Near-field Fourier Ptychographic Phase Retrieval from Local Correlation Measurements

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Online International Conference on Computational Harmonic Analysis - 2021

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Near-field Ptychography

- Ptychography is a particular form of masked phase retrieval in which the collections of masks are generated by taking one physical mask and shifting it in space
- Near-field Ptychography is when the distance between the lens, object, and detector are small (microscopic imaging).

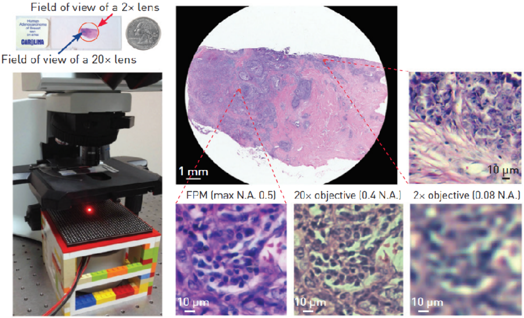


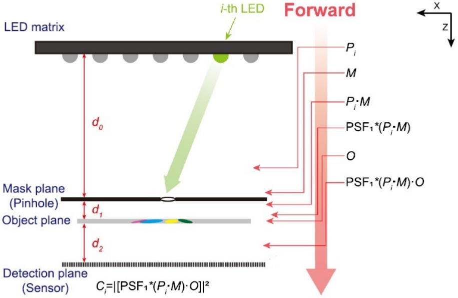
Figure: "Wide-field, high-resolution Fourier ptychographic microscopy" - G. Zheng et al., 2013

Near-field Ptychographic Measurements

- Let $\mathbf{x} \in \mathbb{C}^d$ denote the unknown object
- Let $\mathbf{m}$ denote the known mask, $\mathbf{p}$ the known point spread function (PSF).
- The noisy near-field ptychographic measurements will be of the form

$$Y_{k,\ell} = |(\mathbf{p} * (S_k \mathbf{m} \circ \mathbf{x}))_\ell|^2 + N_{k,\ell}, \quad (k,\ell) \in \mathcal{K} \times \mathcal{L} \subseteq [d] \times [d]$$

- $\mathcal{K}$: set of shifts, $\mathcal{L}$: set of frequencies
- Circular shift: $(S_k \mathbf{m})_n = m_{n+k \text{ mod } d}$
- Discrete convolution: $(\mathbf{u} * \mathbf{v})_n = \sum_{k=0}^{d-1} u_k v_{n-k \text{ mod } d}$
- Pointwise (Hadamard) product: $(\mathbf{u} \circ \mathbf{v})_n = u_n v_n$



Reducing To Inner Product Form

- Consider the noiseless measurements $Y_{k,\ell} = |(\mathbf{p} * (S_k \mathbf{m} \circ \mathbf{x}))_\ell|^2, \ (k, \ell) \in [d] \times [2\delta - 1]$

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- By letting $Y_{k,\ell} = Y_{-k,\ell+k}, \check{\mathbf{m}}_\ell^{(p,m)} = \overline{S_\ell \tilde{\mathbf{p}} \circ \mathbf{m}}, (\tilde{p})_n = p_{-n}$, one can show that

$$Y_{k,\ell} = |\langle \check{\mathbf{m}}_\ell^{(p,m)}, S_k \mathbf{x} \rangle|^2, \quad (k, \ell) \in [d] \times [2\delta - 1].$$

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- Let $D = d(2\delta - 1)$. We then define the block circulant matrix $\breve{\mathbf{M}} \in \mathbb{C}^{D \times D}$ via

$$\breve{\mathbf{M}} := \begin{pmatrix} \breve{\mathbf{M}}_0 & \breve{\mathbf{M}}_1 & \dots & \breve{\mathbf{M}}_{\delta-1} & 0 & 0 & \dots & 0 \\ 0 & \breve{\mathbf{M}}_0 & \breve{\mathbf{M}}_1 & \dots & \breve{\mathbf{M}}_{\delta-1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \breve{\mathbf{M}}_1 & \dots & \breve{\mathbf{M}}_{\delta-1} & 0 & 0 & 0 & \dots & \breve{\mathbf{M}}_0 \end{pmatrix}$$

where the matrices $\breve{\mathbf{M}}_k \in \mathbb{C}^{(2\delta-1)\times(2\delta-1)}$ are defined entry-wise by

$$(\breve{\mathbf{M}}_k)_{ij} := \begin{cases} (\breve{\mathbf{m}}_i)_k \overline{(\breve{\mathbf{m}}_i)_{j+k}}, & 0 \le j \le \delta - k \\ (\breve{\mathbf{m}}_i)_k \overline{(\breve{\mathbf{m}}_i)_{j+k-2\delta}}, & 2\delta - 1 + k \le j \le 2\delta - 2 \wedge k < \delta \\ 0, & \text{otherwise} \end{cases}$$

Condition Number Bound

- Then it is shown that $\text{vec}(\mathbf{Y}) = \check{\mathbf{M}}\mathbf{z}$ for $\mathbf{z} \in \mathbb{C}^d$ being a portion of $\text{vec}(\mathbf{x}\mathbf{x}^*)$

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- In the paper, it is shown that exponential masks $\check{\mathbf{m}}_\ell^{(fpr)}$ defined by

$$(\check{\mathbf{m}}_\ell^{(fpr)})_n = \begin{cases} \frac{e^{-(n+1)/a}}{\sqrt[4]{2\delta-1}} \cdot e^{\frac{2\pi i n \ell}{2\delta-1}}, & n \in [\delta] \\ 0, & \text{otherwise} \end{cases}, \quad a := \max\left\{4, \frac{\delta-1}{2}\right\}$$

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Theorem (Theorem 4 - M. Iwen., et al)

For collection of masks $\check{\mathbf{m}}_\ell^{(fpr)}$, the condition number has the bound

$$\kappa := \kappa(\check{\mathbf{M}}) < \max\left\{144e^2, \frac{9e^2(\delta-1)^2}{4}\right\} \leq C\delta^2, \quad C \in \mathbb{R}^+$$

Furthermore, $\check{\mathbf{M}}$ can be inverted in $\mathcal{O}(\delta \cdot d \log d)$-time.

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Lemma (Admissable Selection of PSF and Mask - Mark R.)

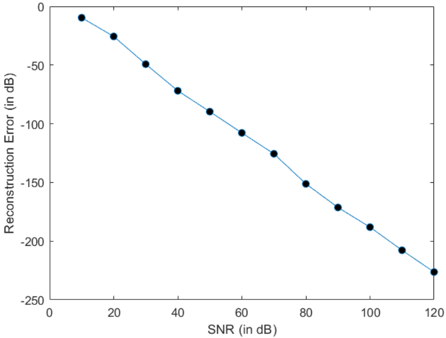
Let $(\mathbf{p})_n := e^{-\frac{\pi i n^2}{2\delta-1}}, n \in [d]$, $(\mathbf{m})_n := \begin{cases} \frac{e^{-n+1}/a}{\sqrt[4]{2\delta-1}} \cdot e^{\frac{\pi i n^2}{2\delta-1}}, & n \in [\delta] \\ 0, & \text{otherwise} \end{cases}$ *for* $a := \max\left\{4, \frac{\delta-1}{2}\right\}$.

Let $\check{\mathbf{m}}_\ell^{(p,m)} = \overline{S_\ell \tilde{\mathbf{p}} \circ \mathbf{m}}$. *Then for* **p** *and* **m** *as defined, we have that*

$$\check{\mathbf{m}}_\ell^{(p,m)} = e^{\frac{\pi i \ell^2}{2\delta-1}} \cdot \check{\mathbf{m}}_\ell^{(fpr)}.$$

Hence the matrices $\breve{\mathbf{M}}_k$ *remain unaltered, thus* $\check{\mathbf{M}}$ *remains unaltered, and the condition number guarantees hold.*

SNR vs. Reconstruction Error



- Let $d = 2^5, \delta = 13$ and let **p** and **m** be our chosen point spread function and mask.
- Then the signal-to-noise ratio is given by $SNR = 10\log_{10}\left(\frac{\|\mathbf{Y}-\mathbf{N}\|_F}{\|\mathbf{N}\|_F}\right)$
- We measure the reconstruction error by $10\log_{10}\left(\frac{\min_\phi \|\mathbf{x} - e^{i\phi}\mathbf{x}_{est}\|_2^2}{\|\mathbf{x}\|_2^2}\right)$ and we plot based on varying levels of signal-to-noise ratio.

Graph Theoretic Approach

- Let $\mathbf{x} = \mathbf{x}^{(mag)} \circ \mathbf{x}^{(\theta)}, \mathbf{x}^{(\theta)} = (e^{i\theta_0} e^{i\theta_1} \ldots e^{i\theta_{d-1}})^T \in \mathbb{C}^d$.

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$$X_{i,j}^{(\theta)} = \begin{cases} e^{\mathbb{i}(\theta_i - \theta_j)}, \\ 0, \end{cases} \quad N_{i,j}^{(\theta)} = \begin{cases} e^{\mathbb{i}\eta_{i,j}} \\ 0 \end{cases} \quad \widehat{X}_{i,j}^{(\theta)} = \begin{cases} e^{\mathbb{i}(\theta_i - \theta_j + \eta_{i,j})}, & |i-j| \bmod d < \delta \\ 0, & \text{otherwise} \end{cases}$$

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Lemma (Theorem 3 - M. Iwen, et al.)

Let $G = (V, E)$ be unweighted graph. Let τ_G denote the spectral gap of the Laplacian $\mathbf{L}_G$. Let $\mathbf{x}_{est}^{(\theta)} \in \mathbb{C}^d$ denote the obtained phase estimate from angular synchronization. Then we have that

$$\min_{\phi \in [0, 2\pi)} \|\mathbf{x}_{est}^{(\theta)} - e^{i\phi}\mathbf{x}^{(\theta)}\|_2 \leq C \cdot \sqrt{2\delta - 1} \cdot \frac{\|\widehat{\mathbf{X}}^{(\theta)} - \mathbf{X}^{(\theta)}\|_F}{\tau_G}, \ C \in \mathbb{R}^+$$

NFP Recovery Theorem

Lemma (Corollary 3 & Corollary 1 - M. Iwen, et al.)

Let $d \geq 4(\delta - 1), \delta \geq 3$. Let $\mathbf{x}_{\min} = \min_{i \in [d]} |x_i|, \mathbf{n} \in \mathbb{C}^D$ be the vectorization of the noise matrix . Then

$$\|\widehat{\mathbf{X}}^{(\theta)} - \mathbf{X}^{(\theta)}\|_F \leq C \frac{\kappa \sqrt{2\delta - 1}\|\mathbf{n}\|_2}{|\mathbf{x}_{\min}|^2}, \ \tau_G > C'\left(\frac{\delta^3}{d^2}\right), \ C, C' \in \mathbb{R}^+$$

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Theorem (Phase NFP Recovery Theorem - M. Roach., et al.)

Suppose $d \geq 4(\delta - 1), \delta \geq 3$. *Then we have that for our admissable choice of* $\mathbf{p}$ *and* $\mathbf{m}$

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Theorem (Theorem 4 - M. Iwen, et al.)

We have that $\min_{\phi \in [0, 2\pi)} \|\mathbf{x}_{est} - e^{i\phi}\mathbf{x}\|_2 \leq \|\mathbf{x}\|_\infty \min_{\phi \in [0, 2\pi)} \|\mathbf{x}_{est}^{(\theta)} - e^{i\phi}\mathbf{x}^{(\theta)}\|_2 + C \cdot d^{1/4} \sqrt{\kappa \|\mathbf{n}\|_2}, \ C \in \mathbb{R}^+$

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Improved NFP Recovery Theorem

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Theorem (Corollary 3 - F. Filbir, F. Krahmer, O. Melnyk)

Consider the associated weighted graph $G_W = (V, E, \mathbf{W})$ *with weight matrix* $\mathbf{W}$ *defined entrywise by*

$$W_{i,j} = \begin{cases} |\widehat{X}_{i,j}|^2, & 0 < |i-j| \text{ mod } d < \delta \\ 0, & \text{otherwise} \end{cases}$$

Then we have that $\min_{\phi \in [0, 2\pi)} \|\mathbf{x}_{est}^{(\theta)} - e^{i\theta}\mathbf{x}^{(\theta)}\|_2 \leq C\sqrt{1 + \|\mathbf{x}\|_\infty} \cdot \frac{\|\mathbf{X} - \widehat{\mathbf{X}}\|_F}{\sqrt{\tau_{G_W}}}$, $C \in \mathbb{R}^+$

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Then we have that $\min_{\phi \in [0,2\pi)} \|\mathbf{x}_{est}^{(\theta)} - e^{i\theta}\mathbf{x}^{(\theta)}\|_2 \leq C\sqrt{1 + \|\mathbf{x}\|_\infty} \cdot \frac{\|\mathbf{X} - \widehat{\mathbf{X}}\|_F}{\sqrt{\tau_{G_W}}}$, $C \in \mathbb{R}^+$

- $\|\mathbf{X} - \widehat{\mathbf{X}}\|_F \leq \kappa \|\mathbf{n}\|_2$

Improved NFP Recovery Theorem

- In *On Recovery Guarantees for Angular Synchronization - Filbir, F., Krahmer, F., Melnyk, O.*, a tighter bound of the phase difference is given, using a weighted graph apporach

Theorem (Corollary 3 - F. Filbir, F. Krahmer, O. Melnyk)

Consider the associated weighted graph $G_W = (V, E, \mathbf{W})$ with weight matrix $\mathbf{W}$ defined entrywise by

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- $\|\mathbf{X}-\widehat{\mathbf{X}}\|_F \leq \kappa \|\mathbf{n}\|_2$

Theorem (Improved NFP Recovery Theorem - M. Roach, et al.)

We have that for our choice of **p** *and* **m**

$$\min_{\phi \in [0,2\pi)} \|\mathbf{x}_{est} - e^{i\phi}\mathbf{x}\|_2 \leq C \cdot \left(\sqrt{\|\mathbf{x}\|_\infty^2 + \|\mathbf{x}\|_\infty^3} \cdot \frac{\delta^2 \|\mathbf{n}\|_2}{\sqrt{\tau_{G_W}}} + d^{1/4} \cdot \delta \cdot \sqrt{\|\mathbf{n}\|_2} \right), \; C \in \mathbb{R}^+$$

Near Field Ptychography via Wirtinger Flow

- In *"Phase Retrieval via Wirtinger Flow: Theory and Algorithms - Candes, Li, Soltanolkotabi"*, quadratic equations of the form $y_n = |\langle \mathbf{m}_\ell, \mathbf{x} \rangle|^2, \; n = 1, 2, \ldots, N$ are solved by applying a gradient descent like algorithm

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- We compute the estimate $\mathbf{z}_0$ via a spectral method and then compute T iterations of $\mathbf{z}_{\tau+1} = \mathbf{z}_\tau - \frac{\mu_{\tau+1}}{||\mathbf{z}_0||^2} \nabla f(\mathbf{z}_\tau)$, where $f(\mathbf{z})$ is a quadratic loss function.

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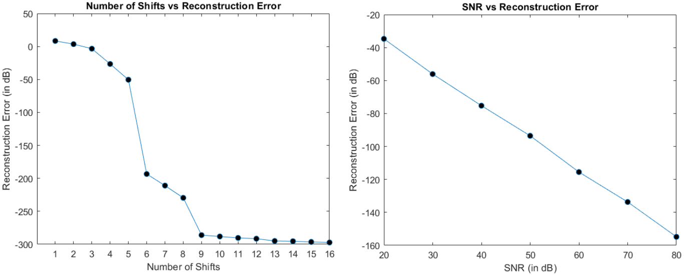
- We then let $\mathbf{x}_{est} = \mathbf{z}_T$



$d = 16$, $\mathbf{p}$ low-pass filter, $\mathbf{m}$ fixed mask, $\mathcal{L} = [d]$, averaged over 20 tests

Open Problems

- Finding recovery guarantees for when **p** is a low pass filter
- In particular, finding what choice of **m** accomplishes this
- Allowing **m** to have full support in spatial domain, with perhaps small support in the frequency domain
- Providing recovery guarantees for when less shifts have been used
- Compute weighted spectral gap for improved recovery theorem

References

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Any questions?

NFP Recovery Algorithm

Algorithm NFP-BlockPR

Input: 1) Variables $d, \delta, D = d(2\delta - 1)$
2) PSF $\mathbf{p} \in \mathbb{C}^d$, mask $\mathbf{m} \in \mathbb{C}^d$, $supp(\mathbf{m}) \subseteq [\delta]$
3) Ptychographic measurement matrix $\mathbf{Y} \in \mathbb{C}^{d \times 2\delta - 1}$

Output: $\mathbf{x}_{est}$ with $\mathbf{x}_{est} \approx e^{i\theta}\mathbf{x}$ for some $\theta \in [0, 2\pi]$

1) Compute collection of masks $\check{\mathbf{m}}_\ell^{(p,m)} = \overline{S_\ell \tilde{\mathbf{p}} \circ \mathbf{m}} \in \mathbb{C}^d$ and corresponding matrix $\check{\mathbf{M}} \in \mathbb{C}^{D \times D}$
2) Rearrange $\mathbf{Y}$ by $Y_{k,\ell} = Y_{-k,\ell+k}$
3) Compute $\mathbf{z} = \check{\mathbf{M}}^{-1}\text{vec}(\mathbf{Y}) \in \mathbb{C}^D$.
4) Reshape $\mathbf{z}$ to obtain a matrix $\mathbf{X}_{est}$ with estimated entries of $\mathbf{x}\mathbf{x}^*$
5) Let $\mathbf{u} \in \mathbb{C}^d$ be the largest eigenvector of $\mathbf{X}_{est}$. Then

$$\mathbf{x}_{est} = \sqrt{diag(\mathbf{X}_{est})} \circ sgn(\mathbf{u})$$

Phase Retrieval via Wirtinger Flow

- In *"Phase Retrieval via Wirtinger Flow: Theory and Algorithms - Candes, Li, Soltanolkotabi"*. quadratic equations of the form

$$y_n = |\langle \mathbf{m}_\ell, \mathbf{x} \rangle|^2, \quad n = 1, 2, \ldots, N$$

  are solved by applying a gradient descent like algorithm to solve the non-convex minimization problem

$$\text{minimize} \;\; f(\mathbf{z}) := \frac{1}{2N} \sum_{\ell=1}^{L} (|\mathbf{m}_\ell^* \mathbf{z}|^2 - y_\ell)^2, \quad \mathbf{z} \in \mathbb{C}^d$$

- For our measurements, by letting $Y_{k,\ell} = Y_{k,\ell+k}$, we can rewrite this system as

$$(vec(\mathbf{Y}))_n = |\langle S_{\mathcal{K}(\lfloor \frac{n}{L} \rfloor)} \widetilde{\mathbf{m}}^{(p,m)}_{\mathcal{L}(n \bmod L)}, \mathbf{x} \rangle|^2, \quad \forall n \in [KL]$$

  where vec represents the row vectorization, $K = |\mathcal{K}|$, $L = |\mathcal{L}|$
- We can then apply the Wirtinger flow algorithm to these set of equations.

Wirtinger Gradient Descent Algorithm

Algorithm NFP Wirtinger Flow

Input: 1) Variable $d \in \mathbb{N}$, set of shifts $\mathcal{K} \subseteq [d]$, set of frequencies $\mathcal{L} \subseteq [d]$, # of iterations T

2) PSF $\mathbf{p} \in \mathbb{C}^d$, mask $\mathbf{m} \in C^d$, $\overline{\widetilde{\mathbf{m}}_\ell^{(p,m)}} = S_\ell \tilde{\mathbf{p}} \circ \mathbf{m}$

3) Measurements $Y_{k,\ell} = |(\mathbf{p} * (S_k \mathbf{m} \circ \mathbf{x}))_\ell|^2,\ (k,\ell) \in \mathcal{K} \times \mathcal{L}, K = |\mathcal{K}|, L = |\mathcal{L}|$

Output: $\mathbf{x}_{est} \in \mathbb{C}^d$ with $\mathbf{x}_{est} \approx e^{\mathrm{i}\theta}\mathbf{x}$ for some $\theta \in [0, 2\pi]$

1) Rearrange measurement matrix such that $Y_{k,\ell} = Y_{k,\ell+k}$

2) Compute initial estimate $\mathbf{z}_0$ via spectral method

3) Compute Wirtinger Flow non-convex optimization with T iterations

for $t \in [T]$ **do**

$$\mathbf{z}_{\tau+1} = \mathbf{z}_\tau - \frac{\mu_{\tau+1}}{||\mathbf{z}_0||^2} \nabla f(\mathbf{z}_\tau), \quad f(\mathbf{z}) := \frac{1}{2KL} \sum_{\ell=1}^{L} (|(S_{\mathcal{K}(\lfloor \frac{n}{L} \rfloor)} \widetilde{\mathbf{m}}^{(p,m)}_{\mathcal{L}(n \bmod L)})^* \mathbf{z}|^2 - y_\ell)^2$$

end for

4) Let $\mathbf{x}_{est} = \mathbf{z}_T$