UNDERACCENT italic_i , italic_j , italic_k italic_A , italic_B , italic_C end_UNDERACCENT start_ARG roman_max end_ARG [italic_S (italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT) - italic_S (italic_ start_POSTSUBSCRIPT italic_i | italic_j end_POSTSUBSCRIPT) + italic_S (italic_ start_POSTSUBSCRIPT italic_k end_POSTSUBSCRIPT) - italic_S (italic_ start_POSTSUBSCRIPT italic_k | italic_i italic_j end_POSTSUBSCRIPT)] , (2)

which has been optimized over the indices i,j,ki,j,kitalic_i , italic_j , italic_k in the set of all the possible permutation of subsystems A,B,C\A,B,C\ italic_A , italic_B , italic_C . Here S(i|j)subscriptconditionalS(\rho_j)italic_S (italic_ start_POSTSUBSCRIPT italic_i | italic_j end_POSTSUBSCRIPT) and S(k|ij)subscriptconditionalS(\rho_k)italic_S (italic_ start_POSTSUBSCRIPT italic_k | italic_i italic_j end_POSTSUBSCRIPT) are relative entropies and i|jsubscriptconditional\rho_iitalic_ start_POSTSUBSCRIPT italic_i | italic_j end_POSTSUBSCRIPT and k|ijsubscriptconditional\rho_kitalic_ start_POSTSUBSCRIPT italic_k | italic_i italic_j end_POSTSUBSCRIPT are the density matrices after a measurement on the subsystem iitalic_i or after a measurement on both subsystems iitalic_i and jjitalic_j, respectively (Buscemi and Bordone, 2013). We refer the reader to the Appendix A for a detailed definition of the relative entropies and their optimization. Like for bipartite systems, the tripartite quantum discord is given by the difference between total and classical correlations:

D()=T()-J().D(\rho)=T(\rho)-J(\rho).italic_D (italic_) = italic_T (italic_) - italic_J (italic_) . (3)

However, among the correlations included in T()T(\rho)italic_T (italic_), a subset is shared by all of the three subsystems (genuine tripartite mutual information), and can be estimated as:

T(3)()=T()-T(2)(),superscript3superscript2T^(3)(\rho)=T(\rho)-T^(2)(\rho),italic_T start_POSTSUPERSCRIPT (3) end_POSTSUPERSCRIPT (italic_) = italic_T (italic_) - italic_T start_POSTSUPERSCRIPT (2) end_POSTSUPERSCRIPT (italic_) , (4)

where T(2)()superscript2T^(2)(\rho)italic_T start_POSTSUPERSCRIPT (2) end_POSTSUPERSCRIPT (italic_) is the maximum amount of mutual information shared by any couple of subsystems:

T(2)()=max[I(A,B),I(A,C),I(B,C)],superscript2subscriptsubscriptsubscriptT^(2)(\rho)=\max[I(\rho_A,B),I(\rho_A,C),I(\rho_B,C)],italic_T start_POSTSUPERSCRIPT (2) end_POSTSUPERSCRIPT (italic_) = roman_max [italic_I (italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT) , italic_I (italic_ start_POSTSUBSCRIPT italic_A , italic_C end_POSTSUBSCRIPT) , italic_I (italic_ start_POSTSUBSCRIPT italic_B , italic_C end_POSTSUBSCRIPT)] , (5)

where I(AB)=S(A)+S(B)-S(AB)subscriptsubscriptsubscriptsubscriptI(\rho_AB)=S(\rho_A)+S(\rho_B)-S(\rho_AB)italic_I (italic_ start_POSTSUBSCRIPT italic_A italic_B end_POSTSUBSCRIPT) = italic_S (italic_ start_POSTSUBSCRIPT italic_A end_POSTSUBSCRIPT) + italic_S (italic_ start_POSTSUBSCRIPT italic_B end_POSTSUBSCRIPT) - italic_S (italic_ start_POSTSUBSCRIPT italic_A italic_B end_POSTSUBSCRIPT). Since DISCORD SERVER that cannot be accounted for by T(2)()superscript2T^(2)(\rho)italic_T start_POSTSUPERSCRIPT (2) end_POSTSUPERSCRIPT (italic_) must be shared between all of the three subsystems, we can conclude that T(3)()superscript3T^(3)(\rho)italic_T start_POSTSUPERSCRIPT (3) end_POSTSUPERSCRIPT (italic_) measures the distance between \rhoitalic_ and the closest product state along any bipartite cut of the system. Indeed it can be shown that T(3)()=min[I(AB,C),I(AC,B),I(BC,A)]superscript3subscriptsubscriptsubscriptT^(3)(\rho)=\min[I(\rho_AB,C),I(\rho_AC,B),I(\rho_BC,A)]italic_T start_POSTSUPERSCRIPT (3) end_POSTSUPERSCRIPT (italic_) = roman_min [italic_I (italic_ start_POSTSUBSCRIPT italic_A italic_B , italic_C end_POSTSUBSCRIPT) , italic_I (italic_ start_POSTSUBSCRIPT italic_A italic_C , italic_B end_POSTSUBSCRIPT) , italic_I (italic_ start_POSTSUBSCRIPT italic_B italic_C , italic_A end_POSTSUBSCRIPT)] (see Ref. (Giorgi et al., 2011)).

Analogously, the genuine tripartite classical correlations reads:

J(3)()=J()-J(2)(),superscript3superscript2J^(3)(\rho)=J(\rho)-J^(2)(\rho),italic_J start_POSTSUPERSCRIPT (3) end_POSTSUPERSCRIPT (italic_) = italic_J (italic_) - italic_J start_POSTSUPERSCRIPT (2) end_POSTSUPERSCRIPT (italic_) , (6)

and GTQD:

D(3)()=D()-D(2)(),superscript3superscript2D^(3)(\rho)=D(\rho)-D^(2)(\rho),italic_D start_POSTSUPERSCRIPT (3) end_POSTSUPERSCRIPT (italic_) = italic_D (italic_) - italic_D start_POSTSUPERSCRIPT (2) end_POSTSUPERSCRIPT (italic_) , (7)

where111In Eqs. (8) and (9) we used the bipartite quantifiers J(A,B)=max[S(A,B)-S(A|B),S(A,B)-S(B|A)]subscriptsubscriptsubscriptconditionalsubscriptsubscriptconditionalJ(\rho_A,B)=\max[S(\rho_A,B)-S(\rho_B),S(\rho_A,B)-S(\rho_B)]italic_J (italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT) = roman_max [italic_S (italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT) - italic_S (italic_ start_POSTSUBSCRIPT italic_A | italic_B end_POSTSUBSCRIPT) , italic_S (italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT) - italic_S (italic_ start_POSTSUBSCRIPT italic_B | italic_A end_POSTSUBSCRIPT)] and D(A,B)=I(A,B)-J(A,B)subscriptsubscriptsubscriptD(\rho_A,B)=I(\rho_A,B)-J(\rho_A,B)italic_D (italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT) = italic_I (italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT) - italic_J (italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT) as they are usually defined in literature for 2-qubits systems (Giorgi et al., 2011; Buscemi and Bordone, 2013; Luo, 2008a).:

J(2)()=max[J(A,B),J(A,C),J(B,C)],superscript2subscriptsubscriptsubscript\displaystyle J^(2)(\rho)=\max[J(\rho_A,B),J(\rho_A,C),J(\rho_B,C)],italic_J start_POSTSUPERSCRIPT (2) end_POSTSUPERSCRIPT (italic_) = roman_max [italic_J (italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT) , italic_J (italic_ start_POSTSUBSCRIPT italic_A , italic_C end_POSTSUBSCRIPT) , italic_J (italic_ start_POSTSUBSCRIPT italic_B , italic_C end_POSTSUBSCRIPT)] , (8)

D(2)()=min[D(A,B),D(A,C),D(B,C)].superscript2subscriptsubscriptsubscript\displaystyle D^(2)(\rho)=\min[D(\rho_A,B),D(\rho_A,C),D(\rho_B,C)].italic_D start_POSTSUPERSCRIPT (2) end_POSTSUPERSCRIPT (italic_) = roman_min [italic_D (italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT) , italic_D (italic_ start_POSTSUBSCRIPT italic_A , italic_C end_POSTSUBSCRIPT) , italic_D (italic_ start_POSTSUBSCRIPT italic_B , italic_C end_POSTSUBSCRIPT)] . (9)

Eqs. (4), (6) and (7) can be significantly simplified for the case of a state \rhoitalic_ symmetrical under any exchange of its subsystems. Indeed it can be shown that (Buscemi and Bordone, 2013):

T(3)()=I(A,BC)=S(A)+S(A,B)-S(),superscript3subscriptsubscriptsubscript\displaystyle T^(3)(\rho)=I(\rho_A,BC)=S(\rho_A)+S(\rho_A,B)-S(\rho),italic_T start_POSTSUPERSCRIPT (3) end_POSTSUPERSCRIPT (italic_) = italic_I (italic_ start_POSTSUBSCRIPT italic_A , italic_B italic_C end_POSTSUBSCRIPT) = italic_S (italic_ start_POSTSUBSCRIPT italic_A end_POSTSUBSCRIPT) + italic_S (italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT) - italic_S (italic_) , (10)

D(3)()=S(A|BC)+S(A,B)-S(),superscript3subscriptconditionalsubscript\displaystyle D^(3)(\rho)=S(\rho_BC)+S(\rho_A,B)-S(\rho),italic_D start_POSTSUPERSCRIPT (3) end_POSTSUPERSCRIPT (italic_) = italic_S (italic_ start_POSTSUBSCRIPT italic_A | italic_B italic_C end_POSTSUBSCRIPT) + italic_S (italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT) - italic_S (italic_) , (11)

J(3)()=S(A)-S(A|BC).superscript3subscriptsubscriptconditional\displaystyle J^(3)(\rho)=S(\rho_A)-S(\rho_BC).italic_J start_POSTSUPERSCRIPT (3) end_POSTSUPERSCRIPT (italic_) = italic_S (italic_ start_POSTSUBSCRIPT italic_A end_POSTSUBSCRIPT) - italic_S (italic_ start_POSTSUBSCRIPT italic_A | italic_B italic_C end_POSTSUBSCRIPT) . (12)

II.2 Tripartite Negativity

In tripartite systems, represented by a state \rhoitalic_, we can detect the presence of entanglement between subsystems by using the negativity NNitalic_N, which is defined as follows (Vidal and Werner, 2002):

N(tC)=i|i(tC)|-1.superscriptsubscriptsubscriptsuperscript1N(\rho^tC)=\sum_i|\lambda_i(\rho^tC)|-1.italic_N (italic_ start_POSTSUPERSCRIPT italic_t italic_C end_POSTSUPERSCRIPT) = start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT | italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT (italic_ start_POSTSUPERSCRIPT italic_t italic_C end_POSTSUPERSCRIPT) | - 1 . (13)

In the previous expression, tCsuperscript\rho^tCitalic_ start_POSTSUPERSCRIPT italic_t italic_C end_POSTSUPERSCRIPT is the partial transpose of \rhoitalic_ with respect to the subsystem CCitalic_C, and i(tC)subscriptsuperscript\lambda_i(\rho^tC)italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT (italic_ start_POSTSUPERSCRIPT italic_t italic_C end_POSTSUPERSCRIPT) are the eigenvalues of tCsuperscript\rho^tCitalic_ start_POSTSUPERSCRIPT italic_t italic_C end_POSTSUPERSCRIPT. The negativity can be equivalently interpreted as the sum of the absolute values of the negative eigenvalues of tCsuperscript\rho^tCitalic_ start_POSTSUPERSCRIPT italic_t italic_C end_POSTSUPERSCRIPT (Vidal and Werner, 2002), and it depends upon the subsystem on which we make the partial transpose of \rhoitalic_.

When negativity is higher than zero, we can conclude that there is an entanglement between the subsystem CCitalic_C and the compound subsystem A-BA-Bitalic_A - italic_B, but the converse is not necessarily true. Starting from this point, we can define the tripartite negativity as follows (Sabín and García-Alcaine, 2008):

N(3)()=N(tA)N(tB)N(tC)3,superscript33superscriptsuperscriptsuperscriptN^(3)(\rho)=\sqrt[3]N(\rho^tA)N(\rho^tB)N(\rho^tC),italic_N start_POSTSUPERSCRIPT (3) end_POSTSUPERSCRIPT (italic_) = nth-root start_ARG 3 end_ARG start_ARG italic_N (italic_ start_POSTSUPERSCRIPT italic_t italic_A end_POSTSUPERSCRIPT) italic_N (italic_ start_POSTSUPERSCRIPT italic_t italic_B end_POSTSUPERSCRIPT) italic_N (italic_ start_POSTSUPERSCRIPT italic_t italic_C end_POSTSUPERSCRIPT) end_ARG , (14)

and this quantifier will be different from zero only when the entanglement is shared among all of the three subsystems, i.e. it is a “full” tripartite entanglement (Sabín and García-Alcaine, 2008). However, apart from pure states, a null negativity could indeed not imply the absence of entanglement. Moreover, we must notice that tripartite negativity cannot distinguish the entanglement of a genuine tripartite entangled state from that of a biseparable state in a generalized sense (Sabín and García-Alcaine, 2008; Cai and Abliz, 2013b). For tripartite systems that are symmetrical under any exchange of their qubits, as in our case of study, the tripartite negativity and the negativity always coincide:

N(3)()=N(tA)=N(tB)=N(tC).superscript3superscriptsuperscriptsuperscriptN^(3)(\rho)=N(\rho^tA)=N(\rho^tB)=N(\rho^tC).italic_N start_POSTSUPERSCRIPT (3) end_POSTSUPERSCRIPT (italic_) = italic_N (italic_ start_POSTSUPERSCRIPT italic_t italic_A end_POSTSUPERSCRIPT) = italic_N (italic_ start_POSTSUPERSCRIPT italic_t italic_B end_POSTSUPERSCRIPT) = italic_N (italic_ start_POSTSUPERSCRIPT italic_t italic_C end_POSTSUPERSCRIPT) . (15)

Another possible quantifier for tripartite entanglement is the three-tangle (Coffman et al., 2000), but in this work we use negativity since the three-tangle is not able to detect tripartite entanglement for all states, e.g. W states (Dür et al., 2000).

III THREE-QUBITS SYMMETRICAL X-STATES

Here, we focus on three qubits X-states (Yu and Eberly, 2004) which, for the particular features of their quantum correlations, have been investigated in the literature, both for bipartite (Vinjanampathy and Rau, 2012; Chen et al., 2011; Ali et al., 2010) and tripartite systems (Weinstein, 2010, 2009; Buscemi and Bordone, 2013). A generic tripartite X-state can be written in the form (Weinstein, 2010):

=(a1000000c10a20000c2000a300c300000a4c4000000c4*b400000c3*00b3000c2*0000b20c1*000000b1).subscript1000000subscript10subscript20000subscript2000subscript300subscript300000subscript4subscript4000000superscriptsubscript4subscript400000superscriptsubscript300subscript3000superscriptsubscript20000subscript20superscriptsubscript1000000subscript1\rho=\left(\beginarray[]cccccccca_1&0&0&0&0&0&0&c_1\\ 0&a_2&0&0&0&0&c_2&0\\ 0&0&a_3&0&0&c_3&0&0\\ 0&0&0&a_4&c_4&0&0&0\\ 0&0&0&c_4^*&b_4&0&0&0\\ 0&0&c_3^*&0&0&b_3&0&0\\ 0&c_2^*&0&0&0&0&b_2&0\\ c_1^*&0&0&0&0&0&0&b_1\endarray\right).italic_ = (start_ARRAY start_ROW start_CELL italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL italic_a start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_a start_POSTSUBSCRIPT 3 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_c start_POSTSUBSCRIPT 3 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_a start_POSTSUBSCRIPT 4 end_POSTSUBSCRIPT end_CELL start_CELL italic_c start_POSTSUBSCRIPT 4 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_c start_POSTSUBSCRIPT 4 end_POSTSUBSCRIPT start_POSTSUPERSCRIPT * end_POSTSUPERSCRIPT end_CELL start_CELL italic_b start_POSTSUBSCRIPT 4 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_c start_POSTSUBSCRIPT 3 end_POSTSUBSCRIPT start_POSTSUPERSCRIPT * end_POSTSUPERSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_b start_POSTSUBSCRIPT 3 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT start_POSTSUPERSCRIPT * end_POSTSUPERSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_b start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT start_POSTSUPERSCRIPT * end_POSTSUPERSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_b start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_CELL end_ROW end_ARRAY) . (16)

In order to simplify the derivation of an analytical expression for GTQD, we limit ourselves to X-states which are symmetrical under any exchange of their subsystems, and invariant under the flip of all of their qubits. This means that \rhoitalic_ can be written in the form:

=18(1-
a1000000c1010000c2000100c2000001c2*000000c2100000c2*001000c2*000010c1*000000
1-

a1),181subscript1000000subscript10subscript10000subscript2000subscript100subscript200000subscript1superscriptsubscript2000000subscript2subscript100000superscriptsubscript200subscript1000superscriptsubscript20000subscript10superscriptsubscript10000001subscript1\rho=\frac18\scriptstyle\left(\beginarray[]cccccccc1-a_1&0&0&0&0&0&% 0&c_1\\ 0&\alpha_1&0&0&0&0&c_2&0\\ 0&0&\alpha_1&0&0&c_2&0&0\\ 0&0&0&\alpha_1&c_2^*&0&0&0\\ 0&0&0&c_2&\alpha_1&0&0&0\\ 0&0&c_2^*&0&0&\alpha_1&0&0\\ 0&c_2^*&0&0&0&0&\alpha_1&0\\ c_1^*&0&0&0&0&0&0&1-a_1\endarray\right),italic_ = divide start_ARG 1 end_ARG start_ARG 8 end_ARG (start_ARRAY start_ROW start_CELL 1 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_CELL start_CELL italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT start_POSTSUPERSCRIPT * end_POSTSUPERSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT end_CELL start_CELL italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT start_POSTSUPERSCRIPT * end_POSTSUPERSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT start_POSTSUPERSCRIPT * end_POSTSUPERSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT start_POSTSUPERSCRIPT * end_POSTSUPERSCRIPT end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 1 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_CELL end_ROW end_ARRAY) , (17) where 1=1+a13subscript11subscript13\alpha_1=1+\fraca_13italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT = 1 + divide start_ARG italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_ARG start_ARG 3 end_ARG (we used the property Tr[]=1Trdelimited-[]1\mathrmTr[\rho]=1roman_Tr [italic_] = 1 to express a2subscript2a_2italic_a start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT in terms of a1subscript1a_1italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT, and then we made the substitutions a11-a18subscript11subscript18a_1\rightarrow\frac1-a_18italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT divide start_ARG 1 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_ARG start_ARG 8 end_ARG, cici8subscriptsubscript8c_i\rightarrow\fracc_i8italic_c start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT divide start_ARG italic_c start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT end_ARG start_ARG 8 end_ARG to get a simpler expression). Now \rhoitalic_ depends only on the parameters (a1,c1,c2)subscript1subscript1subscript2(a_1,c_1,c_2)(italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT , italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT , italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) which, from now on, as a case of study, are assumed to be real.

From the requirement 0i1i0subscript1for-all0\leq\lambda_i\leq 1\;\forall i0 italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT 1 italic_i, where 1,2=18(1-a1c1)subscript1218minus-or-plus1subscript1subscript1\lambda_1,2=\frac18\left(1-a_1\mp c_1\right)italic_ start_POSTSUBSCRIPT 1 , 2 end_POSTSUBSCRIPT = divide start_ARG 1 end_ARG start_ARG 8 end_ARG (1 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) and 3-4-5,6-7-8=124(3+a13c2)subscript345678124minus-or-plus3subscript13subscript2\lambda_3-4-5,6-7-8=\frac124\left(3+a_1\mp 3c_2\right)italic_ start_POSTSUBSCRIPT 3 - 4 - 5 , 6 - 7 - 8 end_POSTSUBSCRIPT = divide start_ARG 1 end_ARG start_ARG 24 end_ARG (3 + italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT 3 italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) are the eigenvalues of \rhoitalic_, we obtain the following constraints for the parameters:

a1subscript1\displaystyle a_1italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT [-3,1],absent31\displaystyle\in\left[-3,1\right], [- 3 , 1] ,

c1subscript1\displaystyle c_1italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT [a1-1,1-a1]absentsubscript111subscript1\displaystyle\in\left[a_1-1,1-a_1\right] [italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT - 1 , 1 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT] (18)

c2subscript2\displaystyle c_2italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT [-1-a13,1+a13].absent1subscript131subscript13\displaystyle\in\left[-1-\fraca_13,1+\fraca_13\right]. [- 1 - divide start_ARG italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_ARG start_ARG 3 end_ARG , 1 + divide start_ARG italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_ARG start_ARG 3 end_ARG] .

IV ESTIMATION OF GENUINE TRIPARTITE QUANTUM DISCORD

IV.1 von Neumann Entropies for \rhoitalic_ and A,Bsubscript\rho_A,Bitalic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT

Now, in order to give an analytical estimation of D(3)()superscript3D^(3)(\rho)italic_D start_POSTSUPERSCRIPT (3) end_POSTSUPERSCRIPT (italic_) for the state \rhoitalic_ described by Eq. (17), we calculate the von Neumann entropies for \rhoitalic_ and for the marginal A,B=TrC[]subscriptsubscriptTrdelimited-[]\rho_A,B=\mathrmTr_C[\rho]italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT = roman_Tr start_POSTSUBSCRIPT italic_C end_POSTSUBSCRIPT [italic_], which appears in the expression of GTQD given by Eq. (11).

From the definition of von Neumann entropy, it follows that:

S()\displaystyle S(\rho)italic_S (italic_) =3+18[2(3+a1)log2(3)-(1-a1-c1)log2(1-a1-c1)-(1-a1+c1)log2(1-a1+c1)fragments318fragments[2fragments(3subscript1)subscript2fragments(3)fragments(1subscript1subscript1)subscript2fragments(1subscript1subscript1)fragments(1subscript1subscript1)subscript2fragments(1subscript1subscript1)\displaystyle=3+\frac18\left[2(3+a_1)\log_2(3)-\left(1-a_1-c_1% \right)\log_2\left(1-a_1-c_1\right)-\left(1-a_1+c_1\right)\log_2% \left(1-a_1+c_1\right)\right.= 3 + divide start_ARG 1 end_ARG start_ARG 8 end_ARG [2 (3 + italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (3) - (1 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT - italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (1 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT - italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) - (1 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT + italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (1 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT + italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT)

-(3+a1-3c2)log2(3+a1-3c2)-(3+a1+3c2)log2(3+a1+3c2)].fragmentsfragments(3subscript13subscript2)subscript2fragments(3subscript13subscript2)fragments(3subscript13subscript2)subscript2fragments(3subscript13subscript2)].\displaystyle\left.-\left(3+a_1-3c_2\right)\log_2\left(3+a_1-3c_2% \right)-\left(3+a_1+3c_2\right)\log_2\left(3+a_1+3c_2\right)\right].- (3 + italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT - 3 italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (3 + italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT - 3 italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) - (3 + italic_a start_POSTSUBSCRIPT

1 end_POSTSUBSCRIPT + 3 italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (3 + italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT + 3 italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT)] . (19)

From Eq. (17) we obtain:

A,B=(3-a11200003+a11200003+a11200003-a112),subscript3subscript11200003subscript11200003subscript11200003subscript112\rho_A,B=\left(\beginarray[]cccc\frac3-a_112&0&0&0\\ 0&\frac3+a_112&0&0\\ 0&0&\frac3+a_112&0\\ 0&0&0&\frac3-a_112\endarray\right),italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT = (start_ARRAY start_ROW start_CELL divide start_ARG 3 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_ARG start_ARG 12 end_ARG end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL divide start_ARG 3 + italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_ARG start_ARG 12 end_ARG end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL divide start_ARG 3 + italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_ARG start_ARG 12 end_ARG end_CELL start_CELL 0 end_CELL end_ROW start_ROW start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL 0 end_CELL start_CELL divide start_ARG 3 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_ARG start_ARG 12 end_ARG end_CELL end_ROW end_ARRAY) , (20)

and after straightforward calculations we find:

S(A,B)subscript\displaystyle S(\rho_A,B)italic_S (italic_ start_POSTSUBSCRIPT italic_A , italic_B end_POSTSUBSCRIPT) =-16(3-a1)log2(3-a1)-16(3+a1)log2(3+a1)+2+log2(3).absent163subscript1subscript23subscript1163subscript1subscript23subscript12subscript23\displaystyle=-\frac16\left(3-a_1\right)\log_2\left(3-a_1\right)-%
\frac16\left(3+a_1\right)\log_2\left(3+a_1\right)+2+\log_2(3).= - divide start_ARG 1 end_ARG start_ARG 6 end_ARG (3 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (3 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) - divide start_ARG 1 end_ARG start_ARG 6 end_ARG (3 + italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (3 + italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) + 2 + roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (3) . (21)

IV.2 Relative entropy minimization

In order to finally evaluate D(3)()superscript3D^(3)(\rho)italic_D start_POSTSUPERSCRIPT (3) end_POSTSUPERSCRIPT (italic_) we need to calculate the relative entropy S(A|BC)subscriptconditionalS(\rho_A)italic_S (italic_ start_POSTSUBSCRIPT italic_A |

italic_B italic_C end_POSTSUBSCRIPT). Following the derivation procedure given in the Appendix A, S(A|BC)subscriptconditionalS(\rho_A)italic_S (italic_ start_POSTSUBSCRIPT italic_A | italic_B italic_C end_POSTSUBSCRIPT) can be written as:

S(A|BC)=mini,iSrel(1,2,1,2)==mini,i1+16[Alog2A+Blog2B]-112i=14ilog2i,subscriptconditionalsubscriptsubscriptitalic-subscriptsubscript1subscript2subscriptitalic-1subscriptitalic-2subscriptsubscriptitalic-116delimited-[]subscriptsubscript2subscriptsubscriptsubscript2subscript112superscriptsubscript14subscriptsubscript2subscriptS(\rho_BC)=\underset\theta_i,\phi_i\min\,S_rel(\theta_1,\theta% _2,\phi_1,\phi_2)=\\ =\underset\theta_i,\phi_i\min\left\1+\frac16\left[\lambda_A\log% _2\lambda_A+\lambda_B\log_2\lambda_B\right]-\frac112\sum_i=1^% 4\lambda_i\log_2\lambda_i\right\,start_ROW start_CELL italic_S (italic_ start_POSTSUBSCRIPT italic_A | italic_B italic_C end_POSTSUBSCRIPT) = start_UNDERACCENT italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT , italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT end_UNDERACCENT start_ARG roman_min end_ARG italic_S start_POSTSUBSCRIPT italic_r italic_e italic_l end_POSTSUBSCRIPT (italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT , italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT , italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT , italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) = end_CELL end_ROW start_ROW start_CELL = start_UNDERACCENT italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT , italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT end_UNDERACCENT start_ARG roman_min end_ARG 1 + divide start_ARG 1 end_ARG start_ARG 6 end_ARG [italic_ start_POSTSUBSCRIPT italic_A end_POSTSUBSCRIPT roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT italic_ start_POSTSUBSCRIPT italic_A end_POSTSUBSCRIPT + italic_ start_POSTSUBSCRIPT italic_B end_POSTSUBSCRIPT roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT italic_ start_POSTSUBSCRIPT italic_B end_POSTSUBSCRIPT] - divide start_ARG 1 end_ARG start_ARG 12 end_ARG start_POSTSUBSCRIPT italic_i = 1 end_POSTSUBSCRIPT start_POSTSUPERSCRIPT 4 end_POSTSUPERSCRIPT italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT , end_CELL end_ROW (22)

where isubscript\theta_iitalic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT and isubscriptitalic-\phi_iitalic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT are optimization parameters (the angles defining the basis vectors: see again Appendix A), and:

Asubscript\displaystyle\lambda_Aitalic_ start_POSTSUBSCRIPT italic_A end_POSTSUBSCRIPT =3+a1cos(21)cos(22),absent3subscript12subscript12subscript2\displaystyle=3+a_1\cos(2\theta_1)\cos(2\theta_2),= 3 + italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT roman_cos (2 italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) roman_cos (2 italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) , Bsubscript\displaystyle\lambda_Bitalic_ start_POSTSUBSCRIPT italic_B end_POSTSUBSCRIPT =3-a1cos(21)cos(22),absent3subscript12subscript12subscript2\displaystyle=3-a_1\cos(2\theta_1)\cos(2\theta_2),= 3 - italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT roman_cos (2 italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) roman_cos (2 italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) ,

Csubscript\displaystyle\lambda_Citalic_ start_POSTSUBSCRIPT italic_C end_POSTSUBSCRIPT =916sin2(21)sin2(22)[(c1-c2)2+4c2(cos(1)+cos(2))(c2cos(1)+c1cos(2))],absent916superscript22subscript1superscript22subscript2delimited-[]superscriptsubscript1subscript224subscript2subscriptitalic-1subscriptitalic-2subscript2subscriptitalic-1subscript1subscriptitalic-2\displaystyle=\frac916\sin^2(2\theta_1)\sin^2(2\theta_2)\left[% \left(c_1-c_2\right)^2+4c_2\left(\cos(\phi_1)+\cos(\phi_2)\right)% \left(c_2\cos(\phi_1)+c_1\cos(\phi_2)\right)\right],= divide start_ARG 9 end_ARG start_ARG 16 end_ARG roman_sin start_POSTSUPERSCRIPT 2 end_POSTSUPERSCRIPT (2 italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) roman_sin start_POSTSUPERSCRIPT 2 end_POSTSUPERSCRIPT (2 italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) [(italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT - italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) start_POSTSUPERSCRIPT 2 end_POSTSUPERSCRIPT + 4 italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (roman_cos (italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) + roman_cos (italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT)) (italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT roman_cos (italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) + italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT roman_cos (italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT))] , (23)

1,2subscript12\displaystyle\lambda_1,2italic_ start_POSTSUBSCRIPT 1 , 2 end_POSTSUBSCRIPT =B±a12(cos(21)+cos(22))2+C,absentplus-or-minussubscriptsuperscriptsubscript12superscript2subscript12subscript22subscript\displaystyle=\lambda_B\pm\sqrta_1^2\left(\cos(2\theta_1)+\cos(2% \theta_2)\right)^2+\lambda_C,= italic_ start_POSTSUBSCRIPT italic_B end_POSTSUBSCRIPT ± square-root start_ARG italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT start_POSTSUPERSCRIPT 2 end_POSTSUPERSCRIPT (roman_cos (2 italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) + roman_cos (2 italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT)) start_POSTSUPERSCRIPT 2 end_POSTSUPERSCRIPT + italic_ start_POSTSUBSCRIPT italic_C end_POSTSUBSCRIPT end_ARG ,

3,4subscript34\displaystyle\lambda_3,4italic_ start_POSTSUBSCRIPT 3 , 4 end_POSTSUBSCRIPT =A±a12(cos(21)-cos(22))2+C.absentplus-or-minussubscriptsuperscriptsubscript12superscript2subscript12subscript22subscript\displaystyl

$$e=\lambda_A\pm\sqrt{a_1^2\left(\cos(2\theta_1)-\cos(2\theta_2)\right)^2+\lambda_C}.$$

The optimization of S_{rel} is an hard task, and cannot be performed fully analytically in a simple way. Indeed, it has been proven that in a bipartite system the optimization of the relative entropy (for a general density matrix) involves the solution of equations containing logarithms of nonlinear quantities, that cannot be obtained analytically (see for instance (Girolami and Adesso, 2011; Javad Akhtarshenas et al., 2013)). This is the reason why we developed a numerical approach to the minimization, whose results have been used as guidelines to give an analytical expression for S_{rel}. A similar method has already been adopted independently to estimate the quantum discord of two-qutrit Werner states in Ref. (Ye et al., 2013).

First, in our procedure, we generate randomly a suitable number of triplets (a_1,c_1,c_2) (obeying to the constraints of Eq. (18)), and then we minimize numerically the corresponding expression of $S_{rel}(\theta_1,\theta_2,\phi_1,\phi_2)$ over a grid of points in the 4D-space $\mathbb{U}=R_{\theta_1}\times R_{\theta_2}\times R_{\phi_1}\times R_{\phi_2}$, where R_{θ_i} and R_{ϕ_i} are the intervals $[0;\pi)$ and $[0;2\pi)$ respectively, given the periodicity of the functions in Eqs. (23). The optimization procedure, which has been shown to be an NP-complete problem (Huang, 2014), was performed using exhaustive enumeration (i.e. brute force search) over a grid in the \mathbbUblackboard_U space, to be sure to find the true absolute minima of SrelsubscriptS_relitalic_S start_POSTSUBSCRIPT italic_r italic_e italic_l end_POSTSUBSCRIPT. Our calculations indicate that the function SrelsubscriptS_relitalic_S start_POSTSUBSCRIPT italic_r italic_e italic_l end_POSTSUBSCRIPT exhibits many equivalent absolute minima, and that the “first” one (i.e. the one with the lowest values of its coordinates) is always reached for 1=2=subscript1subscript2\theta_1=\theta_2=\thetaitalic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT = italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT = italic_ and 1=0subscriptitalic-10\phi_1=0italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT = 0. Specifically, it is found alternatively in one of these three points (1,2,1,2)subscript1subscript2subscriptitalic-1subscriptitalic-2(\theta_1,\theta_2,\phi_1,\phi_2)(italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT , italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT , italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT , italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) of \mathbbUblackboard_U: (0,0,0,0)0000(0,0,0,0)(0 , 0 , 0 , 0), (4,4,0,0)4400(\frac\pi4,\frac\pi4,0,0)(divide start_ARG italic_ end_ARG start_ARG 4 end_ARG , divide start_ARG italic_ end_ARG start_ARG 4 end_ARG , 0 , 0) or (4,4,0,¯2)440subscript¯italic-2(\frac\pi4,\frac\pi4,0,\bar\phi_2)(divide start_ARG italic_ end_ARG start_ARG 4 end_ARG , divide start_ARG italic_ end_ARG start_ARG 4 end_ARG , 0 , over¯ start_ARG italic_ end_ARG start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT), where ¯2subscript¯italic-2\bar\phi_2over¯ start_ARG italic_ end_ARG start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT depends upon (a1,c1,c2)subscript1subscript1subscript2(a_1,c_1,c_2)(italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT , italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT , italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) 222Notice that when i=0subscript0\theta_i=0italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT = 0 other equivalent minima can be found for i=2subscript2\theta_i=\frac\pi2italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT = divide start_ARG italic_ end_ARG start_ARG 2 end_ARG or i=subscript\theta_i=\piitalic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT = italic_, and when i=4subscript4\theta_i=\frac\pi4italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT = divide start_ARG italic_ end_ARG start_ARG 4 end_ARG other equivalent minima can be found for i=34subscript34\theta_i=\frac3\pi4italic_ start_POSTSUBSCRIPT italic_i end_POSTSUBSCRIPT = divide start_ARG 3 italic_ end_ARG start_ARG 4 end_ARG, but we will focus only on the cases =00\theta=0italic_ = 0 or =44\theta=\frac\pi4italic_ = divide start_ARG italic_ end_ARG start_ARG 4 end_ARG, which are the simpler ones.. This means that the minimal relative entropy S(A|BC)subscriptconditionalS(\rho_A)italic_S (italic_ start_POSTSUBSCRIPT italic_A | italic_B italic_C end_POSTSUBSCRIPT) can take only three possible analytical forms (provided that one can find an analytical expression for ¯2subscript¯italic-2\bar\phi_2over¯ start_ARG italic_ end_ARG start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT).

Starting from these numerical results, we performed an analytical study on the specific case of Srel(,,1,2)subscriptsubscriptitalic-1subscriptitalic-2S_rel(\theta,\theta,\phi_1,\phi_2)italic_S start_POSTSUBSCRIPT italic_r italic_e italic_l end_POSTSUBSCRIPT (italic_ , italic_ , italic_ start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT , italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT), which confirmed that this function has two extrema in =00\theta=0italic_ = 0 and =44\theta=\frac\pi4italic_ = divide start_ARG italic_ end_ARG start_ARG 4 end_ARG. Moreover, our analytical approach showed that the function Srel(4,4,0,2)subscript440subscriptitalic-2S_rel(\frac\pi4,\frac\pi4,0,\phi_2)italic_S start_POSTSUBSCRIPT italic_r italic_e italic_l end_POSTSUBSCRIPT (divide start_ARG italic_ end_ARG start_ARG 4 end_ARG , divide start_ARG italic_ end_ARG start_ARG 4 end_ARG , 0 , italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) attains its minimum value for sin(2)=0subscriptitalic-20\sin(\phi_2)=0roman_sin (italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) = 0 or cos(2)=(-c1+c22c1)subscriptitalic-2subscript1subscript22subscript1\cos(\phi_2)=\left(-\fracc_1+c_22c_1\right)roman_cos (italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) = (- divide start_ARG italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT + italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT end_ARG start_ARG 2 italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_ARG), (which holds only if certain conditions are satisfied - see Eq. (60) in the Appendix B). This is consistent with numerical calculations, which give as minimum 2=0subscriptitalic-20\phi_2=0italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT = 0 or 2=¯2=arccos(-c1+c22c1)subscriptitalic-2subscript¯italic-2subscript1subscript22subscript1\phi_2=\bar\phi_2=\arccos\left(-\fracc_1+c_22c_1\right)italic_ start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT = over¯ start_ARG italic_ end_ARG start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT = roman_arccos (- divide start_ARG italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT + italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT end_ARG start_ARG 2 italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_ARG). Further details are given in the Appendix B.

Our derivation leads to the following expressions for the minimum values of SrelsubscriptS_relitalic_S start_POSTSUBSCRIPT italic_r italic_e italic_l end_POSTSUBSCRIPT:

S1=Srel(0,0,0,0)subscript1subscript0000\displaystyle S_1=S_rel(0,0,0,0)italic_S start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT = italic_S start_POSTSUBSCRIPT italic_r italic_e italic_l end_POSTSUBSCRIPT (0 , 0 , 0 , 0) =1-112(a1),absent1112subscript1\displaystyle=1-\frac112\gamma(a_1),= 1 - divide start_ARG 1 end_ARG start_ARG 12 end_ARG italic_ (italic_a start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT) ,

S2=Srel(4,4,0,0)subscript2subscript4400\displaystyle S_2=S_rel(\textstyle\frac\pi4,\textstyle\frac\pi4% ,0,0)italic_S start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT = italic_S start_POSTSUBSCRIPT italic_r italic_e italic_l end_POSTSUBSCRIPT (divide start_ARG italic_ end_ARG start_ARG 4 end_ARG , divide start_ARG italic_ end_ARG start_ARG 4 end_ARG , 0 , 0) =1-

12(3c2+c14),absent1123subscript2subscript14\displaystyle=1-\frac12\varepsilon\left(\textstyle\frac3c_2+c_14% \right),= 1 - divide start_ARG 1 end_ARG start_ARG 2 end_ARG italic_ (divide start_ARG 3 italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT + italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_ARG start_ARG 4 end_ARG) , (24)

S3=Srel(4,4,0,¯2)subscript3subscript440subscript¯italic-2\displaystyle S_3=S_rel(\textstyle\frac\pi4,\textstyle\frac\pi4% ,0,\bar\phi_2)italic_S start_POSTSUBSCRIPT 3 end_POSTSUBSCRIPT = italic_S start_POSTSUBSCRIPT italic_r italic_e italic_l end_POSTSUBSCRIPT (divide start_ARG italic_ end_ARG start_ARG 4 end_ARG , divide start_ARG italic_ end_ARG start_ARG 4 end_ARG , 0 , over¯ start_ARG italic_ end_ARG start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) =1-12(14(c1-c2)3c1),absent11214superscriptsubscript1subscript23subscript1\displaystyle=1-\frac12\varepsilon\left(\textstyle\frac14\sqrt\frac% (c_1-c_2)^3c_1\right),= 1 - divide start_ARG 1 end_ARG start_ARG 2 end_ARG italic_ (divide start_ARG 1 end_ARG start_ARG 4 end_ARG square-root start_ARG divide start_ARG (italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT - italic_c start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT) start_POSTSUPERSCRIPT 3 end_POSTSUPERSCRIPT end_ARG start_ARG italic_c start_POSTSUBSCRIPT 1 end_POSTSUBSCRIPT end_ARG end_ARG) ,

where

(x)=(3+x)log2(3+x)+(3-3x)log2(3-3x)-2(3-x)log2(3-x),3subscript2333subscript23323subscript23\gamma(x)=(3+x)\log_2(3+x)+(3-3x)\log_2(3-3x)-2(3-x)\log_2(3-x),italic_ (italic_x) = (3 + italic_x) roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (3 + italic_x) + (3 - 3 italic_x) roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (3 - 3 italic_x) - 2 (3 - italic_x) roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (3 - italic_x) , (25)

(x)=(1+x)log2(1+x)+(1-x)log2(1-x).1subscript211subscript21\varepsilon(x)=(1+x)\log_2(1+x)+(1-x)\log_2(1-x).italic_ (italic_x) = (1 + italic_x) roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (1 + italic_x) + (1 - italic_x) roman_log start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT (1 - italic_x) . (26)

When both S2subscript2S_2italic_S start_POSTSUBSCRIPT 2 end_POSTSUBSCRIPT and S3subscript3S_3italic_S start_POSTSUBSCRIPT 3 end_POSTSUBSCRIPT are well defined expressions, we found with additional analytical calculations that S3 S(A|BC)=\>\textrmand\>c_1\cdot c_2