5.1 Tensor Representation

This section introduces the idea that geometric-algebra operations are essentially bilinear functions and makes this notion explicit. In later chapters, this representation forms the foundation for the description of random multivector variables and statistical optimizations. Using this form, differentiation and integration in geometric algebra can be related right away to these operations in linear algebra.

Recall that the geometric algebra $\mathbb{G}_{p,q}$ has dimension 2^n. Let $\{\boldsymbol{E}_i\} := \overline{\mathbb{G}}_{p,q}$ denote the canonical algebraic basis, whereby $\boldsymbol{E}_1 \equiv 1$. For example, the algebraic basis of $\mathbb{G}_3$ is given in Table 5.1.

Table 5.1 Algebra basis of $\mathbb{G}_3$, where the geometric product of basis vectors is denoted by combining their indices, i.e. $\boldsymbol{e}_1\,\boldsymbol{e}_2 \equiv \boldsymbol{e}_{12}$

Type	**No.**	**Basis Elements**
Scalar	1	1
Vector	3	$\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$
2-Vector	3	$\boldsymbol{e}_{23}, \boldsymbol{e}_{31}, \boldsymbol{e}_{12}$
3-Vector	1	$\boldsymbol{e}_{123}$

A multivector $\boldsymbol{A} \in \mathbb{G}_{p,q}$ can then be written as $\boldsymbol{A} = a^i\,\boldsymbol{E}_i$, where a^i denotes the ith component of a vector $\mathsf{a} \in \mathbb{R}^{2^n}$ and a sum over the repeated index i is implied. Since the $\{\boldsymbol{E}_i\}$ form an algebraic basis, it follows that the geometric product of two basis blades has to result in a basis blade, i.e.

$$\boldsymbol{E}_i\,\boldsymbol{E}_j = \mathit{\Gamma}^k{}_{ij}\,\boldsymbol{E}_k, \qquad \forall\, i,j \in \{1,\dots,2^2\}, \tag{5.1}$$

where $\mathit{\Gamma}^k{}_{ij} \in \mathbb{R}^{2^n\times 2^n\times 2^n}$ is a tensor encoding the geometric product. Here, the following notation is being used. The expression $\mathit{\Gamma}^k{}_{ij}$ denotes either the element of the tensor at the location (k,i,j) or the ordered set of all elements of the tensor. Which form is meant should be clear from the context in which the symbol is used. All indices in an expression that appear only once and are also not otherwise defined to take on a particular value indicate the ordered set over all values of this index. If an element has one undefined index, it denotes a column vector, and if it has two undefined indices it denotes a matrix, where the first index gives the row and the second index the column.

For example, the expression $b^j\,\mathit{\Gamma}^k{}_{ij}$ has two undefined indices k and i and thus represents a matrix with indices k and i. Since k is the first index, it denotes the row and i denotes the column of the resultant matrix. Similarly, $a^i\,b^j\,\mathit{\Gamma}^k{}_{ij}$ denotes a column vector with the index k.

Table 5.2 Tensor symbols for algebraic operations, and the corresponding Jacobi matrices. For tensors with two indices (i.e. matrices), the first index denotes the matrix row and the second index the matrix column

Operation	Tensor symbol	Jacobi matrices
Geometric Product	${\Gamma^k}_{ij}$	$\Gamma_R(\mathsf{a}) := a^i\, {\Gamma^k}_{ij}$ $\Gamma_L(\mathsf{b}) := b^j\, {\Gamma^k}_{ij}$
Outer Product	${\Lambda^k}_{ij}$	$\Lambda_R(\mathsf{a}) := a^i\, {\Lambda^k}_{ij}$ $\Lambda_L(\mathsf{b}) := b^j\, {\Lambda^k}_{ij}$
Inner Product	${\Theta^k}_{ij}$	$\Theta_R(\mathsf{a}) := a^i\, {\Theta^k}_{ij}$ $\Theta_L(\mathsf{b}) := b^j\, {\Theta^k}_{ij}$
Reverse	${R^j}_i$	$\mathsf{R} := {R^j}_i$
Dual	${D^j}_i$	$\mathsf{D} := {D^j}_i$

5.1.1 Component Vectors

If $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{G}_{p,q}$ are defined as $\boldsymbol{A} := a^i\, \boldsymbol{E}_i$, $\boldsymbol{B} := b^i\, \boldsymbol{E}_i$, and $\boldsymbol{C} := c^i\, \boldsymbol{E}_i$, then it follows from (5.1) that the components of $\boldsymbol{C}$ in the algebraic equation $\boldsymbol{C} = \boldsymbol{A}\,\boldsymbol{B}$ can be evaluated via

$$c^k = a^i\, b^j\, {\Gamma^k}_{ij}, \tag{5.2}$$

where a summation over i and j is again implied. Such a summation of tensor indices is called *contraction*. Equation (5.2) shows that the geometric product is simply a bilinear function. In fact, all products in geometric algebra that are of interest in this text can be expressed in this form, as will be discussed later on.

The geometric product can also be written purely in matrix notation, by defining the matrices

$$\Gamma_R(\mathsf{a}) := a^i\, {\Gamma^k}_{ij} \qquad \text{and} \qquad \Gamma_L(\mathsf{b}) := b^j\, {\Gamma^k}_{ij}. \tag{5.3}$$

The geometric product $\boldsymbol{A}\,\boldsymbol{B}$ can now be written as

$$a^i\, b^j\, {\Gamma^k}_{ij} = \Gamma_R(\mathsf{a})\; \mathsf{b} = \Gamma_L(\mathsf{b})\; \mathsf{a}. \tag{5.4}$$

Note that the matrices $\Gamma_R(\mathsf{a})$ and $\Gamma_L(\mathsf{b})$ are the two Jacobi matrices of the expression $c^k := a^i\, b^j\, {\Gamma^k}_{ij}$. That is,

$$\frac{\partial\, c^k}{\partial\, a^i} = b^j\, {\Gamma^k}_{ij} = \Gamma_L(\mathsf{b}) \qquad \text{and} \qquad \frac{\partial\, c^k}{\partial\, b^j} = a^i\, {\Gamma^k}_{ij} = \Gamma_R(\mathsf{a})\,. \tag{5.5}$$

At this point it is useful to introduce a notation that describes the mapping between multivectors and their corresponding component vectors. For this purpose, the operator $\mathcal{K}$ is introduced. For the algebra $\mathbb{G}_{p,q}$, $\mathcal{K}$ is the bijective mapping

$$\mathcal{K} \;:\; \mathbb{G}_{p,q} \;\longrightarrow\; \mathbb{R}^{2^{p+q}} \qquad \text{and} \qquad \mathcal{K}^{-1} \;:\; \mathbb{R}^{2^{p+q}} \;\longrightarrow\; \mathbb{G}_{p,q}. \tag{5.6}$$

For a multivector $\boldsymbol{A} \in \mathbb{G}_{p,q}$ with $\boldsymbol{A} := a^i\, \boldsymbol{E}_i$, the operator is defined as

$$\mathcal{K} \;:\; \boldsymbol{A} \mapsto \mathsf{a} \qquad \text{and} \qquad \mathcal{K}^{-1} \;:\; \mathsf{a} \mapsto \boldsymbol{A}. \tag{5.7}$$

It follows that

$$\mathcal{K}(\boldsymbol{A}\,\boldsymbol{B}) = \mathit{\Gamma}_R\left(\mathcal{K}(\boldsymbol{A})\right)\,\mathcal{K}(\boldsymbol{B}) = \mathit{\Gamma}_L\left(\mathcal{K}(\boldsymbol{B})\right)\,\mathcal{K}(\boldsymbol{A}) \tag{5.8}$$

and

$$\mathcal{K}^{-1}\big(\mathit{\Gamma}_R\,(\mathsf{a})\;\mathsf{b}\big) = \mathcal{K}^{-1}\big(\mathit{\Gamma}_L\,(\mathsf{b})\;\mathsf{a}\big) = \mathcal{K}^{-1}(\mathsf{a})\,\mathcal{K}^{-1}(\mathsf{b}). \tag{5.9}$$

The mapping $\mathcal{K}$ is therefore an isomorphism between $\mathbb{G}_{p,q}$ and $\mathbb{R}^{2^{(p+q)}}$.

In addition to the geometric product, the inner and outer products and the reverse and dual are important operations in geometric algebra. The corresponding operation tensors are therefore given specific symbols, which are listed in Table 5.2.

5.1.2 Example: Geometric Product in $\mathbb{G}_2$

A simple example of a product tensor is the geometric-product tensor of $\mathbb{G}_2$. The algebraic basis may be defined as

$$\boldsymbol{E}_1 := 1, \quad \boldsymbol{E}_2 := \boldsymbol{e}_1, \quad \boldsymbol{E}_3 := \boldsymbol{e}_2, \quad \boldsymbol{E}_4 := \boldsymbol{e}_{12}. \tag{5.10}$$

The geometric-product tensor $\mathit{\Gamma}^k{}_{ij} \in \mathbb{R}^{4\times4\times4}$ of $\mathbb{G}_2$ then takes on the form

$$\begin{aligned}
\mathit{\Gamma}^1{}_{ij} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \;\mathit{\Gamma}^2{}_{ij} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\
\mathit{\Gamma}^3{}_{ij} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \;\mathit{\Gamma}^4{}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},
\end{aligned} \tag{5.11}$$

where i is the row index and j is the column index. Let $\boldsymbol{A} = \boldsymbol{e}_1$ and $\boldsymbol{B} = \boldsymbol{e}_2$; then $\mathsf{a} = [\,0\,,\ 1\,,\ 0\,,\ 0]^{\mathsf{T}}$ and $\mathsf{b} = [\,0\,,\ 0\,,\ 1\,,\ 0]^{\mathsf{T}}$. Thus

$$\left.\begin{aligned} b^j\,\Gamma^1{}_{ij} &= [\,0\,,\;0\,,\;1\,,\;0\,]^{\mathsf{T}} \\ b^j\,\Gamma^2{}_{ij} &= [\,0\,,\;0\,,\;0\,,\;1\,]^{\mathsf{T}} \\ b^j\,\Gamma^3{}_{ij} &= [\,1\,,\;0\,,\;0\,,\;0\,]^{\mathsf{T}} \\ b^j\,\Gamma^4{}_{ij} &= [\,0\,,\;1\,,\;0\,,\;0\,]^{\mathsf{T}} \end{aligned}\right\} \quad\Longrightarrow\quad b^j\,\Gamma^k{}_{ij} = \begin{bmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{bmatrix}. \tag{5.12}$$

In the resultant matrix, k is the row index and i is the column index. It follows from (5.12) that

$$a^i\,b^j\,\Gamma^k{}_{ij} = [\,0\,,\;0\,,\;0\,,\;1\,]^{\mathsf{T}} \qquad \cong \qquad \boldsymbol{e}_{12} = \boldsymbol{E}_4, \tag{5.13}$$

where $\cong$ denotes isomorphism.

5.1.3 Subspace Projection

Depending on the particular algebra, the corresponding product tensor can become very large. For example, in the algebra $\mathbb{G}_{4,1}$ of conformal space, the geometric-product tensor $\Gamma^k{}_{ij} \in \mathbb{R}^{32\times32\times32}$ has 32 768 components, most of which are zero. When one is performing actual computations with such a tensor on a computer, it can reduce the computational load considerably if the tensor is reduced to only those components that are actually needed. Furthermore, such a projection process can also be used to implement constraints, as will be seen later.

Let $\mathsf{m} \in \{1, \ldots, 2^{(p+q)}\}^r$ denote a vector of r indices that index those basis elements $\{\boldsymbol{E}_i\} \subset \mathbb{G}_{p,q}$ which are needed in a calculation. Also, let us define $\mathsf{e}_u := \mathcal{K}(\boldsymbol{E}_u)$ such that $e^i{}_u = \delta^i{}_u$, where $\delta^i{}_u$ denotes the Kronecker delta, defined as

$$\delta^i{}_u := \begin{cases} 1 \,:\, i = u\,, \\ 0 \,:\, i \neq u\,. \end{cases}$$

A corresponding projection matrix M is then defined as

$$M^j{}_i := \mathcal{K}\big(\boldsymbol{E}_{m^j}\big)^i = e^i{}_{m^j}\,, \qquad M^j{}_i \in \mathbb{R}^{r\times 2^{(p+q)}}, \tag{5.14}$$

where m^j denotes the jth element of m. A multivector $\boldsymbol{A} \in \mathbb{G}_{p,q}$ with $\mathsf{a} = \mathcal{K}(\boldsymbol{A})$ may be mapped to an r-dimensional component vector $\mathsf{a}_M \in \mathbb{R}^r$ related to those basis blades which are indexed by m, via

$$a^j_M = a^i\;M^j{}_i. \tag{5.15}$$

The reduced vector a^j_M can be mapped back to a component vector on the full basis through

$$a^j = a^i_M\;\tilde{M}^j{}_i, \tag{5.16}$$

where $\tilde{M}^j{}_i$ denotes the transpose of $M^j{}_i$.

Suppose $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{G}_{p,q}$ are related by $\boldsymbol{C} = \boldsymbol{A}\,\boldsymbol{B}$. In terms of component vectors $\mathsf{a} = \mathcal{K}(\boldsymbol{A})$, $\mathsf{b} = \mathcal{K}(\boldsymbol{B})$, and $\mathsf{c} = \mathcal{K}(\boldsymbol{C})$, this can be written as $c^k = a^i\, b^j\, \Gamma^k{}_{ij}$. If it can be assumed that only a subset of the elements of a^i and b^j are non-zero, the evaluation of this operation can be reduced as follows. Let $M_a^j{}_i$ and $M_b^j{}_i$ denote the projection matrices for a and b that map only the non-zero components to the reduced component vectors a_M and b_M, respectively. Clearly, only a subset of the components of the resultant vector c will be non-zero. The appropriate projection matrix and reduced component vector are denoted by $M_c^j{}_i$ and c_M. The geometric-product operation for the reduced component vectors then becomes

$$c_M^w = a_M^u\, b_M^v\, M_c^w{}_k\, \tilde{M}_a^i{}_u\, \tilde{M}_b^j{}_v\, \Gamma^k{}_{ij}. \tag{5.17}$$

This can be written equivalently as

$$c_M^w = a_M^u\, b_M^v\, \Gamma_M^w{}_{uv}, \qquad \Gamma_M^w{}_{uv} := M_c^w{}_k\, \tilde{M}_a^i{}_u\, \tilde{M}_b^j{}_v\, \Gamma^k{}_{ij}. \tag{5.18}$$

That is, $\Gamma_M^w{}_{uv}$ encodes the geometric product for the reduced component vectors. In applications, it is usually known which components of the constituent multivectors in an equation are non-zero. Therefore, reduced product tensors can be precalculated.

5.1.4 Example: Reduced Geometric Product

Consider again the algebra $\mathbb{G}_2$ with basis

$$\boldsymbol{E}_1 := 1, \quad \boldsymbol{E}_2 := \boldsymbol{e}_1, \quad \boldsymbol{E}_3 := \boldsymbol{e}_2, \quad \boldsymbol{E}_4 := \boldsymbol{e}_{12}.$$

The geometric-product tensor $\Gamma^k{}_{ij} \in \mathbb{R}^{4\times4\times4}$ of $\mathbb{G}_2$ is given by (5.11). Suppose $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{G}_2^1$; that is, they are linear combinations of $\boldsymbol{E}_2$ and $\boldsymbol{E}_3$. It is clear that the result of the geometric product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is a linear combination of $\boldsymbol{E}_1$ and $\boldsymbol{E}_4$, i.e. $\boldsymbol{C} = \boldsymbol{A}\,\boldsymbol{B} \in \mathbb{G}_2^+$. Therefore,

$$M_a^j{}_i = M_b^j{}_i = \begin{bmatrix} 0\,1\,0\,0 \\ 0\,0\,1\,0 \end{bmatrix}, \qquad M_c^j{}_i = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,0\,1 \end{bmatrix}. \tag{5.19}$$

The reduced geometric-product tensor is thus given by

$$\Gamma_M^w{}_{uv} := M_c^w{}_k\, \tilde{M}_a^i{}_u\, \tilde{M}_b^j{}_v\, \Gamma^k{}_{ij}, \tag{5.20}$$

where

$$\Gamma_M^1{}_{uv} = \begin{bmatrix} 1\;0 \\ 0\;1 \end{bmatrix}, \qquad \Gamma_M^2{}_{uv} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{5.21}$$

If $\boldsymbol{A} = \boldsymbol{e}_2 = \boldsymbol{E}_3$ and $\boldsymbol{B} = \boldsymbol{e}_1 = \boldsymbol{E}_2$, the reduced component vectors are $\mathsf{a}_M = [\,0\,,\;1\,]^\mathsf{T}$ and $\mathsf{b}_M = [\,1\,,\;0\,]^\mathsf{T}$. Thus

$$\left.\begin{aligned} b_M^v\;\Gamma^1_{M\,uv} &= [\,1\,,\;0\,]^\mathsf{T} \\ b_M^v\;\Gamma^2_{M\,uv} &= [\,0\,,\;-1\,]^\mathsf{T} \end{aligned}\right\} \Longrightarrow b_M^v\;\Gamma^w_{M\,uv} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{5.22}$$

It follows from this that

$$c_M^w = a_M^u\;b_M^v\;\Gamma^w_{M\,uv} = [\,0\,,\;-1\,]^\mathsf{T}\,. \tag{5.23}$$

Mapping c_M^w back to the non-reduced form gives

$$c^k = c_M^w\;\tilde{M}^k_{c\;w} = [\,0\,,\;0\,,\;0\,,\;-1\,]^\mathsf{T} \quad \cong \quad -\boldsymbol{e}_{12} = -\boldsymbol{E}_4. \tag{5.24}$$

5.1.5 Change of Basis

Let $\{\boldsymbol{F}_i\} \subset \mathbb{G}_{p,q}$ denote an algebraic basis of $\mathbb{G}_{p,q}$ which is different from the canonical algebraic basis $\{\boldsymbol{E}_i\}$ of $\mathbb{G}_{p,q}$. Given a multivector $\boldsymbol{A} \in \mathbb{G}_{p,q}$, its component vectors in the $\boldsymbol{E}$-basis and $\boldsymbol{F}$-basis are denoted by $\mathsf{a}_E = \mathcal{K}_E(\boldsymbol{A})$ and $\mathsf{a}_F = \mathcal{K}_F(\boldsymbol{A})$, respectively. The tensor that transforms a_E into a_F is given by

$$T^j{}_i := \mathcal{K}_E(\boldsymbol{F}_j)^i \qquad \Rightarrow \qquad a_F^j = a_E^i\;T^j{}_i. \tag{5.25}$$

The inverse of $T^j{}_i$ is denoted by $\bar{T}^j{}_i$, i.e. $a_E^j = a_F^i\;\bar{T}^j{}_i$. If the geometric-product tensor in the $\boldsymbol{E}$-basis is given by $\Gamma^k_{E\,ij}$, the corresponding product tensor in the $\boldsymbol{F}$-basis can be evaluated via

$$\Gamma^w_{F\,uv} = T^w{}_k\;\bar{T}^i{}_u\;\bar{T}^j{}_v\;\Gamma^k_{E\,ij}. \tag{5.26}$$

Such a change of basis finds an application, for example, if an implementation of $\mathbb{G}_{4,1}$ is given where the canonical basis blades are geometric products of the Minkowski basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_+, \boldsymbol{e}_-\}$. Here $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, and $\boldsymbol{e}_+$ square to $+1$ and $\boldsymbol{e}_-$ squares to -1. In many problem settings, however, it is essential to express constraints that involve $\boldsymbol{e}_\infty = \boldsymbol{e}_+ + \boldsymbol{e}_-$ and $\boldsymbol{e}_o = \dfrac{1}{2}(\boldsymbol{e}_- - \boldsymbol{e}_+)$. Therefore, component vectors and the product tensors have to be transformed into the algebraic basis constructed from $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_\infty, \boldsymbol{e}_o\}$.

5.2 Solving Linear Geometric Algebra Equations

In this section, it is shown how geometric-algebra equations of the form $\boldsymbol{A} \circ \boldsymbol{X} = \boldsymbol{B}$ can be solved for $\boldsymbol{X}$ numerically, where $\circ$ stands for any bilinear