Notes of Calcolo delle Variazioni

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Contents

Preface

This work contains the transcription of the notes of the lectures of Calcolo delle Variazioni held at Università Statale di Milano. As these notes weren't initially designed to the students, I did a few additions and modifications in order to standardise the notation to that of the courses I previously attended and provide a more detailed contour of some of the treated topics. Hence, I am the solely responsible for any possible mistake the reader may find in this work. Also, as this work was previously written in Italian, I apologize for any possible mistake due to a quick and inattentive translation. **Finally, keep in mind that these notes are meant to be accessible to all the students for free, not to restricted clientelistic cults** †.

In Chapter 1, the reader will find a brief and systematic introductory treatment of the Classical Variational Calculus, with the sole purpose of introducing a method of solving some optimization problems through the Euler-Lagrange equation.

Chapter 2 contains generic functional analysis topics we use in this notes, notably the definition and the first properties of a differentiation theory in infinte-dimensional Banach spaces, as well as an introduction to Sobolev spaces. For a general treatment of the latest, I refer to [3].

Chapter 3 is devoted to the optimization problem for functionals that are defined on generic Banach spaces, which involves an extension to infinite-dimensional Banach spaces of known results of classical analysis, such as Fermat and Weierstrass theorems and Lagrange multipliers.

In Chapter 4 the deformation lemma is discussed. This is a result that allows to find contradictions in the proofs that will follow; moreover, a compactness-recovery condition and three variants of the mountain pass theorem, which allows to prove the existence of critical points for functionals on Banach spaces, are discussed.

In Chapter 5, a topologic theory related to odd functions is developed, in order to enunciate a fourth and latest version of mountain pass theorem for even functionals.

The "subcritical" growth of some disturbances taken into consideration in these chapters is in a certain sense fundamental to the examples of Chapters 1-5. Therefore, Chapter 6 is finally left to a "critical" growth situation (the adjective "critical" is related to the definition of Sobolev critical exponent). In these cases, sometimes, the compactness condition defined in Chapter 4 is no longer holding, so that a weaker

†Every reference to existent people and facts is absolutely intended.

version of it has to be formulated in order to prove the uniqueness of the solution for the boundary-value problem related to the equation $-\Delta u = \lambda u + u^{\frac{n+2}{n-2}}$.

I wish you a good work,
G. Giacchi.

Chapter 1

Classical Variational Calculus

In order to find critical points in the interior of the domain of a function $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$, one has to find the solutions of the system of equations

$$\nabla f(x) = 0.$$

The purpose of this chapter is that of introducing a model of problems which can be faced and solved through the classical variational calculus.

In particular, we will search the critical points of a certain (non-necessarly linear) functional defined on a space of functions $E \subseteq \mathcal{C}^0([a, b])$, proceeding in the direction of a result that is an analogous version of Fermat theorem for differentiable functions defined on open subsets of $\mathbb{R}^n$. That is, a result that allows to trace the search of solutions $u \in E$ which minimize or maximize a functional J back to solving an equation (as we expect, a differential equation) in the u variable.

The question of which functionals and which function spaces are well-suited for this kind of problem will be clear once the Euler-Lagrance equation is presented.

Remark 1.0.1. As you probably noticed, we intentionally avoided the term "critical point" of the functional J, when J is a functional defined on some functional space. The reason why we did it is that we haven't already defined what the differential of such a functional is.

Let E be a space of properly regular functions, defined on some interval $[a, b]$, in order for the following argument to make sense. Let $J \in E^*$ be a functional in the form

$$J(u) = \int_a^b L\left(x, u(x), u'(x)\right) dx \tag{1.1}$$

(we call such a functional the **action integral**).
L is called the **Lagrangian** of the system and it is a function which is strictly related to the problem we are modelling. In practice, we consider optimization problems for

some quantity which can be expressed as a functional J, which in turn can be expressed as an integral of a function L on some interval $[a, b]$.

For the moment, we suppose that L, as well as every function belonging to E, enjoys every regularity property necessary for the argument that follows to make sense.

Let $u \in E$ be a maximum or a minimum for J and let $h \in \mathcal{C}_C^\infty((a, b))$ (so that the support of h is strictly included in (a, b)). Since u is a maximum/minimum, for $\varepsilon \in \mathbb{R}$, the function

$$\varphi(\varepsilon) = J(u + \varepsilon h)$$

is well defined (as long as E is closed with respect to the sum of its functions with multiples of $\mathcal{C}^\infty$ functions with compact support) and it has a maximum/minimum in correspondence of the value $\varepsilon = 0$. Under the hypothesis on L required for φ to be differentiable in 0, by Fermat theorem:

$$\begin{aligned}
0 = \varphi'(0) = \lim_{\varepsilon \to 0} \frac{\varphi(\varepsilon) - \varphi(0)}{\varepsilon} &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(J(u + h\varepsilon) - J(u) \right) = \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_a^b \big(L(x, u + h\varepsilon, (u + h\varepsilon)') - L(x, u, u') \big)\, dx = \\
&= \lim_{\varepsilon \to 0} \int_a^b h \frac{L(x, u + h\varepsilon, (u + h\varepsilon)') - L(x, u, (u + h\varepsilon)')}{h\varepsilon} dx + \\
&+ \lim_{\varepsilon \to 0} \int_a^b h' \frac{L(x, u, (u + h\varepsilon)') - L(x, u, u')}{h'\varepsilon} dx =
\end{aligned}$$

(changing limit and integral)

$$= \int_a^b h L_u(x, u, u') dx + \int_a^b h' L_{u'}(x, u, u') dx =$$

(integrating by parts and using the hypothesis on the boundary values of h)

$$= \int_a^b h \left[L_u - \frac{d}{dx} L_{u'} \right] dx.$$

The conclusion of the argument will follow from the following lemma:

Lemma 1.0.2. *Let $v \in \mathcal{C}^0([a, b])$ be s.t. $\forall \varphi \in \mathcal{C}_C^\infty((a, b))$*

$$\int_a^b v \varphi dx = 0.$$

Then, $v \equiv 0$ on $[a, b]$.

Proof. Seeking a contradiction, let $x_0 \in [a, b]$ be s.t. $v(x_0) \neq 0$ then, by continuity, it would be $v \neq 0$ in a neighborhood $I \subseteq [a, b]$. Without loss of generality, we could suppose $v > 0$ on I.

We need a function $\varphi \in \mathcal{C}_C^\infty((a,b))$ to turn the integral $\int_a^b v\varphi dx$ on $\int_I v dx$.
Hence, let $\varphi \in \mathcal{C}_C^\infty(a,b)$ be s.t. $0 \leq \varphi \leq 1$, $\text{supp}(\varphi) \subset I$ e $\varphi|_{\tilde{I}} \equiv 1$ for some $\tilde{I} \subset I$. This function exists by the well known Urysohn's lemma. Then, it would be:

$$0 = \int_a^b v\varphi dx = \int_I v\varphi dx > \int_{\tilde{I}} v\varphi dx = \int_{\tilde{I}} v dx > 0.$$

This is a contradiction. □

Remark 1.0.3. A standard density argument allows to extend Lemma (1.0.2) up to consider test functions $v \in \mathcal{C}^0(\bar{\Omega})$, where $\Omega \subset\subset \mathbb{R}^n$ is open (this notation is reserved for subsets of $\mathbb{R}^n$ which have compact closure).

Back to our problem, we showed that for all $h \in \mathcal{C}_C^\infty(a,b)$ it is

$$\int_a^b h \left[L_u - \frac{d}{dx} L_{u'} \right] dx = 0. \tag{1.2}$$

By Lemma (1.0.2) (supposing that L enjoys of all the regularity properties needed to ensure $L_u - d/dx L_{u'}$ to be continuous), u is a solution of the equation

$$L_u = \frac{d}{dx} L_{u'} \tag{E-L}$$

(called the **Euler-Lagrange equation**).

Remark 1.0.4. By this proof it is clear that a space of functions E, for which the argument makes sense, can be

$$E = \{u \in \mathcal{C}^0([a,b]) \text{ and piecewise } \mathcal{C}^1((a,b)) \text{ s.t. } u(a) = \alpha \text{ and } u(b) = \beta\}.$$

(this space is closed with respect to the mapping $u \in E \mapsto u + \varepsilon h$ for ε small enough and $h \in \mathcal{C}_C^\infty$).
In this case, the maxima/minima points for J solve the problem

$$\begin{cases} L_u = \frac{d}{dx} L_{u'} & \text{on } (a,b), \\ u(a) = \alpha, \\ u(b) = \beta. \end{cases}$$

In this example we can already glimpse an interesting flap of the method of variational calculus: starting from the research of maxima/minima points of certain functionals we come across the research of some appropriate solutions of some associated differential equations. Most of the work of these notes will be that of inverting this consideration, getting solutions of PDEs by finding minima/maxima points for some functional that is related to the differential equation.

Remark 1.0.5. In (E-L) we denoted with a subscript the derivatives in the variables of L, i.e. the partial derivatives of $L(x, u, u')$. The symbol d/dx is left to the derivative with respect to x of $L_{u'}(x) = L_{u'}(x, u(x), u'(x))$.

Remark 1.0.6. Euler-Lagrange equation can be simplified in the case in which $L_x = 0$. In fact, by multiplying Euler-Lagrange equation for u', we get

$$u'L_u = u'\frac{d}{dx}L_{u'}.$$

As we already observed, L can be interpreted as a function in the x variables in two ways. Now, we consider $L(x) = L(x, u(x), u'(x))$ and we derive with respect to x:

$$\frac{d}{dx}L = L_x + u'L_u + u''L_{u'} = u'L_u + u''L_{u'},$$

therefore,

$$u'L_u = \frac{d}{dx}L - u''L_{u'}.$$

Using the chain rule for derivatives and Euler-Lagrange equation multiplied by u', we get:

$$\frac{d}{dx}L - u''L_{u'} = u'\frac{d}{dx}L_{u'} \implies \frac{d}{dx}L = \frac{d}{dx}\left(u'L_{u'}\right). \tag{1.3}$$

This equation can be directly integrated, to get:

$$\int_{x_0}^{x} \frac{d}{dt}L(t, u(t), u'(t))dt = \int_{x_0}^{x} u'(t)L_{u'}(t, u(t), u'(t))dt.$$

To simplify the notation, we re-interpret (1.3) in terms of anti-derivatives:

$$L(x, u(x), u'(x)) = u'(x)L_{u'}(x, u(x), u'(x)) + C. \tag{1.4}$$

(1.4) is called the **Beltrami identity**.

In the following three applications of (E-L), we denote with $\dot{g}$ the derivative of g even when we are not deriving with respect to t, instead of g', to lighten the notation (for instance, $(g'')^2$ becomes $\ddot{g}^2$):

Example 1.0.7. **Geodesics of $\mathbb{R}^2$**
Let $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ be two fixed points $\mathbb{R}^2$ with $x_0 \neq x_1$.
In this example, we define

$$E = \{u \in \mathcal{C}^0([x_0, x_1]),\ \text{piecewise } \mathcal{C}^1((x_0, x_1)) \text{ s.t. } u(x_0) = y_0 \text{ and } u(x_1) = y_1\}$$

the space of piecewise $\mathcal{C}^1$ paths which connect P to Q. In the spirit of the definition of the distance induced by a Riemannian metric, we want to minimize the length of such

paths and prove that the curve $\gamma(s) = (s, u(s))$ $(s \in [a,b])$, minimizing the distance from P to Q, is the segment that connects them.
Thus, we define

$$L(x,u,u') = L(u') = |\gamma'| = \sqrt{1+\dot{u}^2}$$

(here, $\dot{u} = du/ds$). We look for the minimizing curve among the solutions of (E-L):

$$0 = L_u = \frac{d}{ds}L_{\dot{u}} = \frac{d}{ds}\frac{\dot{u}}{\sqrt{1+\dot{u}^2}} = \frac{\ddot{u}}{(1+\dot{u}^2)^{3/2}}.$$

the denominator is always positive, hence it must be

$$\ddot{u} = 0,$$

so that, $\dot{u} = m \in \mathbb{R}$, that is $u(x) = mx + q$ for some properly chosen $m, q \in \mathbb{R}$.

Example 1.0.8. **Brachistochrone of $\mathbb{R}^2$**
Let P and Q in $\mathbb{R}^2$ be as in the previous example. We want the optimizing profile held by a track that connects P to Q in order to minimize the comedown time of a point subject to the gravity force which is initially located in P.
In this example, E is the set of the functions $u : [x_0, x_1] \to \mathbb{R}$ for which the following calculation make sense.
By the law of conservation of mechanical energy, the speed of a point at a height $u(x)$ is given by $v(x) = \sqrt{2gu(x)}$, so that the comedown time is given by

$$J(u) = \int_{x_0}^{x_1} \frac{\sqrt{1+\dot{u}^2}}{\sqrt{2gu}}dx$$

Using Euler-Lagrange equation:

$$L_u = \frac{d}{dx}L_{\dot{u}}.$$

In this case:

$$-\frac{\sqrt{1+\dot{u}^2}}{2u^{3/2}\sqrt{2g}} = \frac{d}{dx}\left(\frac{1}{\sqrt{2gu}}\frac{\dot{u}}{\sqrt{1+\dot{u}^2}}\right) = \frac{\ddot{u}\sqrt{u}\sqrt{1+\dot{u}^2} - \dot{u}\left(\frac{1}{2\sqrt{u}}\dot{u}\sqrt{1+\dot{u}^2} + \sqrt{u}\frac{\dot{u}\ddot{u}}{\sqrt{1+\dot{u}^2}}\right)}{\sqrt{2g}u(1+\dot{u}^2)} =$$

$$= \frac{\ddot{u}\sqrt{u}\sqrt{1+\dot{u}^2} - \dot{u}\frac{\dot{u}(1+\dot{u}^2)+2u\dot{u}\ddot{u}}{2\sqrt{u}\sqrt{1+\dot{u}^2}}}{\sqrt{2g}u(1+\dot{u}^2)} = \frac{2u\ddot{u}(1+\dot{u}^2) - \dot{u}\left(\dot{u}(1+\dot{u}^2)+2u\dot{u}\ddot{u}\right)}{2\sqrt{2g}(1+\dot{u}^2)^{3/2}u^{3/2}} =$$

$$= \frac{2u\ddot{u} - \dot{u}^2 - \dot{u}^4}{2\sqrt{2g}(1+\dot{u}^2)^{3/2}u^{3/2}}.$$

That is,

$$2u\ddot{u} - \dot{u}^2 - \dot{u}^4 = -(1+\dot{u}^2)^2 = -1 - 2\dot{u}^2 - \dot{u}^4.$$

By simplifying the previous expression, we get:

$$\ddot{u} = -\frac{1+\dot{u}^2}{2u},$$

that is the equation that defines the cycloid.

Example 1.0.9. **Isoperimetric problems**
This example is typical of a class of problems in which the optimization is subject of a contraint.
We want to maximize the area underlying a regular circuit with a fixed perimeter. In order to simplify the question, we reduce to the case of symmetric curves (with respect to x axis) which pass through the origin. Any of these curves is divided by the x axis into two symmetrical branches. The superior arc is the graph of a concave function $y = y(x)$ defined on $[0, \xi]$ (here, ξ depends on the chosen curve) and such that $y(0) = y(\xi) = 0$. Let L be the (fixed) perimeter.
The area underlying the curve is given by

$$J(\gamma) = \int_0^L y\sqrt{1-\dot{y}^2}ds.$$

We write Euler-Lagrange equations for J:

$$\frac{d}{dx}L_{\dot{y}} = \frac{d}{dx}\left(-\frac{y\dot{y}}{\sqrt{1-\dot{y}^2}}\right) = -\frac{[\dot{y}^2 + y\ddot{y}]\sqrt{1-\dot{y}^2} + y\dot{y}\frac{\dot{y}\ddot{y}}{\sqrt{1-\dot{y}^2}}}{1-\dot{y}^2} =$$
$$= -\frac{[\dot{y}^2 + y\ddot{y}](1-\dot{y}^2) + y\dot{y}^2\ddot{y}}{(1-\dot{y}^2)^{3/2}} = \frac{\dot{y}^4 - \dot{y}^2 - y\ddot{y}}{(1-\dot{y}^2)^{3/2}}.$$

While

$$L_y = \sqrt{1-\dot{y}^2}.$$

Putting the two expressions above together, we get the equation satisfied by y:

$$\ddot{y} = -\frac{1-\dot{y}^2}{y}.$$

We observe immediately that $\forall\alpha \in \mathbb{R}\setminus\{0\}$ the function $y_\alpha(x) = \alpha\sin(x/\alpha)$ is a solution of the Euler-Lagrange equation associated to J.
These functions intersect the x axis in $x = 0, k\pi\alpha$, $k \in \mathbb{Z}$. In particular, the arc we are interested in is the one detected by the graph of $y_\alpha(x)$ in $[0, \alpha\pi]$.
To get the value(s) of α under which the graph of y_α has length $L/2$ we need to solve the following equation:

$$\int_0^{\alpha\pi} \sqrt{1+\cos^2\left(\frac{x}{\alpha}\right)}dx = \frac{L}{2}.$$

Remark 1.0.10. Beltrami identity (1.4) can be used to get the same conclusions as the last two examples. Caution should be exercised, however, when eliminating in the right way the degree of freedom provided by the unknown x_0 in (1.3).

Remark 1.0.11. Consider

$$J(u) = \int_0^1 x^2 \dot{u}(x)^2 dx,$$

defined on

$$E = \{u : [0,1] \to \mathbb{R} \text{ piecewise differentiable s.t. } u(0) = 0 \text{ and } u(1) = 1\}.$$

$\inf_{u \in E} J(u) = 0$, in fact, $J(u) \geq 0$ for all u and it is easy to see that the sequence

$$u_k(x) = \frac{\arctan(kx)}{\arctan k}$$

satisfies $\lim_{k\to\infty} J(u_k(x)) = 0$.
Moreover, observe that

$$\lim_{k\to\infty} u_k(x) = u(x) = \begin{cases} 0 & x = 0 \\ 1 & x \in (0,1] \end{cases}$$

and, clearly, $u \notin E$.
Actually, it is immediate to observe that the infimum cannot be attained on E since, if there were $u \in E$ such that

$$\int_0^1 x^2 \dot{u}^2 dx = 0,$$

then it would be $x^2\dot{u}^2 = 0$ on $[0,1]$, so that $\dot{u}$ would be zero on $[0,1]$ and u would be zero as well, as u is continuous and piecewise constant function with zero integral. But $0 \notin E$, so that $u \notin E$.
In the next chapter we will often be in the situation of proving that the infimum m of a certain functional J defined on a Banach space E is attained by some $u \in E$. The idea is that of starting from a minimizing sequence $\{u_k\}_k$ and producing a converging subsequence u_{k_j} to a certain function $\bar{u} \in E$, then using some lower regularity property of J to prove that $J(\bar{u}) = m$. Through this observation, we made it evident that a powerful theory of this kind cannot be realized choosing E among the classical functional spaces.

Chapter 2

Topics of nonlinear functional analysis

Let E and F be infinite-dimensional normed spaces. We denote with

$$BL(E,F) = \{T : E \to F \text{ linear and bounded}\}.$$

$BL(E,F)$ is a normed linear space with the norm defined by

$$\|f\|_{E\to F} = \sup_{x\in S_E} \|f(x)\|_F .$$

We recall the following important relation between the *Banach-ness* * of $BL(E,F)$ and that of F.

Proposition 2.0.1. *$BL(E,F)$ is a Banach space if and only if F is a Banach space.*

2.1 Derivatives on Banach spaces

Definition 2.1.1. Let $O \subseteq E$ be open and $u \in O$. A mapping $f : O \to E$ is called a **Fréchet-differentiable operator** (or, brefly, an **F-differentialble** operator) in u if there exists $L_u \in BL(E,F)$ s.t.

$$\lim_{v\to 0} \frac{\|f(u+v) - f(u) - L_u v\|_F}{\|v\|_E}. \tag{2.1}$$

The operator $f' : u \mapsto f'(u) = L_u$ is known as **Fréchet derivative** of f and it is denoted by f'.

*I humbly beg your forgiveness.

Example 2.1.2. In finite-dimensional spaces, the previous definition is still well posed. If $E = \mathbb{R}^n$, $F = \mathbb{R}^m$ and $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then f is bounded and F-differentiable in all the $x \in \Omega$. Its Fréchet-derivative in Ω is the operator

$$f'(x)[v] = J(f)(x) \cdot v.$$

A similar argument holds for $f \in \mathcal{C}^1(\Omega)$.

Example 2.1.3. Let $T \in BL(E, F)$, then $T'(u) = T$ for all the $u \in E$. In fact, by the uniqueness of limits in metric spaces, if the limit in (2.1) exists, then it is unique and so is its Fréchet derivative. On the other hand,

$$\lim_{v \to 0} \frac{\|T(u+v) - Tu - Tv\|_F}{\|v\|_E} = 0.$$

Remark 2.1.4. If f is F-differentialble in $u \in O$, then f is continuous in u.

Definition 2.1.5. If f is F-differentiable in u, $\forall u \in O$, and $f' : u \in E \mapsto f'(u)$ is continuous, we write $f \in \mathcal{C}^1(O)$.

Definition 2.1.6. Let E be a Banach space and $J \in \mathcal{C}^1(E, \mathbb{R})$. $u \in E$ is called a **critical point** of J if $J'(u)[v] = 0$, $\forall v \in E$.

Definition 2.1.7. Let $O \subseteq E$ be open. An operator $f : O \to F$ is called a **Gâteaux-differentiable operator** (or, briefly, **G-differentiable**) in $u \in O$ if $\forall v \in E$ the limit

$$f'_G(u)[v] := \lim_{h \to 0} \frac{f(u+hv) - f(u)}{h} \tag{2.2}$$

exists.

Remark 2.1.8. As in $\mathbb{R}^n$

$$\text{differentiable} \implies \text{every directional derivative exists,}$$

F-differentiable $\implies$ G-differentiable. The vice versa is not always holding. However, if f is G-differentiable and $v \in E \mapsto f'_G(u)[v]$ is linear and bounded, then f is F-differentiable in u.

Example 2.1.9. Let $f : \mathcal{C}^2([0,1]) \to \mathcal{C}^0([0,1])$ be the operator

$$f(u) = u'' + u^3,$$

that is, $f = d^2/dx^2 + g$ with $g(u) = u^3$. Then, $f'(u) = d/dx^2 + 3u^2\mathbb{1}$, where $\mathbb{1}(v) = v$. In fact,

$$(u+v)^3 - u^3 = 3u^2v + 3uv^2 + v^3 = 3u^2v + o(\|v\|_{\mathcal{C}^2})$$

and, by the Example (2.1.3), the F-derivative of d^2/dx^2 is d^2/dx^2.

Theorem 2.1.10. *Let $J : L^2 \to L^2$ be an F-differentiable operator. Then, J is linear.*

Example 2.1.11. The functional
$$\int_\Omega |\nabla u|^2 dx$$
does not attains its infimum on $\mathcal{C}^1(\Omega)$.

Remark 2.1.12. We want practical results providing the critical points of differential operators, in order to solve PDEs as we solved ODEs in the examples of the previous chapter. First, we need to decide on which functional spaces E it is convenient to optimize functionals such as the action integrals.
We know that the classical $\mathcal{C}^k$ spaces are not even closed, so that the realization of the infimum is not granted there. On the other hand, Theorem (2.1.10) tells us that even L^2 is not a well-suited space where results, such as Fermat theorem, that allow to establish which the critical points of operators f are by computing the zeros of its Fréchet derivative, can be stated.
Hence, we search for "intermediate space", on which there exist F-differentiable, non-zero and closed operators.

Example 2.1.13. Let $E = \mathcal{C}^1_C(\Omega)$ and $g \in \mathcal{C}^0(\mathbb{R})$. We set
$$G(s) = \int_0^s g(t)dt$$
an antiderivative of g and we consider the operator $I : \mathcal{C}^1_C(\Omega) \to \mathbb{R}$ defined by
$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega G \circ u(x) dx.$$
For all $v \in \mathcal{C}^1_C(\Omega)$,
$$\begin{aligned} I(u+v) - I(u) &= \frac{1}{2}\int_\Omega |\nabla (u+v)|^2 dx + \int_\Omega G \circ (u+v)(x) dx - \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G \circ u(x) dx = \\ &= \int_\Omega \nabla u \cdot \nabla v dx + \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \int_\Omega \left(G \circ (u+v) - G \circ u\right) dx = \\ &= \int_\Omega \nabla u \cdot \nabla v dx + o(\|v\|) + \int_\Omega \left(v g \circ u + o(\|v\|)\right) dx. \end{aligned}$$
So that, I is F-differentiable with
$$I'(u)[v] := \int_\Omega \left(\nabla u \cdot \nabla v + v(g \circ u)\right) dx.$$
The critical points of I are the $u \in E$ s.t.
$$\int_\Omega \left(\nabla u \cdot \nabla v + v g(u)\right) dx = 0 \qquad \left(\forall v \in \mathcal{C}^1_C(\Omega)\right). \tag{2.3}$$

If $u \in \mathcal{C}^2_C(\Omega)$ (we take u compactly supported in order for $-\Delta u + g(u)$ to be continuous up to $\bar{\Omega}$), we can integrate by parts:

$$\int_\Omega (-\Delta u + g \circ u)\, v dx = 0 \tag{2.4}$$

and this relation holds, a fortiori, for all $v \in \mathcal{C}^\infty_C(\Omega)$ (since it holds for all $v \in \mathcal{C}^1_C(\Omega)$). By Lemma (1.0.2), (2.4) is equivalent to searching solutions belonging to $\mathcal{C}^2(\Omega)$ of the problem

$$\begin{cases} -\Delta u + g \circ u = 0 & \text{on } \Omega, \\ u|_{\partial\Omega} \equiv 0. \end{cases}$$

2.2 Sobolev spaces

The best-suited spaces for the development of an optimization-on-Banach-spaces theory are the so-called Sobolev spaces.

We start by weakening the concept of differentiation, defining the "weak derivatives" of any locally integrable function as the function (if it exists) that behaves as its derivative when integrating by parts. More precisely:

Definition 2.2.1. Let $\Omega \subseteq \mathbb{R}^n$ and let $u \in L^1_{loc}(\Omega)$. A function $v \in L^1_{loc}(\Omega)$ is called the **weak i−th derivative of** u in Ω if $\forall \varphi \in \mathcal{C}^\infty_C(\Omega)$,

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega v \varphi dx.$$

In the which case, we write

$$v = \frac{\partial u}{\partial x_i} = D_i u.$$

Remark 2.2.2. If it exists, each weak derivative is unique.

Example 2.2.3. The function $f(x) = |x|$ defined on $\mathbb{R}$ is differentiable in the weak sense on $\mathbb{R}$ and its weak derivative is given by $f'(x) = \operatorname{sgn}(x)$, while it is not *strongly* differentiable (that is, differentiable in the classical sense) in any neighborhood of $x = 0$. Vice versa, if u is differentiable in the classical sense in $\Omega \subseteq \mathbb{R}$, then u is obviously differentiable in the weak sense in Ω and its weak derivative coincides with the classical one.

Definition 2.2.4. Let $p \in [1, +\infty]$. We define the **Sobolev space** $W^{1,p}(\Omega)$ as the space of L^p functions having all their weak derivatives belonging to L^p.

$W^{1,p}(\Omega)$ is a normed space under any of the two equivalent norms

$$\begin{aligned} \|u\|_{W^{1,p}} &= \|u\|_p + \sum_{i=1}^n \|D_i u\|_p\,; \\ &= \left(\|u\|_p^p + \sum_{i=1}^n \|D_i u\|_p^p \right)^{1/p}. \end{aligned}$$

(clearly, as long as the second norm is concerned, it must be $p \neq \infty$). With an abuse of notation we will denote with $\|\cdot\|_{W^{1,p}}$ any of the two equivalent norms defined above. The same abuse of notation will be used over and over in this work when equivalent norms are involved.

Proposition 2.2.5. *The space* $(W^{1,p}(\Omega), \| \ \|_{W^{1,p}})$ *is*

(a) a Banach space if $p \in [1, +\infty]$*;*

(b) reflexive if $p \in (1, +\infty)$*;*

(c) separable if $p \in [1, +\infty)$*.*

Definition 2.2.6. Let $p < n$. We define the **Sobolev critical exponent** of p as

$$p^* = \frac{np}{n-p}.$$

Remark 2.2.7. As it is easy to see,

- $2^* = \frac{2n}{n-2}$ and $2^* - 1 = \frac{n+2}{n-2}$;
- $p^* > 1$ for all $p < n$.

Theorem 2.2.8 (Sobolev embedding theorem)**.** *Let* $\Omega \subset\subset \mathbb{R}^n$ *and* $p < n$*. Then,*

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

for all $q \in [1, p^*]$ *(i.e., the embedding is continuous). In particular, for all* $u \in W^{1,p}(\Omega)$*,*

$$\|u\|_q \leq C \|u\|_{W^{1,p}}$$

for all $q \in [1, p^*]$*.*

Moreover, if $q \neq p^*$, in the previous theorem, the embdedding of $W^{1,p}$ in L^q is compact, as it is stated by the next result.

Theorem 2.2.9 (Rellich-Kondrachov)**.** *Let* $\Omega \subset\subset \mathbb{R}^n$ *and* $p < n$*. Then,*

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$$

for all $p \in [1, p^*)$ *and the embedding is compact, that is any of the following equivalent conditions holds:*

(a) for all the bounded sequences $\{u_n\}_n \subset W^{1,p}(\Omega)$*, there exists a subsequence* $\{u_{n_k}\}_k \subseteq \{u_n\}_n$ *which converges in* $L^q(\Omega)$*.*

(b) for all the $A \subseteq W^{1,p}(\Omega)$ *bounded,* $\bar{A} \subset\subset L^q(\Omega)$*.*

We set

$$H^1(\Omega) := W^{1,2}(\Omega).$$

H^1 is a Hilbert space under the inner product defined by

$$(u,v)_{H^1} = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega uv dx.$$

Remark 2.2.10. It is pretty obvious that $\mathcal{C}_C^\infty(\Omega) \subset W^{1,p}(\Omega)$ for all $p \in [1, +\infty]$.

We set

$$W_0^{1,p}(\Omega) := \overline{\mathcal{C}_C^\infty(\Omega)}^{W^{1,p}}.$$

In particular, for $p = 2$, we set

$$H_0^1(\Omega) := W_0^{1,2}(\Omega).$$

These spaces can be characterized as the spaces of the functions $u \in W^{1,p}$ s.t. $u|_{\partial\Omega} = 0$ almost everywhere.

Lemma 2.2.11. *Let $a, b \in \mathbb{R}$ and let $\varepsilon > 0$. Then,*

$$|ab| \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2.$$

Proof. It's an easy consequence of

$$\left(\sqrt{\varepsilon}|a| - \frac{1}{\sqrt{\varepsilon}}|b|\right)^2 \geq 0.$$

□

Lemma 2.2.12. *Let $(E, \|\cdot\|_E)$ be a normed space, $\{u_n\}_n \subseteq E$ a sequence of elements of E and let $u \in E$. The following conditions are equivalent:*

(a) $u_n \to u$ as $n \to \infty$ (in E);

(b) for every sequence $\{u_{n_k}\}_k \subseteq \{u_n\}_n$ there exists a subsequence that converges to u (in E).

Proof. (a) $\Rightarrow (b)$ it's obvious;

(b) $\Rightarrow (a)$ Seeking a contradiction, if $u_n \not\to u$ in E as $n \to \infty$, then

$$\exists \varepsilon > 0 \ t.c. \ \forall \bar{n} > 0 \ \exists n \geq \bar{n} \ t.c. \ \|u - u_n\| > \varepsilon.$$

In particular, there exists $n_1 \geq 1$ such that $\|u - u_{n_1}\| > \varepsilon$; there exists $n_2 \geq 2$ such that $\|u - u_{n_2}\| > \varepsilon$, etc. The so-defined subsequence is s.t. $\|u - u_{n_k}\| > \varepsilon$ for all k, so that it does not admit any subsequence converging to u. This is a contradiction.

□

Lemma 2.2.13. *Let T be a compact linear operator on the Banach space E taking values in the Banach space F. Let $\{u_k\}_k \subset E$ be a sequence that weakly converges to $u \in E$. Then, $Tu_k \to Tu$ as $k \to \infty$ (in F).*

Proof. Let $\{u_{k_j}\}_j$ be any subsequence of $\{u_k\}_k$. Then, $u_k \rightharpoonup u$ as $k \to \infty$ (in E), $\{u_k\}_k$ is bounded and, a fortiori, $\{u_{k_j}\}_j$ is bounded. The operator T is compact, hence there exists a subsequence Tu_{k_j} that converges in the norm of F to $Tu \in F$. Hence, any subsequence of $\{Tu_k\}_k \subset F$ admits a subsequence that in turns converges to Tu in the norm of F. The assertion follows by the previous lemma. □

In particular, if $\Omega \subset\subset \mathbb{R}^n$, $E = W^{1,p}(\Omega)$ (with $p < n$) and $F = L^q(\Omega)$ (with $q \in [1, p^*)$), by Rellich-Kondrachov's theorem the immersion $\iota \in BL(E, F)$ is compact and the following corollary follows.

Corollary 2.2.14. *Let $\Omega \subset\subset \mathbb{R}^n$, $p < n$ and let $\{u_k\}_k \subset W^{1,p}(\Omega)$ be s.t. $u_k \rightharpoonup u$ in $W^{1,p}(\Omega)$. Then, for all $q \in [1, p^*)$, $u_k \to u$ as $k \to \infty$ in $L^q(\Omega)$.*

At the same time, by the proof of (2.2.13), we get the following corollary.

Corollary 2.2.15. *Under the same assumptions of (2.2.13) on E and F, if $\{u_k\}_k \subset E$ is a bounded sequence, then $\{u_k\}_k$ converges in the norm of F up to subsequences.*

2.3 The space $H^1_0(\Omega)$

We defined the space $H^1_0(\Omega)$ as the closure of $\mathcal{C}^\infty_C(\Omega)$ in the norm of $W^{1,2}(\Omega)$ and said that H^1 is a Hilbert space under the inner product

$$(u, v)_{H^1} = \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega uv dx = (\nabla u, \nabla v)_{L^2} + (u, v)_{L^2}.$$

H^1_0 is closed in H^1, so that H^1_0 inherits from H^1 the structure of a Hilbert space under the same inner product as above. However, when H^1_0 is involved, it is convenient to define an equivalent norm, induced by a slightly different inner product.

Lemma 2.3.1 (Poincaré inequality)**.** *There exists a universal constant $C = C(n, \Omega) > 0$ such that, $\forall u \in H^1_0(\Omega)$,*

$$\|u\|_2 \le C \|\nabla u\|_2 .$$

In particular, if $u \in H^1_0(\Omega)$

$$\|\nabla u\|_2 \le \|u\|_2 + \|\nabla u\|_2 = \|u\|_{H^1} \le (1 + C) \|\nabla u\|_2 ,$$

so that the H^1 norm of u is equivalent to the following one:

$$\|u\|_{H^1_0(\Omega)} := \|\nabla u\|_2 = \int_\Omega |\nabla u|^2 dx.$$

This is actually a norm on H_0^1 that is equivalent to that of H^1 and it is induced by the following inner product [†]

$$(u, v)_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v dx = (\nabla u, \nabla v)_{L^2}$$

for all the $u, v \in H_0^1(\Omega)$.

From now on, we denote with $\|u\|_{H_0^1} = \|\nabla u\|_{L^2}$, for all $u \in H_0^1$, that differs from the norm $\|\cdot\|_{H^1} = \|\cdot\|_2 + \|\nabla \cdot\|_2$.

To sum up:

Proposition 2.3.2. $\left(H_0^1(\Omega), \|\cdot\|_{H_0^1}\right)$ *is a Hilbert space with respect to the inner product* $(\cdot, \cdot)_{H_0^1}$.

[†]We stress the fact that the equivalence holds only on H_0^1, not on the whole H^1.

Chapter 3

Optimization in Banach spaces

3.1 Introduction

Classical variational calculus deals with solving optimization problems involving functionals defined as action integrals, reproducing the standard procedure for the searching of a real-valued function's critical points in the interior of its domain $\Omega \subseteq \mathbb{R}^n$:

1. one defines the function $f : (a, b) \subseteq \mathbb{R} \to \mathbb{R}$ to be optimized;
2. one searches the maxima and/or the minima of f among the solutions of the equation $f'(x) = 0$.

This is essentially the exact procedure we followed in the first chapter, with the action integral $f = J$ and the role of $f' = 0$ played by Euler-Lagrange equation.

Modern variational calculus uses Sobolev spaces instead of the classical $\mathcal{C}^k$ spaces, while preserving the above-mentioned model for the searching of solutions. The power under this subject lies in the possibility of flipping the point of view, searching solutions (initially weak ones, then one studies their regularity) of some differential equation as critical points of a related functional $J : E \to \mathbb{R}$:

1. one considers a certain differential equation, which has the form $F(u) = 0$ for some appropriate differential operator F which usually, in our examples, turns out to be the sum of $-\Delta$ and another differential operator;
2. one associates an *energy* functional to F, say J. In the case in which F is the sum of some elementary differential operators, J is also the sum of "elementary" functionals each of which corresponds to an addendum of F;
3. one finds out that the critical points of J satisfy the initially given differential equation, so that solving $F(x, u, \nabla u, D^2 u) = 0$ is the same as solving $J'(u) = 0$.

Remark 3.1.1. The energy functional associated to the differential equation

$$-\Delta u = 0, \qquad u \in H_0^1(\Omega)$$

(or, equivalently, to the differential operator $-\Delta$), with $\Omega \subset\subset \mathbb{R}^n$, is given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx = \frac{1}{2}\|u\|^2_{H^1_0}.$$

Example 3.1.2. In the example (2.1.13) we observed that, if $g \in \mathcal{C}^0(\mathbb{R})$, the functional associated to

$$-\Delta u + g \circ u = 0$$

on Ω with boundary conditions $u|_{\partial\Omega} = 0$ is given by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega G \circ u dx$$

where G is a suitable-defined antiderivative of g. Under further hypothesis on g, the same argument extends to H^1_0. In fact, the contribute of I related to the addendum $I_1(u) = -\Delta u$ is given by $\frac{1}{2}\int_\Omega |\nabla u|^2 dx$ which is well-defined on H^1_0 and F-differentiable on H^1_0 with Fréchet derivative given by

$$I_1'(u)[v] = \int_\Omega \nabla u \cdot \nabla v dx.$$

We show that, under the following growth condition:

- for certain constants $c_{1,2} \geq 0$, one has $g(t) \leq c_1 + c_2|t|^p$ for some $p \in [1, 2^* - 1]$,

set $G(s) = \int_0^s g(t)dt$, the functional

$$I(u) = \int_\Omega G(u(x))dx$$

belongs to $\mathcal{C}^1(H^1_0(\Omega))$.

1. ***I* is well defined**
   In fact, for all $u \in H^1_0(\Omega)$ and for all $p \in [1, 2^* - 1]$

   $$|I(u)| \leq \int_\Omega \left|\int_0^{u(x)} g(t)dt\right| dx \leq \int_\Omega \left|\int_0^{u(x)} (c_1 + c_2|t|^p)dt\right| dx \leq \int_\Omega \left(c_1|u| + \frac{c_2}{p+1}|u|^{p+1}\right) dx.$$

   By Sobolev embedding theorem, $u \in L^q(\Omega)$ for all $q \in [1, 2^*]$. In particular, $u \in L^1$ and, as $p \in [1, 2^* - 1]$, $u \in L^{p+1}$ (since $p + 1 \in [2, 2^*] \subset [1, 2^*]$). So,

   $$|I(u)| \leq d_1 \|u\|_1 + d_2 \|u\|^{p+1}_{p+1} < \infty.$$

2. ***I* is G-differentiable**
   We prove that $\forall v \in H^1_0(\Omega)$,

   $$I_G'(u)[v] = \int_\Omega v(x)g(u(x))dx.$$

We have to prove that for all $v \in H_0^1(\Omega)$

$$\lim_{t\to 0} \frac{I(u+tv)-I(u)}{t} - I'_G(u)[v] = 0.$$

We fix $v \in H_0^1$ once and for all.
Let

$$\varphi(x,t) = \frac{G(u(x)+tv(x))-G(u(x))}{t} - g(u(x))v(x),$$

so that the assertion takes the following form:

$$\lim_{t\to 0} \int_\Omega \varphi(x,t)dx = 0.$$

If we prove we can change limit and integral, we have finished. In fact, by Lagrange theorem, there exists $\vartheta(x,t) \in [0,1]$ s.t.

$$v(x)\frac{G(u(x)+tv(x))-G(u(x))}{tv} = v(x)g(u(x)+t\vartheta(x,y)v(x)).$$

So,

$$\varphi(x,t) = v(x)\Big[g\big(u(x)+t\vartheta(x,t)v(x)\big) - g\big(u(x)\big)\Big]v(x).$$

By the boundedness of ϑ and the continuity of g, then,

$$\lim_{t\to 0} \varphi(x,t) = 0.$$

Therefore, it is enough to exhibit a dominating integrable function for $|\varphi|$ and change the integral and the limit.
As $t \to 0$,

$$\begin{aligned} |\varphi| &\le |v|\left(|g(u+t\vartheta v)| - |g(u)|\right) \le |v|\left(c_1 + c_2|u+t\vartheta v|^p + c_1 + c_2|u|^p\right) \le \\ &\le C|v|\left(1+(|u|+|v|)^p + |u|^p\right). \end{aligned}$$

We know that $v \in H_0^1 \subseteq L^q$ for all $q \in [1,2^*]$. We need to show that $(1+(|u|+|v|)^p+|u|^p) \in L^{q'}$, where q' is the conjugate exponent of some appropriate q.
We take $q = 2^*$. Then, $v \in L^q(\Omega)$ by the Sobolev embedding theorem and

$$(2^*)' = \frac{2^*}{2^*-1}.$$

Since $u, v \in H_0^1 \subset L^{2^*}$, for $p \in [1, 2^*-1]$ one has $u^p, v^p \in L^{2^*/p}$. Ω is bounded and $p \le 2^*-1$, so that

$$\frac{2^*}{p} \ge \frac{2^*}{2^*-1} \Longrightarrow L^{2^*/p}(\Omega) \subset L^{(2^*)'}(\Omega).$$

Therefore, all of the addenda of $(1+(|u|+|v|)^p+|u|^p)$ belong to $L^{(2^*)'}(\Omega)$. By Hölder's inequality, $|v|\left(1+(|u|+|v|)^p+|u|^p\right) \in L^1(\Omega)$.

3. I **is F-differentiable**
   The operator
$$J_u(v) = I'_G(u)[v] = \int_\Omega v g(u) dx$$
   is linear in the v variable. If we show that it is continuous in its v variable, then the F-differentiability of I will be a consequence of Remark (2.1.8).
   By Hölder's and Sobolev inequalities, that provide an upper bound of the L^{2^*} norm through the H^1 norm (which, in turn, is equivalent to the H_0^1 norm on H_0^1),
$$|J_u(v)| \leq \int_\Omega |g(u)||v| dx \leq \|g \circ u\|_{2^*/2^*-1} \|v\|_{2^*} \leq C \|v\|_{H_0^1} \|g \circ u\|_{2^*/2^*-1}.$$
   But,
$$\|g \circ u\|_{2^*/2^*-1} \leq \left(\int_\Omega (c_1 + c_2|u|^p)^{2^*/2^*-1} dx \right)^{\frac{2^*-1}{2^*}}.$$
   As Ω is bounded, it is enough to check that $u \in \frac{p2^*}{2^*-1}$ to get the desired boundedness (which is uniform in v). As $p \in [1, 2^* - 1]$,
$$1 \leq p \frac{2^*}{2^* - 1} \leq 2^*$$
   and, by Sobolev embedding theorem, $u \in L^q$ for all $q \in [1, p^*]$. This concludes the example.

3.2 Existence of minima and maxima

The purpose of this section is that of providing infinite-dimensional Banach spaces versions of classical results such as Weierstrass and Fermat theorems.

Definition 3.2.1. Let $I : E \to \mathbb{R}$ be a functional on the Banach space E. I is

- **weakly continuous** (WC) if whenever $u_k \rightharpoonup u$ in the norm of E, then $I(u_k) \to I(u)$ in $\mathbb{R}$;
- **weakly lower semicontinuous** (WLS) if whenever $u_k \rightharpoonup u$ in the norm of E, then $I(u) \leq \liminf_{k\to\infty} I(u_k)$.

Example 3.2.2. Let E be a Hilbert space with respect to the inner product $(\cdot, \cdot)$ and let $I : E \to \mathbb{R}$ be defined by
$$I(u) = \|u\|^2 = (u, u).$$
I is WLS. In fact, if $u_k \rightharpoonup u$ in E, then
$$0 \leq \|u_k - u\|^2 = (u - u_k, u - u_k) = \|u\|^2 - 2(u, u_k) + \|u_k\|^2.$$

Therefore,

$$2(u, u_k) - \|u\|^2 \leq \|u_k\|^2 .$$

If $u \rightharpoonup u$ in E, then $(u, u_k) \to (u, u)$ in $\mathbb{R}$ *, so that by taking the lim inf, we get

$$I(u) = \liminf_{k\to\infty} 2(u, u_k) - \|u\|^2 \leq \liminf_{k\to\infty} \|u_k\|^2 = \liminf_{k\to\infty} I(u_k).$$

Example 3.2.3. Let E be a Hilbert space with inner product $(\cdot, \cdot)$ and $L \in BL(E, E)$ be compact. Let $I : E \to \mathbb{R}$ be defined as

$$I(u) = (Lu, u).$$

We show that I is WC.
Let $u_k \rightharpoonup u$ in E, then for Lemma (2.2.13), we have $Lu_k \to Lu$ in E as $k \to \infty$.
Therefore, $(Lu_k, u) \xrightarrow[k\to+\infty]{} (Lu, u)$ by the continuity of the inner product.

$$|(Lu_k, u_k) - (Lu, u)| \leq |(Lu_k, u_k) - (Lu, u_k)| + \underbrace{|(Lu, u_k) - (Lu, u)|}_{= o(1) \text{ by weak conv.}} =$$

$$= |(Lu_k - Lu, u_k)| + o(1) \leq \underbrace{\|Lu_k - Lu\|}_{\substack{= o(1) \text{ for the reasons} \\ \text{indicated above}}} \sup_k \|u_k\| + o(1).$$

Example 3.2.4. We define the 1-dimensional **torus** as the measure space

$$\mathbb{T} := \left([0, 2\pi), dt := \frac{dx}{2\pi} \right)$$

where dx is the Lebesgue measure on $\mathbb{R}$. With an abuse of language, we denote with $\mathbb{T}$ the interval $[0, 2\pi)$ as well. Observe that there exists an obvious identification of $\mathbb{T}$ with S^1.
If $f : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function, it induces a function that, with an abuse of notation, we denote with f, $f : \mathbb{T} \to \mathbb{R}$ through the restriction to $[0, 2\pi)$.
Vice versa, a function f defined on $\mathbb{T}$ induces a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$, though the following definition:

$$f(x) = f(t) \text{ if } x = t + 2k\pi \text{ for some } k \in \mathbb{Z}.$$

We denote with

$$L^q(\mathbb{T}) := L^q([0, 2\pi), dt),$$

so that

$$\int_{\mathbb{T}} f(t)dt = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx.$$

*See the weak convergence theory results.

Hence, the spaces $W^{1,p}(\mathbb{T})$ are well-defined and, in particular, $H^1(\mathbb{T})$ it is well-defined. Let $f \in \mathcal{C}^0(\mathbb{T})$, we define

$$I(u) = \int_{\mathbb{T}} f \circ u dt.$$

We show that I is WC.
By the theory that follows by the Sobolev embedding theorem, in particular by Morrey Theorem, $H^1(\mathbb{T}) \hookrightarrow \mathcal{C}^{0,1/2}(\mathbb{T})$ (the embedding is continuous). In particular, $u \in H^1(\mathbb{T})$ admits a *continuous version* (that is, a continuous representative of u) and, therefore, $f \circ u$ is continuous.
On the other hand, by Ascoli-Arzelà theorem, $\mathcal{C}^{0,1/2}(\mathbb{T}) \subset\subset L^\infty(\mathbb{T}) \subset L^p(\mathbb{T})$ for all p. Using Lemma (2.2.13) again, if $u_k \rightharpoonup u$ in $H^1(\mathbb{T})$, then $u_k \to u$ in L^∞ and, therefore, in L^p for all p [†].
Hence, by uniform continuity [‡],

$$|I(u_k) - I(u)| \leq \int_{\mathbb{T}} |f \circ u_k - f \circ u| dt \leq \sup_{\mathbb{T}} |f| \, \|u_k - u\|_{\infty,\mathbb{T}} \to 0$$

as $k \to \infty$.

Example 3.2.5. Let $E = H_0^1(\Omega)$, where $\Omega \subset\subset \mathbb{R}^n$. Let $p \in [1, 2^*)$ and $I(u) = \|u\|_p^p$ for $u \in E$. If $u_k \rightharpoonup u$ in the norm of E, then $u_k \to u$ in L^p for the values of p considered in Corollary (2.2.14). Then, by the continuity of the p-norm on L^p,

$$\lim_{k\to\infty} \int_\Omega |u_k|^p dx = \int_\Omega |u|^p dx.$$

We will use frequently the following lemma in order to prove that certain functionals are WLS.

Lemma 3.2.6. *Let $I : E \to \mathbb{R}$ be WC and $J : E \to \mathbb{R}$ be WLS. Then, $I + J : E \to \mathbb{R}$ is WLS.*

Proof. Let $u_k \rightharpoonup u$ in E. Then,

$$(I+J)(u) = I(u)+J(u) = \lim_{k\to\infty} I(u_k)+J(u) \leq \lim_{k\to\infty} I(u_k)+\liminf_{k\to\infty} J(u_k) \leq \liminf_{k\to\infty}(I+J)(u_k).$$

□

We recall the following definition:

Definition 3.2.7. Let E be a Banach space. $K \subseteq E$ is called a **weakly closed** subspace of E if whenever $u_k \rightharpoonup u$ in E, one has $u \in K$.

[†] Observe that $\|\cdot\|_{p,\mathbb{T}} \leq \|\cdot\|_{\infty,\mathbb{T}}$.
[‡] The torus is compact.

Theorem 3.2.8. *Let E be a reflexive Banach space and $K \subseteq E$ be weakly closed and bounded. Let $I : K \to \mathbb{R}$ be WLS. Then, I attains its infimum on K.*

Proof. Let $m = \inf_{v \in K} I(v)$ and let $\{u_k\}_k \subset K$ be a minimizing sequence.
K is bounded $\implies \{u_k\}_k$ is bounded $\implies$ there exists a subsequence that converges in E by the Banach-Alaoglu theorem (here is where we use the assumptions on E). Let $\bar{u}$ be the limit of such a subsequence.
K is weakly closed $\implies \bar{u} \in K$.
m is the inf on K of the values of I and $\bar{u} \in K \implies$ it must be $I(\bar{u}) \geq m$.
I is WLS $\implies \liminf_{j\to\infty} I(u_{k_j}) = \lim_{k\to\infty} I(u_k) = m \geq I(\bar{u})$.
We proved ($\geq$) and ($\leq$). The conclusion follows. □

Remark 3.2.9. We recall that Hilbert spaces are always reflexive.

Remark 3.2.10. In the previous theorem, if I were weakly upper semicontinuous (with the obvious modification of WLS definition), we would have the same assertion with appropriate assumptions on the sup. In particular, if I is WC, then I attends both its sup and its inf.

We will often work with operators defined on the whole E. E is not, in general, bounded, hence Theorem (3.2.8) cannot be applied in most of the cases we will deal with.
However, if we add an assumption on the functional I which prevents I "having its inf at infinity", we recover the assertion of Theorem (3.2.8) for reflexive unbounded Banach spaces.

Definition 3.2.11. Let E be a Banach space. A functional $I : E \to \mathbb{R}$ is called a **coercive** functional if $\lim_{\|v\|\to\infty} I(v) = +\infty$.

A coercive functional, therefore, goes to infinity as $\|v\| \to +\infty$. Hence, if it attends its infimum, this must be in a bounded region.

Theorem 3.2.12. *Let E be a reflexive Banach space and I be a WLS coercive functional. Then, $\exists u \in E$ s.t. $I(u) = \min_{v \in E} I(v)$.*

Proof. As I is coercive, by the definition of limit, there exists $R > 0$ s.t.

$$\inf_{\|v\| \geq R} I(v) > m.$$

By Banach-Alaoglu theorem, the closure of $B_R(0)$ is weakly closed, hence we can apply Theorem (3.2.8) using $K = \overline{B_R(0)}$. □

The following result is, instead, a version for infinite-dimensional Banach spaces of Fermat theorem:

Lemma 3.2.13. *Let E be a Banach space and $I : E \to \mathbb{R}$ be F-differentiable in a weakly closed subset $K \subseteq E$. Let $\bar{u} \in \mathring{K}$ be s.t. $I(\bar{u}) = \inf_{u \in K} I(u)$. Then, $I'(\bar{u}) = 0$, i.e. $I'(\bar{u})[v] = 0$ for all $v \in E$.*

Proposition 3.2.14. *Let E be an infinite-dimensional reflexive Banach space and let $K \subseteq E$ be convex and bounded. Let $I : E \to \mathbb{R}$ be WC. Then, $I(\mathring{K}) \subset \overline{I(\partial K)}$.*

Proof. As E is infinite-dimensional, there exists a sequence $\{e_k\}_k \subset E$ s.t. $\|e_k\| = 1$ for all k and $e_k \rightharpoonup 0$.
Let $u \in \mathring{K}$. We must show that there exists a sequence $\{u_k\}_k \subset \partial K$ s.t. $I(u_k) \to I(u)$ as $k \to \infty$.
For all k, let $t_k \in \mathbb{R}$ be s.t. $u_k := u + t_k e_k \in \partial K$.
Since K is bounded, an easy contradiction argument proves that $\{t_k\}_k$ is bounded as well, so that $u_k \rightharpoonup u$.
I is WC, hence $I(u_k) \to I(u)$. This concludes the proof.

□

Corollary 3.2.15. *Let E be a reflexive Banach space. Let $K \subseteq E$ be convex and bounded. Let $I : E \to \mathbb{R}$ be WC. Then,*

$$\sup_{v \in K} I(v) = \sup_{v \in \partial K} I(v)$$

and

$$\inf_{v \in K} I(v) = \inf_{v \in \partial K} I(v).$$

3.3 Applications

We apply the previous section's results.

n-body problem We consider a potential $V \in \mathcal{C}^1(\mathbb{T} \times \mathbb{R}^n, \mathbb{R})$, for some $V = V(t, q)$ 2π-periodic in the variable t.
We put on V a coercivity assumption:

$$\lim_{|q| \to \infty} V(t, q) = +\infty$$

(uniformly in t). We look for solutions of the following system of equations, with a disturbance, in a particular case:

$$\begin{cases} \ddot{q}_1(t) = \frac{d}{dq_1} V(t, q) \\ \vdots \\ \ddot{q}_n(t) = \frac{d}{dq_n} V(t, q) \end{cases}$$

or, shortened,

$$\ddot{q}(t) = \frac{d}{dq} V(t, q).$$

Here, we denoted with the dots the derivatives with respect to t, differently than what we did in Chapter 1.
For instance, we can choose

$$V(t,q) = |q(t)|^2 + f(t),$$

where $f \in \mathcal{C}^0(\mathbb{T})$ is 2π-periodic and $E = (H^1(\mathbb{T}))^n$, i.e. if $q = (q_1, \dots, q_n)$, $q_k \in H^1(\mathbb{T})$ for all $k = 1, \dots, n$.
E is a normed space, with

$$\|q\|_E^2 = \int_{\mathbb{T}} (|\dot{q}(t)|^2 + |q(t)|^2) dt = \frac{1}{2\pi} \int_0^{2\pi} (|\dot{q}(x)|^2 + |q(x)|^2) dx$$

and it is reflexive. We define the operator $I : E \to \mathbb{R}$ by

$$I(q) = \int_{\mathbb{T}} \left(\frac{1}{2} |\dot{q}(t)|^2 + V(t,q) \right) dt = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2} |\dot{q}(t)|^2 + V(t,q) \right) dx$$

I is WLS. In fact, adding and subtracting the remaining term to trace back to the expression of the norm of E, we get:

$$I(q) = \underbrace{\underbrace{\frac{1}{2} \int_{\mathbb{T}} (|\dot{q}(t)|^2 + |q(t)|^2) dt}_{= \|q\|_E^2 \text{ WLS by (3.2.2)}} + \underbrace{\int_{\mathbb{T}} \left(V(t,q) - \frac{1}{2} |q(t)|^2 \right) dt}_{\text{WC by (3.2.4)}}}_{\text{WLS by (3.2.6)}}.$$

In order to prove that there exists a solution $\bar{q}$ of $\ddot{q} = dV/dq$, we need to show that $I \in \mathcal{C}^1$ is coercive.
We define

$$\hat{u}(0) = \int_{\mathbb{T}} u(t) dt = \frac{1}{2\pi} \int_0^{2\pi} u(x) dx$$

the mean of q and proceed with the proof of the coercivity.
Observe that, as V is the sum of an H^1 function and a continuous periodic one, V is bounded. In particular, V is bounded from below and, therefore, $\exists \beta \in \mathbb{R}$ s.t.

$$\frac{1}{2\pi} \int_0^{2\pi} V(t,q) dt > -\beta.$$

Seeking a contradiction, let $\{q_k\} \subset E$ be a sequence s.t. $\|q_k\|_E \to \infty$ and

$$I(q_k) = \frac{1}{2} \int_0^{2\pi} \frac{|\dot{q}_k(t)|^2}{2} dt + \frac{1}{2\pi} \int_0^{2\pi} V(t,q_k) dt \le \alpha$$

for some $\alpha \in \mathbb{R}$.
Using the boundedness of V, we would get

$$\frac{1}{2\pi} \int_0^{2\pi} |\dot{q}_k(t)|^2 dt \le 2(\alpha + \beta) =: C.$$

But,

$$\underbrace{\|q_k\|_E}_{\to\infty} = \|q_k\|_2 + \underbrace{\|\dot{q}_k\|_2}_{\leq C}.$$

Hence, we would have that, if $\|q_k\|_E \to \infty$, then $\|q_k\|_2 \to \infty$ as $k \to \infty$.
Hence, as $k \to \infty$,

$$\lim_{k\to\infty} \int_0^{2\pi} V(t, q_k)dt = \lim_{k\to\infty} \int_0^{2\pi} |q_k|^2 dt + \underbrace{\int_0^{2\pi} f(t)dt}_{= const < \infty} = const + \|q_k\|_2^2 \to \infty.$$

However,

$$I(q_k) = \underbrace{\frac{1}{2\pi} \int_0^{2\pi} \frac{|\dot{q}_k|^2}{2} dt}_{\geq 0} + \underbrace{\frac{1}{2\pi} \int_0^{2\pi} V(t, q_k)dt}_{\to +\infty} \leq \alpha < +\infty,$$

which is clearly a contradiction.
As we want to use the Banach spaces' version of Fermat theorem, we observe that $I \in \mathcal{C}^1(E, \mathbb{R})$:

$$I(q) = \underbrace{\frac{1}{2} \int_{\mathbb{T}} |\dot{q}|^2 dt}_{\substack{\text{we know that} \\ \in \mathcal{C}^1(H^1)}} + \underbrace{\int_{\mathbb{T}} V(t, q)dt}_{\substack{\text{it's easy to see} \\ \text{that } \in \mathcal{C}^1(H^1)}}.$$

This, allows us to conclude that $\exists \bar{q}$ s.t. $I(\bar{q}) = \min_{v \in E} I(v)$ and for all $v \in E$ one has $I'(\bar{q})[v] = 0$. That is, $\forall v \in E$,

$$\int_{\mathbb{T}} \dot{\bar{q}} \dot{v} dt + \int_{\mathbb{T}} V_q(t, \bar{q}) v dt = 0 \Rightarrow \int_{\mathbb{T}} (\ddot{\bar{q}} - V_q) v = 0.$$

This implies that $\bar{q}$ is a weak solution of the equation

$$\ddot{q} = \frac{dV}{dq}$$

as we wanted to prove.

Generalized pendulum We consider the damped pendulum equation:

$$\ddot{q}(t) = A \sin(q(t)) + f(t) \tag{3.1}$$

under the requests for q and f to be $2\pi-$periodic.
We have immediately a necessary condition for a periodic q to be a solution of (3.1):

Lemma 3.3.1. *Let q a 2π-periodic solution of (3.1). Then*[§]*, $|\hat{f}(0)| \leq A$.*

[§]We denoted with $\hat{f}(0) = \frac{1}{2\pi} \int_0^{2\pi} f(x)dx$, the 0-th Fourier coefficient of f.

Proof. q is 2π-periodic $\Longrightarrow \dot{q}$ is 2π-periodic. In particular, by integrating on $[0, 2\pi]$ both the sides of (3.1), we get

$$\underbrace{\int_0^{2\pi} \ddot{q}(t)dt}_{= \dot{q}(2\pi) - \dot{q}(0) = 0} = \underbrace{A \int_0^{2\pi} \underbrace{\sin(q(t))}_{\in [-1,1]} dt}_{\in [-2\pi, 2\pi]} + \int_0^{2\pi} f(t)dt,$$

which implies that

$$\hat{f}(0) = \frac{1}{2\pi} \int_0^{2\pi} f(t)dt \in [-A, A],$$

that is the assertion. □

We still don't know whether this condition is also sufficient. However, we have:

Theorem 3.3.2. *Let $f \in L^2(\mathbb{T})$ be s.t. $\hat{f}(0) = 0$. Then, there exists a solution $q \in H^2(\mathbb{T})$ of (3.1).*

Actually, we want to prove a more general theorem. For, we enunciate and prove some preliminary results:

Theorem 3.3.3 (Wirtinger inequality)**.** *Let $q \in H^1(\mathbb{T})$. Then,*

$$\|q - \hat{q}(0)\|_2 \leq \|\dot{q}\|_2 \,. \tag{W}$$

Proof. If $q \in \mathcal{C}^\infty(\mathbb{T})$, with $q(t) = \sum_{k=-\infty}^{+\infty} \hat{q}(k)e^{ikt}$, using Parseval's inequality:

$$\|q - \hat{q}\|_2^2 = \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} |\hat{q}(k)|^2 \leq \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} |-ik\widehat{(\dot{q})}(k)|^2 = \|\dot{q}\|_2^2$$

and the assertion follows by the density of $\mathcal{C}^\infty(\mathbb{T})$ in $H^1(\mathbb{T})$.

□

Lemma 3.3.4. *Let $F \in \mathcal{C}^1(\mathbb{T} \times \mathbb{R}^n, \mathbb{R})$, $F = F(t, x = (x_1, \dots, x_n))$ be a T_i periodic function in its variables x_i for $i = 1, \dots, n$. Then, the system*

$$\ddot{u}(t) = F_u(t, u) \tag{3.2}$$

i.e.

$$\begin{cases} \ddot{u}_1(t) = F_{u_1}(t, u) \\ \vdots \\ \ddot{u}_n(t) = F_{u_n}(y, u) \end{cases}$$

admits a 2π-periodic solution.

Proof. F is continuous and periodic in all its variables, hence it is bounded: $\exists C_1 \geq 0$ s.t.

$$|F(t,x)| \leq C_1 \quad \forall (t,x) \in \mathbb{T} \times \mathbb{R}^n.$$

We define the energy associated to (3.2) as follows:

$$I(u) = \frac{1}{2}\int_{\mathbb{T}} |\dot{u}(t)|^2 dt + \int_{\mathbb{T}} F(t,u(t))dt = \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \frac{|\dot{u}(t)|^2}{2} dt}_{\geq 0} + \underbrace{\frac{1}{2\pi}\int_0^{2\pi} F(t,u(t))dt}_{\in [-C_1, C_1]}.$$

I is bounded from below by the constant $-C_1$:

$$m = \inf_{u \in E} I(u) \geq -C_1,$$

where $E = (H^1(\mathbb{T}))^n$. Let $\{u^{(k)}\}_k \subset E$ be a minimizing sequence for m, i.e. $I(u^{(k)}) \searrow m$ as $k \to \infty$ and observe that it is a bounded sequence in $\mathbb{R}$, since it converges. By the definition of $I(u^{(k)})$ and by the boundedness of the second addendum, there exists $C_2 \geq 0$ s.t. $\forall k > 0$

$$\frac{1}{2\pi}\int_0^{2\pi} |\dot{u}^{(k)}|^2 dt \leq C_2^2.$$

By Wirtinger inequality applied to $u^{(k)}$,

$$\left\| u^{(k)} - \hat{u}^{(k)}(0) \right\|_2 \leq \left\| \dot{u}^{(k)} \right\|_2 \leq C_2.$$

Then, by the Sobolev inequality,

$$\left\| u^{(k)} - \hat{u}^{(k)}(0) \right\|_E \leq CC_2 = C_3.$$

On the other hand, since F is periodic, for all $i = 1, \dots, n$ and for all $u \in E$, set $\{e_i\}_{i=1,\dots,n}$ the canonic basis of $\mathbb{R}^n$,

$$I(u + T_i e_i) = \frac{1}{2}\int_{\mathbb{T}} |\dot{u}|^2 dt + \int_{\mathbb{T}} F(t, u(t) + T_i e_i) dt = I(u).$$

It follows that, if $\{u^{(k)}\}_k$ is minimizing for I, then

$$\left\{ u^{(k)} + \sum_{i=1}^{n} a_i^k T_i e_i \right\}_k$$

is still minimizing for all $a_1^k, \dots, a_n^k \in \mathbb{Z}$ and, therefore, we can choos a_i^k $(i = 1, \dots, n)$ properly so that $\hat{u}_i^{(k)}(0) + T_i e_i \in [0, T_i)$.
For this new minimizing sequence, that we still call $\{u^{(k)}\}_k$ with an abuse of notation, we have

$$|\hat{u}^{(k)}(0)| \leq C_4^2 \quad \forall k.$$

Using Wirtinger inequality:

$$\begin{aligned}\left\|u^{(k)}\right\|_E^2 &= \frac{1}{2\pi}\int_0^{2\pi}|\dot{u}^{(k)}|^2dt + \frac{1}{2\pi}\int_0^{2\pi}|u^{(k)}|^2dt \le \\ &\le \frac{1}{2\pi}\int_0^{2\pi}|\dot{u}^{(k)}|^2dt + \frac{1}{2\pi}|u^{(k)} - \hat{u}^{(k)}(0)|^2dt + \frac{1}{2\pi}\int_0^{2\pi}|\hat{u}^{(k)}(0)|^2dt \le \\ &\le C_2 + C_2 + C_4\end{aligned}$$

Hence, $\{u^{(k)}\}_k$ is bounded. Since E is reflexive, by Banach-Alaoglu theorem there exists a subsequence $u^{(k_j)}$ which weakly-converges to some $\bar{u} \in E$.
Moreover, it is immediate to observe that I is WLS by Lemma (3.2.6). Then, as $\bar{u} \in E$,

$$m \le I(\bar{u}) \le \liminf_{j\to\infty} I(u^{(k_j)}) = m,$$

so that, $\bar{u}$ is a solution of the Euler-Lagrange equation associated to I that, in our case, is exactly the assertion.

□

We define the measure space of T-periodic functions similarly to how we did for 2π-periodic one, with normalized measure dt/T and so that $\|g\|_{H_0^1([0,T))} = 1/T\int_0^T |\dot{g}|^2dt$.

Lemma 3.3.5. *Let $f \in L^2([0,T),\mathbb{R})$ be a T-periodic function. Then, the equation*

$$\ddot{u}(t) = f(t) \text{ admits 1! } T\text{-periodic solution which belongs to } H_0^1([0,T))$$

if and only if $\int_0^T f(t)dt = 0$.

Proof. ($\Rightarrow$) follows trivially integrating the equation $\ddot{u} = f$.
To prove ($\Leftarrow$), let

$$J(u) = \frac{1}{T}\int_0^T \frac{|\dot{u}(t)|^2}{2}dt + \frac{1}{T}\int_0^T f(t)u(t)dt$$

be defined on $H_0^1(\mathbb{R})$. J is WLS. Since $H_0^1([0,T))$ is a Hilbert space (hence, a reflexive Banach space), in order to prove the existence of a solution it suffices to prove that J is coercive. We have,

$$\begin{aligned}J(u) = J(u-\hat{u}(0)+\hat{u}(0)) &= \frac{1}{T}\int_0^T\frac{|\dot{u}(t)|^2}{2} + \underbrace{\frac{1}{T}\int_0^T f(t)(u(t)-\hat{u}(0))dt}_{|\cdot| \le \|f\|_2\|u-\hat{u}(0)\|_2} + \underbrace{\frac{1}{T}\int_0^T f(t)\hat{u}(0)dt}_{=0 \text{ by assumption}} \ge \\ &\ge \frac{1}{2}\|u\|^2_{H_0^1([0,T))} - \|f\|_2 \underbrace{\|u-\hat{u}(0)\|_2}_{\le C\|\dot{u}\|_2 \text{ by Poincaré}} \ge \frac{1}{2}\|u\|^2_{H_0^1([0,T))} - C\underbrace{\|f\|_2\|\dot{u}\|_2}_{\substack{\le \frac{2C}{2}\|f\|_2^2 + \frac{1}{4C}\|\dot{u}\|_2^2 \\ \text{by (2.2.11)}}} \ge \\ &\ge \frac{1}{4}\|u\|^2_{H_0^1([0,T))} - C^2\|f\|_2^2\end{aligned}$$

Hence,

$$\lim_{\|u\|_{H_0^1([0,T))}\to\infty} J(u) = +\infty.$$

As far as uniqueness is concerned, if u_1 and u_2 are solutions of $\ddot{u} = f$, then

$$\frac{d^2}{dt^2}(u_1 - u_2) = 0.$$

Hence, for all the functions $\varphi \in \mathcal{C}^\infty([0, T))$,

$$0 = \frac{1}{T}\int_0^T \frac{d^2}{dt^2}(u_1 - u_2)\varphi dt = \underbrace{\left[\frac{1}{T}\frac{d}{dt}(u_1 - u_2)\varphi\right]_0^T}_{\substack{= 0 \text{ by the periodicity} \\ \text{of } u_1 - u_2}} - \frac{1}{T}\int_0^T \frac{d}{dt}(u_1 - u_2)\dot{\varphi} dt.$$

Therefore, for $\varphi = u_1 - u_2$

$$0 = \frac{1}{T}\int_0^T |\dot{u}_1 - \dot{u}_2|^2 dt = \|u_1 - u_2\|^2_{H_0^1([0,T))}.$$

This concludes the proof.

□

The more general result (more general than Theorem (3.3.2)) we prove is the following:

Theorem 3.3.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a T-periodic function s.t. $\int_0^T f(t)dt = 0$. Then,*

$$\ddot{u}(t) = F_u(t, u) + f(t) \tag{3.3}$$

admits a solution.

Proof. By the previous lemma, there exists one (and only one) solution $v \in H_0^1([0, T))$ of $\ddot{v} = f$.
Let $u(t) = v(t) + y(t)$. In order for u to be a solution of (3.3), it must be

$$\ddot{v} + \ddot{y} = F_u(t, v + y) + f.$$

Recalling that $\ddot{v} = f$, one has $\ddot{y} = F_u(t, y + v)$. Following the steps of the proof of Lemma (3.3.4), with

$$I(y) = \frac{1}{T}\int_0^T \frac{|\dot{y}|^2}{2} dt + \frac{1}{T}\int_0^T F(t, y + v)dt,$$

we get the existence of y and, therefore, that of u.

□

Elliptic equation with boundary conditions Let $\Omega \subset\subset \mathbb{R}^n$. We look for a $H_0^1(\Omega)$ solution of

$$\begin{cases} -\Delta u + g(x,u) = f & \text{on } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3.4}$$

with the following structural hypothesis: set $G(x,s) = \int_0^s g(x,t)dt$,

- $f \in L^2(\Omega)$;
- $g \in \mathcal{C}^0(\Omega \times \mathbb{R}, \mathbb{R})$;
- there exist constants $C \in \mathbb{R}$, $c, d \geq 0$ s.t. $|G(x,s)| \leq c + d|s|^{p+1}$ for some $p \in [1, 2^* - 1)$ and $G(x,s) \geq -C$.

The functional associated to the equation $-\Delta u + g(x,u) - f = 0$ is

$$I(u) = \underbrace{\frac{1}{2}\int_\Omega |\nabla u|^2 dx}_{= 1/2\, \|u\|^2_{H_0^1} \text{ (WLS)}} + \int_\Omega G(x,u)dx \quad \underbrace{-\int_\Omega fudx}_{\substack{\in BL(H_0^1,\mathbb{R}) \Rightarrow \text{WC} \\ \text{(by def. of the weak top.)}}} .$$

It's easy to see that $I \in \mathcal{C}^1(H_0^1)$ with Fréchet-derivative given for all $v \in H_0^1$ by

$$I'(\bar{u})[v] = \int_\Omega \nabla\bar{u}\nabla v dx + \int_\Omega g(x,\bar{u})vdx - \int_\Omega fvdx.$$

Moreover, the mapping $u \mapsto \int_\Omega G(x,u)dx$ is $\mathcal{C}^1$, hence continuous and, using Rellich-Kondrachov theorem and Lemma (2.2.13), it can be proved that it is WC.
We show that I is coercive:

$$I(u) = \frac{1}{2}\|u\|^2_{H_0^1} + \underbrace{\int_\Omega \underbrace{G(x,u)}_{\geq -C} dx}_{\geq -C|\Omega|} - \underbrace{\int_\Omega fudx}_{\leq \|f\|_2 \|u\|_2} \geq \frac{1}{2}\|u\|^2_{H_0^1} - C|\Omega| \underbrace{- \|f\|_2 \|u\|_2}_{\leq C\|f\|_2 \|u\|_{H_0^1}} \geq$$

$$\geq \frac{1}{2}\|u\|^2_{H_0^1} - C|\Omega| - C\left(C\|f\|_2^2 + \frac{1}{4C}\|u\|^2_{H_0^1}\right) = \frac{1}{4}\|u\|^2_{H_0^1} - C|\Omega| - C^2\|f\|_2^2,$$

which gives the requested coercivity.
Therefore, $\exists \bar{u} \in H_0^1(\Omega)$ s.t. $I(\bar{u}) = \min_{H_0^1} I(u)$. By Lemma (3.2.13), we have $I'(\bar{u}) = 0$. Equivalently, $\bar{u}$ is a weak solution of $-\Delta u + g(x,u) = f$. Finally, since $\bar{u} \in H_0^1(\Omega)$, $\bar{u}|_{\partial\Omega} = 0$.

3.4 Second order elliptic operators

Let $\Omega \subset\subset \mathbb{R}^n$ and let $a_{ij}, c \in L^\infty(\Omega)$ with $a_{ij} = a_{ji}$ for all $i,j = 1,\ldots,n$.
Let $A = A(x)$ be the matrix

$$A(x) = (a_{ij}(x))_{i,j=1}^n \in \mathbb{R}^{n\times n}.$$

We consider the second order differential operator in the divergence form:

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u = -\mathrm{div}(A(x)\nabla u(x)) + c(x)u$$

with uniform ellipticity condition: $\exists \lambda > 0$ s.t. $\forall \xi \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

holds for a.e. $x \in \Omega$.

We consider the problem:

$$\begin{cases} Lu = f & \text{on } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{L_0}$$

with the associated form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$a(u,v) = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} + c(x)uv\right) dx$$

(this is called the **Dirichlet bilinear form**).

Definition 3.4.1. A function $u \in H_0^1(\Omega)$ il called a **weak solution** of (L_0) if

$$a(u,v) = (f,v)_{L^2}.$$

Remark 3.4.2. a is a continuous bilinear form. In fact, using the boundedness of the a_{ij} and that of c:

$$\begin{aligned} |a(u,v)| &\leq C\left(\sum_{ij=1}^{n}\int_\Omega \left|\frac{\partial u}{\partial x_i}\right|\left|\frac{\partial v}{\partial x_j}\right| dx + \int_\Omega |uv| dx\right) \leq \\ &\leq C\Big(\underbrace{\sum_{j=1}^{n}\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}\right\|_2}_{\leq C\,\|u\|_{H_0^1}}\left\|\frac{\partial v}{\partial x_j}\right\|_2 + \underbrace{\|u\|_2\,\|v\|_2}_{\leq C\,\|u\|_{H_0^1}\,\|v\|_{H_0^1}}\Big) \leq \\ &\leq \tilde{C}\left(\|u\|_{H_0^1}\,\|v\|_{H_0^1}\right). \end{aligned}$$

We consider the particular case in which $c = 0$ a.e., in the which case, Dirichlet bilinear formula reads as

$$a_0(u,v) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{x_j}dx$$

for $u, v \in H_0^1(\Omega)$. Moreover, using the uniform ellipticity condition

$$a_0(u,u) = \int_\Omega \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{x_j} dx \geq \lambda \int_\Omega |\nabla u|^2 dx = \lambda \|u\|_{H_0^1}^2$$

and the coercivity of a_0 follows.

Theorem 3.4.3 (Lax-Milgram)**.** *Let H be a Hilbert space and $\phi : H \times H \to \mathbb{R}$ be a bilinear, bounded and coercive operator. Then, for all $T \in BL(H, \mathbb{R})$ there exists 1! $u \in H$ s.t.*

$$Tv = \phi(u, v) \quad \forall v \in H.$$

Applying Lax-Milgram Theorem to the mapping

$$v \in H_0^1(\Omega) \longmapsto \int_\Omega f v dx = (f, v)_{L^2}$$

and a_0, we get the existence of 1! $u \in H_0^1(\Omega)$ s.t.

$$a_0(v, u) = (f, v)_{L^2} \quad \forall v \in H_0^1(\Omega).$$

That is, we have a result concerning the existence and the uniqueness for the solution of (L_0) in the case $c = 0$.
Actually, the procedure above extends immediately to the case in which $c \neq 0$ in $L^\infty(\Omega)$.

Remark 3.4.4. Consider the equation in (3.4) with $g(x, s) = s|s|^{p-1}$, $p \in [1, 2^* - 1]$:

$$-\Delta u + u|u|^{p-1} = f(x). \tag{3.5}$$

In this case,

$$G(x, s) = \frac{1}{p+1} |s|^{p+1} \geq 0$$

and (3.5) admits 1! solution.
However, for $p \in [1, 2^* - 1)$ the system

$$\begin{cases} -\Delta u - u|u|^{p-1} = f & \text{on } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

admits infinite solutions. If, instead, $p = 2^* - 1$ and Ω is a star domain, the unique solution for this last problem is $u \equiv 0$ (see (6.11)).

3.5 Constrained optimization

The purpose of this section is that of studying optimization problems for functionals defined on Banach spaces, along some constraints expressed by equations involving other functionals. The first result we enunciate is a Banach spaces version of the implicit function theorem:

Theorem 3.5.1. *Let X, Y, Z be Banach spaces, $O \subseteq X \times Y$ and $\Phi \in \mathcal{C}^1(O, Z)$.*
Let $(x_0, y_0) \in O$ be s.t.

- $\Phi(x_0, y_0) = 0$;
- $\Phi_y(x_0, y_0) \neq 0$.

Then, there exist a neighborhood $U = U(x_0) \subseteq X$ and a mapping $\varphi \in \mathcal{C}^1(U, Y)$ s.t. $\forall x \in U$

(a) $(x, \varphi(x)) \in O$;

(b) $\Phi(x, \varphi(x)) = 0$;

(c) $\varphi'(x) = -\frac{\Phi_x(x,\varphi(x))}{\Phi_y(x,\varphi(x))}$.

We see an example in order to clarify our goal: we define a functional $\mathcal{E} : H_0^1(\Omega) \to \mathbb{R}$ for $\Omega \subset\subset \mathbb{R}^n$ as

$$\mathcal{E}(u) = \int_\Omega |\nabla u|^2 dx.$$

Theorem 3.5.2. *Let Ω and $\mathcal{E}$ defined as above. Let $\mathcal{S} = \left\{u \in H_0^1(\Omega) : \|u\|_2 = 1\right\}$. Then, there exists $u_0 \in \mathcal{S}$ s.t. $\mathcal{E}(u_0) = \inf_{u \in \mathcal{S}} \mathcal{E}(u)$.*

In other words, we're gonna prove that $\mathcal{E}$ attains its inf not with respect to the whole H_0^1, but on one of its proper subsets, that is $\{u \in H_0^1 : J(u) = 0\}$ with $J = \|\cdot\|_2 - 1$.

Proof. Let $\{u_k\}_k \subset \mathcal{S}$ be a sequence s.t. $\mathcal{E}(u_k) \searrow m = \inf_\mathcal{S} \mathcal{E}$.
Since it converges, $\{\mathcal{E}(u_k)\}_k$ is bounded in $\mathbb{R}$. Moreover, since $\mathcal{E}$ is the Dirichlet norm of H_0^1, the sequence $\{u_k\}_k$ is bounded in H_0^1 with respect to the norm of H_0^1. By the Banach-Alaoglu Theorem, there exists a subsequence $\{u_{k_j}\}_j$ which converges weakly to a function $u_0 \in H_0^1(\Omega)$. As, by Example (3.2.5), $\|\cdot\|_2^2$ is weakly continuous on H_0^1,

$$\|u_0\|_2^2 = \lim_{j\to\infty} \left\|u_{k_j}\right\|_2^2 = 1$$

i.e. $u_0 \in \mathcal{S}$.
On the other hand,

$$m \underset{u_0 \in \mathcal{S}}{\leq} \mathcal{E}(u_0) \leq \liminf_{j\to\infty} \mathcal{E}(u_{k_j}) = m.$$

This concludes the proof. □

Hence, we proved the existence of a minimum for the energy $\mathcal{E}$ along the constraint $\mathcal{S}$. The problem, now, is that of finding a differential equation which is satisfied by this constrained minimum: while the minimum of $\mathcal{E}$ on $H_0^1(\Omega)$ is a weak solution of Laplace equation, we still don't know whether the constraint minimum is a solution of $-\Delta u = 0$ as well.

As a general fact, if the minimum of a functional $I \in \mathcal{C}^1(E, \mathbb{R})$ is the weak solution of some differential equation, there is no guarantee for the minimum of I along a certain constraint to be a solution of the same differential equation. In our example, we'll see that the minimum of $\mathcal{E}$ on $\mathcal{S}$ is a solution of $-\Delta u = \lambda u$ for some $\lambda > 0$, not one of $-\Delta u = 0$. Actually, this result is a consequence of the following theorem:

Theorem 3.5.3 (Lagrange multipliers)**.** *Let E be a Banach space and let $I, F \in \mathcal{C}^1(E, \mathbb{R})$. Let $S = \{u \in E : F(u) = 0\}$ be a constraint and $u_0 \in S$ be a minimizer for $I|_S$ s.t. $F'(u_0) \neq 0$. Then, $\exists \lambda \in \mathbb{R}$ s.t.*

$$I'(u_0) = \lambda F'(u_0). \tag{3.6}$$

Proof. Let u_0 be s.t. $I(u_0) = \min_S I$.

1. **We use the implicit function theorem to "parametrize"** S

   - $\exists w \in E$ s.t. $F'(u_0)[w] = 1$.
     In fact, $F'(u_0) \neq 0$, hence $\exists \tilde{w} \in E$ s.t. $F'(u_0)[\tilde{w}] \neq 0$. $F'(u_0)$ is, by the definition of Fréchet derivative, a linear mapping, so that it is enough to prove that $w = \frac{\tilde{w}}{F'(u_0)[\tilde{w}]}$.
   - $F'(u_0)$ is a functional, so that $E_0 := \ker F'(u_0)$ has codimension 1 in E, i.e.

   $$E = E_0 \oplus \operatorname{span}\{w\} = E_0 \oplus \mathbb{R}w.$$

   Consider the mapping $\Phi : E_0 \times \mathbb{R} \to \mathbb{R}$ defined by

   $$\Phi(v, t) = \Phi(v + tw) = F(u_0 + v + tw).$$

   As far as Φ is concerned, we have:

   (a) $\Phi(0, 0) = F(u_0) = 0$ $(u_0 \in S = \ker(F))$;
   (b) $D_t\Phi(0, 0) = F'(u_0)[w] = 1$ (by the definition of w);
   (c) $D_v\Phi(0, 0) = F'(u_0)[0] = 0$ (since $F'(u_0)$ is linear).

   In particular, we can invoke the Theorem (3.5.1) to grant both the existence of a neighborhood $V = V(0) \subseteq E_0$ and that of a function $\varphi \in \mathcal{C}^1(V, \mathbb{R})$ s.t. $\Phi(v, \varphi(v)) = 0$ $\forall v \in V$, $\varphi(0) = 0$ and, finally,

   $$\varphi'(v) = -\frac{\Phi_v(v, \varphi(v))}{\Phi_t(v, \varphi(v))}.$$

By the definition of Φ, this implies that

$$0 = \Phi(v, \varphi) = F(u_0 + v + w\varphi(v)),$$

that is, $u_0 + v + \varphi(v)w \in \ker(F) = S$ for all $v \in V$.
Hence, we found a neighborhood $U = U(u_0) \subseteq S$ of u_0 (with respect to S) s.t. $\forall u \in U$

$$u = u_0 + v + \varphi(v)w$$

for some $v \in V$. That is, we found a local parametrization of S.

2. **We show that** $\ker(F'(u_0)) \subseteq \ker(I'(u_0))$
Let $\tilde{I} : V \to \mathbb{R}$ be defined as

$$\tilde{I}(v) = I(u_0 + v + \varphi(v)w),$$

the "local restriction" of I to S.
$\tilde{I}$ is F-differentiable as a composition of F-differentiable ones. Since I has a minimum in u_0, $\tilde{I}$ has a minimum in $v = 0$. Therefore, $\tilde{I}'(0)[u] = 0$ for all $u \in E_0 = \ker(F'(u_0))$.
Hence, for all $u \in E_0$,

$$0 = \tilde{I}'(0)[u] = I'(u_0)[u + \underbrace{\varphi'(0)}_{=0} w] = I'(u_0)[u].$$

Therefore, $I'(u_0) = 0$ in E_0, i.e. in the point in which $F'(u_0) = 0$, $I'(u_0) = 0$ as well.
This is equivalent of claiming that $\ker(F'(u_0)) \subseteq \ker(I'(u_0))$.

3. **Conclusion**
We exhibit a $\lambda \in \mathbb{R}$ s.t. $\forall v \in E$,

$$I'(u_0)[v] = \lambda F'(u_0)[v].$$

Recall that $E = E_0 \oplus \mathbb{R}w$. If $v \in E_0$, i.e. if $F'(u_0)[v] = 0$, we have proved that $I'(u_0)[v] = 0$ as well and, therefore, (3.6) trivially follows.
If $v \in \mathbb{R}w \setminus \{0\}$, then, $v = tw$ for some $t \neq 0$. Defined

$$\lambda = \frac{I'(u_0)[w]}{F'(u_0)[w]} = I'(u_0)[w],$$

one has

$$I'(u_0)[v] = tI'(u_0)[w] = t\lambda F'(u_0)[w] = \lambda F'(u_0)[tw] = \lambda F'(u_0)[v].$$

And the assertion follows.

□

The next paragraph consists of an application of Theorem (3.5.3) concerning the research of the minimum of $\mathcal{E}$ on $\mathcal{S} = \{u \in H_0^1(\Omega) : \|u\|_2 = 1\}$.

Eigenvalues of $-\Delta$ Theorem (3.5.3) can be used to derive the differential equation satisfied by the minimum of $\mathcal{E}$ along the constraint

$$\mathcal{S} = \left\{ u \in H_0^1(\Omega) : \|u\|_2^2 = 1 \right\} = \{ u \in H_0^1(\Omega) : F(u) = 0 \}$$

where we set $F(u) := \|u\|_2^2 - 1$.
By Theorem (3.5.2), we know that there exists $u_0 \in \mathcal{S}$ s.t. $\mathcal{S}(u_0) = m = \inf_{\mathcal{S}} \mathcal{E}$.
On the other hand, as $F \in \mathcal{C}^1(H_0^1, \mathbb{R})$, F is G-differentiable and its G-derivative coincide with its F-derivative, so that:

$$F'(u_0)[v] = \lim_{h \to \infty} \frac{F(u_0 + hv) - F(u_0)}{h} = 2 \int_\Omega uv dx.$$

In particular,

$$F'(u_0)[u_0] = 2 \int_\Omega |u|^2 dx = 2 \neq 0.$$

By Lagrange multipliers theorem, there exists $\lambda \in \mathbb{R}$ s.t. $\forall v \in H_0^1(\Omega)$

$$\mathcal{E}'(u_0)[v] = \lambda F'(u_0)[v].$$

Hence, $\forall v \in H_0^1(\Omega)$

$$2 \int_\Omega \nabla u_0 \nabla v dx = \mathcal{E}'(u_0)[v] = \lambda F'(u_0)[v] = 2\lambda \int_\Omega uv dx. \tag{3.7}$$

That is, u_0 is a weak solution of $-\Delta u = \lambda u$.

Remark 3.5.4. By Poincaré inequality, using (3.7) with $v = u_0$, we get

$$\lambda = \int_\Omega |\nabla u_0|^2 dx \geq \frac{1}{C} \|u\|_2 = \frac{1}{C} > 0.$$

Moreover, by linearity

$$\lambda = \|u_0\|_{H_0^1}^2 = \inf_{u \in S} \mathcal{E}(u) = \inf_{u \in H_0^1} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}.$$

We set $\lambda_1 := \lambda$.

Definition 3.5.5. λ_1 is called the **principal eigenvalue of Laplace operator**. The associated eigenfunction $e_1 \in \mathcal{S}$, that is the function $e_1 \in H_0^1(\Omega)$ s.t.

$$\begin{cases} -\Delta e_1 = \lambda_1 e_1, \\ e_1|_{\partial\Omega} = 0 \end{cases}$$

is called the **ground state**.

We look for the other eigenvalues and eigenfunctions: we observe that, if there exists $\lambda \notin \{0, \lambda_1\}$ s.t. $-\Delta v = \lambda v$ for some $v \in H_0^1(\Omega)$, as e_1 is a weak solution of $-\Delta u = \lambda_1 u$ and v is a weak solution of $-\Delta u = \lambda u$, by applying the definitions of weak solution to both of the cases with v and e_1 as test functions respectively, we get:

$$\begin{cases} \int_\Omega \nabla e_1 \nabla v dx = \lambda_1 \int_\Omega e_1 v dx, \\ \int_\Omega \nabla v \nabla e_1 dx = \lambda \int_\Omega v e_1 dx. \end{cases}$$

Hence, as $\lambda_1 \neq \lambda$, it must be:

$$\begin{cases} \int_\Omega v e_1 dx = 0, \\ \int_\Omega \nabla v \nabla e_1 dx = 0. \end{cases}$$

In particular, $v \perp e_1$ in L^2 and in H^1 (hence, in H_0^1).
Let $E_1 = \{u \in \mathcal{S} : \int_\Omega u e_1 dx = 0\}$. Let $\lambda_2 = \inf_{u \in E_1} \mathcal{E}(u)$. Repeating the argument above, we get the existence of $e_2 \in E_1$ (use the weak continuity of the inner product of L^2 on H_0^1 to prove that $e_2 \in E_1$) s.t. $-\Delta e_2 = \lambda_2 e_2$. Obviously,

$$0 < \lambda_1 \leq \lambda_2.$$

By induction, we get the existence of a sequence [¶] of eigenvalues of $-\Delta$ s.t.

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \lambda_k \leq \lambda_{k+1} \leq \ldots .$$

As a non-decreasing real-valued monotonic sequence, $\{\lambda_k\}_k$ is either bounded (and thus, a converging one) or divergent. The second is the one that actually holds.

Proposition 3.5.6. $\lim_{k\to\infty} \lambda_k = +\infty$. *In particular, there are infinitely many eigenvalues of* $-\Delta$.

Proof. We show that $\lim_{k\to\infty} \lambda_k = +\infty$. Obviously, this is enough to grant the fact that $\{\lambda_k\}_k$ is infinite.
We know that $\{\lambda_k\}_k$ is a non-decreasing real-valued monotonic sequence, hence it cannot be oscillating.
Let $\{e_k\}_k$ be the sequence of eigenfunctions related to the family of the eigenvalues of $-\Delta$ we get iterating the above-mentioned argument, so that

$$\begin{aligned} &-\Delta e_k = \lambda_k e_k \ \forall k \geq 1, \\ &\|e_k\|_2 = 1 \ \forall k \geq 1, \\ &\int_\Omega e_i e_j dx = 0 \ \forall i \neq j, \\ &\int_\Omega |\nabla e_k|^2 dx = \|e_k\|_{H_0^1}^2 = \lambda_k \ \forall k \geq 1. \end{aligned}$$

[¶] A possibly definitevely constant sequence, as $E_j = \{u \in \mathcal{S} : \int_\Omega u e_k dx = 0 \ \forall k \leq j\}$ may be empty.

Seeking a contradiction, if $\lambda_k \not\to +\infty$, then $\lambda_k \to \lambda_0 \in \mathbb{R}_+$ as $k \to \infty$. However, $\|e_k\|^2_{H^1_0} = \lambda_k$. Hence, the sequence $\{e_k\}_k$ would be bounded in H^1_0 and, therefore, by Banach-Alaoglu Theorem, there would exist a subsequence $\{e_{k_j}\}_j$ weakly converging to $e_0 \in H^1_0(\Omega)$. On the other hand, by Corollary (2.2.14), $e_{k_j} \to e_0$ as $j \to \infty$ in the norm of $L^2(\Omega)$.
But the e_{k_j} are pairwise orthogonal in L^2 . Hence, $\forall i \neq j$,

$$\left\|e_{k_j} - e_{k_i}\right\|_2^2 = \left\|e_{k_j}\right\|_2^2 + \|e_{k_i}\|_2^2 = 2$$

which contradicts the fact that, as it converges in L^2, the sequence $\{e_{k_j}\}_j$ is Cauchy in L^2.

□

Remark 3.5.7. Observe that, according to the results illustrated in this chapter, set $\forall k > 0$,

$$E_k = \left\{ u \in \mathcal{S} : \int_\Omega e_j u dx = 0 \ \forall j \leq k \right\}$$

and u_k the eigenfunction associated to λ_k, one has

$$\lambda_k = \|e_k\|^2_{H^1_0} = \inf_{u \in E_{k-1}} \mathcal{E}(u).$$

Moreover, even if it's not that essential, it's worth to observe that the k-th eigenfunctions can be thought as belonging to the intersection between the $(k-1)$-th constraint (E_{k-2}) and the constraint $\{u : \int_\Omega e_{k-1} u dx = 0\}$.

Moreover, it's worth to point out the the following characterization for the eigenvalus of $-\Delta$ holds:

Theorem 3.5.8 (Courant-Hilbert). *For all $k = 1, 2, 3, \ldots$, let*

$$\mathcal{E}_k = \left\{ E_k \subseteq H^1_0(\Omega) : E_k \text{ } k\text{-dimensional subspace of } H^1_0 \right\}.$$

Then,

$$\lambda_k = \inf_{E_k \subseteq \mathcal{E}_k} \sup_{u \in E_k \cap \mathcal{S}} \mathcal{E}(u).$$

There are situations in which Lagrange multipliers cannot be used to find solutions of differential equations. We clarify this fact with the following example:

Example 3.5.9. Let $\Omega \subset\subset \mathbb{R}^n$. We consider the nonlinear problem

$$\begin{cases} -\Delta u = u|u|^{p-1} & \text{on } \Omega, \\ u|_{\partial\Omega} \equiv 0, \end{cases} \tag{3.8}$$

for $p \in (1, 2^* - 1)$.
We want to use Lagrange multipliers in a smart way, to provide a non-zero solution of

(3.8). For this purpose, we need to exhibit the minimum of an energy functional on a certain constraint.
Let $\mathcal{E} : H_0^1(\Omega) \to \mathbb{R}$ the usual functional defined by

$$\mathcal{E}(u) = \int_\Omega |\nabla u|^2 dx$$

and let $F(u) = -1 + \|u\|_{p+1}^{p+1}$. We denote with

$$\mathcal{S} = \left\{ u \in H_0^1(\Omega) : F(u) = 0 \right\}.$$

In order to use Theorem (3.5.3), we need to provide a minimum for $\mathcal{E}$ along $\mathcal{S}$.
Let $m = \inf_{\mathcal{S}} \mathcal{E} > 0$ and $\{u_k\}_k \subset \mathcal{S}$ be a sequence s.t. $\mathcal{E}(u_k) \to m$ as $k \to \infty$.
By the definition of $\mathcal{E}$, the sequence $\{u_k\}_k$ is bounded in the norm of $H_0^1(\Omega)$ and, hence, by Banach-Alaoglu Theorem there exists a subsequence $\{u_{k_j}\}_j$ that converges weakly to $u_0 \in H_0^1(\Omega)$ in $H_0^1(\Omega)$.
As $p+1 < 2^*$, $\|\cdot\|_{p+1}^{p+1}$ is WC (see Example (3.2.5)). So, $\|u_0\|_{p+1} = \lim_{j \to +\infty} \left\|u_{k_j}\right\|_{p+1} = 1$. That is, $u_0 \in \mathcal{S}$.
$\mathcal{E}$ is WLS, so that

$$m \leq \mathcal{E}(u_0) \leq \liminf_{j \to \infty} \mathcal{E}(u_{k_j}) = m.$$

Hence, we found a minimizer $u_0 \in H_0^1(\Omega)$ for $\mathcal{S}$. It remains to prove that $F'(u_0) \neq 0$. However, this follows immediately by the fact that

$$F'(u_0)[u_0] = (p+1) \int_\Omega |u_0|^{p-1} u_0^2 dx = p + 1 \neq 0.$$

We can apply Lagrange multipliers to provide a differential equation satisfied by u_0: there exists $\lambda \in \mathbb{R}$ s.t.

$$\mathcal{E}'(u_0)[v] = \underbrace{2 \int_\Omega \nabla u_0 \nabla v dx = \lambda \int_\Omega (p+1)|u_0|^{p-1} u_0 v dx}_{(3.9)} = F'(u_0)[v] \tag{3.9}$$

$\forall v \in H_0^1(\Omega)$. In particular, if $v = u_0$, we get

$$m = \mathcal{E}(u_0) = \|\nabla u_0\|_2^2 = \frac{\lambda}{2}(p+1) \underbrace{\int_\Omega |u_0|^{p+1} dx}_{=1}.$$

Hence,

$$\lambda = \frac{2m}{p+1}$$

and, for all $v \in H_0^1(\Omega)$, substituting in (3.9)

$$2 \int_\Omega \nabla u_0 \nabla v dx = \frac{2m}{p+1}(p+1) \int_\Omega |u_0|^{p-1} u_0 v dx,$$

that is

$$\int_\Omega \nabla u_0 \nabla v dx = m \int_\Omega |u_0|^{p-1} u_0 v dx \tag{3.10}$$

for all $v \in H_0^1(\Omega)$.
For this reason, Lagrange multipliers didn't provide a solution of (3.8), but one of

$$-\Delta u = m \cdot u|u|^{p-1}. \tag{3.11}$$

Luckily, the equation (3.8) is homogeneous, so that it suffices to search for a solution $\bar{u} \in H_0^1(\Omega)$ in the form $\bar{u} = \alpha u_0$ for some $\alpha > 0$, so that

$$\int_\Omega \nabla \bar{u} \nabla v dx = \int_\Omega |\bar{u}|^{p-1} \bar{u} v dx \tag{3.12}$$

for all $v \in H_0^1(\Omega)$ (that is the equation that a solution of (3.8) has to satisfy).
Since u_0 is a solution of (3.11), $\bar{u} = \alpha u_0$ satisfies

$$\alpha^{-1} \int_\Omega \nabla \bar{u} \nabla v dx = m\alpha^{-p} \int_\Omega |\bar{u}|^{p-1} \bar{u} v dx.$$

This implies that the parameter α in correspondence of which (3.12) holds satisfies

$$m = \alpha^{p-1},$$

that is

$$\alpha = m^{\frac{1}{p-1}}.$$

In conclusion, we derived a non-zero solution of (3.8):

$$\bar{u} = m^{\frac{1}{p-1}} u_0.$$

Remark 3.5.10. In this example, it is evident that the homogeneous nature of the equation (3.8) is crucial in order to apply Lagrange multipliers and solve (in H_0^1) an equation as

$$-\Delta u = f(u),$$

finding the minima of $\mathcal{E}(u) = \int_\Omega |\nabla u|^2 dx$ along the constraint $\int_\Omega F(u) dx = 0$ (here, $F(s) = \int_0^s f(t) dt$).

Chapter 4

Minimax theorems

The purpose of this chapter is that of providing several results concerning the localization of functionals' critical points.

4.1 Deformation lemmas

Definition 4.1.1. Let E be a reflexive Banach space and $J \in \mathcal{C}^1(E, \mathbb{R})$.

- The **sublevel of J at the level** $c \in \mathbb{R}$ is the set

$$J^c = \{J < c\} := \{u \in E : J(u) < c\} = J^{-1}\left((-\infty, c)\right)$$

  with the subspace topology inherited by E;

- $c \in \mathbb{R}$ is called a **critical value** of J if $\exists u \in E$ s.t.

$$\begin{cases} J(u) = c, \\ J'(u) = 0; \end{cases} \tag{4.1}$$

- The element $u \in E$ for which (4.1) holds is called a **critical point**.

The following example clarifies the idea under the theorem we will enunciate at the end of this section, after proving a finite-dimensional version of it.

Example 4.1.2. Let $J : \mathbb{R} \to \mathbb{R}$ be defined by $J(x) = x^2$. $c = 0$ is a critical value of J. In fact, for $x_0 = 0$, we have

$$\begin{cases} J(0) = 0, \\ J'(0) = 0. \end{cases}$$

The important remark is the following:

$$J^c = \begin{cases} (-\sqrt{c}, \sqrt{c}) & \text{if } c > 0, \\ \varnothing & \text{if } c \leq 0. \end{cases}$$

The critical value $c = 0$ acts as a "border-value" in the following sense: as $c < 0$, J^c is an interval, while as $c \leq 0$, $J^c = \varnothing$, that is, the sublevels corresponding to $c > 0$ are not homeomorphic to those corresponding to $c \leq 0$.

Lemma 4.1.3 (finite-dimensional deformation lemma)**.** *Let $J \in \mathcal{C}^{1,1}(\mathbb{R}^n, \mathbb{R})$ and $c \in \mathbb{R}$ be a critical value for J. If*

- *$\exists \varepsilon_0 > 0$ s.t. $\mathcal{U} = \{c - \varepsilon_0 \leq J \leq c + \varepsilon_0\} \subseteq \mathbb{R}^n$ is compact;*
- *$\exists \lambda > 0$ s.t. $\forall u \in \mathcal{U}$ one has $|J'(u)| \geq \lambda$;*

then, $\exists \varepsilon > 0$ s.t. $J^{c+\varepsilon}$ is homeomorphic to $J^{c-\varepsilon}$.

Proof. Let $\varepsilon \in (0, \varepsilon_0/2)$.

1. **We exhibit the homeomorphism**
   Let $B = \{c - \varepsilon \leq J \leq c + \varepsilon\}$. We consider

$$f(x) = \frac{\operatorname{dist}(x, \mathcal{U}^C)}{\operatorname{dist}(x, \mathcal{U}^C) + \operatorname{dist}(x, B)}.$$

   f is locally Lipschitz and it's continuous.
   We consider the Cauchy problem

$$\begin{cases} \frac{d}{dt}x(t) = -\frac{2\varepsilon}{\lambda^2} f(x(t)) J'(x(t)), \\ x(0) = x_0. \end{cases} \tag{4.2}$$

   By the theory of the ODEs, we know that this system has 1! solution $\eta(t, x_0) = x(t)$ that is continuous with respect to its variable x_0. Moreover,

$$\frac{d}{dt}J(\eta(t,x_0)) = J'(\eta(t,x_0))\frac{d}{dt}\eta(t,x_0) = -\underbrace{\frac{2\varepsilon}{\lambda^2}}_{\geq 0}\underbrace{f(\eta(t,x_0))}_{\in [0,1]}\underbrace{|J'(\eta(t,x_0))|^2}_{\geq \lambda^2} \leq -2\varepsilon.$$

   Therefore, $J(\eta(t_1, x_0)) \leq J(\eta(t_2, x_0))$ if $t_1 \geq t_2$.
   We define $\Phi(x_0) = \eta(1, x_0)$. Φ is well-defined by the uniqueness of the solution of (4.2) and it is a homeomorphism by the continuity properties of the solution of (4.2).

2. **We prove that** $\Phi(J^{c+\varepsilon}) \subseteq J^{c-\varepsilon}$
   Let $x_0 \in \{J \leq c + \varepsilon\}$.

   - If $\eta(t, x_0) \in \{J \leq c - \varepsilon\}$ for some $t \in [0, 1]$, then $J(\Phi(x_0)) \leq J(\eta(t, x_0)) \leq c - \varepsilon$ as we observed above.

- Seeking a contradiction, if $\eta(t,x_0) \in \{c-\varepsilon < J \le c+\varepsilon\}$ $\forall t \in [0,1]$,

$$J(\Phi(x_0)) = J(x_0) + \int_0^1 \frac{d}{dt} J(\eta(t,x_0))dt = \underbrace{J(x_0)}_{\le c+\varepsilon} - \frac{2\varepsilon}{\lambda^2}\int_0^1 f(\eta(t,x_0)) \underbrace{|J'(\eta(t,x_0))|^2}_{\ge \lambda^2} dt \le$$

$$\le c+\varepsilon - 2\varepsilon \int_0^1 \underbrace{f(\eta(t,x_0))}_{=1 \text{ in } B} dt = c-\varepsilon.$$

In contradiction with the fact that $\eta(1,u) \in \{c-\varepsilon < J \le c+\varepsilon\}$.

This concludes the proof.

□

For the infinite-dimensional case we need to introduce several notions. The first one is a compactness condition whose purpose is that of replacing the assumptions:

- $\|J'(u)\| \ge \lambda$,
- compactness of the strip,

in the finite-dimensional deformation lemma.

Definition 4.1.4. Let E be a Banach space and $J \in \mathcal{C}^1(E,\mathbb{R})$. We say that J satisfies the **Palais-Smale condition** (PS) if all the sequences $\{u_k\}_k \subset E$ s.t.

- $|J(u_k)| \le C$ $\forall k \in \mathbb{N}$ for some $C > 0$;
- $J'(u_k) \to 0$ as $k \to \infty$;

admit a converging (in E) subsequence.

Example 4.1.5. Let $J : \mathbb{R} \to \mathbb{R}$ be the functional $J(x) = e^{-x}$. J does not satisfy (PS). In fact, the sequence $u_k = k$ is s.t.

- $|e^{-k}| \le 1$ for all $k \ge 0$;
- $|-e^{-k}| \to 0$ as $k \to \infty$.

However, $\{k\}_k$ does not admit a converging subsequence.

Example 4.1.6. Let $\Omega \subset\subset \mathbb{R}^n$ and $E = H_0^1(\Omega)$. We consider the functional on $H_0^1(\Omega)$ given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{\alpha}{2}\int_\Omega |u|^2 dx + \int_\Omega fu dx$$

with

- $f \in L^2(\Omega)$;

- $\alpha < \lambda_1$ (λ_1 is the principal eigenvalue of Laplace operator).

We show that J satisfies (PS).
Let $\{u_k\}_k \subset H_0^1(\Omega)$ be a sequence s.t.

- $\forall k > 0$, $|J(u_k)| \leq C_1$ for some $C_1 > 0$;
- $|J'(u_k)| \to 0$ as $k \to \infty$.

We show that $\{u_k\}_k$ admits a subsequence that converges in $H_0^1(\Omega)$.
We use the first assumption:

$$C_1 \geq |J(u_k)| = \left| \frac{1}{2}\int_\Omega |\nabla u_k|^2 dx + \frac{\alpha}{2}\int_\Omega |u_k|^2 dx + \int_\Omega f u_k dx \right| \geq \frac{1}{2}\underbrace{\int_\Omega |\nabla u_k|^2 dx}_{=\|u_k\|^2_{H_0^1}} - \frac{\alpha}{2}\int_\Omega |u_k|^2 dx +$$

$$- \underbrace{\int_\Omega |f u_k| dx}_{\leq \|u_k\|_2 \|f\|_2} \geq \frac{1}{2}\|u_k\|^2_{H_0^1} - \frac{\alpha}{2}\int_\Omega |u_k|^2 dx - \underbrace{\|u_k\|_2}_{\leq C\|u_k\|_{H_0^1}} \|f\|_2$$

By the definition of λ_1,

$$\int_\Omega |u_k|^2 dx \geq \lambda_1 \int_\Omega |u_k|^2 dx$$

so that:

$$C_1 \geq \frac{1}{2}\|u_k\|^2_{H_0^1} - \frac{\alpha}{2\lambda_1}\int_\Omega \|u_k\|^2_{H_0^1} - C\|f\|_2 \|u\|_{H_0^1} = \frac{1}{2}\underbrace{\left(1 - \frac{\alpha}{\lambda_1}\right)}_{>0}\|u_k\|^2_{H_0^1} - const\,\|u_k\|_{H_0^1}$$

Therefore, $\|u_k\|_{H_0^1} \leq C$ for some $C > 0$ (otherwise there would not be a constant upper bound for the left hand side).
By Banach-Alaoglu Theorem there exists a subsequence $\{u_{k_j}\}_j$ which converges weakly in H_0^1 to some function $\bar{u} \in H_0^1$.
By Corollary (2.2.14), $u_{k_j} \to \bar{u}$ as $j \to \infty$ in $L^2(\Omega)$.
Now, we use the second hypothesis on $\{u_k\}_k$: for all $v \in H_0^1(\Omega)$

$$J'(u_{k_j})[v] \to 0$$

as $j \to \infty$.
In particular, for $v = u_{k_j} - \bar{u}$, using the fact that the $J'(u_k)$ are linear functionals,

$$0 \leftarrow J'(u_{k_j})[u_{k_j} - \bar{u}] = \int_\Omega \nabla u_{k_j} \nabla(u_{k_j} - \bar{u}) dx + \frac{\alpha}{2}\int_\Omega u_k(u_{k_j} - \bar{u}) dx + \int_\Omega f(u_{k_j} - \bar{u}) dx =$$

$$= \int_\Omega \nabla u_{k_j} \nabla(u_{k_j} - \bar{u}) dx + \underbrace{\frac{\alpha}{2}(u_{k_j}, u_{k_j} - \bar{u})_{L^2} + (f, u_{k_j} - \bar{u})_{L^2}}_{\to 0 \text{ by the continuity of } (\cdot,\cdot)_{L^2}}$$

as $j \to \infty$. Therefore,

$$\int_\Omega \nabla u_{k_j} \nabla (u_{k_j} - \bar{u}) dx \to 0$$

as $j \to \infty$.
Hence,

$$\left\| u_{k_j} - \bar{u} \right\|^2_{H^1_0} = \int_\Omega |\nabla u_{k_j} - \nabla \bar{u}|^2 dx = \underbrace{\int_\Omega \nabla u_{k_j} (\nabla u_{k_j} - \nabla \bar{u}) dx}_{\to 0} - \underbrace{\int_\Omega \nabla \bar{u} (\nabla u_{k_j} - \nabla \bar{u}) dx}_{\substack{= (\bar{u}, u_{k_j} - \bar{u})_{H^1_0} \to 0 \\ \text{since } u_{k_j} \rightharpoonup \bar{u}}} \to 0$$

as $j \to \infty$.

Remark 4.1.7. For $\alpha = \lambda_1$, the argument above fails. In fact, choosing $f = 0$ for simplicity, one can prove that the sequence $\{u_k + k e_k\}_k$ satisfies (PS), but does not admit any converging subsequence.

The second notion we need, whose purpose is that of replacing the hypothesis that guarantee the existence and uniqueness of the solution of the system of differential equations which provides the homeomorphism between the sublevels in the finite-dimensional deformation lemma, is the following:

Definition 4.1.8. Let E be a Banach space and $J \in \mathcal{C}1(E, \mathbb{R})$.
$v \in E$ is calles a **pseudo-gradient** of J in $u \in E$ if

- $\|v\|_E \leq 2 \|J'(u)\|_{op}$;
- $J'(u)[v] =: \langle J'(u), v \rangle \geq \|J'(u)\|^2_{op}$.

Remark 4.1.9. Pseudo-gradients are not, in general, unique. Moreover, if v_1, v_2 are two pseudo-gradients of J in u, then

$$\vartheta v_1 + (1 - \vartheta) v_2$$

is another pseudo-gradient of J in u for all $\vartheta \in [0, 1]$.

Remark 4.1.10. A pseudo-gradient is an element of E, while $J'(u)$ is a bounded linear functional. If E is a Hilbert space, then $E^* \cong E$, so that the bounded linear functional are all represented by elements of E. In particular, there is an identification between the functional $J'(u)$ and the element $v \in E$ given by Riesz representation theorem, i.e.

$$(v, w)_E = J'(u)[w] \quad \forall w \in E.$$

Then, if E is a Hilbert space *, with an abuse of language, We say that v is the pseudo-gradient of J in u.

*Actually, it suffices for E to be a reflexive Banach space.

Definition 4.1.11. Let E be a Banach space and $J \in \mathcal{C}^1(E, \mathbb{R})$. We denote with

$$\tilde{E} = \{u \in E : J'(u) \neq 0\}.$$

A mapping $V : \tilde{E} \to E$ is called a **vector field of pseudo-gradients** on $\tilde{E}$ if

- $V(u)$ is a pseudo-gradient of J in u for all $u \in \tilde{E}$;
- V is locally Lipschitz.

Lemma 4.1.12 (Paracompactness). *Let (Y, d) be a metric space and let $\mathcal{A} = \{\Omega_\alpha\}_{\alpha \in A}$ be an open cover of Y. Then, there exists a cover $\mathcal{B} = \{\omega_\beta\}_{\beta \in B}$ that is finer than $\mathcal{A}$ and locally finite.*
Moreover, there exists a partition of unity $\{\vartheta_\beta : Y \to \mathbb{R}\}_{\beta \in B}$, that is a collection of functions s.t. $\forall \beta \in B$,

(a) supp$(\vartheta_\beta) \subset \omega_\beta$;

(b) $0 \leq \vartheta_\beta \leq 1$;

(c) $\sum_\beta \vartheta_\beta \equiv 1$ on Y;

(d) ϑ_β is locally Lipschitz.

Lemma 4.1.13. *Let E be a Banach space and $J \in \mathcal{C}^1(E, \mathbb{R})$. Then, there exists a vector field of pseudo-gradients for J on $\tilde{E}$.*

Proof. If $u \in \tilde{E}$, by the definition of $\tilde{E}$,

$$\left\| J'(u) \right\|_{op} = \sup_{\|v\|_E = 1} \langle J'(u), v \rangle > 0.$$

By the definition of sup, $\exists x_u \in E$ s.t. $\|x_u\|_E = 1$ and

$$\langle J'(u), x_u \rangle > \frac{2}{3} \left\| J'(u) \right\|_{op}. \tag{4.3}$$

We define

$$v_u := \frac{3}{2} \left\| J'(u) \right\|_{op} x_u$$

and show that v_u is a pseudo-gradient for J in u:

$$\|v_u\|_E = \frac{3}{2} \left\| J'(u) \right\|_{op} \underbrace{\|x_u\|_E}_{=1} \leq 2 \left\| J'(u) \right\|_{op}. \tag{4.4}$$

Moreover, by the definition of x_u,

$$\langle J'(u), v_u \rangle = \frac{3}{2} \left\| J'(u) \right\|_{op} \langle J'(u), x_u \rangle > \left\| J'(u) \right\|_{op}^2. \tag{4.5}$$

By the continuity of (4.3), (4.4) and (4.5) with respect to u, v_u is a pseudo-gradient of v for all $v \in U \subseteq \tilde{E}$ for an appropriate open neighborhood $U = U(u)$ of u.
$\{U(u)\}_{u \in \tilde{E}}$ is an open cover of $\tilde{E}$, so that there exists a refinement $\{\omega_\beta\}_{\beta \in B}$ and a partition of unity subordinate to it, as in Lemma (4.1.12).
We define $V : \tilde{E} \to E$ as the mapping

$$V(u) = \sum_{\beta \in B} \vartheta_\beta(u) v_{\beta_u}, \tag{4.6}$$

where v_{β_u} is the pseudo-gradient of u in ω_β.
For all $u \in \tilde{E}$, as $\{\omega_\beta\}_\beta$ is locally finite, the series in (4.6) is a finite sum of locally Lipschitz functions (the ϑ_β are locally Lipschitz and the v_{β_u} are bounded), hence locally Lipschitz.
Moreover, for all $u \in \tilde{E}$, $V(u)$ is a convex combination of pseudo-gradients, so that it is a pseudo-gradient for J in u by Remark (4.1.9).
Hence, V is a vector field of pseudo-gradients for J.

□

Lemma 4.1.14. *Let E be a Banach space and $J \in \mathcal{C}^(E, \mathbb{R})$ be s.t. $J(u) = J(-u)$ $\forall u \in E$. Then, there exists a vector field of pseudo-gradients V for J in $\tilde{E}$ s.t. $V(u) = -V(-u)$.*

Proof. Let V be a vector field of pseudo-gradients for J on $\tilde{E}$. We define $W : \tilde{E} \to \mathbb{R}$ as

$$W(u) = \frac{V(u) - V(-u)}{2}.$$

It's easy to see that, if $J(u) = J(-u)$ for all $u \in E$, then $J'(u) = -J'(-u)$ for all $u \in E$.
We show that W is a vector field of pseudo-gradients which satisfies the assertion.

- It is obviously locally Lipschitz and it trivially satisfies $W(u) = -W(-u)$;
- for all $u \in \tilde{E}$,

$$\|W(u)\|_E \le \frac{\|V(u)\|_E + \|V(-u)\|_E}{2} \le \frac{2\,\|J'(u)\|_{op} + 2\,\|J'(-u)\|_{op}}{2} = 2\,\|J'(u)\|_E .$$

- For all $u \in \tilde{E}$,

$$\langle J'(u), W(u)\rangle = \frac{1}{2}\langle J'(u), V(u)\rangle - \frac{1}{2}\langle J'(u), V(-u)\rangle = \frac{\langle J'(u), V(u)\rangle + \langle J'(-u), V(-u)\rangle}{2} \ge$$
$$\ge \frac{1}{2}\,\|J'(u)\|_{op}^2 + \frac{1}{2}\,\|J'(-u)\|_{op}^2 = \|J'(u)\|_{op}^2 .$$

□

We see how (PS) replaces the compactness of $\{c - \varepsilon \le J \le c + \varepsilon\}$ in Lemma (4.1.3).

Lemma 4.1.15. *Let $J \in \mathcal{C}^1(E, \mathbb{R})$ satisfying (PS). Then, for all $c \in \mathbb{R}$ fixed*

(a) the set

$$K_c := \{u \in E : J(u) = c,\ J'(u) = 0\}$$

is compact;

(b) if $I \subset \mathbb{R}$ is compact, then $\bigcup_{c\in I} K_c$ is compact.

Proof. It is clear that $(b) \Longrightarrow (a)$ (choose $I = \{c\}$), so we prove (b): for, we prove that all the sequences of elements of $\bigcup_{c\in I} K_c$ admit a converging subsequence in the union. Let $\{u_k\}_k \subseteq \bigcup_{c\in I} K_c$, i.e. for all k, $J(u_k) \in I$ and $J'(u_k) = 0$. Since $\{J(u_k)\}_k \subseteq I$ and I is bounded,

$$|J(u_k)| \leq C \quad \forall k;$$

moreover,

$$J'(u_k) \equiv 0 \to 0 \quad \text{as } k \to \infty.$$

by (PS), there exists a subsequence $\{u_{k_j}\}_j \subseteq \{u_k\}_k \subseteq \bigcup_{c\in I} K_c$ that converges in E to some function $\bar{u}$. On the other hand, by continuity, $J(\bar{u}) \in I$ and $J'(\bar{u}) = 0$.
This concludes the proof.

□

Lemma 4.1.16 (Deformation lemma)**.** *Let $J \in \mathcal{C}^1(E, \mathbb{R})$ satisfying (PS). Then, for all $c \in \mathbb{R}$, $\forall \bar{\varepsilon} > 0$ and for all the open neighborhoods $O(K_c)$ of K_c there exist $\varepsilon \in (0, \bar{\varepsilon})$ and a deformation $\eta \in \mathcal{C}^0([0,1] \times E, E)$ s.t. $\forall t \in [0,1]$*

(a) $\eta(0, u) = u$ for all $u \in E$;

(b) if $J(u) \notin [c - \bar{\varepsilon}, c + \bar{\varepsilon}]$, then $\eta(t, u) = u$;

(c) $\eta(t, \cdot) : E \to E$ is a homeomorphism;

(d) $J(\eta(t, u)) \leq J(u)$ for all $u \in E$;

(e) $\eta(1, \{J \leq c + \varepsilon\}) \setminus O(K_c) \subseteq \{J \leq c - \varepsilon\}$;

(f) if $K_c = \varnothing$, then $\eta(1, \{J \leq c + \varepsilon\}) \subseteq \{J \leq c - \varepsilon\}$;

(g) if J is even, $\eta(t, \cdot)$ is odd.

This theorem is useful to prove the existence of critical points through the following procedure:

- one supposes, by contradiction, that a candidate critical value $c \in \mathbb{R}$ is not a critical value;
- one uses (f) to prove that a contradictory inequality for $J(u)$ holds for an appropriate u.

Proof. It's all about providing the deformation η and proving that it satisfies all the points of the assertion.
Obviously, we can prove (e) using neighborhoods of K_c of the form

$$N_\delta(K_c) = \left\{ u \in E : \text{dist}(u, K_c) = \inf_{v \in K_c} \text{dist}(u, v) < \delta \right\}.$$

1. **There exist $\beta, \varepsilon_0 > 0$ s.t. $\|J'(u)\|_{op} \geq \beta$ for all** $u \in \{c - \varepsilon_0 \leq J \leq c + \varepsilon_0\} \setminus N_{\delta/8}$ †.
   Seeking a contradiction, we suppose that $\forall \beta, \varepsilon_0 > 0$ there exist $u \in \{c - \varepsilon_0 \leq J \leq c + \varepsilon_0\} \setminus N_{\delta/8}$ s.t. $\|J'(u)\|_{op} < \beta$. Hence, taking any two sequences $\{\beta_k\}_k, \{\varepsilon_k\}_k \in \mathbb{R}_+$ s.t. $\beta_k, \varepsilon_k \searrow 0$ as $k \to \infty$, we would have $\|J'(u_k)\|_{op} < \beta_k$ for an appropriate sequence $\{u_k\}_k$ s.t. $\{c - \varepsilon_k \leq J \leq c + \varepsilon_k\} \setminus N_{\delta/8}$. Then, $\{u_k\}_k$ would satisfy:

   - $|J(u_k)| \leq c + 1$ definitively;
   - $\|J'(u_k)\|_{op} \leq \beta_k \to 0$ as $k \to +\infty$.

   Hence, it satisfies (PS), so that there exists a subsequence $\{u_{k_j}\}_j$ that converges to a certain $\bar{u} \in E$, with

   - $c - \varepsilon_k \leq J(\bar{u}) \leq c + \varepsilon_k$ for all k. Hence, $J(\bar{u}) = c$, by continuity and
   - $\|J'(\bar{u})\|_{op} = 0$ by continuity.

   That is, $\bar{u} \in K_c$. However, $\text{dist}(u_{k_j}, K_c) \geq \delta/8$ for all j at the same time, so that the limit of u_{k_j} cannot be an element of K_c. This is a contradiction.

2. **We exhibit η as a local solution of a certain Cauchy problem**
   First, we need to choose an ODE whose solution is η. We choose ε_0 and β be as above. Up to choose ε_0 even smaller, we can suppose that

   $$0 < \varepsilon_0 < \min\left\{ \bar{\varepsilon}, \frac{\beta\delta}{32}, \frac{\beta^2}{2}, \frac{1}{8} \right\}.$$

   Let $\varepsilon \in (0, \varepsilon_0)$. We define the two sets

   $$A = \{J \geq c + \varepsilon_0\} \cup \{J \leq c - \varepsilon_0\},$$
   $$B = \{c - \varepsilon \leq J \leq c + \varepsilon\}.$$

   A and B are disjoint and closed, by the continuity of J. Now, we define two cut-offs: $f : E \to [0, 1]$ and $g : E \to [0, 1]$ as follows:

   $$f(x) = \frac{\text{dist}(x, N_{\delta/8})}{\text{dist}(x, N_{\delta/8}) + \text{dist}(x, N^C_{\delta/4})}$$

† Actually, this means that we recover the boundedness-from-below hypothesis of $\|J'(u)\|_{op}$ and we do this using the (PS) condition.

($f \equiv 0$ on $N_{\delta/8}$ and, therefore, it is $\neq 0$ away from K_c) and

$$g(x) = \frac{\operatorname{dist}(x, A)}{\operatorname{dist}(x, A) + \operatorname{dist}(x, B)}$$

($g \equiv 0$ on A, so that it takes into account the values for which $\{x - \varepsilon_0 < J < c + \varepsilon_0\}$).
We observe that, under the further hypothesis of (g), that is if J is even, A, B and N_δ are obviously symmetric [‡], so that f and g, in this case, are even functions.
Finally, we define a function whose purpose is that of making the function that we're gonna use in the differential equation locally Lipschitz:

$$h(s) = \chi_{[0,1]}(s) + \frac{1}{s}\chi_{(1,+\infty)}(s).$$

by construction, f, g and h are Lipschitz.
Let V be a vector field of pseudo-gradients for J in $E_1 := E \setminus K_c$ (which can be chosen odd if J is even). We set

$$W(x) = f(x)g(x)h(\|V(x)\|_E)V(x).$$

Then,

- W is defined for all $x \in E$, as $f \equiv 0$ on K_c;
- W is locally Lipschitz since all of its factors are;
- Let $x \in E$, then

$$0 \leq \|W(x)\|_E \leq |f(x)||g(x)||h(V(x))| \, \|V(x)\|_E = h(\|V(x)\|_E) \, \|V(x)\|_E = \\ = \begin{cases} \|V(x)\|_E & \text{if } \|V(x)\|_E \leq 1, \\ 1 & \text{if } \|V(x)\|_E > 1 \end{cases} \leq 1.$$

Let $u \in E$. By the classical ODEs theory, there exists 1! local solution $\eta(\cdot, u) : (t^-(u), t^+(u)) \subseteq \mathbb{R} \to E$ of

$$\begin{cases} \frac{d\eta}{dt}(t, u) = -W(\eta(t, u)), \\ \eta(0, u) = u. \end{cases} \tag{4.7}$$

3. **We show that $\eta(\cdot, u)$ is a global solution of (4.7)**
Seeking a contradiction, let $t^+(u) < \infty$ for some $u \in E$.
In this case, for all the sequences $t_k \nearrow t^+(u)$,

$$\|\eta(t_k) - \eta(t_h, u)\|_E = \left\| \int_{t_h}^{t_k} W(\eta(s, u)) ds \right\|_E \leq \int_{t_h}^{t_k} \underbrace{\|W(\eta(s, u))\|_E}_{\leq 1} ds \leq |t_h - t_k|.$$

[‡]With respect to the origin. That is, $x \in A$ iff $-x \in A$. The same happens for B and N_δ.

Hence, $\{\eta(t_k, u)\}_k$ is Cauchy in E, that is a Banach space. Therefore, there would exist $\bar{u} \in E$ s.t. $\|\eta(t_k, u) - \bar{u}\|_E \to 0$ as $k \to \infty$. Since $\{t_k\}_k$ is arbitrary, we would get

$$\lim_{t \to t^+(u)} \eta(t, u) = \bar{u}.$$

Since $\bar{u} \in E$, we could solve (4.7) with initial data $\bar{u}$ getting a solution $\eta_2(\cdot, \bar{u}) : (t^+(u) - \delta_1, t^+(u) + \delta_1) \to E$ which would coincide with η on the (non-empty) intersection of their domains. This contradicts the maximality of $t^+(u)$.
Similarly, one proves that $t^-(u) = -\infty$.
In conclusion, η is unique and globally defined. In particular, $\eta(t, u)$ is defined on $[0, 1]$.

4. **We show that η satisfies** $(a) - (g)$

   (a) follows immediately by the definition of the initial data (4.7);

   (b) If $J(u) \notin [c - \bar{\varepsilon}, c + \bar{\varepsilon}]$, then $g(u) = 0$. By continuity, $g(u) = 0$ in a neighborhood of u. The function $\eta(t, u) = u$ is, therefore, a local solution of (4.7) with $W = 0$ i.e., it's a global solution.

   (c) η is obviously continuous and it's a homeomorphism, since its inverse is continuous as well, in fact:

   $$\eta(s, \cdot) \circ \eta(t, \cdot) = \eta(s + t, \cdot).$$

   In particular, $\eta(-t, \cdot)$ is the inverse of $\eta(t, \cdot)$. This concludes (c).

   (d) Let $u \in E$.
   We observe that, by the chain rule, it is:

   $$\frac{d}{dt} J(\eta(t, u)) = \left\langle J'(\eta(t, u)), \frac{d}{dt}\eta(t, u) \right\rangle = \left\langle J'(\eta(t, u)), -W(\eta(t, u)) \right\rangle. \quad (4.8)$$

   If $t \in [0, 1]$ is such that $W(\eta(t, u)) = 0$, then

   $$\frac{d}{dt} J(\eta(t, u)) = 0.$$

   While, if $t \in [0, 1]$ is such that $W(\eta(t, u)) \neq 0$, recalling the calculation in (4.8), we have:

   $$\begin{aligned} \frac{d}{dt} J(\eta(t, u)) &= \left\langle J'(\eta(t, u)), \frac{d}{dt}\eta(t, u) \right\rangle = \left\langle J'(\eta(t, u)), -W(\eta(t, u)) \right\rangle = \\ &= -\underbrace{f(\eta(t, u)) g(\eta(t, u)) h(\|V(\eta(t, u))\|_E)}_{\geq 0} \underbrace{\left\langle J'(\eta(t, u)), V(\eta(t, u)) \right\rangle}_{\geq \|J'(\eta(t, u))\|_{op}^2 \geq \beta^2} \leq 0 \end{aligned} \quad (4.9)$$

   In any case, $d\eta(t, u)/dt \leq 0$.

(e) Let $u \in \{J \leq c+\varepsilon\} \setminus N_\delta(K_c)$, we show that $\eta(1,u) \in \{J \leq c-\varepsilon\}$.
If $J(\eta(t,u)) \leq c-\varepsilon$ for some $t \in [0,1]$, the fact that $J(1,u) \leq c-\varepsilon$ follows directly by (d).
We show that, seeking a contradiction, this is the only possible case: let $J(\eta(t,u)) \in (c-\varepsilon, c+\varepsilon]$ for all $t \in [0,1]$. In this case, we would have:

$$0 \underset{(d)}{\leq} \underbrace{J(\eta(0,u))}_{\substack{=J(u) \leq c+\varepsilon \leq \\ \leq c+\varepsilon_0}} - \underbrace{J(\eta(t,u))}_{>c-\varepsilon} \leq c+\varepsilon_0 - c + \underbrace{\varepsilon}_{\leq \varepsilon_0} \leq 2\varepsilon_0. \tag{4.10}$$

We show that $\eta(t,u) \notin N_{\delta/2}$ $\forall t \in [0,1]$. In our case, $g(\eta(t,u)) = 1$ for all $t \in [0,1]$. Hence, we have that $\forall t \in [0,1]$,

$$\begin{aligned}
2\varepsilon_0 &\geq J(u) - J(\eta(t,u)) = \int_t^0 \frac{d}{dt} J(\eta(s,u))ds = \\
&= \int_0^t f(\eta(s,u))h(\|V(\eta(s,u))\|_E) \underbrace{\langle J'(\eta(s,u))V(\eta(s,u))\rangle}_{\geq \|J'(\eta)\|_{op}^2} ds \geq \\
&\geq \int_0^t f(\eta(s,u))h(\|V(\eta(s,u))\|_E) \left\|J'(\eta(s,u))\right\|_{op}^2 ds \underset{\|J'(\eta)\|_{op} \geq \beta}{\geq} \\
&\geq \int_0^1 f(\eta(s,u))h(\|V(\eta(s,u))\|_E) \underbrace{\left\|J'(\eta(s,u))\right\|_{op}}_{\|V(\eta)\|_E \leq 2\|J'(\eta)\|_{op}} ds \geq \\
&\geq \frac{\beta}{2} \int_0^t f(\eta(s,u))h(\|V(\eta(s,u))\|_E) \|V(\eta(s,u))\|_E ds \geq \\
&\geq \frac{\beta}{2} \left\| \int_0^t f(\eta(s,u))h(\|V(s,u)\|_E) V(\eta(s,u))ds \right\|_E = \\
&= \frac{\beta}{2} \left\| \int_0^t -W(\eta(s,u))ds \right\|_E = \frac{\beta}{2} \left\| \int_0^t \frac{d}{ds}\eta(s,u)ds \right\|_E = \frac{\beta}{2} \|\eta(t,u) - u\|_E
\end{aligned}$$

So that, for all $t \in [0,1]$,

$$\|\eta(t,u) - u\|_E \leq \frac{4\varepsilon_0}{\beta} < \frac{4}{\beta}\frac{\beta\delta}{32} = \frac{\delta}{8} < \frac{\delta}{2}.$$

It follows that, $\forall t \in [0,1]$

$$\delta < \operatorname{dist}(u, K_c) \leq \operatorname{dist}(u, \eta(t,u)) + \operatorname{dist}(\eta(t,u), K_c).$$

Therefore, $\forall t \in [0,1]$,

$$\operatorname{dist}(\eta(t,u), K_c) > \delta - \operatorname{dist}(u, \eta(1,u)) > \frac{\delta}{2}.$$

Turning back to the proof by contradiction, if $J(\eta(t,u)) \in (c-\varepsilon, c+\varepsilon]$ for all $t \in [0,1]$, we proved that $\forall t \in [0,1]$ we have $\operatorname{dist}(\eta(t,u), K_c) > \frac{\delta}{2}$. Then,

in this situation, we also have that $f(\eta(t,u)) = 1$ for all $t \in [0,1]$, by the definition of f itself.
Hence, in $[0,1]$, by (4.9) we have

$$\frac{d}{dt}J(\eta(t,u)) \leq -h(\|V(\eta(t,u))\|_E) \left\|J'(\eta(t,u))\right\|_{op}^2 .$$

We divide the two cases:

- $t \in [0,1]$ is such that $\|V(\eta(t,u))\|_E > 1$, then, by the definition h:

$$\frac{d}{dt}J(\eta(t,u)) \leq -\frac{\|J'(\eta(t,u))\|_{op}^2}{\|V(\eta(t,u))\|_E} = -\underbrace{\frac{\|J'(\eta(t,u))\|_{op}^2}{\|V(\eta(t,u))\|_E^2}}_{\substack{\geq \frac{1}{4} \text{ by the def.} \\ \text{of pseudo-grad.}}} \underbrace{\|V(\eta(t,u))\|_E}_{> 1 \text{ in this case}} \leq$$

$$\leq -\frac{1}{4}.$$

- If $t \in [0,1]$ is such that $\|V(\eta(t,u))\|_E \leq 1$, then, using the definition of h once again,

$$\frac{d}{dt}J(\eta(t,u)) \leq -\underbrace{\left\|J'(\eta(t,u))\right\|_{op}^2}_{\geq \beta^2} \leq -\beta^2.$$

We conclude that, for all $t \in [0,1]$ there must be

$$\frac{d}{dt}J(\eta(t,u)) \leq \max\left\{-\beta^2, -\frac{1}{4}\right\} = -\min\left\{\beta^2, \frac{1}{4}\right\}. \tag{4.11}$$

By integration:

$$2\varepsilon_0 < \min\left\{\beta^2, \frac{1}{4}\right\} \leq -\int_0^1 \frac{d}{dt}J(\eta(t,u))dt = J(u) - J(\eta(1,u)) \leq 2\varepsilon_0. \tag{4.12}$$

In contradiction with the choice of ε_0.

(f) follows immediately by the previous point, with $N_\delta(K_c) = \varnothing$.

(g) follows immediately by the previous points and by the particular choice of V we can make under the further assumptions.

□

4.2 A minimax principle

We see how the deformation lemma applies:

Theorem 4.2.1 (Minimax principle)**.** *Let E be a Banach space, $J \in \mathcal{C}^1(E, \mathbb{R})$ satisfying (PS) and η be the deformation whose existence is stated by Lemma (4.1.16). Let $\mathcal{S} = \{A \subseteq E : \forall A \in \mathcal{S}, \eta(1, A) \in \mathcal{S}\}$.*
If

$$-\infty < c := \inf_{A \in \mathcal{S}} \sup_{u \in A} J(u) < +\infty,$$

then c is a critical value of J.

Proof. Seeking a contradiction, if c weren't a critical value, there would exist $\varepsilon > 0$ s.t. $\eta(1, \{J \le c + \varepsilon\}) \subseteq \{J \le c - \varepsilon\}$. By the definition of c, $\exists A_\varepsilon \in \mathcal{S}$ such that

$$\sup_{u \in A_\varepsilon} J(u) \le c + \varepsilon.$$

In particular, for all $u \in A_\varepsilon$, $J(u) \le c + \varepsilon$, so that $J(\eta(1, u)) \le c - \varepsilon$ for all $u \in A_\varepsilon$. As $A_\varepsilon \in \mathcal{S}$, by the definition of $\mathcal{S}$, we would have $\eta(1, A_\varepsilon) \in \mathcal{S}$.

$$c \le \sup_{u \in \eta(1, A_\varepsilon)} J(u) \le c - \varepsilon.$$

This is a contradiction.

□

By the previous theorem, we get the following result:

Corollary 4.2.2. *Let $J \in \mathcal{C}^1(E, \mathbb{R})$ satisfying (PS). Let J be bounded from below, then $\inf_{u \in E} J(u)$ is attended.*

Proof. Let $\mathcal{S} = \{\{u\} : u \in E\}$. By the minimax principle,

$$c := \inf_{\{u\} \in \mathcal{S}} \sup_{v \in \{u\}} J(v) = \inf_{u \in E} J(u)$$

is finite and it's a critical value for J (hence, it is attended by J by the definition of critical value). □

4.3 The mountain pass theorem

In this section, we provide two proofs of the "mountain pass theorem" and one of its applications.
We're talking about a particular version of the minimax principle, in which we consider the infimum on paths connecting the origin of E to a point that satisfies certain conditions.

Theorem 4.3.1 (Mountain Pass)**.** *Let $J \in \mathcal{C}^1(E, \mathbb{R})$ satisfying (PS). We suppose that:*

(i) $J(0) = 0$*;*

(ii) there exist $\rho, \alpha > 0$ s.t. $J|_{\partial B_\rho(0)} \ge \alpha$;

(iii) there exists $e \in E \setminus \overline{B_\rho(0)}$ s.t. $J(e) \leq 0$.

Then, set

$$\Gamma = \{\gamma \in \mathcal{C}^0([0,1], E) : \gamma(0) = 0, \gamma(1) = e\},$$

we have that

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma([0,1])} J(u) \geq \alpha$$

and c is a critical value for J.

As we announced, we provide two proofs of Theorem (4.3.1). In both of the cases, we use the deformation lemma and argue by contradiction.

First proof of (4.3.1). 1. **We show that $c \geq \alpha$**

Let $\gamma \in \Gamma$. By the continuity of γ, there exists a point $w \in \gamma([0,1]) \cap \partial B_\rho(0)$. Then,

$$\sup_{u \in \gamma([0,1])} J(u) \geq J(w) \geq \alpha.$$

Taking the infimum,

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma([0,1])} \geq \alpha.$$

2. **We show that c is a critical value for J**

Seeking a contradiction, if c weren't a critical value for J, using the deformation lemma with $\bar{\varepsilon} = \alpha/2$, there would exists $\varepsilon \in (0, \alpha/2)$ and a deformation η s.t.

- $\eta(1, \{J \leq c + \varepsilon\}) \subseteq \{J \leq c - \varepsilon\}$;
- $\eta(1, u) = u$ if $J(u) \notin (c - \alpha/2, c + \alpha/2)$.

By the definition of infimum, there exists $\gamma_\varepsilon \in \Gamma$ s.t.

$$\sup_{u \in \gamma_\varepsilon([0,1])} J(u) \leq c + \varepsilon.$$

In particular, $J(u) \leq c+\varepsilon$ for all $u \in \gamma_\varepsilon([0,1])$ and, as a consequence, $J(u) \leq c-\varepsilon$ for all $u \in \eta(1, \gamma_\varepsilon([0,1]))$.
Then,

$$\sup_{u \in \eta(1, \gamma_\varepsilon([0,1]))} J(u) \leq c - \varepsilon.$$

If we proved that $\eta(1, \gamma_\varepsilon(\cdot)) \in \Gamma$, we would get the following contradiction and the proof would be concluded:

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma([0,1])} J(u) \leq \sup_{v \in \eta(1, \gamma_\varepsilon([0,1]))} J(v) \leq c - \varepsilon.$$

We show that $\eta(1, \gamma_\varepsilon(\cdot)) \in \{\gamma \in \mathcal{C}^0([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$.

- $\eta(1, \gamma_\varepsilon(\cdot)) = \eta(1, \cdot) \circ \gamma_\varepsilon$ is continuous as a composition of continuous functions.

- $\eta(1, \gamma_\varepsilon(0)) = \eta(1, 0)$. On the other hand, $0 = J(0) \notin (c - \alpha/2, c + \alpha/2)$ since $c \geq \alpha$, so that $\eta(1, 0) = 0$.
- $\eta(1, \gamma_\varepsilon(1)) = \eta(1, e) = e$ for the same reason as the previous point, as

$$J(e) < 0 < c - \frac{\alpha}{2} \implies J(e) \notin \left(c - \frac{\alpha}{2}, c + \frac{\alpha}{2}\right).$$

□

The second proof is not conceptually different from the previous one:

Second proof of (4.3.1). We define

$\mathcal{H} = \{h : E \to E$ homeomorphism s.t. $h(0) = 0, h(e) = e, h(\partial B_\rho(0))$ separates 0 from $e\}$.

We observe that $id_E \in \mathcal{H}$, so that $\mathcal{H} \neq \varnothing$.

1. **We prove that** $c \geq \alpha$
   Let
   $$b = \sup_{h \in \mathcal{H}} \inf_{u \in \partial B_\rho(0)} J(h(u)).$$
   We show that b separates α from c, that is $\alpha \leq b \leq c$.
   As $h(\partial B_\rho(0))$ separates 0 from e, for all $h \in \mathcal{H}$ and all $\gamma \in \Gamma$, there exists $w \in h(\partial B_\rho(0)) \cap \gamma([0, 1])$ by the continuity of each γ.
   As w is a particular element of $h(\partial B_\rho(0))$,
   $$\inf_{u \in \partial B_r ho(0)} J(h(u)) \leq J(w) \leq \sup_{t \in [0,1]} J(\gamma(t))$$
   for some γ in the definition of w.
   The left hand side does not depend on γ as well as the right hand side does not depend on h. Taking the extrema:
   $$\underbrace{\sup_{h \in \mathcal{H}} \inf_{u \in \partial B_\rho(0)} J(h(u))}_{=b} \leq \underbrace{\inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t))}_{=c}.$$
   On the other hand, h is a homeomorphism, hence
   $$\alpha \underset{(ii)}{\leq} \inf_{u \in \partial B_\rho(0)} J(u) \leq \sup_{h \in \mathcal{H}} \inf_{u \in \partial B_\rho(0)} J(h(u)) = b.$$

2. **We prove that** c **is a critical value for** J
   Seeking a contradiction, if c weren't a critical value for J, by the deformation lemma, chosen $\bar{\varepsilon} = \alpha/2$, we would get a deformation η for which all of the assertion of (4.1.16) hold.
   In particular, if for η the assertions of Lemma (4.1.16) hold, for $-\eta$ we have,
   $$\eta(1, \{J \geq c - \varepsilon\}) \subseteq \{J \geq c + \varepsilon\}. \tag{4.13}$$

By the definition of b, there exists $h_\varepsilon \in \mathcal{H}$ s.t.

$$\inf_{u \in \partial B_\rho(0)} J(h_\varepsilon(u)) \geq b - \varepsilon.$$

By (4.13),

$$\inf_{u \in \partial B_\rho(0)} J(\eta(1, h_\varepsilon(u))) \geq c + \varepsilon.$$

If we showed that $\eta(1, h_\varepsilon) \in \mathcal{H}$ we would have the following contradiction:

$$c \geq b = \sup_{h \in \mathcal{H}} \inf_{u \in \partial B_\rho(0)} J(h(u)) \underset{h=\eta(1,h_\varepsilon)}{\geq} \inf_{u \in \partial B_\rho(0)} J(\eta(1, h_\varepsilon(u))) \geq c + \varepsilon.$$

Hence, it suffices to show that $\eta(1, h_\varepsilon) \in \mathcal{H}$:

- $\eta(1, h_\varepsilon) = \eta(1, \cdot) \circ h_\varepsilon$. Therefore, $\eta(1, h_\varepsilon)$ is the composition of two homeomorphism and, therefore, it's a homomorphism itself.
- For the very same reasons as those of the first proof we gave,

$$\eta(1, h_\varepsilon(0)) = \eta(1, 0) = 0,$$
$$\eta(1, h_\varepsilon(1)) = \eta(1, e) = e.$$

- It remains to prove the separation property.
  $0 \in B_\rho(0)$, $e \in \overline{B_\rho(0)}^C$, hence, $\eta(1, h_\varepsilon(0)) \in \eta(1, h_\varepsilon(B_\rho(0)))$, while $\eta(1, h_\varepsilon(e)) \in \eta(1, h_\varepsilon(\overline{B_\rho(0)})^C)$. As $\eta(1, h_\varepsilon)$ is a homeomorphism, it preserves the connected components, so that

$$\eta(1, h_\varepsilon(B_\rho(0))) \cap \eta(1, h_\varepsilon(\overline{B_\rho(0)}^C)) = \varnothing.$$

□

We see an application of the mountain pass theorem:

Example 4.3.2. Let $\Omega \subset\subset \mathbb{R}^n$ be open and let $E = H_0^1(\Omega)$. We consider the problem

$$\begin{cases} -\Delta u(x) = g(x, u(x)) & \forall x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{4.14}$$

with

- $g \in \mathcal{C}^0(\Omega \times \mathbb{R}, \mathbb{R})$;
- $|g(x, r)| \leq c_1 + c_2|r|^p$ for $p \in (1, 2^* - 1)$;
- $g(x, r) = o(r)$ as $r \to 0$;

- $\exists \mu > 2$ and $\exists \bar{r} > 0$ s.t. if $|r| > \bar{r}$,

$$0 < G(x,r) := \int_0^r g(x,t)dt \leq \frac{r}{\mu} g(x,r).$$

Remark 4.3.3. An example for such a function is provided by

$$g(x,r) = r|r|^{p-1}$$

for any $p \in (1, 2^* - 1)$.

As we saw in Example (3.1.2), the functional associated to the equation of (4.14),

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega G(x, u(x))dx,$$

is well-defined and belongs to $\mathcal{C}^1(H_0^1(\Omega), \mathbb{R})$. We check the validity of the assumptions of Theorem (4.3.1):

(i) $J(0) = 0 - \int_\Omega G(x,0)dx = -\int_\Omega \int_0^0 g dt dx = 0$;

(ii) we have that for all the $\varepsilon > 0$ there exists $\rho_\varepsilon > 0$ s.t. if $|r| < \rho_\varepsilon$, then

$$|g(x,r)| < \varepsilon |r|.$$

Hence,

$$|G(x,r)| \leq \int_0^r \varepsilon |t| dt = \frac{\varepsilon}{2} r^2.$$

On the other hand, for $p \in (1, 2^* - 1)$, $\exists k = k_{\rho_\varepsilon}$ s.t. for all $|r| \geq \rho_\varepsilon$,

$$|G(x,r)| \leq k_{\rho_\varepsilon} |r|^{p+1}.$$

Hence, for all $r \in \mathbb{R}$ and all $\varepsilon > 0$,

$$|G(x,r)| \leq \frac{\varepsilon}{2} r^2 + k_{\rho_\varepsilon} |r|^{p+1}.$$

Hence,

$$\left| \int_\Omega G(x,u)dx \right| \leq \int_\Omega \left(\frac{\varepsilon}{2} |u|^2 + k_{\rho_\varepsilon} \int_\Omega |u|^{p+1} \right) dx = \frac{\varepsilon}{2} \int_\Omega |u|^2 dx + k_{\rho_\varepsilon} \underbrace{\int_\Omega |u|^{p+1} dx}_{\substack{H_0^1 \hookrightarrow L^{p+1} \\ \text{by Sobolev}}} \leq$$

$$\leq \frac{\varepsilon}{2} C_1 \|u\|_{H_0^1}^2 + k_{\rho_\varepsilon} C_2 \|u\|_{H_0^1}^{p+1},$$

which gives

$$0 \leq \lim_{\|u\|_{H_0^1} \to 0} \frac{\left| \int_\Omega G(x,u)dx \right|}{\|u\|_{H_0^1}^2} \leq \lim_{\|u\|_{H_0^1} \to 0} \frac{\varepsilon}{2} C_1 + k_{\rho_\varepsilon} C_2 \|u\|_{H_0^1}^{p-1}.$$

But $p \in (1, 2^* - 1) \Longrightarrow p - 1 \in (0, 2^* - 2)$, so that for all the $\varepsilon > 0$,

$$0 \leq \lim_{\|u\|_{H_0^1} \to 0} \frac{\left|\int_\Omega G(x,u)dx\right|}{\|u\|_{H_0^1}^2} \leq \frac{\varepsilon}{2} C_1.$$

That is, as $\|u\|_{H_0^1} \to 0$,

$$\left|\int_\Omega G(x,u)dx\right| = o\left(\|u\|_{H_0^1}^2\right).$$

Hence, recalling the definition of J we have that, as $\|u\|_{H_0^1} \to 0$,

$$J(u) \geq \frac{1}{2}\|u\|_{H_0^1}^2 - \left|\int_\Omega G(x,u)dx\right| = \frac{1}{2}\|u\|_{H_0^1}^2 + o\left(\|u\|_{H_0^1}^2\right).$$

By the definition of limit,

$$\forall \varepsilon > 0 \ \exists \rho = \rho(\varepsilon) : \|u\|_{H_0^1} \leq \rho \Longrightarrow |J(u)| \geq \frac{1}{2}\|u\|_{H_0^1}^2 - \varepsilon\|u\|_{H_0^1}^2.$$

In particular, taking $\varepsilon = \frac{1}{4}$, we have

$$J|_{\partial B_\rho(0)} \geq \frac{1}{2}\rho^2 - \frac{1}{4}\rho^2 = \frac{1}{4}\rho^2 =: \alpha.$$

(iii) We know that there exist $\mu > 2$ and $\bar{r} > 0$ s.t. for all $|r| > \bar{r}$, it is

$$\frac{\mu}{r} \leq \frac{g(x,r)}{G(x,r)} = \frac{G'(x,r)}{G(x,r)}.$$

Integrating in r:

$$\mu \ln|r| + A \leq \ln|G(x,r)| \quad \Longrightarrow \quad A|r|^\mu \leq |G(x,r)|.$$

Given any $\bar{u} \neq 0$,

$$J(\tau\bar{u}) \leq \frac{\tau^2}{2}\|\bar{u}\|_{H_0^1}^2 - \int_\Omega e^A \tau^\mu |\bar{u}|^\mu dx \to -\infty$$

as $\tau \to \infty$, since $\mu > 2$.
On the other hand, the conclusion of (ii) tells us that J is locally positive in some neighborhood of 0, hence there must exist some $e \in H_0^1$ s.t. $J(e) < 0$.

The last assumption to verify in order for Theorem (4.3.1) to be applied is the validity of (PS). To this end, we take $\{u_k\}_k \subset H_0^1(\Omega)$ s.t. $|J(u_k)| \leq C$ for all k and $J'(u_k) \to 0$ as $k \to \infty$.

We show that there exists a subsequence that converges in the norm of H_0^1. Since $J'(u_k) \to 0$, we have that for all the $d > 0$,

$$\left|J'(u_k)[u_k]\right| \le \underbrace{\left\|J'(u_k)\right\|_{op}}_{\to 0} \|u_k\|_{H_0^1} \le d \,\|u_k\|_{H_0^1}$$

as long as k is chosen large enough. Then, for some k large enough,

$$\begin{aligned}
C + \frac{1}{\mu}\|u_k\|_{H_0^1} &\ge J(u_k) - \frac{1}{\mu} \underbrace{J'(u_k)[u_k]}_{=\|\nabla u_k\|_2^2 - \int_\Omega g(x,u_k)u_k dx} = \\
&= \frac{1}{2}\int_\Omega |\nabla u_k|^2 dx - \int_\Omega G(x,u_k)dx - \frac{1}{\mu}\left(\|u_k\|_{H_0^1}^2 - \int_\Omega g(x,u_k)u_k dx\right) = \\
&= \underbrace{\left(\frac{1}{2} - \frac{1}{\mu}\right)}_{\ge 0}\|u_k\|_{H_0^1}^2 + \int_\Omega \left(\frac{1}{\mu}g(x,u_k)u_k - G(x,u_k)\right)dx.
\end{aligned}$$

The last integral can be split as follows:

$$\begin{aligned}
\int_\Omega \left(\frac{u_k}{\mu}g(x,u_k) - G(x,u_k)\right)dx &= \int_{|u_k|\le\bar r}\left(\frac{u_k}{\mu}g(x,u_k) - G(x,u_k)\right)dx + \\
&\quad + \underbrace{\int_{|u_k|>\bar r}\left(\frac{u_k}{\mu}g(x,u_k) - \underbrace{G(x,u_k)}_{\substack{\le \frac{u_k}{\mu}g(x,u_k) \\ \text{for } |u_k(x)|>\bar r}}\right)dx}_{\ge 0} \ge \\
&\ge \int_{|u_k|\le\bar r}\left(\frac{u_k}{\mu}g(x,u_k) - G(x,u_k)\right)dx \ge \\
&\ge \int_{|u_k|\le\bar r}\left(\frac{-\bar r}{\mu}g(x,u_k)dx - \int_{|u_k|\le\bar r}G(x,u_k)dx\right).
\end{aligned}$$

But g and G are bounded on $\Omega \times \{r \le \bar r\}$, hence we deduce that:

$$C + \frac{1}{\mu}\|u_k\|_{H_0^1} \ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_k\|_{H_0^1}^2 - C_2. \tag{4.15}$$

Therefore, there exists $C > 0$ s.t. $\|u_k\|_{H_0^1} \le C$ for all k.

By the Banach-Alaoglu Theorem there exists a subsequence $\{u_{k_j}\}_j \subseteq \{u_k\}_k$ s.t. $u_{k_j} \rightharpoonup \bar u$ in H_0^1. We have to prove that it converges in the norm of H_0^1 as well.
As $J'(u_k)$ is a linear functional and $u_{k_j} \rightharpoonup \bar u$, we have

$$\underbrace{J'(u_{k_j})[\bar u - u_{k_j}]}_{\to 0 \text{ as } j\to\infty} = \int_\Omega \nabla u_{k_j}\nabla(u_{k_j} - \bar u)dx - \int_\Omega g(x,u_k)(u_{k_j} - \bar u)dx. \tag{4.16}$$

We use Hölder inequality with exponents $2^*/p$ and $(2^*/p)'$ (by Sobolev theorem, the H_0^1 norm can be used as an upper bound for all the norms at stake):

$$\left|\int_\Omega g(x,u_{k_j})(u_{k_j}-\bar{u})dx\right| \leq \int_\Omega (c_1+c_2|u_{k_j}|^p)|u_{k_j}-\bar{u}|dx \leq$$
$$\leq c_1 \underbrace{\left\|u_{k_j}-\bar{u}\right\|_1}_{\to 0\ (1<2^*)} + c_2 \underbrace{\left\|u_{k_j}\right\|_{2^*}^p}_{\leq C} \underbrace{\left\|u_{k_j}-\bar{u}\right\|_{\frac{2^*}{2^*-p}}}_{\to 0\ (2^*/2^*-p<2^*)} \to 0$$

as $j\to\infty$. Therefore, by (4.16), as $j\to\infty$,

$$\int_\Omega \nabla u_{k_j}\nabla(u_{k_j}-\bar{u})dx \to 0.$$

We conclude that, as $j\to\infty$,

$$\left\|u_{k_j}-\bar{u}\right\|_{H_0^1}^2 = \int_\Omega |\nabla(u_{k_j}\bar{u})|^2dx = \underbrace{\int_\Omega \nabla u_{k_j}\nabla(u_{k_j}-\bar{u})dx}_{\to 0} - \underbrace{\int_\Omega \nabla\bar{u}\nabla(u_{k_j}-\bar{u})dx \to 0}_{\to 0 \text{ by weak conv.}}.$$

Hence, by Theorem (4.3.1), we have that J has a critical value in

$$c = \inf_{\gamma\in\Gamma}\sup_{u\in\gamma([0,1])} J(u) \geq \frac{\rho^2}{4}.$$

That is, there exists $\bar{u}\in H_0^1(\Omega)$ s.t.

$$J'(\bar{u})[v] = \int_\Omega \nabla\bar{u}\nabla v dx - \int_\Omega g(x,\bar{u})v dx = 0.$$

That is, $\bar{u}$ is a weak solution of (4.14).

4.4 Brouwer and Leray-Schauder degrees

We define the Brouwer degree for $\mathcal{C}^1$ mappings as follows: let $O \subset\subset \mathbb{R}^n$ be open, $b\in\mathbb{R}^n$ and $\varphi\in\mathcal{C}^1(\bar{O},\mathbb{R}^n)$ be s.t. $rk(\mathrm{J}(\varphi(x)))=n$ for all $x\in\varphi^{-1}(b)$.
By these assumptions, φ is a local homeomorphism and, therefore, all the points of $\varphi^{-1}(b)$ are isolated.
Moreover, we suppose that $\varphi^{-1}(b)$ does not include points of ∂O. Under this assumption, there cannot be cluster points of solutions of the equation

$$\varphi(x)=b$$

and, therefore, $\{x\in\bar{O}:\varphi(x)=b\}$ is finite.

Definition 4.4.1. The **Brouwer degree** (or **topological degree**) of φ in O **related to** $b \in \mathbb{R}^n$ is defined as

$$d(\varphi, O, b) = \sum_{x \in \varphi^{-1}(b) \cap O} \operatorname{sgn}[\det \mathrm{J}(\varphi(x))].$$

Let $\varphi \in \mathcal{C}^0(\bar{O}, \mathbb{R}^n)$, let $\{\varphi_k\}_k \subset \mathcal{C}^1(\bar{O}, \mathbb{R}^n)$ be a sequence converging to φ.

Definition 4.4.2. We set

$$d(\varphi, O, b) = \lim_{k \to \infty} d(\varphi_k, O, b).$$

Proposition 4.4.3. *The following properties hold:*

(i) *if $d(\varphi, O, b) \neq 0$, then there exists $x \in O$ s.t. $\varphi(x) = b$. In particular, if $\varphi(x) \neq b$ for all the $x \in O$, then $d(\varphi, O, b) = 0$.*

(ii) *$d(id_O, O, b) = \chi_O(b)$ where χ_O is the characteristic function on O;*

(iii) *$d(\varphi, O, b)$ is continuous in its variable φ in the topology of $\mathcal{C}^1$ and it's, therefore, locally constant;*

(iv) *$d(\varphi, O, b) = d(\varphi - b, O, 0)$ and, by (iii), d is continuous in its b variable;*

(v) *if $O = O_1 \sqcup O_2$, $d(\varphi, O, b) = d(\varphi, O_1, b) + d(\varphi, O_2, b)$.*

Lemma 4.4.4. *The topological degree is homotopy-invariant. That is, if $H : [0, 1] \times \bar{O} \to \mathbb{R}^n$ is an homotopy and it's s.t. $H(t, x) \neq b$ for all $x \in \partial O$ and for all $t \in [0, 1]$, then $d(H(t, \cdot), O, b)$ is constant in its t variable.*

Proof. By the assumptions on H, $d(H(t, \cdot), O, b)$ is well defined.
$d(\cdot, O, b)$ is locally constant, so that for all $t \in [0, 1]$ there exists $\varepsilon_t > 0$ s.t. $d(H(t, \cdot), O, b)$ is constant in $(t - \varepsilon_t, t + \varepsilon_t)$.
$[0, 1]$ is compact and $\{(t - \varepsilon_t, t + \varepsilon_t)\}_{t \in [0,1]}$ is an open cover of $[0, 1]$, so that it admits a finite sub-cover.
On each interval of the sub-cover, H is constant. The assertion follows by the well-known pasting lemma.

□

Corollary 4.4.5. *$d(\varphi, O, b)$ depends only on the values of φ on ∂O. That is, if $\varphi, \psi \in \mathcal{C}^0(\bar{O}, \mathbb{R}^n)$ are s.t. $\psi|_{\partial O} \equiv \varphi|_{\partial O}$, then $d(\varphi, O, b) = d(\psi, O, b)$.*

Proof. The assertion follows by the previous lemma, choosing

$$H(t, x) = t\varphi(x) + (1 - t)\varphi(x),$$

as the homotopy and observing that $\forall t \in [0, 1]$, since $\varphi|_{\partial O} \equiv \psi|_{\partial O}$, it is

$$H(t, \cdot)|_{\partial O} \equiv t\varphi|_{\partial O} + (1 - t)\psi|_{\partial O} \equiv \varphi|_{\partial O} \neq b.$$

□

The definition of the Brouwer degree of φ in $\bar{O}$ is well-posed since, as we saw, the fact that $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ allows to conclude that $\{x \in O : \varphi(x) = b\}$ is finite and, therefore, the series that defines $d(\varphi, O, b)$ is actually a finite sum and there are no problem of convergence.
In the infinite-dimensional case, the definition of topological degree can be recovered as follows: let E be an inifnite-dimensional vector space. Let

$$\mathcal{F} = \{F : \bar{O} \subseteq E \to E \text{ be s.t. } F = id_{\bar{O}} + K \text{ with } K : \bar{O} \to E \text{ compact operator}\}.$$

Now, we take any $F \in \mathcal{F}$ and $b \in E$ s.t. $F(x) \neq b$ for all $x \in \partial O$. By the definition of $\mathcal{F}$, there exists a compact operator K s.t. $F = id_{\bar{O}} + K$. As K is compact, there exists a sequence $\{K_n : O_n = \bar{O} \cap E_n \to E_n\}_n$ of finite rank operators [§] s.t. $\|K_n - K\|_{op} \to 0$ as $n \to \infty$.

For all the n, we define

$$b_n := \mathcal{P}_{E_n}(b)$$

the projection of $b \in E$ on E_n. Hence, $\forall n$, we can compute

$$d(id_{O_n} + K_n, O_n, b_n) \in \mathbb{N}.$$

Definition 4.4.6. We define the **Leray-Schauder degree** of $F \in \mathcal{F}$ in b as

$$deg(F, O, b) = \lim_{n \to \infty} d(id_{O_n} + K_n, O_n, b_n).$$

deg satisfies all the properties in the assertion of Proposition (4.4.3), while the homotopy-invariance is no-longer holding in general. However, if $H : [0,1] \times \bar{O} \to E$ is an homotopy in the form:

$$H(t,x) = x + K(t,x)$$

with $K(t, \cdot) : \bar{O} \to E$ compact, then H preserves the Leray-Schauder degree.

Theorem 4.4.7 (Schauder fixed point theorem)**.** *Let E be a Banach space and $O \subseteq E$ be an open, bounded and convex subset s.t. $0 \in O$. Let $T : \bar{O} \to \bar{O}$ be a compact operator. Then, there exists $x \in \bar{O}$ s.t. $T(x) = x$.*

We only prove the case we're interested in, that is the case in which $O = B_R(0)$ for some $R > 0$:

Proof. If there exists some $x \in \partial O$ s.t. $T(x) = x$, we have finished. Hence, we suppose that $T(x) \neq x$ for all $x \in \partial O$.
Let $\phi_\lambda \in \mathcal{F}$ be the homotopy defined by

$$\phi_\lambda(x) = x - \lambda T(x) \quad \forall \lambda \in [0,1].$$

For all $\lambda \in [0,1]$, ϕ_λ has no zeros on $\partial B_R(0)$. In fact,

[§] That is, each E_n is finite-dimensional.

- if $\lambda = 1$, it is clear as it is the assumption we are assuming;
- if $\lambda < 1$, recalling that $T : \overline{B_R(0)} \to \overline{B_R(0)}$, we have $\|\lambda T(x)\| = \lambda \|T(x)\| \leq \lambda R < R$.

By the homotopy-invariance,

$$deg(\phi_1, B_R(0), 0) = deg(\phi_0, B_R(0), 0) = deg(\mathbb{1}_{B_R(0)}, B_R(0), 0) = 1$$

(as, obviously, $\mathbb{1}(u) = 0 \Leftrightarrow u = 0$).
In particular, there exists $x \in B_R(0)$ s.t. $x = T(x)$. □

4.5 A first generalization of the mountain pass theorem

The following generalization of the mountain pass theorem uses the definition of Brouwer degree:

Theorem 4.5.1 (First generalization of the mountain pass theorem)**.** *Let E be a Banach space s.t.*

$$E = E_1 \oplus E_2$$

with E_1 a finite-dimensional subspace and E_2 an infinite-dimensional one.
Let $I \in \mathcal{C}^1(E, \mathbb{R})$ satisfying (PS). If there exist $O(0) \subseteq E_1$ and $b_1, b_2 \in \mathbb{R}$ s.t.

$$I|_{\partial O} \leq b_1 < b_2 \leq \inf_{u \in E_2} I(u),$$

then there exists a critical value $c \geq b_2$ for I.

Proof. Let

$$\Gamma = \left\{ h \in \mathcal{C}^0(\bar{O}, E) \text{ s.t. } h|_{\partial O} = id_{\partial O} \right\}$$

and let

$$c := \inf_{h \in \Gamma} \sup_{u \in \bar{O}} I(h(u)).$$

1. **We prove that** $h(\bar{O}) \cap E_2 \neq \varnothing$
   Let $\mathcal{P} : E \to E_1$ the projection on E_1. As E_1 has finite dimension, $\mathcal{P}$ is continuous.
   Obviously, for all $u \in \bar{O}$,

   $$h(u) = \underbrace{\mathcal{P} \circ h}_{\in \mathcal{C}^0(\bar{O}, E_1)}(u) + \underbrace{(id_E - \mathcal{P}) \circ h(u)}_{= 0 \text{ on } \partial O}.$$

   E_1 has finite dimension, so that we can use the properties of the Brouwer degree on $\mathcal{P} \circ h$: the value of $d(\mathcal{P} \circ h, O, 0)$ depends only on the values of $\mathcal{P} \circ h|_{\partial O}$.
   But, if $u \in \partial O$, then $\mathcal{P} \circ h(u) = \mathcal{P}(h(u)) = \mathcal{P}(u) = u$, as $O \subseteq E_1$. Hence,

   $$d(\mathcal{P} \circ h, O, 0) = d(id_{\bar{O}}, O, 0) = 1.$$

   Therefore, there exists $w \in O$ s.t. $\mathcal{P}(h(w)) = 0$, that is $h(w) \in E_2$. Hence, we can conclude that $w \in h(\bar{O}) \cap E_2$.

2. **We use w to prove that $b_2 \leq c$**
   We have:
$$\sup_{u\in\bar{O}} I(h(u)) \geq I(h(w)) \geq \inf_{u\in E_2} I(u) \geq b_2.$$
   Hence,
$$b_2 \leq \inf_{h\in\Gamma} \sup_{u\in\bar{O}} I(h(u)) = c.$$

3. **We show that c is a critical value**
   Seeking a contradiction, if c were not a critical value, by the deformation lemma with $\bar{\varepsilon} = \frac{1}{2}(b_2 - b_1) > 0$, there would exist $\varepsilon \in (0, \bar{\varepsilon})$ and a deformation η s.t.
$$\eta(1, \{I \leq c + \varepsilon\}) \subseteq \{I \leq c - \varepsilon\}.$$
   By the definition of c, there exists $h_\varepsilon \in \Gamma$ s.t.
$$\sup_{u\in h_\varepsilon(\bar{O})} I(u) \leq c + \varepsilon,$$
   in particular, $I(u) \leq c + \varepsilon$ for all $u \in h_\varepsilon(\bar{O})$ and, therefore,
$$I(\eta(1, u)) \leq c - \varepsilon$$
   for all $u \in h_\varepsilon(\bar{O})$.
   If we showed that $\eta(1, h_\varepsilon(\cdot)) \in \Gamma$, then we would find the following contradiction:
$$c = \inf_{h\in\Gamma} \sup_{u\in h(\bar{O})} I(u) \leq \sup_{u\in\eta(1,h_\varepsilon(\bar{O}))} I(u) \leq c - \varepsilon.$$
   We show that $\eta(1, h_\varepsilon(\cdot)) \in \Gamma$:
   - $\eta(1, h_\varepsilon(\cdot)) = \eta(1, \cdot) \circ h_\varepsilon$ is continuous;
   - if $u \in \partial O$, then,
$$\eta(1, h_\varepsilon(u)) \underset{h_\varepsilon\in\Gamma}{=} \eta(1, u) = u$$
     where the last equality is a consequence of both the deformation lemma, the fact that
$$I(u) \underset{u\in\partial O}{\leq} b_1 = (c - \bar{\varepsilon}) + b_1 - c + \bar{\varepsilon} = (c - \bar{\varepsilon}) + b_1 - c + \frac{1}{2}(b_2 - b_1) =$$
$$= (c - \bar{\varepsilon}) + \underbrace{\underbrace{\frac{1}{2}(b_1 + b_2)}_{\leq b_2} - c}_{\leq 0 \text{ as } c \geq b_2} \leq c - \bar{\varepsilon}$$
     and that η fixes the elements of $\{c - \bar{\varepsilon} \leq I \leq c + \bar{\varepsilon}\}^C$.

□

Example 4.5.2. Let $\Omega \subset\subset \mathbb{R}^n$ be open. We consider the problem

$$\begin{cases} -\Delta u - \lambda u = f(x,u) & \text{in } \Omega, \\ u|_{\partial\Omega} \equiv 0, \end{cases} \tag{4.17}$$

where

- f is continuous;
- $|f(x,r)| \leq M$ for all $(x,r) \in \Omega \times \mathbb{R}$.

We show that this problem has a solution. As usual, we set $\sigma(-\Delta)$ to denote the spectrum of the operator $-\Delta$.

(a) **if** $\lambda \notin \sigma(-\Delta)$ the existence follows by the Schauder fixed point theorem. In fact, if λ is not an eigenvalue of $-\Delta$, then the mapping $-\Delta - \lambda \mathbb{1}$ is invertible: there exists $(-\Delta - \lambda\mathbb{1})^{-1} : L^2(\Omega) \to L^2(\Omega)$ continuous and, by the regularity theory, we know that the mapping

$$T : f(x,u) \longmapsto (-\Delta - \lambda\mathbb{1})^{-1} f(x,u)$$

takes values in $H^2(\Omega) \cap H_0^1(\Omega)$, hence it's a compact mapping of L^2 into itself by Rellich-Kondrachov theorem. Moreover,

$$\begin{aligned} \|T(f(x,u))\|_2 = \left\|(-\Delta - \lambda\mathbb{1})^{-1} f(x,u)\right\|_2 \leq \left\|(-\Delta - \lambda\mathbb{1})^{-1}\right\|_{op} \|f(x,u)\|_2 \leq \\ \leq C_1 M |\Omega|^{1/2} =: R. \end{aligned}$$

Hence, $T : \overline{B_R(0)} \to \overline{B_R(0)}$ and the existence of the solution we're looking for follows by Schauder fixed point theorem.

(b) **if** $\lambda = \lambda_n \in \sigma(-\Delta)$, we talk about the **resonance problem**. We choose n s.t. $\lambda_{n+1} > \lambda_n$. In this case, the non-invertibility of the operator $-\Delta - \lambda_n\mathbb{1}$ allows us to use Schauder theorem. In order to use Theorem (4.5.1), we add an hypothesis on the decay at infinity of an antiderivative of f: let $F(x,r) = \int_0^r f(x,s)ds$, we suppose that, as $|r| \to \infty$ we have

- $F(x,r) \to +\infty$ uniformly in $x \in \bar{\Omega}$;
- $\frac{F(x,r)}{r} \to 0$.

We define the energy $I : H_0^1(\Omega) \to \mathbb{R}$ as

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda_n}{2}\int_\Omega |u|^2 dx - \int_\Omega F(x,u)dx$$

and observe that I is well-defined by the boundedness of f at infinity we're assuming to hold, in fact

$$|F(x,r)| \leq \left|\int_0^r |f(x,s)|ds\right| \leq M|r|,$$

so that,

$$\left|\int_\Omega F(x,u)dx\right| \leq M\|u\|_1 < \infty$$

by the Sobolev theorem. We write $H_0^1(\Omega)$ as a direct sum:

$$H_0^1(\Omega) = E_1 \oplus E_2$$

where, if $\{e_k\}_k$ is a complete orthonormal system consisting of eigenfunctions related to the eigenvalues $\{\lambda_k\}_k = \sigma(-\Delta)$,

$$E_1 = \overline{\text{span}}(e_1, \dots, e_n), \quad E_2 = E_1^\perp.$$

(n is the same subscript as that of λ_n).

Remark 4.5.3. Thanks to our choice of the $\{e_k\}_k$, $\lambda_k = \frac{1}{\|e_k\|_2^2}$ for all k. In fact, e_k is a weak solution of $-\Delta u = \lambda_k u$, i.e. if $\int_\Omega \nabla e_k \nabla v dx = \lambda_k \int_\Omega e_k v dx$ for all $v \in H_0^1(\Omega)$, then choosing $v = e_k$ we get:

$$1 = \|e_k\|_{H_0^1}^2 = \int_\Omega |\nabla e_k|^2 dx = \lambda_k \int_\Omega |e_k|^2 dx = \lambda_k \|e_k\|_2^2.$$

We prove that the assumptions of Theorem (4.5.1) are satisfied:

(i) $\inf_{u\in E_2} I(u) \geq b_2$ on E_2 for some $b_2 \in \mathbb{R}$, in fact, if $u \in E_2$, then $u = \sum_{k\geq n+1} \alpha_k e_k$ for unique $\{\alpha_k\}_k \subset \mathbb{R}$.
Then,

$$\int_\Omega |\nabla u|^2 dx = (u,u)_{H_0^1}^2 = \sum_{k=n+1}^\infty \alpha_k^2,$$

while

$$\int_\Omega |u|^2 dx = \|u\|_2^2 = (u,u)_{L^2} = \sum_{k=n+1}^\infty \alpha_k^2 \int_\Omega |e_k|^2 dx = \sum_{k=n+1}^\infty \frac{\alpha_k^2}{\lambda_k}$$

Then,

$$\begin{aligned} I(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda_n}{2}\int_\Omega |u|^2 dx - \int_\Omega F(x,u)dx = \\ &= \frac{1}{2}\sum_{k=n+1}^\infty \alpha_k^2 - \frac{\lambda_n}{2}\sum_{k=n+1}^\infty \frac{\alpha_k^2}{\lambda_k} - \int_\Omega F(x,u)dx = \\ &= \frac{1}{2}\sum_{k=n+1}^\infty \left(1 - \frac{\lambda_n}{\lambda_k}\right)\alpha_k^2 - \int_\Omega F(x,u)dx \geq \\ &\geq \frac{1}{2}\sum_{k=n+1}^\infty \alpha_k^2 \underbrace{\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)}_{=:\,\delta>0} - \underbrace{\int_\Omega F(x,u)dx}_{\leq M\|u\|_1} \geq \\ &\geq \frac{\delta}{2}\|u\|_{H_0^1}^2 - M \underbrace{\|u\|_1}_{\leq |\Omega|^{1/2}\|u\|_2} \geq \frac{\delta}{2}\|u\|_{H_0^1}^2 - M|\Omega|^{1/2}\|u\|_2 \geq \frac{\delta}{2}\|u\|_{H_0^1} - C\|u\|_{H_0^1} \end{aligned}$$

by Poincaré inequality. Hence, for all the $u \in E_2$,

$$I(u) \geq \frac{\delta}{2} \|u\|^2_{H^1_0} - C \|u\|_{H^1_0} \geq b_2.$$

The right hand side goes to infinity as $\|u\|_{H^1_0 \to +\infty}$, so that the lower bound holds definitively for all $b_2 > 0$. In particular, $\inf_{u \in E_2} I(u) \geq b_2$ for some appropriate $b_2 \in \mathbb{R}$.

(ii) We have to verify that there exists a neighborhood of 0 in E_1 s.t. I, restricted to that neighborhood, is smaller than b_2. We break E_1 down ad $E_1 = E_0 \oplus E^-$, where

$$E_0 = \overline{\text{span}}\{e_k \text{ eigenfunctions of } \lambda_n\} \quad \text{and}$$

$$E^- = \overline{\text{span}}\{e_k \text{ eigenfunctions of the } \lambda_k < \lambda_n\}.$$

Let $j < n$ be s.t. $E^- = \overline{\text{span}}\{e_1, \ldots, e_j\}$. If $u \in E_1$, then u can be written in a unique way as the sum of its projections on E_0 and E^-: $u = \mathcal{P}_0(u) + \mathcal{P}_-(u)$. The same computations as above tell us that, if $u = \sum_{k \leq n} \alpha_k e_k$, then

$$\begin{aligned}
I(u) &= \frac{1}{2} \sum_{k=1}^{n} \alpha_k^2 \underbrace{\left(1 - \frac{\lambda_n}{\lambda_k}\right)}_{= 0 \text{ per } k > j} - \int_\Omega F(x,u)dx = \frac{1}{2} \sum_{k=1}^{j} \alpha_k^2 \underbrace{\left(1 - \frac{\lambda_n}{\lambda_k}\right)}_{\leq -\delta < 0} - \int_\Omega F(x,u)dx \leq \\
&\leq -\frac{\delta}{2} \|\mathcal{P}_-(u)\|^2_{H^1_0} - \int_\Omega F(x, \mathcal{P}_0(u) + \mathcal{P}_-(u))dx = -\frac{\delta}{2} \|\mathcal{P}_-(u)\|^2_{H^1_0} + \\
&+ \underbrace{\int_\Omega [F(x, \mathcal{P}_0 u) - F(x, \mathcal{P}_0(u) + \mathcal{P}_-(u))]dx}_{\leq \int_\Omega const \, \|\mathcal{P}_-(u)\|_{H^1_0} dx = c \, \|\mathcal{P}_-(u)\|_{H^1_0}} - \int_\Omega F(x, \mathcal{P}_0(u))dx \leq \\
&\leq -\frac{\delta}{2} \|\mathcal{P}_-(u)\|^2_{H^1_0} + c \, \|\mathcal{P}_-(u)\|_{H^1_0} - \int_\Omega F(x, \mathcal{P}_0(u))dx.
\end{aligned}$$

Hence, if $u \in E_1$, for some constant $c > 0$, then

$$I(u) \leq \underbrace{-\frac{\delta}{2} \|\mathcal{P}_-(u)\|^2_{H^1_0} + c \, \|\mathcal{P}_-(u)\|_{H^1_0}}_{\to -\infty \text{ as } \|\mathcal{P}_-(u)\|_{H^1_0} \to \infty} - \int_\Omega F(x, \mathcal{P}_0(u))dx.$$

We show that $\int_\Omega F(x, \mathcal{P}_0(u))dx \to +\infty$ as $\|\mathcal{P}_0(u)\|_{H^1_0} \to +\infty$: let $\{\mathcal{P}_0(u_k)\}_k \subset E_0$ be s.t. $\|\mathcal{P}_0(u_k)\|_{H^1_0} \to +\infty$. For all k, we write

$$\mathcal{P}_0(u_k) = s_k y_k$$

with $s_k = \|\mathcal{P}_0(u_k)\|_{H^1_0}$ and $y_k = \frac{\mathcal{P}_0(u_k)}{\|\mathcal{P}_0(u_k)\|_{H^1 0}} \in \{v \in E_0 : \|v\|_{H^1_0} = 1\} = S_{H^1_0}$. E_0 has finite dimension and the sequence $\{y_k\}_k$ is bounded, hence it admits

a converging subsequence $\{y_{k_j}\}_j$ s.t. $y_{k_j} \to y$ in H_0^1 and in L^∞ as well (since all the norms on E_0 are equivalent).
$y \in \ker(-\Delta - \lambda_n \mathbb{1})$ by continuity, in particular, $y \in \mathcal{C}^\infty(\Omega)$ and it's not identically zero, so that there exist $\delta > 0$ and $B_\rho \subset \Omega$ s.t. $|y(x)| \geq \delta$ for $x \in B_\rho$.
Moreover, for convenience, we can suppose $F(|s|)$ to be an increasing function of $|s|$, so that we can estimate:

$$F(x, s_{k_j}|y_{k_j}|) \geq F(x, s_{k_j}|y| - s_{k_j}|y_{k_j} - y|) \geq F(x, s_{k_j}\delta - s_{k_j}\varepsilon), \quad \forall x \in B_\rho$$

as j is large enough to grant $\|y_{k_kj} - y\|_\infty \leq \varepsilon = \frac{\delta}{2}$. Therefore,

$$\int_\Omega F(x, \mathcal{P}_0(u_{k_j}))dx \geq \int_{B_\rho} F\left(x, s_{k_j}\frac{\delta}{2}\right)dx \to +\infty$$

as $j \to +\infty$.
We have proved that I is bounded from below in E_- and in E_0. By the definition of E_1 as a direct sum, if $u \in E_1$ and $\|u\|_{H_0^1} \to \infty$, then at least one of the two norms $\mathcal{P}_0$ and $\mathcal{P}_-$ must diverge, hence we proved that, if $u \in E_1$ e $\|u\|_{H_0^1} \to \infty$, then $I(u) \to -\infty$. For this reason, there exists a neighborhood O of 0 in E_1 for which $I|_{\partial O} \leq b_1 < b_2$.

(iii) It remains to prove (PS).
Let $\{u_k\}_k \subset H_0^1(\Omega)$ be s.t. $|I(u_k)| \leq C$ for all k and $I'(u_k) \to 0$ as $k \to \infty$. As usual, we prove that $\{\|u_k\|_{H_0^1}\}_k$ is bounded in k in order to grant the existence of a weakly convergence subsequence (by Banach-Alaoglu) and use the standard argument to show that $\int_\Omega \nabla u_{k_j} \nabla(\bar{u} - u_{k_j})dx \to 0$.
To this end, we show that the norms of all the projections are bounded. Let $\mathcal{P}_2$ be the projection of E on E_2. By the assumption, $\|I'(u_k)\|_{op} \xrightarrow[k\to+\infty]{} 0$. In particular, there exists an integer k s.t. $|I'(u_k)[\mathcal{P}_2(u_k)]| \leq \|\mathcal{P}_2(u_k)\|$. Hence,

$$\begin{aligned} &\|\mathcal{P}_2(u_k)\|_{H_0^1} \geq \left|I'(u_k)[\mathcal{P}_2(u_k)]\right| = \\ &= \left|\int_\Omega \nabla u_k \cdot \nabla(\mathcal{P}_2(u_k))dx - \lambda_n \int_\Omega u_k \mathcal{P}_2(u_k)dx - \int_\Omega f(x, \mathcal{P}_2(u_k))\mathcal{P}_2(u_k)dx\right|. \end{aligned} \tag{4.18}$$

Now, we recall that the eigenfunction of $-\Delta$ satisfy

$$\int_\Omega \nabla e_k \nabla e_h dx = \delta_{hk}, \quad \int_\Omega e_h e_k dx = \delta_{hk}\frac{1}{\lambda_h},$$

so that, using $u_k = \sum_{h=1}^\infty \alpha_h e_h$,

$$\int_\Omega \nabla u_k \nabla \mathcal{P}_2(u_k) = (u_k, \mathcal{P}_2(u_k))_{H_0^1} = \|\mathcal{P}_2(u_k)\|^2_{H_0^1} \tag{4.19}$$

and

$$\int_\Omega u_k \mathcal{P}_2(u_k) = (u_k, \mathcal{P}_2(u_k))_2 = \|\mathcal{P}_2(u_k)\|_2^2 . \tag{4.20}$$

Therefore, putting (4.18), (4.19) and (4.20) together, using the boundedness of f and the Sobolev inequality for $q = 1$,

$$\|\mathcal{P}_2(u_k)\|_{H_0^1} \geq \|\mathcal{P}_2(u_k)\|_{H_0^1}^2 - \lambda_n \|\mathcal{P}_2(u_k)\|_2^2 - \underbrace{\int_\Omega |f(x, \mathcal{P}_2(u_k))\mathcal{P}_2(u_k))| dx}_{\leq C \|\mathcal{P}_2(u_k)\|_{H_0^1}} \geq$$

$$\geq \|\mathcal{P}_2(u_k)\|_{H_0^1}^2 \underbrace{\left(1 - \frac{\lambda_n}{\lambda_k}\right)}_{> \frac{1}{2} \text{ definitively}} - C \|\mathcal{P}_2(u_k)\|_{H_0^1} \geq$$

$$\geq \frac{1}{2} \|\mathcal{P}_2(u_k)\|_{H_0^1}^2 - C \|\mathcal{P}_2(u_k)\|_{H_0^1} ,$$

where we used the fact that, by the definition of λ_k, $\lambda_k = \inf_{u \in H_0^1} \|u\|_{H_0^1}^2 / \|u\|_2^2$, $\|\mathcal{P}_2(u_k)\|^2 \leq \frac{1}{\lambda_k} \|\mathcal{P}_2(u_k)\|_{H_0^1}^2$ $1/2 < 1 - \lambda_n/\lambda_k < 1$ as $k \to +\infty$.
Therefore, $\|\mathcal{P}_2(u_k)\|_{H_0^1} \leq C$. Similarly, one proves that $\|\mathcal{P}_-(u_k)\|_{H_0^1} \leq C$.
As far as $\|\mathcal{P}_0(u_k)\|_{H_0^1}$ is concerned, using again $\mathcal{P}_0(u_k) = \sum_{k=j+1}^n \alpha_k e_k$, we get

$$\int_\Omega |\nabla \mathcal{P}_0(u_k)|^2 dx = \lambda_n \int_\Omega |\mathcal{P}_0(u_k)|^2 dx.$$

Hence if, by contradiction, $\|\mathcal{P}_0(u_k)\|_{H_0^1} \to +\infty$ as $k \to +\infty$, we would have

$$\begin{aligned} C \geq |I(u_k)| &= \left| \frac{1}{2} \|u_k\|_{H_0^1}^2 - \frac{\lambda_n}{2} \|u_k\|_2^2 - \int_\Omega F(x, u_k) dx \right| = \\ &= \left| \frac{1}{2} \int_\Omega \left(|\nabla \mathcal{P}_0(u_k)|^2 + |\nabla \mathcal{P}_2(u_k)|^2 + |\nabla \mathcal{P}_-(u_k)|^2 \right) dx + \right. \\ &\quad - \frac{\lambda_n}{2} \int_\Omega \left(|\mathcal{P}_0(u_k)|^2 + |\mathcal{P}_2(u_k)|^2 + |\mathcal{P}_-(u_k)|^2 \right) dx + \\ &\quad \left. - \int_\Omega F\left(x, \mathcal{P}_0(u_k) + \mathcal{P}_2(u_k) + \mathcal{P}_-(u_k)\right) dx \right| = \end{aligned}$$

$$= \left| \frac{1}{2} \int_\Omega \left(|\nabla \mathcal{P}_2(u_k)|^2 + |\nabla \mathcal{P}_-(u_k)|^2 \right) dx - \frac{\lambda_n}{2} \underbrace{\int_\Omega \left(|\mathcal{P}_2(u_k)|^2 + |\mathcal{P}_-(u_k)|^2 \right) dx}_{\leq C \|\mathcal{P}_2(u_k)\|_{H_0^1}^2 + \|\mathcal{P}_-(u_k)\|_{H_0^1}^2} + \right.$$

$$\left. - \underbrace{\int_\Omega [F\left(x, \mathcal{P}_0(u_k) + \mathcal{P}_2(u_k) + \mathcal{P}_-(u_k)\right) - F(x, \mathcal{P}_0(u_k))] dx}_{\leq C(\|\mathcal{P}_2(u_k)\|_{H_0^1} + \|\mathcal{P}_0(u_k)\|_{H_0^1})} - \int_\Omega F(x, \mathcal{P}_0(u_k)) dx \right| \geq$$

$$\geq \left| \tilde{C} \|\mathcal{P}_2(u_k)\|^2_{H^1_0} - C \|\mathcal{P}_2(u_k)\|_{H^1_0} + \tilde{C} \|\mathcal{P}_-(u_k)\|^2_{H^1_0} - C \|\mathcal{P}_-(u_k)\|_{H^1_0} + \right.$$
$$\left. - \int_\Omega F(x, \mathcal{P}_0(u_k))dx \right| \geq \left| C - \int_\Omega F(x, \mathcal{P}_0(u_k))dx \right|,$$

where we used the boundedness from below of $|\tilde{C}x^2 - Cx|$. We've already proved that, if $\|\mathcal{P}_0(u_k)\|_{H^1_0} \to +\infty$, the integral $\int_\Omega F(x, \mathcal{P}_0(u_k))dx \to +\infty$. This is a contradiction, so that $\|u_k\|_{H^1_0} \leq C$. Now, we proceed as usual, using Banach-Alaoglu theorem and proving that the produced subsequence converges in H^1_0.

4.6 A second generalization of the mountain pass theorem

Let E be a Banach space and $S \subseteq E$ be closed. Let $Q \subset E$ a submanifold with boundary ∂Q.

Definition 4.6.1. We say that S and Q are **linked** if

(a) $S \cap \partial Q = \varnothing$;

(b) $\forall h \in \mathcal{C}^0(E, E)$ s.t. $h|_{\partial Q} = id_{\partial Q}$, one has $h(Q) \cap S \neq \varnothing$.

Example 4.6.2. Let $E = E_1 \oplus E_2$ for a finite-dimensional E_1. We define $S := E_2$ e $Q = B_r(0) \subset E_1$. By the definition of direct sum, $\partial Q \cap S = \varnothing$. Let $h \in \mathcal{C}^0(E, E)$ be s.t. $h|_{\partial Q} = id_{\partial Q}$. We consider the projection $\mathcal{P}_1 : E \to E_1$ and show that $h(Q) \cap S \neq \varnothing$. If we proved that $0 \in \mathcal{P}_1 h(Q)$ we would have finished, as we would have proved that there exists a point of $h(Q)$ in correspondence of which the projection on E_1 is 0, that is a point of $h(Q)$ which belongs to E_2 as well.

Let H be the homotopy defined by

$$H(t, u) = t\mathcal{P}_1 h(u) + (1 - t)u \quad t \in [0, 1].$$

We have:

- $H(0, u) = u$;
- $H(1, u) = \mathcal{P}_1 h(u)$;
- $H(t, u) = u$ for all $u \in \partial B_1(0)$ and for all $t \in [0, 1]$, since on ∂Q both $\mathcal{P}_1$ and h behaves like the identity.

Then,

$$d(\mathcal{P}_1 \circ h, Q, 0) = d(H(1, \cdot), Q, 0) = d(H(0, \cdot), Q, 0) = d(id_Q, Q, 0) = 1.$$

Hence, the equation $\mathcal{P}_1 h(u)$ has at least one solution in Q.

Example 4.6.3. Again, let $E = E_1 \oplus E_2$, E_1 being finite-dimensional. Let $e \in E_2$ be s.t. $\|e\|_E = 1$, $0 < \rho < R_1$ and $R_2 > 0$. We set

$$S := \{u \in E_2 \text{ s.t. } \|u\|_E = \rho\};$$

$$Q := \{s \cdot e + u_1 \text{ with } 0 \leq s \leq R_1,\, u_1 \in E_1 \text{ and } \|u_1\|_E \leq R_2\}.$$

By the definition of Q, ∂Q is given by

$$\begin{aligned} \partial Q = &\{s \cdot e + u_1 \in Q \text{ s.t. } 0 \leq s \leq R_1,\, \|u_1\|_E = R_2\} \cup \\ &\cup \{s \cdot e + u_1 \in Q \text{ s.t. } s = 0 \text{ e } \|u_1\|_E \leq R_2\} \cup \\ &\cup \{s \cdot e + u_1 \in Q \text{ s.t. } s = R_1 \text{ and } \|u_1\|_E \leq R_2\}. \end{aligned}$$

We show that S and Q are linked. Once again, let $\mathcal{P}_1$ be the projection of E on E_1, i.e.

$$\mathcal{P}_1(u) = u_1$$

$\forall u \in Q$.
Observe that $S \cap \partial Q = \varnothing$ follows by the explicit definitions of S and ∂Q. Let $h \in \mathcal{C}^0(E, E)$ be s.t. $h|_{\partial Q} = id_{\partial Q}$. Proving that $h(Q) \cap S \neq \varnothing$ is the same as proving the existence of a $u \in Q$ s.t. $\mathcal{P}_1 h(u) = 0$ and $\|h(u)\|_E = \rho$. In fact, under these hypothesis, $h(u) \in S$ and, obviously, $h(u) \in h(Q)$. We define, for $t \in [0,1]$, the mapping $H_t : \mathbb{R} \times E_1 \to \mathbb{R} \times E_1$ as follows: if $(s, u_1) \in \mathbb{R} \times E_1$, we set $u = u_1 + e \cdot s$ and

$$H_t(s, u_1) = (t\,\|h(u) - \mathcal{P}_1 h(u)\|_E + (1-t)s - \rho, t\mathcal{P}_1 h(u) + (1-t)u_1)\,.$$

If $s \cdot e + u_1 \in \partial Q$, by the definition of the mappings at stake, by the explicit definition of ∂Q and recalling that $\|e\|_E = 1$, we get

$$\begin{aligned} H_t(s, u_1) &= (t||\underbrace{\underbrace{h(u)}_{=u} - \underbrace{\mathcal{P}_1 \underbrace{h(u)}_{=u}}_{=u_1}}_{=u-u_1=s\cdot e}||_E - st + s - \rho, t\underbrace{\mathcal{P}_1 \underbrace{h(u)}_{=u}}_{=u_1} + (1-t)u_1) = \\ &= (s - \rho, u_1) \neq (0,0) \end{aligned}$$

and, always by $u_1 + s \cdot e \in \partial Q$, using the properties of Brouwer degree,

$$\begin{aligned} &\underbrace{1 = d((s,u_1), Q, (\rho, 0))}_{\text{unique sol. in } Q:\ s=\rho,\ u_1=0} = d((s-\rho, u_1), Q, 0) = d(H_t(\cdot,\cdot), Q, 0) = d(H_1(\cdot,\cdot), Q, 0) = \\ &= d((\|h(u) - \mathcal{P}_1 h(u)\|_E - \rho, \mathcal{P}_1 h(u)), Q, 0). \end{aligned}$$

That is, there exists $\bar{u} \in Q$ s.t. $\mathcal{P}_1 h(\bar{u}) = 0$ and $\|h(\bar{u}) - \mathcal{P}_1 h(\bar{u})\|_E = \|h(\bar{u})\|_E = \rho$.

Theorem 4.6.4. *Let E be a Banach space and $I \in \mathcal{C}^1(E, \mathbb{R})$ satisfying (PS). Let*

$$\Gamma = \{h \in \mathcal{C}^0(E, E) \text{ be s.t. } h|_{\partial Q} = id_{\partial Q}\}$$

and let S and Q be linked subsets of E. If

$$\alpha := \inf_{u \in S} I(u) > \sup_{u \partial Q} I(u) =: \alpha_0.$$

Then, $c := \inf_{h\in\Gamma} \sup_{u \in h(Q)} I(u) \geq \alpha$ and it is a critical value for I.

Proof. 1. **We prove that $c \geq \alpha$**
By the definition of linked subsets, for all $h \in \Gamma$ there exists $w \in h(Q) \cap S$. Then,

$$\alpha = \inf_{u \in S} I(u) \underset{w \in S}{\leq} I(w) \underset{w \in h(Q)}{\leq} \sup_{u \in h(Q)} I(u).$$

Thus, 1. follows taking the infimum.

2. **We prove that c is a critical value**
Seeking a contradiction, if c weren't a critical value, set $\bar{\varepsilon} = \frac{\alpha - \alpha_0}{2}$, by the deformation lemma we would get the existence of some $\varepsilon \in (0, \bar{\varepsilon})$ and that of a deformation η s.t.

$$\eta(1, \{I \leq c + \varepsilon\}) \subseteq \{I \leq c - \varepsilon\}.$$

By the definition of inf, there exists $h_\varepsilon \in \Gamma$ s.t.

$$\sup_{u \in h_\varepsilon(Q)} I(u) \leq c + \varepsilon.$$

In particular, $I(u) \leq c + \varepsilon \; \forall u \in h_\varepsilon(Q)$, so that $I(u) \leq c - \varepsilon$ for all $u \in \eta(1, h_\varepsilon(Q))$. If we proved that $\eta(1, h_\varepsilon(\cdot)) \in \Gamma$, we would have found a contradiction, as in that case,

$$c \leq \sup_{u \in \eta(1, h_\varepsilon(Q))} I(u) \leq c - \varepsilon.$$

But, $\eta(1, h_\varepsilon(\cdot))$ is obviously continuous (as a composition of continuous functions) and for all $u \in \partial Q$ we have

$$\eta(1, h_\varepsilon(u)) = \eta(1, u) = u,$$

where, the first equality follows by $h_\varepsilon \in \Gamma$ nad we used the fact that

$$I(u) \leq \alpha_0 = c - (c - \alpha_0) \leq c - (\alpha - \alpha_0) < c - \frac{\alpha - \alpha_0}{2} = c - \bar{\varepsilon}$$

in the last step, and $\eta(\cdot, u)$ is the identity on $\{I \leq c - \bar{\varepsilon}\}$.

□

Example 4.6.5. Let $\Omega \subset\subset \mathbb{R}^n$ ed $E = H_0^1(\Omega)$. We consider the problem

$$\begin{cases} -\Delta u = \lambda u + g(x,u) & \text{on } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{4.21}$$

with $\lambda > \lambda_1$ and $\lambda \neq \lambda_k$ for all k, where $\{\lambda_k\}_k = \sigma(-\Delta)$. In particular, $\lambda \in (\lambda_n, \lambda_{n+1})$ for some appropriate integer $n > 1$. Let $G(x,r) = \int_0^r g(x,s)ds$ and suppose that g satisfies the following assumptions:

- g is continuous;
- there exist constants $a_1, a_2 \in (0,+\infty)$ s.t. for some $p \in (1, 2^* - 1)$ one has
$$|g(x,r)| \leq a_1 + a_2|r|^p;$$
- as $r \to 0$ one has
$$g(x,r) = o(|r|);$$
- there exist $\mu > 2$ and $\bar{r} > 0$ s.t.
$$G(x,r) \leq \frac{r}{\mu} g(x,r) \quad \text{for all } r \text{ s.t. } |r| \geq \bar{r};$$
- $r \cdot g(x,r) \geq 0$ for all $r \in \mathbb{R}$. In particular, $G \geq 0$.

We define the functional related to the equation of (4.21):

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \int_\Omega G(x,u)dx.$$

As usual, $I \in \mathcal{C}^1(H_0^1(\Omega), \mathbb{R})$. Let $\{e_k\}_k$ be the complete orthonormal system given by the eigenfunctions of $-\Delta$. We set

$$E_1 = \overline{\text{span}}\{e_1, \dots, e_n\}$$

and $E_2 = E_1^\perp$. Let $e \in E_2$ with $\|e\|_{H_0^1} = 1$. Let S and Q as in Example (4.6.3) for R_1, R_2 and ρ to be fixed in order for the assumptions of Theorem (4.6.4) to hold. We saw that ∂Q is the union of the following three sets:

$$\begin{aligned} \Sigma_1 &= \{v \in E_1 \text{ s.t. } \|v\|_{H_0^1} \leq R_2\}; \\ \Sigma_2 &= \{R_1 e + v \text{ with } v \in E_1 \text{ and } \|v\|_{H_0^1} \leq R_2\}; \\ \Sigma_3 &= \{se + v \text{ with } v \in E_1,\ \|v\|_{H_0^1} = R_2 \text{ and } 0 \leq s \leq R_1\}. \end{aligned}$$

We show that $\inf_{\partial Q} I(u) \leq 0$ and $\sup_S I(u) > 0$. For, we split the estimate for ∂Q on the Σ_k $k = 1, 2, 3$.

1. **estimate on Σ_1**
   $v \in E_1$, so that $v = \sum_{k=1}^{n} \alpha_k e_k$. Repeating the computation in Example (4.5.2), we get

$$I(v) = \frac{1}{2}\sum_{k=1}^{n}\alpha_k^2 - \frac{\lambda}{2}\sum_{k=1}^{n}\frac{1}{\lambda_k}\alpha_k^2 - \underbrace{\int_\Omega G(x,v)dx}_{G\geq 0 \Rightarrow \geq 0} = \frac{1}{2}\sum_{k=1}^{n}\left(1-\frac{\lambda}{\lambda_k}\right)\alpha_k^2 \leq$$

$$\leq \frac{1}{2}\underbrace{\left(1-\frac{\lambda}{\lambda_n}\right)}_{\leq 0\ (\lambda>\lambda_n)}\sum_{k=1}^{n}\alpha_k^2 \leq 0$$

2. **estimate for Σ_2**
   $e \in E_2$, hence if $v \in E_1$ one has $e \perp v$. Therefore,

$$I(R_1 e + v) = \frac{R_1^2}{2}\underbrace{\int_\Omega |\nabla e|^2 dx}_{=1} + \frac{1}{2}\int_\Omega |\nabla v|^2 dx \underbrace{-\frac{\lambda R_1^2}{2}\int_\Omega |e|^2 dx}_{\leq 0} - \frac{\lambda}{2}\int_\Omega |v|^2 +$$

$$- \int_\Omega G(x, R_1 e + v)dx =$$

$$= \underbrace{\frac{1}{2}\int_\Omega |\nabla v|^2 dx - \frac{\lambda}{2}\int_\Omega |v|^2 dx}_{\leq 0 \text{ per } 1.} + \frac{R_1^2}{2} - \int_\Omega G(x, R_1 e + v)dx \leq$$

$$\leq \frac{R_1^2}{2} - \int_\Omega G(x, R_1 e + v)dx.$$

However, as we saw in Example (4.3.2) the growth hypothesis on G

$$G \leq \frac{r}{\mu} g$$

implies that

$$G(x,r) \geq C_1 |r|^\mu.$$

Hence, using the fact that $\mu \geq 2$, so that $\|\cdot\|_\mu \geq \mathit{const}\,\|\cdot\|_2$,

$$I(R_1 e + v) \leq \frac{R_1^2}{2} - C_1\|R_1 e + v\|_\mu^\mu \leq \frac{R_1^2}{2} - C_2\|R_1 e + v\|_2^\mu =$$

$$= \frac{R_1^2}{2} - C_2(\|R_1 e + v\|_2^2)^{\mu/2} = \frac{R_1^2}{2} - C_2 R_1^\mu \|e\|_2^\mu - (2C_2^\mu R_1 (e,v)_2)^{\mu/2} - C_2\|v\|_2^\mu.$$

It is clear that, as long as R_1 is chosen large enough, as $\mu > 2$, the right hand side of the inequality is negative. Therefore, we have proved that

$$I|_{\Sigma_2}(u) \leq 0$$

if R_1 is large enough.

3. **estimate for Σ_3**
Again, set $v = \sum_{k=1}^{n} \alpha_k e_k$ and recall that

$$\int_\Omega |v|^2 dx = \sum_{k=1}^{n} \frac{1}{\lambda_k} \alpha_k^2.$$

We observe that

$$\int_\Omega |v|^2 dx = \sum_{k=1}^{n} \frac{1}{\lambda_k} \alpha_k^2 \geq \frac{1}{\lambda_n} \sum_{k=1}^{n} \alpha_k^2 = \frac{1}{\lambda_n} \|v\|^2_{H_0^1} = \frac{R_2^2}{\lambda_n}.$$

Hence,

$$\begin{aligned} I(s\cdot e + v) = &\underbrace{\frac{s^2}{2}}_{s \leq R_1} \underbrace{\int_\Omega |\nabla e|^2 dx}_{=1} + \frac{1}{2} \underbrace{\int_\Omega |\nabla v|^2 dx}_{=R_2^2} - \underbrace{\frac{\lambda s^2}{2} \int_\Omega |e|^2 dx}_{\geq 0} - \frac{\lambda}{2} \underbrace{\int_\Omega |v|^2 dx}_{\geq \frac{R_2^2}{\lambda_n}} + \\ &- \underbrace{\int_\Omega G(x, s\cdot e + v) dx}_{\geq 0} \leq \frac{R_1^2}{2} + \frac{R_2^2}{2} - \frac{\lambda R_2^2}{2\lambda_n} = \frac{R_1^2}{2} - \frac{1}{2} \underbrace{\left(\frac{\lambda}{\lambda_n} - 1 \right)}_{\geq 0 \ (\lambda > \lambda_n)} R_2^2 \leq 0 \end{aligned}$$

if R_2 is large enough.

Now we show that, for some appropriate ρ, one has $\inf_{\substack{u \in E_2 \\ \|u\|_{H_0^1} = \rho}} I(u) > 0$.

As we saw in Example (4.3.2), as $\|u\|_{H_0^1} \to 0$,

$$\int_\Omega G(x,u) dx = o(\|u\|^2_{H_0^1}).$$

Then, if $u = \sum_{k>n} \alpha_k e_k \in E_2$,

$$\begin{aligned} I(u) &= \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2} \int_\Omega |u|^2 dx - \int_\Omega G(x,u) dx \geq \\ &\geq \frac{\|u\|^2_{H_0^1}}{2} - \frac{\lambda}{2} \sum_{k=n+1}^{\infty} \frac{1}{\lambda_k} \alpha_k^2 + o\left(\|u\|^2_{H_0^1} \right) \geq \\ &\geq \frac{1}{2} \underbrace{\left(1 - \frac{\lambda}{\lambda_{n+1}} \right)}_{>0} \|u\|^2_{H_0^1} + o\left(\|u\|^2_{H_0^1} \right) \end{aligned}$$

as $\|u\|_{H_0^1} \to 0$.
Therefore, if ρ is small enough, $I(u) > 0$.
To show the validity of (PS), let $\{u_k\}_k \subset H_0^1$ be a sequence s.t. $|I(u_k)| \leq C_1$ e $I'(u_k) \to 0$

as $k \to +\infty$. We fix $\beta \in \left(\frac{1}{\mu}, \frac{1}{2}\right)$. With the standard argument, for some k large enough, using the same argument as in (4.15),

$$
\begin{aligned}
C_1 + \|u_k\|_{H_0^1} &\geq I(u_k) - \beta I'(u_k)[u_k] = \\
&= \int_\Omega \left(\frac{1}{2} - \beta\right) |\nabla u_k|^2 dx - \int_\Omega \lambda \left(\frac{1}{2} - \beta\right) |u_k|^2 dx - \int_\Omega G(x, u_k) dx + \\
&+ \beta \underbrace{\int_\Omega g(x, u_k) u_k dx}_{\substack{\geq \mu G(x, u_k) - d \\ \text{for an appropriate } d}} \geq \\
&\geq \underbrace{\left(\frac{1}{2} - \beta\right)}_{=: \delta > 0} \|u_k\|^2_{H_0^1} - C \|u_k\|_2^2 + \underbrace{(\beta\mu - 1)}_{> 0} \int_\Omega \underbrace{\left(G(x, u_k) - \tilde{d}\right)}_{\geq C_2 |u_k|^\mu - \tilde{d}} dx \geq \\
&\geq \delta \|u_k\|^2_{H_0^1} - C \|u_k\|_2^2 + A_1 \|u_k\|_2^\mu - A_2.
\end{aligned}
$$

Using the boundedness from below on $[0, +\infty)$ of the function $h(x) = A_1 x^\mu - C x^2 - A_2$, we deduce that

$$C_1 + \|u_k\|_{H_0^1} \geq \delta \|u_k\|^2_{H_0^1} - D.$$

So that $\|u_k\|_{H_0^1}$ is bounded and the standard argument can be used to provide a subsequence that converges in the norm of H_0^1.

Chapter 5

Symmetric functionals: one last version of the mountain pass theorem

5.1 Krasnoselskii index

The purpose of this section is that of providing topological results that allow to analyze cases of functionals having certain regularities and symmetries. The main result of this section is the first theorem of Borsuk-Ulam, which allows to prove the fact that the so-called *Krasnoselskii index* of a symmetric set which is homeomorphic to the boundary of a bounded neighborhood of $0 \in \mathbb{R}^n$ is equal to n.

Definition 5.1.1. Let E be a Banach space. We say that a subset $A \subseteq E \setminus \{0\}$ is **symmetric** if

$$\forall u \in A, \ -u \in A.$$

We denote with

$$\Sigma(E) := \{A \subseteq E \setminus \{0\} : A \text{ closed and symmetric}\},$$

with

$$\mathcal{C}^k_{odd}(A, B) := \{f \in \mathcal{C}^k(A, B) : f \text{ odd on } A \in \Sigma(E)\}$$

as $k \geq 0$ and with

$$\Theta_{odd}(A, B) := \{\varphi \in \mathcal{C}^0_{odd}(A, B) : \varphi \text{ homeomorphism}\}.$$

Definition 5.1.2. We say that $A \in \Sigma(E)$ has **genus** (or **Krasnoselskii index**) $n \geq 0$ if n is the smallest integer s.t. there exists a function $\varphi \in \mathcal{C}^0_{odd}(A, \mathbb{R}^n \setminus \{0\})$. If such an integer n does not exist, we say that A has **infinite genus**.

To sum up, set $\gamma : \Sigma(E) \to \mathbb{N}$ the function that maps $A \in \Sigma(E)$ to its genus,

$$\gamma(A) = \begin{cases} 0 & \text{if } A = \varnothing; \\ n & \text{if } n = \min\{k \in \mathbb{N} \text{ s.t. there exists } \varphi \in \mathcal{C}^0_{odd}(A, \mathbb{R}^k \setminus \{0\})\} < +\infty; \\ +\infty & \text{otherwise.} \end{cases}$$

We prove some of the genus' properties: first, we enunciate a topological results:

Theorem 5.1.3 (Tietze)**.** *Let (X, τ) be a normal topological space *. Let $C \subseteq X$ be a closed subspace and $f : C \to A \subseteq \mathbb{R}$ be continuous. Then, there exists a continuous function $g : X \to A$ s.t. $g|_C \equiv f$.*

Lemma 5.1.4. *The following properties hold:*

(i) *let $x \in E \setminus \{0\}$, then $\gamma(\{x, -x\}) = 1$;*

(ii) *if there exists $f \in \mathcal{C}^0_{odd}(A, B)$, then $\gamma(A) \leq \gamma(B)$;*

(iii) *if $A \subseteq B$, then $\gamma(A) \leq \gamma(B)$;*

(iv) *if there exists $h \in \Theta_{odd}(A, B)$, then $\gamma(A) = \gamma(B)$;*

(v) *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$;*

(vi) *if $\gamma(B) < +\infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$;*

(vii) *if $A \in \Sigma(E)$ is compact, then $\gamma(A) < +\infty$ and there exists $\delta_0 > 0$ s.t. $\gamma(\overline{N_\delta(A)}) = \gamma(A)$ for all $\delta \in (0, \delta_0)$, where we set $N_\delta(A) = \{x \in E$ s.t. dist$(x, A) < \delta\}$;*

(viii) *if $A \in \Sigma(E)$ is s.t. $\gamma(A) > k$ e $V \subset E$ is a subspace of E having dimension k, then $A \cap V^\perp$;*

(ix) *let $E_n \subset E$ has finite dimension n. Let $S^{n-1} := \{x \in E_n$ with $\|x\|_E = 1\}$, then $\gamma(S^{n-1}) = n$.*

Proof. In what follows, the fact that $\Sigma(E)$ does not contains 0 is fundamental.

(i) is trivial;

(ii) if $\gamma(B) = +\infty$ there's nothing to prove, otherwise, set $\gamma(B) = n$, let $\varphi : B \to \mathbb{R}^n \setminus \{0\}$ be continuous and odd. The function $\varphi \circ f : A \to \mathbb{R}^n \setminus \{0\}$ is continuous and odd, so that $\gamma(A) \leq n$.

(iii) The inclusion $i : A \to B$ is continuous and odd. Hence *(iii)* follows by *(ii)*.

(iv) follows trivially by the previous points.

*Recall that a topological space is called **normal** if any couple of its closed disjoint subsets admits a couple of disjoint open neighborhoods.

(v) If $\gamma(A)$, or $\gamma(B)$, is not finite there is nothing to prove. Otherwise, let $n = \gamma(A)$ and $m = \gamma(B)$. There exist $\varphi \in \mathcal{C}^0_{odd}(A, \mathbb{R}^n \setminus \{0\})$ and $\psi \in \mathcal{C}^0_{odd}(B, \mathbb{R}^m \setminus \{0\})$. by Tietze theorem applied to all the components of the two functions, there exist two continuous extensions to E of φ and ψ taking values in $\mathbb{R}^n \setminus \{0\}$ and $\mathbb{R}^m \setminus \{0\}$ respectively. With abuse of notations we use the same symbols, φ and ψ, to denote the two extensions.
For all $x \in E$ we define

$$\tilde{\varphi}(x) = \frac{\varphi(x) - \varphi(-x)}{2}$$

and

$$\tilde{\psi}(x) = \frac{\psi(x) - \psi(-x)}{2}.$$

Then, $\tilde{\varphi}$ and $\tilde{\psi}$ are two odd extensions of φ and ψ respectively and they take values in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.
$f = (\tilde{\varphi}, \tilde{\psi})$ is defined on E. Its restriction to $A \cup B$ is odd, continuous and it takes values in $\mathbb{R}^{n+m} \setminus \{0\}$. Hence, $\gamma(A \cup B) \leq n + m$ by (ii) and (iii).

(vi) One has $A \subseteq A \cup B \subseteq \overline{A \setminus B} \cup B$ and (vi) follows;

(vii) Choose any $x \in E \setminus \{0\}$ and $0 < r < \|x\|_E$. Since

$$B_r(x) \cap B_r(-x) = \varnothing,$$

$\gamma(\overline{B_r(x)} \cup \overline{B_r(-x)}) = 1$ (choose f continuous, real-valued, defined on one of the two balls and set $f(x) = -f(x)$).
A is compact and symmetric, hence we can cover A with a finite number of such balls:

$$A \subseteq \bigcup_{k=1}^{m} [B_{r_k}(x_k) \cup B_{r_k}(-x_k)] \subseteq \bigcup_{k=1}^{m} \underbrace{\left[\overline{B_{r_k}(x_k)} \cup \overline{B_{r_k}(-x_k)}\right]}_{\gamma\, =\, 1}.$$

Then, $\gamma(A) \leq \sum_{k=1}^{m} \gamma(\overline{B_{r_k}(x_k)} \cup \overline{B_{r_k}(-x_k)}) = m < \infty$.
Let $n = \gamma(A) < \infty$ and $\varphi \in \mathcal{C}^0_{odd}(A, \mathbb{R}^n \setminus \{0\})$.
By Tietze theorem (with the construction of (v)) there exists an odd continuous extension of $\tilde{\varphi}$ and φ to E taking values in $\mathbb{R}^n$.
In general, by "odd-ing" the extension, we lose the fact that the "odd-ed" of the Tietze extension takes values in $\mathbb{R}^n \setminus \{0\}$ as well. However, $\tilde{\varphi}|_A = \varphi$ is nowhere 0 (since it takes values in $\mathbb{R}^n \setminus \{0\}$). Therefore, by the continuity of $\tilde{\varphi}$, there exists a neighborhood $N_{\delta_0}(A) = \{x \in E \text{ s.t. } \operatorname{dist}(x, A) < \delta_0\}$ of A s.t. $\tilde{\varphi}|_{N_{\delta_0}(A)}(x) \neq 0$ for all $x \in N_{\delta_0}(A)$.
For all $\delta \in (0, \delta_0)$, defined $N_\delta(A)$ similarly to $N_{\delta_0}(A)$, one has that $\tilde{\varphi}|_{\overline{N_\delta(A)}} \neq 0$ and it is continuous and odd. Then, $\gamma(\overline{N_\delta(A)}) \leq n$. On the other hand, $A \subset \overline{N_\delta(A)}$, so that $\gamma(\overline{N_\delta(A)}) \geq n$.

(viii) Seeking a contradiction, let $A \cap V^{\perp} = \varnothing$. Then, the projection $\mathcal{P} : A \to V \setminus \{0\}$ is a continuous odd mapping. As V is finite-dimentional, it is homeomorphic to an appropriate subset of $\mathbb{R}^k$. This, however, contradicts the fact that $\gamma(A) \leq k$.

□

Theorem 5.1.5 (Borsuk antipode theorem)**.** *Let $D \subset\subset \mathbb{R}^n$ be a symmetric neighborhood of* 0*. Let $\varphi \in \mathcal{C}^0_{odd}(\bar{D}, \mathbb{R}^n)$ be s.t. $\varphi|_{\partial D} \neq 0$. Then, $d(\varphi, D, 0)$ is an odd number.*

We prove the assertion only in the case of functions $\varphi \in \mathcal{C}^1_{odd}(D, \mathbb{R}^n)$ with just non-degenerate zeros.

Proof. As φ is continuous and odd, $\varphi(0) = 0$ and it is an isolated zero. Hence, there exists $B_\varepsilon(0) \subseteq D$ s.t. $\varphi|_{\overline{B_\varepsilon(0)}\setminus\{0\}} \neq 0$.

$$d(\varphi, D, 0) = d(\varphi, B_\varepsilon(0), 0) + d(\varphi, D \setminus \overline{B_\varepsilon(0)}, 0) = \pm 1 + \sum_{x \in Z} \operatorname{sgn} \det(J\varphi(x))$$

where

$$Z = \varphi^{-1}(0) \cap \left(D \setminus \overline{B_\varepsilon(0)}\right)$$

is the set of the zeros of φ in $D \setminus \overline{B_\varepsilon(0)}$.
φ is odd, hence $\varphi(x) = 0$ if and only if $\varphi(-x) = 0$.
On the other hand, if φ is odd, all of its partial derivatives are even and, therefore, $J(\varphi(x))$ is even.
We conclude that

$$\sum_{x \in Z} \operatorname{sgn} \det(J\varphi(x))$$

is even. This concludes the proof.

□

Corollary 5.1.6 (First theorem of Borsuk-Ulam)**.** *Let $D \subset\subset \mathbb{R}^n$ be an open symmetric neighborhood of 0 and $\psi \in \mathcal{C}^0_{odd}(\partial D, \mathbb{R}^j)$ with $j < n$. Then, $\exists x \in \partial D$ s.t. $\psi(x) = 0$.*

Proof. Seeking a contradiction, if $\psi|_{\partial D} \neq 0$, we could apply the Borsuk antipode theorem to an odd Tietze extension $\tilde{\psi} \in \mathcal{C}^0_{odd}(\bar{D}, \mathbb{R}^n)$. In particular,

$$d(\tilde{\psi}, D, 0) \neq 0,$$

(being odd).
By the continuity of $d(\tilde{\psi}, D, \cdot)$, for all $b \in B_\varepsilon(0) \subset \mathbb{R}^n$ with ε small enough,

$$d(\tilde{\psi}, D, b) \neq 0.$$

At the same time, if we approached 0 choosing $b \in \mathbb{R}^n \setminus \mathbb{R}^j$,

$$0 = d(\tilde{\psi}, D, 0) \underset{\text{continuity}}{=} d(\tilde{\psi}, D, b),$$

where we used the fact that $\tilde{\psi}|_{\partial D} = \psi|_{\partial D} \neq 0$ and d depends only on the boundary values of $\tilde{\psi}$. But this would contradict the fact that $d(\tilde{\psi}, D, 0) \neq 0$. □

Corollary 5.1.7 (Second theorem of Borsuk-Ulam)**.** *Let $D \subset\subset \mathbb{R}^n$ be an open symmetric neighborhood of* 0*. Let $f \in \mathcal{C}^0(\partial D, \mathbb{R}^j)$ with $j < n$.*
Then, $\exists x \in \partial D$ s.t. $f(x) = f(-x)$.

Proof. Let $\psi(x) = f(x) - f(-x)$. ψ satisfies the assumptions of the first theorem of Borsuk-Ulam, the assertion of which, applied to ψ, is exactly our claim.

□

Using the first theorem of Borsuk-Ulam we prove the following:

Theorem 5.1.8. *Let $D \subset\subset \mathbb{R}^n$ be an open symmetric neighborhood of $0 \in \mathbb{R}^n$. Let $A \in \Sigma(\mathbb{R}^n)$.*
If there exists $h \in \Theta_{odd}(A, \partial D)$, then $\gamma(A) = n$.

Proof. $h \in \Theta_{odd}(A, \partial D) \subseteq \mathcal{C}^0_{odd}(A, \mathbb{R}^n \setminus \{0\})$. Hence, $\gamma(A) \leq n$.
Seeking a contradiction, if $\gamma(A) = j < n$, then there would exist $\varphi \in \mathcal{C}^0_{odd}(A, \mathbb{R}^j \setminus \{0\})$.
The function

$$\psi := \varphi \circ h^{-1} : \partial D \to \mathbb{R}^j \setminus \{0\}$$

would be odd and continuous.
By the first theorem of Borsuk-Ulam, there would exist $x \in \partial D$ s.t. $\psi(x) = 0$. But this cannot be possible, as φ takes values in $\mathbb{R}^j \setminus \{0\}$.

□

This theorem has two immediate corollaries, the first of which is a more general version of (5.1.4 (ix)):

Corollary 5.1.9. *Let $A \in \Sigma(E)$. If there exists $h \in \Theta_{odd}(A, S^{n-1})$, then $\gamma(A) = n$.*

Proof. $\gamma(A) = \gamma(S^{n-1})$ since they're homeomorphic.
On the other hand, $S^{n-1} = \partial\{x \in E_n \text{ s.t. } \|x\|_E = 1\}$ is the boundary of a bounded symmetric neighborhood of $0 \in E_n$ for an appropriate finite-dimensional E_n (for this reason, E_n is homeomorphic to $\mathbb{R}^n$). By the previous theorem, $\gamma(S^{n-1}) = n$.

□

Corollary 5.1.10. *Let E be an inifnite-dimensional Banach space. For all $n \in \mathbb{N}$, E contains a subset having genus n.*

Proof. $\gamma(\varnothing) = 0$ and, for all $n > 0$, $\gamma(S^{n-1}) = n$ with S^{n-1} in an appropriate finite-dimensional subspace of E, say E_n. □

5.2 Ljusternik-Schnirelman theory

Krasnoselskii theory has an important employment in the study of even functionals. We start enunciating and proving two results concerning the latest, the first of which is related to the finite-fimensional case, while the second is a simple generalization to the infinite-dimensional case.
We denote with

$$\mathcal{C}^k_{ev}(A,B) = \{f \in \mathcal{C}^k(A,B) : f \text{ is an even function on } A \in \Sigma(E)\}.$$

We have:

Theorem 5.2.1. *Let $I \in \mathcal{C}^1_{ev}(\mathbb{R}^n,\mathbb{R})$. Then, there exist at least n couples of critical points of I in the form $\{x,-x\}$.*

Proof. For all $k = 1,\dots,n$ we define the set

$$\Gamma_k := \{A \in \Sigma(S^{n-1}) \text{ s.t. } \gamma(A) \geq k\}.$$

For all k, as $S^{k-1} \in \Sigma(S^{n-1})$, $\Gamma_k \neq \varnothing$. Hence, it makes sense to consider

$$c_k := \inf_{A\in\Gamma_k} \max_{u\in A} I(u).$$

By the definition of Γ_k, since $\gamma(A) \geq k \Longrightarrow \gamma(A) \geq k-1$, we have

$$\Gamma_n \subset \Gamma_{n-1} \subset \ldots \subset \Gamma_2 \subset \Gamma_1$$

and, therefore, by the monotonicity of the infimum,

$$c_1 \leq c_2 \leq \ldots \leq c_{n-1} \leq c_n.$$

We observe that $\{x,-x\} \in \Gamma_1$, so that (as I is an even functional)

$$c_1 = \inf_{A\in\Gamma_1} \max_{u\in A} I(u) = \inf_{\{-x,x\}\in\Gamma_1} \max_{u\in\{-x,x\}} I(u) = \min_{u\in S^{n-1}} I(u).$$

On the other hand, $\Gamma_n = \{S^{n-1}\}$. In fact, if $A \subset S^{n-1}$ and the inclusion is strict, then chosen y in $S^{n-1} \setminus A$, by the symmetry of A, $-y$ still belongs to $S^{n-1} \setminus A$. The stereographic projection $\mathcal{P}$ of $S^{n-1} \setminus \{y\}$ on the tangent hyperplane in $-y$ is an odd homeomorphism (as A is odd and it does not coincide with the entire sphere) of A into $\mathbb{R}^{n-1} \setminus 0$ (as $\mathcal{P}(-y) = 0$ and $-y \notin A$). Hence,

$$n \leq \gamma(A) = \gamma(\mathcal{P}(A)) \leq n-1$$

that is a contradiction. Therefore,

$$c_n = \max_{u\in S^{n-1}} I(u).$$

As usual, we denote with K_c the set of the critical points with critical value c:

$$K_c := \left\{u \in S^{n-1} \text{ s.t. } I(u) = c \text{ and } \left(I|_{S^{n-1}}\right)'(u) = 0\right\}.$$

We prove that if $c_k = \ldots = c_{k+p-1} =: c$, then $\gamma(K_c) \geq p$.
Seeking a contradiction, suppose $\gamma(K_c) < p$. Since K_c is compact, there exists a neighborhood of K_c, say $N_\delta(K_c) \subseteq \mathbb{R}^n$, s.t. $\gamma(K_c) = \gamma(N_\delta(K_c))$. One would have, by the obvious following inclusions:

$$K_c \subseteq N_\delta(K_c) \cap S^{n-1} \subseteq N_\delta(K_c),$$

that

$$p > \gamma(K_c) = \gamma(N_\delta(K_c)) \geq \gamma(N_\delta(K_c) \cap S^{n-1}) \geq \gamma(K_c).$$

Moreover, $I|_{S^{n-1}} \in \mathcal{C}^{1,1}(S^{n-1}, \mathbb{R})$ as S^{n-1} is compact and I is continuous on S^{n-1}.
By the finite-dimensional deformation lemma, there exist $\varepsilon > 0$ and a homeomorphism η s.t.

$$\eta(1, \{I|_{S^{n-1}} \leq c + \varepsilon\}) \subseteq \{I|_{S^{n-1}} \leq c - \varepsilon\}$$

and, since I is even, $\eta(t, \cdot)$ can be chosen to be odd.
By the definition of $c = c_{k+p-1}$ there exists $A_\varepsilon \in \Sigma(S^{n-1}) \cap \Gamma_{k+p-1}$ s.t.

$$\sup_{u \in A_\varepsilon} I|_{S^{n-1}}(u) \leq c + \varepsilon$$

and $\gamma(A_\varepsilon) \geq k + p - 1$.
Now,

$$\gamma(A_\varepsilon \setminus N_\delta(K_c)) \geq \gamma(A_\varepsilon) - \underbrace{\gamma(\overline{N_\delta(K_c)})}_{\leq p-1} \geq k + p - 1 - (p-1) = k.$$

Hence, $A_\varepsilon \setminus N_\delta(K_c) \subset S^{n-1}$ and $\gamma(A_\varepsilon \setminus N_\delta(K_c)) \geq k$. This tells us that $A_\varepsilon \setminus N_\delta(K_c) \in \Gamma_k$.
Then, since $\eta(1, \cdot)$ is an odd homeomorphism,

$$\gamma\left(\eta(1, A_\varepsilon \setminus N_\delta(K_c))\right) = \gamma(A_\varepsilon \setminus N_\delta(K_c)) \geq k.$$

Therefore, $\eta(1, A_\varepsilon \setminus N_\delta(K_c)) \in \Gamma_k$. This implies the contradiction:

$$\sup_{u \in \eta(1, A_\varepsilon \setminus N_\delta(K_c))} I(u) = c - \varepsilon \geq \inf_{A \in \Gamma_k} \sup_{u \in A} I(u) = c_k = c.$$

We see that the contradiction implies the assertion. In fact, if $c_k = \ldots = c_{k+p-1}$, then $\gamma(K_c) \geq p \geq 2$, so that K_c has to contain at least countable many symmetric couples (by the symmetry of I) of points (otherwise, there would exist an odd continuous real-valued function).
If, instead, $p = 1$, K_c contains at least two (symmetric) points, since there exists an odd function on K_c.

□

For the infinite-dimensional case,

Theorem 5.2.2. *Let E be an infinite-dimensional Banach space. Let $I \in \mathcal{C}^1_{ev}(E, \mathbb{R})$ satisfying (PS).*
Then, $I|_{\partial B_1}$ has infinitely many couples of critical points $\{u, -u\}$.

Proof. The proof goes exactly as in the finite-dimensional case, using (PS) to apply the deformation lemma.

□

We see several simple consequences of these results:

Example 5.2.3. Let $E = H^1_0(\Omega)$ with $\Omega \subset\subset \mathbb{R}^n$.
We consider the equation $-\Delta u = \lambda u$ we already mentioned when we talked about the eigenvalues of Laplace operator.
By repeating the proof of Theorem (5.2.2) for $S^2 = \{u \in H^1_0(\Omega) : \|u\|_2 = 1\}$ one proves that for all k it is

$$\lambda_k = \inf_{A \in \Gamma_k} \sup_{u \in A \subseteq S^2} \int_\Omega |\nabla u|^2 dx$$

and the eigenvalue problem admits infinite solutions.

Example 5.2.4. Let $\Omega \subset\subset \mathbb{R}^n$. We consider the system

$$\begin{cases} -\Delta u = |u|^{p-1} u & \text{on } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

for some $p \in (1, 2^* - 1)$. By repeating the proof of Theorem (5.2.2) with

$$S^{p+1} := \left\{ u \in H^1_0(\Omega) \text{ s.t. } \|u\|_{p+1} = 1 \right\}$$

and, for all $k \geq 1$,

$$\Gamma_k := \left\{ A \in \Sigma\left(S^{p+1}\right) \text{ s.t. } \gamma(A) \geq k \right\}.$$

one can prove that for all k,

$$c_k = \inf_{A \in \Gamma_k} \sup_{u \in A \subset S^{p+1}} \int_\Omega |\nabla u|^2 dx$$

is a collection of critical values for the functional related to the initial system.
In particular, this problem admits infinitely many solutions.
We will see that when $p = 2^* - 1$ and Ω is a star domain, the only solution is $u = 0$ (both in the weak and in the classical sense).

Theorem 5.2.5 (Clark theorem)**.** *Let $I \in \mathcal{C}^1_{ev}(E, \mathbb{R})$ satisfying (PS) and s.t. $I(0) = 0$. Let*

$$\Gamma_k = \{A \in \Sigma(E) \ s.t. \ \gamma(A) \geq k\}.$$

If

$$c_k := \inf_{A \in \Gamma_k} \sup_{u \in A} I(u) \in (-\infty, 0),$$

then c_k is a critical value. Moreover, if $c_k = \ldots = c_{k+p-1} =: c$ for some $p > 1$, then, set as usual

$$K_c = \{u \in E : I(u) = c \text{ and } I'(u) = 0\}$$

one has

$$\gamma(K_c) \geq p.$$

Proof. If we show that, under the same hypothesis as above, $\gamma(K_c) \geq p$, we have finished. In fact, in this case, $K_c \neq \varnothing$, since there would exist at least a continuous odd function defined on K_c, so that c would be a critical value and the assertion would follow.

Seeking a contradiction, let $\gamma(K_c) < p$. By the deformation lemma, there would exist $\varepsilon \in (0, 1)$ s.t.

$$\eta(1, \{I \leq c + \varepsilon\} \setminus N_\delta(K_c)) \subseteq \{I \leq c - \varepsilon\}$$

with $\eta(t, \cdot)$ odd. Let $A_\varepsilon \in \Gamma_{k+p-1}$ be s.t.

$$\sup_{u \in A_\varepsilon} I(u) \leq c + \varepsilon$$

and

$$B := A_\varepsilon \setminus N_\delta(K_c).$$

As

$$\gamma(B) \geq \gamma(A_\varepsilon) - \gamma(\overline{N_\delta(K_c)}) \geq k + p - 1 - (p - 1) = k$$

we have $B \in \Gamma_k$. Moreover, since $\eta(1, \cdot)$ is an odd homeomorphism, $\gamma(\eta(1, B)) = \gamma(B)$, so that $\eta(1, B) \in \Gamma_k$ and

$$c = \inf_{A \in \Gamma_k} \sup_{u \in A} I(u) \leq \sup_{u \in \eta(1,B)} I(u) \leq c - \varepsilon.$$

This is a contradiction. □

Corollary 5.2.6. *Let I be as in the Clark theorem. Let $K \in \Sigma(E)$ with $\gamma(K) = k$. If*

$$\sup_{u \in K} I(u) < 0$$

then, all the conclusions of the Clark theorem hold.

Example 5.2.7. Let $\Omega \subset\subset \mathbb{R}^n$. We consider the problem

$$\begin{cases} -\Delta u = \lambda(u + g(u)) & \text{on } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{5.1}$$

with

- $g \in \mathcal{C}^0_{odd}(\mathbb{R}, \mathbb{R})$;

- $g(s) = o(s)$ as $s \to 0$;
- $\exists s_1 > 0$ s.t. $s_1 + g(s_1) = 0$.

We also define

$$h(s) = s + g(s) \quad \text{and} \quad \tilde{h}(s) = h(s)\chi_{[-s_1,s_1]}(s).$$

We consider the following system:

$$\begin{cases} -\Delta u = \lambda \tilde{h}(u) & \text{on } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{5.2}$$

If u is a solution of (5.2), then $u(\Omega) \subseteq [-s_1, s_1]$. In fact, otherwise, there would exist $\tilde{\Omega} \subseteq \Omega$ and $\tilde{u}$ s.t.

$$\begin{cases} -\Delta \tilde{u} = 0 & \text{on } \tilde{\Omega}, \\ \tilde{u}|_{\partial\tilde{\Omega}} = s_1 \end{cases}$$

and, by the maximum principle for harmonic functions, one would have $\tilde{u} \equiv s_1$, that contradicts the expression that defines h.
In particular, all the solutions u of (5.2) are also solutions of (5.1).
We consider the functional $I \in \mathcal{C}^1_{ev}(H^1_0(\Omega), \mathbb{R})$, that is the one related to the equation of (5.2):

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda \int_\Omega \left(\int_0^u \tilde{h}(s) ds \right) dx.$$

Obviously, $I(0) = 0$ and I is bounded from below.
We show that, if $\lambda > \lambda_k$, then (5.2) (and (5.1)) has at least k couples of solution, using Theorem (5.2.6).
We define

$$K_r = \left\{ v = \sum_{i=1}^k \alpha_i e_i \in H^1_0(\Omega) \ : \ \|v\|^2_{H^1_0 \cap K_r} = \sum_{i=1}^k \alpha_i^2 = r^2 \right\}.$$

K_r is a k-dimensional sphere, so that $\gamma(K_r) = k$. We show that $\sup_{u \in K_r} I(u) < 0$ if r is small enough: for, let $v \in K_r$, as K_r is finite-dimensional, all the norms on K_r are equivalent, so that there exists a constant $C = C(K_r)$ s.t.

$$\|v\|_\infty \leq C \|v\|_{H^1_0 \cap K_r} = Cr.$$

Let $H(s) = \int_0^s \tilde{h}(s) ds$. Then, for $\|v\| = r$ small enough,

$$H(v) = \int_0^v \tilde{h}(s) ds = \int_0^v (s + g(s))\chi_{[-s_1,s_1]}(s) ds = \int_0^v s\chi_{[-s_1,s_1]}(s) ds + o(r^2) = \frac{r^2}{2} + o(r^2).$$

Then, as $\|v\| = r \to 0$, we get

$$\begin{aligned} I(v) &= \frac{1}{2}r^2 - \frac{\lambda}{2}\int_\Omega \left|\sum_{i=1}^k \alpha_i e_i(x)\right|^2 dx + o(r^2) = \\ &= \frac{r^2}{2} - \frac{\lambda}{2}\sum_{i=1}^k \alpha_i^2 \underbrace{\int_\Omega e_i^2 dx}_{= 1/\lambda_i} + o(r^2) = \frac{1}{2}\left(r^2 - \lambda\sum_{i=1}^k \frac{\alpha_i^2}{\lambda_i}\right) + o(r^2) \underset{\lambda_i \le \lambda_k}{\le} \\ &\le \frac{1}{2}\left(r^2 - \frac{\lambda}{\lambda_k}r^2\right) + o(r^2) = \frac{r^2}{2}\underbrace{\left(1 - \frac{\lambda}{\lambda_k}\right)}_{< 0} + o(r^2) < 0 \end{aligned}$$

as $r \to 0$.
If $\lambda > \lambda_k$, then, once defined c_k as in the Clark theorem, one has $c_k < 0$ and, as I is bounded from below, $c_k > -\infty$. Clark theorem applies and we get infinitely many couples of critical values for the functional I. That are, infinitely many solutions for $\lambda > \lambda_k$.

Remark 5.2.8. We observe that as λ grows, the number of couples of solutions grows as well. Indeed, as λ crosses and eigenvalue of Laplace operator, two new solutions add. This phenomenon is called **bifurcation**.

Example 5.2.9. Let $\Omega \subset\subset \mathbb{R}^n$. We consider the problem

$$\begin{cases} -\Delta u = \lambda g(u) & \text{on } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{5.3}$$

with the following assumptions on g:

- $g \in \mathcal{C}^0_{odd}(\mathbb{R}, \mathbb{R})$;
- $g(s) = o(s)$ as $s \to 0$;
- $\exists s_1 > 0$ s.t. $g(s) > 0$ if $s \in (0, s_1)$ and $g(s_1) = 0$.

We show that there exists a collection $\{\mu_k\}_{k\in\mathbb{N}}$ s.t. if $\lambda > \mu_k$, $\exists$ at least k couples of solutions. In the previous example, $\{\mu_k\}_k \equiv \{\lambda_k\}_k$.
We define

$$G(s) = \int_0^s g(t)\chi_{[0,s_1]}(t)dt.$$

The functional $I \in \mathcal{C}^1_{ev}(H^1_0, \mathbb{R})$ associated to the equation of (5.3) is given by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \lambda\int_\Omega G(u)dx.$$

We consider K_r as in the previous example and we show that $\inf_{u \in K_r} \int_\Omega G(u)dx = \alpha_r > 0$ [†].
Seeking a contradiction, let the infimum be 0 and consider a minimizing $\{v_i\}_i$ in K_r. K_r is compact (and finite-dimensional), so that there exists a subsequence $\{v_{i_j}\}_j \subseteq \{v_i\}_i$ s.t.

$$v_{i_j} \to \bar{v} \in K_r$$

as $j \to \infty$ and

$$\int_\Omega G(\bar{v})dx = 0. \tag{5.4}$$

However, $G(s) \neq 0$ if $s \neq 0$, so that it must be $\bar{v} = 0$ a.e. in order for (5.4) to be verified. But $0 \notin K_r$ leading us to a contradiction.
It follows that, for $v \in K_r$,

$$I(v) = \frac{r^2}{2} - \lambda \int_\Omega G(v)dx \leq \frac{r^2}{2} - \lambda \alpha_r < 0$$

for some appropriate $\lambda > \mu_k$.

5.3 Mountain pass theorem for symmetric functionals

The purpose of this section is that of providing a version of the mountain pass theorem that can be applied to prove the existence of an unbounded sequence of critical values of a symmetric functional.

Theorem 5.3.1 (Mountain pass theorem for symmetric functionals). *Let $I \in \mathcal{C}^1_{ev}(E, \mathbb{R})$ satisfying (PS) and s.t.*

(i) $I(0) = 0$;

(ii) there exist $\rho, \alpha > 0$ s.t. $I|_{\partial B_\rho(0)} \geq \alpha$;

(iii) for every finite-dimensional subspace $X \subset E$, the set $\{u \in X : I(u) \geq 0\}$ is bounded. Or, equivalently, for every finite-dimensional subspace $X \subset E$ and every $u \in X$ s.t. $\|u\|_E \geq R$ for some appropriate $R = R(X)$, $I(u) \leq 0$.

Then, there exists an unbounded sequence of critical values for I.

Before proceeding further and prove the two results that will give the proof of Theorem (5.3.1), we show an application.

Example 5.3.2. Let $\Omega \subset\subset \mathbb{R}^n$ be open and consider the problem

$$\begin{cases} -\Delta u = g(x,u) & \text{on } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{5.5}$$

[†] Observe that, by the definition of g, $G(s)$ is positive for all $s \in [-s_1, s_1]$.

with the following assumptions on g (observe that, the latest apart, they are the same hypothesis as the ones of Example (4.3.2):

- $g \in \mathcal{C}^0(\Omega \times \mathbb{R}, \mathbb{R})$;
- $|g(x,r)| \leq c_1 + c_2|r|^p$ for some $p \in (1, 2^* - 1)$;
- $g(x,r) = o(r)$ per $r \to 0$;
- $\exists \mu > 2$, $\bar{r} > 0$ s.t. $\forall |r| \geq \bar{r}$ we have
$$0 < G(x,r) = \int_0^r g(x,t)dt \leq \frac{r}{\mu} g(x,r);$$
- $g(x,s) = -g(x,-s)$.

We already verified the hypothesis (i) and (ii) of (5.5) in Example (4.3.2), we know that
$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(x,u(x))dx$$
and we also verified that I satisfies (PS). It remains to check the hypothesis (5.3.1 (iii)). By the computation in Example (4.3.2), we already know that
$$G(x,s) \geq e^A |s|^\mu.$$
As all the norms on a finite-dimensional vector space are equivalent, for $u \in X$,
$$\|u\|_\mu^\mu \geq c(X)\,\|u\|_2^\mu \geq C(X)\,\|u\|_{H_0^1}^\mu.$$
Therefore,
$$I(u) \leq \frac{1}{2}\int_\Omega |\nabla u|^2 dx - e^A \int_\Omega |u|^\mu dx \leq \frac{1}{2}\|u\|_{H_0^1}^2 - e^A C(X)\,\|u\|_{H_0^1}^\mu \leq 0 \tag{5.6}$$
up to choose $\|u\|_{H_0^1}$ large enough, since $\mu > 2$.

Remark 5.3.3. One can try apply Clark theorem to (5.5), setting
$$\Gamma_k := \{K \in \Sigma(E) : \gamma(K) \geq k\}$$
and
$$c_k := \inf_{K \in \Gamma_k} \sup_{u \in K} I(u).$$
However, one would have $c_k = -\infty$ for all k.
In fact, if we choose $K = \partial B_R \cap E_k$ for some k-dimensional subspace $E_k \subset E$, it's easy to see, by (5.6), that
$$c_k \leq \sup_{u \in K} I(u) = -\infty.$$

The opposite argument would fail as well: if we define $c_k = \sup_{K \in G_k} \inf_{u \in K} I(u)$ with

$$G_k := \{K \in \Sigma(E) : \gamma(K) \leq k\},$$

it would be $c_k = +\infty$. In fact, if we choose $K_n := \partial B_1 \cap E_k$ to be s.t. $K \subset \{e_1, \dots, e_j\}^\perp$ ($\{e_k\}_k$ being the eigenfunctions of Laplace operator) we would have that if $u \in K_n$, then

$$\int_\Omega |\nabla u|^2 dx \geq \lambda_{n+1} \int_\Omega |u|^2 dx \to +\infty$$

as $n \to +\infty$.

Let E be a Banach space and I be a functional that satisfies (5.3.1 (i)-(iii)).
We define

$$A_0 := \{u \in E : I(u) \geq 0\}$$

and

$$\Gamma^* := \{h \in \Theta_{odd}(E, E) : h(\partial B_1) \subseteq A_0\}.$$

Remark 5.3.4. $\Gamma^* \neq \varnothing$. In fact, by the hypothesis (ii), the homeomorphism $u \mapsto \rho u$ belongs to Γ^*.

We also define

$$\Gamma_m := \{K \in \Sigma(E) \text{ compacts s.t. } \gamma(K \cap h(\partial B_1)) \geq m \text{ for all } h \in \Gamma^*\}.$$

Lemma 5.3.5. *Let E be a Banach space and I be a functional satisfying the hypothesis of Theorem (5.3.1). Then:*

(a) $\Gamma_m \neq \varnothing$;

(b) $\Gamma_{m+1} \subseteq \Gamma_m$;

(c) if $K \in \Gamma_m$ *and* $Y \in \Sigma(E)$ *is s.t.* $\gamma(Y) \leq j < m$, *then* $\overline{K \setminus Y} \in \Gamma_{m-j}$;

(d) if $\varphi \in \Theta_{odd}(E, E)$ *is s.t.* $\varphi^{-1}(A_0) \subseteq A_0$, *then* $\varphi(\Gamma_m) \subseteq \Gamma_m$.

Proof. (a) Let $X \subset E$ have dimension equal to m. By the assumption (iii) of Theorem (5.3.1), for some sufficiently large R, as $X \cap A_0$ is bounded, we have

$$K := X \cap \overline{B_R} \supseteq X \cap A_0.$$

We show that $K \in \Gamma_m$.
Since for all $h \in \Gamma^*$, $h(\partial B_1) \subseteq A_0$ we have

$$K \supseteq X \cap h(\partial B_1). \tag{5.7}$$

As h is odd (and $h(0) = 0$) $\Longrightarrow h(B_1)$ is a symmetric neighborhood of 0, so that

$$X \cap h(B_1) \quad \text{is a symmetric neighborhood of } 0 \in X. \tag{5.8}$$

Moreover, as the homeomorphism map boundary points into boundary points, we have

$$\partial(X \cap h(B_1)) \subseteq X \cap h(\partial B_1). \tag{5.9}$$

Putting all together,

$$m \geq \gamma(K \cap h(\partial B_1)) \underset{(5.7)}{\geq} \gamma(X \cap h(\partial B_1)) \underset{(5.9)}{\geq} \gamma(\partial(X \cap h(B_1))) \underset{(5.8)}{=} m.$$

That is, $K \in \Gamma_m$.

(b) follows immediately by the definitions of the Γ_m's.

(c) Let $K \in \Gamma_m$ and $Y \in \Sigma(E)$ be s.t. $\gamma(Y) \leq j < m$. Then, using the properties of sets and γ:

$$\gamma\left(\overline{K \setminus Y} \cap h(\partial B_1)\right) = \gamma\left(\overline{(K \cap h(\partial B_1)) \setminus Y}\right) \geq \gamma(K \cap h(\partial B_1)) - \gamma(Y) \geq m - j.$$

(d) Let $K \in \Gamma_m$, we have to prove that $\varphi(K) \in \Gamma_m$.
Since $K \in \Gamma_m$ and $\varphi \in \Theta_{odd}(E, E)$ is s.t. $\varphi^{-1}(A_0) \subseteq A_0$, $\varphi(K)$ is compact, symmetric and does not contain 0.
Using the homeomorphism-invariance property of γ, we have:

$$\gamma(\varphi(K) \cap h(\partial B_1)) = \gamma(K \cap \varphi^{-1} \circ h(\partial B_1)).$$

By the assumption, however, $\varphi^{-1} \circ h \in \Gamma^*$, in fact

$$\varphi^{-1}(h(\partial B_1)) \subseteq \varphi^{-1}(A_0) \subseteq A_0.$$

Therefore, since $K \in \Gamma_m$, $\gamma(\varphi(K) \cap h(\partial B_1)) \geq m$.

This concludes the proof.

□

Now, Theorem (5.3.1) is a trivial consequence of the following result.

Proposition 5.3.6. *Let E be a Banach space and I be a functional satisfying the hypothesis of Theorem (5.3.1). Let*

$$c_m := \inf_{A \in \Gamma_m} \sup_{u \in A} I(u).$$

Then,

(a) $0 < \alpha \leq c_1 \leq c_2 \leq \ldots$*;*

(b) if $c_m = c_{m+1} = \ldots = c_{m+p-1} =: c$*, then* $\gamma(K_c) \geq p$*. In particular, the* c_m *are critical values for* I*;*

(c) $\lim_{m\to+\infty} c_m = +\infty$.

Proof. (a) We observed that the function $h(u) = \rho u$ is a homeomorphism on Γ^*. Therefore, if $K \in \Gamma_m$,

$$\gamma(K \cap h(\partial B_1)) = \gamma(K \cap \partial B_\rho) \geq m \Longrightarrow K \cap \partial B_\rho \neq \varnothing.$$

Hence, chosen any $K \in \Gamma_m$,

$$c_m := \inf_{K\in\Gamma_m} \sup_{u\in K} I(u) \geq \inf_{u\in\partial B_\rho \cap K} I(u) \geq \alpha.$$

Finally, the monotonicity of the c_m follows obviously by the inclusions of the Γ_m.

(b) Seeking a contradiction, let $\gamma(K_c) < p$.
Since K_c is compact, we know that it admits an open neighborhood $N_\delta(K_c)$ s.t.

$$\gamma(K_c) = \gamma(N_\delta(K_c)) \leq p - 1.$$

By the deformation lemma, with $\bar{\varepsilon} = \alpha/2$: there would exist $\varepsilon \in (0, \bar{\varepsilon})$ and a homeomorphism η s.t. $\eta(1, \cdot) \in \Theta_{odd}(E, E)$ and

$$\eta\left(1, \{I \leq c+\varepsilon\} \setminus N_\delta(K_c)\right) \subset \{I \leq c - \varepsilon\}. \tag{5.10}$$

By the definition of c as c_{m+p-1}, there exists $K_\varepsilon \in \Gamma_{m+p-1}$ s.t.

$$\sup_{u\in K_\varepsilon} I(u) \leq c + \varepsilon. \tag{5.11}$$

Then, $\overline{K_\varepsilon \setminus N_\delta(K_c)} \in \Gamma_m$, in fact, by (5.3.5 (c)), $\overline{K_\varepsilon \setminus N_\delta(K_c)} \in \Gamma_{m+p-1-(p-1)} = \Gamma_m$.
If we showed that $\eta(1, \cdot)^{-1}(A_0) \subseteq A_0$, then using (5.3.5 (d)), we would have

$$\eta(1, \overline{K_\varepsilon \setminus N_\delta(K_c)}) \in \Gamma_m$$

(the image $\eta(1, \Gamma_m)$ would still be included in Γ_m, but we said that all the points in $\overline{K_\varepsilon \setminus N_\delta(K_c)}$ belong to Γ_m). This would give:

$$c = \inf_{A\in\Gamma_m} \sup_{u\in A} I(u) \leq \sup_{u\in\eta(1, \overline{K_\varepsilon \setminus N_\delta(K_c)})} I(u).$$

However, if $v \in \overline{K_\varepsilon \setminus N_\delta(K_c)}$, then $v \in K_\varepsilon$, so that (by (5.11)) $I(v) \leq c+\varepsilon$. Hence, $v \in \{I \leq c + \varepsilon\} \setminus N_\delta(K_c)$. Therefore, if $\eta(1, v) = u$, $I(u) \leq c - \varepsilon$ by (5.10). This would tell us that

$$c \leq \sup_{u\in\eta(1, \overline{K_\varepsilon \setminus N_\delta(K_c)})} I(u) \leq c - \varepsilon.$$

This is a contradiction.
Hence, we show that $\eta(1, \cdot)^{-1}(A_0) \subseteq A_0$.
As we already observe in the deformation lemma, proving that η is a homeomorphism, $\eta(1, \cdot)^{-1} = \eta(-1, \cdot)$.
Then, appealing to the monotonicity of $\eta(\cdot, u)$, for all $u \in A_0$,

$$I(\eta(1, u)^{-1}) = I(\eta(-1, u)) \geq I(\eta(0, u)) = I(u) \geq 0.$$

(c) Seeking a contradiction, if c_m would be bounded, we would have two cases, due to the monotonicity of $\{c_m\}_m$, which prevent the $\{c_m\}_m$ to oscillate:

(i) If the c_m were definitively constant, we would have:

$$\gamma(K_c) < +\infty$$

as K_c is compact and, at the same time, $\gamma(K_c) \geq p$ for all p, by the previous point (definitively $c_m = c_{m+1} = \ldots = c_{m+p-1}$ for all $p \geq 1$). This would be a contradiction.

(ii) If $c_m \nearrow c$ as $m \to \infty$ with $c \neq c_m$ for all m, we define

$$\mathcal{K} := \bigcup_{\ell \in [c_1, c]} K_\ell.$$

$\mathcal{K}$ would be symmetric (as a union of symmetric sets) and compact (by (4.1.15)). Hence, there would exist $j \in \mathbb{N}$ s.t.

$$\gamma(\mathcal{K}) = j$$

and, by the compactness of $\mathcal{K}$, there would exist a neighborhood of $\mathcal{K}$, say $N_\delta(\mathcal{K})$ s.t.

$$\gamma(N_\delta(\mathcal{K})) = \gamma(\mathcal{K}) = j.$$

Using the deformation lemma with $\bar{\varepsilon} = c - c_1$, we would get a $\varepsilon \in (0, \bar{\varepsilon})$ and a deformation η s.t.

$$\eta(1, \{I \leq c + \varepsilon\} \setminus N_\delta(\mathcal{K})) \subseteq \{I \leq c - \varepsilon\}.$$

By the definition of limit, there exists m_0 s.t. $c_{m_0} > c - \varepsilon$. Moreover, by the definition of c_{m_0+j}, there exists $K_\varepsilon \in \Gamma_{m_0+j}$ s.t.

$$\sup_{u \in K_\varepsilon} I(u) \leq \underbrace{c_{m_0+j}}_{< c} + \varepsilon < c + \varepsilon.$$

By (5.3.5(c) and (d)), $\eta(1, K_\varepsilon \setminus N_\delta(\mathcal{K})) \in \Gamma_{m_0}$. This would imply that

$$c - \varepsilon < c_{m_0} = \inf_{K \in \Gamma_{m_0}} \sup_{u \in K} I(u) \leq \sup_{u \in \eta(1, K_\varepsilon \setminus N_\delta(\mathcal{K}))} I(u) \leq c - \varepsilon.$$

This is a contradiction.

□

Chapter 6

Loss of compactness

In all the examples we mentioned up to now, we considered differential equations in the form

$$-\Delta u - g(x, u) = 0,$$

where g satisfied the subcritical growth condition in its second variable:

$$g(x, r) \leq c_1 + c_2|r|^p$$

as $r \to +\infty$, for some $p \in (1, 2^* - 1)$. In particular, we considered the problem

$$\begin{cases} -\Delta u = u|u|^{p-1} & \text{on } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

for $p \in (1, 2^* - 1)$.

The choice $p \in (1, 2^* - 1)$ allowed us to use a consequence of Rellich-Kondrachov theorem, claiming that the p-norm is WC on H_0^1 and proving the validity of (PS).
The purpose of this chapter is that of considering results related to equations in the critical case $p = 2^* - 1$. In this case, we'll see that (PS) has to be replaced with a weaker version of it.

6.1 Pohožaev identity and its applications

Theorem 6.1.1 (Pohožaev). *Let $g \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$ and $G(s) = \int_0^s g(t)dt$. Let $\Omega \subset\subset \mathbb{R}^n$ and $u \in \mathcal{C}^0(\bar{\Omega}) \cap \mathcal{C}^2(\Omega)$ be a classical solution of*

$$\begin{cases} -\Delta u = g(u) & \text{on } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{6.1}$$

Then, if ν is the outward pointing unit vector that is normal to $\partial\Omega$, we have

$$\frac{n-2}{2}\int_\Omega |\nabla u|^2 dx - n\int_\Omega G \circ u dx + \frac{1}{2}\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 x \cdot \nu dx = 0. \tag{P}$$

Proof. We multiply (6.1) by $x \cdot \nabla u$:

$$
\begin{aligned}
0 &= (\Delta u + g(u))(x \cdot \nabla u) = \\
&= \operatorname{div}[\nabla u(x \cdot \nabla u)] - |\nabla u|^2 - x \cdot \nabla \left(\frac{|\nabla u|^2}{2}\right) + x \cdot \nabla G(u) = \\
&= \operatorname{div}\left(\nabla u(x \cdot \nabla u) - x\frac{|\nabla u|^2}{2} + xG(u)\right) + \frac{n-2}{2}|\nabla u|^2 - nG(u).
\end{aligned}
$$

Integrating:

$$
\int_\Omega \left(\operatorname{div}\left(\nabla u(x \cdot \nabla u) - x\frac{|\nabla u|^2}{2} + xG(u)\right) + \frac{n-2}{2}|\nabla u|^2 - nG(u)\right) dx = 0.
$$

We use the divergence theorem on the first addendum:

$$
\int_\Omega \operatorname{div}\left(\nabla u(x \cdot \nabla u) - x\frac{|\nabla u|^2}{2} + xG(u)\right) dx = \int_{\partial\Omega} \nu \cdot \left(\nabla u(x \cdot \nabla u) - x\frac{|\nabla u|^2}{2} + xG(u)\right) dx.
$$

However,

$$
\int_{\partial\Omega} \nu \cdot xG(u)dx = 0
$$

as $u|_{\partial\Omega} = 0$ and $G(0) = 0$.
Since $u|_{\partial\Omega} \equiv 0$, the derivative of u in the direction that is tangent to $\partial\Omega$ in any boundary point is non-zero, that is ∇u is orthogonal to the tangent direction, i.e. ∇u is parallel to ν. Then,

$$
x \cdot \nabla u = |\nu|^2 x \cdot \nabla u = (x \cdot \nu)(\nu \cdot \nabla u) = x \cdot \nu \frac{\partial u}{\partial \nu} \tag{6.2}
$$

and, similarly, $\nu \cdot \nabla u = |\nu|^2 \partial u/\partial\nu = \partial u/\partial \nu$, hence

$$
\nu \cdot (\nabla u(x \cdot \nabla u)) = (\nu \cdot \nabla u)\left(x \cdot \nu \frac{\partial u}{\partial \nu}\right) = \frac{\partial u}{\partial \nu}\left(x \cdot \nu \frac{\partial u}{\partial \nu}\right) = \left|\frac{\partial u}{\partial \nu}\right|^2 x \cdot \nu.
$$

Finally, using (6.2) with $x = \nabla u$

$$
|\nabla u|^2 = \nabla u \cdot \nabla u = (\nu \cdot \nu)\left|\frac{\partial u}{\partial \nu}\right|^2 = \left|\frac{\partial u}{\partial \nu}\right|^2.
$$

Therefore,

$$
\int_{\partial\Omega} \nu \cdot \left(\nabla u(x \cdot \nabla u) - x\frac{|\nabla u|^2}{2} + xG(u)\right) dx = \int_{\partial\Omega} \nu \cdot \left(\left|\frac{\partial u}{\partial \nu}\right|^2 x - \frac{1}{2}x\left|\frac{\partial u}{\partial \nu}\right|^2\right) dx.
$$

And the assertion follows.

□

This theorem has an immediate application which proves the uniqueness of the solution of the following boundary problem:

Theorem 6.1.2. *Let $n \geq 3$ and $\Omega \subset \mathbb{R}^n$ be a smooth star domain with respect to the origing s.t.*

$$x \cdot \nu > 0$$

for all $x \in \partial\Omega$. Let $u \in H_0^1(\Omega)$ be a weak solution of

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{6.3}$$

Then, if $\lambda \leq 0$, $u = 0$ a.e..

We only prove the case $\lambda < 0$. The case $\lambda = 0$ can be treated using the **unique extension principle**.

Proof. Let $\lambda \neq 0$ and $u \in H_0^1(\Omega)$ be a weak solution of (6.3). By the theory of regularity, u is also a classical solution of (6.3), so that we can use Pohožaev identity where, since $g(u) = \lambda u + |u|^{2^*-2}u$,

$$G(u) = \int_0^u g(t)dt = \frac{\lambda}{2}|u|^2 + \frac{1}{2^*}|u|^{2^*}.$$

When applied to G, (P) becomes:

$$\frac{n-2}{2}\int_\Omega |\nabla u|^2 dx - n\int_\Omega \left(\frac{\lambda}{2}|u|^2 + \frac{1}{2^*}|u|^{2^*}\right) dx + \frac{1}{2}\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 x \cdot \nu dx = 0.$$

So that, by multiplying all the terms by $\frac{2}{n-2}$ and observing that $\frac{2n}{n-2} = 2^*$, we get

$$\int_\Omega |\nabla u|^2 dx - 2^*\left(\frac{\lambda}{2}\int_\Omega |u|^2 dx + \frac{1}{2^*}|u|^{2^*} dx\right) + \frac{1}{n-2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 x \cdot \nu dx = 0. \tag{6.4}$$

We consider again the differential equation of (6.3), multiply its bot sides by u and integrate in Ω:

$$\int_\Omega -u\Delta u dx = \lambda \int_\Omega |u|^2 dx + \int_\Omega |u|^{2^*} dx.$$

Using Green's formulas on the first integral, we get

$$\int_\Omega |\nabla u|^2 dx = \lambda \int_\Omega |u|^2 dx + \int_\Omega |u|^{2^*} dx. \tag{6.5}$$

Putting (6.4) and (6.5) together, we get

$$\lambda\left(1 - \frac{n}{n-2}\right)\int_\Omega |u|^2 dx + \frac{1}{n-2}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 x \cdot \nu dx = 0$$

and, working around some computation:

$$\underbrace{\underbrace{-2\lambda}_{>0}\int_\Omega |u|^2 dx}_{\geq 0} + \underbrace{\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 \underbrace{x \cdot \nu}_{>0} dx}_{\geq 0} = 0. \tag{6.6}$$

This sum can be zero if and only if both its addenda are. In particular, $\|u\|_2 = 0$, so that $u = 0$ in L^2, that is $u = 0$ a.e. in Ω. However u has a continuous version that is a classical solution of (6.3), so that $u \equiv 0$ in Ω.

□

Example 6.1.3. We consider the problem of Example (5.2.4) with $p = 2^* - 1$ and a star domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u & \text{on } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

We saw in Example (5.2.4), where the subcritical case was treated, that (in the subcritical case) this problem admits infinitely many solutions whenever $\Omega \subset\subset \mathbb{R}^n$. But, by Theorem (6.1.2), in the case $\lambda = 0$, this problem admits 1! solution in the critical case.

Remark 6.1.4. Observe that the hypothesis of Ω to be a star-domain was crucial to reach the conclusion in (6.6). Without this assumption, in fact, the theorem loses its validity: for instance we consider the annulus

$$A = \{x \in \mathbb{R}^n : R_0 < |x| < R_1\}.$$

It can be proved that there are radial weak solutions of (6.3) on A. That is, solutions

$$u \in H^1_{rad}(A) := \{u \in H^1(A) : u(x) = u(|x|)\}.$$

To find them, it's enough to write $-\Delta$ in polar coordinates, observing that the singularity of $-\Delta$ is irrelevant in the polar coordinates when working on A (as it's an annulus around 0) and observing that

$$H^1_{rad}(A) \subset\subset \mathcal{C}^{0,\alpha}(\bar{A})$$

for all $\alpha \in [0, 1/2)$ and

$$H^1_{rad}(A) \subset\subset L^p(A)$$

for all $p \in [1, +\infty]$ (hence, all the inclusions are compact).
Observe that, for this reasons, the associate energy, defined on $H^1_{rad}(A)$, satisfies (PS).

6.2 The blow-up phenomenon

In the previous section, we saw how (as in Example (6.1.3)), one switches from having infinitely many solutions (in the subcritical case) to the uniqueness of solutions (in the critical case). This is due to the fact that the embedding $H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact.
This is easy to see: consider $\varphi \in \mathcal{C}^\infty_C(\Omega)$ and the dilations

$$\varphi_\lambda(x) = \lambda^{n/2^*}\varphi(\lambda x), \quad (\lambda > 0) \tag{6.7}$$

(observe that $\text{supp}(\varphi_\lambda) = \frac{1}{\lambda}\text{supp}(\varphi)$).
For all $\lambda > 0$, we have that

$$\|\varphi_\lambda\|_{H_0^1} = \|\varphi\|_{H_0^1},$$

so that $\{\varphi_k\}_{k>0}$ is bounded. We show that it cannot admit subsequences that converge in L^{2^*}.
For all $q \in [1, 2^*]$ and $\lambda > 0$ one has $\varphi_\lambda \in L^q(\Omega)$ with

$$\|\varphi_\lambda\|_q^q = \int_\Omega |\varphi_\lambda|^q dx = \lambda^{n\left(\frac{q}{2^*}-1\right)} \|\varphi\|_q^q.$$

It follows that, as $k \to +\infty$

$$\|\varphi_k\|_q^q \longrightarrow \begin{cases} 0 & \text{if } q \in [1, 2^*) \text{ and} \\ \|\varphi\|_{2^*}^{2^*} & \text{if } q = 2^*. \end{cases}$$

While, for a.e. $x \in \Omega$, $\varphi_k(x) \longrightarrow 0$ as $k \to +\infty$ (since $\text{supp}(\varphi_\lambda)$ shrinks around a point as $\lambda \to +\infty$).
We deduce that $\varphi_k \to 0$ in the norm of L^q when $q \in [1, 2^*)$.
When $q = 2^*$, instead, the so-called **blow-up** phenomenon takes place: $\text{supp}(\varphi_k)$ shrinks more and more, while the L^{2^*} norm of φ_k is preserved.
The geometric interpretation is the following: the functions φ_k present a peak around $x_0 \in \Omega$ which grows more and more, while the support shrinks around x_0.
Analytically, if $\{\varphi_k\}_k$ would admit a subsequence $\{\varphi_{k_j}\}_j$ converging in the norm of L^{2^*} to a function g, this would also converge a.e. to 0 by construction, so that $g = 0$ a.e.. However, $\varphi_{k_j} \not\to 0$ in L^{2^*}. This is a contradiction.

6.3 Brezis-Nirenberg theorem

Let $n > 2$. We know that, by Sobolev theorem, if $\Omega \subset\subset \mathbb{R}^n$

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega)$$

for all $q \in [1, 2^*]$. The embedding, still by Sobolev theorem, is continuous: i.e. there exists $C = C(n, \Omega, q)$ s.t.

$$\|u\|_q \leq C \|u\|_{H_0^1}. \tag{6.8}$$

The family of contants $\{C : (6.8) \text{ holds}\}$ is a non-empty subset of $(0, +\infty)$, so that it admits an infimum $S_{n,q}^{-1} = \sup_{\|u\|_{H_0^1}=1} \|u\|_q^2$, that is

$$S_{n,q} := \inf_{u \in H_0^1} \frac{\|u\|_{H_0^1}^2}{\|u\|_q^2}.$$

We ask whether a function that minimizes the Sobolev constant $\bar{u} \in H_0^1$ does exist. More precisely, this function shall have the following property:

$$\|\bar{u}\|_{H_0^1}^2 = S_{n,q} \|\bar{u}\|_q^2.$$

It happens that, if $q \in [1, 2^*)$, this occurs, while if such a minimizing function would exist in the case $q = 2^*$, a contradiction with the case $\lambda = 0$ of (6.1.2) would arise.
However, the equation

$$-\Delta u = |u|^{2^*-2}u \quad (\text{in } \mathbb{R}^n) \tag{6.9}$$

admits an explicit solution:

$$u^*(x) = \frac{[n(n-2)]^{\frac{n-2}{4}}}{(1+|x|^2)^{\frac{n-2}{2}}}$$

called the **Talenti function**. Actually, for all $\varepsilon > 0$ the functions

$$u^*_\varepsilon(x) = \frac{[n(n-2)\varepsilon^2]^{\frac{n-2}{4}}}{(\varepsilon^2+|x|^2)^{\frac{n-2}{2}}} \tag{6.10}$$

still provide solutions of (6.9).

Definition 6.3.1. The functions in (6.10) are called **Talenti functions**.

Observe that, for all $\varepsilon > 0$, under the same notation as (6.7),

$$u^*_\varepsilon(x) = u^*_\lambda(x) \quad \text{with } \lambda = \frac{1}{\varepsilon}$$

and

$$\|\nabla u^*_\varepsilon\|_2^2 = S_{n,2^*}^{n/2} = \|u^*_\varepsilon\|_{2^*}^{2^*},$$

from which we derive

$$\frac{\|\nabla u^*_\varepsilon\|_2^2}{\|u^*_\varepsilon\|_{2^*}^2} = \frac{\|\nabla u^*_\varepsilon\|_2^2}{\left(\|u^*_\varepsilon\|_{2^*}^{2^*}\right)^{2/2^*}} = \frac{S_{n,2^*}^{n/2}}{S_{n,2^*}^{n/2^*}} = S_{n,2^*}.$$

This argument fails if $\Omega \neq \mathbb{R}^n$. In fact, in this case, the u^*_ε are still solutions of (6.9) on Ω, but they must be multiplied by a cut-off function in order for them to belong to $H^1_0(\Omega)$. On the other hand, it can be seen that these new functions still provide good approximations of $S_{n,2^*}$, but the multiplications by the cut-off prevents them to actually *take the value* $S_{n,2^*}$.
We will better discuss the computation needed in the previous discussion in the next result:

Theorem 6.3.2 (Brezis-Nirenberg)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ with $n \geq 3$ and let λ_1 be the principal eigenvalue of Laplace operator. Consider the problem*

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1} & \textit{on } \Omega, \\ u|_{\partial\Omega} = 0, \\ u > 0 & \textit{on } \Omega. \end{cases} \tag{6.11}$$

(a) *if* $n \geq 4$ *and* $\lambda \in (0, \lambda_1)$, *(6.11) admits a solution;*

(b) *if* $n = 3$, *there exists* $\lambda^* > 0$ *s.t.*

- *if* $\lambda \in (\lambda^*, \lambda_1)$, *(6.11) admits a solution;*
- *if* $\lambda \in (0, \lambda^*)$, *(6.11) has no solution.*

The proof of Theorem (6.3.2) relies on two results related to the functional associated to the equation of (6.11) in H_0^1, that is the mapping $I_\lambda \in \mathcal{C}^1(H_0^1(\Omega), \mathbb{R})$ defined by

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx.$$

Remark 6.3.3. I_0 does not satisfy (PS) as (it can be shown) using the Talenti functions $\{u^*_{1/k}\}_k$.
If Ω is bounded, we observed that the Talenti functions, appropriately *cut*, provide a good approximation of the best Sobolev constant $S_{n,2^*}$ with

$$\left\|\nabla u^*_{1/k}\right\|_2^2 = S_{n,2^*}^{n/2} + o(1), \qquad \left\|u^*_{1/k}\right\|_{2^*}^{2^*} = S_{n,2^*}^{n/2} + o(1)$$

as $k \to \infty$. Thanks to this estimates, we get

$$I_0(u^*_{1/k}) = \frac{1}{2}\left(S_{n,2^*}^{n/2} + o(1)\right) - \frac{1}{2^*}\left(S_{n,2^*}^{n/2} + o(1)\right) = \frac{1}{n}S_{n,2^*}^{n/2} + o(1)$$

as $k \to \infty$.
Actually, one can verify that the sequences $\{u_k\}_k$ for which $I_\lambda(u_k) \to c$ with $c < \frac{1}{n}S_{n,2^*}^{n/2}$ s.t. $I'(u_k) \to 0$ admit converging subsequences. In this sense, the compactness can be recovered in the critical case modifying (PS) in order for the sequences for which $I(u_k) \to c$ as $k \to \infty$ to be the only ones taken into account. This is exactly what we do now.

Definition 6.3.4. Let $J \in \mathcal{C}^1(E, \mathbb{R})$ be a functional. We say that J **satisfies the Palais-Smale condition at a level** c and we write "J satisfies $(\mathrm{PS})_c$" if whenever $\{u_k\}_k \subset E$ is a sequence s.t. $J(u_k) \to c$ and $J'(u_k) \to 0$ as $k \to +\infty$, there exists a subsequence $\{u_{k_j}\}_j \subseteq \{u_k\}_k$ that converges in E.

The following is an enhanced version of Fatou's lemma, which "measures" the difference between $\int_\Omega \liminf f_k dx$ and $\liminf \int_\Omega f_k dx$.

Lemma 6.3.5 (Brezis-Lieb)**.** *Let* $q \in [1, +\infty)$ *and* $\{f_k\}_k \subset L^q(\Omega)$ *be a bounded sequence s.t.* $f_k(x) \to f(x)$ *a.e. in* Ω. *Then,* $f \in L^q(\Omega)$ *and*

$$\|f\|_q^q = \lim_{k\to\infty}\left(\|f_k\|_q^q - \|f_k - f\|_q^q\right).$$

Proof. We observe that, by Fatou's Lemma,

$$\int_\Omega |f|^q dx \le \liminf_{k\to+\infty} \int_\Omega |f_k|^q dx \le \sup_{k\ge1} \|f_k\|_q^q .$$

Hence, $f \in L^q(\Omega)$.

1. **We prove that for all $s \in \mathbb{R}$ and all $\varepsilon > 0$, there exists a constant C_ε s.t.**

$$||s+1|^q - |s|^q - 1| \le \varepsilon |s|^q + C_\varepsilon.$$

This inequality, by homogeneity, can be read in the following terms: for all $a, b \in \mathbb{R}$, $\forall \varepsilon > 0$ there exists $C_\varepsilon > 0$ s.t.

$$||a+b|^q - |a|^q - |b|^q| \le \varepsilon |a|^q + C_\varepsilon |b|^q. \tag{6.12}$$

2. **We apply the inequality as follows:**
we fix $\varepsilon > 0$ and let $M := \sup_{k\ge1} \|f_k\|_q$, $u_k := ||f_k|^q - |f_k - f|^q - |f|^q|$ and $v_k := (u_k - \varepsilon |f_k - f|^q)^+$. Observe that $v_k(x), u_k(x) \to 0$ for a.e. $x \in \Omega$ because of the a.e. convergence of f_k to f.
We have to prove that

$$\lim_{k\to\infty} \int_\Omega (|f_k|^q - |f_k - f|^q - |f|^q)\, dx = 0.$$

We have:

$$\left| \int_\Omega (|f_k|^q - |f_k - f|^q - |f|^q) dx \right| \le \int_\Omega u_k dx.$$

Now, $v_k = (u_k - \varepsilon|f_k - f|^q)^+ \ge u_k - \varepsilon|f_k - f|^q$, so that

$$\int_\Omega u_k dx \le \int_\Omega v_k dx + \varepsilon \int_\Omega |f_k - f|^q dx \le \varepsilon \left(\|f_k\|_q + \|f\|_q \right)^q + \int_\Omega v_k dx \le \varepsilon C^q + \int_\Omega v_k dx$$

for some appropriate $C > 0$.
We use (6.12) with $a = f_k - f$ and $b = f$, to get the L^1 dominating function:

$$0 \le v_k = |v_k| \le \varepsilon |f - f_k|^q + C_\varepsilon |f|^q \le C_{(\varepsilon)} |f|^q \in L^1(\Omega).$$

By the dominated convergence theorem, we have

$$\lim_{k\to+\infty} \int_\Omega u_k dx \le \varepsilon C^q$$

$\forall \varepsilon > 0$, hence

$$\lim_{k\to+\infty} \int_\Omega (|f_k|^q - |f_k - f|^q - |f|^q)\, dx = 0$$

by the squeeze theorem.

□

Proposition 6.3.6. *I_λ satisfies (PS)$_c$ for all $c \in \left(0, \frac{1}{n} S_{n,2^*}^{n/2}\right)$.*

Proof. Let $\{u_k\}_k \subset H_0^1(\Omega)$ be a sequence s.t. $I_\lambda(u_k) \to c$ and $I'(u_k) \to 0$ as $k \to +\infty$.

1. **We prove that the sequence $\{u_k\}_k$ is bounded in H_0^1:** as $k \to \infty$, it is

$$2I_\lambda(u_k) - I'_\lambda(u_k)[u_k] \leq \frac{2}{n} S_{n,2^*}^{n/2} + o(1) \|u_k\|_{H_0^1}. \tag{6.13}$$

Outlining the left hand side of the inequality, we get:

$$\begin{aligned} 2I_\lambda(u_k) - I'_\lambda(u_k)[u_k] &= \int_\Omega |\nabla u_k|^2 dx - \lambda \int_\Omega |u_k|^2 dx - \frac{2}{2^*} \int_\Omega |u_k|^{2^*} dx + \\ &- \int_\Omega |\nabla u_k|^2 dx + \lambda \int_\Omega |u_k|^2 dx + \int_\Omega |u_k|^{2^*} dx = \\ &= \left(1 - \frac{2}{2^*}\right) \int_\Omega |u_k|^{2^*} dx. \end{aligned}$$

So that, observing that the first term of the right hand side of (6.13) is constant,

$$\underbrace{\left(1 - \frac{2}{2^*}\right)}_{=:\,\delta > 0} \|u_k\|_{2^*}^{2^*} \leq C + o(1) \|u_k\|_{H_0^1}.$$

Starting from the definition of I_λ and using this latest inequality:

$$\begin{aligned} \|u_k\|_{H_0^1}^2 &= 2I_\lambda(u_k) + \lambda \underbrace{\|u_k\|_2^2}_{\leq C \|u_k\|_{2^*}^2} + \frac{2}{2^*} \|u_k\|_{2^*}^{2^*} \leq 2I_\lambda(u_k) + C \|u_k\|_{2^*}^{2^*} \leq \\ &\leq C + o(1) \|u_k\|_{H_0^1}. \end{aligned}$$

Hence, the $\{u_k\}_k$ cannot be bounded in H_0^1.

2. **We provide the candidate limit:** by the Banach-Alaoglu theorem, there exists a subsequence $\{u_{k_j}\}_j$ that converges weakly to a function $\bar{u} \in H_0^1(\Omega)$.
By Rellich-Kondrachov theorem, $u_{k_j} \to \bar{u}$ in the norm of $L^q(\Omega)$ for all $q \in [1, 2^*)$.
With an abuse of notation, we denote with $\{u_k\}_k$ the above-mentioned subsequence.

3. **We prove that $u_k \to \bar{u}$ as $k \to \infty$ in H_0^1.**
Let $\varphi \in \mathcal{C}_C^\infty(\Omega)$. Then,

$$I'_\lambda(u_k)[\varphi] = \int_\Omega \nabla u_k \nabla \varphi dx - \int_\Omega |u_k|^{2^*-2} u_k \varphi dx - \lambda \int_\Omega u_k \varphi dx.$$

Taking the limit and using the standard density argument of $\mathcal{C}^\infty_C(\Omega) \subset H^1_0(\Omega)$ we get that, for all $v \in H^1_0(\Omega)$:

$$0 = \int_\Omega \nabla\bar{u}\nabla v dx - \int_\Omega |\bar{u}|^{2^*-2}\bar{u}v dx - \lambda\int_\Omega \bar{u}v dx. \tag{6.14}$$

In particular, for $v = \bar{u}$:

$$\int_\Omega |\nabla\bar{u}|^2 dx - \int_\Omega |\bar{u}|^{2^*} dx - \lambda\int_\Omega |\bar{u}|^2 dx = 0.$$

Then, we get

$$\begin{aligned} I_\lambda(\bar{u}) &= \frac{1}{2}\int_\Omega |\nabla\bar{u}|^2 dx - \frac{\lambda}{2}\int_\Omega |\bar{u}|^2 dx - \frac{1}{2^*}\int_\Omega |\bar{u}|^{2^*} dx = \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right)\int_\Omega |\bar{u}|^{2^*} dx = \frac{1}{n}\int_\Omega |\bar{u}|^{2^*} dx \geq 0. \end{aligned} \tag{6.15}$$

Having to prove that $\|\bar{u}_k - u\|_{H^1_0} \to 0$ as $k \to \infty$, we compute $\int_\Omega |\nabla(\bar{u} - u_k)|^2 dx$:

$$\begin{aligned} \int_\Omega |\nabla\bar{u} - \nabla u_k|^2 dx &= \int_\Omega |\nabla u_k|^2 dx + \int_\Omega |\nabla\bar{u}|^2 dx - 2\underbrace{\int_\Omega \nabla u_k\nabla\bar{u} dx}_{\substack{= \int_\Omega |\nabla\bar{u}|^2 dx + o(1) \\ \text{as } k\to\infty}} = \\ &= \int_\Omega |\nabla u_k|^2 dx - \int_\Omega |\nabla\bar{u}|^2 dx + o(1) \end{aligned} \tag{6.16}$$

where we used the weak convergence of $\{u_k\}_k$ in H^1_0.
Moreover, by Brezis-Lieb lemma, as $k \to \infty$,

$$\int_\Omega |u_k|^{2^*} dx = \int_\Omega |u_k - \bar{u}|^{2^*} dx + \int_\Omega |\bar{u}|^{2^*} dx + o(1). \tag{6.17}$$

Using (6.16), the convergence in L^2 and (6.17):

$$\begin{aligned} I_\lambda(u_k) &= \frac{1}{2}\underbrace{\int_\Omega |\nabla u_k|^2 dx}_{\text{we use (6.16)}} - \frac{\lambda}{2}\underbrace{\int_\Omega |u_k|^2 dx}_{= \int_\Omega |\bar{u}|^2 dx + o(1)} - \frac{1}{2^*}\underbrace{\int_\Omega |u_k|^{2^*} dx}_{\text{we use (6.17)}} = \\ &= \frac{1}{2}\left(\underline{\int_\Omega |\nabla(\bar{u} - u_k)|^2 dx} + \underline{\underline{\int_\Omega |\nabla\bar{u}|^2 dx}}\right) + \\ &\underline{\underline{-\frac{\lambda}{2}\int_\Omega |\bar{u}|^2 dx}} - \frac{1}{2^*}\left(\underline{\int_\Omega |u_k - \bar{u}|^{2^*} dx} + \underline{\underline{\int_\Omega |\bar{u}|^{2^*} dx}}\right) + o(1) = \\ &= I_\lambda(\bar{u}) + I_0(u_k - \bar{u}) + o(1). \end{aligned} \tag{6.18}$$

Now, $I'_\lambda(u_k) \to 0$ as $k \to \infty$. Hence, as $k \to \infty$,

$$o(1) = I'_\lambda(u_k)[u_k - \bar{u}] - I'_\lambda(\bar{u})[u_k - \bar{u}] \underset{\text{def. of } I'}{=}$$

$$= \underbrace{\int_\Omega \nabla u_k \nabla(u_k - \bar{u})dx}_{\substack{= \int_\Omega |\nabla(u_k-\bar{u})|^2 dx + o(1) \\ \text{by the weak conv.}}} - \int_\Omega |u_k|^{2^*-2} u_k(u_k - \bar{u})dx - \lambda \underbrace{\int_\Omega u_k(u_k - \bar{u})dx}_{\substack{= \|u_k - \bar{u}\|_2 + (\bar{u}, u_k - \bar{u})_2 \\ = o(1) \text{ by } L^2 \text{ conv.}}} +$$

$$- \underbrace{\int_\Omega \nabla \bar{u} \nabla(u_k - \bar{u})dx}_{\substack{= (\bar{u}, u_k - \bar{u})_{H_0^1} = o(1) \\ \text{by weak conv.}}} + \int_\Omega |\bar{u}|^{2^*-2}\bar{u}(u_k - \bar{u})dx + \lambda \underbrace{\int_\Omega \bar{u}(u_k - \bar{u})dx}_{\substack{= (\bar{u}, u_k - \bar{u})_{L^2} = o(1) \\ \text{by } L^2 \text{ conv.}}} =$$

$$= \int_\Omega |\nabla(u_k - \bar{u})|^2 dx - \int_\Omega \left(|u_k|^{2^*-2}u_k - |\bar{u}|^{2^*-2}\bar{u}\right)(u_k - \bar{u})dx + o(1) =$$

$$= \int_\Omega |\nabla(u_k - \bar{u})|^2 dx - \int_\Omega |u_k|^{2^*}dx + \underbrace{\int_\Omega |u_k|^{2^*-2}u_k\bar{u}dx + \int_\Omega |\bar{u}|^{2^*-2}\bar{u}u_k dx}_{= 2\int_\Omega |\bar{u}|^{2^*}dx + o(1) \text{ by } L^2 \text{ conv.}} +$$

$$- \int_\Omega |\bar{u}|^{2^*}dx + o(1) = \int_\Omega |\nabla(u_k - \bar{u})|^2 dx \underbrace{- \int_\Omega |u_k|^{2^*}dx + \int_\Omega |\bar{u}|^{2^*}dx}_{= \|\bar{u} - u_k\|_{2^*}^{2^*} + o(1) \text{ by Brezis-Lieb}} + o(1).$$

Therefore, as $k \to \infty$

$$\int_\Omega |u_k - \bar{u}|^{2^*}dx = \int_\Omega |\nabla(u_k - \bar{u})|^2 dx + o(1). \tag{6.19}$$

We use (6.19) to write the term $I_0(u_k - \bar{u})$ in (6.18) as follows:

$$I_\lambda(u_k) = I_\lambda(\bar{u}) + \frac{1}{2}\int_\Omega |\nabla(u_k - \bar{u})|^2 dx - \frac{1}{2^*}\int_\Omega |\nabla(u_k - \bar{u})|^2 dx + o(1) =$$
$$= I_\lambda(\bar{u}) + \frac{1}{n}\int_\Omega |\nabla(\bar{u} - u_k)|^2 dx + o(1).$$

Hence, as $k \to \infty$

$$\begin{aligned} \frac{1}{n}\int_\Omega |\nabla(u_k - \bar{u})|^2 dx &= I_\lambda(u_k) - \underbrace{I_\lambda(\bar{u})}_{\geq 0 \text{ by } (6.15)} + o(1) \leq \\ &\leq \underbrace{I_\lambda(u_k)}_{= c + o(1)} + o(1) \leq c + o(1) < \frac{1}{n}S_{n,2^*}^{n/2}. \end{aligned} \tag{6.20}$$

On the other hand, recalling the definition of $S_{n,2^*}$:

$$S_{n,2^*} := \inf_{u \in H_0^1} \frac{\|u\|_{H_0^1}^2}{\|u\|_{2^*}^2} \Longrightarrow S_{n,2^*} \leq \frac{\|u_k - \bar{u}\|_{H_0^1}^2}{\|\bar{u} - u_k\|_{2^*}^2} \Longrightarrow \|u_k - \bar{u}\|_{2^*}^2 \leq \frac{1}{S_{n,2^*}}\|u_k - \bar{u}\|_{H_0^1}^2.$$

Therefore, by (6.19), we have

$$o(1) = \int_\Omega |\nabla(u_k - \bar{u})|^2 dx - \int_\Omega |u_k - \bar{u}|^{2^*} dx \geq \|u_k - \bar{u}\|^2_{H^1_0} - \frac{1}{S^{2^*/2}_{n,2^*}} \|u_k - \bar{u}\|^{2^*}_{H^1_0} =$$

$$= \|u_k - \bar{u}\|^2_{H^1_0} \left(1 - \frac{\|u_k - \bar{u}\|^{2^*}_{H^1_0}}{S^{2^*/2}_{n,2^*}}\right).$$

Hence, as $k \to \infty$,

$$\|u_k - \bar{u}\|_{H^1_0} \to 0,$$

or, as $k \to \infty$

$$\|u_k - \bar{u}\|^2_{H^1_0} = S^{n/2}_{n,2^*} + o(1).$$

However, by (6.20), we know that, as $k \to \infty$

$$\|u_k - \bar{u}\|^2_{H^1_0} \leq nc + o(1) < \beta < S^{n/2}_{n,2^*}$$

for an appropriate β. Hence, the second possibility must be excluded.

And we're done. □

Remark 6.3.7. One may prove that, actually, (6.3.6) holds for $c \neq \frac{k}{n} S^{n/2}_{n,2^*}$ for all the integers $k \geq 1$. In this situations, one talks about **loss of compactness levels quantization**.

Proof of (6.3.2). The idea is that of using an adapted version of the mountain pass theorem (4.3.1) using $(\mathrm{PS})_c$ instead of (PS), an appropriate family of paths Γ and the functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega |u|^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx.$$

We observe that

- $I_\lambda(0) = 0$ trivially;
- $I \in \mathcal{C}^1(H^1_0(\Omega), \mathbb{R})$;
- similarly as what we did in (4.3.2), we can prove that there exist $\alpha, \rho > 0$ s.t.

$$I_\lambda|_{\partial B_\rho(0)} \geq \alpha.$$

1. **We show some useful estimates**.
   We start by turning the Talenti functions

$$u^*_\varepsilon(x) = [n(n-2)]^{\frac{n-2}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{n-2}{2}}$$

into $H_0^1(\Omega)$ functions.
Actually, Talenti functions do not belong to $H_0^1(\Omega)$. For this reason, we multiply them by an appropriate cut-off function $\eta \in \mathcal{C}_C^\infty(\Omega)$ s.t. $\eta|_{B_\rho(0)} \equiv 1$, defining $\forall \varepsilon > 0$,

$$u_\varepsilon := \eta u_\varepsilon^*.$$

Since $u_\varepsilon|_{\partial\Omega} = 0$, for all $\varepsilon > 0$, we have $u_\varepsilon \in H_0^1(\Omega)$.
Now, observing that $\nabla\eta \equiv 0$ on $B_\rho(0)$, since η is constant there,

$$\int_\Omega |\nabla u_\varepsilon|^2 dx \underset{\substack{\text{def.}\\ \text{di } u_\varepsilon}}{=} \int_\Omega |\nabla(\eta u_\varepsilon^*)|^2 dx = \int_\Omega \underbrace{|u_\varepsilon^* \nabla\eta + \eta \nabla u_\varepsilon^*|^2}_{\text{seen as an inner prod.}} \, dx =$$

$$= \underbrace{\int_\Omega \eta^2 |\nabla u_\varepsilon^*|^2 dx}_{=: L} + \underbrace{2\int_{\Omega\setminus B_\rho(0)} \eta u_\varepsilon^* \nabla\eta \cdot \nabla u_\varepsilon^* dx}_{=: J} + \underbrace{\int_{\Omega\setminus B_\rho(0)} |\nabla\eta|^2 |u_\varepsilon^*|^2 dx}_{=: K}.$$

We estimate L, J e K as follows (from now on C *absorbs* all the constants at stake):

$$K \le \max_{\Omega\setminus B_\rho(0)} |\nabla\eta|^2 \int_{\Omega\setminus B_\rho(0)} |u_\varepsilon^*|^2 dx = C\varepsilon^{n-2} \int_\rho^M \left(\frac{1}{\varepsilon^2 + r^2}\right)^{n-2} r^{n-1} dr$$

where $M = \sup_{x\in\Omega} |x| < \infty$ as Ω is bounded.
We observe that, since

$$\int_\rho^M \left(\frac{1}{\varepsilon^2 + r^2}\right)^{n-2} r^{n-1} dr \le \int_\rho^M \frac{r^{n-1}}{r^{2(n-2)}} dr = \int_\rho^M \frac{1}{r^{n-3}} dr \le C$$

we have, as $\varepsilon \to 0$,

$$0 \le K \le C\varepsilon^{n-2} \Longrightarrow K = O(\varepsilon^{n-2}).$$

Similarily, using Cauchy-Schwarz inequality,

$$|J| \le \max_{\Omega\setminus B_\rho(0)} |\eta\nabla\eta| \int_{\Omega\setminus B_\rho(0)} |u_\varepsilon^*||\nabla u_\varepsilon^*| dx \le C\varepsilon^{n-2}$$

that is, as $\varepsilon \to 0$

$$J = O(\varepsilon^{n-2}).$$

Finally, using the fact that $\mathrm{supp}(\eta) \subset \Omega$, observing that $(\eta^2 - 1)|_{B_\rho(0)} \equiv 0$, using Talenti functions' properties e computing:

$$L = \int_{\mathbb{R}^n} \eta^2 |\nabla u_\varepsilon^*|^2 dx = \underbrace{\int_{\mathbb{R}^n} |\nabla u_\varepsilon^*|^2 dx}_{= S_{n,2^*}^{n/2}} + \underbrace{\int_{\mathbb{R}^n\setminus B_\rho(0)} (\rho^2 - 1)|\nabla u_\varepsilon^*|^2 dx}_{= O(\varepsilon^{n-2})}.$$

To sum up, as $\varepsilon \to 0$,

$$\int_\Omega |\nabla u_\varepsilon|^2 dx = S_{n,2^*}^{n/2} + O(\varepsilon^{n-2}).$$

Moreover, using again Talenti functions' properties, one has

$$\int_\Omega |u_\varepsilon|^{2^*} dx = \int_\Omega |u_\varepsilon^*|^{2^*} dx = S_{n,2^*}^{n/2} + O(\varepsilon^n)$$

as $\varepsilon \to 0$.
Finally, using the same argument,

$$\begin{aligned}\int_\Omega |u_\varepsilon|^2 dx &= \int_\Omega |u_\varepsilon^*|^2 dx + \int_\Omega (1-\eta^2)|\nabla u_\varepsilon|^2 dx = \\ &= \int_{B_\rho(0)} |u_\varepsilon^*|^2 dx + \int_{\Omega \setminus B_\rho(0)} |u_\varepsilon^*|^2 dx + \int_\Omega (1-\eta^2)|\nabla u_\varepsilon|^2 dx = \\ &= \int_{B_\rho(0)} |u_\varepsilon^*|^2 dx + O(\varepsilon^{n-2}).\end{aligned}$$

We conclude that, called ω_n the measure of the unit sphere of $\mathbb{R}^n$, as $\varepsilon \to 0$,

$$\begin{aligned}\int_\Omega |u_\varepsilon|^2 dx &= \omega_n \int_0^\rho \frac{C\varepsilon^{n-2}}{(\varepsilon^2+r^2)^{n-2}} r^{n-1} dr + O(\varepsilon^{n-2}) = \\ &= C\varepsilon^{n-2}\left(\int_0^\varepsilon \frac{r^{n-1}}{(\varepsilon^2+r^2)^{n-2}} dr + \int_\varepsilon^\rho \frac{r^{n-1}}{(\varepsilon^2+r^2)^{n-2}} dr\right) + O(\varepsilon^{n-2}) \geq \\ &\geq C\varepsilon^{n-2}\left(\int_0^\varepsilon \frac{r^{n-1}}{(2\varepsilon^2)^{n-2}} dr + \int_\varepsilon^\rho \frac{r^{n-1}}{(2r^2)^{n-2}} dr\right) + O(\varepsilon^{n-2}) \geq \\ &\geq \underbrace{C\varepsilon^{2-n}\left[\frac{r^n}{n}\right]_0^\varepsilon}_{=C\varepsilon^2} + C\varepsilon^{n-2}\int_\varepsilon^\rho r^{3-n} dr + O(\varepsilon^{n-2}) \geq \\ &\geq C\varepsilon^2 + \begin{cases} O(\varepsilon^{n-2}) & \text{if } n \geq 5, \\ C\varepsilon^2 \ln(\varepsilon) + O(\varepsilon^2) & \text{if } n = 4 \text{ and} \\ C\varepsilon + O(\varepsilon) & \text{if } n = 3. \end{cases}\end{aligned} \tag{6.21}$$

2. **We contruct** Γ.
For, we have to choose a function $e \in H_0^1(\Omega)$ acting as an end point for the curves that will belong to Γ.
Let $R > 0$. We observe that

$$I_\lambda(Ru_\varepsilon) = \frac{R^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 dx - \frac{R^2\lambda}{2}\int_\Omega |u_\varepsilon|^2 dx - \frac{R^{2^*}}{2^*}\underbrace{\int_\Omega |u_\varepsilon|^{2^*} dx}_{\geq 0} \to -\infty$$

as $R \to \infty$, since it is sum of powers of R, the biggest of which is R^{2^*} that is multiplied by a constant-in-R negative factor.
Hence, there exists R large enough s.t., set $e = Ru_\varepsilon$, we have

$$I_\lambda(e) < 0.$$

We define

$$\Gamma := \left\{\gamma \in \mathcal{C}^0([0,R], H_0^1(\Omega)) \text{ s.t. } \gamma(0) = 0 \text{ and } \gamma(R) = Ru_\varepsilon\right\}.$$

We observe that the mapping $r \longmapsto ru_\varepsilon$ is an element of Γ (that, therefore, it is non-empty). Moreover,

$$c := \inf_{\gamma\in\Gamma} \sup_{u\in\gamma([0,R])} I_\lambda(u) \le \sup_{r\in[0,R]} I_\lambda(ru_\varepsilon).$$

3. **We verify that** $c < S_{n,2^*}^{n/2}/n$.
If it didn't hold, since I satifies $(\mathrm{PS})_c$, the same argument as that in (4.3.1), that proves that "c" is a critical value for "J", proves that c is a critical value for I_λ. At the end, it would remain to distinguish the cases that appear in the claim of this theorem.
We compute explicitly the sup, observing that it is actually a maximum, as $I_\lambda|_{\partial B_\rho(0)} \ge \alpha > 0$, $I(0) = 0$, $I_\lambda(Ru_\varepsilon) < 0$ and I_λ is continuous.
Since I_λ is differentiable, we will have

$$\frac{d}{dr} I_\lambda(ru_\varepsilon) = 0$$

in the maximum of $\bar{r} \in (0,R)$. We use this observation to search the maximum. For, by

$$I_\lambda(ru_\varepsilon) = \frac{r^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 dx - \frac{r^2\lambda}{2}\int_\Omega |u_\varepsilon|^2 dx - \frac{r^{2^*}}{2^*}\int_\Omega |u_\varepsilon|^{2^*} dx$$

it's easy to deduce that

$$\frac{d}{dr} I_\lambda(ru_\varepsilon) = r\int_\Omega |\nabla u_\varepsilon|^2 dx - \lambda r\int_\Omega |u_\varepsilon|^2 dx - r^{2^*-1}\int_\Omega |u_\varepsilon|^{2^*} dx.$$

Hence, as $\varepsilon \to 0$,

$$r^{2^*-2} = \frac{\int_\Omega |\nabla u_\varepsilon|^2 dx - \lambda\int_\Omega |u_\varepsilon|^2 dx}{\int_\Omega |u_\varepsilon|^{2^*} dx} = \frac{S_{n,2^*}^{n/2} + O(\varepsilon^{n-2}) - \lambda C\varepsilon^2}{S_{n,2^*}^{n/2} + O(\varepsilon^n)} = 1 + \beta\varepsilon^2 + O(\varepsilon^{n-2}).$$

Using the Taylor series of $(1+\varepsilon_k)^a$, as $\varepsilon \to 0$

$$\bar{r} = 1 + \delta\varepsilon^2 + O(\varepsilon^{n-2}).$$

Then, using the estimates in 1.,

$$
\begin{aligned}
I_\lambda(\bar r u_\varepsilon) &= \frac{\bar r^2}{2}\int_\Omega |\nabla u_\varepsilon|^2 dx - \frac{\lambda \bar r^2}{2}\int_\Omega |u_\varepsilon|^2 dx - \frac{\bar r^{2^*}}{2^*}\int_\Omega |u_\varepsilon|^{2^*} dx = \\
&= \frac{1}{2}\underbrace{\left(1+\delta\varepsilon^2+O(\varepsilon^{n-2})\right)^2}_{=1+2\delta\varepsilon^2+O(\varepsilon^{n-2})}\left(S_{n,2^*}^{n/2}+O(\varepsilon^{n-2})\right)+ \\
&- \frac{1}{2^*}\underbrace{\left(1+\delta\varepsilon^2+O(\varepsilon^{n-2})\right)^{2^*}}_{=1+2^*\delta\varepsilon^2+O(\varepsilon^{n-2})}\left(S_{n,2^*}^{n/2}+O(\varepsilon^{n})\right) - \frac{\lambda}{2}\bar r^2\int_\Omega |u_\varepsilon|^2 dx = \\
&= \frac{1}{n}S_{n,2^*}^{n/2} + \left(\delta\varepsilon^2+O(\varepsilon^{n-2})\right)S_{n,2^*}^{n/2} - \left(\delta\varepsilon^2+O(\varepsilon^{n-2})\right)S_{n,2^*}^{n/2} - \frac{\lambda}{2}\bar r^2\int_\Omega |u_\varepsilon|^2 dx.
\end{aligned}
$$

Since, as $\varepsilon \to 0$,

$$\bar r = 1+\delta\varepsilon^2+O(\varepsilon^{n-2}),$$

for ε small enough, $\bar r \geq 1/2$ and we deduce that

$$I_\lambda(\bar r u_\varepsilon) \leq \frac{1}{n}S_{n,2^*}^{n/2} - \frac{\lambda}{4}\int_\Omega |u_\varepsilon|^2 dx.$$

Now, we distinguish the cases, in order to use (6.21):

- If $n \geq 5$,

  $$I_\lambda(\bar r u_\varepsilon) \leq \frac{1}{n}S_{n,2^*}^{n/2} - \frac{\lambda}{4}C\varepsilon^2 < \frac{1}{n}S_{n,2^*}^{n/2}.$$

  So that

  $$\sup_{r\in[0,R]} I_\lambda(r u_\varepsilon) < \frac{1}{n}S_{n,2^*}^{n/2}.$$

  The assertion follows taking the infimum.

- If $n = 4$

  $$I_\lambda(\bar r u_\varepsilon) \leq \frac{1}{4}S_{4,2^*}^{2} - \frac{\lambda}{4}\varepsilon^2|\ln(\varepsilon)| + O(\varepsilon^2) < \frac{1}{4}S_{4,2^*}^{2}.$$

Hence, for $\lambda > 0$, $c < \frac{1}{n}S_{n,2^*}^{n/2}$ whenever $n \geq 4$. Moreover, the request $c > 0$ forces us to choose $\lambda < \lambda_1$.

- if $n = 3$, we need more precise estimates in order to establish the estimate $c < \frac{1}{3}S_{3,2^*}^{3/2}$, so that, using (6.21), we could conclude that

  $$I_\lambda(\bar r u_\varepsilon) = \frac{1}{3}S_{3,2^*}^{3/2} - \frac{\lambda}{4}(C\varepsilon + O(\varepsilon)) = \frac{1}{3}S_{3,2^*}^{3/2} + O(\varepsilon).$$

  In this cases, we need more precise estimates on the behavior of $\int_\Omega |u_\varepsilon|^2 dx$. However, we observe that the dependence of these estimates on λ makes it necessary to consider λ large enough to have $c < S_{3,2^*}^{3/2}$.
  We don't treat this case in details.

4. **Conclusion** In the cases in which the assertion requires the existence of a solution, as already observed, one proceeds as usual proving that c is a critical value (through the deformation lemma, etc.).

$\square$

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