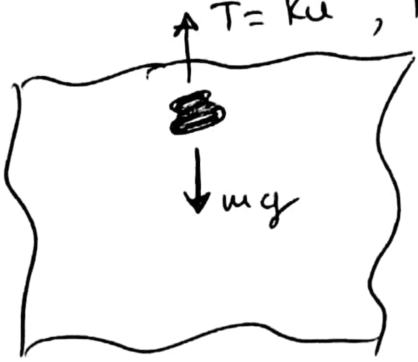
Θέμα 2º

$T = Ku^2$, $K = const. > 0$ and $u = \frac{\partial s}{\partial t}$

$t = 0 \to u = 0$



$$\Sigma F = ma \Rightarrow mg - Ku^2 = m\frac{\partial u}{\partial t} \Rightarrow$$

$$\Rightarrow u^2 - \frac{mg}{K} = -\frac{\partial u}{\partial t} \Rightarrow -\partial t = \frac{1}{u^2 - \frac{mg}{K}}\partial u \Rightarrow$$

$$\Rightarrow -\int_0^t \partial t = \int_0^u \frac{1}{\left(u - \sqrt{\frac{mg}{K}}\right)\left(u + \sqrt{\frac{mg}{K}}\right)}\partial u \Rightarrow$$

$$\Rightarrow -t = \int_0^u \frac{\partial u}{\left(u - \sqrt{\frac{mg}{K}}\right)\left(u + \sqrt{\frac{mg}{K}}\right)}$$

$$\frac{1}{\left(u - \sqrt{\frac{mg}{K}}\right)\left(u + \sqrt{\frac{mg}{K}}\right)} = \frac{A}{u - \sqrt{\frac{mg}{K}}} + \frac{B}{u + \sqrt{\frac{mg}{K}}} \Rightarrow$$

$$\Rightarrow 1 = A\left(u+\sqrt{\frac{mg}{k}}\right) + B\left(u-\sqrt{\frac{mg}{k}}\right) \Rightarrow$$

$$\Rightarrow 1 = Au + A\sqrt{\frac{mg}{k}} + Bu - B\sqrt{\frac{mg}{k}} \Rightarrow$$

$$\Rightarrow 1 = (A+u)u + (A-B)\sqrt{\frac{mg}{k}}$$

$$A+B = 0 \Rightarrow A = -B \Rightarrow \boxed{A = \frac{1}{2\sqrt{\frac{mg}{k}}}}$$

$$(A-B)\sqrt{\frac{mg}{k}} = 1 \Rightarrow -2B\sqrt{\frac{mg}{k}} = 1 \Rightarrow \boxed{B = -\frac{1}{2\sqrt{\frac{mg}{k}}}}$$

$$-t = \int_0^u \frac{\frac{1}{2\sqrt{\frac{mg}{k}}}}{u-\sqrt{\frac{mg}{k}}}\,\partial u - \int_0^u \frac{\frac{1}{2\sqrt{\frac{mg}{k}}}}{u+\sqrt{\frac{mg}{k}}}\,\partial u \Rightarrow$$

$$\Rightarrow -t = \frac{1}{2\sqrt{\frac{mg}{k}}}\int_0^u \frac{1}{u-\sqrt{\frac{mg}{k}}}\,\partial u - \frac{1}{2\sqrt{\frac{mg}{k}}}\int_0^u \frac{1}{u+\sqrt{\frac{mg}{k}}}\,\partial u \rightarrow$$

$$-t = \frac{1}{2\sqrt{\frac{mg}{k}}} \cdot \ln\left|\frac{u-\sqrt{\frac{mg}{k}}}{-\sqrt{\frac{mg}{k}}}\right| - \frac{1}{2\sqrt{\frac{mg}{k}}}\ln\left|\frac{u+\sqrt{\frac{mg}{k}}}{\sqrt{\frac{mg}{k}}}\right| \Rightarrow$$

$$\Rightarrow -t = \frac{1}{2\sqrt{\frac{mg}{k}}} \cdot \left[\ln\left|\frac{\dfrac{u-\sqrt{\frac{mg}{k}}}{-\sqrt{\frac{mg}{k}}}}{\dfrac{u+\sqrt{\frac{mg}{k}}}{\sqrt{\frac{mg}{k}}}}\right|\right] \Rightarrow$$

$$\Rightarrow e^{-2t\sqrt{\frac{mg}{k}}} = \left|\frac{\dfrac{u-\sqrt{\frac{mg}{k}}}{-\sqrt{\frac{mg}{k}}}}{\dfrac{u+\sqrt{\frac{mg}{k}}}{\sqrt{\frac{mg}{k}}}}\right| \Rightarrow$$

$$\Rightarrow e^{-2t\sqrt{\frac{mg}{k}}} = \left|-\frac{u-\sqrt{\frac{mg}{k}}}{u+\sqrt{\frac{mg}{k}}}\right|$$

[redacted]

Case 1

$$e^{-2t\sqrt{\frac{mg}{k}}} = -\frac{u-\sqrt{\frac{mg}{k}}}{u+\sqrt{\frac{mg}{k}}} \Rightarrow$$

$$\Rightarrow u e^{-2t\sqrt{\frac{mg}{k}}} + e^{-2t\sqrt{\frac{mg}{k}}}\sqrt{\frac{mg}{k}} = -u + \sqrt{\frac{mg}{k}} \Rightarrow$$

$$\Rightarrow u\left(e^{-2t\sqrt{\frac{mg}{k}}} + 1\right) = \sqrt{\frac{mg}{k}}\left(1 - e^{-2t\sqrt{\frac{mg}{k}}}\right) \Rightarrow$$

$$\Rightarrow u = \frac{\sqrt{\frac{mg}{k}}\left(1-e^{-2t\sqrt{\frac{mg}{k}}}\right)}{e^{-2t\sqrt{\frac{mg}{k}}}+1}$$

Case 2

$$e^{-2t\sqrt{\frac{mg}{k}}} = \frac{u-\sqrt{\frac{mg}{k}}}{u+\sqrt{\frac{mg}{k}}} \Rightarrow u e^{-2t\sqrt{\frac{mg}{k}}} + \sqrt{\frac{mg}{k}}\, e^{-2t\sqrt{\frac{mg}{k}}} =$$

$$= u - \sqrt{\frac{mg}{k}} \Rightarrow$$

$$\Rightarrow u\left(e^{-2t\sqrt{\frac{mg}{k}}} - 1\right) = -\sqrt{\frac{mg}{k}} - \sqrt{\frac{mg}{k}}\, e^{-2t\sqrt{\frac{mg}{k}}} \Rightarrow$$

$$\Rightarrow u = -\frac{\sqrt{\frac{mg}{k}}\left(1 + e^{-2t\sqrt{\frac{mg}{k}}}\right)}{e^{-2t\sqrt{\frac{mg}{k}}} - 1}$$

For $t \to +\infty \Rightarrow e^{-2t\sqrt{\frac{mg}{k}}} \longrightarrow \emptyset$

$$U_L = \sqrt{\frac{mg}{k}}$$