1.13 COULOMB'S LAW OF ELECTRIC FORCE

21. *State Coulomb's law in electrostatics. Express the same in SI units. Name and define the units of electric charge.*

Coulomb's law. In 1785, the French physicist *Charles Augustin Coulomb* (1736-1806) experimentally measured the electric forces between small charged spheres by using a torsion balance. He formulated his observations in the form of Coulomb's law which is electrical analogue of Newton's law of Universal Gravitation in mechanics.

Coulomb's law *states that the force of attraction or repulsion between two stationary point charges is (i) directly proportional to the product of the magnitudes of the two charges and (ii) inversely proportional to the square of the distance between them. This force acts along the line joining the two charges.*

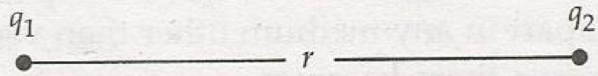

Fig. 1.7 Coulomb's law.

If two point charges q_1 and q_2 are separated by distance r, then the force F of attraction or repulsion between them is such that

$$F \propto q_1 q_2 \quad \text{and} \quad F \propto \frac{1}{r^2}$$

$$\therefore \quad F \propto \frac{q_1 q_2}{r^2} \quad \text{or} \quad F = k\,\frac{q_1 q_2}{r^2}$$

where k is a constant of proportionality, called ***electrostatic force constant***. The value of k depends on the nature of the medium between the two charges and the system of units chosen to measure F, q_1, q_2 and r.

For the two charges located in free space and in SI units, we have

$$k = \frac{1}{4\pi\,\varepsilon_0} = 9 \times 10^9\ \text{Nm}^2\ \text{C}^{-2}$$

where ε_0 is called *permittivity* of free space. So we can express Coulomb's law in SI units as

$$F = \frac{1}{4\pi\,\varepsilon_0} \cdot \frac{q_1 q_2}{r^2}$$

1.14 COULOMB'S LAW IN VECTOR FORM

22. *Write Coulomb's law in vector form. What is the importance of expressing it in vector form ?*

Coulomb's law in vector form. As shown in Fig. 1.8, consider two positive point charges q_1 and q_2 placed in vacuum at distance r from each other. They repel each other.

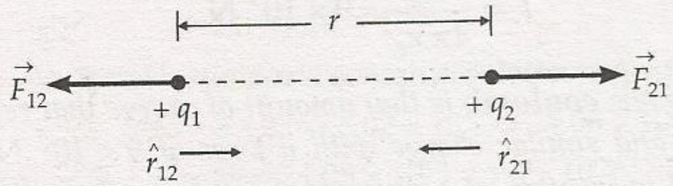

Fig. 1.8 Repulsive coulombian forces for $q_1 q_2 > 0$.

In vector form, Coulomb's law may be expressed as

$$\vec{F}_{21} = \text{Force on charge } q_2 \text{ due to } q_1$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2} \hat{r}_{12}$$

where $\hat{r}_{12} = \frac{\vec{r}_{12}}{r}$, is a unit vector in the direction from q_1 to q_2.

Similarly, $\vec{F}_{12} = \text{Force on charge } q_1 \text{ due to } q_2$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2} \hat{r}_{21}$$

where $\hat{r}_{21} = \frac{\vec{r}_{21}}{r}$, is a unit vector in the direction from q_2 to q_1.

The coulombian forces between unlike charges ($q_1 q_2 < 0$) are attractive, as shown in Fig. 1.9.

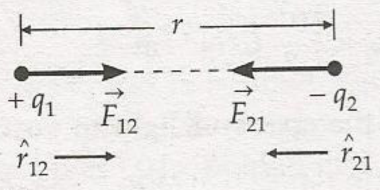

Fig. 1.9 Attractive coulombian forces for $q_1 q_2 < 0$.

1.17 FORCES BETWEEN MULTIPLE CHARGES : THE SUPERPOSITION PRINCIPLE

27. *State the principle of superposition of electrostatic forces. Hence write an expression for the force on a point charge due to a distribution of $N-1$ point charges in terms of their position vectors.*

Principle of superposition of electrostatic forces. Coulomb's law gives force between two point charges. The principle of superposition enables us to find the force on a point charge due to a group of point charges. This principle is based on the property that the forces with which two charges attract or repel each other are not affected by the presence of other charges.

The ***principle of superposition*** *states that when a number of charges are interacting, the total force on a given charge is the vector sum of the forces exerted on it due to all other charges. The force between two charges is not affected by the presence of other charges.*

As shown in Fig. 1.24, consider N point charges $q_1, q_2, q_3, ..., q_N$ placed in vacuum at points whose position vectors w.r.t. origin O are $\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_N$ respectively. According to the principle of superposition, the total force on charge q_1 is given by

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + ... + \vec{F}_{1N}$$

where $\vec{F}_{12}, \vec{F}_{13}, ..., \vec{F}_{1N}$ are the forces exerted on charge q_1 by the individual charges $q_2, q_3, ..., q_N$ respectively.

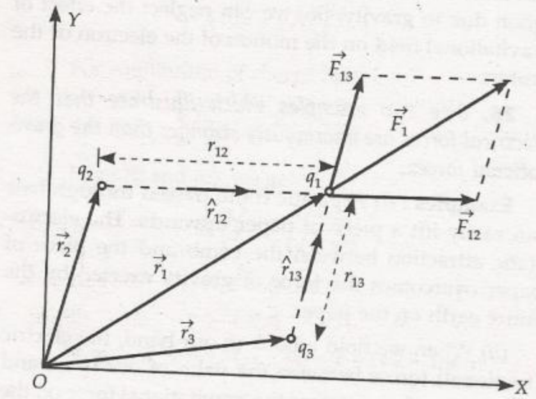

Fig. 1.24 Superposition principle : Force on charge q_1 exerted by q_2 and q_3.

According to Coulomb's law, the force exerted on charge q_1 due to q_2 is

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12}$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2} \cdot \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot q_1 q_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}$$

where $\hat{r}_{12} = \dfrac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$ = a unit vector pointing from q_2 to q_1 and $r_{12} = |\vec{r}_1 - \vec{r}_2|$ = distance of q_2 from q_1.

Hence the total force on charge q_1 is

$$\vec{F}_1 = \frac{1}{4\pi\varepsilon_0} \left[\frac{q_1 q_2}{r_{12}^2} \hat{r}_{12} + \frac{q_1 q_3}{r_{13}^2} \hat{r}_{13} + ... + \frac{q_1 q_N}{r_{1N}^2} \hat{r}_{1N} \right]$$

or $$\vec{F}_1 = \frac{q_1}{4\pi\varepsilon_0} \sum_{i=2}^{N} \frac{q_i}{r_{1i}^2} \hat{r}_{1i}$$

In terms of position vectors,

$$\vec{F}_1 = \frac{1}{4\pi\varepsilon_0} \left[q_1 q_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} + q_1 q_3 \frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|^3} + ... + q_1 q_N \frac{\vec{r}_1 - \vec{r}_N}{|\vec{r}_1 - \vec{r}_N|^3} \right]$$

or $$\vec{F}_1 = \frac{q_1}{4\pi\varepsilon_0} \sum_{i=2}^{N} q_i \frac{\vec{r}_1 - \vec{r}_i}{|\vec{r}_1 - \vec{r}_i|^3}$$

In general, force $\vec{F}_a$ on ath charge q_a located at $\vec{r}_a$ due to all other $(N-1)$ charges may be written as

$\vec{F}_a$ = Total force on ath charge

$$= \frac{q_a}{4\pi\varepsilon_0} \sum_{\substack{b=1 \\ b \neq a}}^{N} \frac{q_b}{r_{ab}^2} \hat{r}_{ab} = \frac{q_a}{4\pi\varepsilon_0} \sum_{\substack{b=1 \\ b \neq a}}^{N} q_b \frac{\vec{r}_a - \vec{r}_b}{|\vec{r}_a - \vec{r}_b|^3}$$

where $a = 1, 2, 3, ..., N$.

It may be noticed that for each choice of a, the summation on b *omits* the value a. This is because summation must be taken only over *other* charges. The above expression can be written in a simpler way as follows :

$\vec{F}$ = Total force on charge q due to many point charges q'

$$= \frac{q}{4\pi\varepsilon_0} \sum_{\substack{\text{all point} \\ \text{charges}}} q' \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

Examples based on Principle of Superposition of Electric Forces

FORMULAE USED

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + ... + \vec{F}_{1N}$$

$$F = \sqrt{F_1^2 + F_2^2 + 2 F_1 F_2 \cos\theta}$$

UNITS USED

Forces are in newton, charges in coulomb and distances in metre.

1.19 ELECTRIC FIELD DUE TO A POINT CHARGE

31. *Obtain an expression for the electric field intensity at a point at a distance r from a charge q. What is the nature of this field ?*

Electric field due to a point charge. A single point charge has the simplest electric field. As shown in Fig. 1.41, consider a point charge q placed at the origin O. We wish to determine its electric field at a point P at

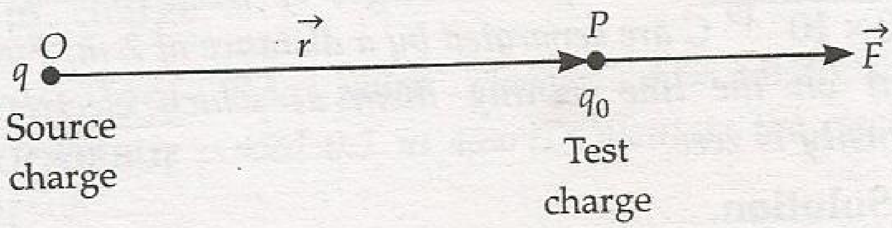

Fig. 1.41 Electric field of a point charge.

a distance r from it. For this, imagine a test charge q_0 placed at point P. According to Coulomb's law, the force on charge q_0 is

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q q_0}{r^2} \hat{r}$$

where $\hat{r}$ is a unit vector in the direction from q to q_0.

Electric field at point P is

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

The magnitude of the field $\vec{E}$ is

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2}$$

Clearly, $E \propto 1/r^2$. This means that at all points on the spherical surface drawn around the point charge, the magnitude of $\vec{E}$ is same and does not depend on the direction of $\vec{r}$. Such a field is called ***spherically symmetric*** or ***radial field,*** *i.e.,* a field which *looks the same* in all directions when seen from the point charge.

1.20 ELECTRIC FIELD DUE TO A SYSTEM OF POINT CHARGES

32. *Deduce an expression for the electric field at a point due to a system of N point charges.*

Electric field due to a system of point charges. Consider a system of N point charges $q_1, q_2, \ldots, q_N$ having position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ with respect to the

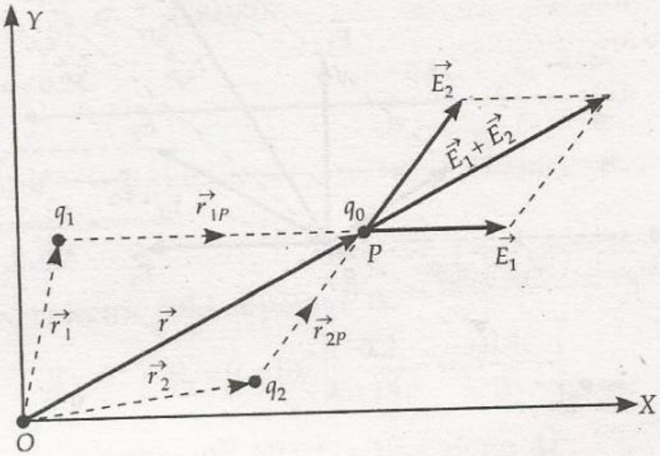

Fig. 1.42 Notations used in the determination of electric field at a point due to two point charges.

origin O. We wish to determine the electric field at point P whose position vector is $\vec{r}$. According to Coulomb's law the force on charge q_0 due to charge q_1 is

$$\vec{F}_1 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_0}{r_{1P}^2} \hat{r}_{1P}$$

where $\hat{r}_{1P}$ is a unit vector in the direction from q_1 to P and r_{1P} is the distance between q_1 and P. Hence the electric field at point P due to charge q_1 is

$$\vec{E}_1 = \frac{\vec{F}_1}{q_0} = \frac{1}{4\pi\varepsilon_0} \frac{q_1}{r_{1P}^2} \hat{r}_{1P}$$

Similarly, electric field at P due to charge q_2 is

$$\vec{E}_2 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_2}{r_{2P}^2} \hat{r}_{2P}$$

According to ***principle of superposition of electric fields***, *the electric field at any point due to a group of charges is equal to the vector sum of the electric fields produced by each charge individually at that point, when all other charges are assumed to be absent.*

Hence, the electric field at point P due to the system of N charges is

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_N$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1}{r_{1P}^2}\hat{r}_{1P} + \frac{q_2}{r_{2P}^2}\hat{r}_{2P} + \ldots + \frac{q_N}{r_{NP}^2}\hat{r}_{NP}\right]$$

or $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_{iP}^2}\hat{r}_{iP}$$

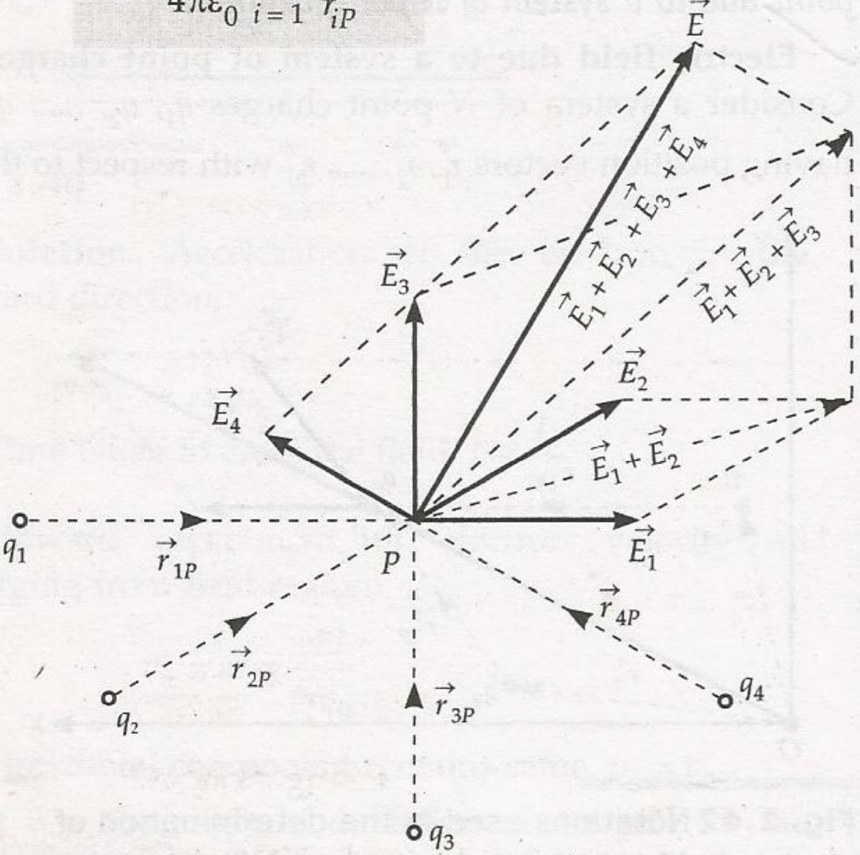

Fig. 1.43 Electric field at a point due to a system of charges is the vector sum of the electric fields at the point due to individual charges.

In terms of position vectors, we can write

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r} - \vec{r}_i|^2}\cdot\frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|}$$

or $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r} - \vec{r}_i|^3}(\vec{r} - \vec{r}_i).$$

1.24 ELECTRIC FIELD AT AN AXIAL POINT OF A DIPOLE

37. *Derive an expression for the electric field at any point on the axial line of an electric dipole.*

Electric field at an axial point of an electric dipole. As shown in Fig. 1.63, consider an electric dipole consisting of charges $+q$ and $-q$, separated by distance $2a$ and placed in vacuum. Let P be a point on the axial line at distance r from the centre O of the dipole on the side of the charge $+q$.

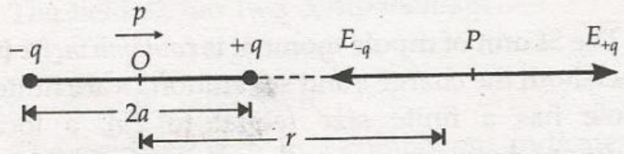

Fig. 1.63 Electric field at an axial point of dipole.

Electric field due to charge $-q$ at point P is

$$\vec{E}_{-q} = \frac{-q}{4\pi\varepsilon_0 (r+a)^2}\hat{p} \qquad \text{(towards left)}$$

where $\hat{p}$ is a unit vector along the dipole axis from $-q$ to $+q$.

Electric field due to charge $+q$ at point P is

$$\vec{E}_{+q} = \frac{q}{4\pi\varepsilon_0 (r-a)^2}\hat{p} \qquad \text{(towards right)}$$

Hence the resultant electric field at point P is

$$\vec{E}_{axial} = \vec{E}_{+q} + \vec{E}_{-q}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{(r-a)^2} - \frac{1}{(r+a)^2}\right]\hat{p}$$

$$= \frac{q}{4\pi\varepsilon_0} \cdot \frac{4ar}{(r^2-a^2)^2}\hat{p}$$

or $$\vec{E}_{axial} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2pr}{(r^2-a^2)^2}\hat{p}$$

Here $p = q \times 2a$ = dipole moment.

For $r >> a$, a^2 can be neglected compared to r^2.

$$\therefore \quad \vec{E}_{axial} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2p}{r^3}\hat{p} \qquad \text{(towards right)}$$

Clearly, electric field at any axial point of the dipole acts along the dipole axis from negative to positive charge *i.e.*, in the direction of dipole moment $\vec{p}$.

1.25 ▼ ELECTRIC FIELD AT AN EQUATORIAL POINT OF A DIPOLE

38. *Derive an expression for the electric field at any point on the equatorial line of an electric dipole.*

Electric field at an equatorial point of a dipole. As shown in Fig. 1.64, consider an electric dipole consisting of charges $-q$ and $+q$, separated by distance $2a$ and placed in vacuum. Let P be a point on the equatorial line of the dipole at distance r from it.

i.e., $OP = r$

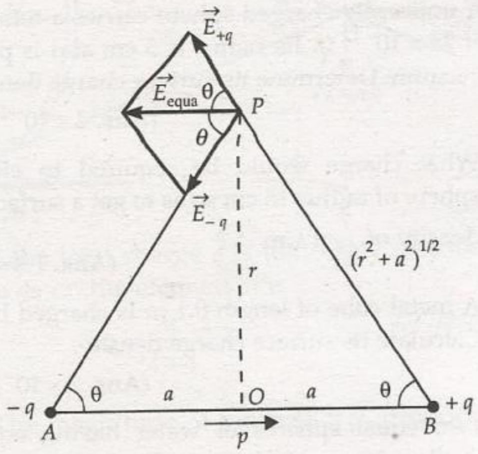

Fig. 1.64 Electric field at an equatorial point of a dipole.

Electric field at point P due to $+q$ charge is

$$\vec{E}_{+q} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2 + a^2}, \text{ directed along } BP$$

Electric field at point P due to $-q$ charge is

$$\vec{E}_{-q} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2 + a^2}, \text{ directed along } PA$$

Thus the magnitudes of $\vec{E}_{-q}$ and $\vec{E}_{+q}$ are equal *i.e.,*

$$E_{-q} = E_{+q} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2 + a^2}$$

Clearly, the components of $\vec{E}_{-q}$ and $\vec{E}_{+q}$ normal to the dipole axis will cancel out. The components parallel to the dipole axis add up. The total electric field $\vec{E}_{equa}$ is opposite to $\vec{p}$.

$$\therefore \quad \vec{E}_{equa} = -(E_{-q}\cos\theta + E_{+q}\cos\theta)\,\hat{p}$$

$$= -2\,E_{-q}\cos\theta\,\hat{p} \qquad [E_{-q} = E_{+q}]$$

$$= -2 \cdot \frac{1}{4\pi\varepsilon_0}\,\frac{q}{r^2 + a^2} \cdot \frac{a}{\sqrt{r^2 + a^2}}\,\hat{p}$$

$$\left[\because \cos\theta = \frac{a}{\sqrt{r^2 + a^2}}\right]$$

or $$\vec{E}_{equa} = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{p}{(r^2 + a^2)^{3/2}} \hat{p}.$$

where $p = 2qa$, is the electric dipole moment.

If the point P is located far away from the dipole, $r >> a$, then

$$\vec{E}_{equa} = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{p}{r^3} \hat{p}$$

Clearly, the direction of electric field at any point on the equatorial line of the dipole will be antiparallel to the dipole moment $\vec{p}$.

...nitudes of electric

1.26 TORQUE ON A DIPOLE IN A UNIFORM ELECTRIC FIELD

40. *Derive an expression for the torque on an electric dipole placed in a uniform electric field. Hence define dipole moment.*

Torque on a dipole in a uniform electric field. As shown in Fig. 1.65(a), consider an electric dipole consisting of charges $+q$ and $-q$ and of length $2a$ placed in a uniform electric field $\vec{E}$ making an angle θ with it. It has a dipole moment of magnitude,

$$p = q \times 2a$$

Force exerted on charge $+q$ by field $\vec{E} = q\vec{E}$

(along $\vec{E}$)

Force exerted on charge $-q$ by field $\vec{E} = -q\vec{E}$

(opposite to $\vec{E}$)

$$\vec{F}_{Total} = +q\vec{E} - q\vec{E} = 0.$$

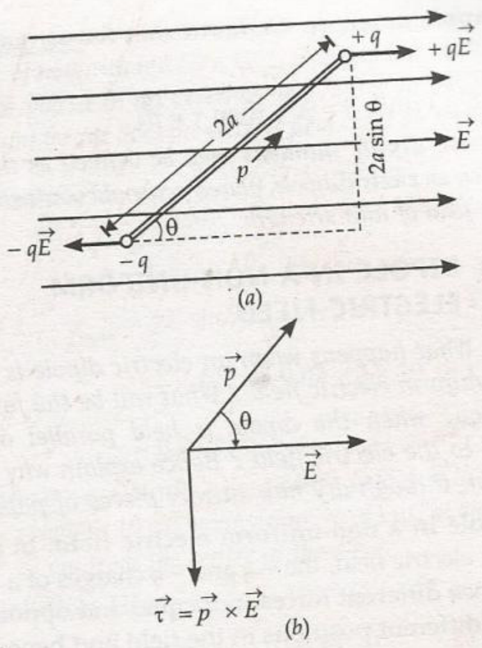

Fig. 1.65 (*a*) Torque on a dipole in a uniform electric field. (*b*) Direction of torque as given by right hand screw rule.

Hence the net translating force on a dipole in a uniform electric field is zero. But the two equal and opposite forces act at different points of the dipole. They form a couple which exerts a torque.

Torque = Either force × Perpendicular distance between the two forces

$$\tau = qE \times 2a \sin\theta = (q \times 2a)\, E \sin\theta$$

or $$\tau = pE \sin\theta \qquad (p = q \times 2a)$$

As the direction of torque $\vec{\tau}$ is perpendicular to both $\vec{p}$ and $\vec{E}$, so we can write

$$\vec{\tau} = \vec{p} \times \vec{E}$$

The direction of vector $\vec{\tau}$ is that in which a right handed screw would advance when rotated from $\vec{p}$ to $\vec{E}$. As shown in Fig. 1.65(*b*), the direction of vector $\vec{\tau}$ is perpendicular to, and points into the plane of paper.

When the dipole is released, the torque $\vec{\tau}$ tends to align the dipole with the field $\vec{E}$ *i.e.*, tends to reduce the angle θ to 0. When the dipole gets aligned with $\vec{E}$, the torque $\vec{\tau}$ becomes zero.

Clearly, the torque on the dipole will be maximum when the dipole is held perpendicular to $\vec{E}$. Thus

$$\tau_{max} = pE \sin 90° = pE.$$

Dipole moment. We know that the torque,

$$\tau = pE \sin \theta$$

If $E = 1$ unit, $\theta = 90°$, then $\tau = p$

Hence ***dipole moment*** *may be defined as the torque acting on an electric dipole, placed perpendicular to a uniform electric field of unit strength.*

1.32 GAUSS'S THEOREM

49. *State and prove Gauss's theorem.*

Gauss's theorem. This theorem gives a relationship between the total flux passing through any *closed* surface and the net charge enclosed within the surface.

Gauss theorem *states that the total flux through a closed surface is $1/\varepsilon_0$ times the net charge enclosed by the closed surface.*

Mathematically, it can be expressed as

$$\phi_E = \oint_S \vec{E} \cdot d\vec{S} = \frac{q}{\varepsilon_0}$$

Proof. For the sake of simplicity, we prove Gauss's theorem for an isolated positive point charge q. As

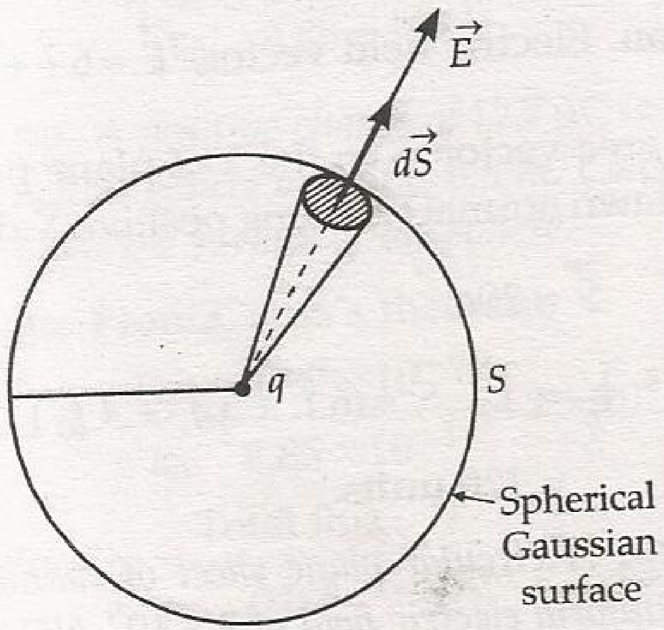

Fig. 1.82 Flux through a sphere enclosing a point charge.

shown in Fig. 1.82, suppose the surface S is a sphere of radius r centred on q. Then surface S is a *Gaussian surface.*

Electric field at any point on S is

$$E = \frac{1}{4\pi\,\varepsilon_0} \cdot \frac{q}{r^2}$$

This field points radially outward at all points on S. Also, any area element points radially outwards, so it is parallel to $\vec{E}$, *i.e.*, $\theta = 0°$.

$\therefore$ Flux through area $\vec{dS}$ is

$$d\phi_E = \vec{E} \,.\, \vec{dS} = E\, dS \cos 0° = EdS$$

Total flux through surface S is

$$\phi_E = \oint_S d\phi_E = \oint_S E\, dS = E \oint_S dS$$

$$= E \times \text{Total area of sphere}$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} \cdot 4\pi r^2$$

or $$\phi_E = \frac{q}{\varepsilon_0}$$

This proves Gauss's theorem.

1.36 ELECTRIC FIELD DUE TO A UNIFORMLY CHARGED INFINITE PLANE SHEET

53. *Apply Gauss's theorem to calculate the electric field due to an infinite plane sheet of charge.*

Electric field due to a uniformly charged infinite plane sheet. As shown in Fig. 1.94, consider a thin, infinite plane sheet of charge with uniform surface charge density σ. We wish to calculate its electric field at a point P at distance r from it.

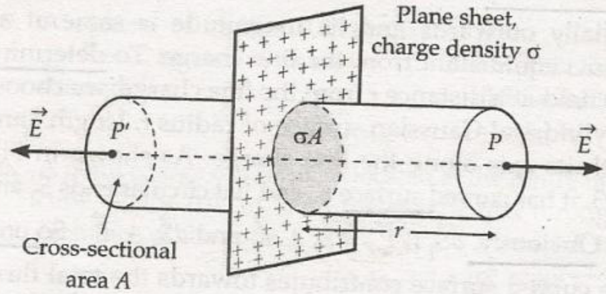

Fig. 1.94 Gaussian surface for a uniformly charged infinite plane sheet.

By symmetry, electric field E points outwards normal to the sheet. Also, it must have same magnitude and opposite direction at two points P and P' equidistant from the sheet and on opposite sides. We choose cylindrical Gaussian surface of cross- sectional area A and length $2r$ with its axis perpendicular to the sheet.

As the lines of force are parallel to the curved surface of the cylinder, the flux through the curved surface is zero. The flux through the plane-end faces of the cylinder is

$$\phi_E = EA + EA = 2\,EA$$

Charge enclosed by the Gaussian surface,

$$q = \sigma A$$

According to Gauss's theorem,

$$\phi_E = \frac{q}{\varepsilon_0}$$

$$\therefore \quad 2\,EA = \frac{\sigma A}{\varepsilon_0} \quad \text{or} \quad E = \frac{\sigma}{2\varepsilon_0}$$

Clearly, E is independent of r, the distance from the plane sheet.

(*i*) If the sheet is *positively* charged ($\sigma > 0$), the field is directed *away* from it.

(*ii*) If the sheet is *negatively* charged ($\sigma < 0$), the field is directed *towards* it.

For a finite large planar sheet, the above formula will be approximately valid in the middle regions of the sheet, away from its edges.

54. *Two infinite parallel planes have uniform charge densities of σ_1 and σ_2. Determine the electric field at points (i) to the left of the sheets, (ii) between them, and (iii) to the right of the sheets.*

Electric field of two positively charged parallel plates. Fig. 1.95 shows two thin plane parallel sheets of charge having uniform charge densities σ_1 and σ_2 with $\sigma_1 > \sigma_2 > 0$. Suppose $\hat{r}$ is a unit vector pointing from left to right.

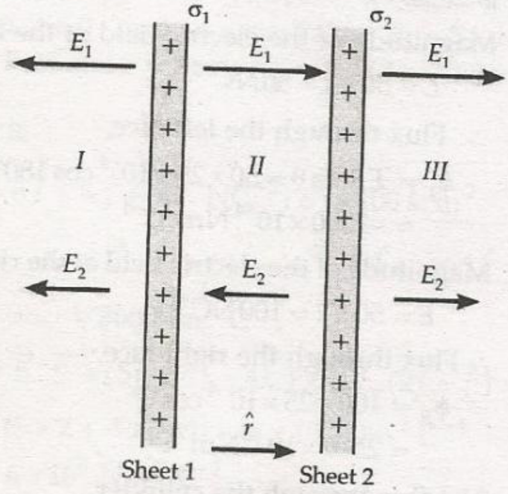

Fig. 1.95

In the region I : Fields due to the two sheets are

$$\vec{E}_1 = -\frac{\sigma_1}{2\varepsilon_0}\hat{r}, \quad \vec{E}_2 = -\frac{\sigma_2}{2\varepsilon_0}\hat{r}$$

From the principle of superposition, the total electric field at any point of region *I* is

$$\vec{E}_I = \vec{E}_1 + \vec{E}_2 = -\frac{\hat{r}}{2\varepsilon_0}(\sigma_1 + \sigma_2)$$

In the region II : Fields due to the two sheets are

$$\vec{E}_1 = \frac{\sigma_1}{2\varepsilon_0}\hat{r}, \quad \vec{E}_2 = -\frac{\sigma_2}{2\varepsilon_0}\hat{r}$$

$\therefore$ Total field, $\quad \vec{E}_{II} = \dfrac{\hat{r}}{2\varepsilon_0}(\sigma_1 - \sigma_2)$

In the region III : Fields due to the two sheets are

$$\vec{E}_1 = \frac{\sigma_1}{2\varepsilon_0}\hat{r}, \quad \vec{E}_2 = \frac{\sigma_2}{2\varepsilon_0}\hat{r}$$

$\therefore$ Total field, $\quad \vec{E}_{III} = \dfrac{\hat{r}}{2\varepsilon_0}(\sigma_1 + \sigma_2)$

densities of $\pm\sigma$. Suppose $\hat{r}$ be a unit vector pointing from left to right.

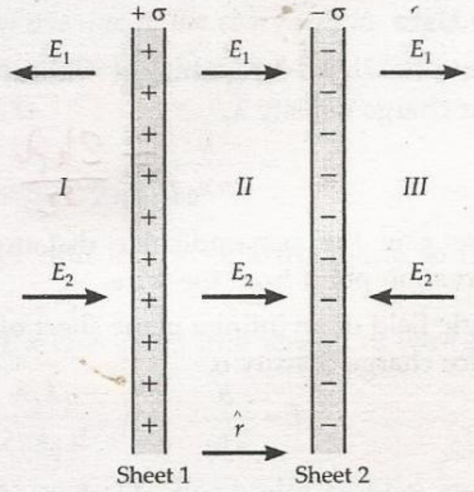

Fig. 1.96

In the region I : Fields due to the two sheets are

$$\vec{E}_1 = -\frac{\hat{r}}{2\,\varepsilon_0}\,\sigma\,, \qquad \vec{E}_2 = \frac{\hat{r}}{2\,\varepsilon_0}\,\sigma$$

Total field, $\vec{E}_I = \vec{E}_1 + \vec{E}_2 = -\frac{\hat{r}}{2\,\varepsilon_0}\,\sigma + \frac{\hat{r}}{2\,\varepsilon_0}\,\sigma = 0$

In the region II : Fields due to the two sheets are

$$\vec{E}_1 = \frac{\hat{r}}{2\,\varepsilon_0}\,\sigma\,, \qquad \vec{E}_2 = \frac{\hat{r}}{2\,\varepsilon_0}\,\sigma$$

Total field, $\vec{E}_{II} = \frac{\hat{r}}{2\,\varepsilon_0}\,\sigma + \frac{\hat{r}}{2\,\varepsilon_0}\,\sigma = \frac{\sigma}{\varepsilon_0}\,\hat{r}$

In the region III : Fields due to the two sheets are

$$E_1 = \frac{\hat{r}}{2\,\varepsilon_0}\,\sigma\,, \quad E_2 = -\frac{\hat{r}}{2\,\varepsilon_0}\,\sigma$$

Total field, $\vec{E}_{III} = 0$.

Thus the electric field between two oppositely charged plates of equal charge density is uniform which is equal to $\frac{\sigma}{\varepsilon_0}$ and is directed from the positive to the negative plate, while the field is zero on the outside of the two sheets. This arrangement is used for producing *uniform electric field.*

1.37 FIELD DUE TO A UNIFORMLY CHARGED THIN SPHERICAL SHELL

56. *Apply Gauss's theorem to show that for a spherical shell, the electric field inside the shell vanishes, whereas outside it, the field is as if all the charge had been concentrated at the centre.*

Electric field due to a uniformly charged thin spherical shell. Consider a thin spherical shell of charge of radius R with uniform surface charge density σ. From symmetry, we see that the electric field $\vec{E}$ at any point is radial and has same magnitude at points equidistant from the centre of the shell *i.e.*, the field is *spherically symmetric*. To determine electric field at any point P at a distance r from O, we choose a concentric sphere of radius r as the Gaussian surface.

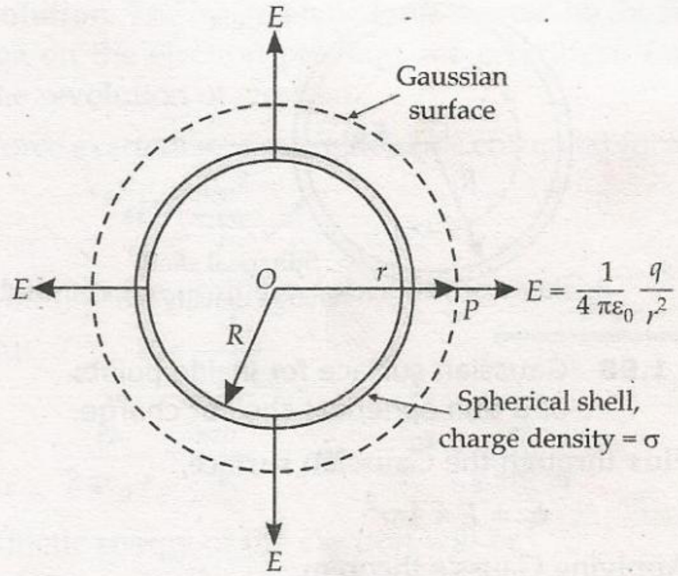

Fig. 1.97 Gaussian surface for outside points of a thin spherical shell of charge.

(*a*) *When point P lies outside the spherical shell.* The total charge q inside the Gaussian surface is the charge on the shell of radius R and area $4\pi R^2$.

$$\therefore \qquad q = 4\pi R^2 \sigma$$

Flux through the Gaussian surface,

$$\phi_E = E \times 4\pi r^2$$

By Gauss's theorem,

$$\phi_E = \frac{q}{\varepsilon_0}$$

$$\therefore \quad E \times 4\pi r^2 = \frac{q}{\varepsilon_0}$$

or $$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} \qquad \text{[For } r > R\text{]}$$

This field is the same as that produced by a charge q placed at the centre O. Hence *for points outside the shell, the field due to a uniformly charged shell is as if the entire charge of the shell is concentrated at its centre.*

(*b*) *When point P lies on the spherical shell.* The Gaussian surface just encloses the charged spherical shell.

Applying Gauss's theorem,

$$E \times 4\pi R^2 = \frac{q}{\varepsilon_0}$$

or $$E = \frac{q}{4\pi \varepsilon_0 R^2} \quad \text{[For } r = R\text{]}$$

or $$E = \frac{\sigma}{\varepsilon_0} \quad [\because\ q = 4\pi R^2\ \sigma]$$

(c) *When point P lies inside the spherical shell.* As is clear from Fig. 1.98, the charge enclosed by the Gaussian surface is zero, *i.e.*,

$$q = 0$$

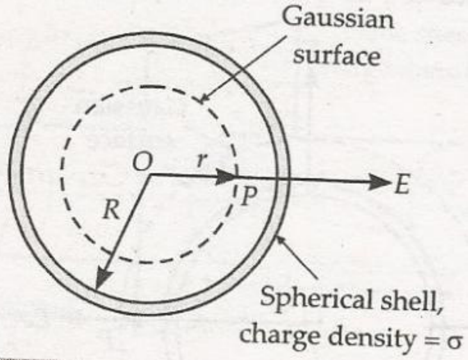

Fig. 1.98 Gaussian surface for inside points of a thin spherical shell of charge.

Flux through the Gaussian surface,

$$\phi_E = E \times 4\pi r^2$$

Applying Gauss's theorem,

$$\phi_E = \frac{q}{\varepsilon_0}$$

$$E \times 4\pi r^2 = 0$$

or $$E = 0 \quad \text{[For } r < R\text{]}$$

Hence electric field due to a uniformly charged spherical shell is zero at all points inside the shell.

Fig. 1.99 shows how E varies with distance r from the centre of the shell of radius r. E is zero from $r = 0$ to $r = R$; and beyond $r = R$, we have

$$E \propto \frac{1}{r^2}.$$

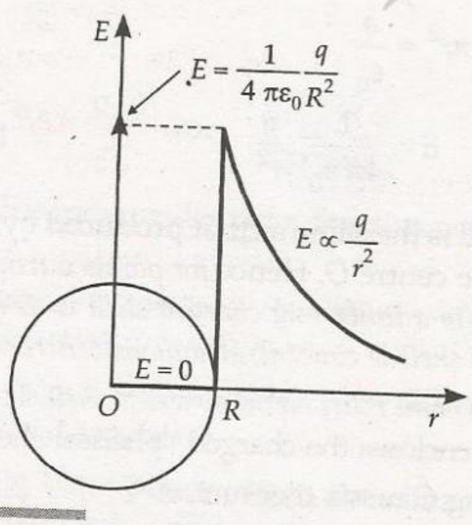

Fig. 1.99 Variation of E with r for a spherical shell of charge.

Examples based on Applications of Gauss's Theorem

Formulae Used

1. Electric field of a long straight wire of uniform linear charge density λ,

$$E = \frac{\lambda}{2\pi \varepsilon_0 r} = \frac{2k\lambda}{r}$$

where r is the perpendicular distance of the observation point from the wire.

2. Electric field of an infinite plane sheet of uniform surface charge density σ,

$$E = \frac{\sigma}{2\varepsilon_0}$$

3. Electric field of two positively charged parallel plates with charge densities σ_1 and σ_2 such that $\sigma_1 > \sigma_2 > 0$,

$$E = \pm \frac{1}{2\varepsilon_0}(\sigma_1 + \sigma_2) \quad \text{(Outside the plates)}$$

$$E = \frac{1}{2\varepsilon_0}(\sigma_1 - \sigma_2) \quad \text{(Inside the plates)}$$

4. Electric field of two equally and oppositely charged parallel plates,

$$E = 0 \quad \text{(For outside points)}$$

$$E = \frac{\sigma}{\varepsilon_0} \quad \text{(For inside points)}$$

5. Electric field of a thin spherical shell of charge density σ and radius R,

$$E = \frac{1}{4\pi \varepsilon_0} \cdot \frac{q}{r^2} \quad \text{For } r > R \text{ (Outside points)}$$

$$E = 0 \quad \text{For } r < R \text{ (Inside points)}$$

$$E = \frac{1}{4\pi \varepsilon_0} \cdot \frac{q}{R^2} \quad \text{For } r = R \text{ (At the surface)}$$

Here $q = 4\pi R^2 \sigma$.

6. Electric field of a solid sphere of uniform charge density ρ and radius R :

$$E = \frac{1}{4\pi \varepsilon_0} \cdot \frac{q}{r^2} \quad \text{For } r > R \text{ (Outside points)}$$

$$E = \frac{1}{4\pi \varepsilon_0} \cdot \frac{qr}{R^3} \quad \text{For } r < R \text{ (Inside points)}$$

$$E = \frac{1}{4\pi \varepsilon_0} \cdot \frac{q}{R^2} \quad \text{For } r = R \text{ (At the surface)}$$

Here $q = \frac{4}{3}\pi R^3 \rho$

Units Used

Here charges are in coulomb, r and R in metre, λ in Cm^{-1}, σ in Cm^{-2}, ρ in Cm^{-3} and electric field E in NC^{-1} or Vm^{-1}.

Solution. Electric field of a line charge from Coulomb's law. Consider an infinite line of charge with uniform line charge density λ, as shown in Fig. 1.60. We wish to calculate its electric field at any point P at a distance y from it. The charge on small element dx of the line charge will be

$$dq = \lambda dx$$

The electric field at the point P due to the charge element dq will be

$$dE = \frac{1}{4\pi\varepsilon_0} \cdot \frac{dq}{r^2} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\lambda dx}{y^2 + x^2}$$

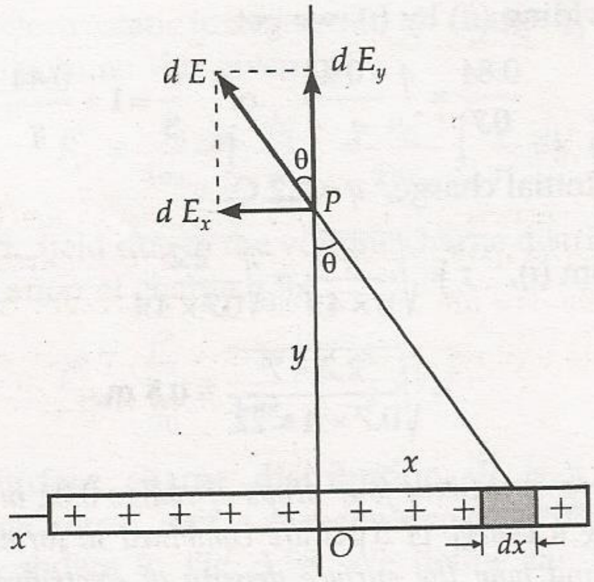

Fig. 1.60 A section of an infinite line of charge.

The field dE has two components :

$$dE_x = -\,dE \sin\theta$$

and $$dE_y = dE \cos\theta$$

The negative sign in x-component indicates that $d\vec{E}_x$ acts in the negative x-direction. Every charge element on the right has a corresponding charge element on the left. The x-components of two such charge elements will be equal and opposite and hence cancel out. The resultant field $\vec{E}$ gets contributions only from y-components and is given by

$$E = E_y = \int dE_y = \int_{x=-\infty}^{x=+\infty} \cos\theta \, dE$$

$$= 2\int_{x=0}^{x=\infty} \cos\theta \cdot \frac{1}{4\pi\varepsilon_0} \frac{\lambda\, dx}{y^2 + x^2}$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \int_{x=0}^{x=\infty} \cos\theta \, \frac{dx}{y^2 + x^2}$$

Now $$x = y \tan\theta$$

$$dx = y \sec^2\theta \, d\theta$$

$$\therefore \quad E = \frac{\lambda}{2\pi\varepsilon_0} \int_{\theta=0}^{\theta=\pi/2} \cos\theta \, \frac{y \sec^2\theta \, d\theta}{y^2(1 + \tan^2\theta)}$$

$$= \frac{\lambda}{2\pi\varepsilon_0 y} \int_{\theta=0}^{\theta=\pi/2} \cos\theta \, d\theta = \frac{\lambda}{2\pi\varepsilon_0 y} [\sin\theta]_0^{\pi/2}$$

$$= \frac{\lambda}{2\pi\varepsilon_0 y} \left(\sin\frac{\pi}{2} - \sin 0 \right)$$

or $$E = \frac{\lambda}{2\pi\varepsilon_0 y}.$$

EXAMPLE 17. A charge is distributed uniformly over a ring