Principle of Virtual Work in terms of finite deformation

$$\int_V S_{ij}\delta F_{ij} dV = \int_{\partial V} T_i \delta u_i dA + \int_V b_i \delta u_i dV$$

Proof:

$$\text{The right hand side of the equation} = \int_{\partial V} T_i \delta u_i dA + \int_V b_i \delta u_i dV$$

considering $\delta u_i = \delta(x_i - X_i) = \delta x_i$

$$RHS = \int_{\partial V} T_i \delta x_i dV + \int_V b_i \delta x_i dV = \int_{\partial V} S_{ij} N_j \delta x_i dA + \int_V b_i \delta x_i dV$$

Using the divergence theorem for the first term above, we have

$$RHS = \int_V (S_{ij}\delta x_i)_{,j} dV + \int_V b_i \delta x_i dV$$

$$= \int_V S_{ij} \frac{\partial \delta x_i}{\partial X_j} dV + \int_V \left(\frac{\partial S_{ij}}{\partial X_j} + b_i\right) \delta x_i dV$$

Considering the equilibrium equation $\frac{\partial S_{ij}}{\partial X_j} + b_i = 0$, the second term vanishes and then it yields

$$RHS = \int_V S_{ij} \frac{\partial \delta x_i}{\partial X_j} dV = \int_V S_{ij} \delta F_{ij} dV = LHS$$